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# A De Bruijn–Erdős theorem for chordal graphs

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## Abstract

A special case of a combinatorial theorem of De Bruijn and Erdős asserts that every noncollinear set of  $n$  points in the plane determines at least  $n$  distinct lines. Chen and Chvátal suggested a possible generalization of this assertion in metric spaces with appropriately defined lines. We prove this generalization in all metric spaces induced by connected chordal graphs.

## 1 Introduction

It is well known that

- (i) *every noncollinear set of  $n$  points in the plane determines at least  $n$  distinct lines.*

As noted by Erdős [11], theorem (i) is a corollary of the Sylvester–Gallai theorem (asserting that, for every noncollinear set  $S$  of finitely many points in

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the plane, some line goes through precisely two points of  $S$ ); it is also a special case of a combinatorial theorem proved later by De Bruijn and Erdős [10].

Theorem (i) involves neither measurement of distances nor measurement of angles: the only notion employed here is incidence of points and lines. Such theorems are a part of *ordered geometry* [7], which is built around the ternary relation of *betweenness*: point  $b$  is said to lie between points  $a$  and  $c$  if  $b$  is an interior point of the line segment with endpoints  $a$  and  $c$ . It is customary to write  $[abc]$  for the statement that  $b$  lies between  $a$  and  $c$ . In this notation, a *line*  $\overline{uv}$  is defined — for any two distinct points  $u$  and  $v$  — as

$$\{u, v\} \cup \{p : [puv] \vee [upv] \vee [uvp]\}. \quad (1)$$

In terms of the Euclidean metric  $dist$ , we have

$$[abc] \Leftrightarrow a, b, c \text{ are three distinct points and } dist(a, b) + dist(b, c) = dist(a, c). \quad (2)$$

In an arbitrary metric space, equivalence (2) defines the ternary relation of *metric betweenness* introduced in [12] and further studied in [1, 3, 8]; in turn, (1) defines the line  $\overline{uv}$  for any two distinct points  $u$  and  $v$  in the metric space. The resulting family of lines may have strange properties. For instance, a line can be a proper subset of another: in the metric space with points  $u, v, x, y, z$  and

$$\begin{aligned} dist(u, v) = dist(v, x) = dist(x, y) = dist(y, z) = dist(z, u) = 1, \\ dist(u, x) = dist(v, y) = dist(x, z) = dist(y, u) = dist(z, v) = 2, \end{aligned}$$

we have

$$\overline{vy} = \{v, x, y\} \quad \text{and} \quad \overline{xz} = \{v, x, y, z\}.$$

Chen [4] proved, using a definition of  $\overline{uv}$  different from (1), that the Sylvester–Gallai theorem generalizes in the framework of metric spaces. Chen and Chvátal [5] suggested that theorem (i), too, might generalize in this framework:

- (ii) *True or false? Every metric space on  $n$  points, where  $n \geq 2$ , either has at least  $n$  distinct lines or else has a line that consists of all  $n$  points.*

They proved that

- every metric space on  $n$  points either has at least  $\lg n$  distinct lines or else has a line that consists of all  $n$  points

and noted that the lower bound  $\lg n$  can be improved to  $\lg n + \frac{1}{2} \lg \lg n + \frac{1}{2} \lg \frac{\pi}{2} - o(1)$ .

Every connected undirected graph induces a metric space on its vertex set, where  $\text{dist}(u, v)$  is defined as the smallest number of edges in a path from vertex  $u$  to vertex  $v$ . Chiniforooshan and Chvátal [6] proved that

- every metric space induced by a connected graph on  $n$  vertices either has  $\Omega(n^{2/7})$  distinct lines or else has a line that consists of all  $n$  vertices;

we will prove that the answer to (ii) is ‘true’ for all metric spaces induced by connected chordal graphs.

**Theorem 1.** *Every metric space induced by a connected chordal graph on  $n$  vertices, where  $n \geq 2$ , either has at least  $n$  distinct lines or else has a line that consists of all  $n$  vertices.*

For graph-theoretic terminology, we refer the reader to Bondy and Murty[2].

## 2 The proof

Given an undirected graph, let us write  $[abc]$  to mean that  $a, b, c$  are three distinct vertices such that  $\text{dist}(a, b) + \text{dist}(b, c) = \text{dist}(a, c)$ ; this is equivalent to saying that  $b$  is an interior vertex of a shortest path from  $a$  to  $c$ .

**Lemma 1.** *Let  $s, x, y$  be vertices in a finite chordal graph such that  $[sxy]$ . If  $\overline{sx} = \overline{sy}$ , then  $x$  is a cut vertex separating  $s$  and  $y$ .*

*Proof.* The set of all vertices  $u$  such that  $\text{dist}(s, u) = \text{dist}(s, x)$  separates  $s$  and  $y$ . Among all its subsets that separate  $s$  and  $y$ , choose a minimal one and call it  $C$ . Since  $x$  is an interior vertex of a shortest path from  $s$  to  $y$ , it belongs to  $C$ . To prove that  $C$  includes no other vertex, assume, to the contrary, that  $C$  includes a vertex  $u$  other than  $x$ .

Our graph with  $C$  removed has distinct connected components  $S$  and  $Y$  such that  $s \in S$  and  $y \in Y$ ; the minimality of  $C$  guarantees that each of its vertices

has at least one neighbour in  $S$  and at least one neighbour in  $Y$ . Since each of  $u$  and  $x$  has at least one neighbour in  $S$ , there is a path from  $u$  to  $x$  with at least one interior vertex and with all interior vertices in  $S$ . Let  $P$  be a shortest such path; note that  $P$  has no chords except possibly the chord  $ux$ . Similarly, there is a path  $Q$  from  $u$  to  $x$  with at least one interior vertex, and with all interior vertices in  $Y$ , that has no chords except possibly the chord  $ux$ . The union of  $P$  and  $Q$  is a cycle of length at least four; since this cycle must have a chord, vertices  $u$  and  $x$  must be adjacent. In turn, the union of  $Q$  and  $ux$  is a chordless cycle, and so  $Q$  has precisely two edges. This means that some vertex  $v$  in  $Y$  is adjacent to both  $u$  and  $x$ .

Write  $i = \text{dist}(s, x)$  and  $j = \text{dist}(x, y)$ . Since all vertices  $t$  with  $\text{dist}(s, t) < i$  belong to  $S$  and since  $v$  has no neighbours in  $S$ , we must have  $\text{dist}(s, v) > i$ ; since  $\text{dist}(x, v) = 1$ , we conclude that  $\text{dist}(s, v) = i + 1$  and that  $v \in \overline{sx}$ . Since  $\overline{sx} = \overline{sy}$ , it follows that  $v \in \overline{sy}$ . Since  $\text{dist}(v, x) = 1$  and  $\text{dist}(x, y) = j$ , we have  $\text{dist}(v, y) \leq j + 1$ . From  $\text{dist}(s, v) = i + 1$ ,  $\text{dist}(s, y) = i + j$ ,  $\text{dist}(v, y) \leq j + 1$ ,  $i \geq 1$ ,  $j \geq 1$ , and  $v \in \overline{sy}$ , we deduce that  $\text{dist}(v, y) = j - 1$ .

Since  $\text{dist}(u, v) = 1$ , it follows that  $\text{dist}(u, y) \leq j$ ; since  $\text{dist}(s, u) = i$  and  $\text{dist}(s, y) = i + j$ , we conclude that  $\text{dist}(u, y) = j$  and  $u \in \overline{sy}$ . Since  $\text{dist}(s, u) = i$ ,  $\text{dist}(s, x) = i$ , and  $\text{dist}(u, x) = 1$ , we have  $u \notin \overline{sx}$ . But then  $\overline{sx} \neq \overline{sy}$ , a contradiction.  $\square$

A vertex of a graph is called *simplicial* if its neighbours are pairwise adjacent.

**Lemma 2.** *Let  $s, x, y$  be three distinct vertices in a finite connected chordal graph. If  $s$  is simplicial and  $\overline{sx} = \overline{sy}$ , then  $\overline{xy}$  consists of all the vertices of the graph.*

*Proof.* Since  $\overline{sx} = \overline{sy}$ , we have  $y \in \overline{sx}$ , and so  $[ysx]$  or  $[syx]$  or  $[sxy]$ ; since  $s$  is simplicial,  $[ysx]$  is excluded; switching  $x$  and  $y$  if necessary, we may assume that  $[sxy]$ . Given an arbitrary vertex  $u$ , we have to prove that  $u \in \overline{xy}$ . Let  $P$  be a shortest path from  $s$  to  $u$  and let  $Q$  be a shortest path from  $u$  to  $y$ . Lemma 1 guarantees that  $x$  is a cut vertex separating  $s$  and  $y$ , and so the concatenation of  $P$  and  $Q$  must pass through  $x$ . This means that  $[sxu]$  or  $[uxy]$  (or both). If  $[uxy]$ , then  $u \in \overline{xy}$ ; to complete the proof, we may assume that  $[sxu]$ , and so  $u \in \overline{sx}$ .

Since  $\overline{sx} = \overline{sy}$ , we have  $[usy]$  or  $[suy]$  or  $[syu]$ ; since  $s$  is simplicial,  $[usy]$  is excluded. If  $[suy]$ , then  $[sxu]$  implies  $[xuy]$ ; if  $[syu]$ , then  $[sxy]$  implies  $[xyu]$ ; in either case,  $u \in \overline{xy}$ .  $\square$

*Proof of Theorem 1.* Consider a connected chordal graph on  $n$  vertices where  $n \geq 2$ . By a theorem of Dirac [9], this graph has at least two simplicial vertices; choose one of them and call it  $s$ . We may assume that the lines  $\overline{sz}$  with  $z \neq s$  are pairwise distinct (else some line consists of all  $n$  vertices by Lemma 2). Since the graph is connected and has at least two vertices,  $s$  has at least one neighbour; choose one and call it  $u$ . If  $u$  is the only neighbour of  $s$ , then every path from  $s$  to another vertex must pass through  $u$ , and so  $\overline{su}$  consists of all  $n$  vertices. If  $s$  has a neighbour  $v$  other than  $u$ , then line  $\overline{uv}$  is distinct from all of the  $n - 1$  lines  $\overline{sz}$  with  $z \neq s$ : since  $s, u, v$  are pairwise adjacent, we have  $s \notin \overline{uv}$ .  $\square$

### 3 Related theorems

In Theorem 1, ‘connected chordal graph’ can be replaced by ‘connected bipartite graph’:

- every metric space induced by a connected bipartite graph on  $n$  vertices, where  $n \geq 2$ , has a line that consists of all  $n$  vertices.

In fact,  $\overline{xy}$  consists of all  $n$  vertices whenever  $x$  and  $y$  are adjacent. To prove this, consider an arbitrary vertex  $u$ . Since the graph is bipartite,  $dist(u, x)$  and  $dist(u, y)$  have distinct parities; since  $dist(x, y) = 1$ , they differ by at most one. We conclude that  $dist(u, x)$  and  $dist(u, y)$  differ by precisely one, and so  $u \in \overline{xy}$ .

In Theorem 1, ‘connected chordal graph’ can be also replaced by ‘sufficiently large graph of diameter two’: Chiniforooshan and Chvátal [6] proved that

- every metric space on  $n$  points where each nonzero distance equals 1 or 2 has  $\Omega(n^{4/3})$  distinct lines and this bound is tight.

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