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Binary Modal Companions for Subintuitionistic Logics

Dick de Jongh and Fatemeh Shirmohammadzadeh Maleki

Abstract The weak subintuitionistic logic WF, for which no standard unary modal companion is known, is found to have a strict implication logic as its binary modal companion. It is also shown that for all modal logics extending the weak logic EN, classical modal logic with necessitation, a strict implication logic exists which is essentially equivalent to it. This logic extends a basic strict implication logic plus an axiom U, and conversely each such logic corresponds to a modal logic extending EN. Among other things this means that any subintuitionistic logic which has a modal companion has a strict implication companion as well.

1 Introduction

Subintuitionistic logics as a theme were first studied by G. Corsi [7], who introduced a basic system F. The system F, which cannot prove formulas like $A \to (B \to A)$ and $A \to (B \to A \wedge B)$, has Kripke frames in which no assumption of preservation of truth is made and which are neither reflexive nor transitive. She also introduced Gödel-type translations of these systems into modal logic. Restall [16] defined a similar system SJ (see also [9]). Basic logic BPC, a much studied extension of F, had already been introduced before by Visser [22] in a study mainly focussed on a further extension FPC of BPC with a provability interpretation. The system BPC has irreflexive Kripke frames with transitivity and preservation. A considerable amount work in

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the area, especially on BPC, has been done by Ardeshir in cooperation with members of his school and with W. Ruitenburg (see e.g. [1, 3]).

In our papers [10, 11, 18] we introduced a basic logic WF much weaker than F, and we developed two types of neighborhood semantics for this logic and its extensions. In [10] we discussed the strength of the various subintuitionistic logics by investigating which part of intuitionistic logic IPC they are able to prove. A translation from IPC into BPC discovered by [2] played an important role.

Furthermore, we discovered modal companions for a number of logics extending $\mathsf{WF_N}$, an extension of WF by a rule. The logic WF did not lend itself to our treatment because its semantics is too different from the usual neighborhood semantics for modal logic. In the present paper we looked for a binary modal companion for WF instead of the usual unary one. This exploration was successful as we will show. It also lead us to investigate the notion of binary modal logic, its neighborhood semantics and its relation to ordinary unary modal logic, and more specifically what we call classical strict implication logic, for which we give a complete basic system $\mathsf{E}^2_{\mathsf{Imp}}$. In fact, we show that all extensions L of the weak logic EN (classical modal logic with necessitation) have a unique counterpart logic L^* with a strict implication. All logics extending $\mathsf{E}^2_{\mathsf{Imp}}$ plus an axiom U are such a counterpart L^*, each is mutually interpretable with L, shares with it the usual logical properties and functions as a modal companion to the same subintuitionistic logics. This result exhibits which conditional logics can be represented in ordinary unary modal logic, at least if one restricts one's attention to modal logics extending EN, which indeed does seem to be a bare minimum.

2 Neighborhood semantics for modal and subintuitionistic logics

In this section we will give in Subsection 2.1 an introduction to the usual neighborhood semantics for modal logic followed in Subsection 2.2 by a quick survey of our neighborhood semantics for subintuitionistic logics and a summary of the results previously obtained by us.

2.1 Neighborhood semantics for modal Logic

Definition 1 The **modal language** $\mathcal{L}^{\Box}(At)$ is the smallest set of formulas generated by the following grammar, where $p \in At$:

$$p \mid \neg A \mid A \wedge B \mid \Box A.$$

The sublanguage $\mathcal{L}_c(At)$ of $\mathcal{L}^{\Box}(At)$ containing its formulas without $\Box$ is the language of (classical) propositional logic. We add to $\mathcal{L}_c(At)$ the symbols $\to$, $\leftrightarrow$, $\top$ and $\bot$ as symbols defined in the usual way.

Definition 2 A pair $\mathfrak{F} = \langle W, N\rangle$ is a ***Neighborhood Frame*** of modal logic if W is a non-empty set and N is a function from W into $\mathcal{P}(\mathcal{P}(W))$.
In a ***Neighborhood Model*** $\mathfrak{M} = \langle W, N, V\rangle$, $V : At \to \mathcal{P}(W)$ is a valuation function on the set of propositional variables At.

Definition 3 Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model and $w \in W$.
Truth of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \models p \quad \Leftrightarrow \quad w \in V(p)$,
2. $\mathfrak{M}, w \models \neg A \quad \Leftrightarrow \quad \mathfrak{M}, w \not\models A$,
3. $\mathfrak{M}, w \models A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \models A$ and $\mathfrak{M}, w \models B$,
4. $\mathfrak{M}, w \models \Box A \quad \Leftrightarrow \quad A^{\mathfrak{M}} \in N(w)$,

where $A^{\mathfrak{M}}$ denotes the truth set of A.

We consider the following axiom schemas and rules.

PC Any axiomatization of propositional calculus

N $\Box\top$

RE $\dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$

MP $\dfrac{A \quad A \to B}{B}$

Nec $\dfrac{A}{\Box A}$

E is the smallest classical modal logic containing all instances of PC which is closed under the rules MP and RE. The logic EN extends E by adding the axiom scheme N, or by adding the rule Nec [15].

Theorem 1

1. *The logic* E *is sound and strongly complete with respect to the class of all neighborhood frames* [15].
2. *The logic* EN *is sound and strongly complete with respect to the class of neighborhood frames that contain the unit, i.e. for all* $w \in W$, $W \in N(w)$ [15].

2.2 Neighborhood semantics for Subintuitionistic Logics

Definition 4 The **language of intuitionistic propositional logic** $\mathcal{L}(At)$ is the smallest set of formulas generated by the following grammar, where $p \in At$:

$$p \mid A \wedge B \mid A \vee B \mid A \to B \mid \bot$$

As usual we consider $\mathcal{L}(At)$ to be an extension of $\mathcal{L}_c(At)$, so we will write $\to$ for both intuitionistic and classical implication. From the context it should be clear which is meant. To $\mathcal{L}(At)$ the the symbols $\neg$ and $\leftrightarrow$ are added as defined symbols in the usual manner. Again this should not create confusion with the symbols of classical propositional logic.

Definition 5 An **NB-Neighborhood Frame** $\mathfrak{F} = \langle W, NB\rangle$ for subintuitionistic logic consists of a non-empty set W, and a function NB from W into $\mathcal{P}((\mathcal{P}(W))^2)$ such that:

$$\forall w \in W,\ \forall X, Y \in \mathcal{P}(W)\ (X \subseteq Y \ \Rightarrow\ (X, Y) \in NB(w)).$$

In an **NB-Neighborhood Model** $\mathfrak{M} = \langle W, NB, V\rangle$, $V : At \to \mathcal{P}(W)$ is a valuation function on the set of propositional variables At.

Definition 6 Let $\mathfrak{M} = \langle W, NB, V\rangle$ be an NB-neighborhood model.
Truth of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \Vdash p \quad \Leftrightarrow \quad w \in V(p)$;
2. $\mathfrak{M}, w \Vdash A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ and $\mathfrak{M}, w \Vdash B$;
3. $\mathfrak{M}, w \Vdash A \vee B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ or $\mathfrak{M}, w \Vdash B$;
4. $\mathfrak{M}, w \Vdash A \to B \quad \Leftrightarrow \quad \left(A^{\mathfrak{M}}, B^{\mathfrak{M}}\right) \in NB(w)$;
5. $\mathfrak{M}, w \nVdash \bot$.

A is **valid** in $\mathfrak{M}$, $\mathfrak{M} \Vdash A$, if for all $w \in W$, $\mathfrak{M}, w \Vdash A$, and A is valid in $\mathfrak{F}$, $\mathfrak{F} \Vdash A$ if for all $\mathfrak{M}$ on $\mathfrak{F}$, $\mathfrak{M} \Vdash A$. We write $\Vdash A$ if $\mathfrak{M} \Vdash A$ for all $\mathfrak{M}$. Also we define $\Gamma \Vdash A$ iff for all $\mathfrak{M}$, $w \in \mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Gamma$ then $\mathfrak{M}, w \Vdash A$.

Definition 7 WF is the logic given by the following axiom schemas and rules,

1. $A \to A \vee B$	2. $B \to A \vee B$	3. $A \to A$
4. $A \wedge B \to A$	5. $A \wedge B \to B$	6. $\frac{A \quad A\to B}{B}$
7. $\frac{A\to B \quad A\to C}{A\to B\wedge C}$	8. $\frac{A\to C \quad B\to C}{A\vee B\to C}$	9. $\frac{A\to B \quad B\to C}{A\to C}$
10. $\frac{A}{B\to A}$	11. $\frac{A\leftrightarrow B \quad C\leftrightarrow D}{(A\to C)\leftrightarrow(B\to D)}$	12. $\frac{A \quad B}{A\wedge B}$
13. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$		14. $\bot \to A$

$\Gamma \vdash_{\mathsf{WF}} A$ iff there is a derivation of A from Γ using the rules 7,8,9,10,11 only when there are no assumptions, and the rule 6, MP, only when the derivation of $A \to B$ contains no assumptions.

For a discussion of the definition of $\Gamma \vdash_{\mathsf{WF}} A$ see Definition 4 of [11] and its introduction.

Theorem 2 (Weak Deduction Theorem, [18] Theorem 2.19)
$A \vdash_{\mathsf{WF}} B$ *iff* $\vdash_{\mathsf{WF}} A \to B$.
$A_1, \dots, A_n \vdash_{\mathsf{WF}} B$ *iff* $\vdash_{\mathsf{WF}} A_1 \wedge \cdots \wedge A_n \to B$.

Theorem 3 *The logic* WF *is sound and strongly complete with respect to the class of* NB-*neighborhood frames.*

We now define a second type of neighborhood semantics for subintuitionistic logics, N-neighborhood frames and models. In fact these are exactly the same frames and models as for modal logic, except of course for the truth definition. This may be confusing but it enables us to compare the logics very comfortably.

Definition 8 $\mathfrak{F} = \langle W, N \rangle$ is an ***N*-Neighborhood Frame** of subintuitionistic logic if W is a non-empty set, N is a function from W into $\mathcal{P}(\mathcal{P}(W))$, and for each $w \in W$, $W \in N(w)$.
Valuation $V : At \to \mathcal{P}(W)$ makes $\mathfrak{M} = \langle W, N, V \rangle$ an ***N*-Neighborhood Model**. **Truth** of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \Vdash p \quad \Leftrightarrow \quad w \in V(p)$;
2. $\mathfrak{M}, w \Vdash A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ and $\mathfrak{M}, w \Vdash B$;
3. $\mathfrak{M}, w \Vdash A \vee B \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash A$ or $\mathfrak{M}, w \Vdash B$;
4. $\mathfrak{M}, w \Vdash A \to B \quad \Leftrightarrow \quad \{v \mid v \Vdash A \ \Rightarrow \ v \Vdash B\} = \overline{A^{\mathfrak{M}}} \cup B^{\mathfrak{M}} \in N(w)$;
5. $\mathfrak{M}, w \nVdash \bot$.

A formula A is **valid** in $\mathfrak{M}$, $\mathfrak{M} \Vdash A$, if for all $w \in W$, $\mathfrak{M}, w \Vdash A$, and A is valid in $\mathfrak{F}$, $\mathfrak{F} \Vdash A$ if for all $\mathfrak{M}$ on $\mathfrak{F}$, $\mathfrak{M} \Vdash A$. We write $\Vdash A$ if $\mathfrak{M} \Vdash A$ for all $\mathfrak{M}$. Also we define $\Gamma \Vdash A$ iff for all $\mathfrak{M}, w \in \mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Gamma$ then $\mathfrak{M}, w \Vdash A$.

The question whether validity in NB-neighborhood frames and N-neighborhood frames is the same was resolved in [11]. The difference resides in the rule N. To the system WF we add this rule to obtain the logic $\mathsf{WF_N}$:

$$\frac{A \to B \vee C \quad C \to A \vee D \quad A \wedge C \wedge D \to B \quad A \wedge C \wedge B \to D}{(A \to B) \leftrightarrow (C \to D)} \qquad (\mathsf{N})$$

Theorem 4 (Weak Deduction Theorem, [11] Theorem 8)
$A \vdash_{\mathsf{WF_N}} B$ *iff* $\vdash_{\mathsf{WF_N}} A \to B$.
$A_1, \dots, A_n \vdash_{\mathsf{WF_N}} B$ *iff* $\vdash_{\mathsf{WF_N}} A_1 \wedge \cdots \wedge A_n \to B$.

Theorem 5 (Completeness of $\mathsf{WF_N}$, [11] Theorem 12)

The logic $\mathsf{WF_N}$ *is sound and strongly complete with respect to the class of* N-*neighborhood frames.*

We consider the translation $\Box$ from $\mathcal{L}(At)$, the language of intuitionistic propositional logic, to $\mathcal{L}^{\Box}(At)$, the language of modal propositional logic (see [7, 10]). It is given by:

1. $p^{\Box} = p$;
2. $\bot^{\Box} = \bot$;
3. $(A \wedge B)^{\Box} = A^{\Box} \wedge B^{\Box}$;
4. $(A \vee B)^{\Box} = A^{\Box} \vee B^{\Box}$;
5. $(A \to B)^{\Box} = \Box(A^{\Box} \to B^{\Box})$.

Note that in (5.) above the first $\to$ is a symbol from $\mathcal{L}$ whereas the second $\to$ is a defined symbol of $\mathcal{L}_c$. This need not cause confusion since from the context in which $\to$ occurs it will always be clear in which language it occurs.

Theorem 6 ([10], Theorem 5.17) *For all formulas A,*

$$\vdash_{\mathsf{WF_N}} A \quad \text{iff} \quad \vdash_{\mathsf{EN}} A^{\Box}.$$

As one says, EN is a *modal companion* of $\mathsf{WF_N}$. For WF the question how to provide it with a modal companion was left open in [10]. It is not easy to imagine a modal logic which weakens EN but leaves N in.

3 A complete basic system for strict implication

In this section we define a neighborhood semantics for modal logic with a binary operator and we introduce a basic system which is sound and complete for this semantics. One might not consider it to be quite proper to call this basic system a system of strict implication since it allows extensions to systems for counterfactuals but it is the best we have come up with.

Definition 9 The **strict implication language** $\mathcal{L}^{\Rightarrow}(At)$ is the smallest set of formulas generated by the following grammar, where $p \in At$:

$$p \mid \neg A \mid A \wedge B \mid A \Rightarrow B.$$

As in the case the modal language the language $\mathcal{L}_c(At)$ is a sublanguage of $\mathcal{L}^{\Rightarrow}(At)$, and we again have the usual defined symbols. The NB-neighborhood frames and models of subintuitionistic logic can be used as frames and models for strict implication logic, again with a different truth definition.

Definition 10 A pair $\mathfrak{F} = \langle W, NB \rangle$ is called a **Neighborhood Frame** of strict implication logic if W is a non-empty set and NB is a neighborhood function from W into $\mathcal{P}((\mathcal{P}(W))^2)$ such that

$$\forall w \in W,\ \forall X, Y \in \mathcal{P}(W),\ (X \subseteq Y \ \Rightarrow\ (X, Y) \in NB(w)).$$

If we delete the final requirement on the neighborhood function we obtain a more general semantics for binary modal logic, but in this article we focus on implication because this is all we are interested in at this point. Generally, our results will stand when we delete this condition. The results will then concern not EN but E.

Definition 11 A **Neighborhood Model** of strict implication logic is a tuple $\mathfrak{M} = \langle W, NB, V \rangle$, where $\langle W, NB \rangle$ is a neighborhood frame of strict implication logic and $V : At \to \mathcal{P}(W)$ a valuation function.

Definition 12 Let $\mathfrak{M} = \langle W, NB, V \rangle$ be a neighborhood model for strict implication logic and $w \in W$. **Truth** of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \models p \quad \Leftrightarrow \quad w \in V(p)$,
2. $\mathfrak{M}, w \models \neg A \quad \Leftrightarrow \quad \mathfrak{M}, w \not\models A$,
3. $\mathfrak{M}, w \models A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \models A$ and $\mathfrak{M}, w \models B$,
4. $\mathfrak{M}, w \models A \Rightarrow B \quad \Leftrightarrow \quad \big(A^{\mathfrak{M}}, B^{\mathfrak{M}}\big) \in NB(w)$,

where $A^{\mathfrak{M}}$ denotes the truth set of A.

Definition 13 A formula A is **valid in a model** $\mathfrak{M}= \langle W, NB, V \rangle$, $\mathfrak{M} \models A$, if for all $w \in W$, $\mathfrak{M}, w \models A$. If all models force A, we write $\models A$ and call A **valid**. A formula A is **valid on a frame** $\mathfrak{F} = \langle W, NB \rangle$, $\mathfrak{F} \models A$ if A is valid in every model based on that frame. We write $\Gamma \models A$, A is a **valid consequence of** Γ, if, for each model $\mathfrak{M}= \langle W, NB, V \rangle$ and $w \in W$, if $\mathfrak{M}, w \models \Gamma$, then $\mathfrak{M}, w \models A$.

The definitions above mean that a model $\mathfrak{M}= \langle W, NB, V \rangle$ will simultaneously be a model for the subintuitionistic language and for the strict implication language. This will enable us to compare the languages and the systems formulated in them directly in Section 4. We will then, to avoid confusion, use different symbols for the two notions of $\models$.

In this section we will be interested in the following axiom schemas and rules.

$$E^2 \quad \frac{A \leftrightarrow B \quad C \leftrightarrow D}{(A \Rightarrow C) \leftrightarrow (B \Rightarrow D)}$$

$$Imp \quad \frac{A \to B}{A \Rightarrow B}$$

Definition 14 $\mathsf{E}^2_{\mathsf{Imp}}$ is the smallest set of formulas containing all instances of PC closed under the rules E^2, Imp and MP. We call it **Classical Strict Implication Logic**.

If one leaves out the rule Imp, then one obtains what one might call *Classical Binary Modal Logic*. In fact, this logic occurs as CK in [5]. We won't discuss it here, but as said, basically our results will extend to that more general case. We will now prove the completeness of $\mathsf{E}^2_{\mathsf{Imp}}$ in a rather standard way (compare [18]).

Definition 15 Let $W_{\mathsf{E}^2_{\mathsf{Imp}}}$ be the set of all $\mathsf{E}^2_{\mathsf{Imp}}$-maximally consistent sets of formulas. Given a formula A, we define the set $[\![A]\!]$ as follows,

$$[\![A]\!] = \left\{ \Delta \mid \Delta \in W_{\mathsf{E}^2_{\mathsf{Imp}}},\ A \in \Delta \right\}.$$

Lemma 1 *Let C and D are formulas. Then*

(a) $[\![C \wedge D]\!] = [\![C]\!] \cap [\![D]\!]$.
(b) $[\![C \vee D]\!] = [\![C]\!] \cup [\![D]\!]$.
(c) If $[\![C]\!] \subseteq [\![D]\!]$ then $\vdash C \to D$.
(d) $[\![C]\!] = [\![D]\!]$ iff $\vdash C \leftrightarrow D$.

Proof The proofs are easy. □

Definition 16 The **Canonical model** $\mathfrak{M}^{\mathsf{E}^2_{\mathsf{Imp}}} = \langle W_{\mathsf{E}^2_{\mathsf{Imp}}}, NB_{\mathsf{E}^2_{\mathsf{Imp}}}, V\rangle$ of $\mathsf{E}^2_{\mathsf{Imp}}$ is defined by:

1. For each $\Gamma \in W_{\mathsf{E}^2_{\mathsf{Imp}}}$ and all formulas A and B,

$$NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma) = \{([\![A]\!], [\![B]\!]) \mid A \Rightarrow B \in \Gamma\} \cup \{(X, Y) \mid X \subseteq Y\}.$$

2. If $p \in At$, then $V(p) = [\![p]\!] = \left\{ \Gamma \mid \Gamma \in W_{\mathsf{E}^2_{\mathsf{Imp}}} \text{ and } p \in \Gamma \right\}$.

In the completeness proof we need to be sure that, if $([\![A]\!], [\![B]\!]) \in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma)$, then $A \Rightarrow B \in \Gamma$.

Lemma 2 *If $NB_{\mathsf{E}^2_{\mathsf{Imp}}} : W_{\mathsf{E}^2_{\mathsf{Imp}}} \to \mathcal{P}((\mathcal{P}(W_{\mathsf{E}^2_{\mathsf{Imp}}}))^2)$ is a function such that for each $\Gamma \in W_{\mathsf{E}^2_{\mathsf{Imp}}}$, $NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma) = \{([\![A]\!], [\![B]\!]) \mid A \Rightarrow B \in \Gamma\} \cup \{(X, Y) \mid X \subseteq Y\}$. Then $([\![A]\!], [\![B]\!]) \in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma)$ implies $A \Rightarrow B \in \Gamma$.*

Proof Assume $([\![A]\!], [\![B]\!]) \in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma)$. This gives us two possibilities:

1. For some C, D, $[\![A]\!] = [\![C]\!]$, $[\![B]\!] = [\![D]\!]$, $C \Rightarrow D \in \Gamma$,
2. $[\![A]\!] \subseteq [\![B]\!]$.

If (1), then by Lemma 1, we have $\vdash A \leftrightarrow C$ and $\vdash B \leftrightarrow D$. Hence by rule E^2 we will have $\vdash (A \Rightarrow B) \leftrightarrow (C \Rightarrow D)$. By assumption, $C \Rightarrow D \in \Gamma$. Hence, $A \Rightarrow B \in \Gamma$.

If (2), then by Lemma 1, we have $\vdash A \to B$. Then by rule Imp we will have $\vdash A \Rightarrow B$. Hence, $A \Rightarrow B \in \Gamma$. □

Theorem 7 (***Truth Lemma***) *For any consistent formula D, if $\mathfrak{M}$ is the canonical model of $\mathsf{E}^2_{\mathsf{Imp}}$, then $D^{\mathfrak{M}} = [\![D]\!]$.*

Proof We only consider the $D := A \Rightarrow B$ case, the other cases are as usual. Let $\Gamma \in W_{\mathsf{E}^2_{\mathsf{Imp}}}$, then,

$$\begin{array}{rcl} \Gamma \models A \Rightarrow B & \iff & (A^{\mathfrak{M}}, B^{\mathfrak{M}}) \in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma) \\ \text{(by induction hypothesis)} & \iff & ([\![A]\!], [\![B]\!]) \in NB_{\mathsf{E}^2_{\mathsf{Imp}}}(\Gamma) \\ \text{(by Lemma 2)} & \iff & A \Rightarrow B \in \Gamma. \end{array}$$

□

Theorem 8 *The classical strict implication logic* $\mathsf{E}^2_{\mathsf{Imp}}$ *is sound and strongly complete with respect to the class of neighborhood frames.*

Proof Soundness is straightforward. For strong completeness, suppose Σ is a consistent set of the classical strict implication logic $\mathsf{E}^2_{\mathsf{Imp}}$. By Lindenbaum's Lemma there is a maximal consistent set Σ^* extending Σ. Then by Lemma 7, $\mathfrak{M}^{\mathsf{E}^2_{\mathsf{Imp}}}, \Sigma^* \models \Sigma$, and we have shown that each consistent set has a model.□

4 Modal Companions

We consider the translation $\Rightarrow$ from $\mathcal{L}$, the language of intuitionistic propositional logic, to $\mathcal{L}^{\Rightarrow}$, the language of classical strict implication logic. It is given by:

1. $p^{\Rightarrow} := p$;
2. $\bot^{\Rightarrow} := \bot$;
3. $(A \wedge B)^{\Rightarrow} := A^{\Rightarrow} \wedge B^{\Rightarrow}$;
4. $(A \vee B)^{\Rightarrow} := A^{\Rightarrow} \vee B^{\Rightarrow}$;
5. $(A \to B)^{\Rightarrow} := (A^{\Rightarrow} \Rightarrow B^{\Rightarrow})$.

As said above we can use neighborhood models to interpret subintuitionistic formulas and modal or strict implication formulas simulatneously. We distinguish these uses by writing $\Vdash$ for truth for subintuitionistic formulas and $\models$ for truth for classical strict implication formulas.

Lemma 3 *Let* $\mathfrak{M} = \langle W, NB, V\rangle$ *be a neighborhood model. Then for all* $w \in W$,

$$\mathfrak{M}, w \Vdash A \quad \text{iff} \quad \mathfrak{M}, w \models A^{\Rightarrow}.$$

Proof The proof is by induction on A. The atomic case holds by induction and the conjunction and disjunction cases are easy. We only check the implication case. So let $A = C \to D$, then

$$\begin{array}{rl} \mathfrak{M}, w \Vdash C \to D & \iff (C^{\mathfrak{M}}, D^{\mathfrak{M}}) \in NB(w) \\ \text{(by induction hypothesis)} & \iff ((C^{\Rightarrow})^{\mathfrak{M}}, (D^{\Rightarrow})^{\mathfrak{M}}) \in NB(w) \\ & \iff \mathfrak{M}, w \models C^{\Rightarrow} \Rightarrow D^{\Rightarrow} \\ & \iff \mathfrak{M}, w \models (C \to D)^{\Rightarrow}. \end{array}$$
□

Theorem 9 *For all formulas A,*

$$\vdash_{\mathsf{WF}} A \quad \text{iff} \quad \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^{\Rightarrow}.$$

Proof By Theorem 8 and Lemma 3. □

Lemma 4 *If* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A \Rightarrow B$ *then* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A \to B$.

Proof Suppose that there is a model $\mathfrak{M} = \langle W, NB, V\rangle$ and a point $w \in W$ such that $\mathfrak{M}, w \nvDash_{\mathsf{E}^2_{\mathsf{Imp}}} A \to B$. Then, $\mathfrak{M}, w \models_{\mathsf{E}^2_{\mathsf{Imp}}} A$ and $\mathfrak{M}, w \nvDash_{\mathsf{E}^2_{\mathsf{Imp}}} B$, therefore $A^{\mathfrak{M}} \nsubseteq B^{\mathfrak{M}}$. Let $\mathfrak{F}'$ be $\mathfrak{F}$ augmented by a g such that $NB(g) = \{(X, Y) \mid X \subseteq Y\}$ and $\mathfrak{M}' = \langle \mathfrak{F}', V\rangle$. Since $A^{\mathfrak{M}} \nsubseteq B^{\mathfrak{M}}$ and hence $\mathfrak{M}', g \nvDash_{\mathsf{E}^2_{\mathsf{Imp}}} A \Rightarrow B$, we have $\nvDash_{\mathsf{E}^2_{\mathsf{Imp}}} A \Rightarrow B$. □

The following theorem, proved by using the Weak Deduction Theorem (2) and Lemmas 9 and 4, shows that the translation works under assumptions.

Theorem 10 $\Gamma \vdash_{\mathsf{WF}} A$ iff $\Gamma^{\Rightarrow} \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^{\Rightarrow}$.

Proof $B_1, \ldots, B_k \vdash_{\mathsf{WF}} A \;\Leftrightarrow\; \vdash_{\mathsf{WF}} B_1 \wedge \cdots \wedge B_k \to A \;\Leftrightarrow$
$\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (B_1 \wedge \cdots \wedge B_k \to A)^{\Rightarrow} \;\Leftrightarrow\; \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (B_1^{\Rightarrow} \wedge \cdots \wedge B_k^{\Rightarrow} \Rightarrow A^{\Rightarrow}) \;\Leftrightarrow$
$\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} B_1^{\Rightarrow} \wedge \cdots \wedge B_k^{\Rightarrow} \to A^{\Rightarrow} \;\Leftrightarrow\; B_1^{\Rightarrow}, \ldots, B_k^{\Rightarrow} \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^{\Rightarrow}$. □

5 Translations

In this section we will show that $\mathsf{E}^2_{\mathsf{Imp}}$ and EN are very closely related by translations. The first section will treat formulas, the second will extend this to logics, and in the third we will show what happens to axiomatizations.

5.1 Translations between $\mathsf{E}^2_{\mathsf{Imp}}$ and EN

Definition 17 The mapping * from $\mathcal{L}^{\Box}$ to $\mathcal{L}^{\Rightarrow}$ is defined by

1. $(p)^* := p$,
2. $(\neg A)^* := \neg A^*$,
3. $(A \wedge B)^* := A^* \wedge B^*$,
4. $(\Box A)^* := \top \Rightarrow A^*$.

Theorem 11 *If* $\vdash_{\mathsf{EN}} A$, *then* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$.

Proof We use induction on the derivation of A. We only consider the rules Nec and E. First rule $\frac{A}{\Box A}$:

1. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$ by induction hypothesis
2. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} \top \to A^*$ by 1
3. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} \top \Rightarrow A^*$ by 2 and rule Imp
4. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (\Box A)^*$ by 3

Rule $\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$:

1. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^* \leftrightarrow B^*$ by induction hypothesis

2. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} \top \leftrightarrow \top$
3. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (\top \Rightarrow A^*) \leftrightarrow (\top \Rightarrow B^*)$ by 1, 2 and rule E^2
4. $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} (\Box A)^* \leftrightarrow (\Box B)^*$ by 3 □

Definition 18 The mapping $^\sharp$ from $\mathcal{L}^{\Rightarrow}$ to $\mathcal{L}^{\Box}$ is defined by

1. $(p)^\sharp := p$,
2. $(\neg A)^\sharp := \neg A^\sharp$,
3. $(A \wedge B)^\sharp := A^\sharp \wedge B^\sharp$,
4. $(A \Rightarrow B)^\sharp := \Box(A^\sharp \to B^\sharp)$.

Theorem 12 *If* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A$*, then* $\vdash_{\mathsf{EN}} A^\sharp$.

Proof We use induction on the derivation of A. We only consider the rules Imp and E^2. First, rule $\frac{A \to B}{A \Rightarrow B}$:

1. $\vdash_{\mathsf{EN}} (A \to B)^\sharp$ by induction hypothesis
2. $\vdash_{\mathsf{EN}} A^\sharp \to B^\sharp$ by 1
3. $\vdash_{\mathsf{EN}} \Box(A^\sharp \to B^\sharp)$ by 2 and rule Nec
4. $\vdash_{\mathsf{EN}} (A \Rightarrow B)^\sharp$ by 3

Rule $\dfrac{A \leftrightarrow B \quad C \leftrightarrow D}{(A \Rightarrow C) \leftrightarrow (B \Rightarrow D)}$:

1. $\vdash_{\mathsf{EN}} A^\sharp \leftrightarrow B^\sharp$ by induction hypothesis
2. $\vdash_{\mathsf{EN}} C^\sharp \leftrightarrow D^\sharp$ by induction hypothesis
3. $\vdash_{\mathsf{EN}} (A^\sharp \to C^\sharp) \leftrightarrow (B^\sharp \to D^\sharp)$ by 1, 2
4. $\vdash_{\mathsf{EN}} \Box(A^\sharp \to C^\sharp) \leftrightarrow \Box(B^\sharp \to D^\sharp)$ by 3 and rule RE
5. $\vdash_{\mathsf{EN}} (A \Rightarrow C)^\sharp \leftrightarrow (B \Rightarrow D)^\sharp$ by 4
6. $\vdash_{\mathsf{EN}} ((A \Rightarrow C) \leftrightarrow (B \Rightarrow D))^\sharp$ □

We can combine the * and $\sharp$-translations:

Lemma 5 $\vdash_{\mathsf{EN}} A \leftrightarrow A^{*\sharp}$.

Proof By induction on A. The atomic case holds by definition and the conjunction and disjunction cases are trivial.

Assume $A = \Box B$, we need to show that $\vdash \Box B \leftrightarrow (\Box B)^{*\sharp}$. By definition, $(\Box B)^{*\sharp}$ is equal to $(\top \Rightarrow B^*)^\sharp$, which is equal to $\Box(\top \to B^{*\sharp})$, which is $\Box(B^{*\sharp})$. Then this is equal to $\Box B$, by the induction hypothesis. □

Theorem 13 *If* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$ *then* $\vdash_{\mathsf{EN}} A$.

Proof Assume $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$, then by Lemma 12, $\vdash_{\mathsf{EN}} A^{*\sharp}$. Again, by Lemma 5, we conclude that $\vdash_{\mathsf{EN}} A$. □

Corollary 1

1. $\vdash_{\mathsf{EN}} A$ *iff* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$

2. $\Gamma \vdash_{\mathsf{EN}} A$ *iff* $\Gamma^* \vdash_{\mathsf{E}^2_{\mathsf{Imp}}} A^*$.

Proof (1) By combining Theorem 13 with Theorem 11.
(2) By applying the weak deduction theorem to (1). □

We call a translation a *faithful interpretation* if provablity is preserved in both directions. So, with this terminology we can say that Corollary 1 states that $*$ is a faithful interpretation of EN into $\mathsf{E}^2_{\mathsf{Imp}}$.

Contrary to this result about $*$ it is not so that $\sharp$ is a faithful interpretation of $\mathsf{E}^2_{\mathsf{Imp}}$ into EN. Clearly $\vdash_{\mathsf{EN}} ((\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q))^\sharp$, but if we consider the neighborhood frame $\mathfrak{F} = \langle W, NB\rangle$ with

$$W = \{w, v\},\ NB(w) = \{(\{v\}, \{w\})\} \cup \{(X, Y) \mid X \subseteq Y\},$$
$$NB(v) = \{(X, Y) \mid X \subseteq Y\},$$

and the valuation $V(p) = \{v\}$, $V(q) = \{w\}$, then it is easy to show that $w \nvDash (\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q)$, that is $\nvdash_{\mathsf{E}^2_{\mathsf{Imp}}} (\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q)$.

5.2 Translations between extensions of $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ and EN

To make $\sharp$ a faifthful interpretation we have to extend $\mathsf{E}^2_{\mathsf{Imp}}$ by an axiom. Let us introduce the axiom U: $(\top \Rightarrow (A \to B)) \leftrightarrow (A \Rightarrow B)$. It characterizes the class of frames closed under equivalence. $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ is the system $\mathsf{E}^2_{\mathsf{Imp}}$ with the axiom U.

Definition 19 Neighborhood frame $\mathfrak{F} = \langle W, NB\rangle$ is closed under **equivalence** if for all $w \in W$, $(X, Y) \in NB(w)$ if and only if $(W, \overline{X} \cup Y) \in NB(w)$.

Lemma 6 *The formula* $(\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q)$ *characterizes the class of neighborhood frames* $\mathfrak{F} = \langle W, NB\rangle$ *satisfying closure under equivalence.*

Proof Let $\mathfrak{F}$ be closed under equivalence and $\mathfrak{M} = \langle W, NB, V\rangle$ be any model based on $\mathfrak{F}$. We have to prove for all $w \in W$, $w \models (\top \Rightarrow (p \to q)) \leftrightarrow (p \Rightarrow q)$. This is easy, because:

$w \models \top \Rightarrow (p \to q)$	iff	$(W, \overline{V(p)} \cup V(q)) \in NB(w)$
by the equivalence condition	iff	$(V(p), V(q)) \in NB(w)$
	iff	$w \models p \Rightarrow q$.

For the other direction, we use contraposition. Suppose that the class is not closed under equivalence. Then there is a frame $\mathfrak{F}$ and $w \in \mathfrak{F}$ such that $(X, Y) \in NB(w)$ but $(W, \overline{X} \cup Y) \notin NB(w)$. Consider the valuation V such that, $V(p) = X$ and $V(q) = Y$. Then, $w \models p \Rightarrow q$ and $w \nvDash \top \Rightarrow (p \to q)$. Therefore $\mathfrak{F} \nvDash (p \Rightarrow q) \to (\top \Rightarrow (p \to q))$. Similarly to this we can show that if $(W, \overline{X} \cup Y) \in NB(w)$ and $(X, Y) \notin NB(w)$ then $\mathfrak{F} \nvDash (\top \Rightarrow (p \to q)) \to (p \Rightarrow q)$. □

The translations $*$ and $\sharp$ have semantical meaning as well of course. This is especially useful in the case of extensions of $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$. That is because NB-neighborhood models satisfying closure under equivalence are essentially equivalent to N-neighborhood models (see [11]). We state the crucial lemmas from that paper.

Lemma 7 *Let $\langle W, N\rangle$ be an* N*-neighborhood frame. Then there exists an equivalent* NB*-neighborhood frame $\langle W, NB\rangle$. This* NB*-frame is closed under* N*-equivalence, i.e., if $(X, Y) \in NB(w)$ and $(X, Y) \equiv (X', Y')$, then $(X', Y') \in NB(w)$. In addition, for all X, Y, w, if $X \subseteq Y$, then $(X, Y) \in NB(w)$.*

Proof The proof is straightforward by considering, for each $w \in W$,
$NB(w) = \left\{(X, Y) \mid \overline{X} \cup Y \in N(w)\right\}$. □

Lemma 8 *Let $\langle W, NB\rangle$ be an* NB*-neighborhood frame closed under* N*-equivalence. Then there exists an equivalent* N*-neighborhood frame $\langle W, N\rangle$.*

Proof The proof is straightforward by considering, for each $w \in W$,
$N(w) = \left\{\overline{X} \cup Y \mid (X, Y) \in NB(w)\right\}$. □

This allows us to interpret strict implication formulas in N-neighborhood models and modal formulas in NB-neighborhood models for $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$. We just state the consequences here without working out the details completely.

Lemma 9

1. *For any N-neighborhood model $\mathfrak{M}$ for modal logic and any modal formula $A(p_1, \ldots, p_n)$, $A^{\mathfrak{M}} = (A^*)^{\mathfrak{M}}$.*
2. *For any neighborhood model for strict implication logic which is closed under* N*-equivalence and any strict implication formula $A(p_1, \ldots, p_n)$, $A^{\mathfrak{M}} = (A^\sharp)^{\mathfrak{M}}$.*

This lemma extends to the Kripke model case when we define $w \models A \Rightarrow B$ as, for all v such that wRv, if $w \models A$, then $w \models B$ (see Definition 22).

Lemma 10

1. *For any Kripke model $\mathfrak{M}$ for modal logic and any modal formula $A(p_1, \ldots, p_n)$, $A^{\mathfrak{M}} = (A^*)^{\mathfrak{M}}$.*
2. *For any Kripke model $\mathfrak{M}$ for strict implication logic and any strict implication formula $A(p_1, \ldots, p_n)$, $A^{\mathfrak{M}} = (A^\sharp)^{\mathfrak{M}}$.*

Theorem 14 *If $\vdash_{\mathsf{E}^2_{Imp}U} A$, then $\vdash_{\mathsf{EN}} A^\sharp$.*

Proof By Theorem 12, we just need to show that $\vdash_{\mathsf{EN}} \mathsf{U}^\sharp$ and this is easy. Because $(\top \Rightarrow (A \to B))^\sharp \leftrightarrow (A \Rightarrow B))^\sharp$ is equal to $\Box(\top \to (A^\sharp \to B^\sharp) \leftrightarrow \Box(A^\sharp \to B^\sharp)$, which is provable in EN. □

Lemma 11 $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A \leftrightarrow A^{\sharp *}$.

Proof By induction on A. The atomic case holds by definition and the conjunction and disjunction cases are trivial.

Assume $A = C \Rightarrow D$, we need to show that $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} (C \Rightarrow D) \leftrightarrow (C \Rightarrow D)^{\sharp *}$. By definition, $(C \Rightarrow D)^{\sharp *}$ is equal to $(\Box(C^{\sharp} \to D^{\sharp}))^*$, which is equal to $(\top \Rightarrow (C^{\sharp *} \to D^{\sharp *}))$, and by axiom U is equal to $(C^{\sharp *} \Rightarrow D^{\sharp *})$. Then this is equal to $(C \Rightarrow D)$, by the induction hypothesis. □

Theorem 15 *If* $\vdash_{\mathsf{EN}} A^{\sharp}$ *then* $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A$.

Proof Assume $\vdash_{\mathsf{EN}} A^{\sharp}$, then by Theorem 11 $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A^{\sharp *}$. Again, by Lemma 11 we conclude that $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A$. □

Corollary 2 $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A$ *iff* $\vdash_{\mathsf{EN}} A^{\sharp}$.

Proof By combining Theorem 15 with Theorem 14. □

So, we have that $\sharp$ is a faithful translation of $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ into EN. We will now see that the classes of logics extending EN and $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ are closely related as well. A logic extending EN will be a set of formulas containing EN closed under its rules and uniform substitution. A logic extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ is similarly defined.

Definition 20

1. Suppose that L is a logic extending EN. We define L^* as the closure of $\{A^* | A \in L\} \cup \{\mathsf{U}\}$ under the rules of $\mathsf{E}^2_{\mathsf{Imp}}$.
2. Suppose that L is a logic extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$. We define $L^{\sharp}$ as the closure of $\{A^{\sharp} | A \in L\}$ under the rules of EN.

Lemma 12 *If* L *is a logic extending* EN*, and* $A \in L^*$*, then* $A^{\sharp} \in L$.

Proof Suppose $A \in L^*$, then there is a finite number of $B_1^*, ..., B_n^*$, with $B_i \in L$, $1 \le i \le n$, such that $B_1^* \wedge ... \wedge B_n^* \vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} A$ and so $\vdash_{\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}} B_1^* \wedge ... \wedge B_n^* \to A$. By Lemma 14 we have $\vdash_{\mathsf{EN}} B_1^{*\sharp} \wedge ... \wedge B_n^{*\sharp} \to A^{\sharp}$. Again, by Lemma 5, we conclude that $\vdash_{\mathsf{EN}} B_1 \wedge ... \wedge B_n \to A^{\sharp}$. Since $B_1 \wedge ... \wedge B_n \in L$, we have $A^{\sharp} \in L$. □

Theorem 16 *If* L *is a logic extending* EN*, then* $L = L^{*\sharp}$.

Proof First we prove $L \subseteq L^{*\sharp}$. Assume $A \in L$ then $A^* \in L^*$ and $A^{*\sharp} \in L^{*\sharp}$. By Lemma 5 $\vdash_{\mathsf{EN}} A \leftrightarrow A^{*\sharp}$. Hence $A \in L^{*\sharp}$.

For the opposite direction assume $A \in L^{*\sharp}$. Then there exist $B_1^{\sharp}, ..., B_n^{\sharp}$, with $B_i \in L^*$, $1 \le i \le n$, such that $B_1^{\sharp} \wedge ... \wedge B_n^{\sharp} \vdash_{\mathsf{EN}} A$. By Lemma 12 each $B_i^{\sharp}$ is in L. Therefore $A \in L$. □

This theorem basically means that each logic extending EN is represented by a logic extending $\mathsf{E}^2_{Imp}U$, by L^*. We can now directly see by an analogous proof that for extensions of $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ we can reverse the order of the translations in Theorem 16.

Theorem 17 *If L is a logic extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$, then $L = L^{\sharp *}$.*

The two theorems together mean that there is a 1-1-correspondence between the logics extending EN and extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$. To find the corresponding logic on the opposite side one only has to check the derivability via the translations on both directions. By the semantic meaning of the translations completeness of the corresponding logic then immediately follows for the same semantics. In fact, this holds for all the usual logical properties since the logics are essentially the same. Also, if one has a unary modal companion one finds in that manner a binary one and vice versa. Of course, this is restricted to logics extending EN or extending $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ respectively.

As an illustration we show directly that the new system $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ is a modal companion of $\mathsf{W}F_N$. First a very straightforward proposition.

Proposition 1 *For all subintuitionistic formulas A, $A^{\Box}$ is identical to $A^{\Rightarrow\sharp}$.*

Theorem 18 $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ *is a modal companion of* $\mathsf{WF_N}$.

Proof We can reason completely syntactically in this case. From Theorem 6 we know that EN is a modal companion of $\mathsf{WF_N}$: $\mathsf{WF_N} \vdash A$ iff $\mathsf{EN} \vdash A^{\Box}$. Thus, by Proposition 1, $\mathsf{WF_N} \vdash A$ iff $\mathsf{EN} \vdash A^{\Rightarrow\sharp}$. Applying Corollary 2 we then immediately get the desired conclusion: $\mathsf{WF_N} \vdash A$ iff $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U} \vdash A^{\Rightarrow}$. □

5.3 Translations, axiomatizations and standard modal logics

In this subsection we consider what happens if a logic extending EN is axiomatized by an axiom A. Then A does not function as a single sentence but it represents all its uniform substitution instances.

Theorem 19 $(\mathsf{EN} + A)^* = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{U} + A^*$.

Proof Obviously $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U} + \mathsf{A}^* \subseteq (\mathsf{EN} + A)^*$. So, we just show the opposite inclusion. Assume $(\mathsf{EN} + A)^* \vdash B$. Then there are substitution instances $A_1, \ldots A_n$ of A such that EN proves $A_1 \wedge \cdots \wedge A_n \to B$. It is a trivial fact of translations and substitution that $(A_1)^*, \ldots (A_n)^*$ are substitution instances of A^*. So, $\mathsf{E}^2_{\mathsf{Imp}}U + A^*$ proves B^*. So, also $(\mathsf{EN} + A)^* \subseteq \mathsf{E}^2_{\mathsf{Imp}}U + A^*$. □

In other words, if L is a logic extending EN axiomatized over EN by A, then L^* is the logic axiomatized over $\mathsf{E}^2_{\mathsf{Imp}}U$ by A^*.

We now apply the results we have obtained to logics having Kripke models. We will find the strict implication variants $\overrightarrow{\mathsf{K}}$, $\overrightarrow{\mathsf{KT}}$, $\overrightarrow{\mathsf{K4}}$ and $\overrightarrow{\mathsf{S4}}$ as the unique correspondents of the logics K, KT, $\mathsf{K4}$ and $\mathsf{S4}$ obtained from the following schemas.

K $\Box(A \to B) \to (\Box A \to \Box B)$
T $\Box A \to A$
4 $\Box A \to \Box\Box A$

Definition 21 A **Kripke frame** $\mathfrak{F}$ is a pair $\langle W, R\rangle$, where W is a nonempty set and R is a binary relation on W. A **Kripke Model** $\mathfrak{M}$ based on a frame $\mathfrak{F}$ is a tuple $\langle W, R, V\rangle$ where $V : At \to 2^W$ is called a valuation function.

Definition 22 (**Truth in Kripke Models**) Let $\mathfrak{M} = \langle W, R, V\rangle$ be a Kripke model and $w \in W$. Truth of a propositional formula in a world w is defined inductively as follows.

1. $\mathfrak{M}, w \models p \quad \Leftrightarrow \quad w \in V(p)$,
2. $\mathfrak{M}, w \models \neg A \quad \Leftrightarrow \quad \mathfrak{M}, w \not\models A$,
3. $\mathfrak{M}, w \models A \wedge B \quad \Leftrightarrow \quad \mathfrak{M}, w \models A$ and $\mathfrak{M}, w \Vdash B$,
4. $\mathfrak{M}, w \models A \Rightarrow B \quad \Leftrightarrow$ for each $w' \in W$ with wRw', if $\mathfrak{M}, w' \models A$, then $\mathfrak{M}, w' \models B$.

By Theorem 19 it is almost immediate that:

Theorem 20

1. $\overrightarrow{\mathsf{K}} = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{UK}^*$,
2. $\overrightarrow{\mathsf{KT}} = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{UK}^*\mathsf{T}^*$,
3. $\overrightarrow{\mathsf{K4}} = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{UK}^*\mathsf{4}^*$,
4. $\overrightarrow{\mathsf{S4}} = \mathsf{E}^2_{\mathsf{Imp}}\mathsf{UK}^*\mathsf{T}^*\mathsf{4}^*$.

Proof We only need to note that EN follows from K. □

Let us just list the $*$-translations here:
$\mathsf{K}^* = (\top \Rightarrow (p \to q)) \to ((\top \Rightarrow p) \to (\top \Rightarrow q))$
$\mathsf{T}^* = (\top \Rightarrow p) \to p$
$\mathsf{4}^* = (\top \Rightarrow p) \to ((\top \Rightarrow (\top \Rightarrow p))$

We do immediately get completeness of each of the systems $\overrightarrow{\mathsf{K}}$, $\overrightarrow{\mathsf{KT}}$, $\overrightarrow{\mathsf{K4}}$, $\overrightarrow{\mathsf{S4}}$ for their Kripke frames and all the regular properties of their correspondents. Surely, these logics can be given more elegant axiomatizations. For example, $\overrightarrow{\mathsf{K}}$ can also be axiomatized as $\mathsf{E}^2_{\mathsf{Imp}} + ((A \to B) \Rightarrow (C \to D)) \to ((A \Rightarrow B) \to (C \Rightarrow D))$.

Also, we immediately get

Theorem 21

1. $\overrightarrow{\mathsf{K}}$ *is a strict implication companion of* F,

2. $\overrightarrow{\mathsf{K4}}$ *is a strict implication companion of* BPC*,*
3. $\overrightarrow{\mathsf{S4}}$ *is a strict implication companion of* IPC*.*

Similarly we obtain that also the correspondent $\overrightarrow{\mathsf{wK4}}$ of wK4 is a strict implication companion of BPC because wK4 is a modal companion of BPC (see [17]).

6 Conclusion

We looked for a binary modal companion of the weak subintuitionistic logic WF and found it in the strict implication logic $\mathsf{E}^2_{\mathsf{Imp}}$. During this search we established also that any extension of the weak modal logic EN can just as well be represented as an equivalent strict implication logic, satisfying a new axiom U and conversely. Among other things this implies that any sub- or superintuitionistic logic which has a standard modal companion has a strict implication companion as well. This is grounded in the fact that $\mathsf{E}^2_{\mathsf{Imp}}\mathsf{U}$ is a strict implication companion of $\mathsf{WF_N}$. A next research goal would be the opposite direction: to find sub- and superintuitionistic logics corresponding to strict implication logics. This of course can only work if the strict implication logics satisfy the rules E^2 and Imp and the axiom U. Most of them do satisfy the rules E^2 and Imp (see [19]). Whether such logics satisfy the axiom U is another matter. Logics with Kripke models do satisfy U, but certainly the interpretability logics IL and its extensions (see e.g. [8]) do not qualify, since $\Box A$ is not definable as $\top \Rightarrow A$, but as $\neg A \Rightarrow \bot$. Also, logics for counterfactuals (see [14, 21]) do not satisfy axiom U. These may be approached differently.

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References

1. M. Ardeshir, *Aspects of Basic Logic*, Ph.D. thesis, Marquette University, Milwaukee (1995).
2. M. Ardeshir, A translation of intuitionistic predicate logic into basic predicate logic, *Studia Logica* 62, 341-352 (1999).

3. M. Ardeshir, W. Ruitenburg, Basic propositional calculus I, *Mathematical Logic Quarterly* 44, 317-343 (1998).
4. S. Celani, R. Jansana, A Closer Look at Some Subintuitionistic Logics, *Notre Dame Journal of Formal Logic*, 42(4), 225-255 (2001).
5. B. Chellas, Basic Conditional Logic, *J Philos Logic* (1975) 4: 133. https://doi.org/10.1007/BF00693270
6. B. Chellas, *Modal logic: An Introduction*, Cambridge University Press (1980).
7. G. Corsi, Weak Logics with strict implication, *Zeitschrift fur Mathematische Logik und Grundlagen der Mathematic*, 33:389-406 (1987).
8. G. Japaridze, D. de Jongh, The Logic of Provability. In: S. Buss, ed., *Handbook of Proof Theory*, 475-546, North-Holland/Elsevier, Amsterdam (1998).
9. D. de Jongh, F. Shirmohammadzadeh Maleki, Subintuitionistic Logics with Kripke Semantics, In 11th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2015, LNCS, pp 333-354, Volume 10148, Springer (2017).
10. D. de Jongh, F. Shirmohammadzadeh Maleki, Subintuitionistic Logics and the Implications they Prove, *Indagationes Mathematicae*, 10.1016/j.indag.2018.01.013.
11. D. de Jongh, F. Shirmohammadzadeh Maleki, Two neighborhood Semantics for Subintuitionistic Logics, In *12th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2018*, LNCS, pp 64-85, Volume 11456, Springer (2019).
12. K. Došen. Duality between modal algebras and neighborhood frames, *Studia Logica*, 48:219-234 (1989).
13. K. Došen, Modal Translation in K and D, *Diamonds and Defaults*, Volume 229 of the series Synthese Library. pp. 103-127 (1994).
14. David Lewis, *Counterfacuals*, Basil Blackwell, Oxford (1973).
15. E. Pacuit, *Neighborhood Semantics for Modal Logic*, Springer (2017).
16. G. Restall, Subintuitionistic Logics, *Notre Dame Journal of Formal Logic*, Volume 35, Number 1, Winter (1994).
17. K. Sano, M. Ma, Alternative semantics for Visser's propositional logics. Aher et al, M. (Eds.) *10th International Tbilisi Symposium on Logic, Language and Computation, TbiLLC 2013*, pp. 257-275, LNCS 8984, Springer (2015).
18. F. Shirmohammadzadeh Maleki, D. de Jongh, Weak Subintuitionistic Logics, *Logic J. IGPL*, doi:10.1093/jigpal/jzw062 (2016).
19. Donald Nute, Conditional Logic, in: D. Gabbay and F. Guenthner, ed., *Handbook of Philosophical Logic, Volume II*, 387-439, Kluwer Academic publishers (1984).
20. A.S. Troelstra, D. Van Dalen, *Constructivism in Mathematics*, Elsevier (2014).
21. F. Veltman, *Logics for Conditionals*, PhD thesis, Universiteit van Amsterdam, ILLC Hsitorical Dissertation Series, HDS-02 (1973).
22. A. Visser, A propositional logic with explicit fixed points. *Studia Logica*, 40, 2, 198, 155-175 (1981).