Chapter

Valuation and Capital Return as Inverse Problems

Petri P. Kärenlampi

Abstract

The capital return rate is the relative time change rate of value. Correspondingly, the current value can be produced in terms of value change rate divided by capital return rate. There is a variety of ways to approximate the expected capital return rate. These are briefly discussed. The approximation of the value change rate is still more variant, depending on the type of businesses discussed. A variety of businesses may appear within a firm, in which case the value change rates must be integrated. An example is provided of a real estate firm benefiting from the growth of multiannual plants of varying age. It is found that the application of a duration-dependent reference capital return rate increases the value increment rate of juvenile stands and decreases that of mature stands, however increasing the valuation result of both.

Keywords: capitalization, capital return rate, value increment rate, expected value

1. Introduction

The capital return rate is the relative time change rate of value. We choose to write

$$r(t) = \frac{d\kappa}{K(t)dt} \tag{1}$$

where κ in the numerator considers value growth, operative expenses, interests and amortizations, but neglects investments and withdrawals. In other words, it is the change of capitalization on an economic profit/loss basis. K in the denominator gives capitalization on a balance sheet basis, being directly affected by any investment or withdrawal.

A significant finding in Eq. (1) is that the capital return rate r depends not only on the value change rate $\frac{d\kappa}{dt}$, but also on the current valuation $K(t)$. Apparently, there is an intimate relationship between the capital return rate and the valuation. A fundamental question then is whether the valuation can be approached by inverting Eq. (1) as

$$K(t) = \frac{d\kappa}{r(t)dt} \tag{2}$$

Obviously, Eq. (2) is mathematically correct. However, in comparison with Eq. (1), it may appear less intuitive. While Eq. (1) provides the definition of capital return rate as a function of observable valuation, does Eq. (2) provide the definition of valuation as a function of observable capital return rate? Or does it possibly provide the definition of valuation as a function of the required capital return rate? Any of these interpretations is possible.

The quantities appearing on the right-hand side of Eq. (1) are observable in a variety of ways, including profit-loss—statement, balance sheet, and market valuation. As the right-hand side has been determined, the left-hand side is naturally known. However, this results in a circular definition in Eq. (2). Alternatively, the capital return rate appearing on the right-hand side of Eq. (2) can be determined from comparable reference investments.

The momentary definitions appearing in Eqs. (1) and (2) provide a highly simplified description of valuation and capital return rate. In reality, there is variability due to a number of factors. Enterprises often contain businesses distributed to a variety of production lines, geographic areas, and markets. In addition, quantities appearing in Eqs. (1) and (2) are not necessarily completely known but may contain probabilistic scatter. Correspondingly, the expected values of capital return rate and valuation can be written as

$$\langle r(t)\rangle = \frac{\int p_{\frac{d\kappa}{dt}} \frac{d\kappa}{dt} d\frac{d\kappa}{dt}}{\int p_K K(t) dK} = \frac{\int p_{\frac{d\kappa}{dt}} r(t) K(t) d\frac{d\kappa}{dt}}{\int p_K K(t) dK} \quad (3)$$

and

$$\langle K(t)\rangle = \int p_K K(t) dK = \int p_K \frac{d\kappa/dt}{r(t)} dK \quad (4)$$

where p_i corresponds to the probability density of quantity i. It is found that while the capital return rate and the capitalization are simply invertible in the absence of any variation (Eqs. (1) and (2)), the same is not the case in the presence of variation, either deterministic or probabilistic (Eqs. (3) and (4)). Here, it is worth noting that the capital return rate in the denominator of Eq. (4) obviously tends to a “reference” capital return rate, rather than a directly observed one.

In the remaining part of this chapter, we will first discuss the practical implementation of the determination of capital return rate and firm value using Eqs. (3) and (4) in the case of a real estate firm benefiting from the growth of multiannual plant stands of varying age. Then, we will discuss the determination of the values of the quantities appearing in Eqs. (3) and (4), as well as factors contributing to them. Finally, a few applications are discussed, as well as interpolation techniques.

2. Application to stands of multiannual plants

In this section, the determination of capital return rate and enterprise value is discussed in the case of a real estate firm benefiting from the growth of multiannual plant stands of varying ages. Conducting a change of variables in Eqs. (3) and (4) results as

$$\langle r(t)\rangle = \frac{\int p_a(t) \frac{d\kappa}{dt}(a,t) da}{\int p_a(t) K(a,t) da} = \frac{\int p_a(t) r(a,t) K(a,t) da}{\int p_a(t) K(a,t) da} \quad (5)$$

and

$$\langle K(t)\rangle = \int p_a(t)K(a,t)da = \int p_a(t)\frac{\frac{d\kappa}{dt}(a,t)}{r(a,t)}da \tag{6}$$

where a refers to stand age. Again, the capital return rate in the denominator of Eq. (6) rather refers to a reference rate than a directly observed one.

It is found from Eqs. (5) and (6) that the probability density of stand age is a function of time, and correspondingly, the capital return rate, as well as the estate value, evolves in time. A significant simplification would appear if the probability densities appearing on the right-hand side of Eqs. (5) and (6) would not change along with time. Within forestry, such a situation would be denoted "normal forest principle." corresponding to evenly distributed stand age determining relevant stand properties [1].

$$\langle r(t)\rangle = \frac{\int \frac{d\kappa}{dt}(a)da}{\int K(a)da} = \frac{\int r(a)K(a)da}{\int K(a)da} \tag{7}$$

and

$$\langle K(t)\rangle = p_a \int K(a,t)da = p_a \int \frac{\frac{d\kappa}{dt}(a,t)}{r(a,t)}da \tag{8}$$

The "normal forest principle" is rather useful when considering silvicultural practices, but seldom applies to the valuation of real-life real estate firms, with generally non-uniform stand age distribution. However, it has recently been shown [2] that the principle is not necessary for the simplification of Eqs. (5) and (6) into (7) and (8). This happens by focusing on a single stand, instead of an entire estate or enterprise, and considering that time proceeds linearly. Then, the probability density function p (a) is constant within an interval [0, τ]. Correspondingly, it has vanished from Eq. (7) and appeared outside of the integral in Eq. (8).

The topic of this chapter, however, is firm valuation. As the relatively simple Eqs. (7) and (8) are useful in the design of silvicultural practices, firm valuation typically happens at a specific instant of time, and the probability density *p(a)* generally is non-uniform. Correspondingly, Eqs. (5) and (6) must be applied. Fortunately, *p (a)* usually is known for any property where recent inventory results are available. It is further fortunate that Eqs. (4), (6), and (8) contain simple summations, unlike Eqs. (3), (5), and (7).

3. Determination of stand capitalization

There is a variety of methods to determine the value K appearing in Eqs. (3)–(6). In the case of an incorporated company or a firm with equivalent reporting, the value can be found from the balance sheet. Such an outcome does depend on applied accounting practices. On the other hand, the value can be determined as a market value. The latter is straightforward in the case of publicly listed companies, or other companies with the established share trading records. A third alternative for firm value determination is the computation of an "intrinsic value," considering the prognosticated future development of the firm [3–5].

In the case of a real estate firm benefiting from the growth of multiannual plants of varying age, the value K within any stand may be approached as the sum of the value of the plants on the stand, the value of bare land, and the value of non-amortized investments. Such computation is problematic if non-mature plants do not have any immediate sale value. In such a case, it is not uncommon to determine stand value by discounting expected future revenues [6–9]. The discount rate is sometimes taken arbitrarily, but often it can be determined as an internal rate of return [10].

We here provide a few examples of the determination of the value of forest stands by interpolation. It is not uncommon that planted seedlings may require several years to mature to young trees of commercial value. However, during those years, expected revenues become closer in time. Correspondingly, it would be unrealistic to assume the growth of saplings would not add value. The capitalization, including such an additional expected value, could be approximated by some kind of a smoothing function. One possibility could be

$$k(a) = \frac{1}{\tau - a} \int_{a}^{\tau} K(t) \exp\left[r(t) * (a - t)\right] dt \tag{9}$$

where r(t) is the capital return rate at stand age t. A simpler version would be

$$k(a) = \frac{1}{\tau - a} \int_{a}^{\tau} K(t) \exp\left[\langle r \rangle * (a - t)\right] dt \tag{10}$$

Both of the above equations converge to terminal capitalization $k(\tau) = K(\tau)$, regardless of the capital return rate $r(t)$ or $\langle r \rangle$. However, there is no guarantee of any definite convergence in a newly established stand. Such convergence $k(initial) = K(initial)$ could be approached by fitting an internal rate of return i, which provides convergence. That would correspond to assuming that the bare land value includes any additional expectation value for a newly established stand, resulting as

$$k(a) = \frac{1}{\tau - a} \int_{a}^{\tau} K(t) \exp\left[i * (a - t)\right] dt \tag{11}$$

It might be possible to determine capitalization indirectly by discounting revenue. This would result as

$$k(a) = BL + \int_{a}^{\tau} R(t) \exp\left[j * (a - t)\right] dt \tag{12}$$

where BL denotes bare land value, and $R(t)$ net revenue at time t. Again, the discount rate j shall be fitted for convergence $k(initial) = K(initial)$.

The functionality of Eqs. (11) and (12) are investigated in **Figure 1**, in the case of a spruce stand established with 1800 saplings/ha, and a wooded stand observed at the age of 35 years. The former initial condition is based on the early application of a growth model on saplings stands [11, 12], the latter on the observations of the wooded stand [13, 12]. The former shows a positive additional expectation value of trees for

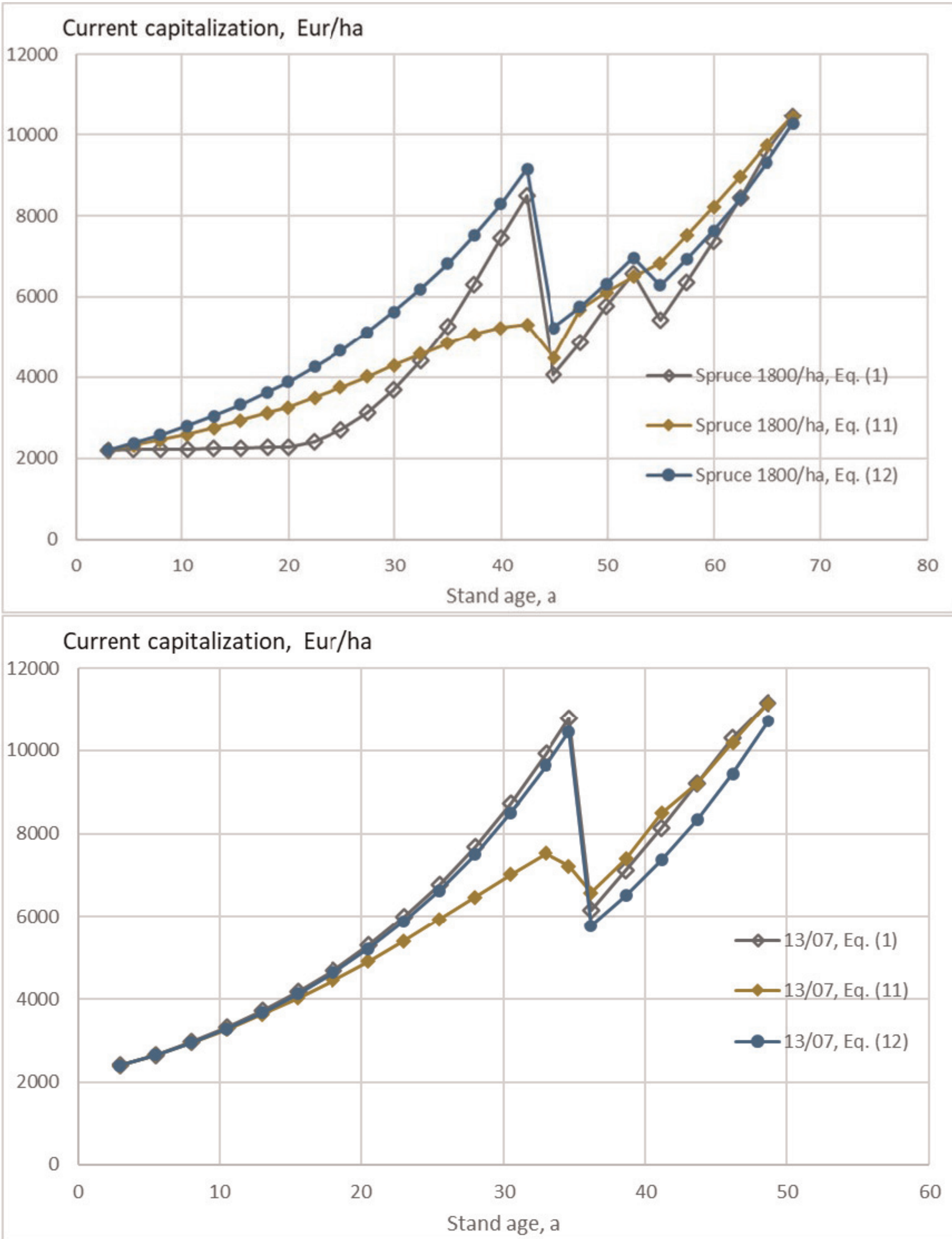

Figure 1.
Capitalization, as appearing in Eq. (1), as well as smoothed capitalization according to Eqs. (11) and (12), in the case of the two example stands. (a) (above) shows a spruce stand established with 1800 saplings/hectare. (b) (below) shows a spruce stand first observed at the age of 35 years.

young stands and after thinnings. If such additional values would be considered, microeconomically optimal rotation ages would be affected. However, Eq. (11) results as the additional value being negative before the first thinning. It is not known how the negative additional expectation value should be considered in management. Within the wooded stand observed at 35 years of age, the additional expectation value of trees of young stands according to Eq. (11) would be negative, being slightly positive only after thinning. Eq. (12) would indicate zero additional value before thinning and somewhat negative after thinning. An explanation for the latter is that regeneration expenses are carried in the balance sheet until the end of the rotation. This results in the capitalization at mature age being greater than the sum of discounted terminal revenue and bare land value.

Figure 1a indicates that in the case of the early application of the growth model, internal rate of return-based interpolation could be useful in the determination of young stand capitalization. In the absence of such adjustment, there would be a negligible value increment for a period of two decades. Eq. (12) can be straightforwardly applied by substituting k(a) from in place of K(a) in Eqs. (5), (6), (7), or (8). However, interpolation is possible only after an initial treatment schedule has been designed using Eqs. (5) or (7). Correspondingly, Eq. (12) must be used iteratively with the other equations.

On the other hand, in the case of **Figure 1b**, interpolation of capitalization appears irrelevant. A natural reason is that the stand has been first observed at the age of 35 years. The capitalization from the stand establishment to the time of observation already has been approximated by exponential interpolation. Correspondingly, results based on the observations of wooded stands do not appear to be in the need of any further interpolation.

4. Determination of a reference capital return rate

In money market theory, increased duration of commitments tends to increase the risk experienced by the borrower [14–16]. However, in **Figure 1**, the discount rates do not depend on the delay time of revenues. This could be corrected by introducing a delay-dependent discount rate. One possibility is a spot discount rate

$$j = \ln\left(u + \frac{d}{s}\right) \tag{13}$$

where d is time to maturity, and u and s are constants. Now, the constants u and s can be adjusted to gain the correspondence $k(initial) = K(initial)$ in Eq. (12). On the other hand, it is only the constant u that determines the discount rate at maturity and that can be determined through matching terminal discount rate to terminal capital return rate, determined as the ratio of terminal value increment rate to terminal capitalization. The outcome, in terms of stand capitalization, is shown in **Figure 2**. At intermediate stand ages, the capitalization becomes higher when Eq. (13) is applied. This is because the revenue is less severely discounted close to maturity. Greater discount rate close to the stand establishment then ensures the correspondence $k(initial) = K(initial)$.

5. Determination of stand value increment rate

There is a variety of methods to determine the value increment rate $^{d\kappa}/_{dt}$ appearing in Eqs. (1)–(8). In the case of an incorporated company, or a firm with equivalent reporting, the value increment rate can be found from a(n annual, quarterly, or prognosticated) profit/loss—statement. The outcome does depend on applied accounting practices.

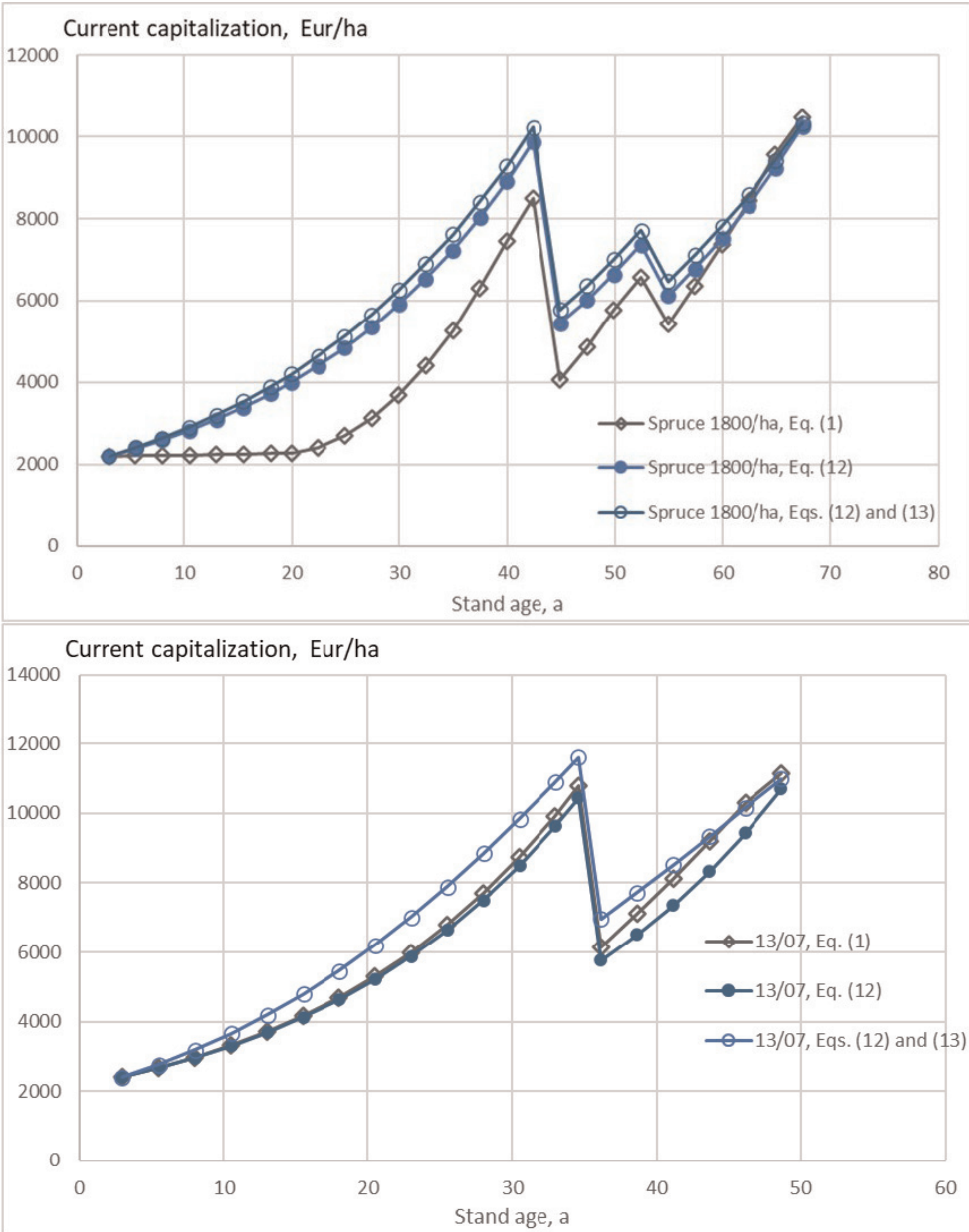

Figure 2.
Capitalization, as appearing in Eq. (1), as well as smoothed capitalization according to Eq. (12) with constant discount rate and (12) together with (13), in the case of the two example stands. (a) (above) shows a spruce stand established with 1800 saplings/hectare. (b) (below) shows a spruce stand first observed at the age of 35 years.

In the case of a real estate firm benefiting from the growth of multiannual plants of varying age, the value increment rate within any stand may be approached by the value increment rate of the plants on the stand, possibly complemented with the increment rate of bare land value. The value increment rate generally is constituted on volumetric increment on the one hand and increment of volumetrically specific value on the other hand [17]. Again, such computation is problematic if non-mature plants do not have any immediate sales value.

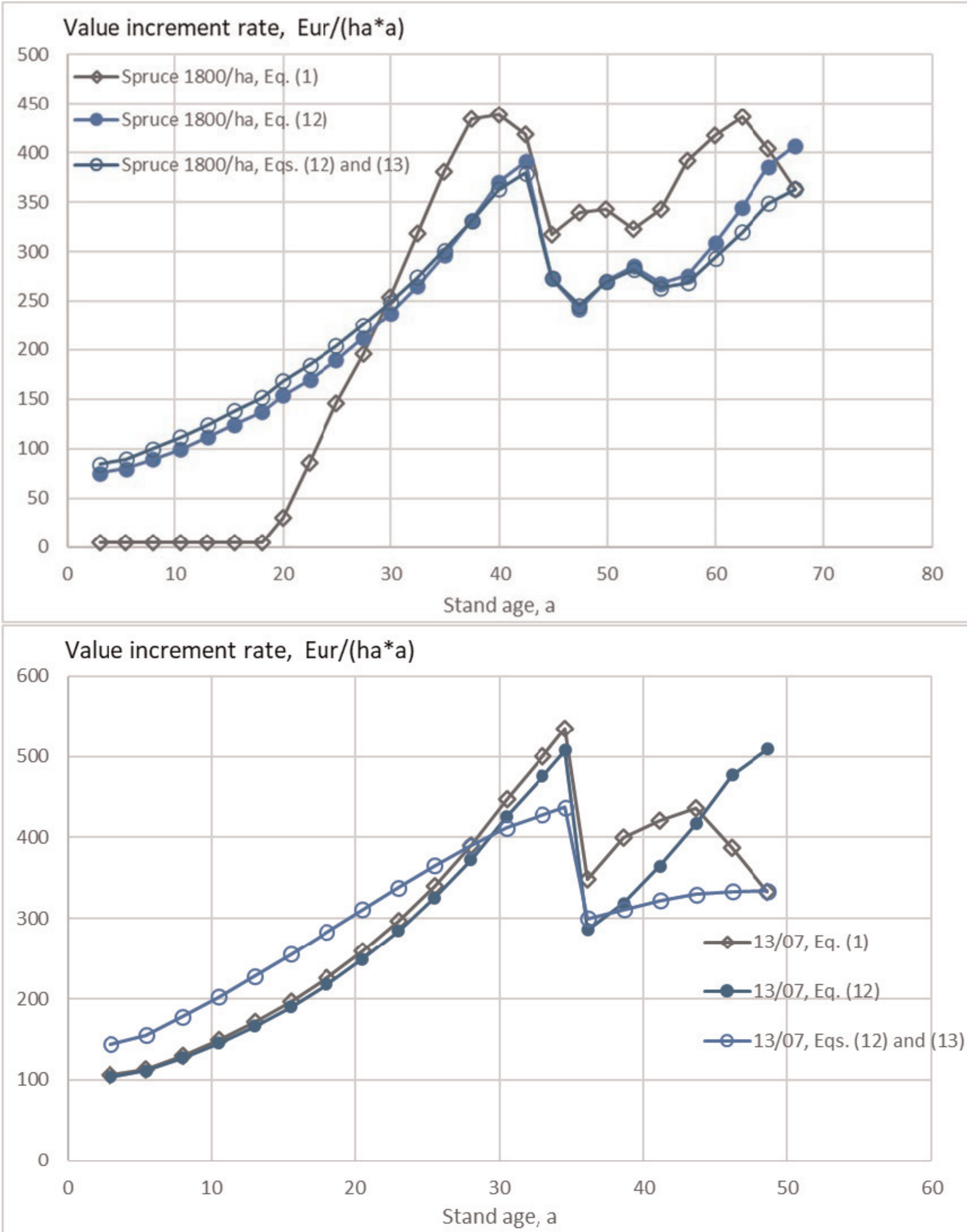

Figure 3.
Annual monetary value increment rate, as appearing in Eq. (1), as well as Eq. (12) with constant discount rate and (12) together with (13), in the case of the two example stands. Fig. 3a (above) shows a spruce stand established with 1800 saplings/hectare. Fig. 3b (below) shows a spruce stand first observed at the age of 35 years.

Figure 3 shows the annual value increment rate per hectare, determined using three different methods. Firstly, the annual value increment is determined directly using the growth model, as often applied in Eq. (1). With the initial condition based on the early application of a growth model on saplings stands (**Figure 3a**), the initial value increment rate is small, increasing rapidly later. Value increment rate based on discounting of revenue with the constant discount rate (Eq. (12)) is smoother, even if not monotonic. Incorporating the delay-dependent discount rate (Eq. (13)) increases the value increment rate at a young age and reduces it at a mature age (**Figure 3**). Similar trends are observable in the case of the example stand first observed at the age of 35 years, except for early stand development according to Eq. (1) is similar to the discounting result with the constant discount rate (Eq. (12) (**Figure 3b**).

The relative annual value increment rate can be readily found by normalizing the monetary increment rate of **Figure 3** with the capitalization appearing in **Figure 2**. The outcome is in **Figure 4**. With the initial condition based on the early application

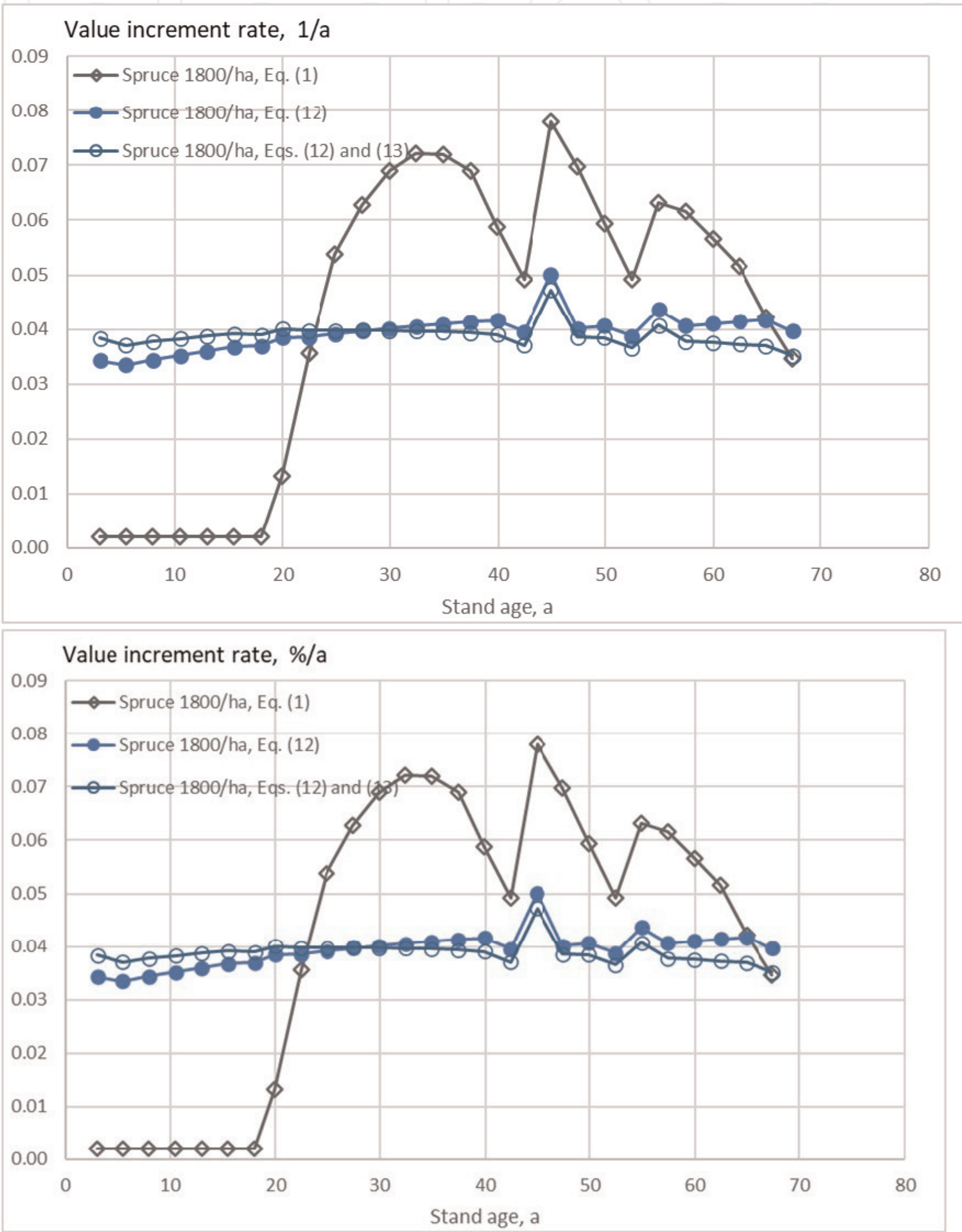

Figure 4.
Annual relative value increment rate, as appearing in Eq. (1), as well as Eq. (12) with constant discount rate and (12) together with (13), in the case of the two example stands. (a) (above) shows a spruce stand established with 1800 saplings/hectare. (b) (below) shows a spruce stand first observed at the age of 35 years.

of a growth model on saplings stands (**Figure 4a**), direct application of the growth model induces a volatile relative value increment rate. On the contrary, capitalization determined by discounting revenue yields stationary value increment rates, except for increases after thinnings. Again, delay-dependent discount rate (Eq. (13)) induces larger value increments at a young age and lower at a mature age. The latter effect is more pronounced in the case of the example stand observed at the wooded state (**Figure 4b**).

6. Further valuation attempts

It is of interest whether Eqs. (4) and (13) can be combined for the valuation of an individual stand. A possibility is

$$K(a) = \frac{\int_a^{\tau} {}^{d\kappa}\!/_{dt} \exp\left[\, j(t-a) * (a-t)\right] dt}{(\tau - a) * r(a)} \tag{14}$$

where reference capital return rate r(a) comprises a numerical solution of Eqs. (12) and (13), appearing in **Figure 4**. The result is shown in **Figure 5**. The reference capital return rate according to Eq. (13) increasing with increasing time to maturity, it is found that mature stands show greater value than the reference curves, whereas juvenile stands show lower value. Eq. (14), however, has an obvious deficiency: The value estimate does match the known terminal value but not the initial value.

The obvious deficiency in Eq. (14) can be simply corrected. The reference capital return rate *r(a)* can be modified, however possibly retaining the simple form of Eq. (13). Eq. (14) can possibly be further simplified by using the same reference rate in the discounting of the value increments. The outcome would be

$$K(a) = \frac{\int_a^{\tau} {}^{d\kappa}\!/_{dt} \exp\left[l(t-a) * (a-t)\right] dt}{(\tau - a) * l(a)} \tag{15}$$

However, there is another deficiency in Eq. (14) and in **Figure 5**. The approximated stand value before the first thinning appears lower than the immediate sales value of the trees. This deficiency does not become corrected by Eq. (15). Correspondingly, Eqs. (14) and (15) cannot be considered appropriate.

7. Discussion

The capital return rate and firm valuation have been discussed as inverse problems. The invertibility is obvious in the absence of probabilistic or deterministic scatter. In the presence of scatter, the invertibility is less straightforward. However, the firm valuation corresponds to a simple sum of units or compartments according to Eqs. (4), (6), and (8).

A real estate firm benefiting from the growth of multiannual plant stands of varying age was discussed as a practical example. Again, the total value is a simple sum of the values of compartments, which transfers the focus to the valuation of

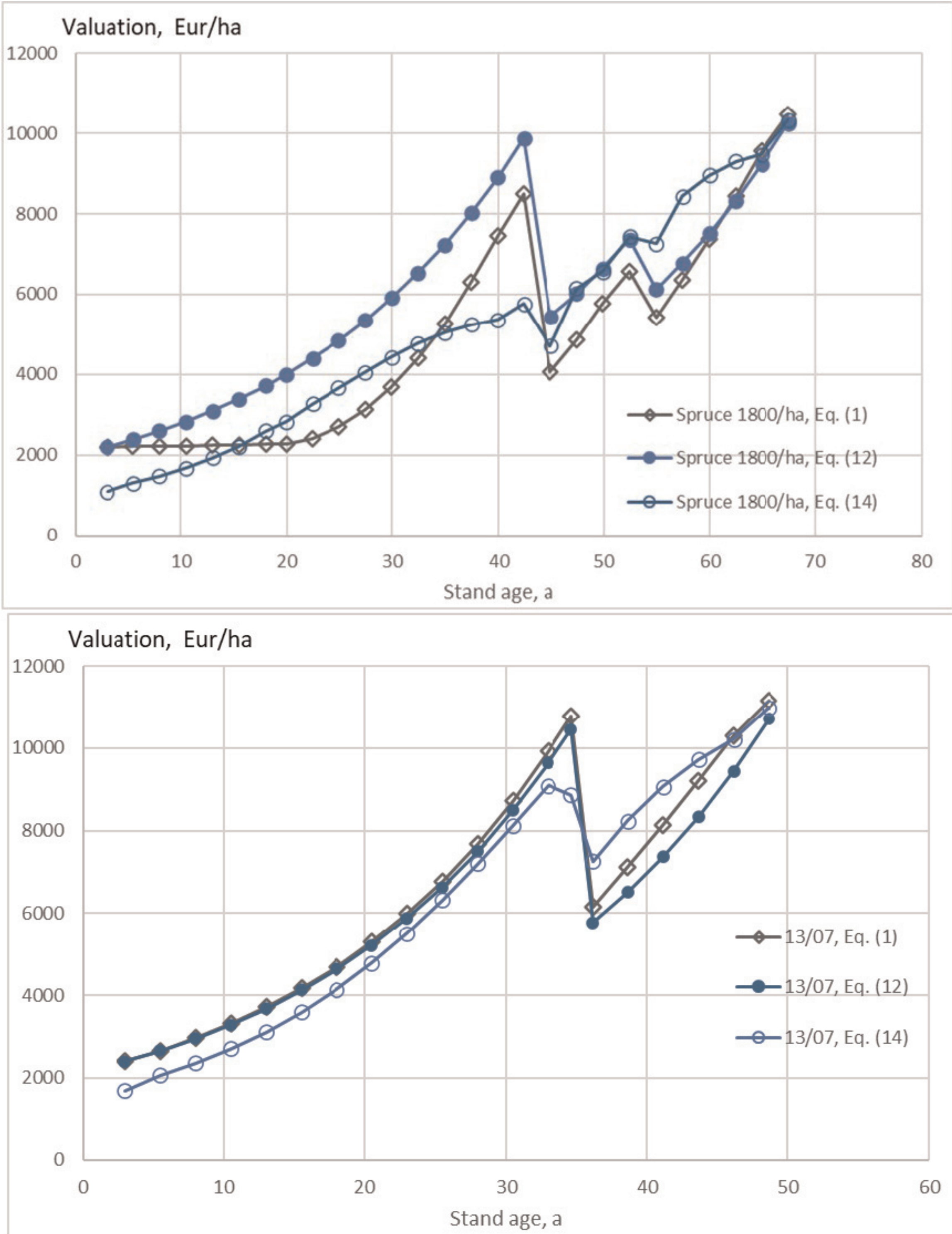

Figure 5.
Capitalization, as appearing in Eq. (1), as well as Eq. (12) with constant discount rate and (12) together with (13), in the case of the two example stands.

individual stands. Young stands with small immediate sales value, or other stands expected to increase in value rapidly in the future, appear to be a problem in stand valuation. Attempts in discounting future capitalization appear to be unsuccessful but attempts in discounting revenue successful. Such discounting appears necessary if the value of the plants is determined through high-resolution observation or computation. Observation of more mature stands, along with interpolation to more juvenile stands, takes care of such interpolation in value (**Figures 1, 2**, and **5**).

Discounting of revenue with constant discount rate, however, appears to be unsatisfactory since it produces stand values slightly lower than the immediate sales value (**Figures 1, 2,** and **5**). This can be corrected by introducing a discount rate that considers the duration effect (or time-to-maturity), according to Eq. (13). Such discount rate slightly increases stand value estimates (**Figure 2**). It increases value increment rate at a young age and reduces it at a mature age (**Figures 3** and **4**).

This chapter, after introducing generic expressions for valuation and capital return as inverse problems, has discussed the valuation of two example cases in a restricted manner. In other words, the initial value and the terminal value are taken as known quantities, and value evolution between these extremes has been interpolated. Such a restricted treatment has some definite benefits: Internal consistency of the results can be relatively easily verified. An important tool in the verification is the non-interpolated sum of value components generally used in Eq. (1). In the case of a forestry firm, that might become

$$K(a) = bare\ land\ value + value\ of\ trees + value\ of\ non\text{-}amortized\ investments \quad (16)$$

The last term in Eq. (16) depends on investment intensity, as well as the amortization schedule. For the purposes of an internal consistency criterion, we define a *reduced current capitalization* as

$$K'(a) = bare\ land\ value + value\ of\ trees \quad (17)$$

Approximations of capitalization established in Eqs. (9) to (12) are designed to include expectations of forthcoming value increment. Consequently, one can take for granted

$$k(a) \geq K'(a) \quad (18)$$

The brief examination above revealed that Eqs. (9), (10), (11), and (14) generally do not satisfy this internal consistency criterion and must thus be rejected. Eq. (12), applied with or without Eq. (13), often does satisfy the consistency criterion. However, this does not happen in all circumstances. Particularly, Eq. (12) with a constant discount rate fails to comply with Eq. (18) with large rotation ages, where the value increment rate becomes essentially non-exponential.

Other kinds of problems relate to Eq. (13). First, there are circumstances where the parameter s turns negative. In particular, this happens in the case of young, productive stands with growth only slightly differing from exponential. Consequently, increased duration of commitments tends to decrease the discount rate or, in other words, invert the yield curve.

Another issue related to Eq. (13) is that it is sensitive to short-range disturbances close to maturity. In particular, recent thinning typically increases the relative value increment rate. Consequently, parameter u in Eq. (13) increases. The estimated capitalization then decreases, and the expected value of capital return rate according to Eqs. (3), (5), or (7) increases. This suggests terminal clear-cutting soon after thinning—a result which obviously is a computational artifact. Instead, Eq. (12) with a constant discount rate appears to remain a valid estimate, provided the internal consistency criterion of Eq. (18) is satisfied.

As mentioned, this chapter has discussed the valuation of two example cases in a restricted manner. A less restricted treatment might open further avenues. A particular possibility might be the introduction of a variety of reference capital return rates in the denominator of Eqs. (2), (4), (6), and (8). Linking the reference rate to alternative investments like interest instruments or shares of listed companies would significantly change valuations—not only intermediate valuations but also the initial and terminal values. The initial value would likely be more affected since the proportion of the bare land value is greater than in the terminal value, the latter including the terminal sales value of mature timber.

Acknowledgements

This work was partially supported by Niemi Foundation. Prof. Dr. Lauri Mehtätalo is gratefully acknowledged for fruitful discussions.

Author details

Petri P. Kärenlampi
Lehtoi Research, Finland

*Address all correspondence to: petri.karenlampi@professori.fi

References

[1] Leslie AJ. A review of the concept of the normal forest. Australian Forestry. 1966;**30**:139-147. DOI: 10.1080/00049158.1966.10675407

[2] Kärenlampi PP. Two Sets of Initial Conditions on Boreal Forest Carbon Storage Economics. Accepted to PLOS Climate. 2021

[3] Tiwari R. Intrinsic value estimates and its accuracy: Evidence from Indian manufacturing industry. Future Business Journal. 2016;**2**(2):138-151. DOI: 10.1016/j.fbj.2016.10.001

[4] Tanjung G. The applications of discount cash flow, abnormal earning, and relative valuation approach. In: (Firm Intrinsic Value Analysis Pada Perusahaan BUMN) Conference: Seminar Nasional Kewirausahaan dan Inovasi Bisnis IVAt: Universitas, Tarumanegara. Bandung, Indonesia: Universitas Widyatama; 2014

[5] Carlin S. How To Calculate Intrinsic Value (Formula – Excel template & AMZN Example). Available from: https://svencarlin.com/how-to-calculate-intrinsic-value-formula/ [Accessed September 14, 2021]

[6] Kilkki P, Väisänen U. Determination of the optimum cutting policy for the forest stand by means of dynamic programming. Acta Forestalia Fennica. 1969;**102**:1-29

[7] Haight RG, Monserud RA. Optimizing any-aged management of mixed-species stands. II: Effects of decision criteria. Forest Science. 1990;**36**:125-144

[8] Pukkala T, Lähde E, Laiho O. Optimizing the structure and management of uneven-sized stands in Finland. Forestry. 2010;**83**:129-142

[9] Tahvonen O. Optimal structure and development of uneven-aged Norway spruce forests. Canadian Journal of Forest Research. 2011;**41**:2389-2402

[10] Kärenlampi PP. Wealth accumulation in rotation forestry– Failure of the net present value optimization? PLoS One. 2019;**14**:e0222918. DOI: 10.1371/journal.pone.0222918

[11] Kärenlampi PP. Capital return rate and carbon storage on forest estates of three boreal tree species. Sustainability. 2021;**13**(12):6675. DOI: 10.3390/su13126675

[12] Kärenlampi PP. Two sets of initial conditions on boreal forest carbon storage economics. Preprint. 2021: 2021070439. DOI: 10.20944/preprints202107.0439.v2

[13] Kärenlampi PP. Diversity of carbon storage economics in fertile boreal spruce (Picea Abies) estates. Sustainability. 2021;**13**:560. Available from: https://www.mdpi.com/2071-1050/13/2/560

[14] Bouchaud J-P, Potters M. Théorie des Risques Financiers (Saclay: Aléa). In: (Engl. transl. 2000) Theory of Financial Risks. Cambridge: Cambridge University Press; 1997. Available from: http://web.math.ku.dk/~rolf/Klaus/bouchaud-book.ps.pdf

[15] Brealey RA, Myers SC, Allen F. Principles of Corporate Finance. Tenth ed. New York, NY: McGraw-Hill Irwin; 2011

[16] Fabozzi FJ. Capital Markets: Institutions, Instruments, and Risk Management. Cambridge, Massachusetts: MIT Press; 2015

[17] Kärenlampi PP. The effect of empirical log yield observations on carbon storage economics. Forests. 2020; **11**:1312