DYADIC WEIGHTS ON $\mathbb{R}^n$ AND REVERSE HÖLDER INEQUALITIES

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Abstract: We prove that for any weight ϕ defined on $[0,1]^n$ that satisfies a reverse Hölder inequality with exponent $p > 1$ and constant $c \geq 1$ upon all dyadic subcubes of $[0,1]^n$, it's non increasing rearrangement ϕ^*, satisfies a reverse Hölder inequality with the same exponent and constant not more than $2^n c - 2^n + 1$, upon all subintervals of $[0,1]$ of the form $[0,t]$, $0 < t \leq 1$. This gives as a consequence, an interval $[p, p_0(p,c)) = I_{p,c}$, such that for any $q \in I_{p,c}$, we have that $\phi \in L^q$.

1. Introduction

The theory of Muckenhoupt's weights has been proved to be an important tool in analysis. One of the most important facts about these is their self improving property. A way to express this is through the so called reverse Hölder inequalities (see [2], [3] and [7]).

Here we will study such inequalities on a dyadic setting. We will say that the measurable function $g : [0,1] \to \mathbb{R}^+$ satisfies the reverse Hölder inequality with exponent $p > 1$ and constant $c \geq 1$ if the inequality

$$\frac{1}{b-a}\int_a^b g^p(u)du \leq c\left(\frac{1}{b-a}\int_a^b g(u)du\right)^p, \tag{1.1}$$

holds for every subinterval of $[0,1]$.

In [1] it is proved the following

Theorem A. *Let g be a non-increasing function defined on $[0,1]$, which satisfies (1.1) on every interval $[a,b] \subseteq [0,1]$. Then if we define $p_0 > p$ as the root of the equation*

$$\frac{p_0 - p}{p_0}\left(\frac{p_0}{p_0 - 1}\right)^p \cdot c = 1, \tag{1.2}$$

we have that $g \in L^q([0,1])$, for any $q \in [p, p_0)$. Additionally g satisfies for every q in the above range a reverse Hölder inequality for possibly another real constant c'. Moreover the result is sharp, that is the value p_0 cannot be increased.

This research has been co-financed by the European Union and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF), aristeia code: MAXBELLMAN 2760, research code:70/3/11913.

MSC Number: 42B25

Now in [4] or [5] it is proved the following

Theorem B. *If $\phi : [0,1] \to \mathbb{R}^+$ is integrable satisfying (1.1) for every $[a,b] \subseteq [0,1]$, then it's non-increasing rearrangement ϕ^*, satisfies the same inequality with the same constant c.*

Here by ϕ^* we denote the non-increasing rearrangement of ϕ, which is defined on $(0,1]$ by

$$\phi^*(t) = \sup_{\substack{E\subseteq[0,1]\\|E|=t}} \left\{ \inf_{x\in E} \mid \phi(x) \mid \right\}, \qquad t \in (0,1].$$

This can be defined also as the unique left continuous, non-increasing function, equimeasurable to $|\phi|$, that is, for every $\lambda > 0$ the following equality holds:

$$\mid \{\phi > \lambda\} \mid = \mid \{\phi^* > \lambda\} \mid,$$

where by $| \cdot |$ we mean the Lesbesgue measure on $[0,1]$.

An immediate consequence of Theorem B, is that Theorem A can be generalized by ignoring the assumption of the monotonicity of the function g.

Recently in [8] it is proved the following

Theorem C. *Let $g : (0,1] \to \mathbb{R}^+$ be non-increasing which satisfies (1.1) on every interval of the form $(0,t]$, $0 < t \leq 1$. That is the following holds*

$$\frac{1}{t}\int_0^t g^p(u)du \leq c \cdot \left(\frac{1}{t}\int_0^t g(u)du\right)^p \tag{1.3}$$

for every $t \in (0,1]$. Then if we define p_0 by (1.2), we have that for any $q \in [p, p_0)$ the following inequality is true

$$\frac{1}{t}\int_0^t g^q(u)du \leq c'\left(\frac{1}{t}\int_0^t g(u)du\right)^q, \tag{1.4}$$

for every $t \in (0,1]$ and some constant $c' \geq c$. Thus $g \in L^q((0,1])$ for any such q. Moreover the result is sharp, that is we cannot increase p_0.

A consequence of Theorem C is that under the assumption that g is non-increasing, the hypothesis that (1.1) is satisfied only on the intervals of the form $(0,t]$ is enough for one to realize the existence of a $p' > p$ fir which $g \in L^{p'}([0,1])$.

In several dimensions, as far as we know, there does not exists any similar result as Theorems A, B and C. All we know is the following, which can be seen in [3].

Theorem D. *Let $Q_0 \subseteq \mathbb{R}^n$ be a cube and $\phi : Q_0 \to \mathbb{R}^+$ measurable that satisfies*

$$\frac{1}{\mid Q \mid}\int_Q \phi^p \leq c \cdot \left(\frac{1}{\mid Q \mid}\int_Q \phi\right)^p \tag{1.5}$$

for fixed constants $p > 1$ and $c \geq 1$ and every cube $Q \subseteq Q_0$. Then there exists $\varepsilon = \varepsilon(n,p,c)$ such that the following inequality holds;

$$\frac{1}{\mid Q \mid}\int_Q \phi^q \leq c'\left(\frac{1}{\mid Q \mid}\int_Q \phi\right)^q \tag{1.6}$$

for every $q \in [p, p+\varepsilon)$, any cube $Q \subseteq Q_0$ and some constant $c' = c'(q,p,n,c)$.

In several dimensions no estimate of the quantity ε, has been found. The purpose of this work is to study the multidimensional case in the dyadic setting. More precisely we consider a measurable function ϕ, defined on $[0,1]^n = Q_0$, which satisfies (1.5) for any Q, dyadic subcube of Q_0. These cubes can be realized by bisecting the sides of Q_0, then bisecting every side of a resulting dyadic cube and so on. We define by $\mathcal{T}_{2^n}$ the respective tree consisting of those mentioned dyadic subcubes of $[0,1]^n$. Then we will prove the following:

Theorem 1. *Let $\phi : Q_0 = [0,1]^n \ \to \ \mathbb{R}^+$ be such that*

$$\frac{1}{\mid Q \mid}\int_Q \phi^p \leq c \cdot \left(\frac{1}{\mid Q \mid}\int_Q \phi\right)^p, \tag{1.7}$$

for any $Q \in \mathcal{T}_{2^n}$ and some fixed constants $p > 1$ and $c \geq 1$. Then, if we set $h = \phi^*$ the non-increasing rearrangement of ϕ, the following inequality is true

$$\frac{1}{t}\int_0^t h^p(u)du \leq (2^n c - 2^n + 1)\left(\frac{1}{t}\int_0^t h(u)du\right)^p, \quad \textit{for any} \quad t \in [0,1]. \tag{1.8}$$

As a consequence $h = \phi^*$ satisfies the assumptions of Theorem C, which can be applied and produce an $\varepsilon_1 = \varepsilon_1(n,p,c) > 0$ such that h belongs to $L^q([0,1])$ for any $q \in [p, p+\varepsilon_1)$. Thus $\phi \in L^q([0,1]^n)$ for any such q. That is we can find an explicit value of ε_1. This is stated as Corollary 3.1 and is presented in the last section of this paper.

As a matter of fact we prove Theorem 1 in a much more general setting. More precisely we consider a non-atomic probability space (X, μ) equipped with a tree $\mathcal{T}_k$, that is a k-homogeneous tree for a fixed integer $k > 1$, which plays the role of dyadic sets as in $[0,1]^n$ (see the definition of Section 2).

As we shall see later, Theorem 1 is independent of the shape of the dyadic sets and depends only on the homogeneity of the tree $\mathcal{T}_k$. Additionally we need to mention that the inequality (1.8) cannot necessarily be satisfied, under the assumptions of Theorem 1, if one replaces the intervals $(0,t]$ by $(t,1]$. That is ϕ^* is not necessarily a weight on $(0,1]$ satisfying a reverse Hölder inequality upon all subintervals of $[0,1]$ (see the related result in [5]).

Additionally we mention that in [6] the study of the dyadic A_1-weights appears, where one can find for any $c > 1$ the best possible range $[1,p)$, for which the following holds: $\phi \in A_1^d(c) \Rightarrow \phi \in L^q$, for any $q \in [1,p)$. All last results that are connected with A_1 dyadic weights ϕ and the behavior of ϕ^* as an A_1-weight on $\mathbb{R}$, can be seen in [9].

2. Preliminaries

Let (X, μ) be a non-atomic probability space. We give the notion of a k-homogeneous tree on X.

Definition 2.1. *Let k be an integer such that $k > 1$. A set $\mathcal{T}_k$ will be called a k-homogeneous tree on X if the following hold*

(i) $X \in \mathcal{T}_k$

(ii) For every $I \in \mathcal{T}_k$, there corresponds a subset $C(I) \subseteq \mathcal{T}_k$ consisting of k subsets of I such that

(a) *the elements of $C(I)$ are pairwise disjoint*

(b) $I = \bigcup C(I)$

(c) $\mu(J) = \frac{1}{k}\mu(I)$, *for every $J \in C(I)$.*

For example one can consider $X = [0,1]^n$, the unit cube of $\mathbb{R}^n$. Define as μ the Lebesque measure on this cube. Then the set $\mathcal{T}_k$ of all dyadic subcubes of X is a tree of homogeneity $k = 2^n$, with $C(Q)$ being the set of 2^n-subcubes of Q, obtained by bisecting the sides, of every $Q \in \mathcal{T}_k$, starting from $Q = X$.

Let now (X,μ) be as above and a tree $\mathcal{T}_k$ on X as in Definition 2.1. From now on, we fix k and write $\mathcal{T} = \mathcal{T}_k$. For any $I \in \mathcal{T}$, $I \neq X$ we set I^* the smallest element of $\mathcal{T}$ such that $I^* \supsetneq I$. That is I^* is the unique element of $\mathcal{T}$ such that $I \in C(I^*)$. We call I^* the father of I in $\mathcal{T}$. Then $\mu(I^*) = k\mu(I)$.

Definition 2.2. *For any (X,μ) and $\mathcal{T}$ as above we define the dyadic maximal operator on X with respect to $\mathcal{T}$, noted as $\mathcal{M}_{\mathcal{T}}$, by*

$$\mathcal{M}_{\mathcal{T}}\phi(X) = \sup\left\{\frac{1}{\mu(I)}\int_I |\ \phi\ |\ d\mu : x \in I \in \mathcal{T}\right\}, \tag{2.1}$$

for any $\phi \in L^1(X,\mu)$.

Remark 2.1. *It is not difficult to see that the maximal operator defined by (2.1) satisfies a weak-type (1,1) inequality, which is the following:*

$$\mu(\{\mathcal{M}_{\mathcal{T}}\phi > \lambda\}) \leq \frac{1}{\lambda}\int_{\{\mathcal{M}_{\mathcal{T}}\phi > \lambda\}} \phi d\mu, \qquad \lambda > 0.$$

It is not difficult to see that the above inequality is best possible for every $\lambda > 0$, and is responsible for the fact that $\mathcal{T}$ differentiates $L^1(X,\mu)$, that is the following holds: For every $\phi \in L^1(X,\mu)$, $\lim\limits_{\substack{x\in I\in\mathcal{T} \\ \mu(I)\to 0}} \frac{1}{\mu(I)}\int_I \phi d\mu = \phi(x)$, μ-almost everywhere on X. This can be seen in [4].

We will also need the following lemma which can be again seen in [4].

Lemma 2.1. *Let ϕ be non-negative function defined on $E \cup \widehat{E} \subseteq X$ such that*

$$\frac{1}{\mu(E)}\int_E \phi d\mu = \frac{1}{\mu(\widehat{E})}\int_{\widehat{E}} \phi d\mu \equiv A, \tag{2.2}$$

Additionally suppose that

$$\phi(x) \leq A, \qquad \textit{for every} \qquad x \notin E \cap \widehat{E}, \tag{2.3}$$

and

$$\phi(x) \leq \phi(y), \qquad \textit{for every} \qquad X \in \widehat{E} \setminus E, \qquad \textit{and} \qquad y \in E, \tag{2.4}$$

Then, for every $p > 1$ the following inequality holds

$$\frac{1}{\mu(E)} \int_E \phi^p d\mu \leq \frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} \phi^p d\mu, \tag{2.5}$$

3. Weights on $(X, \mu, \mathcal{T})$

We proceed now to the

Proof of Theorem 1. We suppose that ϕ is non-negative defined on (X, μ) and satisfies a reverse Hölder inequality of the form

$$\frac{1}{\mu(I)} \int_I \phi^p d\mu \leq c \cdot \left(\frac{1}{\mu(I)} \int_I \phi d\mu \right)^p, \tag{3.1}$$

for every $I \in \mathcal{T}$, where c, p are fixed such that $p > 1$ and $c \geq 1$. We will prove that for any $t \in (0, 1]$ we have that

$$\frac{1}{t} \int_0^t [\phi^*(u)]^p du \leq (kc - k + 1) \left(\frac{1}{t} \int_0^t \phi^*(u) du \right)^p, \tag{3.2}$$

where ϕ^* is the non-increasing rearrangement of ϕ, defined as in Remark **??**, on $(0, 1]$, and k is the homogeneity of $\mathcal{T}$. Fix a $t \in (0, 1]$ and set

$$A_t = \frac{1}{t} \int_0^t \phi^*(u) du.$$

Consider now the following subset of X defined by

$$E_t = \{x \in X : \mathcal{M}_{\mathcal{T}} \phi(x) > A_t\}, \tag{3.3}$$

Then for any $x \in E_t$, there exists an element of $\mathcal{T}$, say I_x, such that

$$x \in I_x \qquad \text{and} \qquad \frac{1}{\mu(I_x)} \int_{I_x} \phi d\mu > A_t. \tag{3.4}$$

For any such I_x we obviously have that $I_x \subseteq E_t$. We set $S_{\phi,t} = \{I_x : x \in E_t\}$. This is a family of elements of $\mathcal{T}$ such that $\bigcup \{I : I \in S_{\phi,t}\} = E_t$. Consider now those $I \in S_{\phi,t}$ that are maximal with respect to the relation of $\subseteq$. We write this subfamily of $S_{\phi,t}$ as $S'_{\phi,t} = \{I_j : j = 1, 2, \ldots\}$ which is possibly finite. Then $S'_{\phi,t}$ is a disjoint family of elements of $\mathcal{T}$, because of the maximality of every I_j and the tree structure of $\mathcal{T}$. (see Definition 2.1).

Then by construction, this family still covers E_t, that is $E_t = \bigcup_{j=1}^{\infty} I_j$. For any $I_j \in S'_{\phi,t}$ we have that $I_j \neq X$, because if $I_j = X$ for some j, we could have from (3.4) that

$$\int_0^1 \phi^*(u) du = \int_X \phi d\mu = \frac{1}{\mu(I_j)} \int_{I_j} \phi d\mu > A_t = \frac{1}{t} \int_0^t \phi^*(u) du,$$

which is impossible, since ϕ^* is non-increasing on $(0,1]$. Thus, for every $I_j \in S'_{\phi,t}$ we have that I_j^* is well defined, but may be common for any two or more elements of $S'_{\phi,t}$. We may also have that $I_j^* \subseteq I_i^*$ for some $I_j, I_i \in S'_{\phi,t}$.

We consider now the family

$$L_{\phi,t} = \{I_j^* : j = 1, 2, \ldots\} \subseteq \mathcal{T}.$$

As we mentioned above, this is not necessarily a pairwise disjoint family. We choose a pairwise disjoint subcollection, by considering those I_j^* that are maximal, with respect to the relation $\subseteq$.

We denote this family as

$$L'_{\phi,t} = \{I_{j_s}^* : s = 1, 2, \ldots\}.$$

Then of course

$$\bigcup J : J \in L_{\phi,t} = \bigcup J : J \in L'_{\phi,t}.$$

Since, each $I_j \in S'_{\phi,t}$ is maximal we should have that

$$\frac{1}{\mu(I_{j_s}^*)} \int_{I_{j_s}^*} \phi d\mu \leq A_t, \tag{3.5}$$

Now note that every $I_{j_s}^*$ contains at least one element of $S'_{\phi,t}$, such that $I \in C(I_{j_s}^*)$. (one such is I_{j_s}). Consider for any s the family of all those I such that $I^* \subseteq I_{j_s}^*$. We write it as

$$S'_{\phi,t,s} = \{I \in S'_{\phi,t} : I^* \subseteq I_{j_s}^*\}.$$

For any $I \in S'_{\phi,t,s}$ we have of course that

$$\frac{1}{\mu(I)} \int_I \phi d\mu > A_t, \qquad \text{so if we set} \qquad K_s = \bigcup\{I : I \in S'_{\phi,t,s}\}.$$

We must have, because of the disjointness of the elements of family $S'_{\phi,t}$, that

$$\frac{1}{\mu(K_s)} \int_{K_s} \phi d\mu > A_t. \tag{3.6}$$

Additionally, $K_s \subseteq I_{j_s}^*$ and by (3.5) and the comments stated above we easily see that

$$\frac{1}{k}\mu(I_{j_s}^*) \leq \mu(K_s) < \mu(I_{j_s}^*), \tag{3.7}$$

By (3.5) and (3.6) we can now choose (because μ is non-atomic) for any s, a measurable set $B_s \subseteq I_{j_s}^* \setminus K_s$, such that if we define $\Gamma_s = K_s \cup B_s$, then $\dfrac{1}{\mu(\Gamma_s)} \displaystyle\int_{\Gamma_s} \phi d\mu = A_t$.

We set now $E_t^* = \bigcup\limits_s I_{j_s}^*$

$$\Gamma = \bigcup_s \Gamma_s, \quad \Delta = \bigcup_s \Delta_s,$$

where $\Delta_s = I_{j_s}^* \setminus \Gamma_s$, for any $s = 1, 2, \ldots$.

Then by all the above, we have that

$$\Gamma \cup \Delta = E_t^* \qquad \text{and} \qquad \frac{1}{\mu(\Gamma)} \int_\Gamma \phi d\mu = A_t,$$

which is true in view of the pairwise disjointness of $\left(I_{j_s}^*\right)_{s=1}^\infty$.

Define now the following function

$$h := (\phi/\Gamma)^* : (0, \mu(\Gamma)] \;\to\; \mathbb{R}^+.$$

Then obviously

$$\frac{1}{\mu(\Gamma)} \int_0^{\mu(\Gamma)} h(u) du = \frac{1}{\mu(\Gamma)} \int_\Gamma \phi d\mu = A_t.$$

By the definition of h we have that $h(u) \leq \phi^*(u)$, for any $u \in (0, \mu(\Gamma)]$. Thus we conclude:

$$\text{(3.8)} \qquad \frac{1}{\mu(\Gamma)} \int_0^{\mu(\Gamma)} \phi^*(u) du \geq \frac{1}{\mu(\Gamma)} \int_0^{\mu(\Gamma)} h(u) du = A_t = \frac{1}{t} \int_0^t \phi^*(u) du,$$

From (3.8), we have that $\mu(\Gamma) \leq t$, since ϕ^* is non-increasing.

We now consider a set $E \subseteq X$ such that $(\phi/E)^* = \phi^*/(0, t]$, with $\mu(E) = t$ and for which $\{\phi > \phi^*(t)\} \subseteq E \subseteq \{\phi \geq \phi^*(t)\}$.

It's existence is guaranteed by the equimeasurability of ϕ and ϕ^*, and the fact that (X, μ) is non-atomic. Then, we see immediately that

$$\frac{1}{\mu(E)} \int_E \phi d\mu = \frac{1}{t} \int_0^t \phi^*(u) du = A_t.$$

We are going now to construct a second set $\widehat{E} \subseteq X$. We first set $\widehat{E}_1 = \Gamma$.

Let now $x \notin \widehat{E}_1$. Since $\Gamma \supseteq \{\mathcal{M}_\mathcal{T}\phi > A_t\}$, we must have that $\mathcal{M}_\mathcal{T}\phi(x) \leq A_t$. But since $\mathcal{T}$ differentiates $L^1(X, \mu)$ we obviously have that for μ-almost every $y \in X$: $\phi(y) \leq \mathcal{M}_\mathcal{T}\phi(y)$. Then the set $\Omega = \{x \notin \widehat{E}_1 : \phi(x) > \mathcal{M}_\mathcal{T}\phi(x)\}$ has μ-measure zero.

At last we set $\widehat{E} = \widehat{E}_1 \cup \Omega = \Gamma \cup \Omega$. Then $\mu(\widehat{E}) = \mu(\Gamma)$ and for every $x \notin \widehat{E}$ we have that $\phi(x) \leq \mathcal{M}_\mathcal{T}\phi(x) \leq A_t$.

Let now $x \notin E$. By the construction of E we immediately see that $\phi(x) \leq \phi^*(t) \leq \frac{1}{t} \int_0^t \phi^*(u) du = A_t$. Thus, if $x \notin E$ or $x \notin \widehat{E}$, we must have that $\phi(x) \leq A_t$, that is (2.3) of Lemma 2.1 is satisfied for these choices of E and $\widehat{E}$. Let now $x \in \widehat{E} \setminus E$ and $y \in E$. Then we obviously have by the above discussion that $\phi(x) \leq \phi^*(t) \leq \phi(y)$. That is $\phi(x) \leq \phi(y)$. Thus (2.4) is also satisfied. Also since $\widehat{E} = \Gamma \cup \Omega$, we obviously have $\frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} \phi d\mu = A_t$, so as a consequence (2.2) is satisfied also.

Applying Lemma 2.1, we conclude that

$$\frac{1}{\mu(E)} \int_E \phi^p d\mu \leq \frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} \phi^p d\mu,$$

or by the definitions of E and $\widehat{E}$ that

$$\frac{1}{t}\int_0^t [\phi^*(u)]^p du \leq \frac{1}{\mu(\Gamma)}\int_\Gamma \phi^p d\mu, \tag{3.9}$$

Our aim is now to show that the right integral average in (3.9) is less or equal that $(kc-k+1)(A_t)^p$. We proceed to this as follows:

We set $\ell_\Gamma = \dfrac{1}{\mu(\Gamma)}\displaystyle\int_\Gamma \phi^p d\mu$. Then by the notation given above, we have that:

$$\begin{aligned}\ell_\Gamma &= \frac{1}{\mu(\Gamma)}\Big(\int_{E_t^*}\phi^p d\mu - \int_\Delta \phi^p d\mu\Big)\\ &= \frac{1}{\mu(\Gamma)}\Big(\sum_{s=1}^\infty \int_{I_{j_s}^*}\phi^p d\mu - \sum_{s=1}^\infty\int_{\Delta_s}\phi^p d\mu\Big)\\ &= \frac{1}{\mu(\Gamma)}\sum_{s=1}^\infty p_s, \qquad (3.10)\end{aligned}$$

where the p_s are given by

$$p_s = \int_{I_{j_s}^*}\phi^p d\mu - \int_{\Delta_s}\phi^p d\mu, \qquad \text{for any} \qquad s = 1,2,\dots .$$

We find now an effective lower bound for the quantity $\displaystyle\int_{\Delta_s}\phi^p d\mu$. By Hölder's inequality:

$$\int_{\Delta_s}\phi^p d\mu \geq \frac{1}{\mu(\Delta_s)^{p-1}}\Big(\int_{\Delta_s}\phi d\mu\Big)^p, \tag{3.11}$$

Since $\Delta_s = I_{j_s}^* \setminus \Gamma_s$, (3.11) can be written as

$$\int_{\Delta_s}\phi^p d\mu \geq \frac{\Big(\int_{I_{j_s}^*}\phi d\mu - \int_{\Gamma_s}\phi du\Big)^p}{(\mu(I_{j_s}^*) - \mu(\Gamma_s))^{p-1}}, \tag{3.12}$$

We now use Hölder's inequality in the form

$$\frac{(\lambda_1+\lambda_2)^p}{(\sigma_1+\sigma_2)^{p-1}} \leq \frac{\lambda_1^p}{\sigma_1^{p-1}} + \frac{\lambda_2^p}{\sigma_2^{p-1}}, \quad \text{for} \quad \lambda_i \geq 0 \quad \text{and} \quad \sigma_i > 0$$

which holds since $p > 1$. Thus (3.12) gives

$$\int_{\Delta_s}\phi^p d\mu \geq \frac{1}{\mu(I_{j_s}^*)^{p-1}}\Big(\int_{I_{j_s}^*}\phi d\mu\Big)^p - \frac{1}{\mu(\Gamma_s)^{p-1}}\Big(\int_{\Gamma_s}\phi d\mu\Big)^p. \tag{3.13}$$

Since $\dfrac{1}{\mu(\Gamma_s)}\displaystyle\int_{\Gamma_s}\phi d\mu = A_t$, (3.13) gives

$$\int_{\Delta_s}\phi^p d\mu \geq \frac{1}{\mu(I_{j_s}^*)^{p-1}}\Big(\int_{I_{j_s}^*}\phi d\mu\Big)^p - \mu(\Gamma_s)\cdot(A_t)^p,$$

so we conclude, by the definition of p_s, that

$$p_s \leq \int_{I_{j_s}^*}\phi^p d\mu - \frac{1}{\mu(I_{j_s}^*)^{p-1}}\Big(\int_{I_{j_s}^*}\phi d\mu\Big)^p + \mu(\Gamma_s)\cdot(A_t)^p, \tag{3.14}$$

Using now (3.1) for $I = I^*_{j_s}$, $s = 1, 2, \dots$ we have as a consequence that:

$$p_s \le (c-1)\frac{1}{\mu(I^*_{j_s})^{p-1}}\left(\int_{I^*_{j_s}} \phi d\mu\right)^p + \mu(\Gamma_s)(A_t)^p. \tag{3.15}$$

Summing now (3.15) for $s = 1, 2, \dots$ we obtain in view of (3.10) that

$$\ell_\Gamma \le \frac{1}{\mu(\Gamma)}\left[(c-1)\sum_{s=1}^{\infty}\frac{1}{\mu(I^*_{j_s})^{p-1}}\left(\int_{I^*_{j_s}} \phi d\mu\right)^p + \left(\sum_{s=1}^{\infty}\mu(\Gamma_s)\right)(A_t)^p\right]. \tag{3.16}$$

Now from $\dfrac{1}{\mu(I^*_{j_s})}\displaystyle\int_{I^*_{j_s}} \phi d\mu \le A_t$, we see that

$$\begin{aligned}\ell_\Gamma &\le \frac{1}{\mu(\Gamma)}\left[(c-1)\sum_{s=1}^{\infty}\mu(I^*_{j_s})\cdot(A_t)^p + \mu(\Gamma)\cdot(A_t)^p\right]\\ &= \left[(c-1)\frac{\mu(E^*_t)}{\mu(\Gamma)} + 1\right]\cdot(A_t)^p, \qquad (3.17)\end{aligned}$$

Since now $E^*_t \supseteq \Gamma \supseteq E_t$, by (3.7) we have that

$$\mu(E^*_t) \le k\mu(E_t) \le k\mu(\Gamma).$$

Thus (3.17) gives

$$\frac{1}{\mu(\Gamma)}\int_\Gamma \phi^p d\mu \le [k(c-1)+1](A_t)^p.$$

Using now (3.9) and the last inequality we obtain the desired result. □

Corollary 3.1. *If ϕ satisfies (3.1) for every $I \in \mathcal{T}$, then $\phi \in L^q$, for any $q \in [p, p_0)$, where p_0 is defined by $\dfrac{p_0 - p}{p_0}\cdot\left(\dfrac{p_0}{p_0-1}\right)^p\cdot(kc-k+1) = 1$.*

Proof. Immediate from Theorem 1 and A. □

Remark 3.1. *All the above hold if we replace the condition (3.1), by the known Muckenhoupt condition of ϕ over the dyadic sets of X. Then the same proof as above gives that the Muckenhoupt condition should hold for ϕ^*, for the intervals of the form $(0, t]$, and for the constant $kc - k + 1$. This is true since there exists analogous lemma as Lemma 2.1 for this case (as can be seen in* [4]*). Also the inequality that is used in order to produce (3.13) from (3.12) is true even for negative exponent $p < 0$. We ommit the details.*

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