PROPERTIES OF EXTREMAL SEQUENCES FOR THE BELLMAN FUNCTION OF THE DYADIC MAXIMAL OPERATOR

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Abstract. We prove a necessary condition that has every extremal sequence for the Bellman function of the dyadic maximal operator. This implies the weak-L^p uniqueness for such a sequence.

1. Introduction

The dyadic maximal operator on $\mathbb{R}^k$ is defined by

$$\mathcal{M}_d\phi(x)=\sup\left\{\frac{1}{|Q|}\int_Q|\phi(u)|du : x\in Q,\ Q\subseteq\mathbb{R}^k \text{ is a dyadic cube}\right\} \tag{1.1}$$

for every $\phi\in L^1_{\text{loc}}(\mathbb{R}^k)$, where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^k$ and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^k$, $N=0,1,2,\ldots$.

It is well known that it satisfies the following weak type (1,1) inequality:

$$|\{x\in\mathbb{R}^k : \mathcal{M}_d\phi(x)\geq\lambda\}|\leq\frac{1}{\lambda}\int_{\{\mathcal{M}_d\phi\geq\lambda\}}|\phi(u)|du, \tag{1.2}$$

for every $\phi\in L^1(\mathbb{R}^k)$ and $\lambda>0$.

From (1.2) it is easy to prove the following L^p-inequality

$$\|\mathcal{M}_d\phi\|_p\leq\frac{p}{p-1}\|\phi\|_p. \tag{1.3}$$

It is easy to see that (1.2) is best possible, while (1.3) is sharp as it can be seen in [W]. (See also [B1] and [B2] for general martingales).

A way of studying the dyadic maximal operator is to find certain refinements of the above inequalities.Concerning (1.2) refinements have been studied in [MN2], [N1] and [N2], while for (1.3) the Bellman function of two variables for $p>1$, has been introduced by the following way:

$$T_p(f,F)=\sup\left\{\frac{1}{|Q|}\int_Q(\mathcal{M}_d\phi)^p : \phi\geq 0,\ \frac{1}{|Q|}\int_Q\phi(u)du=f,\ \frac{1}{|Q|}\int_Q\phi^p(u)du=F\right\} \tag{1.4}$$

where Q is a fixed dyadic cube on $\mathbb{R}^k$ and $0<f^p\leq F$.

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The function given in (1.4) has been explicitely computed. Actually, this is done in a much more general setting of a non-atomic probability measure space (X, μ) where the dyadic sets are now given in a family of sets $\mathcal{T}$, called tree, which satisfies conditions similar to those that are satisfied by the dyadic cubes on $[0,1]^k$.

Then the associated dyadic maximal operator $\mathcal{M}_{\mathcal{T}}$ is defined by

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\}, \tag{1.5}$$

where $\phi \in L^1(X, \mu)$.

Then the Bellman function (for a given $p > 1$) of two variables associated to $\mathcal{M}_{\mathcal{T}}$ is given by

$$S_p(f, F) = \sup\left\{ \int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu : \phi \geq 0,\ \int_X \phi d\mu = f,\ \int_X \phi^p d\mu = F \right\}, \tag{1.6}$$

where $0 < f^p \leq F$.

In [M], (1.6) has been found to be $S_p(f, F) = F\omega_p(f^p/F)^p$ where $\omega_p : [0,1] \to \left[1, \frac{p}{p-1}\right]$ is the inverse function H_p^{-1} of H_p defined on $\left[1, \frac{p}{p-1}\right]$ by $H_p(z) = -(p-1)z^p + pz^{p-1}$.

As a result the Bellman function is independent of the measure space (X, μ) and the underlying tree $\mathcal{T}$. Other approaches for the computation of (1.4) can be seen in [NM] and [SSV].

In this paper we study those sequences of functions: $(\phi_n)_n$, that are extremal for the Bellman function (1.6). That is $\phi_n : (X, \mu) \to \mathbb{R}^+$, $n = 1, 2, \ldots$ satisfy $\int_X \phi_n d\mu = f$, $\int_X \phi_n^p d\mu = F$ and

$$\lim_n \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p. \tag{1.7}$$

In Section 3 we prove the following

Theorem 1.1. *Let $\phi_n : (X, \mu) \to \mathbb{R}^+$ be as above. Then for every $I \in \mathcal{T}$,*

$$\lim_n \frac{1}{\mu(I)} \int_I \phi_n d\mu = f \quad \textit{and} \quad \lim_n \frac{1}{\mu(I)} \int_I \phi_n^p d\mu = F. \tag{1.8}$$

Additionally:

$$\lim_n \frac{1}{\mu(I)} \int_I (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p,$$

for every $I \in \mathcal{T}$. □

This gives as an immediate result that extremal functions do not exist for the Bellman function. Another corollary is the weak-L^p uniqueness of such a sequence in all interesting cases. In other words if $(\phi_n)_n$, $(\psi_n)_n$ are extremal sequences for (1.4), then $\lim_n \int_Q (\phi_n - \psi_n) h d\mu = 0$, for every $h \in L^p(Q)$, where $\frac{1}{p} + \frac{1}{q} = 1$. We need also to mention that related results in connection with Kolmogorov's inequality have been treated in [MN1], while in [N3] it is given a characterization of such extremal sequences. More precisely it s proved there that they actually behave approximately like eigenfunctions of the dyadic maximal operator for a specific eigenvalue.

2. Extremal sequences

Let (X, μ) be a non-atomic probability measure space. We give the following

Definition 2.1. A set $\mathcal{T}$ of measurable subsets of X will be called a *tree* if the following are satisfied:

i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$, $\mu(I) > 0$.

ii) For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I)$ of $\mathcal{T}$ containing at least two elements such that

  (a) the elements of $C(I)$ are disjoint subsets of I

  (b) $I = \cup C(I)$

iii) $\mathcal{T} = \bigcup\limits_{m \geq 0} \mathcal{T}_{(m)}$, where $\mathcal{T}_{(0)} = \{X\}$ and

$$\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} C(I).$$

iv) The following holds $\lim\limits_{m \to \infty} \sup\limits_{I \in \mathcal{T}_{(m)}} \mu(I) = 0$. □

Definition 2.2. Given a tree $\mathcal{T}$ we define the *maximal operator* associated to it as follows:

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\}$$

for every $\phi \in L^1(X, \mu)$. □

From [M] we obtain the following:

Theorem 2.3. *The following holds*

$$\sup\left\{ \int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu : \phi \geq 0,\ \int \phi d\mu = f,\ \int_X \phi^p d\mu = F \right\} = F\omega_p(f^p/F)^p,$$

for $0 < f^p \leq F$. □

At last we give the following

Definition 2.4. Let $(\phi_n)_n$ be a sequence of non-negative measurable functions defined on X and $0 < f^p \leq F$, $p > 1$. $(\phi_n)_n$ is called (p, f, F) *extremal*, or simply *extremal* if the following hold:

$$\int_X \phi_n d\mu = f,\ \int_X \phi_n^p d\mu = F, \quad \text{for every} \quad n = 1, 2, \ldots$$

$$\lim_n \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p.$$

3. Main theorem

Theorem 3.1. *Let $(\phi_n)_n$ be an extremal sequence. Then for every $I \in \mathcal{T}$ the following hold:*

i) $\lim_n \dfrac{1}{\mu(I)} \displaystyle\int_I \phi_n d\mu = f$

ii) $\lim_n \dfrac{1}{\mu(I)} \displaystyle\int_I \phi_n^p d\mu = F$

iii) $\lim \frac{1}{\mu(I)} \int_I (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p$.

Proof. We remind that $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T} = \bigcup_{m\geq 0} \mathcal{T}_{(m)}$. We prove this theorem for $I \in \mathcal{T}_{(1)}$. Then inductively it holds for every $I \in \mathcal{T}_{(m)}$, $m \geq 1$.

Suppose then that $\mathcal{T}_{(1)} = \{I_k,\ k = 1, 2, \ldots\}$ and $I = I_1$. We now set

$$f_{n,1} = \frac{1}{\mu(I_1)} \int_{I_1} \phi_n d\mu, \quad f_{n,2} = \frac{1}{\mu(X \setminus I_1)} \int_{X\setminus I_1} \phi_n d\mu,$$

$$F_{n,1} = \frac{1}{\mu(I_1)} \int_{I_1} \phi_n^p d\mu,\ F_{n,2} = \frac{1}{\mu(X \setminus I_1)} \int_{X\setminus I_1} \phi_n^p d\mu, \text{ for } n = 1, 2, \ldots, \tag{3.1}$$

The above sequences are obviously bounded, so passing to a subsequence we may suppose that

$$\lim_n f_{n,i} = f_i \ \text{ and } \ \lim_n F_{n,i} = F_i, \ \text{ for } \ i = 1, 2.$$

For any $J \in \mathcal{T}$ define

$$\mathcal{M}_J\phi(t) = \sup\left\{ \frac{1}{\mu(K)} \int_K |\phi| d\mu : t \in K \in \mathcal{T}_J \right\}, \ \text{ for } \ t \in J,$$

where $\mathcal{T}_J$ is defined by

$$\mathcal{T}_J = \{K \in \mathcal{T} : K \subseteq J\}.$$

Consider the measure space $\left(J, \frac{\mu(\cdot)}{\mu(J)}\right)$, the tree $\mathcal{T}_J$ and the associated maximal operator $\mathcal{M}_J$. Then using Theorem 2.3, we have that

$$\frac{1}{\mu(J)} \int_J (\mathcal{M}_J\phi)^p d\mu \leq \frac{1}{\mu(J)} \int_J \phi^p d\mu \cdot \omega_p \left(\frac{\left(\frac{1}{\mu(J)} \int_J \phi d\mu\right)^p}{\frac{1}{\mu(J)} \int_J \phi^p d\mu} \right)^p \tag{3.2}$$

for every $\phi \in L^p(J)$, where $\omega_p : [0,1] \to \left[1, \frac{p}{p-1}\right]$ is H_p^{-1}, with

$$H_p(z) = -(p-1)z^p + pz^{p-1}, \ \ z \in \left[1, \frac{p}{p-1}\right].$$

Since H_p is decreasing we conclude from (3.2) that

$$H_p\left(\left[\frac{\int_J(\mathcal{M}_J\phi)^p}{\int_J\phi^p d\mu}\right]^{1/p}\right)\geq\frac{1}{\mu(J)^{p-1}}\frac{\left(\int_J\phi d\mu\right)^p}{\int_J\phi^p d\mu},$$

which gives

$$\begin{aligned}&-(p-1)\int_J(\mathcal{M}_J\phi)^p d\mu+p\left(\int_J\phi^p d\mu\right)^{1/p}\cdot\left(\int_J(\mathcal{M}_J\phi)^p d\mu\right)^{1-\frac{1}{p}}\\ &=\frac{1}{\mu(J)^{p-1}}\left(\int_J\phi d\mu\right)^p+\delta_{\phi,J},\end{aligned}\tag{3.3}$$

for some $\delta_{\phi,J}\geq 0$ positive constant depending on ϕ and J.

For $\phi=\phi_n$ and $J=I_i$, $i=1,2,\dots$ we obtain from (3.3)

$$\begin{aligned}&-(p-1)\int_{I_i}(\mathcal{M}_{I_i}\phi_n)^p d\mu+p\left(\int_{I_i}\phi_n^p d\mu\right)^{1/p}\cdot\left(\int_{I_i}(\mathcal{M}_{I_i}\phi_n)^p d\mu\right)^{1-\frac{1}{p}}\\ &=\frac{1}{\mu(I_i)^{p-1}}\left(\int_{I_i}\phi_n d\mu\right)^p+\delta_{n,i},\ \text{for every } n=1,2,\dots \text{ and } i=1,2,\dots .\end{aligned}\tag{3.4}$$

Summing relations (3.4) for $i\geq 2$ we obtain

$$\begin{aligned}&-(p-1)\sum_{i=2}^{+\infty}\int_{I_i}(\mathcal{M}_{I_i}\phi_n)^p d\mu+p\sum_{i=2}^{+\infty}\left(\int_{I_i}\phi_n^p d\mu\right)^{1/p}\left(\int_{I_i}(\mathcal{M}_{I_i}\phi_n)^p d\mu\right)^{1-\frac{1}{p}}\\ &=\sum_{i=2}^{+\infty}\frac{1}{\mu(I_i)^{p-1}}\left(\int_{I_i}\phi_n d\mu\right)^p+\sum_{i=2}^{+\infty}\delta_{n,i}.\end{aligned}\tag{3.5}$$

In view now of Holder's inequality in its primitive form:

$$\sum_i a_i b_i\leq\left(\sum_i a_i^p\right)^{1/p}\left(\sum_i b_i^q\right)^{1/q},$$

for $a_i,b_i\geq 0$ and $q=p/(p-1)$, (3.5) gives

$$\begin{aligned}&-(p-1)A_2(n)+p\left(\int_{X\setminus I_1}\phi_n^p d\mu\right)^{1/p}\cdot\left[A_2(n)\right]^{1-\frac{1}{p}}\\ &\geq\sum_{i=2}^{+\infty}\frac{1}{\mu(I_i)^{p-1}}\left(\int_{I_i}\phi_n d\mu\right)^p+\sum_{i=2}^{+\infty}\delta_{n,i},\quad\text{where}\end{aligned}\tag{3.6}$$

$$A_2(n)=\sum_{i=2}^{+\infty}\int_{I_i}(\mathcal{M}_{I_i}\phi_n)^p d\mu.\tag{3.7}$$

(In the last inequality we used the fact that $X\setminus I_1=\bigcup_{i=2}^{+\infty}I_i$).

We use now Holder's inequality in the following form:

$$\frac{(\lambda_1+\lambda_2+\cdots+\lambda_m)^p}{(\sigma_1+\sigma_2+\cdots+\sigma_m)^{p-1}}\leq\frac{\lambda_1^p}{\sigma_1^{p-1}}+\frac{\lambda_2^p}{\sigma_2^{p-1}}+\cdots+\frac{\lambda_m^p}{\sigma_m^{p-1}},$$

where σ_i, $\forall\, i = 1, 2, \dots$ and $\lambda_i \geq 0$, and obtain:

$$\sum_{i=2}^{+\infty} \frac{1}{\mu(I_i)^{p-1}} \left(\int_{I_i} \phi_n d\mu \right)^p \geq \frac{1}{\mu(X \setminus I_1)^{p-1}} \left(\int_{X \setminus I_1} \phi_n d\mu \right)^p = \mu(X \setminus I_1) f_{n,2}. \tag{3.8}$$

We also set

$$A_3(n) = \int_{X \setminus I_1} (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu, \quad \text{for} \quad n = 1, 2, \dots . \tag{3.9}$$

Then by definition of $\mathcal{M}_{I_i}$ we have that

$$A_3(n) \geq A_2(n). \tag{3.10}$$

From the above we then have that:

$$-(p-1) A_2(n) + p\mu(X \setminus I_1)^{1/p} (F_{n,2})^{1/p} [A_3(n)]^{1-\frac{1}{p}} = \mu(X \setminus I_1)(f_{n,2})^p + \delta_n^{(1)}, \tag{3.11}$$

where $\delta_n^{(1)} \geq \sum\limits_{i=2}^{+\infty} \delta_{n,i}$.

By passing to a subsequence we may suppose that $\lim\limits_n A_3(n) = A_3$.

We will use now the following Lemma, the proof of which will be given at the end of this section.

Lemma 3.2. *If $(\phi_n)_n$ is extremal then we have that*

$$\lim_n \mu(\{\mathcal{M}_{\mathcal{T}} \phi_n = f\}) = 0.$$

From this Lemma and Definitions (3.7) and (3.9) we easily obtain that $\lim\limits_n A_2(n) = \lim\limits_n A_3(n) = A_3$, in view of the fact that $I_i \in \mathcal{T}_{(1)}$ for $i = 2, 3, \dots$.. Then from (3.11) we conclude that

$$\begin{aligned} &-(p-1) \int_{X \setminus I_1} (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu + p\mu(X \setminus I_1)^{1/p} (F_{n,2})^{1/p} \left(\int_{X \setminus I_1} (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu \right)^{1-\frac{1}{p}} \\ &= \mu(X \setminus I_i)(f_{n,2})^p + \delta_n'', \end{aligned} \tag{3.12}$$

where $\delta_n'' \geq \delta_n'$, for every $n \in N$.

In the same way we have that:

$$\begin{aligned} &-(p-1) \int_{I_1} (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu + p\mu(I_1)^{1/p} (F_{n,1})^{1/p} \cdot \left(\int_{I_1} (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu \right)^{1-\frac{1}{p}} \\ &= \mu(I_1)(f_{n,1})^p + \varepsilon_n'', \end{aligned} \tag{3.13}$$

where ε_n'' is such that $\varepsilon_n'' \geq \delta_{n,1}$, for every $n \in N$.

Summing now (3.12) and (3.13) and using Holder's inequality in both previously mentioned forms we have that:

$$\begin{aligned} &-(p-1) \int_X (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu + pF^{1/p} \left(\int_X (\mathcal{M}_{\mathcal{T}} \phi_n)^p d\mu \right)^{1-\frac{1}{p}} \\ &\geq \mu(I_1)(f_{n,1})^p + \mu(X \setminus I_1)(f_{n,2})^p + \delta_n'' + \varepsilon_n'' \geq f^p + \delta_n'' + \varepsilon_n'', \end{aligned} \tag{3.14}$$

which gives

$$(3.15)\quad -(p-1)\int_X (\mathcal{M}_{\mathcal{T}}\phi_1)^p d\mu + pF^{1/p}\left(\int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu\right)^{1-\frac{1}{p}} = f^p + \vartheta_n,$$

where $\vartheta_n \geq \delta_n'' + \varepsilon_n''$, $n = 1, 2, \dots$.

The hypothesis now for (ϕ_n) is that

$$\lim_n \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p.$$

This gives $\vartheta_n \to 0$ in (3.15 and so

$$\delta_n'' \to 0,\ \varepsilon_n'' \to 0.$$

As a consequence we have

$$\mu(I_1)(f_1)^p + \mu(X \setminus I_1)(f_2)^p = f^p$$

because of equality in (3.14), as $n \to +\infty$.

Since now $\mu(I_1)f_1 + \mu(X \setminus I_1)f_2 = f$ and $t \mapsto t^p$ is strictly convex on $(0, +\infty)$ we have that $f_1 = f_2 = f$.

Additionally $\delta_n'' \to 0$, so because of (3.12) and the fact that $f_2 = f$ we immediately see that

$$(3.16)\qquad \lim_n \frac{1}{\mu(X \setminus I_1)} \int_{X \setminus I_1} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F_2\omega_p(f^p/F_2)^p.$$

Similarly

$$(3.17)\qquad \lim_n \frac{1}{\mu(I_1)} \int_{I_1} (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F_1\omega_p(f^p/F_1)^p.$$

Since $(\phi_n)_n$ is extremal the last two equations give

$$(3.18)\qquad \mu(I_1)\cdot F_1\omega_p(f^p/F_1)^p + \mu(X \setminus I_1)\cdot F_2\omega_p(f^p/F_2)^p = F\omega(f^p/F).$$

But as we shall prove in Lemma 3.3 below the following function $t \mapsto t\omega_p(f^p/t)^p$, $t \in (f^p, +\infty)$ is strictly concave. So since $\mu(I_1)F_1 + \mu(X \setminus I_1)F_2 = F$ we have because of (3.18) that $F_1 = F_2 = F$. Then since (3.17) holds we conclude that

$$\lim_n \frac{1}{\mu(I)} \int_I (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p,$$

and Theorem 3.1 is now proved. $\square$.

We prove now the following

Lemma 3.3. *Let $G : (1, +\infty) \to \mathbb{R}^+$ defined by $G(t) = t\omega_p(1/t)^p$. Then G is strictly concave.*

Proof. It is known from [M] that ω_p satisfies

$$\frac{d}{dx}[\omega_p(x)]^p = -\frac{1}{p-1}\frac{\omega_p(x)}{\omega_p(x)-1}, \quad x \in [0,1].$$

So we can easily see that

$$G'(t) = \omega_p(1/t)^p + \frac{1}{p-1}\frac{1}{t}\frac{\omega_p(1/t)}{\omega_p(1/t)-1}, \quad \text{and}$$

$$G''(t) = \frac{1}{p-1}\cdot\frac{1}{t}\left(\frac{g(t)}{g(t)-1}\right)',$$

where g is defined on $(1,+\infty)$ by $g(t) = \omega_p(1/t)$. Since $g'(t) > 0$, $\forall\ t > 1$, we have that $G''(t) < 0$, $\forall\ t > 1$ and Lemma 3.3 is proved. $\square$

We continue now with

Proof of Lemma 3.1: Let us suppose first that ϕ_n are $\mathcal{T}$-simple functions that is for every n, there exists a m_n such that ϕ_n is constant on each $I \in \mathcal{T}_{(m_n)}$. As a consequence ϕ_n is $\mathcal{T}$-good in the sense of [M], for every n. If we look at the proof of Lemma 9 in [M] p. 324-326 we see that in all inequalities (4.20), (4.22), (4.23), (4.24) we should have equality in the limit. So as a result we must have that $\frac{1}{(\beta+1-\beta\rho_X^n)^{p-1}} - \frac{(p-1)\beta\rho_X^n}{(\beta+1)^p} \to \frac{1}{(\beta+1)^{p-1}}$, for $\beta = \omega_p(f^p/F)-1$, where $\rho_X^n = \frac{a_X^n}{\mu(X)} = a_X^n$, where $a_X^n = \mu(\{\mathcal{M}_\mathcal{T}\phi_n = f\})$. But this can happen only if $a_X^n \to 0$. So the proof is completed in the case of $\mathcal{T}$-simple functions. As for the general case, it is not difficult to see that if $(\phi_n)_n$ is an extremal sequence of measurable functions, then we can construct a sequence of $\mathcal{T}$-simple functions $(\psi_n)_n$ such that $\int_X \psi_n d\mu = f$, $\int_X \psi_n^p d\mu \le F$ and

$$\lim_n \int_X \psi_n^p d\mu = F, \quad \lim_n \int_X (\mathcal{M}_\mathcal{T}\psi_n)^p d\mu = F\omega_p(f^p/F)^p.$$

Additionally, we can arrange everything in such a way that $\{\mathcal{M}_\mathcal{T}\phi_n = f\} \subseteq \{\mathcal{M}_\mathcal{T}\psi_n = f\}$.

Using the same arguments as before for $(\psi_n)_n$ we can prove that $\lim_n \mu(\{\mathcal{M}_\mathcal{T}\psi_n = f\}) = 0$. So $\lim_n \mu(\{\mathcal{M}_\mathcal{T}\phi_n = f\}) = 0$ and Lemma 3.2 is proved. $\square$

We now give some applications of the above.

First we prove the following

Corollary 3.4. *If $0 < f^p < F$ then there do not exist extremal functions for the Bellman function $T_p(f,F)$ described in (1.4).*

Proof. Let ϕ be an extremal function for (1.4). Applying Theorem 3.1 we see that

$$\frac{1}{\mu(I)}\int_I \phi d\mu = f \quad \text{and} \quad \frac{1}{\mu(I)}\int_I \phi^p d\mu = F,$$

for every I dyadic subcube of Q.

As we can see in [G] inequality (1.2) implies that the base of dyadic sets of the tree $\mathcal{T}$ differentiates $L^1(Q)$. That is

$$\begin{aligned}&\phi(x) = f \quad \text{a.e} \quad \text{and}\\ &\phi^p(x) = F \quad \text{a.e.}\end{aligned}$$

This gives $f^p = F$, which is a contradiction. □

We also prove

Corollary 3.5. *Let $T_p(f, F)$ be described by (1.4). Then if $(\phi_n)_n$, $(\psi_n)_n$ are extremal sequences for this function, we must have $\phi_n - \psi_n \overset{w(L^p)}{\longrightarrow} 0$, as $n \to +\infty$.*

Proof. Of course we have that

$$\lim_n \frac{1}{|I|}\int_I \phi_n(u)du = \lim_n \frac{1}{|I|}\int_I \psi_n(u)du = f.$$

So $\lim_n \int_Q (\phi_n - \psi_n)\xi_I(u)du = 0$, f0r every dyadic subcube $I \subseteq Q$.

Since linear combinations of the characteristic functions of the dyadic subcubes of Q are dense in $L^q(Q)$ we should have that $\lim_n \int_Q (\phi_n - \psi_n)h = 0$, for every $h \in L^q(Q)$,where $q = \frac{p}{p-1}$ that is $\phi_n - \psi_n \overset{w(L^p)}{\longrightarrow} 0$, as $n \to +\infty$. □

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