Chapter

Existence, Uniqueness and Stability of Fractional Order Stochastic Delay System

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Abstract

This chapter deals with the problem of fractional higher-order stochastic delay systems. A solution representation is given by using sin and cos matrix functions for different delay intervals. Further, existence and uniqueness results are proved through fixed point theorem. Moreover, finite-time stability criteria are obtained using fractional Gronwall-Bellman inequality lemma. Finally, numerical simulation is carried out to check the proposed theoretical results.

Keywords: existence and uniqueness of solution, fixed point theorem, fractional differential equations (FDEs), stochastic differential system

1. Introduction

Fractional derivatives (FD) initiative concept is quite old and its history spans three centuries. The variety of papers dedicated to FD is multiplied swiftly in the mid-twentieth century and later decades. One of the motives for the full-size interest within the discipline of FD is that it's far feasible to describe the variety of physical [1], synthetic [2], and organic [3] occurrence with fractional differential equations (FDEs). As a new branch of applied mathematics, the field of FD can be seen in many applications. Nevertheless, more and more compelling implementations have been found in various engineering and science fields over the past few decades (see [4]). It is noted that the existing theory of FDEs is committed to a larger part of the research works (see [5–8]). While modeling functional structures, ambient noise and time delays need to be taken into account, which might be very beneficial in building extra sensible fashions of sciences, and so on [9]. It is referred to as the pattern direction houses of the stochastic fractional partial differential gadget powered by way of area time white noise [10].

The problems in a stochastic environment replicate the modeling of single-sever m-mode random queues in computer networks [11], the spatial distribution of mobile users in the telecommunications network coverage area [12], and other anomalies that occurred naturally in many disciplines [13]. Authors in [14] investigated the existence, uniqueness, and large deviation principle solutions to stochastic evolution equations of jump type. Among the many meaningful properties of stochastic stability results describe the maximum vital feature of fractional order stochastic systems and have been investigated in Refs. [15–18]. The notion of finite-time stability for fractional stochastic delay systems occurs a matter of course in stochastic control systems. Without any doubt that this type of fractional stochastic stability is most important in both theory and applications.

However, only few introductions and discussions exist on the definition of finite-time stability in stochastic finite space using fixed point theorem approach. Burton [19] started to analyze the stability characters of dynamical systems broadly using fixed point theorems. Subsequently, few authors applied fixed point approach to establish sufficient conditions for stability of the differential systems (see [20–24]). Based on the above discussions, this chapter provides finite-time stability of the Caputo sense FDEs via fixed point theorems.

The primary contribution of this chapter is defined as follows:

i. A fractional higher-order stochastic delay system (FSDS) is considered in finite-dimensional stochastic settings.

ii. Weaker hypothesis on nonlinear terms and appropriate fixed-point analysis are utilized to obtain the existence and uniqueness of solution.

iii. A new set of generalized sufficient conditions for finite-time stability of a certain FSDS is established by using Generalized Gronwall-Bellman inequality.

Novelties and challenges of this chapter are described through the subsequent statements:

i. Finite-time stability analysis for FSDS is new in literature of finite-dimensional fractional stochastic settings.

ii. It is a challenge to tackle the proposed system with a norm estimation on nonlinear stochastic terms as described in this chapter.

iii. It is more complex to verify the weaker assumptions of the system and the derived result is new, has not been analyzed with the existing literature.

iv. Obtained result is proved in stochastic nature with square norm settings.

Organization of this chapter is as follows: system description and preliminaries are provided in Section 2. Existence and uniqueness of solution are provided in Section 3. Finite-time stability result is proved in Section 4 and Section 5 consists of a numerical example.

Notations: ${}^{C}D^{q}_{-\kappa^{+}}$ represent respectively the Caputo derivative with $q \in (0,1)$; R^n and $R^{n\times n}$ represent the $n-$dimensional Euclidean space and $n \times n$ real matrix; $E(\cdot)$ denotes the mathematical expectation with some probability measure; $\Omega = \left(L^2_{F_0}([0,b], R^n), \|\cdot\|\right)$; for any $y \in R^n$, we define the norm

$$\|y(\eta)\| = \sqrt{\sup_{\eta\in[-\kappa,b]}\left\{e^{-2N\eta}|y(\eta)|^2\right\}};$$

define a column-wise matrix sum

$$\|M\| = \max\left\{\sum_{k=1}^{j}|m_{k1}|, \sum_{k=1}^{j}|m_{k2}|, \dots, \sum_{k=1}^{j}|m_{kjn}|\right\}.$$

Further, let as define the matrix norm

$$\max_{\eta \in [-\kappa, 0]} \left\{ e^{-2N\eta} |\psi(\eta)|^2 \right\} = \|\psi\|^2, \quad \max_{\eta \in [-\kappa, 0]} \left\{ e^{-2N\eta} |\psi'(\eta)|^2 \right\} = \|\psi'\|^2.$$

2. System description and preliminaries

Consider the following system:

$$\begin{cases} {}^C D^q_{-\kappa^+} \left({}^C D^q_{-\kappa^+} y \right)(\eta) + M^2 y(\eta - \kappa) = F(\eta, y(\eta)) + \int_0^\eta \Delta(\zeta, y(\zeta)) dw(\zeta), & \eta \in [0, b], \quad \kappa > 0, \\ y(\eta) = \psi(\eta), \quad y'(\eta) = \psi'(\eta), \quad \eta \in [-\kappa, 0], \quad \kappa > 0, \end{cases} \tag{1}$$

where $y(\eta) \in R^n$ is a state vector. Here, $M \in R^{n \times n}$ is taken as a nonsingular matrix. F is mapping from $[0, b] \times R^n$ to R^n and Δ is a mapping from $[0, b] \times R^n$ to $R^{n \times d}$ are nonlinear continuous function and $\psi \in \mathcal{C}^1([-\kappa, 0], R^n)$ is an initial value function. w denotes $d-$dimensional Wiener process.

Definition 2.1. ([5]) *The Caputo derivative for* $f : [-\kappa, \infty) \to R$, *is*

$$\left({}^C D^q_{-\kappa^+} f \right)(\eta) = \frac{1}{\Gamma(1-q)} \int_{-\kappa}^{\eta} (\eta - \zeta)^{-q} f'(\zeta) d\zeta, \quad q \in (0, 1], \quad \eta > -\kappa, \quad f'(\eta) = \frac{df}{d\eta}.$$

Definition 2.2. *(see* [5]*) Mittag-Leffler function is*

$$E_{q,p}(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(kq + p)} \quad \text{for } q, p > 0.$$

In particular, for $p = 1$,

$$E_{q,1}(\theta u^q) = E_q(\theta u^q) = \sum_{k=0}^{\infty} \frac{\theta^k u^{kq}}{\Gamma(qk + 1)}, \quad \theta, u \in \mathbb{C}.$$

Definition 2.3. *(see* [25]*) The* $2kq$ *degree of polynomial for delayed fractional* cos *matrix is given at* $\eta = k\kappa$, $k = 0, 1, \cdots$

$$\cos_{\kappa,q} M\eta^q = \begin{cases} \Theta, & -\infty < \eta < -\kappa, \\ I, & -\kappa \le \eta < 0, \\ \vdots & \vdots \\ I - M^2 \dfrac{\eta^{2q}}{\Gamma(2q+1)} + \cdots\cdots + (-1)^k M^{2k} \dfrac{(\eta - (k-1)\kappa)^{2k\kappa}}{\Gamma(2kq+1)}, & (k-1)\kappa \le \eta < k\kappa, \\ \vdots & \vdots \end{cases}$$

where Θ and I represent the zero and identity matrices.

Definition 2.4. *(*[25]*) The* $(2k+1)q$ *degree of polynomial for a delayed fractional* sin *matrix is given at* $\eta = k\kappa$, $k = 0, 1, \cdots$

$$\sin_{\kappa,q} M\eta^q = \begin{cases} \Theta, & -\infty < \eta < -\kappa, \\ M\dfrac{(\eta+\kappa)^q}{\Gamma(q+1)}, & -\kappa \le \eta < 0, \\ \vdots & \vdots \\ M\dfrac{(\eta+\kappa)^q}{\Gamma(q+1)} + \cdots\cdots + (-1)^k M^{2k+1}\dfrac{(\eta-(k-1)\kappa)^{(2k+1)q}}{\Gamma[(2k+1)q+1]}, & (k-1)\kappa \le \eta < k\kappa. \\ \vdots & \vdots \end{cases}$$

We have the following square norm estimations:

i. $\| \cos_{\kappa,q} M\eta^q \|^2 = \left(\sum_{k=0}^{\infty} \frac{\left(\|M\|\eta^{2q}\right)^k}{\Gamma(2kq+1)} \right)^2 \le \left[\mathbb{E}_{2q}\left(\|M\|^2\eta^{2q}\right)\right]^2, \ \ \eta \in [(k-1)\kappa, k\kappa),$ $k = 0, 1, 2, \cdots n$

ii.

$$\begin{aligned} &\| \sin_{\kappa,q} M\eta^q \|^2 \\ &= \sum_{k=0}^{\infty} \frac{(\|M\|(\eta+\kappa)^q)^{2k}}{(\Gamma(kq+1))^2} + \sum_{k=0}^{\infty} \frac{\left(\|M\|^2(\eta+\kappa)^{2q}\right)^{2k}}{(\Gamma(2kq+1))^2} \\ &\quad +2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(\|M\|(\eta+\kappa)^q)^{k_1}}{\Gamma(k_1 q+1)} \frac{\left(\|M\|^2(\eta+\kappa)^{2q}\right)^{k_2}}{\Gamma(2k_2 q+1)} \\ &\le \left[\mathbb{E}_q(\|M\|(\eta+\kappa)^q)\right]^2 + \left[\mathbb{E}_{2q}\left(\|M\|^2(\eta+\kappa)^{2q}\right)\right]^2 \\ &\quad +2\mathbb{E}_q(\|M\|(\eta+\kappa)^q)\mathbb{E}_{2q}\left(\|M\|^2(\eta+\kappa)^{2q}\right), \ \ \eta \in [(k-1)\kappa, k\kappa), \ \ k = 0, 1, 2, \cdots n. \end{aligned}$$

Definition 2.5. *System* (1) *satisfying* $y(\eta) \equiv \psi(\eta)$ *and* $y'(\eta) \equiv \psi'(\eta)$ *for* $-\kappa \le \eta \le 0$ *is finite-time stable in mean square with respect to* $\{0, [0, b], \delta, \varepsilon, \kappa\}$, *if and only if* $\delta_1 < \delta \ \ (\delta > 0)$ *implies* $E\|y(\eta)\|^2 < \varepsilon \ \ (\varepsilon > 0), \ \ \forall \ \ \eta \in [0, b]$ *where* $\delta_1 = \max\left\{\|\psi\|^2, \|\psi'\|^2\right\}$ *denotes the initial time of observation of the system.*

Lemma 2.1. *[26] (Generalized Gronwall-Bellman inequality) Let* $v(\eta), \ \ b(\eta)$ *be nonnegative and locally integrable on* $0 \le \eta < b$ *and let* $h(\eta)$ *be a nonnegative, nondecreasing continuous function defined on* $0 \le \eta < b, \ \ h(\eta) \le M,$ *and let M be a real constant,* $q > 0$ *with*

$$v(\eta) \le b(\eta) + h(\eta)\int_0^{\eta} (\eta-\zeta)^{q-1} v(\zeta) d\zeta$$

and then

$$v(\eta) \le b(\eta) + \int_0^{\eta} \left[\sum_{k=1}^{\infty} \frac{(h(\eta)\Gamma(q))^k}{\Gamma(kq)} (\eta-\zeta)^{kq-1} v(\zeta) d\zeta \right].$$

Moreover, if $b(\eta)$ *is a nondecreasing function on* $[0, b]$. *Then*

$$v(\eta) \le b(\eta) E_{q,1}(h(\eta)\Gamma(q)\eta^q), \ \ \eta \in [0, b],$$

where $E_{q,1}(\cdot)$ is the one parameter Mittag-Leffler function.

Assumption 1: Let $x, y \in R^n$, then we take

$$\sup_{\eta \in [-\kappa, b]} e^{-2N\eta}|x(\eta) - y(\eta)|^2 = E\|x(\eta) - y(\eta)\|^2.$$

Lemma 2.2. *For a nonsingular matrix M, the solution of the inhomogeneous system is*

$$\begin{cases} {}^C D^q_{-\kappa^+}\left({}^C D^q_{-\kappa^+} y\right)(\eta) + M^2 y(\eta - \kappa) = f(\eta), \ \eta \in [0, b], \quad \kappa > 0, \\ y(\eta) = \psi(\eta), \\ y'(\eta) = \psi'(\eta), \quad \eta \in [-\kappa, 0], \end{cases} \tag{2}$$

for zero initial value has the below form:

$$y(\eta) = \int_0^\eta \cos_{\kappa,q} M(\eta - \kappa - \zeta)^q f(\zeta) d\zeta, \quad \eta \in [0, b].$$

Proof. Consider

$$y(\eta) = \int_0^\eta \cos_{\kappa,q} M(\eta - \kappa - \zeta)^q C(\zeta) d\zeta$$

where $C(\zeta)$ (unknown) $\zeta \in [0, \eta]$. By applying ${}^C D^q_{-\kappa^+}\left({}^C D^q_{-\kappa^+}\right)$ on both sides of the above equation one can obtain

$$\begin{aligned} {}^C D^q_{-\kappa^+}\left({}^C D^q_{-\kappa^+} y\right)(\eta) &= \left(\cos_{\kappa,q} M\eta^q\right) C(\eta) - M^2 \int_0^\eta \cos_{\kappa,q} M(\eta - 2\kappa - \zeta)^q C(\zeta) d\zeta \\ &= C(\eta) - M^2 \int_0^\eta \cos_{\kappa,q} M(\eta - 2\kappa - \zeta)^q C(\zeta) d\zeta \\ &\quad + M^2 \int_0^{\eta - \kappa} \cos_{\kappa,q} M(\eta - 2\kappa - \zeta)^q C(\zeta) d\zeta. \end{aligned}$$

Substitute the above expression into (2), one can get

$$C(\eta) - M^2 \int_0^\eta \cos_{\kappa,q} M(\eta - 2\kappa - \zeta)^q C(\zeta) d\zeta + M^2 \int_0^{\eta - \kappa} \cos_{\kappa,q} M(\eta - 2\kappa - \zeta)^q C(\zeta) d\zeta = f(\eta),$$

since $\int_{\eta-\kappa}^{\eta} \cos_{\kappa,q} M(\eta - 2\kappa - \zeta)^q C(\zeta) d\zeta = 0$. Hence the proof. □

Using [25] and Lemma 2.2, the solution of (1) is

$$\begin{aligned} y(\eta) &= \left(\cos_{\kappa,q} M\eta^q\right)\psi(-\kappa) + M^{-1}\left(\sin_{\kappa,q} M(\eta - \kappa)^q\right)\psi'(0) + \int_{-\kappa}^{0} \cos_{\kappa,q} M(\eta - \kappa - \zeta)^q \psi'(\zeta) d\zeta \\ &+ \int_0^\eta \cos_{\kappa,q} M(\eta - \kappa - \zeta)^q F(\zeta, y(\zeta)) d\zeta \\ &+ \int_0^\eta \cos_{\kappa,q} M(\eta - \kappa - \zeta)^q \left(\int_0^\zeta \Delta(\lambda, y(\lambda)) dw(\lambda)\right) d\zeta. \end{aligned}$$

Define the nonlinear operator $\mathcal{P} : R^n \to R^n$ by

$$(\mathcal{P}y)(\eta) = \left(\cos_{\kappa,q} M\eta^q\right)\psi(-\kappa) + M^{-1}\left(\sin_{\kappa,q} M(\eta-\kappa)^q\right)\psi'(0)$$
$$+\int_{-\kappa}^{0} \cos_{\kappa,q} M(\eta-\kappa-\zeta)^q \psi'(\zeta)d\zeta + \int_0^{\eta} \cos_{\kappa,q} M(\eta-\kappa-\zeta)^q F(\zeta, y(\zeta))d\zeta$$
$$+\int_0^{\eta} \cos_{\kappa,q} M(\eta-\kappa-\zeta)^q \left(\int_0^{\zeta} \Delta(\lambda, y(\lambda))dw(\lambda)\right)d\zeta, \quad \eta \in [0b].$$

3. Existence and uniqueness results

Theorem 3.1. *Assume that Assumption 1 hold. Then the system* (1) *has a unique solution and following inequality is satisfied*

$$K := 2\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2 \left[\left(\frac{e^{-Nb}-1}{Nb}\right)^2 + 4b\left(\frac{e^{-2Nb}-1}{2Nb}\right)\right] < 1.$$

Proof. Let $x, y \in R^n$. From Assumption 1 for each $\eta \in [0, b]$, we have

$$|(\mathcal{P}x)(\eta) - (\mathcal{P}y)(\eta)|^2 \le 2\left|\int_0^{\eta} \cos_{\kappa,q} M(\eta-\kappa-\zeta)^q [F(\zeta, x(\zeta)) - F(\zeta, y(\zeta))]d\zeta\right|^2$$
$$+2\left|\int_0^{\eta} \cos_{\kappa,q} M(\eta-\kappa-\zeta)^q \left(\int_0^{\zeta} [\Delta(\lambda, x(\lambda)) - \Delta(\lambda, y(\lambda))]dw(\lambda)\right)d\zeta\right|^2.$$

Multiply by $e^{-2N\eta}$ on both sides, we get

$$e^{-2N\eta}|(\mathcal{P}x)(\eta) - (\mathcal{P}y)(\eta)|^2$$
$$\le 2\left|\int_0^{\eta} \cos_{\kappa,q} M(\eta-\kappa-\zeta)^q e^{-N\zeta}[F(\zeta, x(\zeta)) - F(\zeta, y(\zeta))]d\zeta\right|^2$$
$$+2\left|\int_0^{\eta} \cos_{\kappa,q} M(\eta-\kappa-\zeta)^q e^{-N\zeta}\left(\int_0^{\zeta} [\Delta(\lambda, x(\lambda)) - \Delta(\lambda, y(\lambda))]dw(\lambda)\right)d\zeta\right|^2 := 2[(i) + (ii)]. \tag{3}$$

First, we estimate (i):

$$\left|\int_0^{\eta} \cos_{\kappa,q} M(\eta-\kappa-\zeta)^q e^{-N\zeta}[F(\zeta, x(\zeta)) - F(\zeta, y(\zeta))]d\zeta\right|^2$$
$$\le \left(\int_0^{\eta} |\cos_{\kappa,q} M(\eta-\kappa-\zeta)^q|^2 e^{-N(\eta-\zeta)}d\zeta\right)$$
$$\times\left(\int_0^{\eta} e^{-N(\eta-\zeta)}e^{-2N\zeta}|F(\zeta, x(\zeta)) - F(\zeta, y(\zeta))|^2 d\zeta\right)$$
$$\le \left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2 \left(\int_0^{\eta} e^{-N(\eta-\zeta)}d\zeta\right)$$
$$\times\left(\int_0^{\eta} e^{-N(\eta-\zeta)}d\zeta\right)e^{-2N\eta}|F(\eta, x(\eta)) - F(\eta, y(\eta))|^2$$
$$\le \left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2 \left(\int_0^{\eta} e^{-N(\eta-\zeta)}d\zeta\right)^2 E\|x(\eta) - y(\eta)\|^2$$
$$\le \left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2 \left(\frac{e^{-Nb}-1}{Nb}\right)^2 E\|x(\eta) - y(\eta)\|^2.$$

By Burkholder-Davis-Gundy inequality and Assumption 1, the estimate for (ii) is given by

$$\left|\int_0^\eta \cos_{\kappa,q} M(\eta-\kappa-\zeta)^q e^{-N\zeta}\left(\int_0^\zeta [\Delta(\lambda,x(\lambda))-\Delta(\lambda,y(\lambda))]dw(\lambda)\right)d\zeta\right|^2$$

$$\leq 4b\int_0^\eta |\cos_{\kappa,q} M(b-\kappa-\zeta)^q|^2 e^{-2N(b-\zeta)}e^{-2N\zeta}|\Delta(\zeta,x(\zeta))-\Delta(\zeta,y(\zeta))|^2 d\zeta$$

$$\leq 4b\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2\left(\int_0^b e^{-2N(b-\zeta)}d\zeta\right)e^{-2N\eta}|\Delta(\eta,x(\eta))-\Delta(\eta,y(\eta))|^2$$

$$\leq 4b\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2\left(\frac{e^{-2Nb}-1}{2Nb}\right)E\|x(\eta)-y(\eta)\|^2.$$

From the above two estimates of (i) and (ii), Eq. (3) becomes

$$E\|(\mathcal{P}x)(\eta)-(\mathcal{P}y)(\eta)\|^2$$

$$\leq 2\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2\left(\frac{e^{-Nb}-1}{Nb}\right)^2 E\|x(\eta)-y(\eta)\|^2$$

$$+8b\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2\left(\frac{e^{-2Nb}-1}{2Nb}\right)E\|x(\eta)-y(\eta)\|^2$$

$$\leq 2\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2\left[\left(\frac{e^{-Nb}-1}{Nb}\right)^2+4b\left(\frac{e^{-2Nb}-1}{2Nb}\right)\right]E\|x(\eta)-y(\eta)\|^2.$$

This implies that

$$E\|(\mathcal{P}x)(\eta)-(\mathcal{P}y)(\eta)\|^2 \leq KE\|x(\eta)-y(\eta)\|^2.$$

Hence, from statement of the Theorem 3.1, the nonlinear operator $(\mathcal{P})$ is a contraction. Hence the nonlinear operator $(\mathcal{P})$ has a unique solution $y(\cdot)\in R^n$, which is nothing but solution of Eq. (1). Hence the proof. □

4. Finite-time stability

Theorem 4.1. *If Assumption 1 hold and provided that*

$$5\left[K_1+|M^{-1}|^2\left(K_3+K_2-2\sqrt{K_3}\sqrt{K_2}\right)+\kappa^2K_2\right]E_{q,1}\left[5\frac{b^{2-q}}{2-q}K_2\Gamma(q)\eta^q\right]\leq\frac{\varepsilon}{\delta}.$$

Then the system (1) is finite-time stable in mean square.

Proof. By multiplying $e^{-2N\eta}$ on both sides of the solution of system (1), we derive

$$\begin{aligned}
e^{-2N\eta}|y(\eta)|^2 \leq\; & 5\left|\left(\cos_{\kappa,q}M\eta^q\right)e^{-N\eta}\psi(-\kappa)\right|^2 + 5\left|M^{-1}\left(\sin_{\kappa,q}M(\eta-\kappa)^q\right)e^{-N\eta}\psi'(0)\right|^2 \\
& +5\left|\int_{-\kappa}^{0}\cos_{\kappa,q}M(\eta-\kappa-\zeta)^q e^{-N\zeta}\psi'(\zeta)d\zeta\right|^2 \\
& +5\left|\int_0^{\eta}\cos_{\kappa,q}M(\eta-\kappa-\zeta)^q e^{-N\zeta}F(\zeta,y(\zeta))d\zeta\right|^2 \\
& +5\left|\int_0^{\eta}\cos_{\kappa,q}M(\eta-\kappa-\zeta)^q e^{-N(\eta-\zeta)}e^{-N\zeta}\left(\int_0^{\zeta}\Delta(\lambda,y(\lambda))dw(\lambda)\right)d\zeta\right|^2 \\
\leq\; & 5\left|\left(\cos_{\kappa,q}M\eta^q\right)\right|^2 e^{-2N\eta}|\psi(-\kappa)|^2 \\
& +5\left|M^{-1}\right|^2\left|\left(\sin_{\kappa,q}M(\eta-\kappa)^q\right)\right|^2 e^{-2N\eta}|\psi'(0)|^2 \\
& +5\kappa^2\left|\cos_{\kappa,q}M(\eta+\kappa)^q\right|^2 e^{-2N\eta}|\psi'(\eta)|^2 + 5\left(\int_0^{\eta}(\eta-\zeta)^{q-1}d\zeta\right) \\
& \times\int_0^{\eta}(\eta-\zeta)^{1-q}\left|\cos_{\kappa,q}M(\eta-\kappa-\zeta)^q\right|^2 e^{-2N\zeta}|F(\zeta,y(\zeta)) \\
& -F(\zeta,0)|^2 d\zeta + 20\left(\int_0^{\eta}(\eta-\zeta)^{q-1}d\zeta\right) \\
& \times\int_0^{\eta}(\eta-\zeta)^{1-q}\left|\cos_{\kappa,q}M(\eta-\kappa-\zeta)^q\right|^2 e^{-2N\zeta}|\Delta(\zeta,y(\zeta)-\Delta(\zeta,0)|^2 d\zeta \\
\|y(\eta)\|^2 \leq\; & 5\left[E_{2q}\left(\|M\|^2 b^{2q}\right)\right]^2\|\psi\|^2 + 5\left|M^{-1}\right|^2(\left[E_q(\|M\|(b+\kappa)^q)\right]^2 \\
& +\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2 - 2E_q(\|M\|(b+\kappa)^q)E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right))\|\psi'\|^2 \\
& +5\kappa^2\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2\|\psi'\|^2 \\
& +5\frac{b^{2-q}}{2-q}\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2\left(\int_0^{\eta}(\eta-\zeta)^{q-1}\|y(\zeta)\|^2 d\zeta\right) \\
& +20\frac{b^{2-q}}{2-q}\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2\left(\int_0^{\eta}(\eta-\zeta)^{q-1}\|y(\zeta)\|^2 d\zeta\right) \\
\leq\; & 5\Bigg\{K_1\delta_1 + \left|M^{-1}\right|^2\left(K_3+K_2-2\sqrt{K_3}\sqrt{K_2}\right)\delta_1 + \kappa^2K_2\delta_1 \\
& +\frac{b^{2-q}}{2-q}K_2\left(\int_0^{\eta}(\eta-\zeta)^{q-1}\|y(\zeta)\|^2 d\zeta\right)+4\frac{b^{2-q}}{2-q}K_2\left(\int_0^{\eta}(\eta-\zeta)^{q-1}\|y(\zeta)\|^2 d\zeta\right)\Bigg\} \\
\leq\; & 5\Bigg\{\left[K_1 + \left|M^{-1}\right|^2\left(K_3+K_2-2\sqrt{K_3}\sqrt{K_2}\right) + \kappa^2K_2\right]\delta \\
& +\frac{b^{2-q}}{2-q}K_2\left(\int_0^{\eta}(\eta-\zeta)^{q-1}\|y(\zeta)\|^2 d\zeta\right) \\
& +4\frac{b^{2-q}}{2-q}K_2\left(\int_0^{\eta}(\eta-\zeta)^{q-1}\|y(\zeta)\|^2 d\zeta\right)\Bigg\}.
\end{aligned}$$

According to Lemma 2.1, let as take

$$b(\eta) = 5\left[K_1 + |M^{-1}|^2\left(K_3 + K_2 - 2\sqrt{K_3}\sqrt{K_2}\right) + \kappa^2 K_2\right]\delta$$

and

$$h(\eta) = 5\frac{b^{2-q}}{2-q}K_2.$$

Moreover, $b(\eta)$ is a nondecreasing function on $[0, b]$, then

$$v(\eta) \le b(\eta)E_{q,1}[h(\eta)\Gamma(q)\eta^q]$$
$$\|y(\eta)\|^2 \le 5\left[K_1 + |M^{-1}|^2\left(K_3 + K_2 - 2\sqrt{K_3}\sqrt{K_2}\right) + \kappa^2 K_2\right]\delta E_{q,1}\left[5\frac{b^{2-q}}{2-q}K_2\Gamma(q)\eta^q\right]$$

Then from the statement of Theorem 4.1, we get

$$\|y(\eta)\|^2 \le \varepsilon.$$

Hence the system (1) is finite-time stable in mean square. Hence the proof. □

Remark 4.1. *By using fixed-point rule, existence and uniqueness of solution, and controllability results have been investigated in* [27]. *Some well-known results on relative controllability of semilinear delay differential system with linear parts defined by permutable matrices are studied in* [28]. *In this chapter, we proved some new results of finite-time stability criteria in finite-dimensional space by employing Generalized Gronwall-Bellman inequality and suitable assumption on nonlinear terms.*

5. An example

Consider Eq. (1) in the below matrix form:

$$\begin{cases} {}^CD^{0.5}_{-0.75^+}\left({}^CD^{0.5}_{-0.75^+}y_1\right)(\eta) + 0.01y_1(\eta - 0.75) = -(3-\eta)\dfrac{y_1^2(\eta)}{1-\eta} + \displaystyle\int_0^\eta (\zeta y_1(\zeta)\Delta_1 dB_1(\zeta), \\ y_1(\eta) = 2\eta, y_1'(\eta) = 2, \quad \eta \in [-0.75, 0]; \\ {}^CD^{0.5}_{-0.75^+}\left({}^CD^{0.5}_{-0.75^+}y_2\right)(\eta) + 0.01y_2(\eta - 0.75) = -(3-\eta)\dfrac{y_2^2(\eta)}{1-\eta} + \displaystyle\int_0^\eta (\zeta y_2(\zeta)\Delta_2 dB_s(\zeta), \\ y_2(\eta) = 4\eta, y_2'(\eta) = 4, \quad \eta \in [-0.75, 0], \end{cases} \tag{4}$$

where $q = 0.5, \ \kappa = 0.75, \Delta_1 = 0.3, \ \Delta_2 = 0.5$

$$A = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad F(\eta, y(\eta)) = \begin{pmatrix} -(3-\eta)\dfrac{y_1^2(\eta)}{1-\eta}e^{2N\eta} \\ -(3-\eta)\dfrac{y_2^2(\eta)}{1-\eta}e^{2N\eta} \end{pmatrix}.$$

$$\Delta(\eta, y(\eta)) = \begin{pmatrix} -\eta y_1(\eta)e^{2N\eta}\sigma_1 dB_1 \\ -\eta y_2(\eta)e^{2N\eta}\sigma_2 dB_2 \end{pmatrix}, \quad \psi(\eta) = \begin{pmatrix} 2\eta \\ 4\eta \end{pmatrix}, \quad \psi'(\eta) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Further, we have the following fractional delayed cos matrices:

$$\cos_{0.75,0.65}\left(M\eta^{0.65}\right)=\begin{cases}\Theta, & -\infty<\eta<-0.75,\\ I, & -0.75\leq\eta<0,\\ I-M^2\dfrac{\eta^{1.3}}{\Gamma(2.3)}, & 0\leq\eta<0.75,\\ I-M^2\dfrac{\eta^{1.3}}{\Gamma(2.3)}+M^4\dfrac{(\eta-0.75)^{2.6}}{\Gamma(3.6)}, & 0.75\leq\eta<1.5.\end{cases} \tag{5}$$

From Eq.(5) and using basic calculation, one can get $\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2=0.9712$, $\left(\frac{e^{-Nb}-1}{Nb}\right)^2=0.2683$ and $4b\left(\frac{e^{-2Nb}-1}{2Nb}\right)=-1.9004$. Using the above-obtained values, one can easily verify that

$$2\left[E_{2q}\left(\|M\|^2(b+\kappa)^{2q}\right)\right]^2\left[\left(\frac{e^{-Nb}-1}{Nb}\right)^2+4b\left(\frac{e^{-2Nb}-1}{2Nb}\right)\right]<1.$$

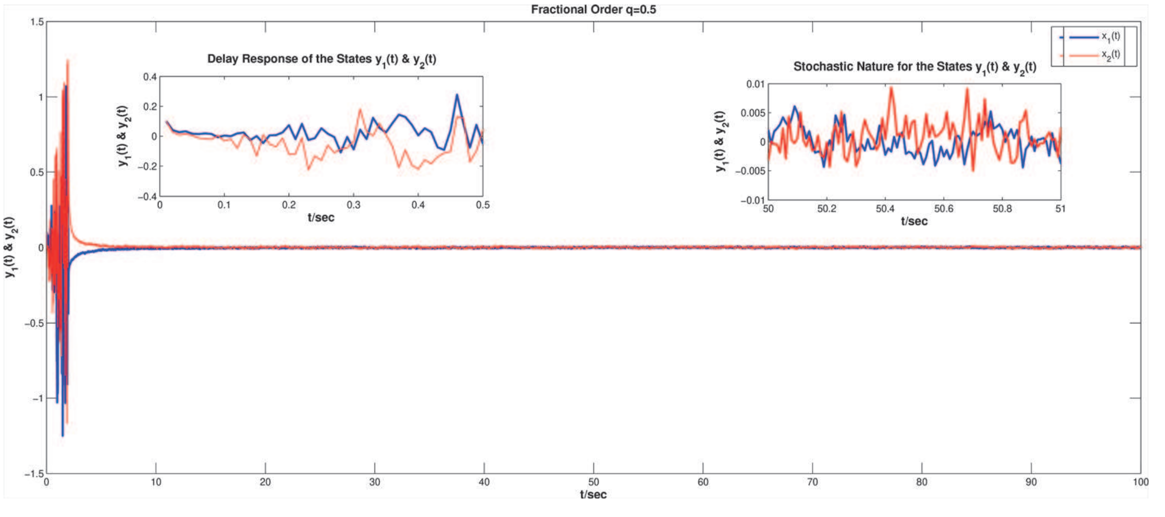

Figure 1.
The system (4) is stable at q = 0.5.

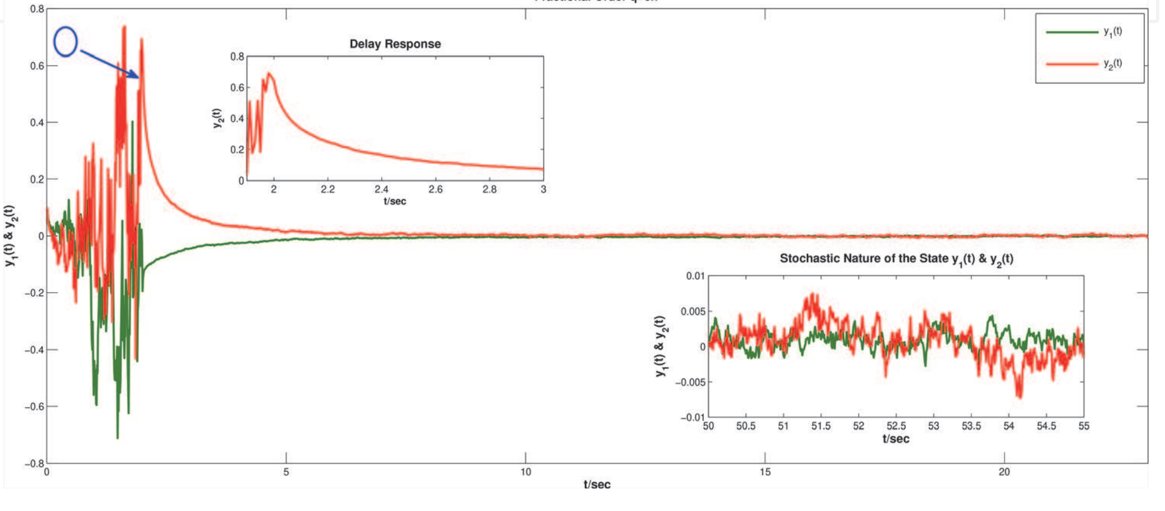

Figure 2.
The system (4) is stable at q = 0.7.

Hence we verified Theorem 3.1. Further, it is easy to verify that for any $x(\eta), y(\eta) \in R^2$.

$$e^{-2N\eta}|F(\eta,x(\eta)) - F(\eta,y(\eta))|^2 \le -(3-\eta)\mathbb{E}\|x(\eta) - y(\eta)\|^2$$
$$e^{-2N\eta}|\Delta(\eta,x(\eta)) - \Delta(\eta,y(\eta))|^2 \le -0.5\eta\ \mathbb{E}\|x(\eta) - y(\eta)\|^2$$

Hence, F and Δ satisfies Assumption 1. In **Figures 1** and **2**, we showed the stable response of the system (4) with fractional order $q = 0.5$ and $q = 0.7$, respectively. From the above verification, one can conclude that the system (4) is finite-time stable in mean square.

6. Conclusion

In this chapter, we have derived some meaningful and general results for finite-time stability of nonlinear fractional stochastic delay systems. Existence, uniqueness of solution and stability analysis of FSDS have been proved in finite-dimensional stochastic fractional higher-order differential system. Finally, a numerical simulation test is carried out to validate the obtained theoretical results. Derived result generalizes many existing results with integer and fractional-order systems.

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