# How unique is Lovász's theta function?

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***Abstract*—The famous Lovász's $\vartheta$ function is computable in polynomial time for every graph, as a semi-definite program (Grötschel, Lovász and Schrijver, 1981 [5]). The chromatic number and the clique number of every perfect graph $G$ are computable in polynomial time, since they are equal to $f_\vartheta(G) = \vartheta(\overline{G})$. Despite numerous efforts since the last three decades, recently stimulated by the Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour and Thomas, 2006 [2]), no combinatorial proof of this result is known.**

**In this work, we try to understand why the "key properties" of Lovász's $\vartheta$ function make it so "unique". We introduce an infinite set of convex functions, which includes the clique number $\omega$ and $f_\vartheta$. This set includes a sequence of linear programs which are monotone increasing and converging to $f_\vartheta$. We provide some evidences that $f_\vartheta$ is the unique function in this setting allowing to compute the chromatic number of perfect graphs in polynomial time.**

***Keywords—semi-definite programming;theta function.***

## I. Introduction

Berge introduced perfect graphs [1] in the early sixties, motivated from Shannon's problem of finding the zero-error capacity of a discrete memoryless channel [13]. A graph $G$ is a *perfect* graph if and only if $\omega(G') = \chi(G')$ holds for all induced subgraphs $G' \subseteq G$ (where the order of a largest clique of $G$ is its *clique number* $\omega(G)$, and the least number of colors required to assign different colors to adjacent nodes is its *chromatic number* $\chi(G)$).

Berge conjectured that a graph $G$ is perfect if and only if its complement $\overline{G}$ is perfect (the complement $\overline{G}$ has the same nodes as $G$, but two nodes are adjacent in $\overline{G}$ if and only if they are non-adjacent in $G$). This was proved by Lovász [9], who gave two short and elegant proofs.

A further conjecture of Berge was proved by Chudnovsky et al. [2] who characterized perfect graphs as precisely the graphs without chordless cycles $C_{2k+1}$ with $k \geq 2$, termed *odd holes*, or their complements, the *odd antiholes* $\overline{C}_{2k+1}$.

Perfect graphs have been extensively studied and turned out to be an interesting and important class of graphs with a rich structure. Most notably, the two in general hard to compute graph parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if $G$ is perfect [4].

The latter result relies on the following polyhedral characterization of perfect graphs. The *stable set polytope* $\mathrm{STAB}(G)$ is defined as the convex hull of the incidence vectors of all stable sets of $G$.

A canonical relaxation of $\mathrm{STAB}(G)$ is the *clique constraint stable set polytope*

$$\mathrm{QSTAB}(G) = \{\mathbf{x} \in \mathbb{R}_+^{|V|} : \sum_{i \in Q} x_i \leq 1, \ Q \subseteq G \text{ clique}\}.$$

We have $\mathrm{STAB}(G) \subseteq \mathrm{QSTAB}(G)$ in general and equality for perfect graphs [3] only. However, solving the stable set problem for a perfect graph $G$ by maximizing a linear objective function over $\mathrm{QSTAB}(G)$ does not work directly [4], but only via a detour involving a geometric representation of graphs [10] and the resulting *theta-body* $\mathrm{TH}(G)$ introduced by Lovász et al. [6].

An orthonormal representation of a graph $G = (V, E)$ is a sequence $(\mathbf{u_i} : i \in V)$ of $|V|$ unit-length vectors $\mathbf{u_i} \in \mathbb{R}^N$, where $N$ is some positive integer, such that $\mathbf{u_i}^T \mathbf{u_j} = 0$ for all $ij \notin E$. For any orthonormal representation of $G$ and any additional unit-length vector $\mathbf{c} \in \mathbb{R}^N$, the corresponding orthonormal representation constraint is $\sum_{i \in V} (\mathbf{c}^T \mathbf{u_i})^2 x_i \leq 1$. $\mathrm{TH}(G)$ denotes the convex set of all vectors $\mathbf{x} \in \mathbb{R}_+^{|V|}$ satisfying all orthonormal representation constraints for $G$. For any graph $G$, we have

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G).$$

The key property of $\mathrm{TH}(G)$ is that, for any graph $G$, the optimization problem

$$\vartheta(G) = \max\{\mathbb{1}^T \mathbf{x} : \mathbf{x} \in \mathrm{TH}(G)\}$$

can be solved in polynomial time [4]. This deep result relies on the fact that $\vartheta(G)$ can be characterized in many equivalent ways, e.g., as the

- optimum value of a semidefinite program,
- largest eigenvalue of a certain set of symmetric matrices,
- maximum value of a function involving orthonormal representation constraints,

see [5] for further details.

For perfect graphs, $\mathrm{STAB}(G)$ and $\mathrm{TH}(G)$ coincide which allows to compute the clique number by $\omega(G) = \vartheta(\overline{G})$ and the chromatic number by $\chi(G) = \omega(G)$ for perfect graphs $G$ in polynomial time.

Denote by $f_\vartheta$ the function defined by $f_\vartheta(G) = \vartheta(\overline{G})$ for every graph $G$. We shall call $f_\vartheta$ "the theta function", though it is actually the usual theta function applied to the complement of the input graph. Then $f_\vartheta$ satisfies the three assertions:

$P_1$) $f_\vartheta$ is computable in polynomial time for any graph $G$;

$P_2$) $f_\vartheta$ is monotonic with respect to homomorphism: if $G$ is homomorphic to $H$ then $f_\vartheta(G) \leq f_\vartheta(H)$;

$P_3$) $f_\vartheta$ is strictly monotonic on cliques: for every integer $i \geq 1$, $f_\vartheta(K_i) < f_\vartheta(K_{i+1})$ and the difference has a polynomial space encoding.

Graph homorphisms is a crucial concept in this paper as it has a prominent role with respect to clique and chromatic number. Recall that a graph $G$ is said to be homomorphic to $H$ if there is a mapping from the nodes of $G$ to the nodes of $H$, preserving adjacency. Then the clique number (resp. the chromatic number) of a graph $G$ is equal to the biggest (resp. smallest) integer $k$ such that $K_k$ (resp. $G$) is homomorphic to $G$ (resp. $K_k$).

The proof that the chromatic number of perfect graphs is computable in polynomial time relies on the three main properties introduced above. Indeed, take any real function $g$ satisfying $P_1$, $P_2$ and $P_3$. Let $G$ be a perfect graph with clique number $\omega$ and chromatic number $\chi$: $G$ is homomorphic to $K_\chi = K_\omega$ and $K_\omega$ is homomorphic to $G$. From property $P_2$, it follows that $g(G) = g(K_\omega)$. Let $n$ be the number of nodes of $G$. From property $P_1$, we may compute $g(G)$, $g(K_1), \ldots, g(K_n)$ in polynomial time. From property $P_3$, there is a unique index, say $k$, such that $g(G) = g(K_k)$ and we may determine it in polynomial time. Thus $\omega = k$ is computable in polynomial time.

Notice that is is easy to get functions satisfying two of the properties $P_1$, $P_2$ and $P_3$. Indeed, any constant function satisfies $P_1$ and $P_2$ (but not $P_3$), the function returning the number of nodes of a graph satisfies $P_1$ and $P_3$ (but not $P_2$), the clique number satisfies $P_2$ and $P_3$ (but not $P_1$).

However, there does not seem to be many functions satisfying $P_1$, $P_2$ and $P_3$, though $f_\vartheta$ is not the unique one, as some of its variants, such as the vectorial chromatic number [7] and the strong vectorial chromatic number [12], for instance, also satisfy these three properties.

The purpose of this work (which continues the considerations presented in [11]) is to investigate "how unique" the theta function is, by considering a more general setting, based on some convex supersets of SDP matrices.

The paper is organized as follows:

- In the second section, we define for every set of reals $X$ including $\{0,1\}$, a real function $f_X$. We give the basic properties of every function $f_X$, and establish that $f_{\{0,1\}} = \omega$ and $f_\mathbb{R} = f_\vartheta$.
- In the third section, we study functions $f_X$, such that $X$ is infinite.
- In the fourth section, we focus on the case of $X$ being finite and exhibit a sequence of linear programs monotone increasing and converging to $f_\vartheta$.

The results of sections 2 and 3 are the content of the third section of [11].

## II. Notations and basic properties

Let $\{0,1\} \subseteq X \subseteq \mathbb{R}$. For every graph $G = (V, E)$ with at least one edge, denote by $n$ its number of nodes and by $f_X(G)$ the value $1 - \frac{1}{s}$ where $s$ is the optimum of the following program:

$$\begin{aligned} \min \quad & s \\ s.t. \quad & \exists M \in \mathcal{M}_X \\ & M \text{ is symmetric} \\ & M_{ii} = 1, \forall i \in V \\ & M_{ij} = s, \forall ij \in E \end{aligned}$$

where $\mathcal{M}_X$ is defined as the following set of matrices:

$$\mathcal{M}_X \quad = \quad \{M \in \mathbb{R}^{V \times V}, \text{ s.t. }, \mathbf{u}^T M \mathbf{u} \geq 0, \forall \mathbf{u} \in X^V\}$$

If $G$ does not have any edge, we let $f_X(G) = 1$. If $M$ is a matrix of $\mathcal{M}_X$, we say that $M$ is feasible. A feasible matrix which yields the value $f_X(G)$ is called optimal.

Here are some basic observations, for every graph $G$:

- $f_\mathbb{R}(G) = \vartheta(\overline{G})$ (Lovász's theta function [10]), and thus $f_\mathbb{R}$ is computable in polynomial time with given accuracy;
- if $X \subseteq X'$ then $\mathcal{M}_{X'} \subseteq \mathcal{M}_X$ and thus $f_X(G) \leq f_{X'}(G)$.
- for every $\lambda \in \mathbb{R}^+$, $f_{\lambda X}(G) = f_X(G)$ as $\mathcal{M}_{\lambda X} = \mathcal{M}_X$.

Table 1 presents some numerical values $f_X(G)$ for some small graphs $G$ and the sets $X$ in $\{\{0,1\}, \{-1,0,1\}, \{-2,-1,0,1,2\}, \mathbb{R}\}$.

| | {0,1} | {-1,0,1} | {-2,-1,0,1,2} | $\mathbb{R}$ |
|---|---|---|---|---|
| $C_9 = K_{9/4}$ | 2 | 2.061 | | 2.064 |
| $C_7 = K_{7/3}$ | 2 | 2.103 | 2.1096 | 2.1099 |
| $C_5 = K_{5/2}$ | 2 | 2.200 | 2.231 | 2.236 |
| $K_{8/3}$ | 2 | 2.333 | | 2.343 |
| $K_{11/4}$ | 2 | 2.3996 | | 2.408 |
| $K_{10/3}$ | 3 | 3.125 | | 3.167 |
| $\overline{C_7} = K7/2$ | 3 | 3.222 | 3.294 | 3.318 |
| $K_{11/3}$ | 3 | 3.400 | | 3.452 |
| $\overline{C_9} = K_{9/2}$ | 4 | 4.231 | | 4.360 |
| Petersen | 2 | 2.5 | 2.5 | 2.5 |
| $\overline{\text{Petersen}}$ | 4 | 4 | 4 | 4 |
| $C_5$ + 1 multiplied node | 2 | 2.210526 | | 2.236 |

TABLE I

Some numerical result for $f_X$,
$X \in \{\{0,1\}, \{-1,0,1\}, \{-2,-1,0,1,2\}, \mathbb{R}\}$

**Lemma 1.** *$\mathcal{M}_X$ is a convex cone and a superset of the set of semi-definite positive matrices of size $n \times n$.*

We first compute the value $f_X$ for cliques:

**Lemma 2.** *$f_X(K_i) = i$ for every $i$.*

It follows from Lemma 2 that every function $f_X$ satisfies property $P_3$. We now establish in the following lemma that every function $f_X$ partially satisfies property $P_2$.

**Lemma 3.** *If $H$ is a subgraph of $G$ then $f_X(H) \leq f_X(G)$.*

This implies the so-called sandwich-property:

**Corollary 4.** $\omega(G) \leq f_X(G) \leq \vartheta(\overline{G}) \leq \chi(G)$

*Proof:* Due to Lemma 2 and Lemma 3, we have $\omega(G) \leq f_X(G)$. Furthermore, $f_X(G) \leq \vartheta(\overline{G})$ by definition of $\mathcal{M}_X$. ■

## III. $X$ INFINITE: THE ROLES OF THE CLIQUE NUMBER AND THE THETA FUNCTION

Multiplying a node $v$ of a graph $G$ means to replace $v$ by a stable set $S$ such that all nodes in $S$ have the same neighbors in $G$ as the original node $v$. Thus, multiplying a node of a graph $G$ gives a homomorphically equivalent graph $H$. Hence if $X$ is a set of reals such that $f_X$ satisfies the monotonic property $P_2$, then $f_X(G) = f_X(H)$. Thus $f_{\{-1,0,1\}}$ does not satisfy $P_2$ as multiplying a node of a $C_5$ yields a different value (see Table II). Therefore, additional constraints are needed for sets $X$ in order to ensure that property $P_2$ is fulfilled. We next show that being closed with respect to addition is such a sufficient condition:

**Lemma 5.** *Assume that $X$ is closed with respect to addition. If $G$ is homomorphic to $H$ then $f_X(G) \leq f_X(H)$ (monotonic property).*

If $X$ contains 0 and positive reals only then $f_X$ is the clique number:

**Lemma 6.** *For every graph $G$, $f_{\mathbb{R}+}(G) = \omega(G)$.*

As an obvious consequence of Lemma 6, we get:

**Corollary 7.** *$f_{\{0,1\}}$ is NP-hard to compute.*

Due to Lemma 6, the base set $X$ has to have one negative element, say -1, in order to get a function $f_X$ which is different from the clique number. If we apply the requirement of Lemma 5 to get a function satisfying the monotonic property then $X$ contains all integers. We next establish that this implies that $f_X$ has to be the theta function:

**Lemma 8.** *If $\mathbb{Z} \subseteq X$ or $[-1, 1] \subseteq X$ then $f_X(G) = \vartheta(\overline{G})$ for every graph $G$.*

These results show that the clique number and $f_\vartheta$ are two prominent functions when $X$ is infinite: we do not know whether there is a function $f_X$, with $X$ infinite, distinct of the clique number and $f_\vartheta$.

## IV. X FINITE: A SEQUENCE OF LINEAR PROGRAMS CONVERGING TO THE THETA FUNCTION

For every positive integer $k$, let $X_k$ denote the set of integers $\{-k, -(k-1), \dots, -1, 0, 1, \dots, k-1, k\}$, and $f_k$ be the function $f_{X_k}$. Notice that for every graph $G$ with $n$ nodes, the value $f_k(G)$ is the output of a linear program with exponentially many constraints (approximatively $(2k+1)^n$ constraints). Furthermore, $f_k(G)$ is a rational for every $k$ and graph $G$ (and thus distinct of $f_\vartheta$). The sequence $f_k(G)_{k\geq 1}$ is an increasing sequence, as $X_{k-1} \subsetneq X_k$ (for every $k \geq 2$).

Hence we have, for every graph $G$,

$$\omega(G) \leq f_1(G) \leq f_2(G) \leq \ldots \leq f_k(G) \leq f_\vartheta(G).$$

We establish that

$$\lim_k f_k(G) = f_\vartheta(G)$$

holds for every graph $G$ as a consequence of the following lemma and Lemma 8:

**Lemma 9.** *Let $Y_1 \subset Y_2 \subset \cdots$ be a monotonous chain of subsets containing $\{0, 1\}$ and set $Y = \cup_k Y_k$. For every graph $G$ we have*

$$f_Y(G) = \lim_k f_{Y_k}(G).$$

Due to Lemma 9, the sequence $f_k$ is converging to $f_\vartheta$.

Notice that for graphs $G$ such that $f_1(G) = f_\vartheta(G)$ (e.g. perfect graphs) then $f_1(G) = f_2(G) = \ldots = f_k(G)$ holds for every $k$. We do not know whether the sequence is strictly increasing for graphs $G$ such that $f_1(G) \neq f_\vartheta(G)$, but suspect that it is. In particular, computer experiments suggest that $f_k(C_5) < f_{k+1}(C_5)$ for every positive integer $k$.

We believe that none of the functions $f_k$ is monotonic with respect to homorphisms but were not yet able to prove it.

In Lemma 5, the set $X$ is assumed to be closed with respect to addition, a property which is satisfied by none of the sets $X_k$. We used this assumption in the proof of Lemma 5 by constructing an optimal matrix for a graph $G$ with a duplicated node: the construction consists of duplicating one row and one column.

The next lemma shows that if $f_k$ is monotonic with respect to homorphism, then every optimal matrix for a graph is such a matrix with "one duplicated row and one duplicated column". This suggests that $X_k$ has "somehow" to be closed with respect to addition, a contradiction.

**Lemma 10.** *Let $H$ be a circulant graph (that is a Cayley on a cyclic group) and let $G$ be obtained from $H$ by duplicating a node. If $f_k(G) = f_k(H)$ then every optimal matrix of $M$ is obtained from an optimal matrix of $H$ by duplicating one row and column.*

## V. CONCLUDING REMARKS

Our study seems to indicate that the clique number function and the theta function are the only functions in our setting that satisfy the monotonic requirement with respect to homomorphism (property $P_2$). Hence in this sense, the theta function is really unique, since it is also computable in polynomial time (property $P_1$).

As of the sandwich property, we point out that it holds even if the monotonic property is not satisfied (Corollary 4): there are many different functions $f_X$ in between the clique and the chromatic number, all of them being a lower bound for the theta function.

For further works, it is worth to notice that the numerical values presented in Table II suggest that the function $f_{\{-1,0,1\}}$ gives already good lower bounds for the theta function.

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