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OF OKOUNKOV'S BC-TYPE INTERPOLATION
MACDONALD POLYNOMIALS

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Abstract. Nonsymmetric interpolation Laurent polynomials in n variables are introduced, with the interpolation points depending on q and on a n-tuple of parameters $\tau = (\tau_1, \ldots, \tau_n)$. When $\tau_i = st^{n-i}$, Okounkov's 3-parameter BC_n-type interpolation Macdonald polynomials are recovered from the nonsymmetric interpolation Laurent polynomials through Hecke algebra symmetrisation with respect to a type C_n Hecke algebra action. In the Appendix we give some conjectures about extra vanishing, based on Mathematica computations in rank two.

1. Introduction

The goal of this paper is to introduce and solve a special class of Newton type interpolation problems for Laurent polynomials in several variables. An important special case leads to nonsymmetric analogs of Okounkov's [26], [27] BC_n-type interpolation Macdonald polynomials.

1.1. Background: (non)symmetric interpolation polynomials

Denote by $\mathbb{C}[y]^{S_n}$ the ring of symmetric polynomials in n variables $y = (y_1, \ldots, y_n)$. Let Λ_n^+ be the set of partitions of length at most n. Associate to a function

$$\Omega^+ : \{1, \ldots, n\} \times \mathbb{Z}_{\geq 0} \to \mathbb{C}$$

(called a grid) the interpolation points $\Upsilon^+(\lambda) := (\Omega^+(1, \lambda_1), \ldots, \Omega^+(n, \lambda_n)) \in \mathbb{C}^n$ ($\lambda \in \Lambda_n^+$). Okounkov [27] showed that under explicit generic conditions on the grid function Ω^+ there exists, for each $\lambda \in \Lambda_n^+$, a $I_\lambda^+(y) \in \mathbb{C}[y]^{S_n}$ of total degree at most $|\lambda| := \lambda_1 + \cdots + \lambda_n$ satisfying

$$I_\lambda^+(\Upsilon^+(\mu)) = 0$$

for all $\mu \in \Lambda_n^+ \setminus \{\lambda\}$ with $|\mu| \leq |\lambda|$ and satisfying $I_\lambda^+(\Upsilon^+(\lambda)) \neq 0$. Imposing the normalization condition $I_\lambda^+(\Upsilon^+(\lambda)) = 1$ fixes $I_\lambda^+(y)$ uniquely (often a different normalization is used, but this one is the most convenient choice in the present paper).

Okounkov [27, Def. 4.4] calls a grid perfect when the extra vanishing property $I_\lambda^+(\Upsilon^+(\mu)) = 0$ holds true for all $\lambda, \mu \in \Lambda_n^+$ such that $\lambda \not\subseteq \mu$, where $\subseteq$ is the natural inclusion order on Λ_n^+. For generic $q, s, t \in \mathbb{C}^*$ the grid

$$\Omega^+(i, m) := st^{n-i}q^m + \frac{1}{st^{n-i}q^m} \tag{1.1}$$

is perfect, and all other perfect grids can be obtained from (1.1) by replacing the grid values $\Omega^+(i, m)$ by $a\Omega^+(i, m) + b$ where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$ are independent of i and m (but they may depend on the parameters) and by taking limits (see [27]).

The interpolation polynomials $I_\lambda^+(y)$ ($\lambda \in \Lambda_n^+$) associated to the grid (1.1) are Okounkov's [26] 3-parameter family of BC_n-type interpolation Macdonald polynomials. They admit explicit q-integral and combinatorial representations (see [26], [28]) while a generalized binomial formula provides an explicit expansion of the Koornwinder polynomial in the $I_\lambda^+(y)$; see [26], [28]. Rains [31, Thm. 3.2] characterized the $I_\lambda^+(y)$ as eigenfunctions of a linear q-difference operator acting on $y_1, \ldots, y_n$ as well as the parameter s.

An important degenerate perfect grid is

$$\Omega^+(i, m) := t^{n-i}q^m. \tag{1.2}$$

In this case, the corresponding interpolation polynomials $I_\lambda^+(y)$ ($\lambda \in \Lambda_n^+$) are the interpolation Macdonald polynomials of Knop and Sahi [35], [10], having the Macdonald polynomials as their top homogeneous components. Their fundamental properties (q-difference equations, extra vanishing, q-integral formula, combinatorial formula, binomial formula) were obtained at the end of the past century [10], [24], [25], [35].

The theory of symmetric interpolation polynomials associated to perfect grids originates from the study of the Capelli identity and its generalisations; see [14], [34]. By now various classes of symmetric interpolation polynomials $I_\lambda^+(y)$ associated with perfect grids have been realised as eigenvalues of Capelli operators or as

images under the Harish-Chandra isomorphism of quantum immanants (see, e.g., [19], [20], [23], [28], [34], [38] and references therein). They have found applications in the theory of multivariable special functions associated to classical root systems (see, e.g., [1], [26], [31], [33]), exactly solvable models (see, e.g., [20], [28], [35]), and infinite dimensional harmonic analysis on Lie groups and symmetric spaces [29]. An important recent development is the generalisation of the BC_n-type interpolation Macdonald polynomials to the elliptic level; see [4], [32].

Macdonald and Koornwinder polynomials have natural nonsymmetric counterparts, see [2], [21], [37]. They are the joint polynomial eigenfunctions of Cherednik's commuting q-difference reflection operators, which in turn constitute part of Cherednik's [3] polynomial representation of the double affine Hecke algebra in terms of Demazure–Lusztig operators.

Nonsymmetric counterparts $I_\alpha(y) \in \mathbb{C}[y]$ ($\alpha \in \mathbb{Z}_{\geq 0}^n$) of the interpolation polynomials $I_\lambda^+(y)$ were introduced in [10], [35] for grids of the form

$$\Omega(i,m) := \tau_i q^m. \tag{1.3}$$

The associated interpolation points are $\Upsilon(\beta) := \boldsymbol{u_\beta}(\Upsilon^+(\beta^+)) \in (\mathbb{C}^*)^n$ ($\beta \in \mathbb{Z}_{\geq 0}^n$), where $\beta^+ \in \Lambda_n^+$ denotes the unique partition in the S_n-orbit of β and $u_\beta \in S_n$ is the element of smallest length such that $u_\beta(\beta^+) = \beta$ (here we follow the notations from Section 2 for the permutation actions of S_n on $(\mathbb{C}^*)^n$ and $\mathbb{Z}^n$). Up to normalization, $I_\alpha(y) \in \mathbb{C}[y]$ is characterized as the nonzero polynomial of degree at most $|\alpha|$ such that $I_\alpha(\Upsilon(\beta)) = 0$ for all $\beta \in \mathbb{Z}_{\geq 0}^n \setminus \{\alpha\}$ satisfying $|\beta| \leq |\alpha|$.

Note that in the principal specialization $\tau_i := t^{n-i}$, the grid (1.3) reduces to the grid (1.2). In this case, the $I_\alpha(y)$ ($\alpha \in \mathbb{Z}_{\geq 0}^n$) are nonsymmetric analogs of the interpolation Macdonald polynomials whose top degree components are nonsymmetric Macdonald polynomials. They satisfy extra vanishing conditions, have a natural duality property, are simultaneous eigenfunctions on inhomogeneous versions of Cherednik operators, and they admit explicit binomial and evaluation formulas, see [10], [36], [39].

1.2. Nonsymmetric interpolation Laurent polynomials

In this paper, we introduce nonsymmetric counterparts of the interpolation polynomials $I_\lambda^+(y)$ for the grid

$$\Omega^+(i,m) := \tau_i q^m + \frac{1}{\tau_i q^m}. \tag{1.4}$$

In the principal specialization $\tau_i := st^{n-i}$, the grid (1.4) reduces to the perfect grid (1.1) underlying the BC_n-type interpolation Macdonald polynomials.

In case of the grid (1.4), it is instrumental to look for nonsymmetric analogs of the symmetric interpolations within the space $\mathbb{C}[x^{\pm 1}]$ of *Laurent* polynomial in n variables $x = (x_1, \ldots, x_n)$. Consider $\mathbb{C}[y]$ as the subalgebra of $\mathbb{C}[x^{\pm 1}]$ generated by $y_i := x_i + x_i^{-1}$. Then $\mathbb{C}[y]^{S_n}$ is the algebra $\mathbb{C}[x^{\pm 1}]^{W_n}$ of W_n-invariant Laurent polynomials, where W_n is the type C_n Weyl group acting by permutations and inversions of the variables. Note that the interpolation points for $I_\lambda^+(y)$ in terms of the x-variables are given by

$$\Upsilon(\mu) := (\Omega(1,\mu_1), \ldots, \Omega(n,\mu_n)) \qquad (\mu \in \Lambda_n^+),$$

with Ω the type A_{n-1} grid defined by (1.3).

The Weyl group W_n acts on $(\mathbb{C}^*)^n$ by permutations and inversions, and on $\mathbb{Z}^n$ by permutations and sign changes. Write $\beta^+ \in \Lambda_n^+$ for $\beta \in \mathbb{Z}^n$ the unique partition in the W_n-orbit of β, and $w_\beta \in W_n$ for the element of smallest length such that $w_\beta(\beta^+) = \beta$. We call $|\alpha| := |\alpha_1| + \cdots + |\alpha_n|$ the degree of the monomial $x^\alpha \in \mathbb{C}[x^{\pm 1}]$ ($\alpha \in \mathbb{Z}^n$).

Our main result is as follows (see Section 4).

Theorem 1.1. *Assume $q \in \mathbb{C}^*$ is not a root of unity, and denote by $\mathcal{T}_n$ the generic set of parameters $\tau = (\tau_1, \ldots, \tau_n)$ defined by (3.3). For $\tau \in \mathcal{T}_n$ and $\alpha \in \mathbb{Z}^n$ there exists a unique Laurent polynomial $G_\alpha(x) = G_\alpha(x; q, \tau) \in \mathbb{C}[x^{\pm 1}]$ of degree at most $|\alpha|$ satisfying $G_\alpha(\overline{\alpha}) = 1$ and*

$$G_\alpha(\overline{\beta}) = 0 \qquad (\beta \in \mathbb{Z}^n \setminus \{\alpha\} : |\beta| \leq |\alpha|),$$

where $\overline{\beta} := \boldsymbol{w}_\beta(\Upsilon(\beta^+))$ (see (2.3) for the notation in bold). Furthermore,

$$I_\lambda^+(y) = \sum_{\alpha \in W_n \lambda} G_\alpha(x) \qquad (\lambda \in \Lambda_n^+)$$

with $I_\lambda^+(y)$ the symmetric interpolation polynomial relative to the grid (1.4).

The proof of the existence of the interpolation polynomials $I_\lambda^+(y)$ and $I_\alpha(y)$ are based on explicit recursion relations, which allow a direct proof by induction to the degree (see [27, Prop. 2.7] and [35, Cor. 4.4]). We revisit the proof for $I_\lambda^+(y)$ with the grid (1.4) from the Laurent polynomial perspective in Section 3. It forms a convenient starting point for the much more elaborate inductive proof of Theorem 1.1, which is given in Section 4.

The nonsymmetric interpolation polynomials $G_\alpha(x; q, s, t)$ ($\alpha \in \mathbb{Z}^n$) with parameters $\tau = (\tau_1, \ldots, \tau_n)$ specialized to the $\tau_i := st^{n-i}$ are nonsymmetric analogs of Okounkov's BC_n-type interpolation Macdonald polynomials. In this case, the BC_n-type interpolation Macdonald polynomials can alternatively be reobtained from the $G_\alpha(x; q, s, t)$ by symmetrising with respect to a type C_n Hecke algebra action on $\mathbb{C}[x^{\pm 1}]$ in terms of Demazure–Lusztig type operators (see Section 5). It is a first indication that the $G_\alpha(x; q, s, t)$ are amenable to the Hecke algebra techniques from [10], [36], [39]. The missing ingredient from this perspective is the interpretation of the $G_\alpha(x; q, s, t)$ as simultaneous eigenfunctions of commuting inhomogeneous Cherednik-type operators. This would allow one to involve double affine Hecke algebra techniques in deriving a binomial formula for nonsymmetric Koornwinder polynomials, and in deriving nonsymmetric analogs of extra vanishing and duality (compare with [10, 39] for type A). We expect this to be the key step towards further applications of the nonsymmetric BC_n-type interpolation polynomials in the theory of nonsymmetric Macdonald–Koornwinder polynomials, algebraic combinatorics, and exactly solvable models.

In the Appendix, we give a conjecture about the extra vanishing, based on Mathematica computations in rank two.

2. Preliminaries

Throughout the paper, we assume that $q \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is not a root of unity. For $a \in \mathbb{C}$, the *q-shifted factorial* is given by $(a;q)_k := (1-a)(1-qa)\cdots(1-q^{k-1}a)$ $(k = 1, 2, \ldots)$ and $(a;q)_0 := 1$. We also write $(a_1, \ldots, a_r; q)_k := (a_1; q)_k \cdots (a_r; q)_k$.

Let $n \in \mathbb{Z}_{>0}$. Write

$$\Lambda_n := \mathbb{Z}^n, \qquad [1,n] := \{1, \ldots, n\}, \qquad [1,n) := \{1, \ldots, n-1\},$$

where $[1,n)$ is taken to be empty if $n = 1$.

For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda_n$ put

$$
\begin{aligned}
|\alpha| &:= |\alpha_1| + \cdots + |\alpha_n| && (\textit{weight of } \alpha), \\
n_a(\alpha) &:= \#\{i \mid \alpha_i = a\} && \text{if } a \in \mathbb{Z}, \\
x^\alpha &:= x_1^{\alpha_1} \cdots x_n^{\alpha_n} && (\text{Laurent monomial}), \\
x_I &:= \prod_{i \in I} x_i && \text{if } I \subseteq [1,n].
\end{aligned}
$$

When we write $I = \{i_1, \ldots, i_k\} \subseteq [1,n]$ for a subset of $[1,n]$ of cardinality k, then we will always assume the ordering $i_1 < i_2 < \cdots < i_k$ of its elements. We write I^{c} for the complement of I in $[1,n]$.

A *Laurent polynomial* f in the n complex variables x has the form

$$f(x) = \sum_{\alpha \in \Lambda_n} c_\alpha x^\alpha \tag{2.1}$$

with $c_\alpha \in \mathbb{C}$ and $c_\alpha \neq 0$ for only finitely many α. The *degree* of f in (2.1) is defined by

$$\deg(f) := \max\{|\alpha| \mid c_\alpha \neq 0\}; \qquad \deg(f) := -\infty \quad \text{if } f \text{ is identically zero}.$$

Note that $\deg(fg) \leq \deg(f) + \deg(g)$, but equality does not necessarily hold. This means that the degree function defines a filtration on the algebra $\mathcal{P}_n := \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of Laurent polynomials in x, but not a grading. The filtration is $\mathcal{P}_n = \bigcup_{d=0}^{\infty} \mathcal{P}_{n,d}$ with $\mathcal{P}_{n,d}$ the subspace of $\mathcal{P}_n$ consisting of Laurent polynomials of degree at most d. Write $\mathcal{G}(\mathcal{P}_n) = \bigoplus_{d=0}^{\infty} \mathcal{G}_{n,d}$ for the associated graded algebra, with the dth graded piece given by

$$\mathcal{G}_{n,d} := \mathcal{P}_{n,d}/\mathcal{P}_{n,d-1}$$

with $\mathcal{P}_{n,-1} := \{0\}$. We write $[f]_d := f + \mathcal{P}_{n,d-1} \in \mathcal{G}_{n,d}$ for $f \in \mathcal{P}_{n,d}$.

Let $W_n = \{\pm 1\}^n \rtimes S_n$ be the Weyl group associated with the root system of type C_n (and B_n and BC_n). Then $\sigma \in \{\pm 1\}^n$ and $\pi \in S_n$ act on Λ_n (and $\mathbb{C}^n$) by

$$(\sigma \alpha)_j := \sigma_j \alpha_j, \quad (\pi \alpha)_j := \alpha_{\pi^{-1}(j)} \qquad (\alpha \in \Lambda_n \text{ or } \mathbb{C}^n, \ j = 1, \ldots, n). \qquad (2.2)$$

Equivalently, if $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$ then $\sigma(e_j) = \sigma_j \, e_j$ and $\pi(e_j) = e_{\pi(j)}$. Since $2\pi i \Lambda_n$ is invariant under the action of W_n, we can exponentiate its action on $\mathbb{C}^n$ to an action on $(\mathbb{C}^*)^n$. We will write this action in bold. Then

$$(\boldsymbol{\sigma} x)_j := x_j^{\sigma_j}, \quad (\boldsymbol{\pi} x)_j := x_{\pi^{-1}(j)} \qquad (x \in (\mathbb{C}^*)^n, \ j = 1, \ldots, n) \qquad (2.3)$$

and $(\boldsymbol{w} x)^\alpha = x^{w^{-1}\alpha}$ $(x \in (\mathbb{C}^*)^n, \ w \in W_n, \ \alpha \in \Lambda_n)$. The action of W_n on $(\mathbb{C}^*)^n$ induces an action of W_n on Laurent polynomials (2.1) by

$$(wf)(x) := f(\boldsymbol{w}^{-1} x) \qquad (f \in \mathcal{P}_n, \ w \in W_n, \ x \in (\mathbb{C}^*)^n). \qquad (2.4)$$

Thus $(wf)(x) := x^{w\alpha}$ if $f(x) = x^\alpha$. Note that $|w\alpha| = |\alpha|$ for $w \in W_n$ and $\alpha \in \Lambda_n$. In particular, $\deg(wf) = \deg(f)$ for $w \in W_n$ and $f \in \mathcal{P}_n$. Hence the W_n-action on $\mathcal{P}_n$ is an action by filtered algebra automorphisms and induces a W_n-action by graded algebra automorphisms on the associated graded algebra $\mathcal{G}(\mathcal{P}_n)$. We write $\mathcal{P}_n^{W_n}$, $\mathcal{P}_{n,d}^{W_n}$ and $\mathcal{G}_{n,d}^{W_n}$ for the subspaces of W_n-invariant elements in $\mathcal{P}_n$, $\mathcal{P}_{n,d}$ and $\mathcal{G}_{n,d}$, respectively. By construction the associated graded algebra $\mathcal{G}(\mathcal{P}_n^{W_n})$ of the filtered algebra $\mathcal{P}_n^{W_n} = \bigcup_{d=0}^{\infty} \mathcal{P}_{n,d}^{W_n}$ is isomorphic to $\mathcal{G}(\mathcal{P}_n)^{W_n} = \bigoplus_{d=0}^{\infty} \mathcal{G}_{n,d}^{W_n}$.

If $n > 1$ then we write

$$x' := (x_1, \ldots, x_{n-1})$$

for the $n - 1$ complex variables obtained from x by removing x_n. Similarly, if $\tau = (\tau_1, \ldots, \tau_n)$ is a n-tuple of complex numbers, then we write

$$\tau' := (\tau_1, \ldots, \tau_{n-1}).$$

Sometimes we also need to remove an arbitrary complex variable x_k from x. In that case we write

$$x^{(k)} := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n).$$

A similar notation will be employed for n-tuples of complex numbers. Note that $x' = x^{(n)}$ and $\tau' = \tau^{(n)}$.

For the root system R of type C_n we take $\beta^i := e_i - e_{i+1}$ $(i = 1, \ldots, n-1)$ and $\beta^n := 2e_n$ as the simple roots. Then the set R^+ of positive roots consists of the vectors $e_i \pm e_j$ $(i < j)$ and $2e_i$ $(i = 1, \ldots, n)$. Write $R^- = -R^+$ for the set of negative roots. Denote the simple reflections corresponding to the simple roots by $s_1, \ldots, s_n$. Each $w \in W_n$ can be written as a product of simple reflections. The minimal number of factors in such a product representing w is called the *length* $\ell(w)$ of w. The length of w is also equal to the number of positive roots sent to negative roots by w; see [7, Lem. 10.3A] or [8, Cor. 1.7]. If $w = s_{i_1} \cdots s_{i_r}$ with $r = \ell(w)$ (a *reduced expression* of $w \in W_n$) then $w_k := s_{i_k} \cdots s_{i_r}$ gives a reduced expression for w_k $(k = 1, \ldots, r)$ and the $r - k + 1$ positive roots sent by w_k to

negative roots are precisely the positive roots $s_{i_r} s_{i_{r-1}} \cdots s_{i_{j+1}} \beta^{i_j}$ $(j = k, \ldots, r)$; see [8, Sect. 1.7]. The element w_0 of maximal length in W_n is $w_0 = -\operatorname{id}_{\mathbb{R}^n}$.

By a *partition* we mean $\lambda \in \Lambda_n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The *length* $\ell(\lambda) \in \{0, 1, \ldots, n\}$ of λ is the index such that $\lambda_i = 0$ iff $i > \ell(\lambda)$. Denote the set of partitions of length at most n by Λ_n^+. For $\alpha \in \Lambda_n$ there exists a unique partition $\alpha^+ \in \Lambda_n^+$ in the W_n-orbit $\{w\alpha\}_{w \in W_n}$. If $\alpha \in \mathbb{Z}_{\geq 0}^n$ then α^+ can also be characterized as the unique partition in the S_n-orbit $\{\pi\alpha\}_{\pi \in S_n}$. For $m \leq n$, we embed $\Lambda_m^+ \hookrightarrow \Lambda_n^+$ by $(\lambda_1, \ldots, \lambda_m) \mapsto (\lambda_1, \ldots, \lambda_m, 0, \ldots, 0)$.

On Λ_n^+ the *dominance partial ordering* $\leq$ and *inclusion partial ordering* $\subseteq$ are defined by

$$\lambda \leq \mu \quad \text{iff} \quad \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \quad (i = 1, \ldots, n), \qquad (2.5)$$
$$\lambda \subseteq \mu \quad \text{iff} \quad \lambda_i \leq \mu_i \quad (i = 1, \ldots, n).$$

A partition λ has weight $|\lambda| = \lambda_1 + \cdots + \lambda_n$. If $\lambda \leq \mu$ then $|\lambda| \leq |\mu|$.

For $\lambda \in \Lambda_n^+$ write $W_{n,\lambda} \subseteq W_n$ for the stabilizer subgroup of λ. Then by [7, Lem. 10.3B], the subgroup $W_{n,\lambda}$ $(\lambda \in \Lambda_n^+)$ is a *parabolic subgroup* of W_n; i.e., $W_{n,\lambda}$ is generated by the simple reflections it contains. Write W_n^λ for the set of $u \in W_n$ satisfying $\ell(uv) = \ell(u) + \ell(v)$ for all $v \in W_{n,\lambda}$. Then W_n^λ is a complete set of representatives of $W_n/W_{n,\lambda}$ (see [8, Sect. 1.10] for details).

Definition 2.1. For $\alpha \in \Lambda_n$ denote by $w_\alpha = \sigma_\alpha \pi_\alpha$ $(\sigma_\alpha \in \{\pm 1\}^n, \pi_\alpha \in S_n)$ the unique element in $W_n^{\alpha^+}$ such that $w_\alpha(\alpha^+) = \alpha$.

Define the function $\operatorname{sgn} \colon \mathbb{R} \to \{\pm 1\}$ by $\operatorname{sgn}(a) := 1$ if $a \geq 0$ and $\operatorname{sgn}(a) := -1$ if $a < 0$. The following explicit description of σ_α and π_α, due to Sahi [37, p. 277], plays an important role in what follows.

Lemma 2.2. *Let* $\alpha \in \Lambda_n$. *Then*

$$\sigma_\alpha = (\operatorname{sgn}(\alpha_1), \ldots, \operatorname{sgn}(\alpha_n)),$$

and $\pi_\alpha \in S_n$ *is the unique permutation satisfying the following two properties:*

(a) $\alpha^+ = (|\alpha_{\pi_\alpha(1)}|, \ldots, |\alpha_{\pi_\alpha(n)}|)$;
(b) *Let* $a \in \mathbb{Z}_{\geq 0}$ *such that* $n_a(\alpha^+) > 0$. *Denote by* $j_1 < j_2$ *the indices such that* $\{j \mid \alpha_j^+ = a\} = \{j_1, j_1 + 1, \ldots, j_2 - 1\}$.
  (1) *If* $j_1 \leq j < j_1 + n_a(\alpha)$ *then* $\alpha_{\pi_\alpha(j)} = a \geq 0$ *and* $\pi_\alpha(j)$ *is increasing in* j,
  (2) *if* $j_1 + n_a(\alpha) \leq j < j_2$ *then* $\alpha_{\pi_\alpha(j)} = -a < 0$ *and* $\pi_\alpha(j)$ *is decreasing in* j.

Remark 2.3. Alternatively, $\pi_\alpha \in S_n$ is the unique permutation such that, for $i < j$,

$$\pi_\alpha^{-1}(i) < \pi_\alpha^{-1}(j) \iff |\alpha_i| > |\alpha_j| \text{ or } 0 \leq \alpha_i = \pm\alpha_j. \qquad (2.6)$$

See also [37, p. 277] for a short formulation of what is essentially the description of π_α in Lemma 2.2, and for an example.

3. Interpolation theorem for BC_n-symmetric Laurent polynomials

Following [30], we call a Laurent polynomial (2.1) BC_n-*symmetric* if it is invariant under the Weyl group W_n. A basis of the linear space of BC_n-symmetric Laurent polynomials is given by the *symmetrized monomials*

$$m_\lambda(x) := \sum_{\mu \in W_n \lambda} x^\mu \qquad (\lambda \in \Lambda_n^+).$$

The Laurent polynomial m_λ has degree $|\lambda|$, and

$$m_\lambda = \frac{|W_n^\lambda|}{|W_n|} \tilde{m}_\lambda, \quad \text{where} \quad \tilde{m}_\lambda(x) := \sum_{w \in W_n} x^{w\lambda}.$$

Lemma 3.1. *Let* $a \in \mathbb{C} \backslash \{0\}$. *Let* $\lambda \in \Lambda_{n-1}^+ \hookrightarrow \Lambda_n^+$ $(n > 1)$.

(a) *There are constants* c_μ $(\mu \in \Lambda_{n-1}^+, \mu \leq \lambda)$ *with* $c_\lambda \neq 0$ *such that*

$$m_\lambda(x', a) = \sum_{\mu \leq \lambda} c_\mu m_\mu(x') \qquad (3.1)$$

   as identity in $\mathcal{P}_{n-1}^{W_{n-1}}$.

(b) *There are constants* d_μ $(\mu \in \Lambda_{n-1}^+, \mu \leq \lambda)$ *with* $d_\lambda \neq 0$ *such that*

$$m_\lambda(x') = \sum_{\mu \leq \lambda} d_\mu m_\mu(x', a) \qquad (3.2)$$

   as identity in $\mathcal{P}_{n-1}^{W_{n-1}}$.

(c) *For every* BC_{n-1}-*symmetric Laurent polynomial* f *in* x' *of degree* d, *there exists a* BC_n-*symmetric Laurent polynomial* g *in* x *of degree* d *such that* $g(x', a) = f(x')$.

Proof. (a) Denote the length of $\lambda \in \Lambda_{n-1}^+$ by ℓ. Then

$$\tilde{m}_\lambda(x', a) = \sum_{j=1}^n (a^{\lambda_j} + a^{-\lambda_j}) \tilde{m}_{\lambda^{(j)}}(x') = 2(n-\ell)\tilde{m}_\lambda(x') + \sum_{j=1}^\ell (a^{\lambda_j} + a^{-\lambda_j}) \tilde{m}_{\lambda^{(j)}}(x').$$

The result now follows from the fact that $\lambda^{(j)} < \lambda$ for $j = 1, 2, \ldots, \ell$.

(b) From (3.1) we get

$$m_\lambda(x') = c_\lambda^{-1} m_\lambda(x', a) - \sum_{\mu < \lambda} c_\lambda^{-1} c_\mu m_\mu(x').$$

Now (3.2) follows by induction on the weight of the partition.

(c) By (3.2), if $f = m_\lambda \in \mathcal{P}_{n-1}^{W_{n-1}}$ then we can take $g = \sum_{\mu \in \Lambda_{n-1}^+; \, \mu \leq \lambda} d_\mu m_\mu$. $\square$

Write $R_A := \{e_i - e_j \mid 1 \leq i \neq j \leq n\} \subset R$, which is a root subsystem in R of type A_{n-1}. Then we define the parameter domain $\mathcal{T}_n$ by

$$\mathcal{T}_n := \{\tau \in (\mathbb{C}^*)^n \mid \tau^\beta \notin q^{\mathbb{Z}} \quad \forall \beta \in R \setminus R_A \quad \& \quad \tau^\beta \notin q^{\mathbb{Z}\setminus\{0\}} \quad \forall \beta \in R_A\}, \quad (3.3)$$

where $\tau^\alpha \in \mathbb{C}^*$ ($\alpha \in \Lambda_n$) stands for the monomial x^α evaluated at $\tau \in (\mathbb{C}^*)^n$. In other words, $\tau \in (\mathbb{C}^*)^n$ belongs to $\mathcal{T}_n$ if $\tau_i \tau_j \notin q^{\mathbb{Z}}$ for $1 \leq i \leq j \leq n$ and $\tau_i \tau_j^{-1} \notin q^{\mathbb{Z}\setminus\{0\}}$ for $1 \leq i < j \leq n$. Note that $\mathbf{s} := (s,\ldots,s) \in \mathcal{T}_n$ for $s \in \mathbb{C}^*$ satisfying $s^2 \notin q^{\mathbb{Z}}$, and $\mathcal{T}_n$ is invariant under scalar multiplication by q. For $\mu \in \Lambda_n^+$ define $\overline{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_n) \in (\mathbb{C}^*)^n$ by

$$\overline{\mu}_i := q^{\mu_i} \tau_i. \quad (3.4)$$

The map $\mu \mapsto \overline{\mu}$ is injective on Λ_n^+. Sometimes we write $\overline{\mu} = \overline{\mu}(q, \tau)$ and $\overline{\mu}_i = \overline{\mu}_i(q, \tau)$ if it is important to specify the dependence on q, τ.

Remark 3.2. We will develop the theory of symmetric and nonsymmetric interpolation Laurent polynomials for parameters (q, τ) with $q \in \mathbb{C}^*$ not a root of unity and $\tau \in \mathcal{T}_n$ (see (3.3)). It is easy to check that the results also hold true with q and τ_j rational indeterminates, and for (q, τ) with $0 < |q| < 1$ and $0 < |\tau_1| < \cdots < |\tau_n| < 1$. The latter case requires straightforward adjustments to the proofs of Corollary 4.3 and Lemma 4.9.

The following two properties of the interpolation points will play an important role in what follows:

$$\begin{aligned}
\overline{\mu}(q, \tau) &= (\overline{\mu}(q, \tau'), \tau_n) & (\mu \in \Lambda_{n-1}^+ \hookrightarrow \Lambda_n^+), \\
\overline{\mu}(q, \tau) &= \overline{(\mu - \mathbf{1})}(q, q\tau) & (\mu \in \Lambda_n^+ : \mu_n > 0),
\end{aligned} \quad (3.5)$$

with $\overline{\mu}(q, \tau') = (q^{\mu_1}\tau_1, \ldots, q^{\mu_{n-1}}\tau_{n-1})$ for $\mu \in \Lambda_{n-1}^+$ the interpolation point in $(\mathbb{C}^*)^{n-1}$. Put

$$\Lambda_{n,d}^+ := \{\mu \in \Lambda_n^+ \mid |\mu| \leq d\} \qquad (d \in \mathbb{Z}_{\geq 0}).$$

Proposition 3.3. *Let $n \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}_{\geq 0}$ and $\tau \in \mathcal{T}_n$. For every map $\overline{f} : \Lambda_{n,d}^+ \to \mathbb{C}$ there exists a unique BC_n-symmetric Laurent polynomial f of degree $\leq d$ such that $f(\overline{\mu}(q, \tau)) = \overline{f}(\mu)$ for all $\mu \in \Lambda_{n,d}^+$.*

Proof. First note that both the space $\mathcal{P}_{n,d}^{W_n}$ of BC_n-symmetric Laurent polynomials in x of degree at most d and the space of complex-valued functions on the set $\{\overline{\mu} \mid \mu \in \Lambda_{n,d}^+\}$ have dimension $|\Lambda_{n,d}^+|$. Therefore, surjectivity of the linear map which restricts a BC_n-symmetric Laurent polynomial to the set of interpolation points $\{\overline{\mu} \mid \mu \in \Lambda_{n,d}^+\}$ implies injectivity, so that existence implies uniqueness.

To prove existence we will use induction on $n + d$. If $n + d = 1$, so $(n, d) = (1, 0)$, then $\Lambda_{1,0}^+ = \{(0)\}$ and $\overline{0} = \tau_1$ and there is nothing to prove (take f to be the appropriate constant function). Suppose that the existence of f, with (n, d) replaced by $(\widetilde{n}, \widetilde{d})$, is true for $\widetilde{n} + \widetilde{d} < n + d$ for all possible parameters in $\mathcal{T}_{\widetilde{n}}$ and all possible maps $\Lambda_{\widetilde{n},\widetilde{d}}^+ \to \mathbb{C}$. Fix $\overline{f} : \Lambda_{n,d}^+ \to \mathbb{C}$ and $\tau \in \mathcal{T}_n$ and let $\overline{\mu}$ be $\overline{\mu}(q, \tau)$. To

establish the induction step, we need to prove the existence of $f \in \mathcal{P}_{n,d}^{W_n}$ satisfying $f(\overline{\mu}) = \overline{f}(\mu)$ for all $\mu \in \Lambda_{n,d}^+$.

We first construct $g \in \mathcal{P}_{n,d}^{W_n}$ satisfying the partial interpolation property

$$g(\overline{\mu}) = \overline{f}(\mu) \qquad (\mu \in \Lambda_{n,d}^+ \text{ with } \mu_n = 0). \qquad (3.6)$$

First assume $n > 1$. By induction, there exists a BC_{n-1}-symmetric Laurent polynomial $\widetilde{g}$ in x' of degree at most d such that

$$\widetilde{g}\big(\overline{\mu}(q, \tau')\big) = \overline{f}(\mu, 0) \qquad (\mu \in \Lambda_{n-1,d}^+). \qquad (3.7)$$

By Lemma 3.1(c) there exists a BC_n-symmetric Laurent polynomial g in x of degree at most d such that $g(x', \tau_n) = \widetilde{g}(x')$. Then

$$g\big(\overline{\mu}(q, \tau'), \tau_n\big) = \overline{f}(\mu, 0) \qquad (\mu \in \Lambda_{n-1,d}^+).$$

Now the first formula of (3.5) gives (3.6). If $n = 1$ then put $g(x) := \overline{f}(0)$, where g has degree $d \geq 0$. Then, in particular, $g(\tau_1) = \overline{f}(0)$. This concludes the proof of (3.6) in all cases.

Note that (3.6) already concludes the proof of the induction step when $n > d$. Indeed, in this case we can simply take $f = g$ since $\mu_n = 0$ for all $\mu \in \Lambda_{n,d}^+$.

To complete the induction step we thus may and will assume from now on that $d \geq n$. We make the Ansatz that the symmetric interpolation Laurent polynomial f we are searching for is of the form

$$f(x) = g(x) + h(x) \prod_{i=1}^{n}(x_i - \tau_n)(x_i^{-1} - \tau_n) \qquad (3.8)$$

with g as constructed above and $h \in \mathcal{P}_{n,d-n}^{W_n}$. Then $f(\overline{\mu}) = \overline{f}(\mu)$ for all $\mu \in \Lambda_{n,d}^+$ with $\mu_n = 0$. The identity $f(\overline{\mu}) = \overline{f}(\mu)$ will also hold for $\mu_n > 0$ if h satisfies

$$h(\overline{\mu}) = \frac{\overline{f}(\mu) - g(\overline{\mu})}{\prod_{i=1}^{n}(\overline{\mu}_i - \tau_n)(\overline{\mu}_i^{-1} - \tau_n)} \qquad (\mu \in \Lambda_{n,d}^+ \text{ with } \mu_n > 0). \qquad (3.9)$$

Note that, since $\tau \in \mathcal{T}_n$, no factors in the above denominator vanish. Hence what remains to show is the existence of a BC_n-symmetric Laurent polynomial $h \in \mathcal{P}_{n,d-n}^{W_n}$ satisfying (3.9).

Note that we have a bijection

$$\{\mu \in \Lambda_{n,d}^+ \mid \mu_n > 0\} \xrightarrow{\sim} \Lambda_{n,d-n}^+$$

given by $\mu \mapsto \mu - \mathbf{1} := (\mu_1 - 1, \ldots, \mu_n - 1)$. By the induction hypothesis, there exists a $h \in \mathcal{P}_{n,d-n}^{W_n}$ such that

$$h(\overline{\nu}(q, q\tau)) = \frac{\overline{f}(\nu + \mathbf{1}) - g(\overline{\nu + \mathbf{1}})}{\prod_{i=1}^{n}\big((\nu + \mathbf{1})_i - \tau_n\big)\big((\nu + \mathbf{1})_i^{-1} - \tau_n\big)} \qquad (\nu \in \Lambda_{n,d-n}^+).$$

By the second formula of (3.5) we have

$$\overline{\nu + \mathbf{1}} = \overline{\nu}(q, q\tau) \qquad (\nu \in \Lambda_{n,d-n}^+),$$

hence we conclude that $h \in \mathcal{P}_{n,d-n}^{W_n}$ satisfies the desired interpolation property (3.9). This concludes the proof of the induction step. $\square$

In view of Proposition 3.3 we can give the following definition.

Definition 3.4. Fix $\tau \in \mathcal{T}_n$. The BC_n-*symmetric interpolation Laurent polynomial of degree* $\lambda \in \Lambda_n^+$ is the unique BC_n-symmetric Laurent polynomial $R_\lambda(x; q, \tau)$ in n variables x of degree at most $|\lambda|$ such that $R_\lambda(\overline{\lambda}; q, \tau) = 1$ and

$$R_\lambda(\overline{\mu}; q, \tau) = 0 \qquad \forall \mu \in \Lambda_{n,|\lambda|}^+ \setminus \{\lambda\},$$

where $\overline{\mu} := \overline{\mu}(q, \tau)$.

It follows from Proposition 3.3 that $\{R_\lambda(x; q, \tau) \mid \lambda \in \Lambda_{n,d}^+\}$ is a linear basis of $\mathcal{P}_{n,d}^{W_n}$.

The following two properties, which correspond to [27, Prop. 2.4] and [27, Prop. 2.7], respectively, in Okounkov's setup, easily follow from Definition 3.4, Proposition 3.3 and the two special properties (3.5) of the interpolation points.

Proposition 3.5. *Let* $\tau \in \mathcal{T}_n$.

(a) *For* $\lambda \in \Lambda_{n-1}^+ \hookrightarrow \Lambda_n^+$ *we have*

$$R_\lambda((x', \tau_n); q, \tau) = R_\lambda(x'; q, \tau'),$$

with on the left-hand side the interpolation Laurent polynomial in n variables and on the right-hand side the interpolation Laurent polynomial in $n - 1$ variables.

(b) *For* $\lambda \in \Lambda_n^+$ *with* $\lambda_n > 0$ *we have*

$$R_\lambda(x; q, \tau) = R_{\lambda - \mathbf{1}}(x; q, q\tau) \prod_{i=1}^{n} \frac{(x_i - \tau_n)(x_i^{-1} - \tau_n)}{(\overline{\lambda}_i - \tau_n)(\overline{\lambda}_i^{-1} - \tau_n)}.$$

Set $\Lambda_{n,-1}^+ := \varnothing$ and write $\widehat{\Lambda}_{n,d}^+ := \Lambda_{n,d}^+ \setminus \Lambda_{n,d-1}^+$ for the partitions of length at most n and weight d. Then $\{[m_\lambda]_d \mid \lambda \in \widehat{\Lambda}_{n,d}^+\}$ is a linear basis of $\mathcal{G}_{n,d}^{W_n}$. Furthermore, by Proposition 3.3, $\{[R_\lambda]_d \mid \lambda \in \widehat{\Lambda}_{n,d}^+\}$ is also a linear basis of $\mathcal{G}_{n,d}^{W_n}$. In particular,

$$\deg(R_\lambda(x; q, \tau)) = |\lambda| \qquad (\lambda \in \Lambda_n^+).$$

The following important property is less immediate.

Proposition 3.6. *Let* $\tau \in \mathcal{T}_n$ *and* $\lambda \in \widehat{\Lambda}_{n,d}^+$. *The coefficient* $c_{\lambda,\lambda}$ *in the expansion* $R_\lambda(x; q, \tau) = \sum_{\mu \in \Lambda_{n,d}^+} c_{\lambda,\mu} m_\mu(x)$ $(c_{\lambda,\mu} \in \mathbb{C})$ *is nonzero.*

Proof. It suffices to show that the coefficient of $[m_\lambda]_d$ in the expansion of $[R_\lambda]_d$ in terms of the linear basis $\{[m_\mu]_d \mid \mu \in \widehat{\Lambda}_{n,d}^+\}$ of $\mathcal{G}_{n,d}^{W_n}$ is nonzero. We prove this by induction on $n + d$. For $n = 1$, the result follows from Example 3.7. To prove the induction step we need to consider two cases.

If $\lambda_n > 0$ then $d \geq n$ and

$$[R_\lambda]_d = \left(\prod_{i=1}^{n} \frac{\tau_n}{(\overline{\lambda}_i - \tau_n)(\tau_n - \overline{\lambda}_i^{-1})} \right) [R_{\lambda - \mathbf{1}}]_{d-n} [m_\mathbf{1}]_n \qquad (3.10)$$

in $\mathcal{G}(\mathcal{P}_n^{W_n})$ by Proposition 3.5(b). The result now immediately follows from the induction hypothesis.

If $\lambda_n = 0$ then first consider $R_\lambda(x'; q, \tau')$. By the induction hypothesis, $m_\lambda(x')$ occurs with nonzero coefficient in the linear expansion of $R_\lambda(x'; q, \tau')$ in the basis $\{m_\mu(x') \mid \mu \in \Lambda_{n-1,d}^+\}$ of $\mathcal{P}_{n-1,d}^{W_{n-1}}$. By the Proof of Lemma 3.1(c), there exists $g \in \mathcal{P}_{n,d}^{W_n}$ such that

$$g(x', \tau_n) = R_\lambda(x'; q, \tau')$$

and such that $m_\lambda(x)$ occurs with nonzero coefficient in the linear expansion of $g(x)$ in the basis $\{m_\mu(x) \mid \mu \in \Lambda_{n,d}^+\}$ of $\mathcal{P}_{n,d}^{W_n}$. Fixing this choice of g, there exists by (the proof of) Proposition 3.3 a unique $h \in \mathcal{P}_{n,d-n}^{W_n}$ such that

$$R_\lambda(x; q, \tau) = g(x) + h(x) \prod_{i=1}^{n} (x_i - \tau_n)(x_i^{-1} - \tau_n) \tag{3.11}$$

in $\mathcal{P}_{n,d}^{W_n}$ (if $d < n$, then (3.11) should be read as $R_\lambda(x; q, \tau) = g(x)$ and the proof below goes through with the obvious adjustments). Hence

$$[R_\lambda]_d = [g]_d + (-\tau_n)^n [h]_{d-n} [m_{\mathbf{1}}]_n$$

in $\mathcal{G}(\mathcal{P}_n^{W_n})$, and the result follows from the fact that the linear expansion of $[h]_{d-n}[m_{\mathbf{1}}]_n \in \mathcal{G}_{n,d}^{W_n}$ in the basis $\{[m_\mu]_d \mid \mu \in \widehat{\Lambda}_{n,d}^+\}$ of $\mathcal{G}_{n,d}^{W_n}$ only involves the basis elements $[m_\mu]_d$ with $\mu_n > 0$. □

Example 3.7. If $n = 1$ then the interpolation parameter $\tau \in \mathcal{T}_1$ is a complex number $s \in \mathbb{C}^*$ satisfying $s^2 \notin q^{\mathbb{Z}}$. We denote the corresponding symmetric interpolation Laurent polynomial $R_{(\ell)}(x; q, \tau)$ in one variable x by $R_\ell(x; q, s)$ for $\ell \in \mathbb{Z}_{\geq 0}$. Then

$$R_\ell(x; q, s) = \frac{(sx, sx^{-1}; q)_\ell}{(q^\ell s^2, q^{-\ell}; q)_\ell} \tag{3.12}$$

and the coefficient of $m_\ell(x)$ in the linear expansion of $R_\ell(x; q, s)$ with respect to the basis $\{m_k(x) \mid 0 \leq k \leq \ell\}$ of $\mathcal{P}_{1,\ell}^+$ is

$$\frac{(-s)^\ell q^{\ell(\ell-1)/2}}{(q^\ell s^2, q^{-\ell}; q)_\ell}.$$

Write $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_i := t^{n-i}$, and $st^\rho := (st^{\rho_1}, \ldots, st^{\rho_n})$ for $s, t \in \mathbb{C}^*$. Note that $\rho + (1, \ldots, 1) = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. Furthermore, if $st^\rho \in \mathcal{T}_n$ then $t \notin q^{\mathbb{Z} \setminus \{0\}}$ and $s^2 \notin q^{\mathbb{Z}}$.

Definition 3.8 (cf. [26]). Let $s, t \in \mathbb{C}^*$ such that $st^\rho \in \mathcal{T}_n$. Then we call

$$R_\lambda(x; q, s, t) := R_\lambda(x; q, st^\rho)$$

the BC_n-*type interpolation Macdonald polynomial* of degree $\lambda \in \Lambda_n^+$.

The specialization $\tau = st^\rho$ of the parameters τ is called the *principal specialization*. By Definition 3.4 and (3.4), $R_\lambda(x; q, s, t)$ is the BC_n-symmetric Laurent polynomial of degree $|\lambda|$ such that

$$R_\lambda(q^\mu st^\rho; q, s, t) = \delta_{\lambda, \mu} \qquad (\mu \in \Lambda_n^+, |\mu| \leq |\lambda|). \tag{3.13}$$

It is related to Okounkov's polynomial $P_\lambda^*(x; q, s, t)$ from [26, Def. 1.1] by the formula

$$R_\lambda(x; q, s, t) = \frac{P_\lambda^*(xt^{-\rho}s^{-1}; q, t, s)}{P_\lambda^*(q^\lambda; q, t, s)}.$$

For the BC_n-type interpolation Macdonald polynomials, part (a) of Proposition 3.5 is [26, Prop. 2.2], part (b) of Proposition 3.5 is [26, Prop. 2.1], and Proposition 3.6 is a special case of [26, Cor. 5.4].

The BC_n-type interpolation Macdonald polynomials form a distinguished class of BC_n-symmetric Laurent interpolation polynomials. They satisfy various special properties, such as the *extra vanishing property*

$$R_\lambda(q^\mu st^\rho; q, s, t) = 0 \quad \text{if} \quad \lambda \nsubseteq \mu, \tag{3.14}$$

and they admit explicit binomial, combinatorial and integral formulas; see [26], [27]. The combinatorial formula [26, Thm. 5.2] allows to obtain more precise information on the expansion components of $R_\lambda(x; q, s, t)$ in symmetric monomials, while the binomial formula [26, Thm. 7.1] provides the explicit expansion of Koornwinder polynomials in terms of BC_n-type interpolation Macdonald polynomials.

Remark 3.9. The interpolation grid for the BC_n-type interpolation Macdonald polynomials naturally appears in the theory of Koornwinder polynomials in the following way. Koornwinder polynomials are the BC_n-symmetric Laurent polynomial eigenfunctions of the commuting Koornwinder–van Diejen q-difference operators [6, 13], depending on five parameters a, b, c, d, t. These operators generate a commutative algebra isomorphic to $\mathcal{P}_n^{W_n}$ through the Harish-Chandra isomorphism (cf. [15, §2]). Through this isomorphism, the eigenvalues of the Koornwinder–van Diejen q-difference operators are described by the evaluation morphisms $\mathcal{P}_n^{W_n} \to \mathbb{C}$, $p \mapsto p(q^\lambda st^\rho)$ ($\lambda \in \Lambda_n^+$), where $s = \sqrt{q^{-1}abcd}$.

4. Interpolation theorem for nonsymmetric Laurent polynomials

We extend definition (3.4) of the interpolation points $\overline{\mu}$ from $\mu \in \Lambda_n^+$ to $\mu \in \Lambda_n$ as follows. Put $\tau \in \mathcal{T}_n$, with $\mathcal{T}_n$ defined by (3.3). For $\alpha \in \Lambda_n$, we define $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_n) \in \mathbb{C}^n$ by

$$\overline{\alpha}_i := q^{\alpha_i} \big(\tau_{\pi_\alpha^{-1}(i)}\big)^{\operatorname{sgn}(\alpha_i)}. \tag{4.1}$$

Here π_α is as in Lemma 2.2. We write $\overline{\alpha} = \overline{\alpha}(q, \tau)$ and $\overline{\alpha}_i = \overline{\alpha}_i(q, \tau)$ if we need to emphasize the dependence of $\overline{\alpha}$ on the parameters.

Recall the actions (2.2), (2.3), (2.4) of W_n on Λ_n, $(\mathbb{C}^*)^n$ and $\mathcal{P}_n$, respectively. The resulting action of W_n on the interpolation points $\overline{\alpha} \in (\mathbb{C}^*)^n$ can be described as follows.

Lemma 4.1. *Let $\alpha \in \Lambda_n$.*

(a) *Let $j \in [1, n]$. If $s_j\alpha \neq \alpha$ then $\boldsymbol{s_j}\overline{\alpha} = \overline{s_j\alpha}$.*

(b) *If $j \in [1, n)$ and $\alpha_j = \alpha_{j+1}$ then*

$$\overline{\alpha}_j / \overline{\alpha}_{j+1} = \big(\tau_{\pi_\alpha^{-1}(j)} / \tau_{\pi_\alpha^{-1}(j) + \operatorname{sgn}(\alpha_j)}\big)^{\operatorname{sgn}(\alpha_j)}. \tag{4.2}$$

(c) *If $\alpha_n = 0$ then $\overline{\alpha}_n = \tau_n$.*

Remark 4.2. In case of the special specialisation $\tau = st^\rho$, part (a) of Lemma 4.1 was observed by Sahi [37, Proof of Thm. 5.3], and parts (b) and (c) were observed in [40, Remark 4.7].

Proof of Lemma 4.1. First we prove (a). For $j = n$, this reduces by (4.1) and the assumption $s_n\alpha \neq \alpha$ to showing that $\pi^{-1}_{s_n\alpha} = \pi^{-1}_\alpha$. It follows immediately from Lemma 2.2 that these two permutations are equal. For $j < n$, the statement of (a) reduces by (4.1) to showing that $\pi^{-1}_{s_j\alpha} = \pi^{-1}_\alpha \circ s_j$ if $s_j\alpha \neq \alpha$. Also the equality of these two permutations under the given condition follows immediately from Lemma 2.2.

Similarly the proofs of (b) and (c) reduce by (4.1) to respectively showing that $\pi^{-1}_\alpha(j+1) = \pi^{-1}_\alpha(j) + \operatorname{sgn}\alpha_j$ and $\pi^{-1}_\alpha(n) = n$. Both statements immediately follow from Lemma 2.2 taking into account the assumption. $\square$

Corollary 4.3. *Let $\tau \in \mathcal{T}_n$.*

(a) *For all $\alpha \in \Lambda_n$ we have $\overline{\alpha} = \boldsymbol{w_\alpha}\overline{\alpha^+}$.*

(b) *The map $\Lambda_n \to (\mathbb{C}^*)^n$, $\alpha \mapsto \overline{\alpha}$, is injective.*

(c) *We have $\overline{\alpha}^{\beta^j} \neq 1$ for $j \in [1,n]$ and $\alpha \in \Lambda_n$ such that $s_j\alpha \neq \alpha$.*

Proof. (a) By Definition 2.1, equation (3.4), Lemma 2.2(a), and equation (4.1) we have

$$
(\boldsymbol{w_\alpha}\overline{\alpha^+})_i = (\boldsymbol{\sigma_\alpha}\pi_\alpha\overline{\alpha^+})_i = \left(\pi_\alpha\overline{\alpha^+}\right)_i^{\operatorname{sgn}\alpha_i} = \left(\overline{\alpha^+}\right)^{\operatorname{sgn}\alpha_i}_{\pi^{-1}_\alpha(i)}
$$
$$
= \left(q^{\alpha_{\pi^{-1}_\alpha(i)}}\tau_{\pi^{-1}_\alpha(i)}\right)^{\operatorname{sgn}\alpha_i} = \left(q^{|\alpha_i|}\right)^{\operatorname{sgn}\alpha_i}\left(\tau_{\pi^{-1}_\alpha(i)}\right)^{\operatorname{sgn}\alpha_i}
$$
$$
= q^{\alpha_i}\left(\tau_{\pi^{-1}_\alpha(i)}\right)^{\operatorname{sgn}\alpha_i} = \overline{\alpha}_i .
$$

(b) This follows from the explicit expression (4.1) using the fact that q is not a root of unity and that $\tau \in \mathcal{T}_n$ (see (3.3)).

(c) By part (b) of the Corollary and Lemma 4.1(a) we have for $s_j\alpha \neq \alpha$ that $\mathbf{s}_j\overline{\alpha} = \overline{s_j\alpha} \neq \overline{\alpha}$. Hence $\overline{\alpha}^{\beta^j} = \overline{\alpha}_j/\overline{\alpha}_{j+1} \neq 1$ for $j < n$ and $\overline{\alpha}^{\beta^j} = \overline{\alpha}_n^2 \neq 1$ for $j = n$. $\square$

For $d \in \mathbb{Z}_{\geq 0}$ and $I = \{i_1, \dots, i_k\} \subseteq [1,n]$ put

$$
\Lambda_{n,d} := \{\mu \in \Lambda_n \mid |\mu| \leq d\},
$$
$$
R(n,d,I) := \{\alpha \in \Lambda_{n,d} \mid \alpha_j \neq 0 \text{ for all } j \text{ and } \alpha_j \neq -1 \text{ if } j \in I^c\},
$$
$$
T(n,d,I) := \{\alpha \in \Lambda_{n,d} \mid \alpha_j \neq 0 \text{ if } j \in I^c\}.
$$

Note that
$$
R(n, d, [1,n]) = T(n, d, \varnothing), \qquad T(n, d, [1,n]) = \Lambda_{n,d}.
$$

Furthermore,

$$
R(n,d,I) = \varnothing \quad \text{if} \quad d - n < 0; \qquad T(n,d,I) = \varnothing \quad \text{if} \quad d - n + |I| < 0.
$$

Proposition 4.4. *Let $n \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}_{\geq 0}$, $\tau \in \mathcal{T}_n$. Let $I = \{i_1, \dots, i_k\} \subseteq [1,n]$ be a set of cardinality k.*

(a) *For every map* $\overline{f} \colon R(n, d, I) \to \mathbb{C}$ *there exists a Laurent polynomial* $f \in \mathcal{P}_n$ *such that*

$$f(\overline{\alpha}(q, \tau)) = \overline{f}(\alpha) \qquad \forall\, \alpha \in R(n, d, I),$$
$$\deg\big(x_J f(x)\big) \le d - n + k \qquad \forall\, J \subseteq I. \tag{4.3}$$

(b) *For every map* $\overline{f} \colon T(n, d, I) \to \mathbb{C}$ *there exists a Laurent polynomial* $f \in \mathcal{P}_n$ *such that*

$$f(\overline{\alpha}(q, \tau)) = \overline{f}(\alpha) \qquad \forall\, \alpha \in T(n, d, I),$$
$$\deg\big(x_J f(x)\big) \le d \qquad \forall J \subseteq I^c. \tag{4.4}$$

Remark 4.5. Note that statement (b) for $I = \varnothing$ is statement (a) for $I = [1, n]$.

Proof of Proposition 4.4. If $\overline{f}$ is a map on an empty set then choose f identically zero. Thus statement (a) holds trivially when $d < n$ and statement (b) holds trivially when $d < n - k$.

If $(n, d) = (1, 0)$ then statement (a), and statement (b) for $I = \varnothing$, are true by the remark in the previous paragraph. For statement (b) with $I = \{1\}$ note that $T(1, 0, \{1\}) = \{0\}$, hence we can take $f(x)$ to be the constant polynomial $\overline{f}(0)$.

Now let $n + d \ge 2$. Suppose that all the statements of the Proposition, with (n, d) replaced by $(\widetilde{n}, \widetilde{d})$, are true for all subsets $I \subset [1, \widetilde{n}]$ and all $\tau \in \mathcal{T}_{\widetilde{n}}$ when $\widetilde{n} + \widetilde{d} < n + d$. We will then successively prove statements (a) and (b) by induction on the cardinality $|I|$ of the subset $I \subseteq [1, n]$.

Proof of statement (a). We may assume that $d \ge n$.

First consider the case $I = \varnothing$. Note that

$$R(n, d, \varnothing) = \{\alpha \in \Lambda_{n,d} \mid \alpha_j \ne 0, -1 \text{ for all } j\}.$$

Fix a map $\overline{f} \colon R(n, d, \varnothing) \to \mathbb{C}$ and $\tau \in \mathcal{T}_n$. We prove the existence of a Laurent polynomial $f \in \mathcal{P}_{n,d-n}$ such that $f(\overline{\alpha}(q, \tau)) = \overline{f}(\alpha)$ for all $\alpha \in R(n, d, \varnothing)$ by solving a related interpolation problem on the set $T(n, d - n, [1, n]) = \Lambda_{n,d-n}$, using statement (b) with shifted parameters $q\tau \in \mathcal{T}_n$.

Consider for this the bijection

$$\alpha \mapsto \beta \colon R(n, d, \varnothing) \xrightarrow{\ \sim\ } T(n, d - n, [1, n]), \qquad \beta_i := \alpha_i - \operatorname{sgn}(\alpha_i). \tag{4.5}$$

Note that $\operatorname{sgn}(\beta_i) = \operatorname{sgn}(\alpha_i)$ and, using (2.6), $\pi_\beta = \pi_\alpha$. Hence, by (4.1), $\overline{\alpha}(q, \tau) = \overline{\beta}(q, q\tau)$. By the induction hypothesis, statement (b) with τ replaced by $q\tau$ is valid for the function $\beta \mapsto \overline{f}(\alpha)$ $(\beta \in T(n, d - n, [1, n]))$. Hence there exists $f \in \mathcal{P}_{n,d-n}$ such that $f(\overline{\beta}(q, q\tau)) = \overline{f}(\alpha)$ for all $\beta \in T(n, d - n, [1, n])$. But $f \in \mathcal{P}_{n,d-n}$ then also satisfies $f(\overline{\alpha}(q, \tau)) = \overline{f}(\alpha)$ for all $\alpha \in R(n, d, \varnothing)$, which completes the proof of statement (a) for $I = \varnothing$.

Let $k > 0$ and assume that statement (a) is true for all $\overline{f}^{\vee} \colon R(n^\vee, d^\vee, I^\vee) \to \mathbb{C}$ when $n^\vee + d^\vee \le n + d$, $\tau^\vee \in \mathcal{T}_{n^\vee}$ and with $I^\vee \subseteq [1, n^\vee]$ of cardinality $< k$. (Note that for $n^\vee + d^\vee < n + d$ this assumption already holds by our earlier induction hypothesis.) Let $I = \{i_1, \ldots, i_k\} \subseteq [1, n]$ be a set of cardinality k, $\tau \in \mathcal{T}_n$, and

consider a function $\overline{f}\colon R(n, d, I) \to \mathbb{C}$. We prove the existence of an interpolation Laurent polynomial $f \in \mathcal{P}_n$ satisfying (4.3) by splitting the interpolation problem in two pieces. For this we use the disjoint union

$$R(n, d, I) = R^{\vee}(n, d, I) \sqcup R(n, d, I\backslash\{i_1\})$$

with

$$R^{\vee}(n, d, I) := \{\alpha \in R(n, d, I) \mid \alpha_{i_1} = -1\}.$$

The first step is to prove the existence of a Laurent polynomial $g \in \mathcal{P}_n$ such that

$$
\begin{aligned}
g(\overline{\alpha}(q, \tau)) &= \overline{f}(\alpha) &&\forall\, \alpha \in R^{\vee}(n, d, I), \\
\deg\big(g(x)x_K\big) &\le d - n + k - 1 &&\forall\, K \subseteq I \setminus \{i_1\}.
\end{aligned}
\tag{4.6}
$$

For $n = 1$, we have $d \ge 1$ and $I = \{1\}$, hence $R^{\vee}(1, d, \{1\}) = \{-1\}$. In this case, we can take $g(x)$ to be the constant polynomial $\overline{f}(-1)$. Assume that $n > 1$. In this case, we solve the interpolation problem (4.6) by rewriting it as an interpolation problem for a function on $R(n-1, d-1, J)$ with $J := \{i_2 - 1, i_3 - 1, \ldots, i_k - 1\} \subseteq [1, n-1]$.

Consider the bijection

$$\alpha \mapsto \gamma\colon R^{\vee}(n, d, I) \xrightarrow{\sim} R(n - 1, d - 1, J)$$

with $\gamma := \alpha^{(i_1)} = (\alpha_1, \ldots, \alpha_{i_1-1}, \alpha_{i_1+1}, \ldots, \alpha_n)$. In other words, $\gamma_i = \alpha_{i^{\vee}}$ for $i \in [1, n)$ with $i^{\vee} := i$ if $i < i_1$ and $i^{\vee} := i + 1$ if $i_1 \le i < n$.

The interpolation points behave under this bijection in the following manner. By the explicit description of π_α (see Lemma 2.2) we have $\pi_\alpha^{-1}(i_1) = n$ and $\pi_\alpha^{-1}(i^{\vee}) = \pi_\gamma^{-1}(i)$ $(i \in [1, n))$ for $\alpha \in R^{\vee}(n, d, I)$. Then, by (4.1) we have

$$
\begin{aligned}
\overline{\alpha}_{i^{\vee}}(q, \tau) &= \overline{\gamma}_i(q, \tau') &&(1 \le i < n), \\
\overline{\alpha}_{i_1}(q, \tau) &= q^{-1}\tau_n^{-1}
\end{aligned}
\tag{4.7}
$$

for $\alpha \in R^{\vee}(n, d, I)$.

Consider the function $\overline{g}^{\vee}\colon R(n - 1, d - 1, J) \to \mathbb{C}$, defined by $\overline{g}^{\vee}(\gamma) := \overline{f}(\alpha)$. Since $|J| < k$ the induction hypothesis (either the one on the sum of the number of variables and the weight, or the one on the size of the subset) implies that statement (a) is true for $\overline{g}^{\vee}$ and $\tau' \in \mathcal{T}_{n-1}$. Hence there exists a Laurent polynomial $g^{\vee} \in \mathcal{P}_{n-1}$ such that $g^{\vee}(\overline{\gamma}(q, \tau')) = \overline{f}(\alpha)$ for all $\gamma \in R(n - 1, d - 1, J)$, satisfying the degree conditions $\deg\big(g^{\vee}(x')x_K\big) \le d - n + k - 1$ for all $K \subseteq J$. Define $g \in \mathcal{P}_n$ by $g(x) := g^{\vee}(x^{(i_1)})$, then it follows that g satisfies (4.6). This completes the first step.

As a second step we add an appropriate term to g to obtain the desired interpolation properties for the full set $R(n, d, I)$. Note that Laurent polynomials f of the form

$$f(x) = g(x) + (x_{i_1}^{-1} - q\tau_n)h(x) \tag{4.8}$$

with $h \in \mathcal{P}_n$ all satisfy $f(\overline{\alpha}(q, \tau)) = \overline{f}(\alpha)$ for $\alpha \in R^{\vee}(n, d, I)$ in view of (4.7). The interpolation property $f(\overline{\alpha}(q, \tau)) = \overline{f}(\alpha)$ is then also satisfied for $\alpha \in R(n, d, I \backslash \{i_1\})$ if

$$h(\overline{\alpha}(q, \tau)) = \frac{\overline{f}(\alpha) - g(\overline{\alpha}(q, \tau))}{\overline{\alpha}_{i_1}(q, \tau)^{-1} - q\tau_n} \qquad \left(\alpha \in R(n, d, I \backslash \{i_1\})\right). \tag{4.9}$$

Note that the right-hand side is well defined, since the conditions (3.3) on the parameters together with (4.1) and the fact that $\alpha_{i_1} \neq -1$ imply that the denominator is nonzero.

Due to the induction hypothesis, we are allowed to apply statement (a) to the function

$$\overline{h}(\alpha) := \frac{\overline{f}(\alpha) - g(\overline{\alpha}(q, \tau))}{\overline{\alpha}_{i_1}(q, \tau)^{-1} - q\tau_n} \qquad \left(\alpha \in R(n, d, I \backslash \{i_1\})\right).$$

This gives a Laurent polynomial $h \in \mathcal{P}_n$ fulfilling (4.9) and satisfying the degree conditions $\deg\bigl(h(x)x_K\bigr) \leq d - n + k - 1$ for all $K \subseteq I \setminus \{i_1\}$. Then f given by (4.8) satisfies $f(\overline{\alpha}(q, \tau)) = \overline{f}(\alpha)$ for all $\alpha \in R(n, d, I)$. Furthermore, by the degree properties of g and h the degree conditions $\deg(f(x)x_J) \leq d - n + k$ for all $J \subseteq I$ are satisfied. Hence f satisfies (4.3) as desired.

Proof of statement (b). The proof is along the same lines as the proof of statement (a), but there are subtle differences in the combinatorics.

We may assume that $d \geq n - k$. If $I = \varnothing$ then the statement is correct due to Remark 4.5.

Let $k > 0$ and assume that statement (b) is true for all $\overline{f}^{\wedge} \colon R(n^{\wedge}, d^{\wedge}, I^{\wedge}) \to \mathbb{C}$ when $n^{\wedge} + d^{\wedge} \leq n + d$, $\tau^{\wedge} \in \mathcal{T}_{n^{\wedge}}$ and $I^{\wedge} \subseteq [1, n^{\wedge}]$ is a set of cardinality $< k$. (Note that for $n^{\vee} + d^{\vee} < n + d$ this assumption already holds by our earlier induction hypothesis.) Let $I = \{i_1, \ldots, i_k\} \subseteq [1, n]$ be a set of cardinality k and consider a function $\overline{f} \colon T(n, d, I) \to \mathbb{C}$. We have to prove the existence of an interpolation Laurent polynomial $f \in \mathcal{P}_n$ satisfying (4.4).

Consider this time the decomposition

$$T(n, d, I) = T^{\wedge}(n, d, I) \sqcup T(n, d, I \backslash \{i_k\})$$

with

$$T^{\wedge}(n, d, I) := \{\alpha \in T(n, d, I) \mid \alpha_{i_k} = 0\}.$$

We claim that there exists $g \in \mathcal{P}_n$ such that

$$\begin{aligned} g(\overline{\alpha}(q, \tau)) &= \overline{f}(\alpha) && \forall \alpha \in T^{\wedge}(n, d, I), \\ \deg\bigl(g(x)x_K\bigr) &\leq d && \forall K \subseteq I^c. \end{aligned} \tag{4.10}$$

For $n = 1$, we have $d \geq 1$ and $I = \{1\}$, hence $T^{\wedge}(1, d, \{1\}) = \{0\}$ and we can take $g(x)$ to be the constant polynomial equal to $\overline{f}(0)$. For $n > 1$, consider the bijection

$$\alpha \mapsto \delta \colon T^{\wedge}(n, d, I) \xrightarrow{\sim} T(n - 1, d, I \setminus \{i_k\})$$

with $\delta := \alpha^{(i_k)}$. In other words, $\delta_i := \alpha_{i^{\wedge}}$ $(i \in [1, n))$ with $i^{\wedge}$ defined by $i^{\wedge} := i$ if $i < i_k$ and $i^{\wedge} := i + 1$ if $i_k \leq i < n$. As in the proof of statement (a) one then shows that

$$\begin{aligned} \overline{\alpha}_{i^{\wedge}}(q, \tau) &= \overline{\delta}_i(q, \tau') && i \in [1, n), \\ \overline{\alpha}_{i_k}(q, \tau) &= \tau_n \end{aligned} \tag{4.11}$$

for $\alpha \in T^\wedge(n, d, I)$. By the induction hypothesis (either the induction hypothesis on the sum of the number of variables and the weight, or the induction hypothesis on the size of the subset), there exists $g^\wedge \in \mathcal{P}_n$ such that $g^\wedge(\overline{\delta}(q, \tau')) = \overline{f}(\alpha)$ for $\delta \in T(n-1, d, I \setminus \{i_k\})$ and satisfying the degree conditions $\deg(g^\wedge(x')x_K) \leq d$ for all $K \subseteq (I^c \cup \{i_k\}) \cap [1, n-1]$. Then $g \in \mathcal{P}_n$, defined by $g(x) := g^\wedge(x^{(i_k)})$, satisfies (4.10).

Now define $\overline{h} \colon T(n, d, I \setminus \{i_k\}) \to \mathbb{C}$ by

$$\overline{h}(\alpha) := \frac{\overline{f}(\alpha) - g(\overline{\alpha}(q, \tau))}{\overline{\alpha}_{i_k}(q, \tau) - \tau_n} \qquad (\alpha \in T(n, d, I \setminus \{i_k\})).$$

Note that the right-hand side is well defined, since the conditions (3.3) on the parameters together with (4.1) and the fact that $\alpha_{i_k} \neq 0$ imply that the denominator is nonzero. By the induction hypothesis, there exists $h \in \mathcal{P}_n$ such that $h(\overline{\alpha}(q, \tau)) = \overline{h}(\alpha)$ for all $\alpha \in T(n, d, I \setminus \{i_k\})$ which satisfies the degree conditions $\deg(h(x)x_K) \leq d$ for all $K \subseteq I^c \cup \{i_k\}$. Furthermore, with this choice of h and (4.11) it is clear that

$$f(x) := g(x) + (x_{i_k} - \tau_n)h(x)$$

satisfies the desired interpolation property $f(\overline{\alpha}(q, \tau)) = \overline{f}(\alpha)$ for all $\alpha \in T(n, d, I)$. By the degree conditions on $g(x)$ and $h(x)$, we have $\deg(f(x)x_J) \leq d$ for all $J \subseteq I^c$, which completes the proof of statement (b). $\square$

Theorem 4.6. *Let $\tau \in \mathcal{T}_n$. For every map $\overline{f} \colon \Lambda_{n,d} \to \mathbb{C}$ there exists a unique Laurent polynomial $f \in \mathcal{P}_{n,d}$ such that $f(\overline{\alpha}(q, \tau)) = \overline{f}(\alpha)$ for all $\alpha \in \Lambda_{n,d}$.*

Proof. Denote by $\mathcal{F}_{n,d}^{q,\tau}$ the space of complex-valued functions on

$$\mathcal{S}_{n,d}^{q,\tau} := \{\overline{\alpha}(q, \tau) \mid \alpha \in \Lambda_{n,d}\}.$$

Then, by Corollary 4.3(b), the space $\mathcal{F}_{n,d}^{q,\tau}$ has dimension $|\Lambda_{n,d}|$. Define the linear map $\phi_{n,d}^{q,\tau} \colon \mathcal{P}_{n,d} \to \mathcal{F}_{n,d}^{q,\tau}$ by $\phi_{n,d}^{q,\tau}(f) := f|_{\mathcal{S}_{n,d}^{q,\tau}}$. Proposition 4.4(b) with $I = [1, n]$ implies that $\phi_{n,d}^{q,\tau}$ is surjective. Then $\phi_{n,d}^{q,\tau}$ is also injective, since both vector spaces $\mathcal{P}_{n,d}$ and $\mathcal{F}_{n,d}^{q,\tau}$ are of dimension $|\Lambda_{n,d}|$. Hence $\phi_{n,d}^{q,\tau}$ is a linear isomorphism, which implies the theorem. $\square$

In the remainder of this section, we fix $\tau \in \mathcal{T}_n$ and write $\overline{\alpha} := \overline{\alpha}(q, \tau)$ for $\alpha \in \Lambda_n$. In view of Theorem 4.6, we can give the following definition.

Definition 4.7. The (nonsymmetric) interpolation Laurent polynomial of degree $\alpha \in \Lambda_n$ is the unique Laurent polynomial $G_\alpha(x; q, \tau)$ in n variables x of degree at most $|\alpha|$ such that $G_\alpha(\overline{\alpha}; q, \tau) = 1$ and

$$G_\alpha(\overline{\beta}; q, \tau) = 0 \qquad \forall \beta \in \Lambda_{n,|\alpha|} \setminus \{\alpha\}.$$

Theorem 4.6 implies that $\{G_\alpha(x; q, \tau) \mid \alpha \in \Lambda_{n,d}\}$ is a linear basis of $\mathcal{P}_{n,d}$.

Example 4.8. Recall from Example 3.7 that for $n = 1$ the interpolation para-
meter $\tau \in \mathcal{T}_1$ is given by a complex number s satisfying $s^2 \notin q^{\mathbb{Z}}$. We write
$G_{(\ell)}(x; q, \tau)$ for $\ell \in \mathbb{Z}$ by $G_\ell(x; q, s)$. Then the Laurent polynomial $G_\ell(x; q, s)$ has
degree at most $|\ell|$ and is characterized by the equations $G_\ell(q^k s^{\mathrm{sgn}(k)}; q, s) = \delta_{\ell,k}$
for $k \in \mathbb{Z}$ with $|k| \leq |\ell|$. It follows that

$$
\begin{aligned}
G_\ell(x; q, s) &= \frac{(qsx, sx^{-1}; q)_\ell}{(q^{1+\ell} s^2, q^{-\ell}; q)_\ell}, & \ell &\in \mathbb{Z}_{\geq 0}, \\
G_{-\ell}(x; q, s) &= \frac{q^\ell sx (qsx; q)_{\ell-1} (sx^{-1}; q)_{\ell+1}}{(q^\ell s^2; q)_{\ell+1} (q^{1-\ell}; q)_{\ell-1}}, & \ell &\in \mathbb{Z}_{>0}.
\end{aligned}
\tag{4.12}
$$

Furthermore, for $n > 1$,

$$
G_\alpha(x; q, \mathbf{s}) := \prod_{i=1}^n G_{\alpha_i}(x_i; q, s) \qquad (\alpha \in \Lambda_n)
$$

with (recall) $\mathbf{s} = (s, \dots, s) \in \mathcal{T}_n$.

As in Section 3, one concludes from Theorem 4.6 that $\{[G_\alpha]_d \mid \alpha \in \widehat{\Lambda}_{n,d}\}$ is a
linear basis of $\mathcal{G}_{n,d}$, where $\widehat{\Lambda}_{n,d} := \Lambda_{n,d} \setminus \Lambda_{n,d-1}$ and $\Lambda_{n,-1} := \varnothing$. In particular,

$$
\deg\big(G_\alpha(x; q, \tau)\big) = |\alpha| \qquad (\alpha \in \Lambda_n).
$$

Recall from Proposition 3.6 that the coefficient of $m_\lambda(x)$ in the linear expansion of
the symmetric interpolation Laurent polynomial $R_\lambda(x; q, \tau)$ in symmetric mono-
mials $m_\mu(x)$ ($\mu \in \Lambda_n^+$) is nonzero. For the nonsymmetric interpolation Laurent
polynomial $G_\alpha(x; q, \tau)$ we have the following result.

Lemma 4.9. *Let $\alpha \in \Lambda_n$. The coefficients $c_\gamma^\alpha(q, \tau) \in \mathbb{C}$ in the linear expansion*

$$
G_\alpha(x; q, \tau) = \sum_{\gamma \in \Lambda_{n,|\alpha|}} c_\gamma^\alpha(q, \tau) x^\gamma
\tag{4.13}
$$

*are rational functions in the variables $q, \tau_1, \dots, \tau_n$. The rational function represent-
ing $c_\alpha^\alpha(q, \tau)$ for q not a root of unity and $\tau \in \mathcal{T}_n$, is nonzero.*

Proof. We will prove that the $c_\gamma^\alpha(q, \tau)$ are rational in q, τ by considering them
for fixed α as solutions of a linear system with rational coefficients. We use the
notations introduced in the proof of Theorem 4.6. Let $d \in \mathbb{Z}_{\geq 0}$ and set $m := |\Lambda_{n,d}|$.
Identify $\Lambda_{n,d}$ with $[1, m]$ by fixing an enumeration of the elements in $\Lambda_{n,d}$. This
provides vector space identifications

$$
\sum_{\gamma \in \Lambda_{n,d}} d_\gamma x^\gamma \mapsto (d_\gamma)_\gamma : \mathcal{P}_{n,d} \xrightarrow{\sim} \mathbb{C}^m, \qquad f \mapsto \big(f(\overline{\gamma}(q, \tau))\big)_\gamma : \mathcal{F}_{n,d}^{q,\tau} \xrightarrow{\sim} \mathbb{C}^m.
$$

The linear isomorphism $\phi_{n,d}^{q,\tau} : \mathcal{P}_{n,d} \xrightarrow{\sim} \mathcal{F}_{n,d}^{q,\tau}$ is then represented by the invertible
matrix

$$
A(q, \tau) := \big(\overline{\beta}(q, \tau)^\gamma\big)_{\beta,\gamma} \in \mathrm{GL}_m(\mathbb{C}).
$$

Let $\{e^\alpha \mid \alpha \in \Lambda_{n,d}\}$ be the standard basis of $\mathbb{C}^m$. Then Definition 4.7 implies that

$$\left(c_\gamma^\alpha(q,\tau)\right)_\gamma = A(q,\tau)^{-1} e_\alpha \qquad \forall\, \alpha \in \Lambda_{n,d},$$

and Example 4.8 shows that $c_\alpha^\alpha(q,\mathbf{s}) \neq 0$ if $s \in \mathbb{C}^*$ and $s^2 \notin q^{\mathbb{Z}}$. The result now follows from the fact that the matrix coefficients of $A(q,\tau)$ are rational functions in $q, \tau_1, \ldots, \tau_n$. $\square$

For $n = 1$ we find from (3.12) and (4.12) that

$$R_\ell(x; q, s) = G_\ell(x; q, s) + G_{-\ell}(x; q, s) \qquad (\ell \in \mathbb{Z}_{\geq 0}).$$

This generalizes to arbitrary $n \geq 1$ as follows.

Theorem 4.10. *Let $R_\lambda(x) = R_\lambda(x; q, \tau)$ and $G_\alpha(x) = G_\alpha(x; q, \tau)$ be the symmetric and nonsymmetric interpolation polynomials as given by Definitions 3.4 and 4.7, respectively. For $\lambda \in \Lambda_n^+$ and $\alpha \in \Lambda_n$ we have*

$$R_\lambda(\overline{\alpha}) = R_\lambda(\overline{\alpha^+}), \qquad R_\lambda(x) = \sum_{\beta \in W_n \lambda} G_\beta(x). \qquad (4.14)$$

Proof. The first formula in (4.14) follows immediately from part (a) of Corollary 4.3.

Let $\lambda \in \Lambda_n^+$ and write $H_\lambda := \sum_{\beta \in W_n \lambda} G_\beta$. Then, by the definitions of R_λ and G_β, and by part (a) of the theorem, R_λ and H_λ are Laurent polynomials of degree at most $|\lambda|$ satisfying $R_\lambda(\overline{\alpha}) = \delta_{\alpha^+, \lambda} = H_\lambda(\overline{\alpha})$ $(\alpha \in \Lambda_{n,|\lambda|})$. By Theorem 4.6, we conclude that $R_\lambda = H_\lambda$. $\square$

Remark 4.11. An analogous statement as Theorem 4.10 holds true for Sahi's [35] symmetric and nonsymmetric interpolation polynomials, with essentially the same proof.

Consider the principal specialization $\tau_i = st^{n-i}$ of $\tau \in \mathcal{T}_n$. The explicit formula (4.1) for the interpolation point $\overline{\alpha}$ $(\alpha \in \Lambda_n)$ then takes the form

$$\overline{\alpha}_i = q^{\alpha_i} \left(st^{n - \pi_\alpha^{-1}(i)}\right)^{\text{sgn}(\alpha_i)} \qquad (i \in [1, n]). \qquad (4.15)$$

If $s = \sqrt{q^{-1}abcd}$ then $\overline{\alpha}_i$ (4.15) corresponds to the eigenvalue of Noumi's Y-operator Y_i for the nonsymmetric Koornwinder polynomial $E_\alpha(x; a, b, c, d; q, t)$ of degree α, see [37, §6] (compare with Remark 3.9 for the symmetric theory). Note that for $\tau = st^\rho$, the second formula in (4.14) gives the expansion of the BC_n-type interpolation Macdonald polynomial $R_\lambda(x; q, s, t)$ in terms of the interpolation Laurent polynomials $G_\beta(x; q, s, t)$ $(\beta \in W_n \lambda)$. This is an analogue for interpolation polynomials of the expansion formula expressing Koornwinder polynomials as linear combination of nonsymmetric Koornwinder polynomials, see [40, Thm. 6.6].

Definition 4.12. Let $s, t \in \mathbb{C}^*$ such that $st^\rho \in \mathcal{T}_n$. Then we call $G_\alpha(x; q, s, t) := G_\alpha(x; q, st^\rho)$ *the nonsymmetric* BC_n-*type interpolation Macdonald polynomial of degree* $\alpha \in \Lambda_n$.

Observe that $G_\alpha(x; q, s, t)$ is the unique Laurent polynomial of degree $\leq |\alpha|$ satisfying $G_\alpha(\overline{\alpha}; q, s, t) = 1$ and $G_\alpha(\overline{\beta}; q, s, t) = 0$ ($\beta \in \Lambda_{n,|\alpha|} \setminus \{\alpha\}$), with $\overline{\alpha}$ given by (4.15).

In the following section and in the Appendix, we present first steps towards answering the question whether the nonsymmetric BC_n-type interpolation Macdonald polynomials satisfy extra vanishing properties and admit explicit binomial formulas. For Knop's [10] type A_{n-1} nonsymmetric interpolation Macdonald polynomials, extra vanishing and explicit binomial formulas were derived in [10], [35], [39]. Their proofs lean on a generalization of Cherednik's action of the double affine Hecke algebra on polynomials in n variables for which the type A_{n-1} nonsymmetric interpolation Macdonald polynomials are common eigenfunctions of the resulting Y-operators.

It is not known whether the nonsymmetric BC_n-type interpolation Macdonald polynomials $G_\alpha(x; q, s, t)$ satisfy extra vanishing properties. In the Appendix, we will present the outcome of computer algebra computations describing extra vanishing for $G_\alpha(x; q, s, t)$ when $n = 2$ and $|\alpha| = 4$.

5. The action of Demazure–Lusztig operators

We introduce an action of the type C_n Hecke algebra on the space of Laurent polynomials in n variables, defined in terms of Demazure–Lusztig operators. We explicitly compute its action on nonsymmetric BC_n-type interpolation Macdonald polynomials. Similar to the type A_{n-1} case in [10], [35], the Hecke algebra techniques in this section can only be applied when taking the principal specialization $\tau = st^\rho$.

Recall our notations associated with root system C_n in Section 2.

Definition 5.1 (Hecke algebra of type B_n or C_n)**.** Let $\mathcal{H}_n(t, t_n)$ be the complex unital associative algebra with generators $T_1, \ldots, T_n$, parameters $t, t_n \in \mathbb{C}^*$, and defining relations

$$
\begin{aligned}
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & i &\in [1, n-2], \\
T_{n-1} T_n T_{n-1} T_n &= T_n T_{n-1} T_n T_{n-1}, & & \\
T_i T_j &= T_j T_i, & |i - j| &> 1, \\
(T_i - t_i)(T_i + 1) &= 0, & i &\in [1, n],
\end{aligned}
$$

with $t_i := t$ for $i \in [1, n)$.

Remark 5.2. The relations $(T_i - t_i)(T_i + 1) = 0$ are related to the usual Hecke relations $(\widetilde{T}_i - \widetilde{t}_i)(\widetilde{T}_i + \widetilde{t}_i^{-1}) = 0$ [18, (4.1.1)] by the substitutions $\widetilde{t}_i = t_i^{1/2}$, $\widetilde{T}_i = t_i^{-1/2} T_i$. (Sahi [37, §2.3] has Hecke relations as in [18] with $\widetilde{t}_i = t_i^{1/2}$.)

The trivial one-dimensional representation χ of $\mathcal{H}_n(t, t_n)$ is characterized by $\chi(T_i) = t_i$ for $i \in [1, n]$. For a reduced expression $w = s_{i_1} \cdots s_{i_r}$ of $w \in W_n$

define $T_w \in \mathcal{H}_n(t, t_n)$ by $T_w := T_{i_1} \cdots T_{i_r}$. This is independent of the choice of the reduced expression; see [9, Prop. 1.15]. Define the Hecke symmetrizer of $\mathcal{H}_n(t, t_n)$ by

$$C_+ := \sum_{w \in W_n} T_w, \tag{5.1}$$

then

$$hC_+ = \chi(h)C_+ = C_+h \qquad (h \in \mathcal{H}_n(t, t_n)); \tag{5.2}$$

cf., e.g., [18, (5.5.7), (5.5.9)].

Noumi [21] introduced a one-parameter family of representations of $\mathcal{H}_n(t, t_n)$ on $\mathcal{P}_n$ in terms of Demazure–Lusztig type operators [16, Prop. 3.6]. Concretely, it is given by

$$T_j \mapsto t + \frac{x_j - tx_{j+1}}{x_j - x_{j+1}} (s_j - 1), \qquad\qquad j \in [1, n),$$

$$T_n \mapsto t_n + \frac{(1 - ax_n^{-1})(1 - bx_n^{-1})}{1 - x_n^{-2}} (s_n - 1),$$

with $a, b \in \mathbb{C}$ such that $ab = -t_n$. As we shall see in Proposition 5.5, the specialization of the Hecke parameters t, t_n and the representation parameters a, b that is needed for the application to nonsymmetric BC_n-type interpolation Macdonald polynomials is $t_n = -1$ and $a = s$, $b = s^{-1}$ with $s \in \mathbb{C}^*$. Noumi's representation then takes the following form.

Lemma 5.3. *Let* $s, t \in \mathbb{C}^*$. *The maps* $T_j \mapsto H_j^{(t)}$ ($j \in [1, n)$) *and* $T_n \mapsto H_n^{(s)}$ *with*

$$H_j^{(t)} := t + \frac{x_j - tx_{j+1}}{x_j - x_{j+1}} (s_j - 1), \qquad\qquad j \in [1, n), \tag{5.3}$$

$$H_n^{(s)} := -1 + \frac{(1 - sx_n^{-1})(1 - s^{-1}x_n^{-1})}{1 - x_n^{-2}} (s_n - 1) \tag{5.4}$$

define a one-parameter family of representations $\pi_s \colon \mathcal{H}_n(t, -1) \to \mathrm{End}(\mathcal{P}_n)$ *on* $\mathcal{P}_n$.

In the following lemma, we show that π_s preserves the degree-filtration on $\mathcal{P}_n$.

Lemma 5.4. *Let* $s, t \in \mathbb{C}^*$. *Then* $\mathcal{P}_{n,d}$ ($d \in \mathbb{Z}_{\geq 0}$) *is a* $\mathcal{H}_n(t, -1)$-*submodule of* $\mathcal{P}_n$ *with respect to the action* π_s.

Proof. We have to prove that $H_n^{(s)}(x^\alpha)$ and $H_j^{(t)}(x^\alpha)$ ($j \in [1, n)$) are Laurent polynomials of degree at most $|\alpha|$. Clearly, it is sufficient to prove the first claim for $n = 1$ and the second claim for $n = 2$ and $j = 1$.

A straightforward computation gives that

$$H_1^{(s)}(x^k) = -\sum_{i=0}^{k} x^{2i-k} - \sum_{i=1}^{k} x^{2i-k} + (s + s^{-1}) \sum_{i=1}^{k} x^{2i-k-1}$$

for $k \in \mathbb{Z}$, where we use the convention that $\sum_{i=k}^{\ell} f_i$ is equal to 0 if $k = \ell + 1$ and equal to $-\sum_{i=\ell+1}^{k-1} f_i$ if $k > \ell + 1$ (then formally, $\sum_{i=k}^{\ell} f_i = \sum_{i=k}^{\infty} f_i - \sum_{i=\ell+1}^{\infty} f_i$ for all $k, \ell \in \mathbb{Z}$). Similarly,

$$H_1^{(t)}(x_1^k x_2^\ell) = -\sum_{i=1}^{k-\ell} x_1^{i+\ell} x_2^{-i+k} + t \sum_{i=0}^{k-\ell} x_1^{i+\ell} x_2^{-i+k}.$$

It follows from these formulas that $H_1^{(s)}(x^k)$ and $H_1^{(t)}(x_1^k x_2^\ell)$ are Laurent polynomials of degree $|k|$ and $|k| + |\ell|$, respectively (for the second case observe that the set $\{(\xi, \eta) \in \mathbb{R}^2 \mid |\xi| + |\eta| \leq d\}$ is convex). $\square$

The action of Demazure–Lusztig type operators on normalized nonsymmetric Koornwinder polynomials was determined explicitly in [40, Prop. 7.8(ii)], and on nonsymmetric interpolation Macdonald polynomials in [39, Lem. 10(1)]. For nonsymmetric BC_n-type interpolation Macdonald polynomials, we have the following result.

Proposition 5.5. *Let $\alpha \in \Lambda_n$ such that $st^\rho \in \mathcal{T}_n$. Write $G_\alpha = G_\alpha(\cdot; q, s, t)$ for the nonsymmetric BC_n-type interpolation Macdonald polynomial of degree α, and $\overline{\alpha}$ for the interpolation point (4.15).*

(a) *Let $j \in [1, n)$. Then*

$$\begin{aligned}
H_j^{(t)} G_\alpha &= t G_\alpha & \text{if } s_j \alpha = \alpha, \\
H_j^{(t)} G_\alpha &= -G_\alpha + \frac{\overline{\alpha}_{j+1} - t\overline{\alpha}_j}{\overline{\alpha}_{j+1} - \overline{\alpha}_j} \left(G_{s_j \alpha} + G_\alpha\right) & \text{if } s_j \alpha \neq \alpha.
\end{aligned} \tag{5.5}$$

(b) *We have*

$$H_n^{(s)} G_\alpha = -G_\alpha + \frac{(1 - s\overline{\alpha}_n)(1 - s^{-1}\overline{\alpha}_n)}{1 - \overline{\alpha}_n^2} \left(G_{s_n \alpha} + G_\alpha\right). \tag{5.6}$$

In particular, if $s_n \alpha = \alpha$ then $H_n^{(s)} G_\alpha = -G_\alpha$ by Lemma 4.1(c).

Remark 5.6. Note that the right-hand sides of (5.5) and (5.6) are well defined by part (c) of Corollary 4.3. Note furthermore that in (5.5) the second formula does not reduce to the first formula if we would assume $s_j \alpha = \alpha$. Indeed, part (b) of Lemma 4.1 in case of the principal specialization $\tau_i = st^{n-i}$ ($i \in [1, n]$) implies that

$$\overline{\alpha}_j / \overline{\alpha}_{j+1} = t \qquad (j \in [1, n), \; \alpha \in \Lambda_n \text{ with } s_j \alpha = \alpha). \tag{5.7}$$

Hence the right-hand side of the first formula in (5.5) equals $(2t + 1)G_\alpha$ when $s_j \alpha = \alpha$ ($j \in [1, n)$).

Proof of Proposition 5.5. (a) The starting point is the formula (from (5.3))

$$\left(H_j^{(t)} G_\alpha\right)(\overline{\beta}) = t G_\alpha(\overline{\beta}) + \frac{\overline{\beta}_j - t\overline{\beta}_{j+1}}{\overline{\beta}_j - \overline{\beta}_{j+1}} \left(G_\alpha(\boldsymbol{s}_j \overline{\beta})\right) - G_\alpha(\overline{\beta})) \tag{5.8}$$

for $\beta \in \Lambda_n$. For $\beta \in \Lambda_{n,|\alpha|} \setminus \{\alpha\}$, we then have by Lemma 4.1 that

$$\left(H_j^{(t)} G_\alpha\right)(\overline{\beta}) = \begin{cases} 0 & \text{if } s_j\beta \neq \alpha, \\ \dfrac{\overline{\beta}_j - t\,\overline{\beta}_{j+1}}{\overline{\beta}_j - \overline{\beta}_{j+1}} & \text{if } s_j\beta = \alpha. \end{cases} \tag{5.9}$$

Now take $\beta = \alpha$ and first consider the case that $s_j\alpha = \alpha$. Then (5.9) implies that $H_j^{(t)} G_\alpha \in \mathcal{P}_{n,|\alpha|}$ vanishes at the interpolation points $\overline{\beta}$ with $\beta \in \Lambda_{n,|\alpha|} \setminus \{\alpha\}$, while we get from (5.8) and Lemma 4.1 that

$$\left(H_j^{(t)} G_\alpha\right)(\overline{\alpha}) = t G_\alpha(\overline{\alpha}) = t.$$

This proves the first formula of (5.5).

Now suppose that $s_j\alpha \neq \alpha$. Then (5.9) implies that $H_j^{(t)} G_\alpha \in \mathcal{P}_{n,|\alpha|}$ vanishes at the interpolation points $\overline{\beta}$ for $\beta \in \Lambda_{n,|\alpha|} \setminus \{\alpha, s_j\alpha\}$. Furthermore, by (5.8) and Lemma 4.1,

$$\left(H_j^{(t)} G_\alpha\right)(\overline{s_j\alpha}) = \frac{\overline{\alpha}_{j+1} - t\overline{\alpha}_j}{\overline{\alpha}_{j+1} - \overline{\alpha}_j}, \qquad \left(H_j^{(t)} G_\alpha\right)(\overline{\alpha}) = \frac{(t-1)\overline{\alpha}_j}{\overline{\alpha}_j - \overline{\alpha}_{j+1}}.$$

Hence the Laurent polynomial

$$F := H_j^{(t)} G_\alpha - \frac{\overline{\alpha}_{j+1} - t\overline{\alpha}_j}{\overline{\alpha}_{j+1} - \overline{\alpha}_j} G_{s_j\alpha} - \frac{(t-1)\overline{\alpha}_j}{\overline{\alpha}_j - \overline{\alpha}_{j+1}} G_\alpha$$

of degree at most $|\alpha|$ vanishes at all the interpolation points $\overline{\beta}$ with $\beta \in \Lambda_{n,|\alpha|}$, which forces $F \equiv 0$. We conclude that

$$H_j^{(t)} G_\alpha = \frac{\overline{\alpha}_{j+1} - t\overline{\alpha}_j}{\overline{\alpha}_{j+1} - \overline{\alpha}_j} G_{s_j\alpha} + \frac{(t-1)\overline{\alpha}_j}{\overline{\alpha}_j - \overline{\alpha}_{j+1}} G_\alpha.$$

Rewriting the right-hand side yields the second formula of (5.5).

(b) The proof proceeds similar to the proof of (a), now starting with the formula (from (5.3))

$$\left(H_n^{(s)} G_\alpha\right)(\overline{\beta}) = -G_\alpha(\overline{\beta}) + \frac{\left(1 - s\overline{\beta}_n^{-1}\right)\left(1 - s^{-1}\overline{\beta}_n^{-1}\right)}{1 - \overline{\beta}_n^{-2}} \left(G_\alpha(s_n\overline{\beta}) - G_\alpha(\overline{\beta})\right) \tag{5.10}$$

for $\beta \in \Lambda_n$. By Lemma 4.1, the formula reduces for $\beta \in \Lambda_{n,|\alpha|} \setminus \{\alpha\}$ to

$$\left(H_n^{(s)} G_\alpha\right)(\overline{\beta}) = \begin{cases} 0 & \text{if } s_n\beta \neq \alpha, \\ \dfrac{\left(1 - s\overline{\beta}_n^{-1}\right)\left(1 - s^{-1}\overline{\beta}_n^{-1}\right)}{1 - \overline{\beta}_n^{-2}} & \text{if } s_n\beta = \alpha. \end{cases} \tag{5.11}$$

If $s_n\alpha = \alpha$, then it follows from part (c) of Lemma 4.1 in case of principal specialization that $\overline{\alpha}_n = s$, hence $(H_n^{(s)} G_\alpha)(\overline{\alpha}) = -G_\alpha(\overline{\alpha}) = -1$ by (5.10). We

conclude that $H_n^{(s)}G_\alpha = -G_\alpha$ if $s_n\alpha = \alpha$, which agrees with (5.6). If $s_n\alpha \neq \alpha$, then observe that, by (5.10) and Lemma 4.1,

$$\left(H_n^{(s)}G_\alpha\right)(\overline{s_n\alpha}) = \frac{(1 - s\overline{\alpha}_n)(1 - s^{-1}\overline{\alpha}_n)}{1 - \overline{\alpha}_n^2}, \quad \left(H_n^{(s)}G_\alpha\right)(\overline{\alpha}) = \frac{2\overline{\alpha}_n^2 - (s + s^{-1})\overline{\alpha}_n}{1 - \overline{\alpha}_n^2}.$$

Continuing the proof as in part (a) readily leads to the formula (5.6). □

Recall from Lemma 5.3 the representation π_s of $\mathcal{H}(t, -1)$ on $\mathcal{P}_n$ and from (5.1) the Hecke symmetrizer C_+.

Lemma 5.7. *Let* $s, t \in \mathbb{C}^*$. *The map* $\pi_s(C_+)$ *restricts to a map* $\mathcal{P}_{n,d} \to \mathcal{P}_{n,d}^{W_n}$.

Proof. The fact that $f^+ := \pi_s(C_+)f$ is W_n-invariant for $f \in \mathcal{P}_n$ follows from the following standard argument in Cherednik–Macdonald theory: by (5.2), f^+ satisfies $(H_j^{(t)} - t)f^+ = 0$ ($j \in [1, n)$) and $(H_n^{(s)} + 1)f^+ = 0$. By the explicit forms (5.3), (5.4) of the Demazure–Lusztig operators, this is equivalent to $s_j f^+ = f^+$ for $j \in [1, n)$ and $s_n f^+ = f^+$. The map $\pi_s(C_+)$ preserves $\mathcal{P}_{n,d}$ by Lemma 5.4. □

Lemma 5.8. *Let* $s, t \in \mathbb{C}$ *such that* $st^\rho \in \mathcal{T}_n$. *Then*

$$\pi_s(C_+)G_\alpha = \mathrm{cst}_\alpha\, R_{\alpha^+} \tag{5.12}$$

for all $\alpha \in \Lambda_n$, *with* $\mathrm{cst}_\alpha := \left(\pi_s(C_+)G_\alpha\right)(\overline{\alpha^+})$.

Proof. To prove (5.12) it suffices, in view of the previous lemma, to show that

$$\left(\pi_s(C_+)G_\alpha\right)(\overline{\mu}) = 0$$

for all $\mu \in \Lambda_{n,|\alpha|}^+ \setminus \{\alpha^+\}$. But by Proposition 5.5 we have

$$\pi_s(C_+)G_\alpha = \sum_{\gamma \in W_n\alpha} d_\gamma^\alpha\, G_\gamma$$

for certain coefficients $d_\gamma^\alpha \in \mathbb{C}$. Each G_γ vanishes at $\overline{\mu}$ ($\mu \in \Lambda_{n,|\alpha|}^+ \setminus \{\alpha^+\}$) since

$$\Lambda_{n,|\alpha|}^+ \setminus \{\alpha^+\} \subseteq \Lambda_{n,|\gamma|}^+ \setminus \{\gamma\} \qquad \forall\, \gamma \in W_n\alpha. □$$

We end this section by computing $\mathrm{cst}_\lambda = (\pi_s(C_+)G_\lambda)(\overline{\lambda})$ for $\lambda \in \Lambda_n^+$. Fix $\lambda \in \Lambda_n^+$. Put $C_+^\lambda := \sum_{u \in W_n^\lambda} T_u$ and $C_{+,\lambda} := \sum_{v \in W_{n,\lambda}} T_v$. Since $\ell(uv) = \ell(u) + \ell(v)$ for $u \in W_n^\lambda$ and $v \in W_{n,\lambda}$, it follows that

$$C_+ = C_+^\lambda C_{+,\lambda} \tag{5.13}$$

in $\mathcal{H}_n(t, -1)$. With χ the trivial one-dimensional representation of $\mathcal{H}_n(t, -1)$ we see by Proposition 5.5 that

$$\pi_s(C_{+,\lambda})G_\lambda = \chi(C_{+,\lambda})G_\lambda. \tag{5.14}$$

What remains is to compute $(\pi_s(C_+^\lambda)G_\lambda)(\overline\lambda)$. Let $R = R_s \cup R_\ell$ be the decomposition of the root system R of type C_n in short and long roots (by convention, for $n = 1$ we write $R_s = \varnothing$ and $R_\ell = R$). We define two W_n-invariant functions $R \to \mathbb{C}$ by

$$
\kappa_\beta := \begin{cases} t & \text{if } \beta \in R_s, \\ s & \text{if } \beta \in R_\ell, \end{cases} \qquad\qquad
\upsilon_\beta := \begin{cases} 1 & \text{if } \beta \in R_s, \\ -s^{-1} & \text{if } \beta \in R_\ell. \end{cases}
$$

Define for $\beta \in R$ the rational function

$$
c_\beta(x) := \frac{(x^{\beta^\vee} - \kappa_\beta)(x^{\beta^\vee} + \upsilon_\beta)}{x^{2\beta^\vee} - 1},
$$

where $\beta^\vee := 2\beta/\|\beta\|^2$ denotes the co-root of β. Then $H_j^{(t)} = t + c_{\beta^j}(x)(s_j - 1)$ for $j \in [1, n)$ and $H_n^{(s)} = -1 + c_{\beta^n}(x)(s_n - 1)$. Furthermore, for $\alpha \in \Lambda_n$ with $s_j\alpha \neq \alpha$ we have

$$
c_{e_{j+1} - e_j}(\overline\alpha) = \frac{\overline\alpha_{j+1} - t\overline\alpha_j}{\overline\alpha_{j+1} - \overline\alpha_j}, \qquad\qquad j \in [1, n),
$$

$$
c_{-e_n}(\overline\alpha) = \frac{(1 - s\overline\alpha_n)(1 - s^{-1}\overline\alpha_n)}{1 - \overline\alpha_n^2},
$$

which are exactly the coefficients appearing in (5.5) and (5.6). Hence we have

$$
\pi_s(T_j)G_\alpha = -G_\alpha + c_{-\beta^j}(\overline\alpha)\big(G_{s_j\alpha} + G_\alpha\big) \quad (j \in [1, n] \; \alpha \in \Lambda_n \text{ with } s_j\alpha \neq \alpha) \qquad (5.15)
$$

by Proposition 5.5.

Recall the longest element $w_0 = -\mathrm{id}_{\mathbb{R}^n}$ in W_n. Let $w_0^\lambda \in W_n^\lambda$ be the minimal coset representative of the coset $w_0 W_{n,\lambda}$.

Theorem 5.9. *Let* $s, t \in \mathbb{C}$ *such that* $st^\rho \in \mathcal{T}_n$. *Then*

$$
\pi_s(C_+)G_\lambda = \mathrm{cst}_\lambda R_\lambda
$$

for $\lambda \in \Lambda_n^+$, *with*

$$
\mathrm{cst}_\lambda = \chi(C_{+,\lambda}) \prod_{\beta \in R^+ \cap (w_0^\lambda)^{-1} R^-} c_{-\beta}(\overline\lambda).
$$

Proof. Fix $\lambda \in \Lambda_n^+$ and write $\lambda^- := w_0(\lambda) = w_0^\lambda(\lambda)$ for the antidominant element in the orbit $W_n\lambda$. By (5.13), (5.14), Lemma 5.8 and Theorem 4.10 we have

$$
\chi(C_{+,\lambda})\pi_s(C_+^\lambda)G_\lambda = \pi_s(C_+)G_\lambda = \mathrm{cst}_\lambda R_\lambda = \mathrm{cst}_\lambda \sum_{\alpha \in W_n\lambda} G_\alpha,
$$

so it suffices to show that

$$
\big(\pi_s(C_+^\lambda)G_\lambda\big)(\overline{\lambda_-}) = \prod_{\beta \in R^+ \cap (w_0^\lambda)^{-1} R^-} c_{-\beta}(\overline\lambda).
$$

Using Proposition 5.5, the coefficient $e^\lambda_{\lambda_-}$ in the expansion

$$\pi_s(T_{w_0^\lambda})G_\lambda = \sum_{\gamma \in W_n\lambda} e^\lambda_\gamma G_\gamma$$

is the same as the coefficient of G_{λ_-} in the expansion of $\pi_s(C^\lambda_+)G_\lambda$ in nonsymmetric BC_n-type interpolation Macdonald polynomials. Hence it suffices to show that

$$e^\lambda_{\lambda_-} = \prod_{\beta \in R^+ \cap (w_0^\lambda)^{-1}R^-} c_{-\beta}(\bar\lambda).$$

Choose a reduced expression $w_0^\lambda = s_{i_1} \cdots s_{i_r}$. By the proof of Corollary 4.3(a), the elements $\lambda_k := s_{i_k} \cdots s_{i_r}\lambda \in W_n\lambda$ ($1 \le k \le r+1$, with $\lambda_{r+1} := \lambda$) are pairwise distinct. In particular, $\lambda_k = s_{i_k}\lambda_{k+1} \ne \lambda_{k+1}$ ($1 \le k \le r$) and $\lambda_1 = \lambda_-$. It follows from Corollary 4.3(c) that $c_{-\beta^{i_k}}(\bar\lambda_{k+1})$ is well defined for $k \in [1, r]$, and part (a) of Lemma 4.1 implies that

$$c_{-\beta^{i_k}}(\bar\lambda_{k+1}) = c_{-s_{i_r}\cdots s_{i_{k+1}}\beta^{i_k}}(\bar\lambda).$$

The coefficient $e^\lambda_{\lambda_-}$ can now be computed using Proposition 5.5 and (5.15),

$$\begin{aligned}
e^\lambda_{\lambda_-} &= c_{-\beta^{i_1}}(\bar\lambda_2)\cdots c_{-\beta^{i_{r-1}}}(\bar\lambda_r)c_{-\beta^{i_r}}(\bar\lambda_{r+1})\\
&= c_{-s_{i_r}\cdots s_{i_2}\beta^{i_1}}(\bar\lambda)\cdots c_{-s_{i_r}\beta^{i_{r-1}}}(\bar\lambda)c_{-\beta^{i_r}}(\bar\lambda)\\
&= \prod_{\beta \in R^+ \cap (w_0^\lambda)^{-1}R^-} c_{-\beta}(\bar\lambda),
\end{aligned}$$

where the third equality follows from the well-known description of the set of positive roots mapped by w_0^λ to negative roots in terms of the reduced expression $w_0^\lambda = s_{i_1}\cdots s_{i_r}$ (see Section 2). $\square$

A. Appendix

For $n = 2$, $|\alpha| \le 4$, and pseudo-random parameters q, s, t from $\{1/100, 2/100, \ldots \ldots, 99/100\}$, we computed the zeros of $\bar\beta \mapsto G_\alpha(\bar\beta; q, s, t)$ for $\beta = (\beta_1, \beta_2) \in \Lambda_2$ satisfying $\beta_1, \beta_2 \in \{-10, \ldots, 10\}$ using *Wolfram Mathematica* [41], and checked the results by computing the zeros once more for a second choice of pseudo-random parameters. We present in Figures 1–9 the outcome of these computations for $|\alpha| = 4$.

The meaning of the colours of the dots in the pictures is as follows:

- cross: $\alpha = (\alpha_1, \alpha_2)$.
- circle: $(0, 0)$.
- black dots: $\beta \ne (0, 0)$ for which $\bar\beta$ is an interpolation point; i.e., $\beta \in \Lambda_{2, |\alpha|}$ and $\beta \ne \alpha$.
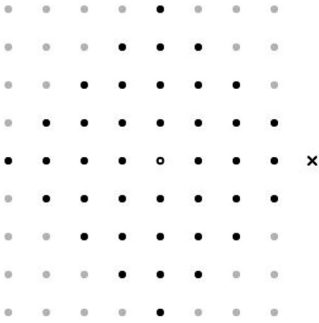- grey dots: $\beta \in \{-10, \ldots, 10\}^{\times 2}$ for which $\bar\beta$ is an extra vanishing point.
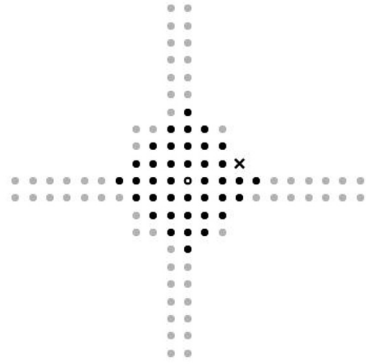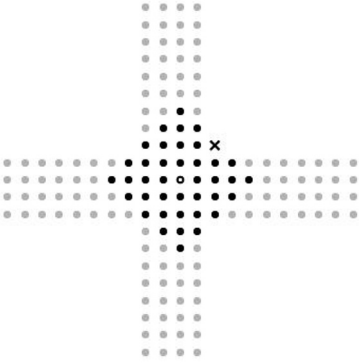
FIGURE 1. $\alpha = (4,0)$
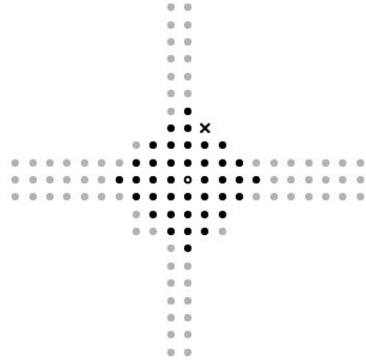
FIGURE 2. $\alpha = (3,1)$
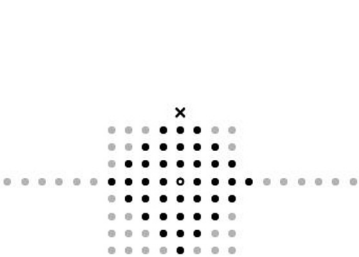
FIGURE 3. $\alpha = (2,2)$

FIGURE 4. $\alpha = (1,3)$

FIGURE 5. $\alpha = (0,4)$

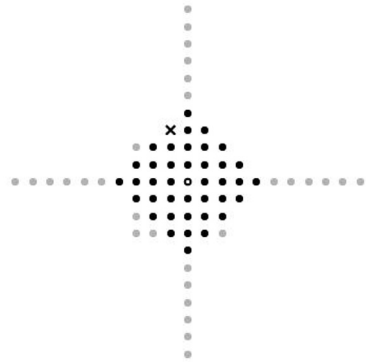FIGURE 6. $\alpha = (-1,3)$

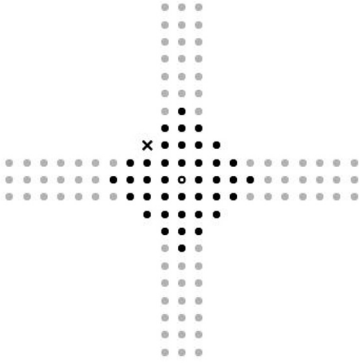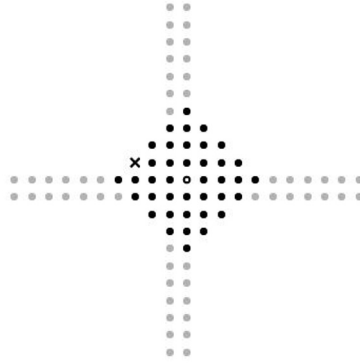FIGURE 7. $\alpha = (-2, 2)$

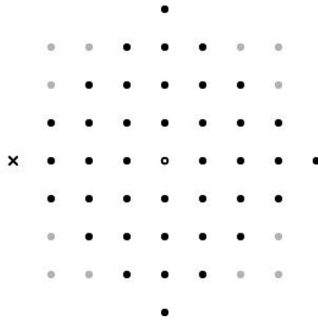

FIGURE 8. $\alpha = (-3, 1)$

FIGURE 9. $\alpha = (-4, 0)$

We have not included pictures for (α_1, α_2) with $\alpha_2 < 0$ because the following symmetry

$$G_{(\alpha_1, -\alpha_2)}(\overline{(\beta_1, \beta_2)}; q, s, t) = 0 \quad \text{iff} \quad G_{(\alpha_1, \alpha_2)}(\overline{(\beta_1, -\beta_2)}; q, s, t) = 0$$

holds true in all the computed cases.

All our pictures, including the ones not displayed here, are in agreement with the following conjecture about the zero set

$$Z_\alpha := \{\beta \in \Lambda_n \mid G_\alpha(\overline{\beta}; q, s, t) = 0\}$$

for $q, s, t \in \mathbb{C}^*$ with q not a root of unity and $st^\rho \in \mathcal{T}_n$.

Conjecture A.1. *Let* $\alpha \in \Lambda_n$, $\beta \in W_n\alpha$. *Let* $\overline{V_\beta}$ *consist of all* $\mu \in \Lambda_n$ *such that, for* $i = 1, \ldots, n$, $\mu_i \geq \beta_i$, $\mu_i \leq \beta_i$ *or* $\mu_i \in \mathbb{Z}$ *according to whether* $\beta_i > 0$, $\beta_i < 0$ *or* $\beta_i = 0$, *respectively. Let* V_β^0 *consist of all* $\mu \in \Lambda_n$ *such that, for* $i = 1, \ldots, n$, $\mu_i > \beta_i$, $\mu_i < \beta_i$ *or* $\mu_i \in \mathbb{Z}\backslash\{0\}$ *according to whether* $\beta_i > 0$, $\beta_i < 0$ *or* $\beta_i = 0$, *respectively. Then there are sets* $V_\beta(\alpha)$ *with* $V_\beta^0 \subseteq V_\beta(\alpha) \subseteq \overline{V_\beta}$ *(*$\beta \in W_n\alpha$*) and* $V_\alpha(\alpha) = \overline{V_\alpha}$ *such that* Z_α *is the complement in* Λ_n *of* $\bigcup_{\beta \in W_n\alpha} V_\beta(\alpha)$.

Remark A.2. If $\alpha \in \Lambda_n^+$ (i.e., α is a partition) then for all $\beta \in W_n\alpha$,

$$\overline{V_\beta} \cap \Lambda_n^+ \subseteq \{\mu \in \Lambda_n^+ \mid \mu \supseteq \alpha\}, \text{ hence } V_\beta(\alpha) \cap \Lambda_n^+ \subseteq V_\alpha(\alpha) \cap \Lambda_n^+ = \{\mu \in \Lambda_n^+ \mid \mu \supseteq \alpha\}.$$

Thus Conjecture A.1 implies that, for α a partition, $Z_\alpha \cap \Lambda_n^+$ consists of all partitions μ which do not include the partition α. Compare with the case of Okounkov's BC_n-type interpolation Macdonald polynomials; see (3.14).

Our pictures suggest possible characterizations of the sets $V_\beta(\alpha)$ in Conjecture A.1. These seem to be quite similar to the case of root system of type A (see [10, Thm. 4.5]) if $n = 2$, $\alpha_1 > 0$, $\alpha_2 \geq 0$, or possibly for general n, $\alpha_1, \ldots, \alpha_{n-1} > 0$, $\alpha_n \geq 0$. In contrast, in Figures 6 and 8, where $\alpha_1 < 0$, we see that $\lambda \in V_\alpha(\beta)$ is not always given by one set of inequalities for λ_1, λ_2.

In [10, §4] Knop introduced a new partial order on $\mathbb{Z}_{\geq 0}^n$ to describe the extra vanishing of the type A_{n-1} nonsymmetric interpolation Macdonald polynomials. Knop's order relation between two elements $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$ can be described in terms of inequalities of the entries of the corresponding partitions $\alpha^+, \beta^+ \in \Lambda_n^+$, with the strictness or non-strictness of the inequalities depending on the defining permutation $\pi = u_\alpha u_\beta^{-1}$ (here $u_\alpha, u_\beta \in S_n$ are the permutations of shortest lengths such that $u_\alpha(\alpha^+) = \alpha$ and $u_\beta(\beta^+) = \beta$). One may wonder whether the extra vanishing of the nonsymmetric BC_n-type interpolation Macdonald polynomials $G_\alpha(x; q, s, t)$ can be formulated in terms of a hyperoctahedral version of Knop's partial order, with the strictness or nonstrictness of the entries of the associated partitions $\alpha^+, \beta^+ \in \Lambda_n^+$ now described in terms of $w_\alpha w_\beta^{-1} \in W_n$ for $\alpha, \beta \in \Lambda_n$.

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