2022

On the Frequency Module of the Hull of a Primitive Substitution Tiling

April Lynne D. Say-Awen

Dirk Frettlöh

Ma. Louise Antonette N. De Las Peñas

FOUNDATIONS
ADVANCES

ISSN: 2053-2733
journals.iucr.org/a

On the frequency module of the hull of a primitive substitution tiling

April Lynne D. Say-awen, Dirk Frettlöh and Ma. Louise Antonette N. De Las Peñas

Acta Cryst. (2022). A**78**, 36–55

Acta Cryst A
FOUNDATIONS
ADVANCES

ISSN 2053-2733

On the frequency module of the hull of a primitive substitution tiling

April Lynne D. Say-awen,[a,b]* Dirk Frettlöh[c] and Ma. Louise Antonette N. De Las Peñas[a]

[a]Department of Mathematics, Ateneo de Manila University, Loyola Heights, Quezon City, 1108, Philippines, [b]Mathematics and Statistics Department, De La Salle University, Malate, Manila, 1004, Philippines, and [c]Faculty of Technology, Bielefeld University, Postfach 100131, 33501 Bielefeld, Germany. *Correspondence e-mail: asay-awen@ateneo.edu

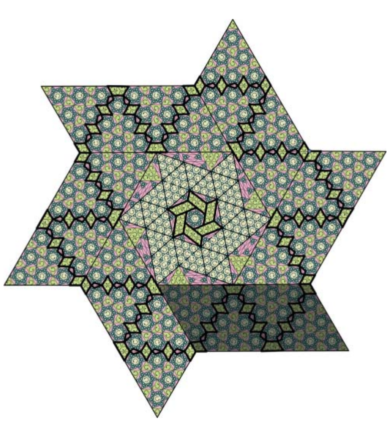
Received 24 June 2021
Accepted 25 November 2021

Edited by M. I. Aroyo, Universidad del País Vasco, Spain

Keywords: frequency module; primitive substitution tilings; aperiodic tilings; quasicrystals.

Supporting information: this article has supporting information at journals.iucr.org/a

Understanding the properties of tilings is of increasing relevance to the study of aperiodic tilings and tiling spaces. This work considers the statistical properties of the hull of a primitive substitution tiling, where the hull is the family of all substitution tilings with respect to the substitution. A method is presented on how to arrive at the frequency module of the hull of a primitive substitution tiling (the minimal $\mathbb{Z}$-module, where $\mathbb{Z}$ is the set of integers) containing the absolute frequency of each of its patches. The method involves deriving the tiling's edge types and vertex stars; in the process, a new substitution is introduced on a reconstructed set of prototiles.



1. Introduction

Tiling theory has evolved in recent years owing to the increasing relevance of aperiodic tilings and tiling spaces to various fields in mathematics such as algebra, geometry, topology, dynamical systems, computer science and statistics [see for instance Grünbaum & Shephard (1986), Solomyak (1997), Frettlöh (2008), Baake & Grimm (2013), Frettlöh & Richard (2014) and Moustafa (2010), and references therein]. Moreover, tiling theory gained more significance during the 1980s as theories on aperiodic tilings were used as a basis for understanding structural properties of quasicrystals.

One of the methods for constructing an aperiodic tiling is via a substitution. In a nutshell, a substitution is a rule on how to inflate a given prototile (or a set of several prototiles) and dissect it into equivalent copies of the prototiles (see Fig. 1). Iterating this rule fills larger and larger regions of space.

Many studies deal with substitutions that are primitive (defined in Section 2). This is primarily because the primitivity of a substitution σ paves the way to interesting properties of the hull $\mathbb{X}_\sigma$, the space which contains all tilings corresponding to σ. One of these properties is the frequency module of the hull $\mathbb{X}_\sigma$. By a frequency module, we mean the minimal $\mathbb{Z}$-module that contains the absolute frequency of every patch in a tiling $\mathcal{T} \in \mathbb{X}_\sigma$.

For a one-dimensional substitution tiling with substitution factor $\lambda > 1$ ($\lambda \in \mathbb{R}$), the frequency module is known to be the $\mathbb{Z}[1/\lambda]$-module generated by the frequencies of the tiles together with the frequencies of the vertex stars (Bellissard, 1992). This article demonstrates the two-dimensional analogue of this result, which states that the frequency module is the $\mathbb{Z}[1/\lambda^2]$-module generated by the frequencies of the tiles, edge types and vertex stars.

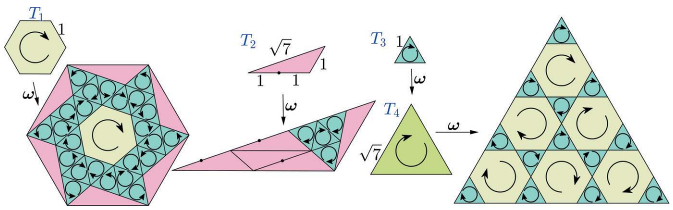


Figure 1
The substitution ω with substitution factor $7^{1/2}$. The prototiles T_1, T_3 and T_4 are regular polygons. The circular arrows indicate the orientations of symmetric tiles. The dot at the midpoint on the middle edge of T_2 is a pseudo-vertex. (A larger version of this figure is available in the supporting information.)

Embedding techniques and projection methods are known methods for deriving frequencies of vertex stars (Baake & Grimm, 2013). However, these methods fail for some primitive substitutions such as the pinwheel substitution. The frequencies of the vertex stars of a pinwheel tiling are derived via the introduction of a variation of the pinwheel substitution called the kite-domino substitution (Baake *et al.*, 2007; Baake & Grimm, 2013), where its construction is based on the observation that the triangles in the pinwheel tiling always match face to face along their hypotenuse. The kite-domino substitution gives rise to a tiling which is closely related to a pinwheel tiling.

In this article, we extend the ideas employed in extracting the frequencies of the vertex stars of a pinwheel tiling (Baake *et al.*, 2007; Baake & Grimm, 2013) to the computation of the frequencies of edge types and vertex stars of any primitive substitution tiling of $\mathbb{R}^2$, and to the extraction of the respective frequency module. More precisely, we give conditions that lead to the construction of variations of a given primitive substitution σ, which gives rise to the computation of the frequencies of the edge types and vertex stars of any tiling in the hull $\mathbb{X}_\sigma$. We illustrate the method on the primitive substitution ω (see Fig. 1).

An alternative method to derive the frequencies of the patches of a primitive substitution tiling is by means of K-theory, discussed in the article by Moustafa (2010) in which the frequency module of a pinwheel tiling was derived. The process involved determining the 108 collared prototiles corresponding to a pinwheel tiling together with their frequencies. This method, however, can be tedious for substitution tilings with more than one prototile compared to the method presented here, since the number of collared tiles is expected to be huge.

In related work, deriving patch frequencies requires a finite number of prototiles up to translations only (Solomyak, 1997; Baake & Grimm, 2013). In the present article, we consider a finite number of prototiles up to Euclidean motions.

It is interesting to note that the frequency module of the hull of a tiling $\mathcal{T}$ has implications on the diffraction spectrum of an aperiodic solid associated to $\mathcal{T}$. By the well known gap-labelling conjecture (Bellissard, 1992; van Elst, 1994; Bellissard *et al.*, 2001), the frequency module of the hull of a tiling (with finite local complexity) is equal to the gap-labelling group associated to the tiling, which is the set of possible gap labels for the spectrum of a Schrödinger operator describing the electronic motion in the aperiodic solid (Bellissard *et al.*, 2000; Kellendonk, 2021).

This article is organized as follows. Section 2 contains some basic definitions and facts about substitution tilings. The method for deriving the frequency module of the hull of a primitive substitution and its application to a primitive substitution is discussed in Section 3. Finally, Section 4 provides a summary and considers possible future work.

2. Preliminaries

A *tiling* $\mathcal{T}$ of $\mathbb{R}^2$ is a collection of sets called *tiles* which have pairwise disjoint interiors and whose union is the entire $\mathbb{R}^2$. A finite set $P \subset \mathcal{T}$ of tiles is called a *patch* of $\mathcal{T}$. A *vertex star* of $\mathcal{T}$ is a patch of all tiles intersecting some vertex in $\mathcal{T}$. An *edge type* of (an edge-to-edge tiling) $\mathcal{T}$ is a patch consisting of two tiles that intersect along an edge. Two tiles or patches of $\mathcal{T}$ are *equivalent* if the tiles or patches, respectively, can be made to coincide with each other by an isometry in $\mathbb{R}^2$. Let $\mathcal{F} := \{T_1, T_2, \ldots, T_m\}$ be a finite set of tiles and $\lambda > 1$ a real number. For each tile $T_i \in \mathcal{F}, 1 \le i \le m$, let $\lambda T_i = \cup_{j=1}^{n(i)} T_{i_j}$ such that each T_{i_j} is equivalent to a tile in $\mathcal{F}$ and the tiles $T_{i_1}, T_{i_2}, \ldots, T_{i_{n(i)}}$ have pairwise disjoint interiors. A *substitution* σ is the mapping $\sigma : \mathcal{F} \to \mathscr{S}$ defined by $\sigma(T_i) = \{T_{i_1}, T_{i_2}, \ldots, T_{i_{n(i)}}\}$, where $\mathscr{S}$ is the collection of all sets containing tiles equivalent to tiles in $\mathcal{F}$. The substitution σ has *prototiles* T_1, T_2, $\ldots$, T_m and *substitution factor* λ. This mapping naturally extends to any equivalent copy of a prototile as follows: an equivalent copy T of a prototile T_i of σ can be written as $T = RT_i + \mathbf{t}$, where R is an isometry fixing the origin and $\mathbf{t} \in \mathbb{R}^2$. The image of T under σ is defined as $\sigma(T) = \sigma(RT_i + \mathbf{t}) = R\sigma(T_i) + \lambda\mathbf{t}$. Moreover, given $\mathcal{A} \in \mathscr{S}$, σ extends to any set in $\mathscr{S}$ by $\sigma(\mathcal{A}) = \{\sigma(T) \mid T \in \mathcal{A}\}$. Hence, $\sigma : \mathscr{S} \to \mathscr{S}$ is a well defined mapping. One can iterate σ on a prototile T_i to obtain the *k-order supertile* $\sigma^k(T_i)$ of T_i. A tiling $\mathcal{T}$ is called a *substitution tiling* with respect to σ if for each patch $\mathcal{P} \subset \mathcal{T}$ there is a $k \in \mathbb{N}$ and $i \in \{1, 2, \ldots, m\}$ such that an equivalent copy of $\mathcal{P}$ is contained in $\sigma^k(T_i)$. The family of all substitution tilings with respect to the substitution σ is called the *hull* of σ, denoted by $\mathbb{X}_\sigma$.

An example of a substitution is the substitution ω (Frettlöh *et al.*, 2017; Say-awen *et al.*, 2018; Say-awen, 2016) shown in Fig. 1. It is defined using four prototiles T_1, T_2, T_3 and T_4 with substitution factor $7^{1/2}$. One of the tilings in the hull $\mathbb{X}_\omega$ of ω is obtained as follows: let R_α denote the rotation about the origin by α, where α is the smallest interior angle of T_2. Let T_1 be centred at the origin. Then $T_1 \subset R_\alpha\omega(T_1)$. Consequently, $(R_\alpha\omega)^{k-1}(T_1) \subset (R_\alpha\omega)^k(T_1)$ (see Fig. 2 for $k = 1, 2, 3$). Thus, $\big((R_\alpha\omega)^k(T_1)\big)_{k\in\mathbb{N}}$ is a nested sequence that converges to a tiling in $\mathbb{X}_\omega$. Other tilings in $\mathbb{X}_\omega$, particularly those tilings with rotational symmetry, are derived in Say-awen *et al.* (2018).

A substitution σ is *primitive* if there is a $k \in \mathbb{N}$ such that the k-order supertile of every prototile of σ contains an equivalent copy of every prototile. A substitution tiling with respect to a primitive substitution is called a *primitive substitution tiling*. A way to determine primitivity of a substitution is through its

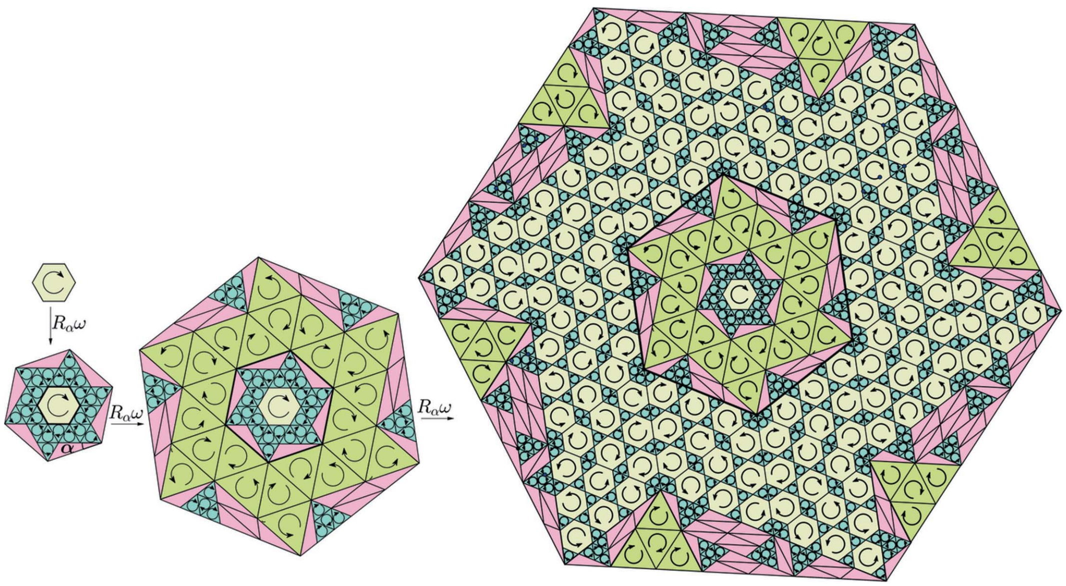


Figure 2
The first four terms of the nested sequence $\left((R_\alpha\omega)^k(T_1)\right)_{k\in\mathbb{N}}$, which converges to a tiling in $\mathbb{X}_\omega$.

corresponding substitution matrix. The *substitution matrix* of σ with prototiles $T_1, T_2, \ldots, T_m$ is the square matrix $M_\sigma = (a_{ij})_{1\le i,j\le m}$, where the entry a_{ij} is the number of tiles equivalent to T_i in the 1-order supertile $\sigma(T_j)$, $i, j \in \{1, 2, \ldots, m\}$. A substitution σ is *primitive* if its substitution matrix M_σ is primitive, that is, if there exists a $k \in \mathbb{N}$ such that ${M_\sigma}^k$ contains positive entries only (Baake & Grimm, 2013).

For example, the substitution ω given earlier is primitive. Its substitution matrix M_ω is given by

$$M_\omega = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 6 & 5 & 0 & 0 \\ 24 & 4 & 0 & 13 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

It can be checked from Fig. 1 that $\omega(T_1)$ contains 1 copy of T_1, 6 copies of T_2 and 24 copies of T_3, and no copy of T_4. This yields the first column entries of M_ω. The other entries can be obtained similarly. It can be checked that the matrix ${M_\omega}^3$ contains positive entries only.

It is well known that if σ is primitive, then the hull $\mathbb{X}_\mathcal{T}$ of any tiling $\mathcal{T}$ in $\mathbb{X}_\sigma$ is $\mathbb{X}_\sigma$ itself (Frettlöh, 2008; Baake & Grimm, 2013). The *hull* $\mathbb{X}_\mathcal{T}$ of $\mathcal{T}$ is the closure of the set $\{x\mathcal{T} \mid x \in G\}$ in the local topology, where G is the group of all rigid motions in $\mathbb{R}^2$. The local topology can be defined via a metric. In this metric, two tilings are ϵ-close if they agree on a large ball of radius $1/\epsilon$ around the origin after a small translation or direct Euclidean motion.

3. Frequency module

The *frequency module* of the hull $\mathbb{X}_\sigma$ of a primitive substitution σ is the minimal $\mathbb{Z}$-module that contains the absolute frequency of every patch in a tiling $\mathcal{T} \in \mathbb{X}_\sigma$. The *absolute frequency* or *frequency* of a patch $\mathcal{P} \subset \mathcal{T}$, denoted by freq($\mathcal{P}$), is defined as the average number of equivalent copies of $\mathcal{P}$ per unit area in $\mathcal{T}$. That is,

$$\operatorname{freq}(\mathcal{P}) = \lim_{r\to\infty} \frac{\#\{\mathcal{P}' \in \mathcal{T} \cap B_r \mid \mathcal{P}' \text{ is equivalent to } \mathcal{P}\}}{\operatorname{Area}(B_r)}, \tag{1}$$

where B_r is the ball of radius $r > 0$ around the origin and Area(B_r) is the area of B_r.

Because the hull $\mathbb{X}_\mathcal{T}$ of any tiling $\mathcal{T} \in \mathbb{X}_\sigma$ is $\mathbb{X}_\sigma$ itself as σ is primitive, we consider the frequency module of the hull of $\mathcal{T}$.

The frequencies of the prototiles in $\mathcal{T}$ can be computed via its substitution matrix M_σ, which satisfies the condition of the Perron–Frobenius theorem (Perron, 1907). The Perron–Frobenius theorem (Theorem 1, which we state below) asserts that M_σ has a positive real eigenvalue which is greater than the absolute value of any other eigenvalue of M_σ.

Theorem 1. Let M be a primitive non-negative square matrix. Then M has a real eigenvalue $\lambda > 0$ which is simple. Moreover, $\lambda > |\lambda'|$ for any eigenvalue $\lambda' \neq \lambda$ of M. This eigenvalue is called the Perron–Frobenius-eigenvalue or PF-eigenvalue. Moreover, the associated left and right eigenvectors of λ can be chosen to have positive entries. Such eigenvectors are called the left PF-eigenvector and right PF-eigenvector of M.

A significant consequence of the Perron–Frobenius theorem is stated in the following theorem (Pytheas Fogg, 2002; Baake & Grimm, 2013), which is essential in our calculations.

Theorem 2. Let σ be a primitive substitution in $\mathbb{R}^2$ with substitution factor λ and prototiles $T_1, T_2, \ldots, T_m$; let M_σ be the substitution matrix of σ. Then the PF-eigenvalue of M_σ is λ^2. The left PF-eigenvector $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ contains the areas of the different prototiles, up to scaling. The normalized right PF-eigenvector $\mathbf{v} = (v_1, v_2, \ldots, v_m)^T$ of M_σ contains the relative frequencies of the prototiles of any tiling $\mathcal{T} \in \mathbb{X}_\sigma$ in the following sense: The entry v_i is the relative frequency of T_i in $\mathcal{T}$.

The relative frequencies from the above theorem and equation (1) can be used to compute the absolute frequency of the prototile T_i, as follows:

$$\mathrm{freq}(T_i) = \frac{v_i}{\sum_{j=1}^{m} u_j v_j} = \frac{v_i}{\mathbf{u} \cdot \mathbf{v}}, \qquad (2)$$

where $\mathbf{u} \cdot \mathbf{v}$ is the dot product of the left PF-eigenvector $\mathbf{u}$ and normalized right PF-eigenvector $\mathbf{v}$ of M_σ. We note that the absolute frequency of the prototile T_i is the average number of equivalent copies of T_i per unit area in $\mathcal{T}$, whereas the relative frequency v_i of T_i is the ratio of the number of equivalent copies of T_i to the number of tiles in $\mathcal{T}$ (Baake *et al.*, 2007; Frettlöh, 2005).

An important result in determining the frequency module of $\mathcal{T} \in \mathbb{X}_\sigma$ is stated in the next theorem. For convenience in the discussion, we introduce the following terminologies. Given an edge type E and vertex star $\mathcal{V}$ of $\mathcal{T}$, the patches $\sigma^k(E)$ and $\sigma^k(\mathcal{V})$, $k \in \mathbb{N}$, will be referred to as the *k-order super-edge type* and *k-order super-vertex star* of E and $\mathcal{V}$, respectively. Moreover, the boundary between the two k-order supertiles in a k-order super-edge type will be called a *k-order super-edge* and the common vertex of the k-order supertiles in a k-order super-vertex star will be called a *k-order super-vertex*.

Theorem 3. The frequency module of the hull of a primitive substitution tiling with convex prototiles and substitution factor λ is the $\mathbb{Z}[1/\lambda^2]$-module generated by the absolute frequencies of its prototiles, edge types and vertex stars.

Proof. Given a primitive substitution σ on convex prototiles, consider a tiling $\mathcal{T} \in \mathbb{X}_\sigma$. To derive the absolute frequencies of patches in $\mathcal{T}$ we need to count copies of patches such that each copy is counted exactly once. For this we use the notion of s-order supertiles, s-order super-vertex stars and s-order super-edges.

In order to count the equivalent copies of a patch $\mathcal{P} \subset \mathcal{T}$ we choose s such that the following are satisfied for an equivalent copy of $\mathcal{P}$: (a) it contains at most one s-order super-vertex and (b) it is entirely contained in at least one s-order super-vertex star.

The natural number s is obtained in the following manner. Let $\mathcal{P}'$ be an equivalent copy of $\mathcal{P}$ in $\mathcal{T}$. Since $\mathcal{T}$ is a substitution tiling, $\mathcal{P}'$ is contained by a $q_{\mathcal{P}'}$-order supertile $L_{\mathcal{P}'}$ for some $q_{\mathcal{P}'} \in \mathbb{N}$. $\mathcal{P}'$ is entirely contained by at least one $s_{\mathcal{P}'}$-order super-vertex star, which either occurs in the interior of $L_{\mathcal{P}'}$ ($s_{\mathcal{P}'} < q_{\mathcal{P}'}$) or contains $L_{\mathcal{P}'}$ ($s_{\mathcal{P}'} = q_{\mathcal{P}'}$). If $\mathcal{P}'$ is entirely contained by an $s_{\mathcal{P}'}$-order super-vertex star, then it should be contained by a super-vertex star with a higher order than $s_{\mathcal{P}'}$. So, we take $s \geq \max(\{s_{\mathcal{P}'} \mid \mathcal{P}' \text{ is an equivalent copy of } \mathcal{P}\})$ so then every equivalent copy $\mathcal{P}'$ of $\mathcal{P}$ is contained by at least one s-order super-vertex star. At the same time s is taken so that $\mathcal{P}'$ contains at most one s-order super-vertex.

Now having chosen s, we count equivalent copies of $\mathcal{P}$ as follows. First, count each equivalent copy of $\mathcal{P}$ that is entirely contained in some s-order supertile. The set of all s-order supertiles is a partition of $\mathbb{R}^2$ (since the set of all tiles in $\mathcal{T}$ is a partition of $\mathbb{R}^2$), thus each copy is counted at most once. Next, we count each equivalent copy of $\mathcal{P}$ that contains an s-order super-vertex. Each copy is counted at most once since it contains exactly one s-order super-vertex by condition (a). The remaining equivalent copies of $\mathcal{P}$ cross some s-order super-edges, but contain no s-order super-vertex. Suppose $\mathcal{P}'$ is one of these copies and that it crosses more than one s-order super-edge. Since the prototiles are convex, the intersection of the boundaries containing these s-order super-edges is a single s-order super-vertex. Because $\mathcal{P}'$ is entirely contained in one s-order super-vertex star S [by condition (b)] and all the tiles are convex (since prototiles are convex), these s-order super-edges must intersect in the single s-order super-vertex defining S. We account for $\mathcal{P}'$ by this s-order super-vertex star S. Lastly, suppose $\mathcal{P}'$ crosses a single s-order super-edge. We count $\mathcal{P}'$ by counting the s-order super-edge type that contains $\mathcal{P}'$.

Now we consider the absolute frequencies of s-order supertiles, s-order super-vertex stars containing the patches considered earlier and for the last case the s-order super-edge types corresponding to the s-order super-edges.

If T_i is a prototile of $\mathcal{T}$, then the absolute frequency of $\sigma^s(T_i)$ [using Theorem 2 and equations (1) and (2)] is given by

$$\mathrm{freq}(\sigma^s(T_i)) = \frac{\mathrm{freq}(T_i)}{\lambda^{2s}}. \qquad (3)$$

Using equation (3) and its analogue computations for absolute frequencies for s-order super-vertex stars and s-order super-edge types, the absolute frequency of each patch in $\mathcal{T}$ is contained in the $\mathbb{Z}[1/\lambda^2]$-module generated by the absolute frequencies of the prototiles together with the absolute frequencies of the vertex stars and the absolute frequencies of the edge types. □

In view of Theorem 3, the derivation of the frequency module requires finding the edge types and vertex stars of $\mathcal{T} \in \mathbb{X}_\sigma$ and their frequencies, and the frequencies of the prototiles. We begin by discussing how to determine the edge types and vertex stars of $\mathcal{T}$.

Note that the number of non-equivalent edge types and vertex stars is finite.

3.1. Determining the complete list of edge types

Let E be an edge type of $\mathcal{T}$. Since $\mathcal{T}$ is a substitution tiling with respect to σ, there exist a natural number k and a prototile T_i of σ such that an equivalent copy E' of E is contained in $\sigma^k(T_i)$. Now, we note that $\sigma^k(T_i)$ can be partitioned into 1-order supertiles. Hence, either E' is contained by a 1-order supertile or E' crosses a 1-order super-edge whose corresponding edge type is contained in $\sigma^{k-1}(T_i)$.

Based on the above remarks, the complete list of non-equivalent edge types of $\mathcal{T}$ can be found by performing the following steps.

(1) List the edge types that are entirely contained by 1-order supertiles.

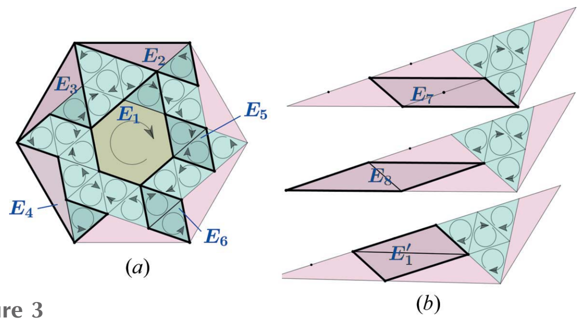


Figure 3
Edge types (*a*) $E_1, E_2, \ldots, E_6$ in $\omega(T_1)$; and (*b*) E_7, E_8 and E'_1 in $\omega(T_2)$.

(2) Apply σ on every edge type E obtained in step (1) and list the new edge types crossing the 1-order super-edge(s) corresponding to $\sigma(E)$.

(3) Repeat step (2) to every new edge type until no more new edge types are found.

To illustrate the given steps, consider the substitution ω (Fig. 1) introduced earlier. Let $\mathcal{T}_\omega \in \mathbb{X}_\omega$. We denote an edge type by E_i ($i \in \mathbb{N}$) if it corresponds to an edge of length 1 or two edges of length 1 (intersecting at a pseudo-vertex) and by E'_j ($j \in \mathbb{N}$) if it corresponds to an edge of length $7^{1/2}$.

The first step is to list the edge types which are entirely contained by 1-order supertiles as shown in Fig. 3. These are $E_1, E_2, \ldots, E_8$ and $E_1{}'$.

We proceed by applying ω to every edge type E in the set $\{E_1, E_2, \ldots, E_8, E'_1\}$ and listing, if it exists, every new edge type that crosses the 1-order super-edge/s of $\omega(E)$. Let us begin with E_1. Applying ω on E_1 yields the edge type E'_2, which crosses the 1-order super-edge of $\omega(E_1)$ [Fig. 4(*a*)]. Applying ω on the edge types E_2, E_3 and E_4 yields copies of E'_2, which we previously found [Fig. 4(*b*)]. Applying ω on E_5 yields the edge type E'_3 [Fig. 4(*c*)]. Applying ω on E_6 yields the edge type E'_4 [Fig. 4(*d*)]. Applying ω on E_7 and E_8 yields copies of E'_1 [Fig. 4(*e*)]. Applying ω on E'_1 yields the edge types E_9 and E_{10} [Fig. 4(*f*)].

We continue by applying ω to each of the edge types E'_2, E'_3, E'_4, E_9, E_{10} found in the previous step. The edge types E'_2, E'_3 and E'_4 yield E_{11}, E_{12} and E_{13}, respectively; while E_9 and E_{10} yield copies of E'_1 (Fig. 5). So, in this step, we have obtained the new edge types E_{11}, E_{12}, E_{13}.

We repeat the process with E_{11}, E_{12} and E_{13}. The edge types E_{11} and E_{12} yield copies of E'_1, and the edge type E_{13} gives rise to the new edge type E'_5 (Fig. 6).

The edge type E'_5 gives rise to the new edge types E_{14} and E_{15} (Fig. 7).

The edge types E_{14} and E_{15} yield copies of E'_5 (Fig. 8). In this step, no new edge types are found, which means that we have arrived at the complete list of non-equivalent edge types.

3.2. Determining the complete list of vertex stars

Let $\mathcal{V}$ be a vertex star of $\mathcal{T}$. Just as in the case of an edge type, there exist a natural number k and a prototile T_i of σ such that an equivalent copy $\mathcal{V}'$ of $\mathcal{V}$ is contained by $\sigma^k(T_i)$. Since $\sigma^k(T_i)$ can be partitioned into 1-order supertiles, $\mathcal{V}'$ satisfies one of the following: (*a*) $\mathcal{V}'$ is contained by a 1-order

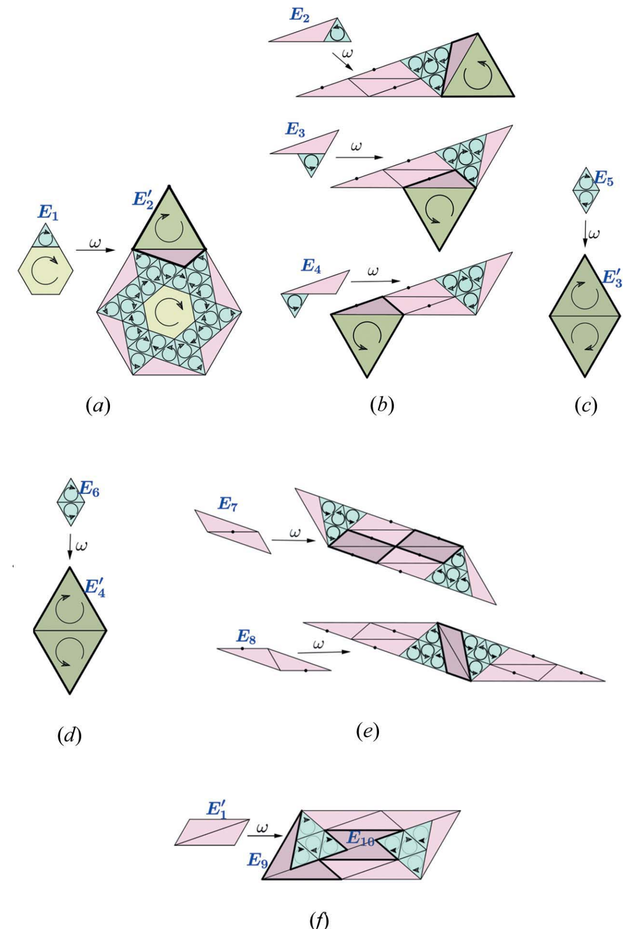


Figure 4
(*a*) $E'_2 \subset \omega(E_1)$; (*b*) copies of E'_2 in $\omega(E_2)$, $\omega(E_3)$, $\omega(E_4)$; (*c*) $E'_3 \subset \omega(E_5)$; (*d*) $E'_4 \subset \omega(E_6)$; (*e*) copies of E'_1 in $\omega(E_7)$ and $\omega(E_8)$; and (*f*) E_9 and E_{10} in $\omega(E'_1)$. (A larger version of this figure is available in the supporting information.)

supertile in $\sigma^k(T_i)$; (*b*) $\mathcal{V}'$ crosses a single 1-order super-edge in $\sigma^k(T_i)$ whose corresponding edge type is contained in $\sigma^{k-1}(T_i)$; or (*c*) the defining vertex of $\mathcal{V}'$ is a 1-order super-vertex whose corresponding vertex star is contained by $\sigma^{k-1}(T_i)$.

The location of the vertex stars allows us to determine the complete list of non-equivalent vertex stars of $\mathcal{T}$ as follows.

(1) List the vertex stars that are entirely contained by 1-order supertiles and vertex stars that cross single 1-order super-edges.

(2) Let $\mathcal{V}_{1,1}, \mathcal{V}_{2,1}, \ldots, \mathcal{V}_{q,1}$ be the vertex stars obtained in step (1). For each $i \in \{1, 2, \ldots, q\}$, apply σ on $\mathcal{V}_{i,1}$ and list the vertex star $\mathcal{V}_{i,2}$ whose defining vertex is the defining 1-order super-vertex of $\sigma(\mathcal{V}_{i,1})$. We repeat this process to $\mathcal{V}_{i,2}$ to obtain the vertex star $\mathcal{V}_{i,3}$ whose defining vertex is the defining 1-order super-vertex of $\sigma(\mathcal{V}_{i,2})$. We iterate this process until we obtain the vertex star $\mathcal{V}_{i,n(i)}$, where the vertex star whose defining vertex is the defining 1-order super-vertex of $\sigma(\mathcal{V}_{i,n(i)})$ is equivalent to a vertex star that has already been found.

To illustrate the method, we find the vertex stars of $\mathcal{T}_\omega \in \mathbb{X}_\omega$. First, we list the vertex stars $\mathcal{V}_{1,1}, \mathcal{V}_{2,1}, \ldots, \mathcal{V}_{6,1}$ that are entirely contained by 1-order supertiles and vertex stars $\mathcal{V}_{7,1}, \mathcal{V}_{8,1}, \ldots, \mathcal{V}_{21,1}$ that cross single 1-order super-edges.

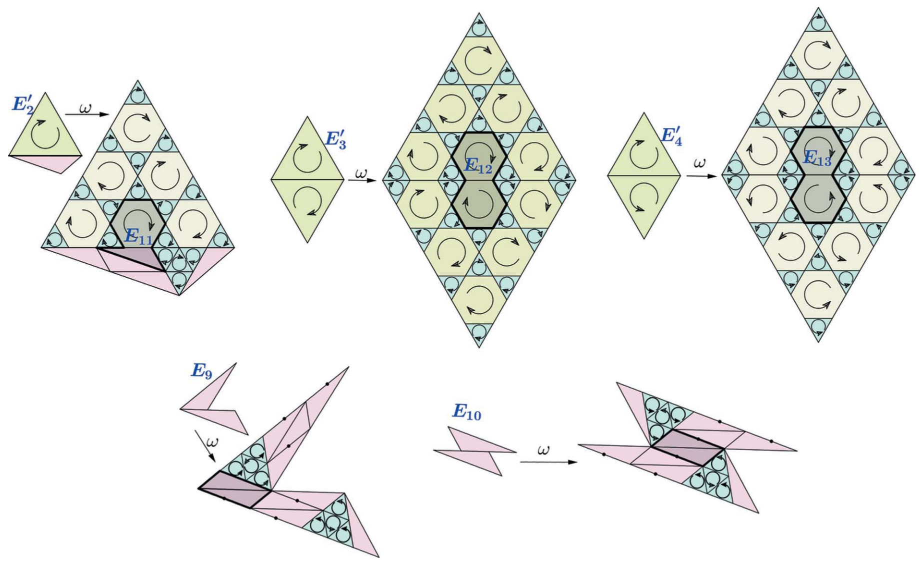


Figure 5
$E_{11} \subset \omega(E_2')$, $E_{12} \subset \omega(E_3')$, $E_{13} \subset \omega(E_4')$, and copies of E_1' in $\omega(E_9)$ and $\omega(E_{10})$.

These vertex stars are shown in Table 3. For example, Fig. 9 shows that vertex star $\mathcal{V}_{1,1}$ is contained by $\omega(T_1)$ and vertex star $\mathcal{V}_{7,1}$ crosses the 1-order super-edge of $\omega(E_2')$.

Next, we apply ω on $\mathcal{V}_{1,i}$, $i \in \{1, 2, \ldots, 21\}$. Consider the vertex star $\mathcal{V}_{1,1}$. Applying ω on $\mathcal{V}_{1,1}$ yields the vertex star $\mathcal{V}_{1,2}$ whose defining vertex is the defining 1-order super-vertex of $\omega(\mathcal{V}_{1,1})$ [Fig. 10(*a*)]. Applying ω on $\mathcal{V}_{1,2}$ yields the vertex star $\mathcal{V}_{1,3}$ whose defining vertex is the defining 1-order super-vertex of $\omega(\mathcal{V}_{1,2})$ [Fig. 10(*b*)]. Applying ω on $\mathcal{V}_{1,3}$ yields an equivalent copy of $\mathcal{V}_{1,2}$ [Fig. 10(*c*)]. The other vertex stars (see Table 3) can be obtained by iterating the process to $\mathcal{V}_{2,1}, \mathcal{V}_{3,1}, \ldots, \mathcal{V}_{21,1}$.

The next step in deriving the frequency module of $\mathbb{X}_\sigma$ is to compute the frequencies of the prototiles, edge types and vertex stars.

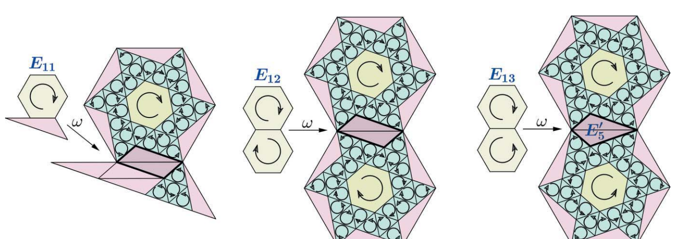


Figure 6
Copies of E_1' in $\omega(E_{11})$ and $\omega(E_{12})$, and $E_5' \subset \omega(E_{13})$. (A larger version of this figure is available in the supporting information.)

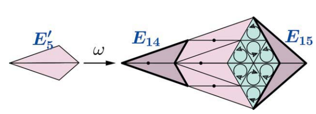


Figure 7
E_{14} and E_{15} in $\omega(E_5')$.

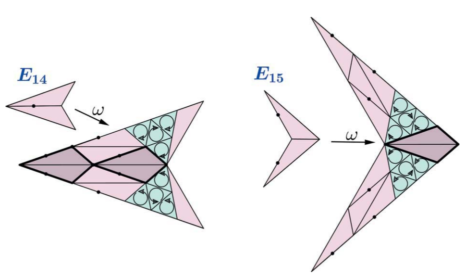


Figure 8
Copies of E_5' in $\omega(E_{14})$ and $\omega(E_{15})$.

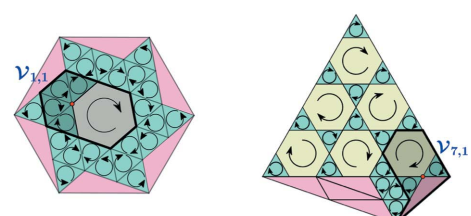


Figure 9
$\omega(T_1)$ contains $\mathcal{V}_{1,1}$ and $\omega(E_2')$ contains $\mathcal{V}_{7,1}$.

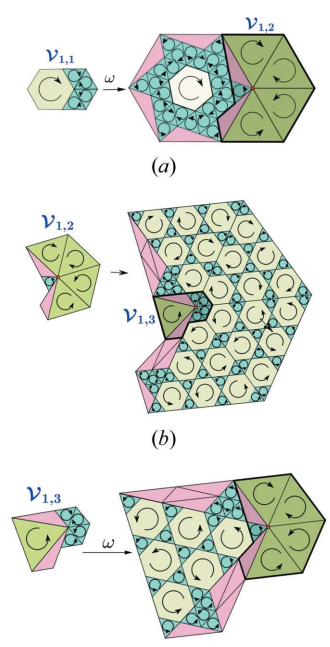


Figure 10
(*a*) $\mathcal{V}_{1,1}$ yields the vertex star $\mathcal{V}_{1,2}$; (*b*) $\mathcal{V}_{1,2}$ yields the vertex star $\mathcal{V}_{1,3}$; and (*c*) $\mathcal{V}_{1,3}$ yields an equivalent copy of $\mathcal{V}_{1,2}$.

3.3. Determining the absolute frequencies of the prototiles

To extract the absolute frequencies of the prototiles of $\mathcal{T} \in \mathbb{X}_\sigma$, we apply Theorem 2 and equation (2). We discuss the process by computing the absolute frequencies of the prototiles of $\mathcal{T}_\omega \in \mathbb{X}_\omega$.

It can be computed that the PF-eigenvalue of M_ω is 7, and the corresponding left PF-eigenvector and normalized right PF-eigenvector are respectively given by $\mathbf{u} = (6, 2, 1, 7)$ and $\mathbf{v} = ((1/12), (1/4), (7/12), (1/12))^T$. By Theorem 2, the left PF-eigenvector contains the areas of the tiles up to scaling, and the normalized right PF-eigenvector contains the relative frequencies of the prototiles. For example, we can say that one-twelfth of the tiles in $\mathcal{T}_\omega$ are equivalent to T_1 and one-quarter of the tiles are equivalent to T_2. Now, using equation (2), the average number of tiles in $\mathcal{T}_\omega$ equivalent to T_1 per unit area is given by $\mathrm{freq}(T_1) = (1/12)/(\mathbf{u} \cdot \mathbf{v}) = 1/26$. Similarly, it can be shown that $\mathrm{freq}(T_2) = 3/26$, $\mathrm{freq}(T_3) = 7/26$ and $\mathrm{freq}(T_4) = 1/26$.

We now focus on the frequencies of the edge types and vertex stars in the following.

3.4. Determining the absolute frequencies of the edge types

The equivalent copies of an edge type E are accounted for by counting 1-order supertiles containing equivalent copies of E and 1-order super-edge types corresponding to 1-order super-edges crossed by equivalent copies of E. Then we compute the absolute frequencies of these 1-order supertiles and 1-order super-edge types containing equivalent copies of E.

Unlike in the case of a prototile, the frequency of an edge type cannot always be extracted directly from σ. The frequency of an edge type can be derived from σ if every copy of the edge type is contained by a 1-order supertile (or a k-order supertile for a fixed natural number k). However, in any substitution, this is not the case for all edge types because super-edges are necessarily crossed by edge types.

A technique or method we introduce is to put together edge types of $\mathcal{T}$ in a set $\mathcal{E}$ satisfying a certain condition. For each $\mathcal{E}$, we derive a new substitution tiling $\mathcal{T}'$ wherein patches of $\mathcal{T}$ containing copies of edge types in $\mathcal{E}$ are treated as tiles of $\mathcal{T}'$. Given that the substitution corresponding to $\mathcal{T}'$ is primitive, we can use Theorem 2 to find the frequencies of these patches, which contribute to the frequencies of the edge types in $\mathcal{E}$. If necessary, we repeat the process of deriving a new substitution tiling until all copies of every edge type in $\mathcal{E}$ are accounted for.

After obtaining the frequencies of edge types contained by several sets $\mathcal{E}$, it is possible that the frequencies of the remaining edge types can be derived from these known frequencies of edge types together with the frequencies of the prototiles. In this case, it is no longer necessary to derive a new substitution tiling.

We now discuss the method. The first step is to put together edge types of $\mathcal{T}$ in a set $\mathcal{E}$ satisfying the following.

(*) For each edge type $E^* \in \mathcal{E}$, there exists a natural number l (we choose the smallest l) such that every edge type crossing the l-order super-edge/s of $\sigma^l(E^*)$ is equivalent to an edge type in $\mathcal{E}$.

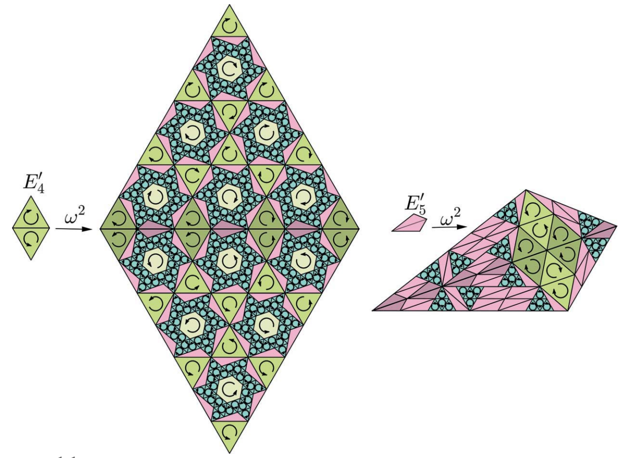


Figure 11
Every edge type along the 2-order super-edge corresponding to $\omega^2(E_4')$ or $\omega^2(E_5')$ is either equivalent to E_4' or E_5'.

We note that $\mathcal{E}$ always exists because the number of non-equivalent edge types is finite.

Defining $\mathcal{E}$ helps in counting equivalent copies of E^* in the tiling $\mathcal{T}$. A copy of $E^* \in \mathcal{E}$ occurs in the interior of an l-order supertile or crosses an l-order super-edge. For the latter case, the edge type crosses an l-order super-edge corresponding to an edge type in $\mathcal{E}$ by condition (*). Now, Theorem 2 together with equations (2) and (3) can only account for the copies of E^* that occur in the interior of l-order supertiles.

To account for the copies of E^* that cross l-order super-edges, we reconstruct or 'rewrite' $\mathcal{T}$ as a tiling $\mathcal{T}'$ using σ^l as a substitution and treating some patch $\mathcal{P}$ containing an equivalent copy of an edge type in $\mathcal{E}$ as a prototile. So $\sigma^l(\mathcal{P})$ does not only contain equivalent copies of some edge types in $\mathcal{E}$ that occur in the l-order supertiles in $\sigma^l(\mathcal{P})$ but even those edge types along an l-order super-edge in the interior of $\sigma^l(\mathcal{P})$.

A reconstruction of $\mathcal{T}$ using σ^l is arrived at by finding a partition $\mathcal{H}$ of $\mathcal{T}$ consisting of patches which act as tiles of $\mathcal{T}'$. The partition $\mathcal{H}$ of $\mathcal{T}$ should satisfy the following conditions:

(A) the number of non-equivalent patches of $\mathcal{H}$ is finite;

(B) for each patch $\mathcal{P} \in \mathcal{H}$, $\sigma^l(\mathcal{P})$ is a union of patches belonging to $\mathcal{H}$;

(C) if two patches $\mathcal{P}$ and $\mathcal{P}'$ in $\mathcal{H}$ are equivalent, then $\sigma^l(\mathcal{P})$ and $\sigma^l(\mathcal{P}')$ are partitioned in the same way; and

(D) an equivalent copy of every edge type in $\mathcal{E}$ is contained by a patch of $\mathcal{H}$.

We note that since $\mathcal{H}$ is a partition of $\mathcal{T}$, the collection of patches $\mathcal{S} = \{\sigma^l(\mathcal{P}) | \mathcal{P} \in \mathcal{H}\}$ is also a partition of $\mathcal{T}$.

Let us illustrate the ideas above using the substitution ω. The smallest set of edge types of $\mathcal{T}_\omega$ satisfying (*) is the set $\mathcal{E}_1 = \{E_4', E_5'\}$. Fig. 11 shows that every edge type lying on the 2-order super-edge of $\omega^2(E_i')$, $i = 4, 5$, is equivalent to either E_4' or E_5'. Hence, the set $\mathcal{E}_1 = \{E_4', E_5'\}$ satisfies condition (*) with $l = 2$ (which is also the smallest natural number for which $\mathcal{E}_1$ satisfies the condition).

By analysing large patches of $\mathcal{T}_\omega$, we have found the partition $\mathcal{H}_1$ of $\mathcal{T}_\omega$ satisfying the conditions (A)–(D). A portion of $\mathcal{H}_1$ is shown in Fig. 12.

The number of non-equivalent patches in $\mathcal{H}_1$ is finite. A complete set of non-equivalent patches in $\mathcal{H}_1$ consists of $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_8$ as shown in Fig. 13. For each $\mathcal{P} \in \mathcal{H}_1$, $\omega^2(\mathcal{P})$ is a union of patches belonging to $\mathcal{H}_1$. For example, the union of patches in the shaded portion of $\mathcal{H}_1$ (Fig. 12) is an equivalent copy of $\omega^2(\mathcal{P}_7)$. Moreover, it is evident in the figure that equivalent copies of $\omega^2(\mathcal{P}_7)$ are partitioned in the same way. Lastly, an equivalent copy of $E_i' \in \mathcal{E}_1$, $i = 4, 5$, is contained by a patch of $\mathcal{H}_1$.

After finding the partition $\mathcal{H}$, we define the tiling $\mathcal{T}'$ consisting of tiles obtained by deleting interior edge/s of each patch in $\mathcal{H}$. Owing to conditions (A)–(C), $\mathcal{T}'$ is a substitution tiling with prototiles $T_1', T_2', \ldots, T_n'$, where T_i' is the tile arising from deleting interior edge/s of $\mathcal{P}_i \in \mathcal{A} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n\}$, where $\mathcal{A}$ is a complete set of non-equivalent patches in $\mathcal{H}$. The substitution σ' corresponding to $\mathcal{T}'$ behaves the same way as σ^l.

Now, because $\mathcal{T}$ can be obtained by replacing every tile of $\mathcal{T}'$ by a patch in $\mathcal{H}_1$, and vice versa, $\text{freq}(\mathcal{P}_i) = \text{freq}(T_i')$, $i \in \{1, 2, \ldots, n\}$. Moreover, owing to condition (D), $\sum_{i=1}^n k_i \text{freq}(\mathcal{P}_i) = \sum_{i=1}^n k_i \text{freq}(T_i')$ contributes to $\text{freq}(E^*)$, where k_i is the number of equivalent copies of E^* in $\mathcal{P}_i$, $i \in \{1, 2, \ldots, n\}$ ($k_i = 0$ if $\mathcal{P}_i$ does not contain an equivalent copy of E^*).

Provided that σ' is primitive, we can use Theorem 2 and equation (2) to compute the frequencies of $T_1', T_2', \ldots, T_n'$.

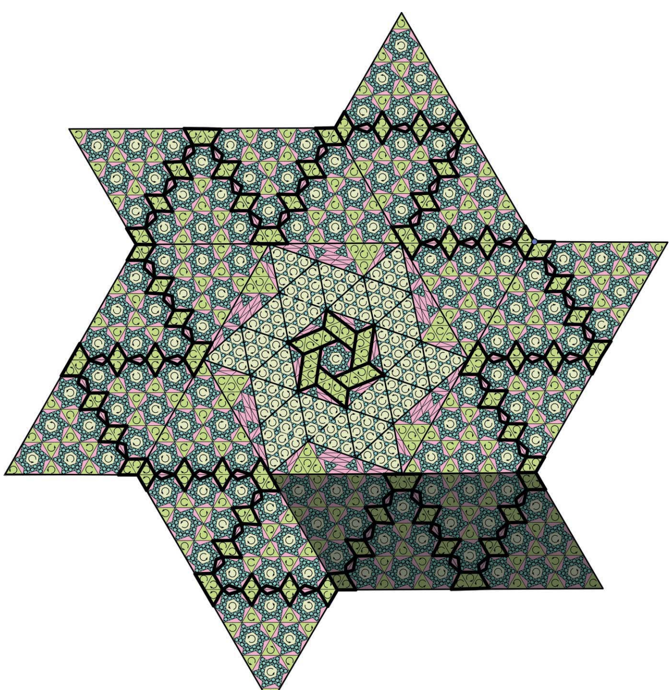


Figure 12
A portion of the partition $\mathcal{H}_1$ of $\mathcal{T}_\omega$. A patch in $\mathcal{H}_1$ consisting of more than one tile is enclosed by thick black edges. The partition of a patch equivalent to $\omega^2(\mathcal{P}_7)$ is shaded. (A larger version of this figure is available in the supporting information.)

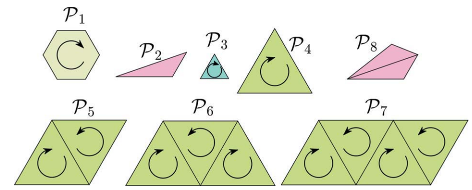


Figure 13
The complete list of non-equivalent patches in the partition $\mathcal{H}_1$ of $\mathcal{T}_\omega$.

If every equivalent copy of $E^* \in \mathcal{E}$ in $\mathcal{T}$ is contained by a patch in the partition $\mathcal{H}$, then $\text{freq}(E^*) = \sum_{i=1}^n k_i \text{freq}(T_i')$. Otherwise, $\sum_{i=1}^n k_i \text{freq}(T_i')$ contributes only partially to $\text{freq}(E^*)$. In this case, we look for another partition $\mathcal{H}'$ satisfying conditions (A)–(D), where equivalent copies of $E^* \in \mathcal{E}$ that are not taken into account in the previous partition $\mathcal{H}$ are contained by some of the patches of the new partition $\mathcal{H}'$. To avoid double counting, a copy of E^* that is contained by a patch in $\mathcal{H}$ should not be contained by any patch in $\mathcal{H}'$. Then we derive its corresponding substitution. We repeat the search for partitions and the derivation of substitutions associated to those partitions until every equivalent copy of $E^* \in \mathcal{E}$ in $\mathcal{T}$ is accounted for. To obtain the frequency of $E^* \in \mathcal{E}$, we add

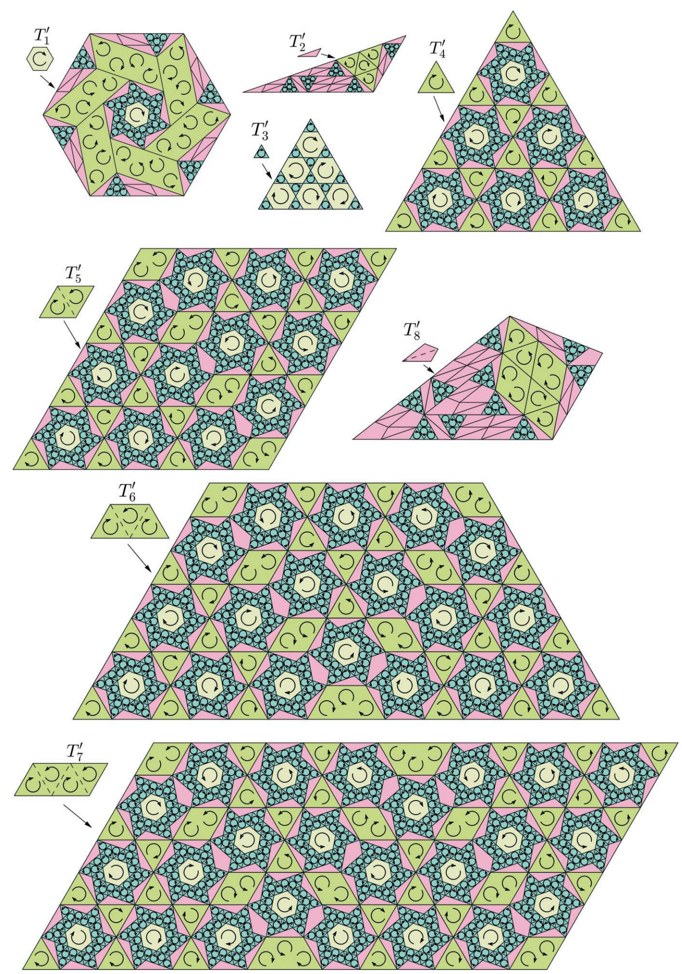
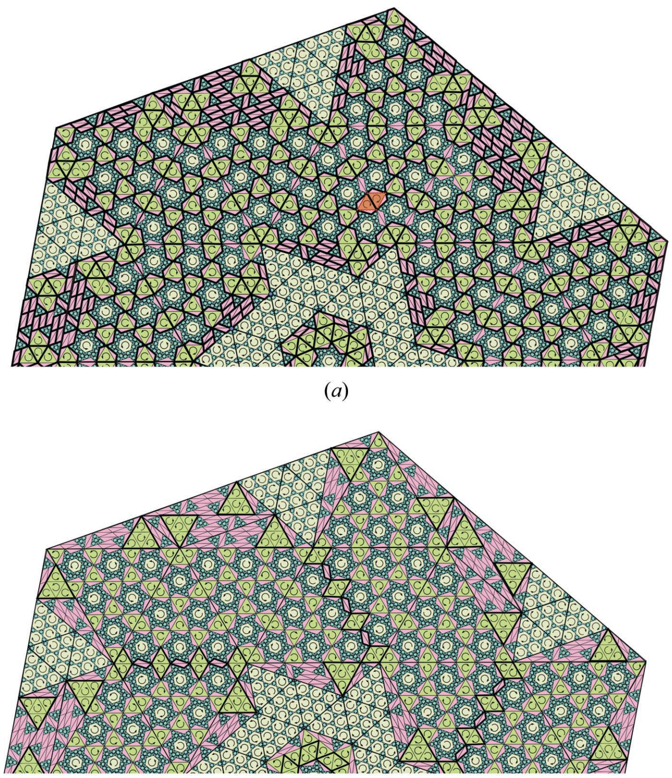


Figure 14
The substitution ω'. (A larger version of this figure is available in the supporting information.)



(a)

(b)

Figure 15
Portions of the partitions (a) $\mathcal{H}_{2,1}$ and (b) $\mathcal{H}_{2,2}$ of $\mathcal{T}_\omega$. A patch (in $\mathcal{H}_{2,1}$ or $\mathcal{H}_{2,2}$) consisting of more than one tile is enclosed by thick edges. (A larger version of this figure is available in the supporting information.)

$\sum_{i=1}^{n} k_i \mathrm{freq}(T'_i)$ and its analogue sums corresponding to the other partitions of $\mathcal{T}$.

Let us apply the process to finding the frequencies of the edge types E'_4 and E'_5. As discussed above, we define the tiling $\mathcal{T}'_\omega$ consisting of tiles obtained by deleting interior edge/s of each patch in $\mathcal{H}_1$. The prototiles of $\mathcal{T}'_\omega$ are $T'_1, T'_2, \ldots, T'_8$, where T'_i is the tile arising from deleting, if it exists, every interior edge of P_i, $i \in \{1, 2, \ldots, 8\}$. The substitution ω' corresponding to $\mathcal{T}'_\omega$ is shown in Fig. 14, which behaves the same way as ω^2.

Now, because $\mathcal{T}_\omega$ can be obtained by replacing every tile of $\mathcal{T}'_\omega$ by a patch in $\mathcal{H}_1$, and vice versa, $\mathrm{freq}(\mathcal{P}_i) = \mathrm{freq}(T'_i)$, $i \in \{1, 2, \ldots, 8\}$. Observe from Fig. 13 that the patches $\mathcal{P}_5$, $\mathcal{P}_6$ and $\mathcal{P}_7$ contain one, two and three equivalent copies of E'_4, respectively; and $\mathcal{P}_8$ contains one copy of E'_5. Thus, $\mathrm{freq}(T'_5) + 2\mathrm{freq}(T'_6) + 3\mathrm{freq}(T'_7)$ contributes to $\mathrm{freq}(E_4{}')$ and $\mathrm{freq}(T'_8)$ contributes to $\mathrm{freq}(E_5{}')$. But because each equivalent copy of $E'_i \in \mathcal{E}_1$, $i = 4, 5$, in the tiling $\mathcal{T}_\omega$ is contained by a patch of $\mathcal{H}_1$, we have $\mathrm{freq}(E'_4) = \mathrm{freq}(T'_5) + 2\mathrm{freq}(T'_6) + 3\mathrm{freq}(T'_7)$ and $\mathrm{freq}(E'_5) = \mathrm{freq}(T'_8)$.

We now compute the frequencies of $T'_1, T'_2, \ldots, T'_8$. The substitution matrix of ω' is given by

$$M_{\omega'} = \begin{bmatrix} 1 & 0 & 6 & 6 & 12 & 18 & 24 & 0 \\ 36 & 25 & 0 & 36 & 66 & 96 & 126 & 40 \\ 48 & 20 & 13 & 144 & 288 & 432 & 576 & 40 \\ 0 & 4 & 0 & 13 & 18 & 24 & 30 & 4 \\ 0 & 0 & 0 & 0 & 4 & 6 & 8 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 6 & 9 & 5 \end{bmatrix}.$$

It can be verified that ω' is primitive since $M_{\omega'}{}^4$ contains positive entries only. The primitivity of ω' allows us to use Theorem 2 to solve the frequencies of T'_5, T'_6, T'_7, T'_8.

The PF-eigenvector and normalized right PF-eigenvector of $M_{\omega'}$ are equal to $\mathbf{u}' = (6, 2, 1, 7, 14, 21, 28, 4)$ and $\mathbf{v}' = (7/81, 323/1269, 49/81, 589/15228, 109/53298, 1/2268, 2/189, 1/423)^T$, respectively. By Theorem 2, $\mathbf{u}'$ and $\mathbf{v}'$ encode the areas and the relative frequencies of the prototiles, respectively. Using equation (2), the absolute frequencies of T'_5, T'_6, T'_7, T'_8 can be extracted. For instance,

$$\mathrm{freq}(T'_5) = \frac{\mathbf{v}_5{}'}{\mathbf{u}' \cdot \mathbf{v}'} = \frac{109/53298}{182/81} = \frac{109}{2^2 \times 7^2 \times 13 \times 47}.$$

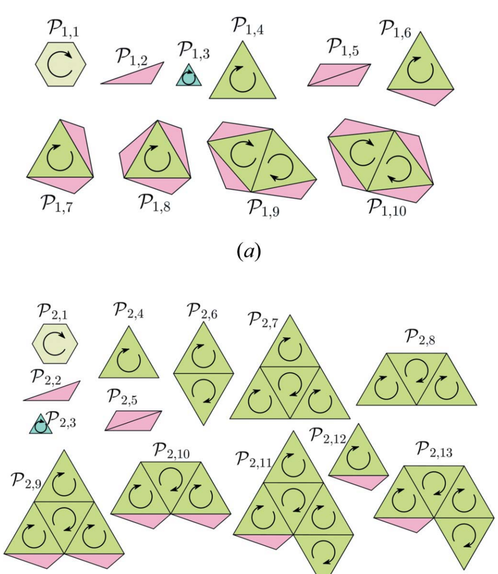
The frequencies of the other prototiles can be derived similarly. We have

$$\mathrm{freq}(T'_6) = \frac{1}{2^3 \times 7^2 \times 13}, \quad \mathrm{freq}(T'_7) = \frac{3}{7^2 \times 13}$$
$$\text{and } \mathrm{freq}(T'_8) = \frac{9}{2 \times 7 \times 13 \times 47}.$$

Therefore,

$$\begin{aligned} \mathrm{freq}(E'_4) &= \mathrm{freq}(T'_5) + 2\mathrm{freq}(T'_6) + 3\mathrm{freq}(T'_7) \\ &= \frac{109}{2^2 \times 7^2 \times 13 \times 47} + 2 \times \frac{1}{2^3 \times 7^2 \times 13} + 3 \times \frac{3}{7^2 \times 13} \\ &= \frac{66}{7 \times 13 \times 47} \end{aligned}$$

and



(a)

(b)

Figure 16
The complete lists of non-equivalent patches of partitions (a) $\mathcal{H}_{2,1}$ and (b) $\mathcal{H}_{2,2}$.

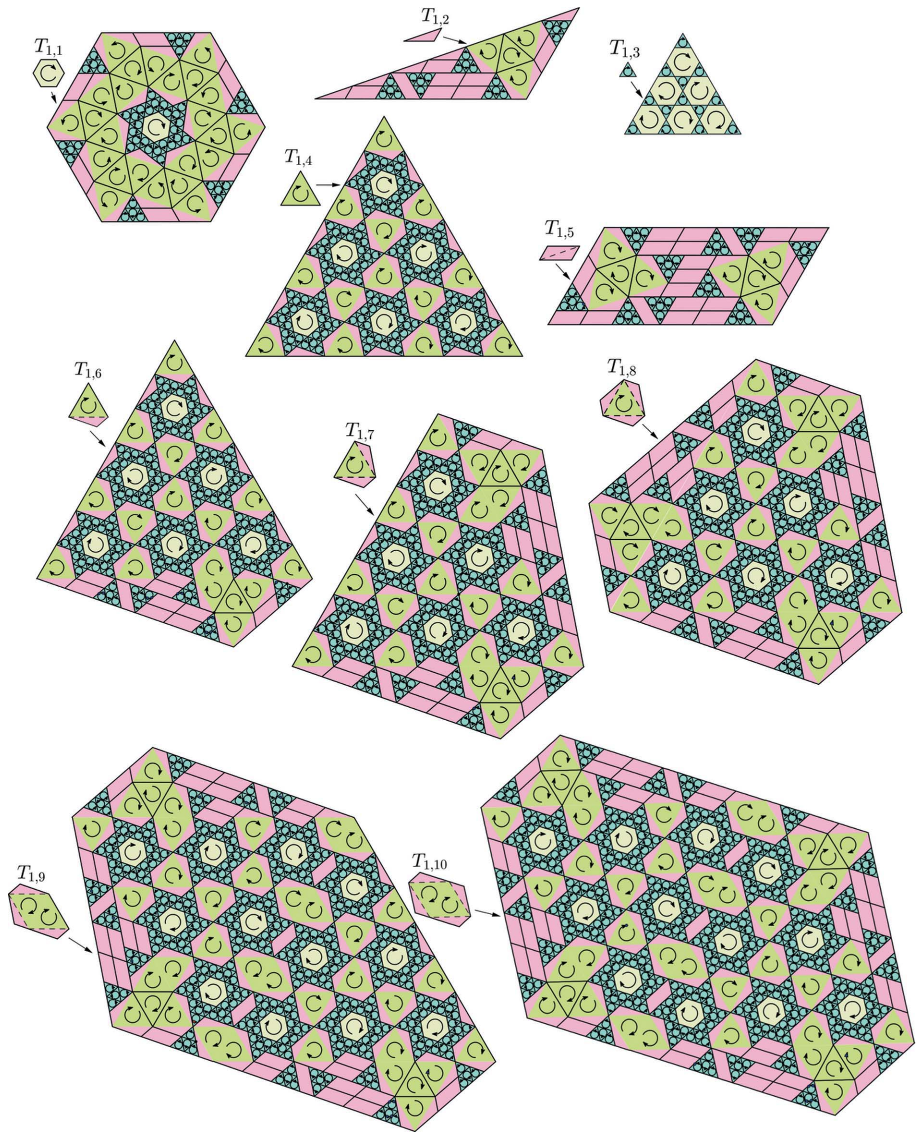


Figure 17
The substitution ω_1. (A larger version of this figure is available in the supporting information.)

$$\mathrm{freq}(E'_5) = \mathrm{freq}(T'_8) = \frac{9}{2 \times 7 \times 13 \times 47}.$$

We summarize these results in the following lemma.

Lemma 4. The absolute frequencies of the edge types E'_4 and E'_5 are respectively equal to

$$\mathrm{freq}(E'_4) = \frac{66}{7 \times 13 \times 47} \text{ and } \mathrm{freq}(E'_5) = \frac{9}{2 \times 7 \times 13 \times 47}.$$

In the case of $\mathcal{E}_1 = \{E'_4, E'_5\}$, one partition is sufficient to obtain the frequencies of the edge types. We discuss in the following lemma the case of $\mathcal{E}_2 = \{E'_1, E'_2, E'_3\}$, where in two partitions are defined to obtain the frequencies of E'_1, E'_2, E'_3.

Lemma 5. The absolute frequencies of the edge types E'_1, E'_2 and E'_3 are respectively equal to

$$\mathrm{freq}(E'_1) = \frac{11439}{2 \times 7 \times 13 \times 47^2}, \quad \mathrm{freq}(E'_2) = \frac{22665}{2 \times 7 \times 13 \times 47^2}$$
$$\text{and } \mathrm{freq}(E'_3) = \frac{2829}{7 \times 13 \times 47^2}.$$

Proof. It can be shown that the set $\mathcal{E}_2 = \{E'_1, E'_2, E'_3\}$ satisfies condition (*) with $l = 2$. We first consider the partition $\mathcal{H}_{2,1}$. The partition $\mathcal{H}_{2,1}$ satisfies conditions (A)–(D). A portion of $\mathcal{H}_{2,1}$ is shown in Fig. 15(a). A complete set of non-

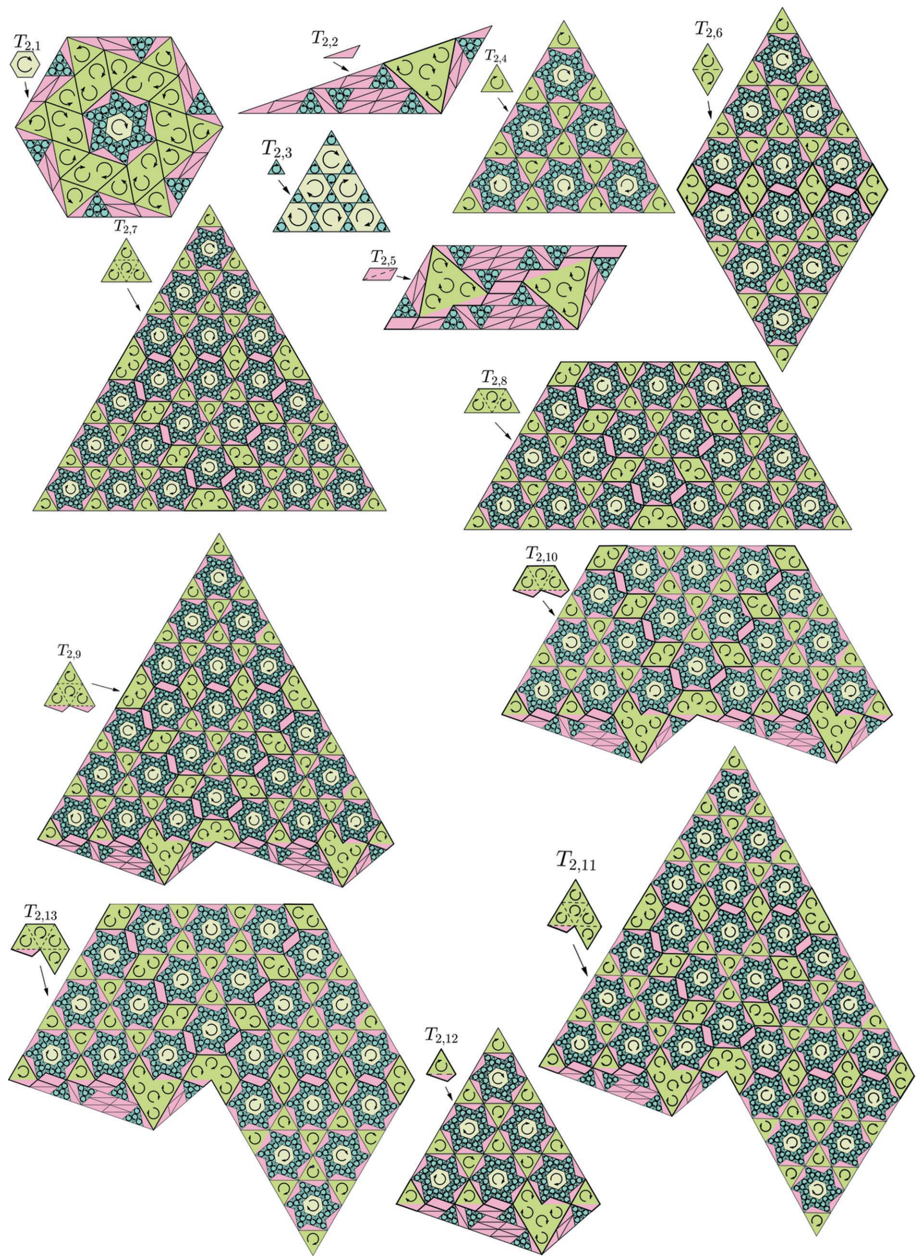


Figure 18
The substitution ω_2. (A larger version of this figure is available in the supporting information.)

equivalent patches in $\mathcal{H}_{2,1}$ consists of $\mathcal{P}_{1,1}, \mathcal{P}_{1,2}, \ldots, \mathcal{P}_{1,10}$ as shown in Fig. 16(*a*).

Now, some equivalent copies of E_1', E_2' and E_3' in $\mathcal{T}$ are not contained by patches of $\mathcal{H}_{2,1}$. For example, an equivalent copy of E_3' that is not contained by a patch of $\mathcal{H}_{2,1}$ is shaded red in Fig. 15(*a*); its edge lies on a boundary between two patches of $\mathcal{H}_{2,1}$. To take into account such equivalent copies, we introduce the second partition $\mathcal{H}_{2,2}$ satisfying conditions (*A*)–(*D*). A portion of $\mathcal{H}_{2,2}$ is shown in Fig. 15(*b*). A complete set of non-equivalent patches in $\mathcal{H}_{2,2}$ consists of $\mathcal{P}_{2,1}, \mathcal{P}_{2,2}, \ldots, \mathcal{P}_{2,13}$ as shown in Fig. 16(*b*).

It can be verified that each copy of an edge type in $\mathcal{E}_2 = \{E_1', E_2', E_3'\}$ in $\mathcal{T}$ is contained by a patch in either $\mathcal{H}_{2,1}$ or $\mathcal{H}_{2,2}$ but not both.

Following a similar process to that discussed earlier, the partitions $\mathcal{H}_{2,1}$ and $\mathcal{H}_{2,2}$ along with ω^2 give rise to the subtitutions ω_1 and ω_2 shown in Fig. 17 and Fig. 18 with prototile sets $\{T_{1,1}, T_{1,2}, \ldots, T_{1,10}\}$ and $\{T_{2,1}, T_{2,2}, \ldots, T_{2,13}\}$, respectively.

In the following, we compute the frequencies of the prototiles of ω_1 and ω_2, which are equal to the frequencies of $\mathcal{P}_{1,1}, \mathcal{P}_{1,2}, \ldots, \mathcal{P}_{1,10}$ and $\mathcal{P}_{2,1}, \mathcal{P}_{2,2}, \ldots, \mathcal{P}_{2,13}$, respectively.

The substitution matrix of ω_1 is given by

$$M_{\omega_1} = \begin{bmatrix} 1 & 0 & 6 & 6 & 0 & 6 & 6 & 6 & 12 & 12 \\ 0 & 5 & 0 & 9 & 0 & 6 & 3 & 0 & 3 & 0 \\ 48 & 20 & 13 & 144 & 40 & 164 & 184 & 204 & 348 & 368 \\ 6 & 1 & 0 & 0 & 2 & 1 & 2 & 3 & 3 & 4 \\ 6 & 8 & 0 & 0 & 19 & 10 & 20 & 30 & 33 & 43 \\ 12 & 2 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 6 & 6 & 8 & 8 & 6 & 9 & 8 \\ 0 & 0 & 0 & 4 & 0 & 5 & 7 & 10 & 12 & 14 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 \end{bmatrix},$$

which is primitive because ${M_{\omega_1}}^4$ contains positive entries only. The PF-eigenvector and normalized right PF-eigenvector are respectively given by $\mathbf{u}_1 = (6, 2, 1, 7, 4, 9, 11, 13, 20, 22)$ and

$$\mathbf{v}_1 = \left(\frac{14}{137}, \frac{66}{6439}, \frac{98}{137}, \frac{18}{957}, \frac{22203}{302633}, \frac{9789}{360584}, \frac{620507}{16947448}, \frac{110249}{8473724}, \frac{7545}{2421064}, \frac{503}{2421064} \right)^T.$$

By equation (2),

$$\mathrm{freq}(T_{1,5}) = \frac{\mathbf{v}_{15}}{\mathbf{u}_1 \cdot \mathbf{v}_1} = \frac{22203/302633}{364/137} = \frac{22203}{2^2 \times 7 \times 13 \times 47^2}.$$

The frequencies of the other prototiles, which we list in Table 1, can be derived similarly.

We proceed by extracting the frequencies of E_1', E_2' and E_3'. Each of the patches $\mathcal{P}_{1,5}$ and $\mathcal{P}_{2,5}$ contains one equivalent copy of E_1'. Hence,

$$\begin{aligned} \mathrm{freq}(E_1') &= \mathrm{freq}(T_{1,5}) + \mathrm{freq}(T_{2,5}) \\ &= \frac{22203}{2^2 \times 7 \times 13 \times 47^2} + \frac{675}{2^2 \times 7 \times 13 \times 47^2} \\ &= \frac{11439}{2 \times 7 \times 13 \times 47^2}. \end{aligned}$$

The patches $\mathcal{P}_{1,6}$, $\mathcal{P}_{2,11}$, $\mathcal{P}_{2,12}$ and $\mathcal{P}_{2,13}$ each contain one equivalent copy of E_2'; $\mathcal{P}_{1,7}$, $\mathcal{P}_{2,9}$ and $\mathcal{P}_{2,10}$ each contain two equivalent copies of E_2'; $\mathcal{P}_{1,8}$ and $\mathcal{P}_{1,9}$ each contain three equivalent copies of E_2'; and $\mathcal{P}_{1,10}$ contains four equivalent copies of E_2'. Therefore,

$$\begin{aligned} \mathrm{freq}(E_2') = {} & \frac{753}{2^5 \times 7^2 \times 47} + \frac{15}{7^3 \times 13 \times 47^2} + \frac{5}{2^3 \times 7 \times 13 \times 47^2} \\ & + \frac{5}{2^4 \times 7^3 \times 13 \times 47^2} + 2 \times \frac{620507}{2^5 \times 7^2 \times 13 \times 47^2} \\ & + 2 \times \frac{675}{2 \times 7^3 \times 13 \times 47^2} + 2 \times \frac{225}{2^5 \times 7^3 \times 13 \times 47^2} \\ & + 3 \times \frac{110249}{2^4 \times 7^2 \times 13 \times 47^2} + 3 \times \frac{7545}{2^5 \times 7 \times 13 \times 47^2} \\ & + 4 \times \frac{503}{2^5 \times 7 \times 13 \times 47^2} \\ = {} & \frac{22665}{2 \times 7 \times 13 \times 47^2}. \end{aligned}$$

Finally, the patches $\mathcal{P}_{1,9}$, $\mathcal{P}_{1,10}$ and $\mathcal{P}_{2,6}$ each contain one equivalent copy of E_3'; $\mathcal{P}_{2,8}$ and $\mathcal{P}_{2,10}$ each contain two equivalent copies of E_3'; $\mathcal{P}_{2,7}$, $\mathcal{P}_{2,9}$ and $\mathcal{P}_{2,13}$ each contain three equivalent copies of E_3'; and $\mathcal{P}_{2,11}$ contains four equivalent copies of E_3'. Therefore,

$$\begin{aligned} \mathrm{freq}(E_3') = {} & \frac{7545}{2^5 \times 7 \times 13 \times 47^2} + \frac{503}{2^5 \times 7 \times 13 \times 47^2} \\ & + \frac{61485}{2^3 \times 7^2 \times 13 \times 47^2} + 2 \times \frac{491}{2^4 \times 7^3 \times 13 \times 47} \\ & + 2 \times \frac{225}{2^5 \times 7^3 \times 13 \times 47^2} + 3 \times \frac{486}{7^3 \times 13 \times 47} \\ & + 3 \times \frac{675}{2 \times 7^3 \times 13 \times 47^2} + 3 \times \frac{5}{2^4 \times 7^3 \times 13 \times 47^2} \\ & + 4 \times \frac{15}{7^3 \times 13 \times 47^2} \\ = {} & \frac{2829}{7 \times 13 \times 47^2}. \end{aligned}$$

□

Table 1
The absolute frequencies of the patches $T_{1,5}, T_{1,6}, \ldots, T_{1,10}$, $T_{2,5}, T_{2,6}, \ldots, T_{2,13}$.

i, j	$\mathrm{freq}(T_{i,j})$	i, j	$\mathrm{freq}(T_{i,j})$
1, 5	$\frac{22203}{2^2 \times 7 \times 13 \times 47^2}$	2, 7	$\frac{486}{7^3 \times 13 \times 47}$
1, 6	$\frac{753}{2^5 \times 7^2 \times 47}$	2, 8	$\frac{491}{2^4 \times 7^3 \times 13 \times 47}$
1, 7	$\frac{620507}{2^5 \times 7^2 \times 13 \times 47^2}$	2, 9	$\frac{675}{2 \times 7^3 \times 13 \times 47^2}$
1, 8	$\frac{110249}{2^4 \times 7^2 \times 13 \times 47^2}$	2, 10	$\frac{225}{2^5 \times 7^3 \times 13 \times 47^2}$
1, 9	$\frac{7545}{2^5 \times 7 \times 13 \times 47^2}$	2, 11	$\frac{15}{7^3 \times 13 \times 47^2}$
1, 10	$\frac{503}{2^5 \times 7 \times 13 \times 47^2}$	2, 12	$\frac{5}{2^3 \times 7 \times 13 \times 47^2}$
2, 5	$\frac{675}{2^2 \times 7 \times 13 \times 47^2}$	2, 13	$\frac{5}{2^4 \times 7^3 \times 13 \times 47^2}$
2, 6	$\frac{61485}{2^3 \times 7^2 \times 13 \times 47^2}$		

For some edge types, it may not be necessary to derive a new substitution. This is when each copy of that edge type is contained by a 1-order super-tile or crosses a 1-order super-edge corresponding to an edge type with a known frequency. This is the case for the edge types $E_1, E_2, \ldots, E_{15}$ of $\mathcal{T}_\omega$.

We note that either an equivalent copy of E_i, $i \in \{1, 2, \ldots, 15\}$, in $\mathcal{T}_\omega$ is contained by a 1-order supertile or it crosses a 1-order super-edge corresponding to a 1-order super-edge type equivalent to $\omega(E'_k)$ for some $k \in \{1, 2, 3, 4, 5\}$ (see Figs. 4–8). Thus, each copy of E_i, $i \in \{1, 2, \ldots, 15\}$, in $\mathcal{T}_\omega$ is contained by either a 1-order supertile or a 1-order super-edge type equivalent to $\omega(E'_k)$ for some $k \in \{1, 2, 3, 4, 5\}$. Therefore,

$$\begin{aligned}\mathrm{freq}(E_i) &= \sum_{j=1}^{4} a_j \mathrm{freq}\big(\omega(T_j)\big) + \sum_{k=1}^{5} b_k \mathrm{freq}(\omega(E'_k)) \\ &= \sum_{j=1}^{4} a_j \frac{\mathrm{freq}(T_j)}{7} + \sum_{k=1}^{5} b_k \frac{\mathrm{freq}(E'_k)}{7}, \end{aligned} \tag{4}$$

where a_j is the number of equivalent copies of E_i in $\omega(T_j)$ and b_k is the number of equivalent copies of E_i crossing the 1-order super-edge of $\omega(E'_k)$.

For example, $\omega(T_1)$, $\omega(T_4)$ and $\omega(E'_2)$ respectively contain six equivalent copies, 27 equivalent copies and one equivalent copy of E_1 (see Fig. 1 and Fig. 5). Hence,

$$\begin{aligned}\mathrm{freq}(E_1) &= 6 \times \frac{\mathrm{freq}(T_1)}{7} + 27 \times \frac{\mathrm{freq}(T_4)}{7} + \frac{\mathrm{freq}(E'_2)}{7} \\ &= 6 \times \frac{1/(2\times 13)}{7} + 27 \times \frac{1/(2\times 13)}{7} \\ &\quad + \frac{22665/(2\times 7\times 13\times 47^2)}{7} \\ &= \frac{266472}{7^2\times 13\times 47^2}.\end{aligned}$$

We list in Table 2 the frequency, which can be obtained using equation (4), of each E_i, $i \in \{1, 2, \ldots, 15\}$, together with the 1-order supertile/s and 1-order super-edge type/s containing equivalent copies of E_i.

3.5. Determining the absolute frequencies of the vertex stars

We count equivalent copies of a vertex star $\mathcal{V}$ by counting 1-order supertiles containing equivalent copies of $\mathcal{V}$, 1-order super-edge types corresponding to 1-order super-edges crossed by equivalent copies of $\mathcal{V}$ and 1-order super-vertex stars whose defining vertices are also the defining vertices of some equivalent copies of $\mathcal{V}$. Then we compute the absolute frequencies of these 1-order supertiles, 1-order super-edge types and 1-order super-vertex stars containing equivalent copies of $\mathcal{V}$.

We illustrate the steps on the vertex stars $\mathcal{V}_{1,1}$, $\mathcal{V}_{1,2}$ and $\mathcal{V}_{1,3}$ of $\mathcal{T}_\omega$. Six equivalent copies of $\mathcal{V}_{1,1}$ are contained by $\omega(T_1)$ and one equivalent copy crosses the 1-order super-edge of $\omega(E'_2)$ (see Fig. 1 and Fig. 5). Thus,

$$\begin{aligned}\mathrm{freq}\big(\mathcal{V}_{1,1}\big) &= 6 \times \mathrm{freq}(\omega(T_1)) + \mathrm{freq}(\omega(E'_2)) \\ &= 6 \times \frac{\mathrm{freq}(T_1)}{7} + \frac{\mathrm{freq}(E'_2)}{7} \\ &= 6 \times \frac{1/(2\times 13)}{7} + \frac{22665/(2\times 7\times 13\times 47^2)}{7} \\ &= \frac{115443}{2\times 7^2\times 13\times 47^2}.\end{aligned}$$

The vertex star $\mathcal{V}_{1,2}$ is contained by $\omega(\mathcal{V}_{1,1})$ [where the defining vertex of $\mathcal{V}_{1,2}$ is also the defining 1-order super-vertex of $\omega(\mathcal{V}_{1,1})$] and one equivalent copy of $\mathcal{V}_{1,2}$ is contained by $\omega(\mathcal{V}_{1,3})$ (see Fig. 10). Hence,

$$\mathrm{freq}\big(\mathcal{V}_{1,2}\big) = \mathrm{freq}\big(\omega\big(\mathcal{V}_{1,1}\big)\big) + \mathrm{freq}\big(\omega\big(\mathcal{V}_{1,3}\big)\big). \tag{5}$$

One equivalent copy of $\mathcal{V}_{1,3}$ is contained by $\omega(\mathcal{V}_{1,2})$ (see Fig. 10). Thus,

$$\mathrm{freq}\big(\mathcal{V}_{1,3}\big) = \mathrm{freq}\big(\omega\big(\mathcal{V}_{1,2}\big)\big). \tag{6}$$

By equations (5) and (6), we have

$$\begin{aligned}\mathrm{freq}\big(\mathcal{V}_{1,2}\big) &= \frac{49}{48} \times \frac{\mathrm{freq}\big(\mathcal{V}_{1,1}\big)}{7} \\ &= \frac{49}{48} \times \frac{115443/(2\times 7^2\times 13\times 47^2)}{7} \\ &= \frac{38481}{2^5\times 7\times 13\times 47^2}\end{aligned}$$

and

$$\begin{aligned}\mathrm{freq}\big(\mathcal{V}_{1,3}\big) &= \mathrm{freq}\big(\omega\big(\mathcal{V}_{1,2}\big)\big) = \frac{\mathrm{freq}\big(\mathcal{V}_{1,2}\big)}{7} \\ &= \frac{38481/(2^5\times 7\times 13\times 47^2)}{7} \\ &= \frac{38481}{2^5\times 7^2\times 13\times 47^2}.\end{aligned}$$

We summarize in Table 3 the frequency of each vertex star $\mathcal{V}_{i,j}$ together with the 1-order supertile/s, 1-order super-edge type/s and 1-order super-vertex star/s containing equivalent copies of $\mathcal{V}_{i,j}$.

It can be verified that the sum $s_\mathcal{V}$ of the absolute frequencies of the vertex stars is equal to 9/26.

We can confirm the validity of the sum $s_\mathcal{V}$ by the following arguments. The sum of the interior angles of the hexagonal prototile T_1 is 4π. Hence, every copy of T_1 possesses a total of 2 vertices. The sum of the interior angles of T_2 is 2π. (Note that there is a pseudo-vertex at the midpoint of the edge of length 2.) So every copy of T_2 possesses a total of 1 vertex. Both T_3 and T_4 are equilateral triangles, so the sums of their interior angles are both equal to π. Hence, every copy of T_i ($i = 3, 4$) possesses a total of half a vertex. Therefore,

Table 2
The absolute frequencies of the edge types $E_1, E_2, \ldots, E_{15}$.

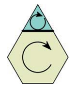
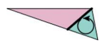
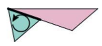
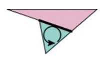
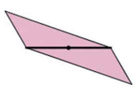
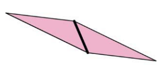
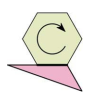
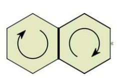
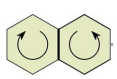
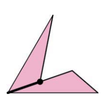
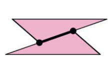
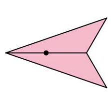
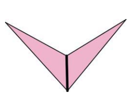
E_i		E_i has	freq(E_i)
E_1		6 copies in $\omega(T_1)$, 27 copies in $\omega(T_4)$ and 1 copy in $\omega(E'_2)$	$\frac{266472}{7^2 \times 13 \times 47^2}$
E_2		6 copies in $\omega(T_1)$, 2 copies in $\omega(T_2)$ and 1 copy in $\omega(E'_2)$	$\frac{16017}{2 \times 7^2 \times 47^2}$
E_3		6 copies in $\omega(T_1)$, 1 copy in $\omega(T_2)$, 2 copies in $\omega(E'_1)$ and 2 copies in $\omega(E'_2)$	$\frac{29625}{2 \times 7 \times 13 \times 47^2}$
E_4		6 copies in $\omega(T_1)$, 1 copy in $\omega(T_2)$ and 2 copies in $\omega(E'_1)$	$\frac{12465}{2 \times 7^2 \times 47^2}$
E_5		6 copies in $\omega(T_1)$, 3 copies in $\omega(T_2)$, 1 copy in $\omega(E'_2)$ and 4 copies in $\omega(E'_3)$	$\frac{2829}{13 \times 47^2}$
E_6		18 copies in $\omega(T_1)$, 4 copies in $\omega(E'_4)$ and 2 copies in $\omega(E'_5)$	$\frac{66}{13 \times 47}$
E_7		1 copy in $\omega(T_2)$	$\frac{3}{2 \times 7 \times 13}$
E_8		1 copy in $\omega(T_2)$	$\frac{3}{2 \times 7 \times 13}$
E_9		2 copies in $\omega(E'_2)$	$\frac{22665}{7^2 \times 13 \times 47^2}$
E_{10}		2 copies in $\omega(E'_3)$	$\frac{8487}{7^2 \times 13 \times 47^2}$
E_{11}		3 copies in $\omega(E'_4)$	$\frac{198}{7^2 \times 13 \times 47}$
E_{12}		2 copies in $\omega(E'_1)$	$\frac{11439}{7^2 \times 13 \times 47^2}$
E_{13}		1 copy in $\omega(E'_1)$	$\frac{11439}{2 \times 7^2 \times 13 \times 47^2}$
E_{14}		2 copies in $\omega(E'_5)$	$\frac{9}{7^2 \times 13 \times 47}$
E_{15}		1 copy in $\omega(E'_5)$	$\frac{9}{2 \times 7^2 \times 13 \times 47}$

Table 3
The absolute frequencies of the vertex stars.

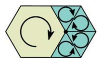
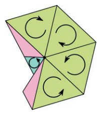
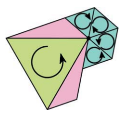
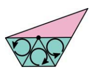
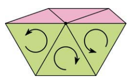
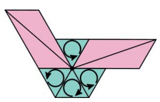
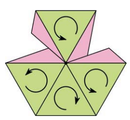
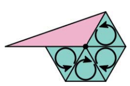
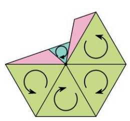
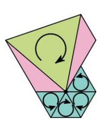
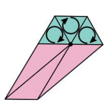
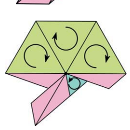
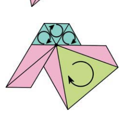
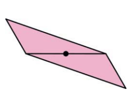
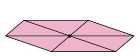
Vertex star		$\mathcal{V}_{i,j}$ has	Frequency
$\mathcal{V}_{1,1}$		6 copies in $\omega(T_1)$ and 1 copy in $\omega(E_2')$	$\frac{115443}{2 \times 7^2 \times 13 \times 47^2}$
$\mathcal{V}_{1,2}$		1 copy in $\omega(\mathcal{V}_{1,1})$ and 1 copy in $\omega(\mathcal{V}_{1,3})$	$\frac{38481}{2^5 \times 7 \times 13 \times 47^2}$
$\mathcal{V}_{1,3}$		1 copy in $\omega(\mathcal{V}_{1,2})$	$\frac{38481}{2^5 \times 7^2 \times 13 \times 47^2}$
$\mathcal{V}_{2,1}$		6 copies in $\omega(T_1)$ and 1 copy in $\omega(T_2)$	$\frac{9}{2 \times 7 \times 13}$
$\mathcal{V}_{2,2}$		1 copy in $\omega(\mathcal{V}_{2,1})$	$\frac{9}{2 \times 7^2 \times 13}$
$\mathcal{V}_{2,3}$		1 copy in $\omega(\mathcal{V}_{2,2})$ and 1 copy in $\omega(\mathcal{V}_{2,4})$	$\frac{3}{2^5 \times 7 \times 13}$
$\mathcal{V}_{2,4}$		1 copy in $\omega(\mathcal{V}_{2,3})$	$\frac{3}{2^5 \times 7^2 \times 13}$
$\mathcal{V}_{3,1}$		6 copies in $\omega(T_1)$	$\frac{3}{7 \times 13}$
$\mathcal{V}_{3,2}$		1 copy in $\omega(\mathcal{V}_{3,1})$ and 1 copy in $\omega(\mathcal{V}_{3,3})$	$\frac{1}{2^4 \times 13}$
$\mathcal{V}_{3,3}$		1 copy in $\omega(\mathcal{V}_{3,2})$	$\frac{1}{2^4 \times 7 \times 13}$
$\mathcal{V}_{4,1}$		1 copy in $\omega(T_2)$	$\frac{3}{2 \times 7 \times 13}$
$\mathcal{V}_{4,2}$		1 copy in $\omega(\mathcal{V}_{4,1})$ and 1 copy in $\omega(\mathcal{V}_{4,3})$	$\frac{1}{2^5 \times 13}$
$\mathcal{V}_{4,3}$		1 copy in $\omega(\mathcal{V}_{4,2})$	$\frac{1}{2^5 \times 7 \times 13}$
$\mathcal{V}_{5,1}$		1 copy in $\omega(T_2)$	$\frac{3}{2 \times 7 \times 13}$
$\mathcal{V}_{5,2}$		1 copy in $\omega(\mathcal{V}_{5,1})$	$\frac{3}{2 \times 7^2 \times 13}$

Table 3 (continued)

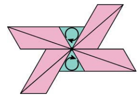
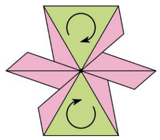
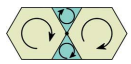
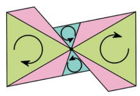
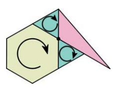
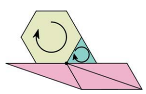
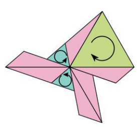
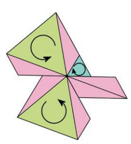
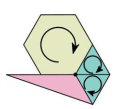
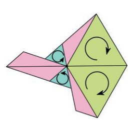
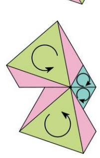
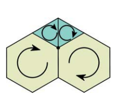
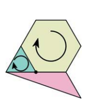
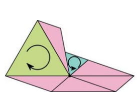
Vertex star		$\mathcal{V}_{i,j}$ has	Frequency
$\mathcal{V}_{5,3}$		1 copy in $\omega(\mathcal{V}_{5,2})$ and 1 copy in $\omega(\mathcal{V}_{5,4})$	$\frac{1}{2^5 \times 7 \times 13}$
$\mathcal{V}_{5,4}$		1 copy in $\omega(\mathcal{V}_{5,3})$	$\frac{1}{2^5 \times 7^2 \times 13}$
$\mathcal{V}_{6,1}$		9 copies in $\omega(T_4)$	$\frac{9}{2 \times 7 \times 13}$
$\mathcal{V}_{6,2}$		1 copy in $\omega(\mathcal{V}_{6,1})$, 1 copy in $\omega(\mathcal{V}_{6,2})$, and 1 copy in $\omega(\mathcal{V}_{7,1})$	$\frac{13486}{7^2 \times 13 \times 47^2}$
$\mathcal{V}_{7,1}$		1 copy in $\omega(E_2')$	$\frac{22665}{2 \times 7^2 \times 13 \times 47^2}$
$\mathcal{V}_{8,1}$		1 copy in $\omega(E_2')$	$\frac{22665}{2 \times 7^2 \times 13 \times 47^2}$
$\mathcal{V}_{8,2}$		1 copy in $\omega(\mathcal{V}_{8,1})$ and 1 copy in $\omega(\mathcal{V}_{8,3})$	$\frac{7555}{2^5 \times 7 \times 13 \times 47^2}$
$\mathcal{V}_{8,3}$		1 copy in $\omega(\mathcal{V}_{8,2})$	$\frac{7555}{2^5 \times 7^2 \times 13 \times 47^2}$
$\mathcal{V}_{9,1}$		1 copy in $\omega(E_2')$	$\frac{22665}{2 \times 7^2 \times 13 \times 47^2}$
$\mathcal{V}_{9,2}$		1 copy in $\omega(\mathcal{V}_{9,1})$, 1 copy in $\omega(\mathcal{V}_{9,3})$, and 1 copy in $\omega(\mathcal{V}_{10,1})$	$\frac{18871}{2^5 \times 7 \times 13 \times 47^2}$
$\mathcal{V}_{9,3}$		1 copy in $\omega(\mathcal{V}_{9,2})$	$\frac{18871}{2^5 \times 7^2 \times 13 \times 47^2}$
$\mathcal{V}_{10,1}$		6 copies in $\omega(E_3')$	$\frac{16974}{7^2 \times 13 \times 47^2}$
$\mathcal{V}_{11,1}$		2 copies in $\omega(E_2')$	$\frac{22665}{7^2 \times 13 \times 47^2}$
$\mathcal{V}_{11,2}$		1 copy in $\omega(\mathcal{V}_{11,1})$ and 1 copy in $\omega(\mathcal{V}_{12,1})$	$\frac{696}{7 \times 13 \times 47^2}$

Table 3 (continued)

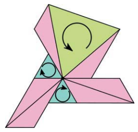
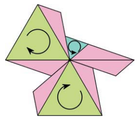
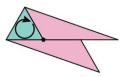
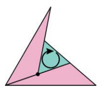
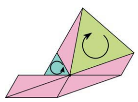
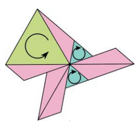
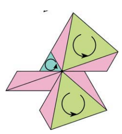
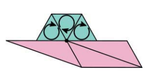
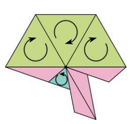
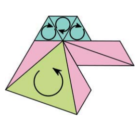
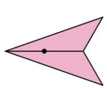
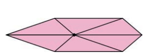
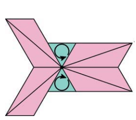
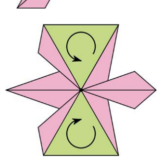
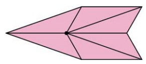
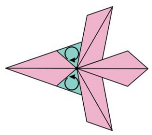
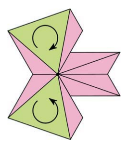
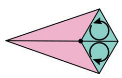
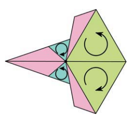
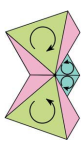
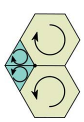
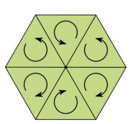
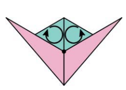
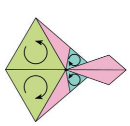
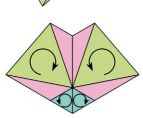
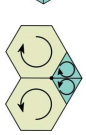
Vertex star		$\mathcal{V}_{i,j}$ has	Frequency
$\mathcal{V}_{11,3}$		1 copy in $\omega(\mathcal{V}_{11,2})$ and 1 copy in $\omega(\mathcal{V}_{12,4})$	$\frac{29}{2 \times 13 \times 47^2}$
$\mathcal{V}_{11,4}$		1 copy in $\omega(\mathcal{V}_{11,3})$	$\frac{29}{2 \times 7 \times 13 \times 47^2}$
$\mathcal{V}_{12,1}$		2 copies in $\omega(E_1')$	$\frac{11439}{7^2 \times 13 \times 47^2}$
$\mathcal{V}_{13,1}$		2 copies in $\omega(E_1')$	$\frac{11439}{7^2 \times 13 \times 47^2}$
$\mathcal{V}_{13,2}$		1 copy in $\omega(\mathcal{V}_{13,1})$	$\frac{11439}{7^3 \times 13 \times 47^2}$
$\mathcal{V}_{13,3}$		1 copy in $\omega(\mathcal{V}_{13,2})$ and 1 copy in $\omega(\mathcal{V}_{13,4})$	$\frac{3813}{2^4 \times 7^2 \times 13 \times 47^2}$
$\mathcal{V}_{13,4}$		1 copy in $\omega(\mathcal{V}_{13,3})$	$\frac{3813}{2^4 \times 7^3 \times 13 \times 47^2}$
$\mathcal{V}_{14,1}$		1 copy in $\omega(E_1')$	$\frac{11439}{7^2 \times 13 \times 47^2}$
$\mathcal{V}_{14,2}$		1 copy in $\omega(\mathcal{V}_{14,1})$ and 1 copy in $\omega(\mathcal{V}_{14,3})$	$\frac{3813}{2^4 \times 7 \times 13 \times 47^2}$
$\mathcal{V}_{14,3}$		1 copy in $\omega(\mathcal{V}_{14,2})$	$\frac{3813}{2^4 \times 7^2 \times 13 \times 47^2}$
$\mathcal{V}_{15,1}$		2 copies in $\omega(E_5')$	$\frac{9}{7^2 \times 13 \times 47}$
$\mathcal{V}_{15,2}$		1 copy in $\omega(\mathcal{V}_{15,1})$	$\frac{9}{7^3 \times 13 \times 47}$
$\mathcal{V}_{15,3}$		1 copy in $\omega(\mathcal{V}_{15,2})$ and 1 copy in $\omega(\mathcal{V}_{15,4})$	$\frac{3}{2^4 \times 7^2 \times 13 \times 47}$
$\mathcal{V}_{15,4}$		1 copy in $\omega(\mathcal{V}_{15,3})$	$\frac{3}{2^4 \times 7^3 \times 13 \times 47}$

Table 3 (continued)

Vertex star		$\mathcal{V}_{i,j}$ has	Frequency
$\mathcal{V}_{16,1}$		1 copy in $\omega(E_5')$	$\frac{9}{2\times7^2\times13\times47}$
$\mathcal{V}_{16,2}$		1 copy in $\omega(\mathcal{V}_{16,1})$ and one copy in $\omega(\mathcal{V}_{16,3})$	$\frac{3}{2^5\times7\times13\times47}$
$\mathcal{V}_{16,3}$		1 copy in $\omega(\mathcal{V}_{16,2})$	$\frac{3}{2^5\times7^2\times13\times47}$
$\mathcal{V}_{17,1}$		1 copy in $\omega(E_5')$	$\frac{9}{2\times7^2\times13\times47}$
$\mathcal{V}_{17,2}$		1 copy in $\omega(\mathcal{V}_{17,1})$, 1 copy in $\omega(\mathcal{V}_{17,3})$, and 1 copy in $\omega(\mathcal{V}_{18,1})$	$\frac{135}{2^5\times7\times13\times47}$
$\mathcal{V}_{17,3}$		1 copy in $\omega(\mathcal{V}_{17,2})$	$\frac{135}{2^5\times7^2\times13\times47}$
$\mathcal{V}_{18,1}$		3 copies in $\omega(E_4')$	$\frac{198}{7^2\times13\times47}$
$\mathcal{V}_{19,1}$		1 copy in $\omega(E_5')$ and 1 copy in $\omega(\mathcal{V}_{19,2})$	$\frac{3}{2^5\times13\times47}$
$\mathcal{V}_{19,2}$		1 copy in $\omega(\mathcal{V}_{19,1})$	$\frac{3}{2^5\times7\times13\times47}$
$\mathcal{V}_{20,1}$		1 copy in $\omega(E_5')$	$\frac{9}{2\times7^2\times13\times47}$
$\mathcal{V}_{20,2}$		1 copy in $\omega(\mathcal{V}_{20,1})$, 1 copy in $\omega(\mathcal{V}_{20,3})$ and 1 copy in $\omega(\mathcal{V}_{21,1})$	$\frac{135}{2^5\times7\times13\times47}$
$\mathcal{V}_{20,3}$		1 copy in $\omega(\mathcal{V}_{20,2})$	$\frac{135}{2^5\times7^2\times13\times47}$
$\mathcal{V}_{21,1}$		3 copies in $\omega(E_4')$	$\frac{198}{7^2\times13\times47}$

$$s_{\mathcal{V}} = 2 \times \text{freq}(T_1) + \text{freq}(T_2) + \frac{1}{2} \times \text{freq}(T_3) + \frac{1}{2} \times \text{freq}(T_4)$$
$$= 2 \times \frac{1}{2 \times 13} + 1 \times \frac{3}{2 \times 13} + \frac{1}{2} \times \frac{7}{2 \times 13} + \frac{1}{2} \times \frac{1}{2 \times 13}$$
$$= \frac{9}{26}.$$

We note that sum of the absolute frequencies of the vertex stars is not equal to 1 because the areas of the prototiles are not equal.

3.6. Determining the frequency module of $\mathbb{X}_\omega$

From Theorem 3, the frequency module of $\mathbb{X}_\omega$ is the $\mathbb{Z}[1/7]$-module generated by the frequencies of the prototiles, edge types and vertex stars of $\mathcal{T}_\omega$. By considering all the frequencies of the prototiles, edge types (Lemmas 4 and 5, Table 2) and vertex stars (Table 3), we have the following result:

Theorem 6. The frequency module of $\mathbb{X}_\omega$ is

$$\mathbf{F} = \frac{1}{2^5 \times 13 \times 47^2}\mathbb{Z}\left[\frac{1}{7}\right] = \left\{\frac{m}{2^5 \times 13 \times 47^2 \times 7^k} | m, k \in \mathbb{N}\right\}.$$

Therefore, the minimal $\mathbb{Z}$-module that contains the absolute frequency of every patch in a tiling in the hull of the primitive substitution ω is $\mathbf{F} = [1/(2^5 \times 13 \times 47^2)]\mathbb{Z}[1/7]$.

4. Conclusion and outlook

In this article we have discussed how to obtain the frequency module of the hull $\mathbb{X}_\sigma$ of a primitive substitution tiling $\mathcal{T} \in \mathbb{X}_\sigma$ with convex prototiles. This is done by determining the absolute frequencies of the prototiles, edge types and vertex stars of $\mathcal{T} \in \mathbb{X}_\sigma$. The following are highlighted in this work.

The frequencies of the prototiles can be derived via the well known result of the Perron–Frobenius theorem and its consequence (Theorem 2). However, we cannot always directly apply these results to edge types. So to compute the frequencies of the edge types, we introduce a method which involves defining new substitution/s. We put together edge types into a set $\mathcal{E}$ satisfying condition (*). Then we look for a partition $\mathcal{H}$ of patches of $\mathcal{T}$, wherein an equivalent copy of every edge type in $\mathcal{E}$ occurs in the interior of a patch in $\mathcal{H}$. This partition must satisfy certain conditions, which ensure that we can reconstruct $\mathcal{T}$ as a substitution tiling $\mathcal{T}'$ by deleting the interior edges of the patches in $\mathcal{H}$. Given that the corresponding substitution σ' of $\mathcal{T}'$ is primitive, one can obtain the frequencies of the prototiles of σ' using Theorem 2. These frequencies are equal to the frequencies of the non-equivalent patches of $\mathcal{H}$, which contribute to the frequencies of the edge types in $\mathcal{E}$.

Now, it may happen that some copies of the edge types in $\mathcal{E}$ in the tiling $\mathcal{T}$ are not contained by patches in $\mathcal{H}$. In this case, we repeat the search for a partition and the derivation of its corresponding substitution. We iterate the process until every equivalent copy of $\mathcal{E}$ in $\mathcal{T}$ is accounted for. Then we extract the frequencies of the edge types in $\mathcal{E}$ from the frequencies of the patches of the partition/s.

For some set of edge types, it may not be necessary to apply the procedure of finding a partition. The frequencies of such edge types can be obtained from the frequencies of the prototiles and edge types with known frequencies.

Finally, for the first set of vertex stars (the set of vertex stars whose equivalent copies appear in 1-order supertiles, 1-order super-edge types, or both), the frequencies are obtained from the frequencies of the prototiles and edge types. The frequencies of the remaining vertex stars are obtained from the frequencies of the vertex stars from the first set.

The above method has been applied to the primitive substitution ω. The primitive substitution tiling ω is introduced and discussed in Frettlöh *et al.* (2017) and Say-awen (2016) along with other primitive substitutions which give rise to tilings that are invariant under n-fold rotation for $n \in \{3, 4, 5, 6, 7, 8\}$, with finite local complexity and dense tile orientations. The substitution ω is the substitution corresponding to $n = 6$.

We chose the substitution tiling with the least value of n ($n = 6$) to be discussed in this article since using this tiling can illustrate the ideas put forward by the method requiring only two partitions or two new substitutions to extract the frequencies of the edge types.

There are some points for investigation arising from this work. For instance, it would be interesting to determine whether a partition $\mathcal{H}$ corresponding to a set $\mathcal{E}$ always exists, or to identify properties of the substitution that can guarantee the existence of $\mathcal{H}$ without constructing large patches of the tiling. Another aspect that could be explored is to determine whether the new substitution/s arising from partition/s $\mathcal{H}$ corresponding to $\mathcal{E}$ is/are always primitive. It can be shown that if one partition is sufficient to extract the frequencies of the edge types in $\mathcal{E}$, then the corresponding substitution of such a partition is primitive. We suggest that the scenario wherein there are at least two partitions should be studied in more depth in the future.

Acknowledgements

ALDS would like to thank the Ateneo de Manila University for funding obtained through the University Research Council in 2020 and 2021, and the De La Salle University for funding obtained through the University Research Coordination Office in 2019 and 2020.

References

Baake, M., Frettlöh, D. & Grimm, U. (2007). *J. Geom. Phys.* **57**, 1331–1343.
Baake, M. & Grimm, U. (2013). *Aperiodic Order: A Mathematical Invitation*, Vol. 1. Cambridge University Press.
Bellissard, J. (1992). *From Number Theory To Physics*, edited by M. Waldschmidt, P. Moussa, J.-M. Luck & C. Itzykson, pp. 538–630. Berlin: Springer-Verlag.

Bellissard, J., Hermann, D. J. L. & Zarrouti, M. (2000). *Directions in Mathematical Quasicrystals* (CRM Monograph Series), Vol. 13, edited by M. Baake & R. V. Moody, pp. 207–258. Providence, RI: American Mathematical Socitey.
Bellissard, J., Kellendonk, J. & Legrand, A. (2001). *C. R. Acad. Sci. Ser. I Math.* **332**, 521–525.
Elst, A. van (1994). *Rev. Math. Phys.* **6**, 319–342.
Frettlöh, D. (2005). *Rev. Roum. Math. Pures Appl.* **50**, 619–639.
Frettlöh, D. (2008). *Eur. J. Combin.* **29**, 1881–1893.
Frettlöh, D. & Richard, C. (2014). *Discrete Contin. Dyn. Syst.* **34**, 531–556.
Frettlöh, D., Say-awen, A. L. D. & De Las Peñas, M. L. A. N. (2017). *Indag. Math.* **28**, 121–131.
Grünbaum, B. & Shephard, G. C. (1986). *Tilings and Patterns.* New York: W. H. Freeman and Company.
Kellendonk, J. (2021). *arXiv*:2107.00737v1.
Moustafa, H. (2010). *Commun. Math. Phys.* **298**, 369–405.
Perron, O. (1907). *Math. Ann.* **64**, 248–263.
Pytheas Fogg, N. (2002). *Substitutions in Dynamics, Arithmetics and Combinatorics*. *Lecture Notes in Mathematics*, Vol. 1794. Berlin: Springer-Verlag.
Say-awen, A. L. D. (2016). PhD thesis, Ateneo de Manila University, Philippines.
Say-awen, A. L. D., De Las Peñas, M. L. A. N. & Frettlöh, D. (2018). *Acta Cryst.* A**74**, 388–398.
Solomyak, B. (1997). *Ergod. Th. Dyn. Sys.* **17**, 695–738.