EXTREMAL SEQUENCES FOR THE BELLMAN FUNCTION OF THE DYADIC MAXIMAL OPERATOR

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Abstract: We give a characterization of the extremal sequences for the Bellman function of the dyadic maximal operator. In fact we prove that they behave approximately like eigenfunctions of this operator for a specific eigenvalue.

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is a usefull tool in analysis and is defined by

$$\mathcal{M}_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|du : x \in Q,\ Q \subseteq \mathbb{R}^n \text{ is a dyadic cube}\right\} \tag{1.1}$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ where $|\cdot|$ is the Lesbesgue measure on $\mathbb{R}^n$ and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, $N = 0, 1, 2, \dots$.

It is well known that it satisfies the following weak type (1,1) inequality

$$|\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) > \lambda\}| \le \frac{1}{\lambda}\int_{\{\mathcal{M}_d\phi > \lambda\}} |\phi(u)|du, \tag{1.2}$$

for every $\phi \in L^1(\mathbb{R}^n)$ and $\lambda > 0$.

From (1.2) it is not difficult to prove the following L^p-inequality

$$\|\mathcal{M}_d\phi\|_p \le \frac{p}{p-1}\|\phi\|_p, \tag{1.3}$$

for every $p > 1$ and $\phi \in L^p(\mathbb{R}^n)$, and this can be proved by using the well known Doob's method on the dyadic maximal operator.

It is also easy to see that (1.2) is best possible, while (1.3) is also best possible as can be seen in [15]. (See [1] and [2] for general martingales).

Our aim is to study this maximal operator. One way to do this is to find certain refinements of the inequalities satisfied by it such as (1.2) and (1.3). Concerning (1.2) refinements have been done in [8], [10] and [12]. Refinements of (1.3) can be found in [5] or even more general in [6].

In order to study (1.3) the following function has been introduced in [5]

$$B_p^Q(f, F) = \sup\left\{\frac{1}{|Q|}\int_Q (\mathcal{M}_d\phi)^p : \phi \ge 0,\ Av_Q(\phi) = f,\ Av_Q(\phi^p) = F\right\} \tag{1.4}$$

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where Q is a fixed dyadic cube in $\mathbb{R}^n$, $\phi \in L^p(Q)$ and

$$Av_Q(h) = \frac{1}{|Q|}\int_Q |h(u)|du,$$

for every $h \in L^1(Q)$. This is the so-called Bellman function of two variables associated to the dyadic maximal operator.

This function given has been explicitly computed. Actually this is done in a much more general setting of a non-atomic probability measure space (X, μ), where the dyadic sets are now given in a family of sets $\mathcal{T}$, (called tree), which satisfies conditions similar to those that are satisfied by the dyadic cubes on $[0,1]^n$ (for details see section 2). Then the associated dyadic maximal operator $\mathcal{M}_\mathcal{T}$ is defined by

$$\mathcal{M}_\mathcal{T}\phi(x) = \sup\left\{\frac{1}{\mu(I)}\int_I |\phi|d\mu : x \in I \in \mathcal{T}\right\}, \tag{1.5}$$

for $\phi \in L^1(X, \mu)$.

The Bellman function of two variables for $p > 1$ associated to $\mathcal{M}_\mathcal{T}$ is now given by

$$B_p^\mathcal{T}(f, F) = \sup\left\{\int_X (\mathcal{M}_\mathcal{T}\phi)^p d\mu : \phi \geq 0,\ \int_X \phi d\mu = f,\ \int_X \phi^p d\mu = F\right\}, \tag{1.6}$$

where $0 < f^p \leq F$.

In [5], (1.6) has been found equal to $B_p^\mathcal{T}(f, F) = F\omega_p(f^p/F)^p$ where $\omega_p : [0,1] \longrightarrow \left[1, \frac{p}{p-1}\right]$ is the inverse function H_p^{-1} of H_p defined for $z \in \left[1, \frac{p}{p-1}\right]$ by $H_p(z) = -(p-1)z^p + pz^{p-1}$. This gives us as a result that it is independent of the measure space (X, μ) and the tree structure $\mathcal{T}$.

For the evaluation of this function the author in [5] introduced a technique which enabled him to compute it. This is based on an effective linearization of the dyadic maximal operator that holds for an adequate set of functions, called $\mathcal{T}$-good. Certain sharp inequalities were proved in [5] by using Holder's inequality upon suitable subsets of X in an effective way. After the evaluation of (1.6) he was also able to evaluate other more general Bellman functions of $\mathcal{M}_\mathcal{T}$ that involve three parameters. The evaluations of these new Bellman functions, which are connected with the Dyadic Carleson Imbedding Theorem and others, are based on the result of (1.6) entirely and are proved by its application on certain elements of the tree $\mathcal{T}$.

The next step for studying the dyadic maximal operator is to investigate the opposite problem for the Bellman function related to Kolmogorov's inequality which has been studied in [7]. More precisely the following function

$$B_q(f, h) = \sup\left\{\int_X (\mathcal{M}_\mathcal{T}\phi)^q d\mu : \phi \geq 0,\ \int_X \phi d\mu = f,\ \int_X \phi^q d\mu = h\right\}, \tag{1.7}$$

has been computed there, where $0 < h \leq f^q$ and $q \in (0,1)$ is a fixed constant.

In [7] the authors precisely evaluated the above function by using the linearization technique introduced in [5]. The situation is now different and new methods were found in order that (1.7) be evaluated.

Additionally the following has been proved in [11]

Proposition: *Let* $(\phi_n)_n$ *be a sequence of nonnegative functions in* $L^1(X,\mu)$ *such that* $\int_X \phi_n d\mu = f$ *and* $\int_X \phi_n^p d\mu = F$ *for all* $n \in N$. *If* $(\phi_n)_n$ *is extremal for (1.6), then for every* $I \in \mathcal{T}$ *we have that* $\lim_n \frac{1}{\mu(I)} \int_I \phi_n d\mu = f$ *and* $\lim_n \frac{1}{\mu(I)} \int_I \phi_n^p d\mu = F$. *Moreover* $\lim_n \frac{1}{\mu(I)} \int_I (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = B_p^{\mathcal{T}}(f,F)$.

This gives as an immediate result that there do not exist extremal functions for (1.7). This is true because if $\mathcal{T}$ differentiates $L^1(X,\mu)$ we would have for any extremal ϕ that it should be constant almost everywhere on X so that $F = f^p$ which is a trivial case that we do not consider.

Thus our interest is for those sequences of functions $(\phi_n)_n$ that are extremal for this Bellman function. That is $\phi_n : (X,\mu) \to \mathbb{R}^+$, $n \in N$ must satisfy

$$\int_X \phi_n d\mu = f, \ \int_X \phi_n^p d\mu = F \ \text{ and } \ \lim_n \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p.$$

Our aim in this paper is to give a characterization of these extremal sequences of functions. For this reason we restrict ourselves to the class of $\mathcal{T}$-good functions, that is enough to describe the problem as it was described in [5] (see Section 3). We give now the statement of our main result

Theorem A: *Let* $(\phi_n)_n$ *be a sequence of nonnegative,* $\mathcal{T}$*-good functions such that* $\int_X \phi_n d\mu = f$ *and* $\int_X \phi_n^p d\mu = F$. *Then it is extremal for (1.6), if and only if*

$$\lim_n \int_X |\mathcal{M}_{\mathcal{T}}\phi_n - c\phi_n|^p d\mu = 0,$$

for $c = \omega_p(f^p/F)$.

For the proof of the above theorem we use the technique introduced in [5] for the evaluation (1.6). In fact we generalize it in two directions (Theorem 3.1 and 3.2) and by using these we prove Theorem 3.3 for the extremal sequences we are interested in. This theorem is in fact a weak form of Theorem A. It is proved by producing two inequalities that involve $\int_A (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu$ and $\int_A \phi^p d\mu$ on measurable subsets of $A \subset X$ that have a certain form with respect to the tree $\mathcal{T}$ and the function ϕ. More precisely A is a union of certain elements of S_ϕ or a complement of such a set, where S_ϕ is a subtree of $\mathcal{T}$ that depends on X and gives all the information we need for $\mathcal{M}_{\mathcal{T}}\phi$ (for the definition of S_ϕ see section 2). Using these two inequalities we prove Theorem 3.3.

In order to prove Theorem A we need to apply Theorem 3.3 to a new extremal sequence (g_{ϕ_n}) which satisfies the following relation $\lim_n \int_X |g_{\phi_n} - \phi_n|^p d\mu = 0$. g_{ϕ_n} is defined properly on suitable subsets of X where ϕ_n is defined. The number of different values of g_{ϕ_n} on each of those subsets are at most two with the one being zero. Then we prove that the measure of the set where g_{ϕ_n} is zero tends to zero by using the fact that (g_{ϕ_n}) is extremal sequence for (1.6). Thus we can arrange everything so that this new extremal sequence is constant on those suitable sets. We rename this sequence as (g'_{ϕ_n}). Because of the just mentioned property that it satisfies and the certain form that have these suitable subsets of X we can apply Theorem 3.3 to it and produce Lemma 3.5 . By using the last mentioned result and the fact that $\lim_n \int_X |g_{\phi_n} - g'_{\phi_n}|^p d\mu = 0$, we complete the proof for the characterization of the extremal sequences for the Bellman function of the dyadic maximal operator.

We mention also that additional work concerning the Bellman functions and certain symmetrization principles for the dyadic maximal operator can be seen in [6] and [13]. It is also worth saying that in [14] it has been given an alternative method for the evaluation of the Bellman function (1.6). Also we need to say that the phenomenon that the norm of a maximal operator is attained by a sequence of eigenfunctions of such a maximal operator can be seen in [4] and [3] . So by considering the results of this paper one might guess that it shouldn't be rare and and may occur in other settings such as square functions or other dyadic operators. Finally we need to mention that the extremizers for the Bellman function of three variables related to Kolmogorov's inequality have been characterized in [9].

2. Preliminaries

Let (X, μ) be a non-atomic probability measure space. We now give the following from [5].

Definition 2.1. *A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following are satisfied*

i) $X \in \mathcal{T}$ *and for every* $I \in \mathcal{T}$, $\mu(I) > 0$.

ii) *For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I)$ of $\mathcal{T}$ containing at least two elements such that*

  a) *the elements of $C(I)$ are pairwise disjoint subsets of I*

  b) $I = \cup C(I)$.

iii) $\mathcal{T} = \bigcup_{m\geq 0} \mathcal{T}_{(m)}$, *where* $\mathcal{T}_{(0)} = \{X\}$ *and* $\mathcal{T}_{(m+1)} = \bigcup_{I\in\mathcal{T}_{(m)}} C(I)$.

iv) *The following holds*

$$\lim_{m\to\infty} \sup_{I\in\mathcal{T}_{(m)}} \mu(I) = 0.$$

We state the following lemma as is given in [5].

Lemma 2.1. *For every $I \in \mathcal{T}$ and every $a \in (0, 1)$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of pairwise disjoint subsets of I such that*

$$\mu\Big(\bigcup_{J\in\mathcal{F}(I)} J\Big) = \sum_{J\in\mathcal{F}(I)} \mu(J) = (1-a)\mu(I).$$

Given a tree $\mathcal{T}$ we define the maximal operator associated to it as follows

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\left\{\frac{1}{\mu(I)}\int_I |\phi| d\mu : \ x \in I \in \mathcal{T}\right\},$$

for every $\phi \in L^1(X,\mu)$. From [5] now we recall

Theorem 2.1. *The following is true*

$$\sup\left\{(\mathcal{M}_{\mathcal{T}}\phi)^p d\mu : \ \phi \geq 0, \ \int_X \phi d\mu = f, \ \int_X \phi^p d\mu = F\right\} = F\omega_p(f^p/F)^p,$$

for every f, F such that $0 < f^p \leq F$.

Additionally we give the notion of the extremal sequence as

Definition 2.2. *Let $(\phi_n)_n$ be a sequence of μ-measurable nonnegative functions defined on X, $p > 1$ and $0 < f^p \leq F$. Then $(\phi_n)_n$ is called (p, f, F) extremal or simply extremal if the following hold:*

$$\int_X \phi_n d\mu = f, \ \int_X \phi_n^p d\mu = F \quad \textit{and} \quad \lim_n \int_X (\mathcal{M}_{\mathcal{T}}\phi_n)^p d\mu = F\omega_p(f^p/F)^p.$$

3. Characterization of the extremal sequences

For the proof of Theorem 2.1 an effective linearization for the operator $\mathcal{M}_{\mathcal{T}}$ was introduced in [5] valid for certain functions ϕ. We describe it as appears there and use it in the sequel.

For every $\phi \in L^1(X,\mu)$ nonnegative and $I \in \mathcal{T}$ we define $Av_I(\phi) = \dfrac{1}{\mu(I)}\int_I \phi d\mu$.

We will say that ϕ is $\mathcal{T}$-good if the set

$$\mathcal{A}_\phi = \{x \in X : \mathcal{M}_{\mathcal{T}}\phi(x) > Av_I(\phi) \ \text{ for all } \ I \in \mathcal{T} \ \text{ such that } \ x \in I\}$$

has μ-measure zero.

Let now ϕ be $\mathcal{T}$-good and $x \in X \setminus \mathcal{A}_\phi$. We define $I_\phi(x)$ to be the largest in the nonempty set

$$\{I \in \mathcal{T} : \ x \in I \text{ and } \ \mathcal{M}_{\mathcal{T}}\phi(x) = Av_I(\phi)\}.$$

Now given $I \in \mathcal{T}$ let

$$A(\phi, I) = \{x \in X \setminus \mathcal{A}_\phi : \ I_\phi(x) = I\} \subseteq I \quad \text{and}$$

$$S_\phi = \{I \in \mathcal{T} : \ \mu(A(\phi, I)) > 0\} \cup \{X\}.$$

Obviously then $\mathcal{M}_{\mathcal{T}}\phi = \sum_{I \in S_\phi} Av_I(\phi) J_{A(\phi,I)}$, μ-a.e. where J_E is the characteristic function of E.

We define also the following correspondence $I \to I^*$ by: I^* is the smallest element of $\{J \in S_\phi : \ I \subsetneq J\}$. It is defined for every $I \in S_\phi$ except X. It is obvious that the $A(\phi, I)$'s are then pairwise disjoint and that $\mu\Big(\bigcup_{I \notin S_\phi} (A(\phi, I))\Big) = 0$, so that $\bigcup_{I \in S_\phi} A(\phi, I) \approx X$, where by $A \approx B$ we mean that $\mu(A \setminus B) = \mu(B \setminus A) = 0$.

Now the following is a consequence of the above.

Lemma 3.1. *Let ϕ be $\mathcal{T}$-good and $I \in \mathcal{T}$, $I \neq X$. Then $I \in S_\phi$ if and only if for every $J \in \mathcal{T}$ that contains properly I we have that $Av_J(\phi) < Av_I(\phi)$.*

Proof. Suppose that $I \in S_\phi$. Then $\mu(A(\phi, I)) > 0$. Thus $A(\phi, I) \neq \emptyset$, so there exists $x \in A(\phi, I)$. By the definition of $A(\phi, I)$ we have that $I_\phi(x) = I$, that is I is the largest element of $\mathcal{T}$ such that $\mathcal{M}_\mathcal{T}\phi(x) = Av_I(\phi)$. As a consequence the implication stated in our Lemma holds.

Conversely suppose that $I \in \mathcal{T}$ and for every $J \in \mathcal{T}$ that contains properly I we have that $Av_J(\phi) < Av_I(\phi)$. Then since ϕ is $\mathcal{T} - good$, for every $x \in I \setminus \mathcal{A}_\phi$ there exists $J_x = I_\phi(x)$ in S_ϕ such that $\mathcal{M}_\mathcal{T}\phi(x) = Av_{J_x}(\phi)$ and $x \in J_x$. By our hypothesis we must have that $J_x \subseteq I$. Consider the family $S^1 = (J_x)_{x \in I \setminus \mathcal{A}_\phi}$. This has the property that $\bigcup_{x \in I \setminus \mathcal{A}_\phi} J_x \approx I$. Choose a pairwise disjoint subfamily $S^2 = (J_i)_i$ with $X \approx \cup J_i$. We just need to consider those $J_x \in S^1$ maximal under $\subseteq$ relation. Then by our construction $Av_{J_i}(\phi) \geq Av_I(\phi)$. Suppose now that $I \notin S_\phi$. This means that $\mu(A(\phi, I)) = 0$, that is we must have for every $x \in I \setminus \mathcal{A}_\phi$ that $J_x \subsetneq I$. Since J_x belongs to S_ϕ for every such x, by the first part of the proof of this Lemma we conclude that $Av_{J_x}(\phi) > Av_I(\phi)$ and as a consequence $Av_{J_i}(\phi) > Av_I(\phi)$ for every i. Since S^2 is a decomposition of X and because of the last inequality we reach to a contradiction. In this way we derive the proof of our Lemma. □

Now the following is true (see [5]).

Lemma 3.2. *Let ϕ be $\mathcal{T}$-good*

i) *If I, $J \in S_\phi$ then either $A(\phi, J) \cap I = \emptyset$ or $J \subseteq I$.*
ii) *If $I \in S_\phi$ then there exists $J \in C(I)$ such that $J \notin S_\phi$.*
iii) *For every $I \in S_\phi$ we have that*

$$I \approx \bigcup_{\substack{J \in S_\phi \\ J \subseteq I}} A(\phi, J).$$

iv) *For every $I \in S_\phi$ we have that*

$$A(\phi, I) = I \setminus \bigcup_{\substack{J \in S_\phi \\ J^* \in I}} J, \quad \textit{so that}$$

$$\mu(A(\phi, I)) = \mu(I) - \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J).$$

From the above we see that

$$Av_I(\phi) = \frac{1}{\mu(I)} \sum_{\substack{J \in S_\phi \\ J \subseteq I}} \int_{A(\phi, J)} \phi \, d\mu =: y_I$$

where $I \in S_\phi$, and for these $I's$ we also define

$$\chi_I = a_I^{-1+\frac{1}{p}} \int_{A(\phi,I)} \phi d\mu, \quad \text{where} \quad a_I = \mu(A(\phi,I)).$$

We prove now the following

Theorem 3.1. *Let ϕ be $\mathcal{T}$-good function such that $\int\limits_X \phi d\mu = f$. Let also $B = \{I_j\}$ be a family of pairwise disjoint elements of S_ϕ, which is maximal on S_ϕ under $\subseteq$ relation. That is if $I \in S_\phi$ then $I \cap (\cup I_j) \neq \emptyset$.*

Then the following inequality holds:

$$\int_{X\setminus\bigcup\limits_j I_j} \phi^p d\mu \geq \frac{f^p - \sum\limits_j \mu(I_j) y_{I_j}^p}{(\beta+1)^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p} \int_{X\setminus\bigcup\limits_j I_j} (\mathcal{M}_\mathcal{T}\phi)^p d\mu$$

for every $\beta > 0$, where $y_{I_j} = Av_{I_j}(\phi)$.

Proof. We follow [5]. We have that

$$\int_{X\setminus\cup I_j} \phi^p d\mu = \sum_{\substack{I \supsetneq \text{piece}(B) \\ I \in S_\phi}} \int_{A(\phi,I)} \phi^p d\mu, \tag{3.1}$$

where by writing $I \supsetneq \text{piece}(B)$ we mean that $I \supsetneq I_j$ for some j. Of course (3.1) is true since $X \setminus \bigcup\limits_j I_j \approx \bigcup\limits_{\substack{J \in S_\phi \\ I \supsetneq \text{piece}(B)}} A(\phi,I)$ in view of the maximality of B and Lemma 3.2.

Now from (3.1) we have by Holder's inequality that

$$\int_{X\setminus\bigcup\limits_j I_j} \phi^p d\mu \geq \sum_{\substack{I \in S_\phi \\ I \supsetneq \text{piece}(B)}} x_I^p = \sum_{\substack{I \in S_\phi \\ I \supsetneq \text{piece}(B)}} \frac{\left(\int\limits_{A(\phi,I)} \phi d\mu\right)^p}{a_I^{p-1}}. \tag{3.2}$$

It is true that

$$\mu(I) y_I = \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) y_J + \int_{A(\phi,I)} \phi d\mu, \quad \text{for every} \quad I \in S_\phi.$$

So by using Holder's inequality in the form

$$\frac{(\lambda_1 + \cdots + \lambda_m)^p}{(\sigma_1 + \cdots + \sigma_m)^{p-1}} \leq \frac{\lambda_1^p}{\sigma_1^{p-1}} + \frac{\lambda_2^p}{\sigma_2^{p-1}} + \cdots + \frac{\lambda_n^p}{\sigma_m^{p-1}}, \quad \text{we have}$$

$$\int_{X\setminus\cup I_j} \phi^p d\mu \geq \sum_{\substack{I \in S_\phi \\ I \supsetneq \text{piece(B)}}} \frac{\left(\mu(I) y_I - \sum\limits_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) y_J\right)^p}{\left(\mu(I) - \sum\limits_{\substack{J \in S_\phi \\ J^* = I}} \mu(J)\right)^{p-1}}$$

$$\geq \sum_{\substack{I \in S_\phi \\ I \supsetneq \text{piece(B)}}} \left\{ \frac{(\mu(I) y_I)^p}{(\tau_I \mu(I))^{p-1}} - \sum_{\substack{J \in S_\phi \\ J^* = I}} \frac{(\mu(J) y_J)^p}{((\beta+1)\mu(J))^{p-1}} \right\}, \tag{3.3}$$

where $\tau_I = (\beta+1) - \beta\rho_I$, $\rho_I = \dfrac{a_I}{\mu(I)}$, $\beta > 0$.

Thus by (3.3) we have because of the maximality of B that

$$\int_{X\setminus\bigcup_j I_j} \phi^p d\mu \geq \sum_{\substack{I\in S_\phi\\ I\supsetneq \mathrm{piece}(B)}} \frac{\mu(I)y_I^p}{\tau_I^{p-1}} - \sum_{(*)} \frac{\mu(I)y_I^p}{(\beta+1)^{p-1}}, \tag{3.4}$$

where the summation in $(*)$ is extended to:

(a) $I \in S_\phi$: $I \supsetneq \mathrm{piece}(B)$ with $I \neq X$ and (b)$I \in S_\phi$ is a piece of B ($I = I_j$, for some j).

So we can write:

$$\int_{X\setminus\cup I_j} \phi^p d\mu \geq \frac{y_x^p}{\tau_x^{p-1}} + \sum_{\substack{I\in S_\phi\\ I\neq X\\ I\supsetneq \mathrm{piece}(B)}} \frac{1}{\rho_I}\left(\frac{1}{\tau_I^{p-1}} - \frac{1}{(\beta+1)^{p-1}}\right) a_I y_I^p -$$

$$- \frac{1}{(\beta+1)^{p-1}} \sum_j \mu(I_j) y_{I_j}^p. \tag{3.5}$$

It is easy now to see that

$$\frac{1}{(\beta+1-\beta x)^{p-1}} - \frac{1}{(\beta+1)^{p-1}} \geq \frac{(p-1)\beta x}{(\beta+1)^p}, \tag{3.6}$$

for any $x \in [0,1]$, in view of the mean value theorem on derivatives.

Then (3.5) becomes

$$\int_{X\setminus\cup I_j} \phi^p d\mu \geq \frac{y_x^p}{\tau_x^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p} \sum_{\substack{I\neq X\\ I\in S_\phi\\ I\supsetneq \mathrm{piece}(B)}} a_I y_I^p - \frac{1}{(\beta+1)^{p-1}} \sum_j \mu(I_j) y_{I_j}^p$$

$$= \left[\frac{1}{((\beta+1)-\beta\rho_x)^{p-1}} - \frac{(p-1)\beta\rho_x}{(\beta+1)^p}\right] f^p + \frac{(p-1)\beta}{(\beta+1)^p} \sum_{\substack{I\in S_\phi\\ I\supsetneq \mathrm{piece}(B)}} a_I y_I^p$$

$$- \frac{1}{(\beta+1)^{p-1}} \sum_j \mu(I_j) y_{I_j}^p, \tag{3.7}$$

On the other hand $\sum\limits_{\substack{I\in S_\phi\\ I\supsetneq \mathrm{piece}(B)}} a_I y_I^p = \sum\limits_{X\setminus\cup I_j} (\mathcal{M}_\mathcal{T}\phi)^p d\mu$, so in view of (3.6) we must have that

$$\int_{X\setminus\cup I_j} \phi^p \geq \frac{f^p - \sum \mu(I_j) y_{I_j}^p}{(\beta+1)^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p} \int_{X\setminus\cup I_j} (\mathcal{M}_\mathcal{T}\phi)^p d\mu,$$

for every $\beta > 0$, and the proof of the theorem is now complete. □

If we follow the same proof as above but now work inside any of the I_j we obtain

Theorem 3.2. *Let ϕ be $\mathcal{T}$-good and $\mathcal{A} = \{I_j\}$ be a pairwise disjoint family of elements of S_ϕ. Then for every $\beta > 0$ we have that:*

$$\int_{\bigcup\limits_j I_j} \phi^p d\mu \geq \frac{\sum \mu(I_j) y_{I_j}^p}{(\beta+1)^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p} \int_{\bigcup\limits_j I_j} (\mathcal{M}_\mathcal{T}\phi)^p d\mu.$$

We have now the following generalization of Theorem 3.1

Corollary 3.1. *ϕ be a $\mathcal{T}$-good and $A = \{I_j\}$ be a pairwise disjoint family of elements of S_ϕ. Then for every $\beta > 0$*

$$\int_{X \setminus \bigcup\limits_j I_j} \phi^p d\mu \geq \frac{f^p - \sum\limits_j \mu(I_j) y_{I_j}^p}{(\beta+1)^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p} \int_{X \setminus \bigcup\limits_j I_j} (\mathcal{M}_\mathcal{T}\phi)^p d\mu,$$

where $f = \int\limits_X \phi d\mu$.

Proof. This is true since there exist families B, Γ of pairwise disjoint elements of S_ϕ with B as in the statement of Theorem 3.1, such that $B = \bigcup\limits_j I'_j$, $\Gamma = \bigcup\limits_i J_i$ with $\bigcup\limits_j I'_j = \Big(\bigcup\limits_j I_j\Big) \cup \Big(\bigcup\limits_i J_i\Big)$ and the additional property that I_j is disjoint to J_i for every j, i. Applying Theorem 3.1 for B and Theorem 3.2 for Γ we obtain, by summing the respective inequalities, the proof of Corollary 3.1. □

As a consequence of the above we have

Theorem 3.3. *Let $(\phi_n)_n$ an extremal sequence consisting of $\mathcal{T}$-good functions. Consider for every $n \in \mathbb{N}$ a pairwise disjoint family $\mathcal{A}_n = \{I_j^n\}$ of elements of S_{ϕ_n} such that the following limit exists*

$$\lim_n \sum_{I \in \mathcal{A}_n} \mu(I) y_{I,n}^p, \quad \textit{where} \quad y_{I,n} = Av_I(\varphi_n), \quad I \in \mathcal{A}_n.$$

Then

$$\lim_n \int_{\cup \mathcal{A}_n} (\mathcal{M}\phi_n)^p d\mu = \omega_p(f^p/F)^p \lim_n \int_{\cup \mathcal{A}_n} \phi_n^p d\mu$$

meaning that if one of the limits on the above relation exists then the other also does and we have the stated equality.

Proof. In view of Theorem 3.2 and Corollary 3.1 we have that

$$\int_{X \setminus \cup \mathcal{A}_n} \phi_n^p d\mu \geq \frac{f^p - \sum\limits_{I \in \mathcal{A}_n} \mu(I) y_{I,n}^p}{(\beta+1)^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p} \int_{X \setminus \cup \mathcal{A}_n} (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu, \quad \text{and} \tag{3.8}$$

$$\int_{\cup \mathcal{A}_n} \phi_n^p d\mu \geq \frac{\sum_{I \in \mathcal{A}_n} \mu(I) y_{I,n}^p}{(\beta+1)^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p} \int_{\cup \mathcal{A}_n} (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu, \tag{3.9}$$

for every $\beta > 0$ and $n \in \mathbb{N}$.

Summing relations (3.8) and (3.9) for every $n \in \mathbb{N}$ we obtain

$$F = \int_X \phi_n^p d\mu \geq \frac{f^p}{(\beta+1)^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p} \int_X (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu, \tag{3.10}$$

Since $(\phi_n)_n$ is extremal we have equality in the limit in (3.10) for $\beta = \omega_p(f^p/F) - 1$ (see [5], relation (4.24)).

So we must have equality on (3.8) and (3.9) in the limit for this value of β. Suppose now that $h_n = \sum_{I \in \mathcal{A}_n} \mu(I) y_{I,n}^p$ and that $h_n \to h$. (3.9) now can be written in the form

$$\int_{\cup \mathcal{A}_n} (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu \leq \left(1 + \frac{1}{\beta}\right) \frac{(\beta+1)^{p-1} \int_{\cup \mathcal{A}_n} \phi_n^p d\mu - h_n}{p-1}, \tag{3.11}$$

(see [5], relations (4.24) and (4.25)), for every $\beta > 0$. The right hand side of (3.11), $n \in \mathbb{N}$, is minimized for $\beta = \beta_n = \omega_p\Big(h_n / \int_{\cup \mathcal{A}_n} \phi_n^p d\mu\Big) - 1$, as can be seen at the end of the proof of Lemma 9 in [5], or by making the related simple calculations.

Since, we have equality in the limit in (3.11) we must have that

$$\lim_n \frac{h_n}{\int_{\cup \mathcal{A}_n} \phi_n^p d\mu} = \frac{f^p}{F}, \tag{3.12}$$

Thus (3.12) and (3.11) give

$$\lim_n \int_{\cup \mathcal{A}_n} (\mathcal{M}_\mathcal{T}\phi_n)^p d\mu = \omega_p(f^p/F)^p \lim_n \int_{\cup \mathcal{A}_n} \phi_n^p$$

and this holds in the sense stated above. This completes the proof of Theorem 3.3. □

We need now some additional Lemmas that we are going to state and prove below. First we prove the following.

Lemma 3.3. *Let ϕ be $\mathcal{T}$-good. Then we can associate to ϕ, a measurable function defined on X, g_ϕ, which attains two at most values (c_J^ϕ or 0) on certain subsets of $A(\phi, J)$, that decompose it, for every $J \in S_\phi$, and which is defined in a way that for every $I \in \mathcal{T}$ which contains an element of S_ϕ (that is it is not contained in any of the A_J) we must have that $\int_I g_\phi d\mu = \int_I \phi d\mu$. Additionally for any $I \in S_\phi$ we will have that $\int_{A_I} g_\phi^p d\mu = \int_{A_I} \phi^p d\mu$ and $\mu(\{\phi = 0\} \cap A_I) \leq \mu(\{g_\phi = 0\} \cap A_I)$.*

Proof. We define g_ϕ inductively using Lemma 3.2. Note that $A(\phi, X) = A_X = X \setminus \cup_{I \in S_\phi, I^* = X} I$. We define first a function $g_\phi^{(1)} : X \to \mathbb{R}^+$ such that the integral relation mentioned above holds for this function and additionally $g_\phi^{(1)}/A_X$ attains at most two values on certain subsets of A_X, which are in fact unions of elements of $\mathcal{T}$, and which decompose A_X. For this proof we proceed as follows. We set $g_\phi^{(1)}(x) = \phi(x)$, for $x \in X \setminus A_X$. We write $A_X = \cup_j I_{j,X}$, where $(I_{j,X})_j$ is a family of elements of $\mathcal{T}$, maximal with respect to the relation $I_{j,X} \subseteq A_X$. For every $I_{j,X}$ there exists an integer $k_j > 0$, such that $I_{j,X} \in \mathcal{T}_{(k_j)}$. Then we consider the unique $I'_{j,X}$ such that

$I_{j,X} \in C(I'_{j,X})$, that is $I'_{j,X} \in \mathcal{T}_{(k_j-1)}$ and $I'_{j,X} \supsetneq I_{j,X}$. By the maximality of $I_{j,X}$ for any j we have that $I'_{j,X} \cap (X \setminus A_X) \neq \emptyset$, thus by Lemma 3.2 iv) there exists $I \in S_\phi$ such that $I^* = X$ and $I'_{j,X} \cap I \neq \emptyset$. Since $I'_{j,X} \cap A_X \neq \emptyset$, we conclude that $I'_{j,X} \supsetneq I$, for any such $I \in S_\phi$. We consider now a maximal disjoint subfamily of $(I'_{j,X})_j$, denoted by $(I'_{j_N,X})_N$, which still covers $\cup_j I'_{j,X}$. By the above discussion we have that for every N, we can write $I'_{j_N,X} = D_{j_N} \cup B_{j_N}$, where $B_{j_N} = I'_{j_N,X} \cap A_X$ and D_{j_N} is a union of some of the elements J, of S_ϕ for which $J^* = X$. Obviously we have $\cup_N B_{j_N} = A_X$ and each B_{j_N} is a union of elements of certain elements of the family $(I_{j,X})_j$. Now fix a j_N. For any $a \in (0,1)$ which will be chosen later, using Lemma 2.1, we construct a family $\mathcal{A}^X_{\phi,j_N}$, of elements of $\mathcal{T}$, all of which are contained in B_{j_N}, and such that

$$\sum_{J \in \mathcal{A}^X_{\phi,j_N}} \mu(J) = a\mu(B_{j_N}). \tag{3.13}$$

Define the function $g_{N,\phi,X} : B_{j_N} \to \mathbb{R}^+$ by setting

$$\begin{aligned} g_{N,\phi,X} &:= c^\phi_{N,X}, \quad \text{on} \quad \cup \mathcal{A}^X_{\phi,j_N} \\ &:= 0, \qquad\quad \text{on} \quad B_{j_N} \setminus \cup \mathcal{A}^X_{\phi,j_N} \end{aligned} \tag{3.14}$$

where the constants $c^\phi_{N,X}$ and $\gamma^\phi_{N,X} := \mu(\cup \mathcal{A}^X_{\phi,j_N})$ satisfy

$$\left.\begin{aligned} &\int_{B_{j_N}} g_{N,\phi,X}\, d\mu = c^\phi_{N,X}\gamma^\phi_{N,X} = \int_{B_{j_N}} \phi\, d\mu \quad \text{and} \\ &\int_{B_{j_N}} g^p_{N,\phi,X}\, d\mu = (c^\phi_{N,X})^p \gamma^\phi_{N,X} = \int_{B_{j_N}} \phi^p\, d\mu, \end{aligned}\right\}, \tag{3.15}$$

It is easy to see that such choices for $c^\phi_{N,X}$ and $\gamma^\phi_{N,X}$ are possible.

In fact (3.15) give

$$\gamma^\phi_{N,X} = \left[\frac{\left(\int_{B_{j_N}} \phi\, d\mu \right)^p}{\int_{B_{j_N}} \phi^p\, d\mu} \right]^{1/(p-1)} \leq \mu(B_{j_N}), \quad \text{by Holder's inequality}$$

so we just need to set

$$a = \frac{\gamma^\phi_{N,X}}{\mu(B_{j_N})}.$$

Then we set $c^\phi_{N,X} = \dfrac{\int_{B_{j_N}} \phi\, d\mu}{\gamma^\phi_{N,X}}$. Define now $g^{(1)}_\phi$ on $A_X = \cup_N B_{j_N}$ by $g^{(1)}_\phi(t) = g_{N,\phi,X}(t) = c^\phi_{N,X}$, for $t \in B_{j_N}$, for any N. Note now that $g^{(1)}_\phi$ may attain more than one positive values on A_X. It is easy then to see that there exists a common positive value, denoted by c^ϕ_X and measurable sets $L_N \subseteq B_{j_N}$, such that if we define $g_\phi(t) = c^\phi_X$ for $t \in L_N$, and $g_\phi(t) = 0$, for $t \in B_{j_N} \setminus L_N$ and for any N, we still have that $\int_{B_{j_N}} g_\phi\, d\mu = \int_{B_{j_N}} \phi\, d\mu = c^\phi_X \mu(L_N)$ and $\int_{A_X} g^p_\phi\, d\mu = \int_{A_X} \phi^p\, d\mu$. For the construction of L_N

and c_X^ϕ, we just need to find first the subsets L_N of B_{j_N} such that the first of the integral equalities mentioned right above are true, and this can be done for arbitrary c_X^ϕ, since the space (X,μ) is nonatomic. Then we just need to find the constant c_X^ϕ for which the second integral equality is also true. Note that for these choices of L_N and c_X^ϕ we may not have $\int_{B_{j_N}} g_\phi^p d\mu = \int_{B_{j_N}} \phi^p d\mu$, for every N, but the respective equality with A_X in place of B_{j_N} should be true.

Until now we have defined g_ϕ on A_X. We set now $g_\phi = \phi$ on $X \setminus A_X$. It is immediate then, by the construction of g_ϕ, that if $I \in \mathcal{T}$ is such that $I \cap A_X \neq \emptyset$, and $I \cap (X \setminus A_X) \neq \emptyset$, we must have that $\int_I g_\phi d\mu = \int_I \phi d\mu$. This is true since then I can be written as a certain union of some subfamily of $I'_{j_N,X}$ and of some class of $J's$, where J is such that $J^* = X$. We continue then inductively and change the values of g_ϕ on the sets A_I, for I is such that $I^* = X$, in the same way as was done before, but now working inside those $I's$. In the limit we have defined the function g_ϕ in all X, which obviously has the desired properties. Moreover the inequality $\mu(\{\phi = 0\} \cap A_I) \leq \mu(\{g_\phi = 0\} \cap A_I)$ is easily verified if we work as above in $B_{j_N} \cap \{\phi > 0\}$ instead of B_{j_N}. In this way by passing from ϕ to g_ϕ we increase or leave unchanged the measure of the set where the corresponding function is zero.

Let now $(\phi_n)_n$ be an extremal sequence consisting of $\mathcal{T}$-good functions and let $g_n = g_{\phi_n}$. We are now ready to prove the following

Lemma 3.4. *With the above notation for an extremal $(\phi_n)_n$ sequence of $\mathcal{T}$-good functions we have that* $\lim_n \mu(\{\phi_n = 0\}) = 0$.

Proof. Fix $n \in \mathbb{N}$ and let $\phi = \phi_n$ and $g_\phi = g_{\phi_n}$ and $S = S_\phi$ the respective subtree of ϕ.

We consider two cases:

i) $p \geq 2$

We set $P_I = \dfrac{\int_{A_I} \phi^p d\mu}{a_I}$, for every $I \in S_\phi$.

We obviously have $\sum_{I \in S_\phi} a_I P_I = F$. We consider then the sum $\Sigma_\phi = \sum_{I \in S_\phi} \gamma_I P_I$, where $\gamma_I = \gamma_I^\phi$ as above. We must have

$$\Sigma_\phi = \sum_{I \in S_\phi} \gamma_I \frac{\int_{A_I} \phi^p}{a_I} = \sum_{I \in S_\phi} \gamma_I \frac{\gamma_I \cdot c_I^p}{a_I} = \sum_{I \in S_\phi} \gamma_I^2 \frac{c_I^p}{a_I} = \sum_{I \in S_\phi} \frac{\gamma_I^2 a_I^{p-2} c_I^p}{a_I^{p-1}}$$

$$\overset{p\geq 2}{\geq} \sum_{I \in S_\phi} \frac{(\gamma_I c_I)^p}{a_I^{p-1}} = \sum_{I \in S_\phi} \frac{\left(\int_{A_I} \phi\right)^p}{a_I^{p-1}}.$$

From the first inequality in (4.20) in [5], and since ϕ_n is extremal we have that the last sum in the last inequality tends to F, as ϕ moves along $(\phi_n)_n$. We conclude

$$\sum_{I \in S_\phi} \gamma_I P_I \approx F \tag{3.16}$$

since $\Sigma_\phi \leq F$. Consider now for every $R > 0$ and every ϕ the following set

$$S_{\phi,R} = \cup\{A_I = A(\phi, I) : \; I \in S_\phi, \quad P_I < R\}.$$

For every $I \in S_\phi$ such that $P_I < R$ we have that $\int\limits_{A_I} \phi^p < Ra_I$. Summing for all such I we obtain

$$\int_{S_{\phi,R}} \phi^p d\mu < R\mu(S_{\phi,R}). \tag{3.17}$$

Additionally we have that

$$\left| \sum_{\substack{I \in S_\phi \\ P_I \geq R}} a_I P_I - F \right| = \int_{S_{\phi,R}} \phi^p d\mu, \quad \text{and} \tag{3.18}$$

$$\sum_{\substack{I \in S_\phi \\ P_I < R}} \gamma_I P_I \leq \sum_{\substack{I \in S_\phi \\ P_I < R}} a_I P_I \leq \int_{S_{\phi,R}} \phi^p d\mu. \tag{3.19}$$

From (3.15) and (3.19) we have that

$$\limsup_\phi \left| \sum_{\substack{I \in S_\phi \\ P_I \geq R}} \gamma_I P_I - F \right| \leq \lim_\phi \int_{S_{\phi,R}} \phi^p d\mu, \tag{3.20}$$

where we have supposed that the last limit exists (in the opposite case we just pass to a subsequence of $(\phi_n)_n$). From (3.18) and (3.20) we conclude that

$$\limsup_\phi \sum_{\substack{I \in S_\phi \\ P_I \geq R}} (a_I - \gamma_I) P_I \leq 2 \lim_\phi \int_{S_{\phi,R}} \phi^p d\mu. \tag{3.21}$$

By using now Theorem 3.3 we have that

$$\lim_\phi \int_{K_\phi} (\mathcal{M}_\mathcal{T} \phi)^p d\mu = \omega_p (f^p/F)^p \lim_\phi \int_{K\phi} \phi^p d\mu,$$

whenever the limits exist, where K_ϕ is a union of pairwise disjoint elements of S_ϕ. (The conditions of Theorem 3.3 are satisfied because of the boundedness of the sequences mentioned there).

Now for a fixed $R > 0$, $S_{\phi,R}$ is a union of sets of the form A_I, for certain $I \in S_\phi$. Each A_I can be written in view of Lemma 3.2 as $A_I = I \setminus \bigcup\limits_{J \in S_\phi} J$. Using then a diagonal argument and passing if necessary to a subsequence we can suppose that

$$\lim_{\phi} \int_{S_{\phi,R}} (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu = \omega_p(f^p/F)^p \lim_{\phi} \int_{S_{\phi,R}} \phi^p. \tag{3.22}$$

Since $\mathcal{M}_{\mathcal{T}}\phi(t) \geq f$, for every $t \in X$, we have that

$$\lim_{\phi} \int_{S_{\phi,R}} (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu \geq (\limsup_{\phi} \mu(S_{\phi,R}))f^p, \tag{3.23}$$

and because of (3.17) we have that

$$\lim_{\phi} \int_{S_{\phi,R}} \phi^p d\mu \leq \limsup_{\phi} R\mu(S_{\phi,R}), \tag{3.24}$$

for any $R > 0$. Combining the last two relations (in view of (3.22)) we obtain that

$$f^p(\limsup_{\phi} \mu(S_{\phi,R})) \leq R\omega_p(f^p/F)^p \cdot (\limsup_{\phi} \mu(S_{\phi,R})), \tag{3.25}$$

so by choosing $R > 0$ suitable small depending only on f, F we have that

$$\limsup_{\phi} \mu(S_{\phi,R}) = 0. \tag{3.26}$$

Using now (3.21) and (3.24) we obtain, for this R that

$$R \limsup_{\phi} \sum_{\substack{I \in S_\phi \\ P_I \geq R}} (a_I - \gamma_I) \leq 2 \lim_{\phi} \int_{S_{\phi,R}} \phi^p d\mu \leq 2R \lim_{\phi} \mu(S_{\phi,R}) = 0$$

Thus

$$\lim_{\phi} \sum_{\substack{I \in S_\phi \\ P_I \geq R}} (a_I - \gamma_I) = 0. \tag{3.27}$$

Since now $\sum\limits_{I \in S_\phi} a_I = 1$, $\mu(S_{\phi,R}) = \sum\limits_{\substack{I \in S_\phi \\ P_I < R}} a_I$ we easily obtain from (3.27) that:

$$\lim_{\phi} \left[1 - \mu(S_{\phi,R}) - \sum_{\substack{I \in S_\phi \\ P_I \geq R}} \gamma_I \right] = 0 \Rightarrow$$

$$\lim_{\phi} \sum_{\substack{I \in S_\phi \\ P_I \geq R}} \gamma_I = 1, \quad \text{which gives of course:}$$

$$\lim_{\phi} \sum_{I \in S_\phi} (a_I - \gamma_I) = 0. \quad \text{But then we have that}$$

$$\mu(\{\phi = 0\}) \leq \mu(\{g_\phi = 0\}) = \sum_{I \in S_\phi} (a_I - \gamma_I) \xrightarrow{\phi} 0,$$

Lemma 3.2 is proved in the first case.

ii) The case $1 < p < 2$ is treated in a similar way:

Here we define $P_I = \dfrac{\int_{A_I} \phi^p}{a_I^{p-1}}$ and prove in the same manner that

$$\lim_\phi \sum_{I \in S_\phi} (a_I^{p-1} - \gamma_I^{p-1}) P_I = 0.$$

Using then the inequality $x^q - y^q > q(x-y)$, for $1 > x > y$ and $0 < q < 1$, we conclude that:

$$\begin{aligned}
&\lim_\phi \sum_{I \in S_\phi} (a_I - \gamma_I) = 0, \quad \text{that is} \\
&\lim_\phi \mu(\{g_\phi = 0\}) = 0, \quad \text{and so} \\
&\lim_\phi \mu(\{\phi = 0\}) = 0,
\end{aligned}$$

and by this we end the proof of Lemma 3.4. □

Suppose now that $(\phi_n)_n$ is extremal. For every $\phi \in \{\phi_n,\ n = 1, 2, \ldots\}$ we define $g'_\phi : x \to \mathbb{R}^+$ by $g'_\phi(t) = c_I^\phi$, $t \in A_I$ for $I \in S_\phi$, that is we ignore the zero values of g_ϕ. Then we easily see because of Lemma 3.2 that

$$\lim_\phi \int_X g'_\phi d\mu = f, \quad \lim_\phi \int_X (g'_\phi)^p d\mu = F \quad \text{and}$$

$$\lim_\phi \int_X |g_\phi - g'_\phi|^p d\mu = 0. \tag{3.28}$$

Additionally because of $\int_{A_I} g_\phi d\mu = \int_{A_I} \phi d\mu$, $I \in S_\phi$ and $I \approx \bigcup_{\substack{J \in S_\phi \\ J \subseteq I}} A(\phi, J)$ we have that for every $I \in S_\phi$

$$Av_I(g_\phi) = Av_I(\phi). \tag{3.29}$$

From (3.29) we have that $\mathcal{M}_\mathcal{T} g_\phi \geq \mathcal{M}_\mathcal{T} \phi$ on $X \Rightarrow \lim_\phi \int_X (\mathcal{M}_\mathcal{T} g_\phi)^p d\mu = F \omega_p(f^p/F)^p$, in view of (3.15) and Theorem 2.1.

Since $\int_X g_\phi d\mu = f$, $\int_X (g_\phi)^p d\mu = F$ we have that $(g_\phi)_\phi$ is an extremal sequence. Suppose now that we have proved the following

$$\lim_\phi \int_X |g'_\phi - \phi|^p d\mu = 0, \tag{3.30}$$

and that

$$\lim_\phi \int_X |\mathcal{M}_\mathcal{T} g_\phi - c g_\phi|^p d\mu = 0, \quad \text{for} \quad c = \omega_p(f^p/F) \tag{3.31}$$

Then because of (3.28) we would have that

$$\lim_{\phi} \int_X |\phi - g_\phi|^p d\mu = 0 \overset{(3.31)}{\Rightarrow}$$
$$\lim_{\phi} \int_X |\mathcal{M}_{\mathcal{T}}\phi - c\phi|^p d\mu = 0$$

that is the result we need to prove. We proceed to the proof of (3.30) and (3.31).

Lemma 3.5. *With the above notation*

$$\lim_{\phi} \int_X |\mathcal{M}_{\mathcal{T}} g_\phi - c g_\phi|^q d\mu = 0.$$

Proof. We recall that $c = \omega_p(f^p/F)$. We set for each $\phi \in \{\phi_n,\, n = 1, 2, \ldots\}$

$$\Delta_\phi = \{t \in X : \mathcal{M}_{\mathcal{T}} g_\phi(t) > c g_\phi(t)\}$$

It is obvious by passing if necessary to a subsequence that

$$\lim_{\phi} \int_{\Delta_\phi} (\mathcal{M}_{\mathcal{T}} g_\phi)^p d\mu \geq \omega_p(f^p/F)^p \lim_{\phi} \int_{\Delta_\phi} g_\phi^p d\mu. \tag{3.32}$$

We consider now for every $I \in S_\phi$ the set $(X \setminus \Delta_\phi) \cap A_I$. We distinguish now two cases:

(i)$Av_I(\phi) = y_I > c c_I^\phi$, where c_I^ϕ is the positive value of g_ϕ on A_I (if it exists). Then because of Lemma 3.3 we have that $\mathcal{M}_{\mathcal{T}} g_\phi(t) \geq Av_I(g_\phi) = Av_I(\phi) > c c_I^\phi \geq c g_\phi(t)$, for each $t \in A_I$. Thus $(X \setminus \Delta_\phi) \cap A_I = \emptyset$ in this case. We study now the second one.

(ii)$y_I \leq c c_I^\phi$. Let now $t \in A_I$ with $g_\phi(t) > 0$, that is $g_\phi(t) = c_I^\phi$. We prove that for such t we have $\mathcal{M}_{\mathcal{T}} g_\phi(t) \leq c g_\phi(t) = c c_I^\phi$. Suppose now that for some t we have the opposite inequality. Then there exists J_t such that $t \in J_t$ and $Av_{J_t}(g_\phi) > c c_I^\phi$. Then one of the following hold

(a)$J_t \subseteq A_I$. Then by the form of g_ϕ/A_I (equals 0 or c_I^ϕ), we have that $Av_{J_t}(g_\phi) \leq c_I^\phi < c c_I^\phi$, which is a contradiction. Thus this case is excluded.

(b)J_t is not a subset of A_I. Then two subcases can occur.

b_1) $J_t \subseteq I$ and contains properly an element of S_ϕ, J', for which $(J')^* = I$. Since now (ii) holds, $t \in J_t$ and $Av_{J_t}(g_\phi) > c c_I^\phi$, we must have that $J' \subsetneq J_t \subsetneq I$. We choose now an element of $\mathcal{T}$, $J'_t \subsetneq I$, which contains J_t, with maximum value on the average $Av_{J'_t}(\phi)$. Then by it's choice we have that for each $K \in \mathcal{T}$ such that $J'_t \subseteq K \subsetneq I$ the following holds $Av_K(\phi) \leq Av_{J'_t}(\phi)$. Since now $I \in S_\phi$ and $Av_I(\phi) \leq c c_I^\phi$ by Lemma 3.1 and the choice of J'_t we have that $Av_K(\phi) < Av_{J'_t}(\phi)$ for every $K \in \mathcal{T}$ such that $J'_t \subsetneq K$. So again by Lemma 3.1 we conclude that $J'_t \in S_\phi$. But this is impossible since $J' \subsetneq J'_t \subsetneq$, $J', I \in S_\phi$ and $(J')^* = I$. We turn now to the last subcase.

b_2) $I \subsetneq J_t$. Then by an application of Lemma 3.3 we have that $Av_{J_t}(\phi) = Av_{J_t}(g_\phi) > c c_I^\phi \geq y_I = Av_I(\phi)$ which is impossible by Lemma 3.1, since $I \in S_\phi$.

In any of the two cases b_1) and b_2) we have proved that we have $(X \setminus \Delta_\phi) \cap A_I = A_I \setminus (g_\phi = 0)$, while we showed that in case (i), $(X \setminus \Delta_\phi) \cap A_I = \emptyset$.

We remind that $\sum\limits_{I\in S_\phi}(a_I-\gamma_I)\overset{\phi}{\longrightarrow}0$. Since $\bigcup_{I\in S_\phi}A_I\approx X$ we conclude by the above discussion that $X\setminus\Delta_\phi\approx\left(\bigcup_{I\in S_{1,\phi}}A_I\right)\setminus E_\phi$, where $\mu(E_\phi)\to 0$ and $S_{1,\phi}$ is a subset of the subtree S_ϕ. Since now each $A_I, I\in S_{1,\phi}\subseteq S_\phi$ is written by Lemma 3.2 as a set difference of unions of elements of S_ϕ and Theorem 3.3 holds for such unions, we conclude by a diagonal argument and by passing if necessary to a subsequence, that

$$\lim_\phi\int_{\underset{I\in S_{1,\phi}}{\cup A_I}}(\mathcal{M}_\mathcal{T}\phi)^p d\mu=\omega_p(f^p/F)^p\cdot\lim_\phi\int_{\underset{I\in S_{1,\phi}}{\cup A_I}}\phi^p d\mu,\quad\text{so since}$$

$$\mu(E_\phi)\ \to\ 0\Longrightarrow\lim_\phi\int_{X\setminus\Delta_\phi}(\mathcal{M}_\mathcal{T}\phi)^p d\mu=\omega_p(f^p/F)^p\lim_\phi\int_{X\setminus\Delta_\phi}\phi^p d\mu.$$

Because now of the relation $\mathcal{M}_\mathcal{T}g_\phi\geq\mathcal{M}_\mathcal{T}\phi$,which holds μ-almost everywhere on X we have as a result that

$$\lim_\phi\int_{X\setminus\Delta_\phi}(\mathcal{M}_\mathcal{T}g_\phi)^p d\mu\geq\omega_p(f^p/F)^p\lim_\phi\int_{X\setminus\Delta_\phi}g_\phi^p d\mu. \tag{3.33}$$

Adding the relations (3.32) and (3.33) we have obtain $\lim\limits_\phi\int\limits_X(\mathcal{M}_\mathcal{T}g_\phi)^p d\mu\geq\omega_p(f^p/F)F$, which in fact is an equality since (g_ϕ) is an extremal sequence. So we must have equality in both (3.32) and (3.33). By using then the elementary inequality $x^p-y^p>(x-y)^p$ which holds for every $x>y>0$ and $p>1$, in view of the inequality $\mathcal{M}_\mathcal{T}g_\phi\geq cg_\phi$ on Δ_ϕ we must have that

$$\lim_\phi\int_{\Delta_\phi}|\mathcal{M}_\mathcal{T}g_\phi-cg_\phi|^p d\mu=0 \tag{3.34}$$

Similarly for $X\setminus\Delta_\phi$. That is

$$\lim_\phi\int_{\Delta_\phi}|\mathcal{M}_\mathcal{T}g_\phi-cg_\phi|^p d\mu=0 \tag{3.35}$$

Adding (3.34) and (3.35) we derive $lim_\phi||\mathcal{M}_\mathcal{T}g_\phi-cg_\phi||_{L^p}=0$, and by this we end the proof of our Lemma. □

We now proceed to

Lemma 3.6. *Under the above notation (3.30) is true.*

Proof. We just need to prove that

$$\lim_\phi\int_{\{g'_\phi\leq\phi\}}[\phi^p-(g'_\phi)^p]d\mu=0. \tag{3.36}$$

Then since

$$\lim_\phi\int_{\{g'_\phi\leq\phi\}}[\phi^p-(g'_\phi)^p]d\mu=\lim_\phi\int_{\phi\leq g'_\phi}[(g'_\phi)^p-\phi^p],\quad\text{and}\quad p>1$$

we have the desired result, in view of the inequality $(x-y)^p<x^p-y^p$, for $0<y<x$ and $p>1$.

We use the inequality

$$t \leq \frac{t^p}{p} + \frac{1}{q}, \text{ for every } t > 0 \text{ where } p, q > 1 \text{ such that } \frac{1}{p} + \frac{1}{q} = 1, \tag{3.37}$$

We set

$$\Delta^{(1)}_{I,\phi} = \{g'_\phi \leq \phi\} \cap A(\phi, I)$$
$$\Delta^{(2)}_{I,\phi} = \{\phi < g'_\phi\} \cap A(\phi, I).$$

Because of (3.37) if we write $c_{I,\phi}$ instead of c_I^ϕ and suppose that $c_{I,\phi} > 0$, we have that

$$\frac{1}{c_{I,\phi}}\phi(x) \leq \frac{1}{p}\frac{1}{c^p_{I,\phi}}\phi^p(x) + \frac{1}{q}, \quad \text{for every } \ x \in A_I = A(\phi, I).$$

Integrating over $\Delta^{(1)}_{I,\phi}$, and $\Delta^{(2)}_{I,\phi}$ we have that

$$\frac{1}{c_{I,\phi}} \int_{\Delta^{(j)}_{I,\phi}} \phi d\mu \leq \frac{1}{p}\frac{1}{c^p_{I,\phi}} \int_{\Delta^{(j)}_{I,\phi}} \phi^p d\mu + \frac{1}{q}\mu(\Delta^{(j)}_{I,\phi}), \quad \text{for } \ j = 1, 2, \quad I \in S_\phi$$

which gives

$$c^{p-1}_{I,\phi} \int_{\Delta^{(j)}_{I,p}} \phi d\mu \leq \frac{1}{p} \int_{\Delta^{(j)}_{I,\phi}} \phi^p d\mu + \frac{1}{q}\mu(\Delta^{(j)}_{I,\phi})c^p_{I,\phi}.$$

Note that the last inequality is satisfied even if $c_{I,\phi} = 0$. Summing the above for $I \in S_\phi$ we obtain

$$\sum_{I \in S_\phi} c^{p-1}_{I,\phi} \int_{\Delta^{(j)}_{I,\phi}} \phi d\mu \leq \frac{1}{p} \int_{\bigcup\limits_I \Delta^{(j)}_{I,\phi}} \phi^p d\mu + \frac{1}{q} \sum_{I \in S_\phi} \mu(\Delta^{(j)}_{I,\phi})c^p_{I,\phi}, \tag{3.38}$$

for $j = 1, 2 \Rightarrow$ (by adding the above to inequalities)

$$\sum_{I \in S_\phi} c^{p-1}_{I,\phi} \int_{A(\phi,I)} \phi d\mu \leq \frac{1}{p}F + \frac{1}{q} \sum_{I \in S_\phi} \mu(A(\phi, I))c^p_{I,\phi}. \tag{3.39}$$

The left hand side of (3.39) is equal to

$$\sum_{I \in S_\phi} c^{p-1}_{I,\phi}(c_{I,\phi}\gamma^\phi_I) = \sum_{I \in S_\phi} \gamma^\phi_I c^p_{I,\phi} = \int_X g^p_\phi d\mu$$

while the right hand side is equal to $\frac{1}{p}F + \frac{1}{q}\int\limits_X (g'_\phi)^p d\mu$. In the limit we have equality in the limit on (3.39), because of (3.28). This gives equality on (3.38) for $j = 1, 2$ in the limit. Thus for $j = 1$ we have that

$$\sum_{I \in S_\phi} c^{p-1}_{I,\phi} \int_{\Delta^{(1)}_{I,\phi}} \phi d\mu \approx \frac{1}{p} \sum_{I \in S_\phi} \int_{\Delta^{(1)}_{I,\phi}} \phi^p d\mu + \frac{1}{q} \sum_{I \in S_\phi} c^p_{I,\phi}\mu(\Delta^{(1)}_{I,\phi}) \Rightarrow$$

$$\int_{\{g'_\phi \leq \phi\}} \phi(g'_\phi)^{p-1} d\mu \approx \frac{1}{p} \int_{\{g'_\phi \leq \phi\}} \phi^p d\mu + \frac{1}{q} \int_{\{g'_\phi \leq \phi\}} (g'_\phi)^p d\mu. \tag{3.40}$$

We set

$$t_\phi = \left(\int_{\{g'_\phi \le \phi\}} \phi^p d\mu \right)^{1/p}, \quad S_\phi = \left(\int_{\{g'_\phi \le \phi\}} (g'_\phi)^p d\mu \right)^{1/p}.$$

Then

$$\int_{\{g'_\phi \le \phi\}} \phi (g'_\phi)^{p-1} d\mu \le t_\phi \cdot S_\phi^{p-1}, \quad \text{so (3.40) gives:}$$

$$\frac{1}{p} t_\phi^p + \frac{1}{q} S_\phi^p \underset{\phi}{\le} t_\phi \cdot S_\phi^{p-1}$$

so as a result we have because of (3.37) that

$$\frac{1}{p} t_\phi^p + \frac{1}{q} S_\phi^p \underset{\phi}{\approx} t_\phi \cdot S_\phi^{p-1}.$$

Since now in (3.37) we have equality only for $t = 1$, and t_ϕ, S_ϕ are bounded we conclude that

$$\frac{t_\phi^p}{S_\phi^p} \underset{\phi}{\longrightarrow} 1, \quad \text{so} \quad t_\phi^p - S_\phi^p \xrightarrow{\phi} 0 \Rightarrow \int_{\{g'_\phi \le \phi\}} [\phi^p - (g'_\phi)^p] d\mu \xrightarrow{\phi} 0,$$

which is (3.36). □

We have thus proved Theorem A. We mention it as

Theorem 3.4. *Let $(\phi_n)_n$ be a sequence of $\mathcal{T}$-good functions such that $\int_X \phi_n d\mu = f$ and $\int_X \phi_n^p d\mu = F$. Then $(\phi_n)_n$ is extremal if and only if*

$$\lim_n \int_X |\mathcal{M}_\mathcal{T} \phi_n - c\phi_n|^p d\mu = 0, \quad \textit{where} \quad c = \omega_p(f^p/F).$$

At last we mention that since $\mathcal{T}$-good functions include $\mathcal{T}$-step functions, in the case of $\mathbb{R}^n$, where the Bellman function is given by (1.4) for a fixed dyadic cube Q, we obtain the result in Theorem 3.4 for every sequence of Lesbesgue measurable functions $(\phi_n)_n$. In general in all interesting cases we do not need the hypothesis for ϕ_n to be $\mathcal{T}$-good since $\mathcal{T}$-simple functions are dense on $L^p(X, \mu)$.

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