Chapter

Quaternion MPCEP, CEPMP, and MPCEPMP Generalized Inverses

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Abstract

A generalized inverse of a matrix is an inverse in some sense for a wider class of matrices than invertible matrices. Generalized inverses exist for an arbitrary matrix and coincide with a regular inverse for invertible matrices. The most famous generalized inverses are the Moore–Penrose inverse and the Drazin inverse. Recently, new generalized inverses were introduced, namely the core inverse and its generalizations. Among them, there are compositions of the Moore–Penrose and core inverses, MPCEP (or MP–Core–EP) and EPCMP (or EP–Core–MP) inverses. In this chapter, the notions of the MPCEP inverse and CEPMP inverse are expanded to quaternion matrices and introduced new generalized inverses, the right and left MPCEPMP inverses. Direct method of their calculations, that is, their determinantal representations are obtained within the framework of theory of quaternion row-column determinants previously developed by the author. In consequence, these determinantal representations are derived in the case of complex matrices.

Keywords: Moore–Penrose inverse, Drazin inverse, generalized inverse, core-EP inverse, quaternion matrix, noncommutative determinant

1. Introduction

The field of complex (or real) numbers is designated by $\mathbb{C}$ ($\mathbb{R}$). The set of all $m \times n$ matrices over the quaternion skew field

$$\mathbb{H} = \left\{h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k} | \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, h_0, h_1, h_2, h_3 \in \mathbb{R}\right\},$$

is represented by $\mathbb{H}^{m\times n}$, while $\mathbb{H}_r^{m\times n}$ is reserved for the subset of $\mathbb{H}^{m\times n}$ with matrices of rank r. If $h = h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k} \in \mathbb{H}$, its conjugate is $\overline{h} = h_0 - h_1\mathbf{i} - h_2\mathbf{j} - h_3\mathbf{k}$, and its norm $\|h\| = \sqrt{h\overline{h}} = \sqrt{\overline{h}h} = \sqrt{h_0^2 + h_1^2 + h_2^2 + h_3^2}$. For $\mathbf{A} \in \mathbb{H}^{m\times n}$, its rank and conjugate transpose are given by $\mathrm{rank}(\mathbf{A})$ and $\mathbf{A}^*$, respectively. A matrix $\mathbf{A} \in \mathbb{H}^{n\times n}$ is said to be Hermitian if $\mathbf{A}^* = \mathbf{A}$. Also,

- $\mathcal{C}_r(\mathbf{A}) = \left\{\mathbf{c} \in \mathbb{H}^{m\times 1} : \mathbf{c} = \mathbf{Ad}, \mathbf{d} \in \mathbb{H}^{n\times 1}\right\}$ is the right column space of $\mathbf{A}$;
- $\mathcal{R}_l(\mathbf{A}) = \left\{\mathbf{c} \in \mathbb{H}^{1\times n} : \mathbf{c} = \mathbf{dA}, \mathbf{d} \in \mathbb{H}^{1\times m}\right\}$ is the left row space of $\mathbf{A}$;

- $\mathcal{N}_r(\mathbf{A}) = \{\mathbf{d} \in \mathbb{H}^{n\times 1} : \mathbf{Ad} = 0\}$ is the right null space of $\mathbf{A}$;
- $\mathcal{N}_l(\mathbf{A}) = \{\mathbf{d} \in \mathbb{H}^{1\times m} : \mathbf{dA} = 0\}$ is the left null space of $\mathbf{A}$.

Let us recall the definitions of some well-known generalized inverses that can be extend to quaternion matrices as follows.

Definition 1.1. *The Moore–Penrose inverse of* $\mathbf{A} \in \mathbb{H}^{n\times m}$ *is the unique matrix* $\mathbf{A}^{\dagger} = \mathbf{X}$ *determined by equations*

$$(1)\ \mathbf{AXA} = \mathbf{A};\ (2)\ \mathbf{XAX} = \mathbf{X};\ (3)\ (\mathbf{AX})^* = \mathbf{AX};\ (4)\ (\mathbf{XA})^* = \mathbf{XA}. \tag{1}$$

Definition 1.2. *The Drazin inverse of* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is the unique* $\mathbf{A}^d = \mathbf{X}$ *that satisfying Eq.(2) from* (1) *and the following equations,*

$$(5)\ \mathbf{A}^k = \mathbf{XA}^{k+1}, \quad (6)\ \mathbf{XA} = \mathbf{AX},$$

where $k = \mathrm{Ind}(\mathbf{A})$ is the index of $\mathbf{A}$, i.e. the smallest positive number such that $\mathrm{rank}\left(\mathbf{A}^{k+1}\right) = \mathrm{rank}\left(\mathbf{A}^k\right)$. If $\mathrm{Ind}(\mathbf{A}) \le 1$, then $\mathbf{A}^d = \mathbf{A}^{\#}$ is the group inverse of $\mathbf{A}$. If $\mathrm{Ind}(\mathbf{A}) = 0$, then $\mathbf{A}^{\#} = \mathbf{A}^{\dagger} = \mathbf{A}^{-1}$.

A matrix $\mathbf{A}$ satisfying the conditions $(i), (j), \ldots$ is called an $\{i, j, \ldots\}$-inverse of $\mathbf{A}$, and is denoted by $\mathbf{A}^{(i,j,\ldots)}$. In particular, $\mathbf{A}^{(1)}$ is called the *inner inverse*, $\mathbf{A}^{(2)}$ is called the *outer inverse*, and $\mathbf{A}^{(1,2)}$ is called the *reflexive inverse*, and $\mathbf{A}^{(1,2,3,4)}$ is the Moore–Penrose inverse, etc.

Note that the Moore–Penrose inverse inducts the orthogonal projectors $\mathbf{P}_A = \mathbf{AA}^{\dagger}$ and $\mathbf{Q}_A = \mathbf{A}^{\dagger}\mathbf{A}$ onto the right column spaces of $\mathbf{A}$ and $\mathbf{A}^*$, respectively.

In [1], the core-EP inverse over the quaternion skew field was presented similarly as in [2].

Definition 1.3. *The core-EP inverse of* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is the unique matrix* $\mathbf{A}^{\dagger} = \mathbf{X}$ *which satisfies*

$$\mathbf{X} = \mathbf{XAX}, \quad \mathcal{C}_r(\mathbf{X}) = \mathcal{C}_r\left(\mathbf{A}^d\right) = \mathcal{C}_r(\mathbf{X}^*).$$

According to [3], (Theorem 2.3), for $m \ge \mathrm{Ind}(\mathbf{A})$, we have that $\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^d\mathbf{A}^m(\mathbf{A}^m)^{\dagger}$. In a special case that $\mathrm{Ind}(\mathbf{A}) \le 1$, $\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^{\textcircled{\#}}$ is the core inverse of $\mathbf{A}$ [4].

Definition 1.4. *The dual core-EP inverse of* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is the unique matrix* $\mathbf{A}_{\textcircled{\dagger}} = \mathbf{X}$ *for which*

$$\mathbf{X} = \mathbf{XAX}, \quad \mathcal{R}_l(\mathbf{X}) = \mathcal{R}_l\left(\mathbf{A}^d\right) = \mathcal{R}_l(\mathbf{X}^*).$$

Recall that, $\mathbf{A}_{\textcircled{\dagger}} = (\mathbf{A}^m)^{\dagger}\mathbf{A}^m\mathbf{A}^d$ for $m \ge \mathrm{Ind}(\mathbf{A})$.

Since the quaternion core-EP inverse $\mathbf{A}^{\dagger}$ is related to the right space $\mathcal{C}_r(\mathbf{A})$ of $\mathbf{A} \in \mathbb{H}^{n\times n}$ and the quaternion dual core-EP inverse $\mathbf{A}_{\dagger}$ is related to its left space $\mathcal{R}_l(\mathbf{A})$. So, in [1], they are also named the right and left core-EP inverses, respectively.

Various representations of core-EP inverse can be found in [1, 5–7]. In [8], continuity of core-EP inverse was investigated. Bordering and iterative methods to find the core-EP inverse were proved in [9, 10], and its determinantal representation for complex matrices was derived in [2]. New determinantal representations of the complex core-EP inverse and its various generalizations were obtained in [11]. The core-EP inverse was generalized to rectangular matrices [12], Hilbert space operators [13], Banach algebra elements [14], tensors [15], and elements of rings [3]. Combining the core-EP inverse or the dual core-EP inverse with the Moore–Penrose inverse, the MPCEP inverse and CEPMP inverse were introduced in [16] for bounded linear Hilbert space operators.

In the last years, interest in quaternion matrix equations is growing significantly based on the increasing their applications in various fields, among them, robotic manipulation [17], fluid mechanics [18, 19], quantum mechanics [20–22], signal processing [23, 24], color image processing [25–27], and so on.

The main goals of this chapter are investigations of the MPCEP and CEPMP inverses, introductions and representations of new right and left MPCEPMP inverses over the quaternion skew field, and obtaining of their determinantal representations as a direct method of their constructions. The chapter develops and continues the topic raised in a number of other works [28–33], where determinantal representations of various generalized inverses were obtained.

The remainder of our chapter is directed as follows. In Section 2, we introduce of the quaternion MPCEP and CEPMP inverses and give characterizations of new generalized inverses, namely left and right MPCEPMP-inverses. In Section 3, we commence with introducing determinantal representations of the projection matrices inducted by the Moore–Penrose inverse and of core-EP inverse previously obtained within the framework of theory of quaternion row-column determinants and, based of them, determinantal representations of the MPCEP, CEPMP, and left and right MPCEPMP inverses are derived. Finally, the conclusion is drawn in Section 4.

2. Characterizations of the quaternion MPCEP, CEPMP, and MPCEPMP inverses

Analogously as in [16], the MPCEP inverse and CEPMP inverse can be defined for quaternion matrices.

Definition 2.1. *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$*. The MPCEP (or MP-Core-EP) inverse of* $\mathbf{A}$ *is the unique solution* $\mathbf{A}^{\dagger,\textcircled{\dagger}} = \mathbf{X}$ *to the system*

$$\mathbf{X} = \mathbf{X}\mathbf{A}\mathbf{X}, \qquad \mathbf{X}\mathbf{A} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}, \qquad \mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{A}^{\textcircled{\dagger}}.$$

The CEPMP (or Core-EP-MP) inverse of $\mathbf{A}$ is the unique solution $\mathbf{A}^{\textcircled{\dagger},\dagger} = \mathbf{X}$ to the system

$$\mathbf{X} = \mathbf{X}\mathbf{A}\mathbf{X}, \qquad \mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger}, \qquad \mathbf{X}\mathbf{A} = \mathbf{A}_{\textcircled{\dagger}}\mathbf{A}.$$

We can represent the MPCEP inverse and CEPMP inverse, by [16], as

$$\mathbf{A}^{\dagger,\textcircled{\dagger}} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}, \tag{2}$$

$$\mathbf{A}^{\textcircled{\dagger},\dagger} = \mathbf{A}_{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger}. \tag{3}$$

According to our concepts, we can define the left and right MPCPMP inverses.

Definition 2.2. *Suppose* $\mathbf{A} \in \mathbb{H}^{n\times n}$*. The right MPCEPMP inverse of* $\mathbf{A}$ *is defined as*

$$\mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,r} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger}.$$

The left MPCEPMP inverse of $\mathbf{A}$ is defined as

$$\mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,l} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}_{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger}.$$

The following gives the characteristic equations of these generalized inverses.

Theorem 2.3. *Let* $\mathbf{A}, \mathbf{X} \in \mathbb{H}^{n\times n}$. *The following statements are equivalent:*

i. $\mathbf{X}$ *is the right MPCEPMP inverse of* $\mathbf{A}$.

ii.
$$\mathbf{X} = \mathbf{A}^{\dagger,\textcircled{\dagger}}\mathbf{P}_{A}. \tag{4}$$

iii. $\mathbf{X}$ *is the unique solution to the three equations:*

$$1.\mathbf{X} = \mathbf{XAX}, \qquad 2.\mathbf{XA} = \mathbf{A}^{\dagger,\textcircled{\dagger}}\mathbf{A}, \qquad 3.\mathbf{AX} = \mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{P}_{\mathbf{A}}. \tag{5}$$

Proof. $[(\mathrm{i})] \mapsto [(\mathrm{ii})]$. By Eq. (2) and the denotation of $\mathbf{P}_A$, it is evident that

$$\mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,r} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger,\textcircled{\dagger}}\mathbf{P}_{A}.$$

$[(\mathrm{i})] \mapsto [(\mathrm{iii})]$. Now, we verify the condition (5). Let $\mathbf{X} = \mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,r} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger}$. Then, from the Definition 1.1 and the representation (2), we have

$$\mathbf{XAX} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}\left(\mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\dagger}\right)\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\left(\mathbf{A}\mathbf{A}^{\dagger}\mathbf{A}\right)\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger} =$$

$$= \mathbf{A}^{\dagger}\mathbf{A}\left(\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\right)\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{X},$$

$$\mathbf{XA} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\left(\mathbf{A}\mathbf{A}^{\dagger}\mathbf{A}\right) = \mathbf{A}^{\dagger,\textcircled{\dagger}}\mathbf{A},$$

$$\mathbf{AX} = \left(\mathbf{A}\mathbf{A}^{\dagger}\mathbf{A}\right)\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{P}_{A}.$$

To prove that the system (5) has unique solution, suppose that $\mathbf{X}$ and $\mathbf{X}_1$ are two solutions of this system. Then $\mathbf{XA} = \mathbf{A}^{\dagger,\textcircled{\dagger}}\mathbf{A} = \mathbf{X}_1\mathbf{A}$ and $\mathbf{AX} = \mathbf{A}\mathbf{A}^{\textcircled{\dagger}}\mathbf{P}_A = \mathbf{AX}_1$, which give $\mathbf{X}(\mathbf{AX}) = (\mathbf{XA})\mathbf{X}_1 = \mathbf{X}_1\mathbf{AX}_1 = \mathbf{X}_1$. Therefore, $\mathbf{X}$ is the unique solution to the system. □

The next theorem can be proved in the same way.

Theorem 2.4. *Let* $\mathbf{A}, \mathbf{X} \in \mathbb{H}^{n\times n}$. *The following statements are equivalent:*

i. $\mathbf{X}$ *is the left MPCEPMP inverse of* $\mathbf{A}$.

ii.
$$\mathbf{X} = \mathbf{Q}_{A}\mathbf{A}^{\textcircled{\dagger},\dagger}. \tag{6}$$

iii. $\mathbf{X}$ *is the unique solution to the system:*

$$1.\mathbf{X} = \mathbf{XAX}, \qquad 2.\mathbf{AX} = \mathbf{A}\mathbf{A}^{\textcircled{\dagger},\dagger}, \qquad 3.\mathbf{XA} = \mathbf{A}_{\textcircled{\dagger}}\mathbf{A}.$$

3. Determinantal representations of the quaternion MPCEP and *CEPMP inverses

It is well known that the determinantal representation of the regular inverse is given by the cofactor matrix. The construction of determinantal representations of generalized inverses is not so evident and unambiguous even for matrices with complex or real entries. Taking into account the noncommutativity of quaternions, this task is more complicated due to a problem of defining the determinant of a matrix with noncommutative elements (see survey articles [34–36] for detail). Only now, the solving this problem begins to be decided thanks to the theory of noncommutative column-row determinants introduced in [37, 38].

For arbitrary quaternion matrix $\mathbf{A} \in \mathbb{H}^{n\times n}$, there exists an exact technique to generate n row determinants ($\mathfrak{R}$-determinants) and n column determinants ($\mathfrak{C}$-determinants) by stating a certain order of factors in each term.

Definition 3.1. *Let* $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$.

- *For an arbitrary row index* $i \in I_n$, *the ith* $\mathfrak{R}$*-determinant of* $\mathbf{A}$ *is defined as*

$$\mathrm{rdet}_i \mathbf{A} := \sum_{\sigma \in S_n} (-1)^{n-r} \left(a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i} \right) \dots \left(a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}} \right),$$

in which S_n denotes the symmetric group on $I_n = \{1, \dots, n\}$, while the permutation σ is defined as a product of mutually disjunct subsets ordered from the left to right by the rules

$$\sigma = (i\, i_{k_1} i_{k_1+1} \dots i_{k_1+l_1})(i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}),$$
$$i_{k_t} < i_{k_t+s}, \quad i_{k_2} < i_{k_3} < \dots < i_{k_r}, \quad \forall\ t = 2, \dots, r, \quad s = 1, \dots, l_t.$$

- *For an arbitrary column index* $j \in I_n$, *the jth* $\mathfrak{C}$*-determinant of* $\mathbf{A}$ *is defined as the sum*

$$\mathrm{cdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} \left(a_{j_{k_r} j_{k_r+l_r}} \cdots a_{j_{k_r+1} j_{k_r}} \right) \cdots \left(a_{j j_{k_1+l_1}} \cdots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j} \right),$$

in which a *permutation* τ is ordered from the right to left in the following way:

$$\tau = \left(j_{k_r+l_r} \cdots j_{k_r+1} j_{k_r} \right) \cdots \left(j_{k_2+l_2} \cdots j_{k_2+1} j_{k_2} \right) \left(j_{k_1+l_1} \cdots j_{k_1+1} j_{k_1} j \right), j_{k_t} < j_{k_t+s}, \quad j_{k_2} < j_{k_3} < \dots < j_{k_r}.$$

It is known that all $\mathfrak{R}$- and $\mathfrak{C}$-determinants are different in general. However, in [37], the following equalities are verified for a Hermitian matrix $\mathbf{A}$ that introduce a determinant of a Hermitian matrix: $\mathrm{rdet}_1 \mathbf{A} = \dots = \mathrm{rdet}_n \mathbf{A} = \mathrm{cdet}_1 \mathbf{A} = \dots = \mathrm{cdet}_n \mathbf{A} := \det \mathbf{A} \in \mathbb{R}$.

$\mathfrak{D}$-Representations of various generalized inverses were developed by means of the theory of $\mathfrak{R}$- and $\mathfrak{C}$-determinants (see e.g. [28–31]).

The following notations are used for determinantal representations of generalized inverses.

Let $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$ be subsets with $1 \le k \le \min\{m, n\}$. Suppose that $\mathbf{A}^{\alpha}_{\beta}$ is a submatrix of $\mathbf{A} \in \mathbb{H}^{m\times n}$ whose rows and columns are indexed by α and β, respectively. Then, $\mathbf{A}^{\alpha}_{\alpha}$ is a principal submatrix of $\mathbf{A}$

whose rows and columns are indexed by α. If $\mathbf{A}$ is Hermitian, then $|\mathbf{A}|_{\alpha}^{\alpha}$ stands for a principal minor of $\det\mathbf{A}$. The collection of strictly increasing sequences of $1 \le k \le n$ integers chosen from $\{1, \ldots, n\}$ is denoted by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \le \alpha_1 < \cdots < \alpha_k \le n\}$. For fixed $i \in \alpha$ and $j \in \beta$, put $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}, J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

Let $\mathbf{a}_{.j}$ and $\mathbf{a}_{.j}^*$ be the jth columns, $\mathbf{a}_{i.}$ and $\mathbf{a}_{i.}^*$ be the ith rows of $\mathbf{A}$ and $\mathbf{A}^*$, respectively. Suppose that $\mathbf{A}_{i.}(\mathbf{b})$ and $\mathbf{A}_{.j}(\mathbf{c})$ stand for the matrices obtained from $\mathbf{A}$ by replacing its ith row with the row vector $\mathbf{b} \in \mathbb{H}^{1\times n}$ and its jth column with the column vector $\mathbf{c} \in \mathbb{H}^m$, respectively.

Based on determinantal representations of the Moore–Penrose inverse obtained in [28], we have determinantal representations of the projections.

Lemma 3.2. [28] *If $\mathbf{A} \in \mathbb{H}_r^{m\times n}$, then the determinantal representations of the projection matrices $\mathbf{A}^\dagger\mathbf{A} =: \mathbf{Q}_A = \left(q_{ij}^A\right)_{n\times n}$ and $\mathbf{A}\mathbf{A}^\dagger =: \mathbf{P}_A = \left(p_{ij}^A\right)_{m\times m}$ can be expressed as follows*

$$q_{ij}^A = \frac{\sum_{\beta\in J_{r,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.j})\right)_\beta^\beta}{\sum_{\beta\in J_{r,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta} = \frac{\sum_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}^*\mathbf{A})_{j.}(\dot{\mathbf{a}}_{i.})\right)_\alpha^\alpha}{\sum_{\alpha\in I_{r,n}} |\mathbf{A}^*\mathbf{A}|_\alpha^\alpha}, \tag{7}$$

$$p_{ij}^A = \frac{\sum_{\alpha\in I_{r,m}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{i.})\right)_\alpha^\alpha}{\sum_{\alpha\in I_{r,m}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha} = \frac{\sum_{\beta\in J_{r,m}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}\mathbf{A}^*)_{.i}(\ddot{\mathbf{a}}_{.j})\right)_\beta^\beta}{\sum_{\beta\in J_{r,m}} |\mathbf{A}\mathbf{A}^*|_\beta^\beta}, \tag{8}$$

where $\dot{\mathbf{a}}_{.i}$ and $\dot{\mathbf{a}}_{.j}$, $\ddot{\mathbf{a}}_{i.}$ and $\ddot{\mathbf{a}}_{.j}$ are the ith rows and the jth columns of $\mathbf{A}^*\mathbf{A} \in \mathbb{H}^{n\times n}$ and $\mathbf{A}\mathbf{A}^* \in \mathbb{H}^{m\times m}$, respectively.

Recently, $\mathfrak{D}$-representations of the quaternion core-EP inverses were obtained in [1] as well.

Lemma 3.3. [1] *Suppose that $\mathbf{A} \in \mathbb{H}^{n\times n}$, $\operatorname{Ind}(\mathbf{A}) = k$ and $\operatorname{rank}\left(\mathbf{A}^k\right) = s$. Then $\mathbf{A}^{\dagger} = \left(a_{ij}^{\dagger,r}\right)$ and $\mathbf{A}_{\dagger} = \left(a_{ij}^{\dagger,l}\right)$ possess the determinantal representations, respectively,*

$$a_{ij}^{\dagger,r} = \frac{\sum_{\alpha\in I_{s,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\hat{\mathbf{a}}_{i.})\right)_\alpha^\alpha}{\sum_{\alpha\in I_{s,n}} \left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\alpha^\alpha}, \tag{9}$$

$$a_{ij}^{\dagger,l} = \frac{\sum_{\beta\in J_{s,n}\{i\}} \operatorname{cdet}_i\left(\left(\left(\mathbf{A}^{k+1}\right)^*\mathbf{A}^{k+1}\right)_{.i}(\breve{\mathbf{a}}_{.j})\right)_\beta^\beta}{\sum_{\beta\in J_{s,n}} \left|\left(\mathbf{A}^{k+1}\right)^*\mathbf{A}^{k+1}\right|_\beta^\beta}, \tag{10}$$

where $\hat{\mathbf{a}}_{i.}$ is the ith row of $\hat{\mathbf{A}} = \mathbf{A}^k\left(\mathbf{A}^{k+1}\right)^*$ and $\breve{\mathbf{a}}_{.j}$ is the jth column of $\breve{\mathbf{A}} = \left(\mathbf{A}^{k+1}\right)^*\mathbf{A}^k$.

Theorem 3.4. *Let $\mathbf{A} \in \mathbb{H}_s^{n\times n}$, $\operatorname{Ind}(\mathbf{A}) = k$ and $\operatorname{rank}\left(\mathbf{A}^k\right) = s_1$. Then its MPCEP inverse $\mathbf{A}^{\dagger,\dagger} = \left(a_{ij}^{\dagger,\dagger}\right)$ is expressed by componentwise*

$$a_{ij}^{\dagger,\dagger} = \frac{\sum_{\alpha\in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}\left(\mathbf{v}_{i.}^{(1)}\right)\right)_\alpha^\alpha}{\sum_{\beta\in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \sum_{\alpha\in I_{s_1,n}} \left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\alpha^\alpha} \tag{11}$$

$$= \frac{\sum_{\beta\in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{u}_{.j}^{(1)}\right)\right)_\beta^\beta}{\sum_{\beta\in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \sum_{\alpha\in I_{s_1,n}} \left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\alpha^\alpha}, \tag{12}$$

where

$$\mathbf{v}_{i.}^{(1)} = \left[\sum_{\beta\in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\tilde{\mathbf{a}}_{.l})\right)_\beta^\beta\right] \in \mathbb{H}^{1\times n},\ l = 1, \dots, n, \tag{13}$$

$$\mathbf{u}_{.j}^{(1)} = \left[\sum_{\alpha\in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\tilde{\mathbf{a}}_{f.})\right)_\alpha^\alpha\right] \in \mathbb{H}^{n\times 1},\ f = 1, \dots, n,$$

and $\tilde{\mathbf{a}}_{.l}$ and $\tilde{\mathbf{a}}_{f.}$ are the lth column and the fth row of $\tilde{\mathbf{A}} = \mathbf{A}^*\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*$.

Proof. By (2), we have

$$a_{ij}^{\dagger,\dagger} = \sum_{l=1}^n q_{il} a_{lj}^{\dagger,r}. \tag{14}$$

Using (7) and (9) for the determinantal representations of $\mathbf{Q}_A = \mathbf{A}^\dagger\mathbf{A} = \left(q_{ij}\right)$ and $\mathbf{A}^\dagger$, respectively, from (14) it follows

$$a_{ij}^{\dagger,\dagger} = \sum_{l=1}^n \sum_{f=1}^n \frac{\sum_{\beta\in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.f})\right)_\beta^\beta}{\sum_{\beta\in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta} \times \frac{\sum_{\alpha\in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\hat{\mathbf{a}}_{l.})\right)_\alpha^\alpha}{\sum_{\alpha\in I_{s_1,n}} \left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\alpha^\alpha} =$$

$$= \sum_{l=1}^n \sum_{f=1}^n \frac{\sum_{\beta\in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.f})\right)_\beta^\beta}{\sum_{\beta\in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta} \tilde{a}_{fl} \frac{\sum_{\alpha\in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\mathbf{e}_{l.})\right)_\alpha^\alpha}{\sum_{\alpha\in I_{s_1,n}} \left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\alpha^\alpha},$$

where $\mathbf{e}_{.f}$ and $\mathbf{e}_{l.}$ are the fth column and the lth row of the unit matrix $\mathbf{I}_n$, $\hat{\mathbf{a}}_{l.}$ is the lth row of $\hat{\mathbf{A}} = \mathbf{A}^k\left(\mathbf{A}^{k+1}\right)^*$, and $\tilde{a}_{fl}$ is the (fl)th element of $\tilde{\mathbf{A}} = \mathbf{A}^*\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*$.

If we denote by

$$v_{il}^{(1)} := \sum_{f=1}^n \sum_{\beta\in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.f})\right)_\beta^\beta \tilde{\mathbf{a}}_{fl} = \sum_{\beta\in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\tilde{\mathbf{a}}_{.l})\right)_\beta^\beta$$

the lth component of a row-vector $\mathbf{v}_{i.}^{(1)} = \left[v_{i1}^{(1)}, \dots, v_{in}^{(1)}\right]$, then

$$\sum_{l=1}^n v_{il}^{(1)} \sum_{\alpha\in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\mathbf{e}_{l.})\right)_\alpha^\alpha$$

$$= \sum_{\alpha\in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}\left(\mathbf{v}_{i.}^{(1)}\right)\right)_\alpha^\alpha.$$

So, we have (11). By putting

$$
\begin{aligned}
u_{fj}^{(1)} &:= \sum_{l=1}^{n} \tilde{a}_{fl} \sum_{\alpha \in I_{s_1,n}\{j\}} \mathrm{rdet}_j \left(\left(\mathbf{A}^{k+1} \left(\mathbf{A}^{k+1} \right)^* \right)_{j.} (\mathbf{e}_{l.}) \right)_{\alpha}^{\alpha} \\
&= \sum_{\alpha \in I_{s_1,n}\{j\}} \mathrm{rdet}_j \left(\left(\mathbf{A}^{k+1} \left(\mathbf{A}^{k+1} \right)^* \right)_{j.} (\tilde{\mathbf{a}}_{f.}) \right)_{\alpha}^{\alpha}
\end{aligned}
$$

as the fth component of a column-vector $\mathbf{u}_{.j}^{(1)} = \left[u_{1j}^{(1)}, \dots, u_{nj}^{(1)} \right]^T$, it follows

$$
\sum_{f=1}^{n} \sum_{\beta \in J_{s,n}\{i\}} \mathrm{cdet}_i \left((\mathbf{A}^* \mathbf{A})_{.i} (\mathbf{e}_{.f}) \right)_{\beta}^{\beta} u_{fj}^{(1)} = \sum_{\beta \in J_{s,n}\{i\}} \mathrm{cdet}_i \left((\mathbf{A}^* \mathbf{A})_{.i} \left(\mathbf{u}_{.j}^{(1)} \right) \right)_{\beta}^{\beta}.
$$

Hence, we obtain (12). □

Determinantal representations of a complex MPCEP inverse are obtained by substituting row-column determinants for usual determinants in (11)–(12).

Corollary 3.5. *Let* $\mathbf{A} \in \mathbb{C}_s^{n \times n}$, $\mathrm{Ind}(\mathbf{A}) = k$ *and* $\mathrm{rank}\left(\mathbf{A}^k \right) = s_1$. *Then its MPCEP inverse* $\mathbf{A}^{\dagger,\dagger} = \left(a_{ij}^{\dagger,\dagger} \right)$ *has the following determinantal representations*

$$
\begin{aligned}
a_{ij}^{\dagger,\dagger} &= \frac{\sum_{\alpha \in I_{s_1,n}\{j\}} \left| \left(\mathbf{A}^{k+1} \left(\mathbf{A}^{k+1} \right)^* \right)_{j.} \left(\mathbf{v}_{i.}^{(1)} \right) \right|_{\alpha}^{\alpha}}{\sum_{\beta \in J_{s,n}} |\mathbf{A}^* \mathbf{A}|_{\beta}^{\beta} \sum_{\alpha \in I_{s_1,n}} \left| \mathbf{A}^{k+1} \left(\mathbf{A}^{k+1} \right)^* \right|_{\alpha}^{\alpha}} \\
&= \frac{\sum_{\beta \in J_{s,n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{.i} \left(\mathbf{u}_{.j}^{(1)} \right) \right|_{\beta}^{\beta}}{\sum_{\beta \in J_{s,n}} |\mathbf{A}^* \mathbf{A}|_{\beta}^{\beta} \sum_{\alpha \in I_{s_1,n}} \left| \mathbf{A}^{k+1} \left(\mathbf{A}^{k+1} \right)^* \right|_{\alpha}^{\alpha}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{v}_{i.}^{(1)} &= \left[\sum_{\beta \in J_{s,n}\{i\}} \left| (\mathbf{A}^* \mathbf{A})_{.i} (\tilde{\mathbf{a}}_{.l}) \right|_{\beta}^{\beta} \right] \in \mathbb{C}^{1 \times n}, \; l = 1, \dots, n, \\
\mathbf{u}_{.j}^{(1)} &= \left[\sum_{\alpha \in I_{s_1,n}\{j\}} \left| \left(\mathbf{A}^{k+1} \left(\mathbf{A}^{k+1} \right)^* \right)_{j.} (\tilde{\mathbf{a}}_{f.}) \right|_{\alpha}^{\alpha} \right] \in \mathbb{C}^{n \times 1}, \; f = 1, \dots, n,
\end{aligned} \tag{15}
$$

and $\tilde{\mathbf{a}}_{.l}$ *and* $\tilde{\mathbf{a}}_{f.}$ *are the lth column and the fth row of* $\tilde{\mathbf{A}} = \mathbf{A}^* \mathbf{A}^{k+1} \left(\mathbf{A}^{k+1} \right)^*$.

Theorem 3.6. *Let* $\mathbf{A} \in \mathbb{H}_s^{n \times n}$, $\mathrm{Ind}(\mathbf{A}) = k$ *and* $\mathrm{rank}\left(\mathbf{A}^k \right) = s_1$. *Then its CEPMP inverse* $\mathbf{A}^{\dagger,\dagger} = \left(a_{ij}^{\dagger,\dagger} \right)$ *has the following determinantal representations*

$$a_{ij}^{\dagger,\dagger} = \frac{\sum_{\alpha \in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}\mathbf{A}^*)_{j.} \left(\mathbf{v}_{i.}^{(2)} \right) \right)_\alpha^\alpha}{\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha \sum_{\beta \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right|_\beta^\beta} \tag{16}$$

$$= \frac{\sum_{\beta \in J_{s_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} \left(\mathbf{u}_{.j}^{(2)} \right) \right)_\beta^\beta}{\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha \sum_{\beta \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right|_\beta^\beta}, \tag{17}$$

where

$$\begin{aligned} \mathbf{v}_{i.}^{(2)} &= \left[\sum_{\beta \in J_{s_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} (\hat{\mathbf{a}}_{.l}) \right)_\beta^\beta \right] \in \mathbb{H}^{1\times n}, \ l = 1, \dots, n, \\ \mathbf{u}_{.j}^{(2)} &= \left[\sum_{\alpha \in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}\mathbf{A}^*)_{j.} (\hat{\mathbf{a}}_{f.}) \right)_\alpha^\alpha \right] \in \mathbb{H}^{n\times 1}, \ f = 1, \dots, n. \end{aligned} \tag{18}$$

Here $\hat{\mathbf{a}}_{.l}$ and $\hat{\mathbf{a}}_{f.}$ are the lth column and the fth row of $\hat{\mathbf{A}} = \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1}\mathbf{A}^*$.

Proof. The proof is similar to the proof of Theorem 3.4 by using the representation (3) for the CEPMP inverse.

Corollary 3.7. *Let* $\mathbf{A} \in \mathbb{C}_s^{n\times n}$, $\mathrm{Ind}(\mathbf{A}) = k$ *and* $\mathrm{rank}\left(\mathbf{A}^k\right) = s_1$. *Then its CEPMP inverse* $\mathbf{A}^{\dagger,\dagger} = \left(a_{ij}^{\dagger,\dagger} \right)$ *has the following determinantal representations*

$$a_{ij}^{\dagger,\dagger} = \frac{\sum_{\alpha \in I_{s,n}\{j\}} \left| (\mathbf{A}\mathbf{A}^*)_{j.} \left(\mathbf{v}_{i.}^{(2)} \right) \right|_\alpha^\alpha}{\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha \sum_{\beta \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right|_\beta^\beta}$$

$$= \frac{\sum_{\beta \in J_{s_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} \left(\mathbf{u}_{.j}^{(2)} \right) \right)_\beta^\beta}{\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha \sum_{\beta \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right|_\beta^\beta},$$

where

$$\begin{aligned} \mathbf{v}_{i.}^{(2)} &= \left[\sum_{\beta \in J_{s_1,n}\{i\}} \left| \left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} (\hat{\mathbf{a}}_{.l}) \right|_\beta^\beta \right] \in \mathbb{C}^{1\times n}, \ l = 1, \dots, n, \\ \mathbf{u}_{.j}^{(2)} &= \left[\sum_{\alpha \in I_{s,n}\{j\}} \left| (\mathbf{A}\mathbf{A}^*)_{j.} (\hat{\mathbf{a}}_{f.}) \right|_\alpha^\alpha \right] \in \mathbb{C}^{n\times 1}, \ f = 1, \dots, n. \end{aligned} \tag{19}$$

Here $\hat{\mathbf{a}}_{.l}$ and $\hat{\mathbf{a}}_{f.}$ are the lth column and the fth row of $\hat{\mathbf{A}} = \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1}\mathbf{A}^*$.

Theorem 3.8. *Let* $\mathbf{A} \in \mathbb{H}_s^{n\times n}$, $\mathrm{Ind}(\mathbf{A}) = k$ *and* $\mathrm{rank}\left(\mathbf{A}^k\right) = s_1$. *Then its right MPCEPMP inverse* $\mathbf{A}^{\dagger,\dagger,\dagger,r} = \left(a_{ij}^{\dagger,\dagger,\dagger,r} \right)$ *has the following determinantal representations*

$$a_{ij}^{\dagger,\dagger,\dagger,r} = \frac{\sum_{\alpha \in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}\mathbf{A}^*)_{j.} \left(\phi_{i.}^{(1)} \right) \right)_\alpha^\alpha}{\left(\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha \right)^2 \sum_{\beta \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right|_\beta^\beta} = \tag{20}$$

$$= \frac{\sum_{\beta \in J_{s_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} \left(\psi_j^{(1)} \right) \right)_\beta^\beta}{\left(\sum_{\beta \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\beta^\beta \right)^2 \sum_{\alpha \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right|_\alpha^\alpha}, \tag{21}$$

where

$$\phi_{i.}^{(1)} = \left[\sum_{\beta \in J_{s_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} (\hat{\mathbf{u}}_{.l}) \right)_\beta^\beta \right] \in \mathbb{H}^{1 \times n}, \ l = 1, \dots, n,$$

$$\psi_j^{(1)} = \left[\sum_{\alpha \in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}\mathbf{A}^*)_{j.} (\hat{\mathbf{u}}_{f.}) \right)_\alpha^\alpha \right] \in \mathbb{H}^{n \times 1}, \ f = 1, \dots, n.$$

Here $\hat{\mathbf{u}}_{.l}$ *and* $\hat{\mathbf{u}}_{f.}$ *are the lth column and the fth row of* $\hat{\mathbf{U}} = \mathbf{U}_2\mathbf{A}\mathbf{A}^*$*, and the matrix* $\mathbf{U}_2$ *is constructed from the columns* (18).

Proof. Owing to (4), we have

$$a_{ij}^{\dagger,\dagger,\dagger,r} = \sum_{t=1}^n a_{it}^{\dagger,\dagger} p_{tj}. \tag{22}$$

Applying (8) for the determinantal representation of $\mathbf{P}_A = \mathbf{A}\mathbf{A}^\dagger = \left(p_{ij} \right)$ and (17) for the determinantal representation of $\mathbf{A}^{\dagger,\dagger}$ in (22), we obtain

$$a_{ij}^{\dagger,\dagger,\dagger,r} = \sum_{t=1}^n \frac{\sum_{\beta \in J_{s_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} \left(\mathbf{u}_{.t}^{(2)} \right) \right)_\beta^\beta}{\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha \sum_{\beta \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right|_\beta^\beta} \times \frac{\sum_{\alpha \in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}\mathbf{A}^*)_{j.} (\ddot{\mathbf{a}}_{t.}) \right)_\alpha^\alpha}{\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha} =$$

$$= \sum_{l=1}^n \sum_{f=1}^n \frac{\sum_{\beta \in J_{s_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} (\mathbf{e}_{.f}) \right)_\beta^\beta}{\sum_{\beta \in J_{s_1,n}} \left| \left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right|_\beta^\beta} \hat{u}_{fl} \times \frac{\sum_{\alpha \in I_{s,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}\mathbf{A}^*)_{j.} (\mathbf{e}_{l.}) \right)_\alpha^\alpha}{\left(\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_\alpha^\alpha \right)^2},$$

where $\mathbf{e}_{.f}$ and $\mathbf{e}_{l.}$ are the fth column and the lth row of the unit matrix $\mathbf{I}_n$, and $\hat{u}_{fl}$ is the (fl)th element of $\hat{\mathbf{U}} = \mathbf{U}_2\mathbf{A}\mathbf{A}^*$. The matrix $\mathbf{U}_2 = \left[\mathbf{u}_{.1}^{(2)}, \dots, \mathbf{u}_{.n}^{(2)} \right]$ is constructed from the columns (18). If we denote by

$$\phi_{il}^{(1)} \coloneqq \sum_{f=1}^n \sum_{\beta \in J_{s_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} (\mathbf{e}_{.f}) \right)_\beta^\beta \hat{u}_{fl}$$

$$= \sum_{\beta \in J_{s_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\left(\mathbf{A}^{k+1} \right)^* \mathbf{A}^{k+1} \right)_{.i} (\hat{\mathbf{u}}_{.l}) \right)_\beta^\beta$$

the lth component of a row-vector $\phi_{i.}^{(1)} = \left[\phi_{i1}^{(1)}, \dots, \phi_{in}^{(1)}\right]$, then

$$\sum_{l=1}^{n} \phi_{il}^{(1)} \sum_{\alpha \in I_{s,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}(\mathbf{e}_{l.})\right)_{\alpha}^{\alpha} = \sum_{\alpha \in I_{s,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\phi_{i.}^{(1)}\right)\right)_{\alpha}^{\alpha}.$$

Therefore, (20) holds.

By putting

$$\psi_{fj}^{(1)} := \sum_{l=1}^{n} \hat{a}_{fl} \sum_{\alpha \in I_{s,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}(\mathbf{e}_{l.})\right)_{\alpha}^{\alpha} = \sum_{\alpha \in I_{s,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}(\hat{\mathbf{u}}_{f.})\right)_{\alpha}^{\alpha}$$

as the fth component of a column-vector $\psi_{j}^{(1)} = \left[\psi_{1j}^{(1)}, \dots, \psi_{nj}^{(1)}\right]^T$, it follows

$$\sum_{f=1}^{n} \sum_{\beta \in J_{s_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\left(\mathbf{A}^{k+1}\right)^* \mathbf{A}^{k+1}\right)_{.i}(\mathbf{e}_{.f})\right)_{\beta}^{\beta} \psi_{fj}^{(1)} = \sum_{\beta \in J_{s_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\left(\mathbf{A}^{k+1}\right)^* \mathbf{A}^{k+1}\right)_{.i}\left(\psi_{.j}^{(1)}\right)\right)_{\beta}^{\beta}.$$

Thus, Eq. (21) holds.

Corollary 3.9. *Let* $\mathbf{A} \in \mathbb{C}_s^{n\times n}$, $\operatorname{Ind}(\mathbf{A}) = k$ *and* $\operatorname{rank}\left(\mathbf{A}^k\right) = s_1$. *Then its right MPCEPMP inverse* $\mathbf{A}^{\dagger,\dagger,\dagger,r} = \left(a_{ij}^{\dagger,\dagger,\dagger,r}\right)$ *has the following determinantal representations*

$$a_{ij}^{\dagger,\dagger} = \frac{\sum_{\alpha \in I_{s,n}\{j\}} \left|\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\phi_{i.}^{(1)}\right)\right|_{\alpha}^{\alpha}}{\left(\sum_{\alpha \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_{\alpha}^{\alpha}\right)^2 \sum_{\beta \in J_{s_1,n}} \left|\left(\mathbf{A}^{k+1}\right)^* \mathbf{A}^{k+1}\right|_{\beta}^{\beta}}$$

$$= \frac{\sum_{\beta \in J_{s_1,n}\{i\}} \left|\left(\left(\mathbf{A}^{k+1}\right)^* \mathbf{A}^{k+1}\right)_{.i}\left(\psi_{.j}^{(1)}\right)\right|_{\beta}^{\beta}}{\left(\sum_{\beta \in I_{s,n}} |\mathbf{A}\mathbf{A}^*|_{\beta}^{\beta}\right)^2 \sum_{\alpha \in J_{s_1,n}} \left|\left(\mathbf{A}^{k+1}\right)^* \mathbf{A}^{k+1}\right|_{\alpha}^{\alpha}},$$

where

$$\phi_{i.}^{(1)} = \left[\sum_{\beta \in J_{s_1,n}\{i\}} \left|\left(\left(\mathbf{A}^{k+1}\right)^* \mathbf{A}^{k+1}\right)_{.i}(\hat{\mathbf{u}}_{.l})\right|_{\beta}^{\beta}\right] \in \mathbb{C}^{1\times n},\ l = 1, \dots, n,$$

$$\psi_{.j}^{(1)} = \left[\sum_{\alpha \in I_{s,n}\{j\}} \left|\left(\mathbf{A}\mathbf{A}^*\right)_{j.}(\hat{\mathbf{u}}_{f.})\right|_{\alpha}^{\alpha}\right] \in \mathbb{C}^{n\times 1},\ f = 1, \dots, n.$$

Here $\hat{\mathbf{u}}_{.l}$ *and* $\hat{\mathbf{u}}_{f.}$ *are the lth column and the fth row of* $\hat{\mathbf{U}} = \mathbf{U}_2\mathbf{A}\mathbf{A}^*$, *and the matrix* $\mathbf{U}_2$ *is constructed from the columns* (19).

Theorem 3.10. *Let* $\mathbf{A} \in \mathbb{H}_s^{n\times n}$, $\operatorname{Ind}(\mathbf{A}) = k$ *and* $\operatorname{rank}\left(\mathbf{A}^k\right) = s_1$. *Then its left MPCEPMP inverse* $\mathbf{A}^{\dagger,\dagger,\dagger,l} = \left(a_{ij}^{\dagger,\dagger,\dagger,l}\right)$ *has the following determinantal representations*

$$a_{ij}^{\dagger,\dagger,\dagger,l}=\frac{\sum_{\alpha\in I_{s_1,n}\{j\}}\mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}\left(\phi_{i.}^{(2)}\right)\right)_\alpha^\alpha}{\left(\sum_{\beta\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_\beta^\beta\right)^2\sum_{\alpha\in I_{s_1,n}}\left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\alpha^\alpha}= \tag{23}$$

$$=\frac{\sum_{\beta\in J_{s,n}\{i\}}\mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\psi_{.j}^{(2)}\right)\right)_\beta^\beta}{\left(\sum_{\beta\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_\beta^\beta\right)^2\sum_{\alpha\in I_{s_1,n}}\left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\alpha^\alpha}, \tag{24}$$

where

$$\phi_{i.}^{(2)}=\left[\sum_{\beta\in J_{s,n}\{i\}}\mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\tilde{\mathbf{v}}_{.t})\right)_\beta^\beta\right]\in\mathbb{H}^{1\times n},\quad t=1,\dots,n,$$

$$\psi_{.j}^{(2)}=\left[\sum_{\alpha\in I_{s_1,n}\{j\}}\mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\tilde{\mathbf{v}}_{f.})\right)_\alpha^\alpha\right]\in\mathbb{H}^{n\times 1},\quad f=1,\dots,n,$$

and $\tilde{\mathbf{v}}_{.l}$ *and* $\tilde{\mathbf{v}}_{f.}$ *are the lth column and the fth row of* $\tilde{\mathbf{V}}=\mathbf{A}^*\mathbf{A}\mathbf{V}_1$*, where the matrix* $\mathbf{V}_1$ *is determined from the rows* (13).

Proof. Due to (6),

$$a_{ij}^{\dagger,\dagger,\dagger,l}=\sum_{t=1}^n q_{it}a_{tj}^{\dagger,\dagger}. \tag{25}$$

Using (7) for the determinantal representation of $\mathbf{Q}_A=\mathbf{A}^\dagger\mathbf{A}=\left(q_{ij}\right)$ and (9) for the determinantal representation of $\mathbf{A}^{\dagger,\dagger}$ in (14), we obtain

$$\begin{aligned}a_{ij}^{\dagger,\dagger,\dagger,l}&=\sum_{t=1}^n\frac{\sum_{\beta\in J_{s,n}\{i\}}\mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.t})\right)_\beta^\beta}{\sum_{\beta\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_\beta^\beta}\times\frac{\sum_{\alpha\in I_{s_1,n}\{j\}}\mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}\left(\mathbf{v}_{t.}^{(1)}\right)\right)_\alpha^\alpha}{\sum_{\beta\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_\beta^\beta\sum_{\alpha\in I_{s_1,n}}\left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\alpha^\alpha}\\&=\sum_{t=1}^n\sum_{f=1}^n\frac{\sum_{\beta\in J_{s,n}\{i\}}\mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.f})\right)_\beta^\beta}{\left(\sum_{\beta\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_\beta^\beta\right)^2}\tilde{v}_{ft}\\&\times\frac{\sum_{\alpha\in I_{s_1,n}\{j\}}\mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\mathbf{e}_{t.})\right)_\alpha^\alpha}{\sum_{\alpha\in I_{s_1,n}}\left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\alpha^\alpha},\end{aligned}$$

where $\tilde{v}_{ft}$ is the (ft)th element of $\tilde{\mathbf{V}}=\mathbf{A}^*\mathbf{A}\mathbf{V}_1$ and the matrix $\mathbf{V}_1$ is constructed from the rows (13). If we put

$$\begin{aligned}\phi_{it}^{(2)}&:=\sum_{f=1}^n\sum_{\beta\in J_{s,n}\{i\}}\mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{.f})\right)_\beta^\beta\tilde{v}_{ft}\\&=\sum_{\beta\in J_{s,n}\{i\}}\mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\tilde{\mathbf{v}}_{.t})\right)_\beta^\beta\end{aligned}$$

as the lth component of a row-vector $\phi_{i.}^{(2)} = \left[\phi_{i1}^{(2)}, \dots, \phi_{in}^{(2)}\right]$, then

$$\sum_{t=1}^{n} \phi_{it}^{(2)} \sum_{\alpha \in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\mathbf{e}_{t.})\right)_{\alpha}^{\alpha}$$
$$= \sum_{\alpha \in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}\left(\phi_{i.}^{(2)}\right)\right)_{\alpha}^{\alpha},$$

then Eq. (23) holds. If we denote by

$$\psi_{fj}^{(2)} := \sum_{t=1}^{n} \tilde{u}_{ft} \sum_{\alpha \in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\mathbf{e}_{t.})\right)_{\alpha}^{\alpha} =$$
$$= \sum_{\alpha \in I_{s_1,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\tilde{\mathbf{u}}_{f.})_{\alpha}^{\alpha}\right.$$

the fth component of a column-vector $\psi_{.j}^{(2)} = \left[\psi_{1j}^{(2)}, \dots, \psi_{nj}^{(2)}\right]^T$, then

$$\sum_{f=1}^{n} \sum_{\beta \in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{e}_{f})\right)_{\beta}^{\beta} \psi_{fj}^{(2)} = \sum_{\beta \in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\psi_{.j}^{(2)}\right)\right)_{\beta}^{\beta}.$$

Hence, we obtain (24). □

Corollary 3.11. *Let* $\mathbf{A} \in \mathbb{C}_s^{n\times n}$, $\mathrm{Ind}(\mathbf{A}) = k$ *and* $\mathrm{rank}\left(\mathbf{A}^k\right) = s_1$. *Then its left MPCEPMP inverse* $\mathbf{A}^{\dagger,\dagger,\dagger,l} = \left(a_{ij}^{\dagger,\dagger,\dagger,l}\right)$ *has the following determinantal representations*

$$a_{ij}^{\dagger,\dagger,\dagger,l} = \frac{\sum_{\alpha \in I_{s_1,n}\{j\}} \left|\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}\left(\phi_{i.}^{(2)}\right)\right|_{\alpha}^{\alpha}}{\left(\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta}\right)^2 \sum_{\alpha \in I_{s_1,n}} \left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_{\alpha}^{\alpha}} =$$
$$= \frac{\sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}\left(\psi_{.j}^{(2)}\right)\right|_{\beta}^{\beta}}{\left(\sum_{\beta \in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta}\right)^2 \sum_{\alpha \in I_{s_1,n}} \left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_{\alpha}^{\alpha}},$$

where

$$\phi_{i.}^{(2)} = \left[\sum_{\beta \in J_{s,n}\{i\}} \left|(\mathbf{A}^*\mathbf{A})_{.i}(\tilde{\mathbf{v}}_{.t})\right|_{\beta}^{\beta}\right] \in \mathbb{C}^{1\times n}, \quad t = 1, \dots, n,$$
$$\psi_{.j}^{(2)} = \left[\sum_{\alpha \in I_{s_1,n}\{j\}} \left|\left(\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right)_{j.}(\tilde{\mathbf{v}}_{f.})\right|_{\alpha}^{\alpha}\right] \in \mathbb{C}^{n\times 1}, \quad f = 1, \dots, n,$$

and $\tilde{\mathbf{v}}_{.t}$ *and* $\tilde{\mathbf{v}}_{f.}$ *are the tth column and the fth row of* $\tilde{\mathbf{V}} = \mathbf{A}^*\mathbf{A}\mathbf{V}_1$, *where the matrix* $\mathbf{V}_1$ *is determined by* (13).

4. Conclusions

In this chapter, notions of the MPCEP and CEPMP inverses are extended to quaternion matrices, and the new right and left MPCEPMP inverses are introduced and their characterizations are explored. Their determinantal representations are obtained within the framework of the theory of noncommutative column-row determinants previously introduced by the author. Also, determinantal representations of these generalized inverses for complex matrices are derived by using regular determinants. The obtained determinantal representations give new direct methods of calculations of these generalized inverses.

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