Chapter

Elements of the Nonlinear Theory of Elasticity Based on Tensor-Nonlinear Equations

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Abstract

The chapter contains information that forms the basis of a new direction in the nonlinear theory of elasticity. The theory, having adopted the mathematical apparatus obtained in the middle of the last century, after its analysis, is used with significant changes. This concept allows us to more accurately reveal the mechanism of deformation of materials, the elastic nature of which significantly depends on the type of stress state, due to the growth of additional volumetric deformation associated with the accumulation of defects, called dilatation. The work is original — after abandoning the elasticity characteristics in the form of modules - constants, the main role is assigned to material functions, which represent statistical characteristics. Their relation can be considered a coefficient of variation and a parameter of tensor nonlinearity, which makes it possible to represent the deformation in the form of two parts, different in origin.

Keywords: dilatancy, volume deformation, shape change, phase similarity of deviators, volume deformation, coefficient of variation, tensor nonlinearity, anisotropy, variable elasticity parameter

1. Introduction

Experimental studies of well-known mechanics with various materials already in the eighteenth century revealed numerous nonlinear effects described in the book [1]. From the standpoint of the linear theory of elasticity, many of them could not be explained, so they were called second-order effects, as not significant. However, in the middle of the twentieth century, they pushed M. Rayner [2], and a little later, V. V. Novozhilov [3], to the need to develop a theory based on a new concept of tensor-nonlinear equations [4, 5] that more accurately reflect the nonlinearity of materials. The widespread introduction of composite media and the study of their mechanical properties began at the end of the last century. In the same years, a lot of experimental works appeared to study the mechanical properties of various composites, illuminating the properties of not only reinforced materials, but also grain composites, which differ in different reactions to tension and compression. This property is possessed by media whose longitudinal modulus of elasticity and other characteristics depend on the type of stress state, determined at values of deformations close to zero. It should be called the work of Tolokonnikov L. A., Makarov E. S. [6] and many others who have devoted research to the properties of these media, in which the presence of damage to internal connections and loosening, that is, the development of dilatancy, is stated. The theories put forward by them are based on tensor-linear equations. As a rule, in them all the characteristics of different-modulus media are determined from the condition of the existence of a specific deformation potential.

In this paper, in continuation of the study [7], to take into account the noted effects, such a transformation equations was found, which made it possible to develop methods for determining the elasticity characteristics. These equations presented for the main deformations made it possible not only to describe the deformation of the shape change, the coefficients of transverse deformations along different axes, to determine the volume deformation depending on the average stress, but also the dilatancy associated with the shape change.

2. About of different-module materials

The development of methods was carried out based on the results of studies of grain composite [8], and in earlier works of gray cast iron, using the research of [9]. The first is a hardened mechanical mixture of a mineral filler with a polymer matrix, the test results and information about its properties are published in [8, 10–12]. These materials have not only the presence of divergence of the initial longitudinal modules under tension and compression, but also show the dependence of elastic properties on time; therefore, in this work, the test results obtained at a single strain rate are used. The nonlinearity of the diagrams of a grain composite is clearly represented by the results of testing cross-shaped samples under repeated static stretching. It has a high malleability at normal temperature. The main purpose of testing such samples was to more fully reveal the mechanism of deformation of different-modulus materials. **Figure 1a** shows the curve 1—the ascending branch at the first cycle of active deformation along the axis 1–1 represents the initial properties of the material. Where P is the force in H, Δl is the elongation in millimeters. When unloading, the curve decreases sharply, which indicates a significant decrease in the number of bonds that break down with small deformations. The residual deformation does not represent plastic properties, but a residual dilatancy, from which it is possible to make a quantitative assessment of the initial deformation anisotropy for the next loading cycle. Curve 2—the ascending branch of the second cycle illustrates the resistance of the restored "short" and remaining "long" bonds. In **Figure 1b**, curve 1 is the ascending branch of the test at the first cycle along the axis 2–2. For comparison, a diagram (dashed) is shown, marking the initial properties of the composite. The difference in the curves of the first cycles in different directions suggests that the connection break occurs in the transverse direction as well. The first curve shows that the "short" connections in the direction 2–2 are partially preserved.

The difference between the ascending branches of the first and second stretching cycles along the 1–1 and 2–2 axes is a real one, called [3] by V. V. Novozhilov "real" anisotropy. The second cycle shows that the material has noticeably softened, the slope of the curve has decreased, but the tangential longitudinal elastic modules manifest themselves on the second part of the branch as increasing, differing from the first cycle. This emphasizes the fact that the links are divided into "short" and "long"—stronger, although in [13] a more detailed gradation of links is given, which will be superfluous for this work.

Both in [8, 12], it is noted that stretching is accompanied by a noticeable increase in volume. The same is observed with compression, although to a lesser extent. The loss of bonds and softening are the cause of the loss of elastic energy, which is taken

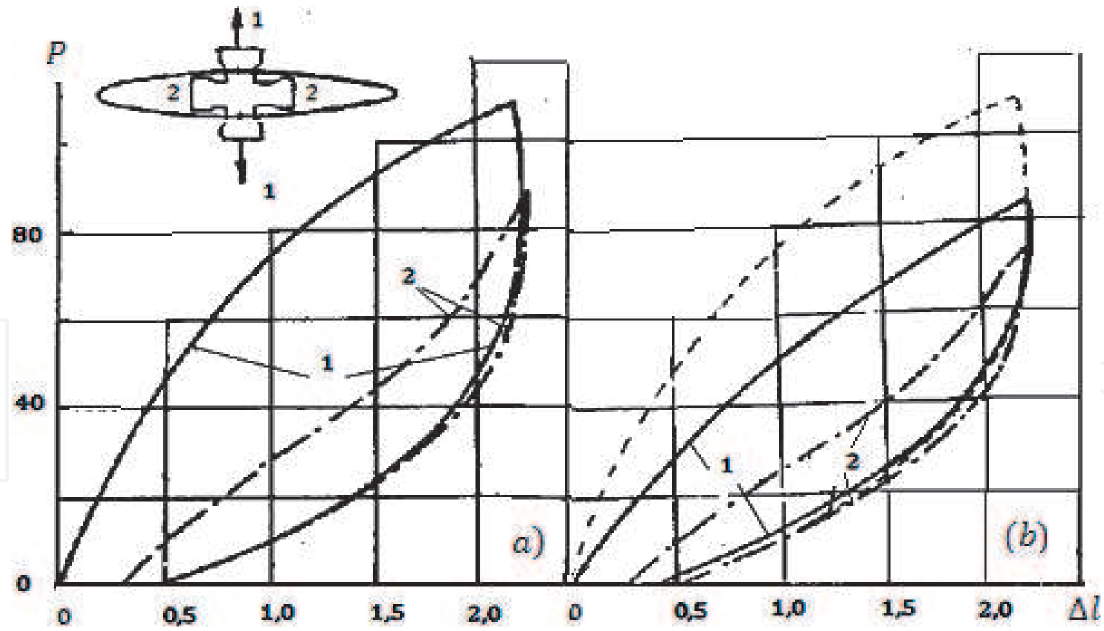

Figure 1.
a—Curve 1—The ascending branch at the first cycle of active deformation on the axis 1–1, curve 2—The ascending branch of the second cycle; b—Curve 1—The ascending branch at the first cycle on the axis 2–2.

into account by the mathematical model with a proportional increase in stresses only by the growth of additional volume deformation, as in the deformation theory, plastic shifts. For practical calculations, test diagrams of standard samples were used according to the method described in [8]. The tensile diagram for testing along the 1–1 axis, curve 1, **Figure 1a**, is a sequence of limit values of groups of bonds that are close in strength. The same is true for other types of loading, but to a lesser extent.

The purpose of this work is to fully reveal the possibilities tensor-nonlinear equations: transformed to a form convenient for the formulation of material functions, analysis, and processing of test results. On their basis, to develop methods for calculating all characteristics, including the coefficients of transverse deformations, elastic modulus, and compliance, as well as parameters that characterize the loosening of the structure and the change in elastic properties both with increasing load and with a change in the type of stress state.

3. On tensor-nonlinear equations

To describe the deformation of different-modulus materials, considering them isotropic, we used tensor-nonlinear equations of the connection of the strain deviator D_e with the stress deviator D_σ by V. V. Novozhilov [3], which, unlike the equations of M. Reiner [2], do not yet require the equation of the connection of the average strain with the average stress:

$$\varepsilon_{ij} - \frac{1}{3}\hat{e}_1\delta_{ij} = \frac{1}{2G}\left[\frac{\cos(2\xi+\psi)}{\cos 3\xi}S_{ij} + \sqrt{\frac{3}{\hat{s}_2}}\frac{\sin\omega}{\cos 3\xi}\left(S_{i\alpha}S_{\alpha j} - \frac{2}{3}\hat{s}_2\delta_{ij}\right)\right]. \tag{1}$$

In the left part: $e_{ij} = \varepsilon_{ij} - \varepsilon_0\delta_{ij}-$ components of the strain deviator; $\varepsilon_0 = (\varepsilon_{ii})/3 = \hat{e}_1/3-$ average strain; $\hat{e}_1$ – the first, $\hat{e}_2 = 3e_0^2/4$ – the second and $\hat{e}_3 = 3\det|D_e|-$ the third invariants of the strain tensor;

$$e_0 = \left(2/3e_{ij}e_{ij}\right)^{1/2} \tag{2}$$

Strain intensity. In the right part: $S_{ij} = \sigma_{ij} - \sigma_0\delta_{ij}-$ components of the stress deviator; $\sigma_0 = (\sigma_{ii})/3 = \hat{s}_1/3-$ medium voltage, $\hat{s}_1-$ first, $\hat{s}_2 = S_0^2/3-$ second and $\hat{s}_3 = -3\det|D_\sigma|-$third invariants of the stress tensor; $S_0 = \left(32S_{ij}S_{ij}\right)^{1/2}-$ is the intensity of the stress; $S_i = S_0c_i/3$- principal values of the stress deviator; $e_i = e_0d_i/2-$ the main values of the deviator of the strain used in [3]; $c_1 = 2\cos\xi,\ c_2 = \sqrt{3}\sin\xi - \cos\xi, c_3 = -\left(\sqrt{3}\sin\xi + \cos\xi\right)-$ trigonometric values that relate the main stresses to the stress intensities and similar d_i to the strain intensities.

Abandoning the constancy of the phase similarity diverters ω, which was proposed in [4], the generalized modulus G and the phase can be expressed through the coefficients of the tensor arguments:

$$X = \frac{1}{2G}\frac{\cos(2\xi+\psi)}{\cos 3\xi},\quad Y = \frac{1}{2G}\sqrt{\frac{3}{\hat{s}_2}}\frac{\sin\omega}{\cos 3\xi} = \frac{1}{2G}\frac{3}{S_0}\frac{\sin\omega}{\cos 3\xi} \tag{3}$$

For this we can use Eq. (1) presented for the main component of the deviator of the strain

$$e_i = XS_i + Y\left(S_i^2 - 2/9S_0^2\right). \tag{4}$$

The coefficients X and Y can be given an unambiguous physical meaning and formulas for determining them can be derived. Using three shear pliabilities $\varphi_i = \gamma_i/\tau_i$ i in sites with principal tangential stresses $\tau_i = \left(S_j - S_\alpha\right)/2$, where $\gamma_i = e_j - e_\alpha-$ a are the principal shifts, Eq. (12) allow us to find three shear pliabilities $\varphi_i = 2\left(X - \hat{Y}c_i\right)$, where $\hat{Y} = YS_0/3$. Given that the sum $(c_i) = 0$, from the relations for the pliabilities we find their average value and standard deviation:

$$\Phi_m = (\phi_i)/3 = 2X;\quad \Phi_d = \left\{\left[\left(\phi_j - \phi_\alpha\right)^2\right]/8\right\}^{1/2} \tag{5}$$

Thus, the analysis of the Eq. (1) allows, without any assumptions, to be free from uncertainty and to find an approach to the characterization of the deformation Φ_m and Φ_d that are already used for different materials, therefore will continue to remain the same notation, naming the material features:

$$\Phi_m = 2X = \frac{\cos(2\xi+\psi)}{G\ \cos 3\xi};.\ \Phi_d = \frac{1}{3}\hat{Y} = \frac{\sin\omega}{2G\ \cos 3\xi} \tag{6}$$

The sum of the squares of the differences of the main values of the deformation deviator

$$\left(e_i - e_j\right)^2 = \frac{{S_0}^2}{9}\left(c_i - c_j\right)^2\left(X^2 + 2X\hat{Y}c_\alpha + \hat{Y}^2c_\alpha^2\right) \tag{7}$$

leads to the need to calculate the relations: $\sum\left(c_i - c_j\right)^2 = 18$, $\sum c_\alpha\left(c_i - c_j\right)^2 = 18\sin 3\xi$, $\sum c_\alpha^2\left(c_i - c_j\right)^2 = 18$; i, j, $\alpha = 1, 2, 3$; $i \neq j \neq \alpha$. Finally, the relationship between the strain intensity (2) and the stress intensity is reduced to the equation:

$$e_0 = \frac{2S_0}{3}\left[\left(X^2 + 2X\hat{Y}\sin 3\xi + \hat{Y}^2\right)\right]^{1/2}. \tag{8}$$

It leads to generalized malleability:

$$\Phi_\xi = \frac{3e_0}{S_0} = \left[\Phi_m^2 + (4/3)\Phi_m\Phi_d\sin 3\xi + (4/9)\Phi_d^2\right]^{1/2}, \tag{9}$$

as a function of the angle ξ, and the inverse of the malleability to the generalized modulus of elasticity under shear:

$$G = 1/\Phi_\xi = \frac{1}{2}\sqrt{\hat{s}_2/\hat{e}_2} = 1/\left\{2\left[\left(X^2 + 2X\hat{Y}\sin 3\xi + \hat{Y}^2\right)\right]^{1/2}\right\}. \tag{10}$$

It follows from this relation that the modulus clearly depends on the type of stress state, and it can be a constant value only in the special case, as it was envisaged in [4]. After replacing the second invariants on the stress intensity and strain intensity and replacing the sequence of main stresses: $\sigma_1 \geq \sigma_2 \geq \sigma_3$, it is possible to give the original (1) equations of V. V. Novozhilov a form that was used without simplifications in the work [7]. After replacing the second invariants on the stress intensity and strain intensity and replacing the sequence of main stresses:, it is possible to give the original equations of V. V. Novozhilov (1) the form, without any simplifications, which was also used in the work [7]:

$$e_{ij} = \Phi_m S_{ij}/2 + \Phi_d\left(S_{i\alpha}S_{\alpha j} - 2S_0^2/9\delta_{ij}\right)/S_0. \tag{11}$$

After replacing the third invariants, the formulas for the angles take the form: the first $\xi = 1/3\arccos\left[27S_{ij}S_{j\alpha}S_{\alpha i}/\left(2S_0^3\right)\right]$ – is the angle of the stress state view, and the second one is $\psi = 1/3\arccos\left[4e_{ij}e_{j\alpha}e_{\alpha i}/\left(3e_0^3\right)\right]$ – the angle of the view of the deformed state, which change already in other limits: $0 \leq \xi\ u\ \psi \leq \pi/3$; $i, j, \alpha = 1, 2, 3$; $i \neq j \neq \alpha$. The coefficients for tensor arguments (6) make it possible to find a formula for determining the phase the similarity of deviators:

$$\omega = \xi - \psi. \tag{12}$$

The exact definition of which is given below. Performing trigonometric transformations taking into account the new sequence of principal stresses, the material functions in Eq. (8) can be represented:

$$\Phi_m = \Phi_\xi \sin(3\xi - \omega)/\sin 3\xi = \varphi_i/3; \tag{13}$$

$$\Phi_d = 3\Phi_\xi \sin(\omega)/(2\sin 3\xi) = \left\{3/8\left[(\Phi_m - \varphi_i)^2\right]\right\}^{1/2}. \tag{14}$$

where they acquire values that have a physical meaning of average and standard compliance, manifesting themselves by statistical characteristics. The deviatory part of M. Rayner's equations [2] leads to the same results of the functions φ_i.

4. Initial data

Due to the lack of proven methods, the first calculations in [8] used only the results of tensile and compression tests. Generalized compliance is determined by the relation (6), which for these states is taken by simple expressions:

$$\Phi_{\xi p} = \Phi_m + 2/3\Phi_d, \quad \Phi_{\xi c} = \Phi_m - 2/3\Phi_d \tag{15}$$

Assuming the independence of these functions from the type of stress state, we find a simple way to approximate the calculation of the shear modulus and the phase similarity of deviators according to the formula (6). The form change for any stress state, although approximate, can be described. To refine it, you can use the same ratio, but for a pure shift. At the same time, difficulties arose due to the fact that the tests were usually carried out on other equipment and other means of measuring deformations, so the lack of initial data was compensated by algorithms that were derived from the same equations converted to equations for anisotropic media [8, 10].

Experimental data obtained by tensile testing and compression of grain composite [8, 11], which has the maximum deformation under compression $\varepsilon_c > 10\%$ in the form of primary charts $\sigma_i - \varepsilon$ and graphs for the coefficients of lateral deformations, $\nu_i - \varepsilon$, $\nu_i = -\varepsilon_n/\varepsilon$, stress and strain have to specify, according to the formulas of [4], which can be given as follows:

$$\sigma^* = \sigma\left[\left(1 + 2\varepsilon_n^*\right)^2/(1 + 2\varepsilon^*)\right]^{1/2}, \tag{16}$$

where $\varepsilon^* = \varepsilon(1 + \varepsilon/2)$; $\varepsilon_n^* = \varepsilon_n(1 + \varepsilon_n/2)$; $\nu_i' = -\varepsilon_n^*/\varepsilon^*$ – are the coefficients of transverse deformations; $i = p, c$; index n – indicates transverse deformation. The stresses σ^* and deformations ε^* are called reduced [4]. Then the asterisk above the given stresses and deformations is removed. The ratio of material functions can be considered a coefficient of variation [14, p. 544]:

$$p = \Phi_d/\Phi_m = 3\sin\omega/2\sin(3\xi - \omega), \tag{17}$$

Since the material functions exhibit a statistical character, and its values correspond to the condition: $p < 1$. The study of its extremum shows that the derivative with respect to the angle ξ is zero if the phase of similarity of deviators obeys equality:

$$\omega = \mathrm{arctg}[2p\sin 3\xi/(3 + 2p\cos 3\xi)]. \tag{18}$$

The graphs for the phase differ slightly from the half-wave of the sine wave when the angle ξ changes from zero to $\pi/3$.

For phase values other than zero, the ratios of the deviator components belonging to the same stress state are not equal: $e_1/S_1 \neq e_2/S_2 \neq e_3/S_3$– is the condition of their disproportionality. However, for the states of tension and compression, this inequality becomes an equality: $S_1/S_2 = e_1/e_2 = 1$, since similarity conditions are implemented for them, since $S_2 = S_3$ and $e_2 = e_3$, so the phase is zero regardless of the properties of the This conclusion is consistent with the relation (13), which directly follows from the formulas (13) and (10). The material functions are similar: $\Phi_d = p\Phi_m$– for all states. This connection of functions allows us to consider both shape-changing deformations and volumetric deformations in the form of two parts. The first part should be associated with a change in the intermolecular distances in the rigid elements of the structure, and the second part of the deformation, including the coefficient of variation, should be attributed to the loss of bonds [8]. These deformations, despite their different physical origin, are included in the model as elastic. The initial data were taken based on the results of tests [8] obtained during tests of grain composites, the diagrams of which are shown in **Figure 2a** with dashed lines.

Solid lines represent two diagrams, after the refinement performed according to formulas (11). The dependence of $S_{0\tau}$ on the strain, taken as a diagram for pure shear (fine stroke), is obtained from diagrams for stretching and compression, according to an algorithm [7] using transformed equations. The stress values along the ordinate axis in **Figure 2a** in MPa.

Graphs for the coefficient of variation p (dashed line), the maximum values of the similarity phase of the deviators ω_{max} (small stroke), and the functions by which they are determined are shown in **Figure 2b**. These functions include Φ_d and Φ_ξ for stretching and compressing (solid lines).

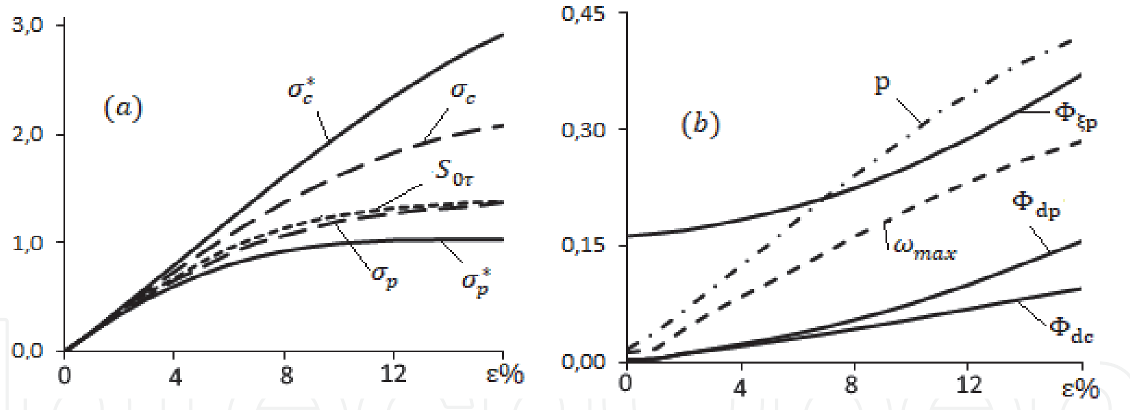

Figure 2.
a: Test diagrams of granular composites: Curve σ_p– During the tensile test, curve σ_c – for compression, curve $S_{0\tau}$ –according to the algorithm using data on tension and compression; curves σ_p^* and σ_c^* – after the transition to the reduced stresses. b: Curves based on the results of calculations: The change in the p – Coefficient of variation and the ω–phase of the similarity of deviators and the curves Φ_d, Φ_m, and Φ_ξ for the characteristics of the shape change with increasing deformation.

5. On the equations for form-changing

The rejection of the constancy of the phase gives the ratio of (6), which after the transition to the second sequence of the principal stresses is the law of deformation:

$$e_0 = \Phi_\xi S_0/3, \tag{19}$$

where the main characteristic becomes generalized compliance (7):

$$\Phi_\xi = \Phi_m\left[1 + (4/3)\mathrm{p}\cos 3\xi + (4/9)\mathrm{p}^2\right]^{1/2} = 1/\mathrm{G}, \tag{20}$$

as the inverse of the generalized shift modulus of G, they are represented in a discrete (digital) form by a mathematical model, as well as material functions. After replacing the sequence of main stresses, sin3ξ in the expression (6) is transformed in the ratio (15) into $\cos 3\xi$. If the relation (15) is simplified by getting rid of the square root, then the second part with the coefficient of variation can be represented as:

$$e_0^* = \Phi_\xi^* S_0/3, \quad \Phi_\xi^* \approx \mathrm{p}\Phi_m[\cos 3\xi + (1/3)\mathrm{p}] \tag{21}$$

where the compliance for the second part is the value Φ_ξ^*.
From the ratio (15) for stretching and compression, it also follows:

$$\Phi_{\xi i} = \Phi_{mi}(1 \pm 2/3\mathrm{p}), \tag{22}$$

where $i = p, c$; (p– stretching, c - compression). The functions Φ_{mi} and Φ_{di}, as the characteristics of the shape change, are determined for these states using the first Cauchy sign [14]. On this basis, their values follow from the relations (13) and (10), if the angle ξ is shifted by a small deviation from the original angles. The second variant of determining the coefficient of variation follows from the relations (16):

$$\mathrm{p} = 3(\kappa - \kappa_m)/2(\kappa + \kappa_m). \tag{23}$$

It protects the characteristics of the shape change from errors in their calculations: Φ_m, Φ_d and Φ_ξ, where $\kappa = \Phi_{\xi p}/\Phi_{\xi c}$– is the ratio of generalized and $\kappa_m = \Phi_{mp}/\Phi_{mc}$– is the average compliance. Calculation of material functions by formulas (13) and (10), or rather by their second equalities, cannot be carried out, since there is no initial information about the functions $\varphi_i = \gamma_i/\tau_i$ for the same state. This obstacle can be overcome if we use the following postulates: the first one states that the values of the functions φ_i can be considered the values of the malleability $\varphi_i = 3e_0/S_0 = \Phi_{\xi i}$ for three stress states: stretching, net shear, and compression. According to the second one, the functions $\varphi_i = \Phi_{\xi i}$are equal.

The results of calculations for two variants according to the formulas (12) and (17) showed that they differ only by the fifth significant digit after the decimal point for any loading stage. It is for checking the postulates that duplication is necessary. If there is a coefficient of variation, the calculation of material functions for any other states is significantly simplified: first, Φ_m by relation (13) is determined, and then $\Phi_d = p\Phi_m$, as a function of the angle ξ and the load level, since the coefficient of variation is the only value independent of the type of stress state.

6. On the equation for volumetric deformation

The derivation of equality (21), as an additional part of the deformation of the form change, is proposed as an unknown formula for dilatancy, as a part of the volume deformation, consistent with the previously expressed idea that the parameter p allows the deformation, divided into two parts. This thought, the results of experimental studies and already published works allow us to propose an equation for the volumetric strain in the following form:

$$\varepsilon_0 = \varepsilon_y + \varepsilon_g = \sigma_0/3K_\xi + 2p\Phi_m æ S_0(1 + k\zeta)/9. \tag{24}$$

The first part ε_y– linearly dependent on the mean stress refers to the deformation of the stiffer elements of the structure, where the value K_ξ is the theoretical bulk elasticity modulus. The formula for linear-elastic deformation is inherited from linear elasticity theory, and the second part ε_g– dilatancy with the parameter p, including œ – the loosening parameter and Φ_d– the function reflecting the dependence of the volume strain on the form change. The coefficient k in formula (18) was introduced in order to take into account the influence of average stress on dilatancy as well as for convenience of checking the proposed relation. So, at $k = 0$ the formula for dilatancy takes the form that has already been used in several works of the author, including [7, 15], because at $k = 0.3$ the curves for volume deformations under tension and compression are well superposed on the experimental curves.

The process of transformation of the tensor-nonlinear equations mentioned above is covered in sufficient detail in [7 , p. 56] and probably first implemented in [10]. The equations for coupling the strain tensor to the stress tensor (8), together with the equation for average strain with average stress (18), lead to the equations for coupling the strain tensor to the stress tensor

$$\varepsilon_{ij} = \frac{(3\Phi|k + 2œ\Phi_d k)\sigma_0\delta_{ij}}{9} + \frac{\Phi_m S_{ij}}{2} + \frac{\Phi_d}{S_0}\left[S_{i\alpha}S_{\alpha j} - \frac{2(1-œ)S_0^2\delta_{ij}}{9}\right]. \tag{25}$$

The equations reduced to the principal deformations are used for the matrix transformation: $\varepsilon_i = a_{ij}\sigma_i$, which can then be reduced to the form of equations characteristic of anisotropic media:

$$\varepsilon_i = \sigma_i/E_i - \nu_{ji}\sigma_j/E_j - \nu_{\alpha i}\sigma_\alpha/E_\alpha, \tag{26}$$

with the known specifications for the diagonal components:

$$\mathrm{a_{ii}} = [3\Phi_m + \phi_h + \Phi_d c_{\mathrm{ii}}] = \mathrm{E_i^{-1}} \tag{27}$$

and non-diagonal matrix components:

$$\mathrm{a_{ij}} = \left[\frac{3\Phi_m}{2} - \phi_h - \Phi_d c_{\mathrm{ij}}\right] = -\nu_{\mathrm{ij}}\mathrm{E_j^{-1}}, \tag{28}$$

where $\phi_h = 1/\mathrm{K}_\xi + 2\Phi_d æ\mathrm{k}/9 = (\phi_\xi + a\Phi_d)/3$; $a = 2\mathrm{k}œ/3$; E_i—moduli of longitudinal elasticity, ν_{ij}— coefficients of transverse deformation.
Reconciliation of Eqs. (7.6) leads to the equation for the relation of average strain with stresses:

$$\varepsilon_0 = (\sigma_1\phi_{k1} + \sigma_2\phi_{\mathrm{k2}} + \sigma_3\phi_{k3})/3, \tag{29}$$

where $\phi_{ki} = 1/\mathrm{K_i}$ is the bulk elasticity yield

$$\phi_{ki} = 3(1 - \nu_{ij} - \nu_{i\alpha})/E_i = 3(1 - \nu_{\mathrm{i}})/E_i. \tag{30}$$

Pairs of coefficients $\nu_{\mathrm{i}} = (\nu_{\mathrm{ij}} + \nu_{\mathrm{i\alpha}})/2$ determine the transverse deformations in three directions of the main stresses and volumetric deformations; where $\mathrm{i} = 1, 2, 3; \mathrm{i} \neq \mathrm{j} \neq \alpha$. The relations (23) are an integral part of the methodology of determining K_ξ— theoretical bulk modulus of elasticity and œ— the loosening parameter. In this process, the most critical importance is assigned to the procedure of matching theoretical curves for transverse strain coefficients [7].

7. Supplement to the methodology

The high values of the theoretical modulus of volumetric elasticity, but low for compliance with tension, and low for compression, can be explained by a simple transformation of the ratio (18), if we isolate from it $\varepsilon_{\mathrm{y}} = \varepsilon_{0\mathrm{i}} - \varepsilon_{\mathrm{gi}} = \sigma_0\phi_{ki}/3$—linear volumetric deformation. It allows you to find the pliability ϕ_{ki} for stretching and compression, which are required to combine experimental curves with theoretical curves during the transformation. Taking $\zeta_c = -\zeta_p$, $1/\zeta_i \cong \pm 3$; $1/\zeta = 3,0009$ и $K_i = E_i/3(1 - 2\nu_i)$, simple actions lead to the formulas:

$$\phi_{kp} = \frac{1}{K_p} - 2œ_p\Phi_{\mathrm{dp}}(1/\zeta + \mathrm{k}) \cong \frac{1}{K_p} - 6.6œ_p\Phi_{\mathrm{dp}}, \tag{31}$$

$$\phi_{kc} = \frac{1}{K_c} + 2œ_c\Phi_{\mathrm{dc}}(1/\zeta - \mathrm{k}) \cong \frac{1}{K_c} + 5.4œ_c\Phi_{\mathrm{dc}}. \tag{32}$$

It follows from the first that the second term reduces the flexibility for stretching, and the value of the theoretical module, on the contrary, increases as an inverse value. In the second formula, the second term increases the malleability for compression, although dilatancy is present. The second terms in these relations allow us to quantify its influence on the values of theoretical compliance. From the second formula, for compression, greater malleability is required, although dilatancy is present. The second terms in these relations allow us to quantify its influence on the values of theoretical compliance. Since the pliability of ϕ_{kp} is determined by the initial value of the function Φ_{dp}, it makes sense to refine it by redefining the loosening parameter $œ_p \cong \left(\phi_p - \phi_{kp}\right)/6.6\Phi_{\mathrm{dp}}$ and then dilatancy. The mean stress in pure shear is zero, but given that, $\zeta_p + \zeta_c = 0$, as the value of the parameter ζ_τ, it is suggested algorithm, as a response to the question about the significance of the theoretical module, and for this condition: $\phi_{k\tau} = \frac{1}{K_\tau} + œ_\tau \Phi_{\mathrm{d}\tau} \approx K_{\xi i}/2$, where $i = p, c$; ($p-$ stretching, $c-$ compression).

It follows from the relations (24) and (25) that in the process of converting tensor-nonlinear equations to matrix equations, the pliabilities $\phi_{ki} = 1/K_{\xi i}$ are realized, the values of which along the axes 2 and 3 satisfy the conditions of continuity and smoothness, as functions of the main stresses. These formulas contain an answer to the reasons for the large difference in the values of the theoretical module. It is called theoretical, because its values correspond to the inequality with respect to the classical module: $K_\xi > K$. The considered technique made it possible to find such values of the theoretical elastic modulus that lead to more accurate values of the linear elastic volume deformation.

8. On deformation anisotropy

V. V. Novozhilov in his work [3] expressed his opinion about this phenomenon, for the description of which the mathematical apparatus of tensor-nonlinear equations can be used, as an "important phenomenon," without emphasizing on what characteristics it manifests itself. The studies show that the effect of dilatancy on the longitudinal elastic moduli E_i is not significant. Their divergence with different indices is less than 5%, but leads to appreciable strain anisotropy of the transverse strain coefficients. In the history of the mechanics of materials described in the book [1], much space is devoted to the research of its initial value (the Poisson's ratio), since not only modules, but also the theories of authoritative scientists depended on it. However, the latter values, for example, at destructive stresses, are not given due attention, especially in other areas of the main stresses. In this paper, perhaps for the first time, graphs of the theoretical coefficients of transverse deformations are given. They are easier to describe not by formulas, but by graphs for: $\nu_{12}, \nu_{31}, \ \nu_p, \nu_c, \$ and ν_i, $\Sigma\nu_i/3$. The line in **Figure 3a**, represented by points, here repeats the curves for $\nu_{12} = \nu_{13}$, which are combined with the values of the coefficient ν_p by the method. The deviation of the curve for the coefficient ν_p from its initial value should be considered the main "source" of dilatancy and all other coefficients. If this curve for the coefficients ν_p and $\nu_1$coincided with the graph for $\Sigma\nu_i/3$, then all the curves presented in

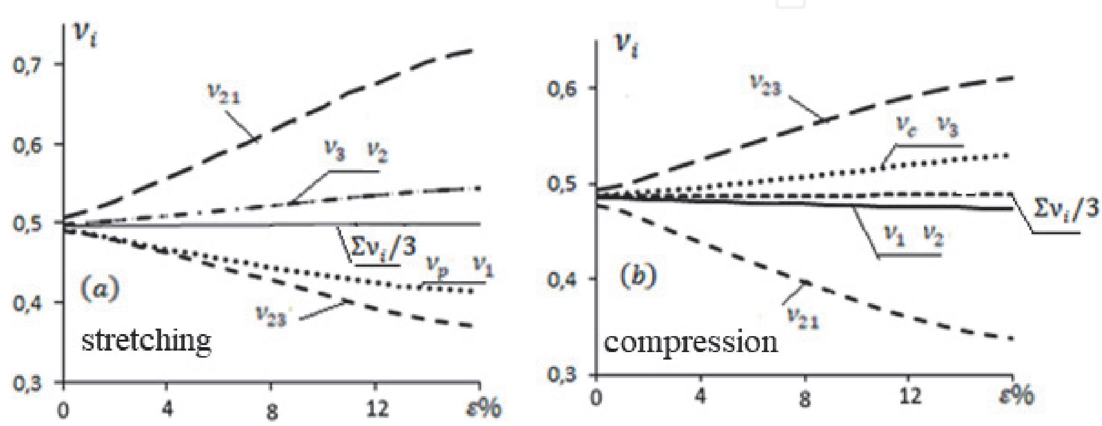

Figure 3.
The change in the coefficients of transverse deformations: a – Under tension; b – Under compression.

Figure 3a, would merge into one curve, and there would be no dilatancy. The main direction is the voltage σ_1.

The lower the values of the last points of the curve for ν_p fall, the greater the dilatancy takes on and the higher the values of the coefficients of the other two pairs, ν_2 and ν_3, which overlap each other, rise. Since the dilatancy is stretched in the direction of stretching, it is transverse for deformations of other directions. The coefficients of the first pair have the same values, $\nu_{12} = \nu_{13}$, but the coefficients of the other two pairs, $\nu_2$and ν_3, differ significantly. The graphs that make up the second pair of coefficients, ν_3 and ν_{23}, reveal their behavior—the values of ν_{23}, exceed the number 0.5.

Figure 3b shows graphs of the dependence of transverse deformations during compression. The line shown by the dots refers to the main direction coinciding with the voltage σ_3, and the graphs with the symbols ν_c and $\nu_3$should be considered the main "source" of dilatancy. As they increase, they cross the value of 0.5, which is typical for many loosening materials. The graphs for the coefficients with the symbols ν_1 and ν_2 coincide, slightly deviating from the graph for the curve $\Sigma\nu_i/3$, although the curves that make up them, ν_{21} and ν_{23}, are almost symmetrical.

The deformation anisotropy is more clearly shown on the graphs for the pliability of the bulk elasticity in the direction of the main stresses. The total volume deformation is determined by the formula (22), where $\phi_{ki} = \phi_k + æ(a + c_i)\Phi_d = 3(1-\nu_i)/E_i$– the compliance of the volume elasticity in the directions of the main stresses. In contrast to the theoretical volume compliance of ϕ_k the characteristics of ϕ_{ki} are smooth and continuous functions of stresses. Its first term is the pliability ϕ_k, established by the methodology, the second with a coefficient $a = 2k/3$, which is responsible for taking into account the dependence of the average voltage, and the third with a coefficient $c_{i.}$

which determines the directions of the axes. Give ϕ_{ki}, value (reverse module), to allow any state to find the values of three parameters changing of elasticity:

$$\vartheta_{\xi i} = \frac{\phi_k}{\phi_{ki}} = \frac{K_{\xi i}}{K_\xi}, \tag{33}$$

defining them as the degree of deviation from the theoretical volumetric compliance, which is the average, $\phi_k = \phi_{ki}/3$, for three compliance ϕ_{ki}. Each of them refers to the main stress, in the direction of which the initial values of the volume elasticity modules $K_{\xi i} = 1/\phi_{ki}$ are calculated (for $\sigma_i = 0$). In **Figure 4** curves 1, 2, and 3 represent graphs of these parameters $\vartheta_{\xi 1}$ по оси. The value of the parameter $\vartheta_{\xi 1}$ on curve 1 exceeds the values of other curves with a rapid decrease along the axis ξ to the value $\vartheta_0 = 1$. Judging by the shape of these curves, the elementary volume acquires the greatest deformation anisotropy in the direction with index 1. Curves with indices 2 and 3, having at first equal and small values compliance with the growth of the

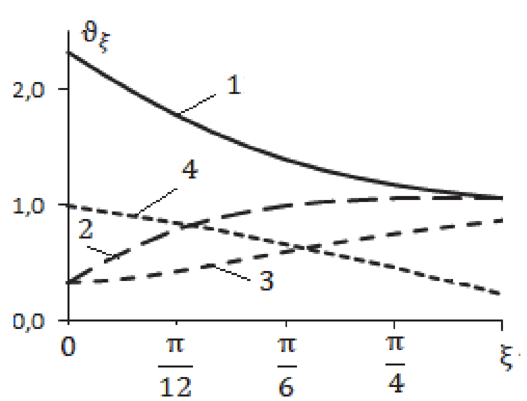

Figure 4.
Curves of changes in the values of the parameters of the changing elasticity $\vartheta_{\xi i}$.

angle ξ, increase slowly and in different ways. The third curve is affected by the presence of negative stress along the axis 3, given that the curves for these coefficients of transverse deformations overlap each other. Curve 4, denoted by the symbol $\vartheta_{\xi}^{*} = \phi_k/\phi_{kmax}$, is the ratio of the pliability of $\phi_k = \phi_{\mathrm{ki}}/3$ to its first value.

The behavior of the curves for the parameters $\vartheta_{\xi i}$ can be associated with the behavior of the interstructural connections involved in creating the dilatancy for each state. The ordinates of the points of the curves, as it were, show the number of lost connections related to dilatancy. Numerical values of parameters can be useful for comparing the behavior of different materials, which is an important procedure for their analysis and practical selection of materials that differ, for example, in the binding matrix. At the same time, the material more clearly exerts a real deformation anisotropy [16, p. 151].

Briefly still on the shape change, it should be noted that the initial values of shear moduli $G_{\xi i}$ or their yields $\Phi_{\xi i}$ during the shape change deformation have no such features as the bulk yields, although the different values of the transverse strain ratios ν_{ij}, analyzed above, naturally influence their behavior. Nevertheless, the ratios of the strain intensities found, as from the initial data associated with the experimental results, to the strain intensities found after the matrix transformation of Eq. (19) are equal to 1. The high accuracy of each strain is especially valuable in determining the Lode parameters [15] when processing the results of experimental studies carried out in the 30–50 years of the last century. In order to estimate the nonlinear properties of the materials used, researchers resorted to constructing Lode diagrams based on the results of experimental studies, for example, in [17–19] by testing tubular specimens. In the test process, two strains are most often measured: axial and circumferential. And the researchers had to calculate the radial strain from the condition of "incompressibility," considering the sum of these three strains equal to zero. This led to a noticeable discrepancy in the results of each author, so that the author of the already quoted book [1] placed in it a diagram of the S-shaped curve with a minimum and a maximum.

The solution to this problem is formulated using tensor-nonlinear Equations [15]. Using the material functions of the proposed equations, finding the difference of Lode parameters, $\Delta\lambda = \lambda_\sigma - \lambda_\varepsilon$, without assumptions, diagrams with one minimum were obtained. The first λ_σ for stresses and the second for deformations:

$$\lambda_\varepsilon = 3(\varepsilon_2 - \varepsilon_1 - \varepsilon_3)/(\varepsilon_1 - \varepsilon_3), \tag{34}$$

where the former repeats the same fraction with the principal stresses by which it is determined. The problem of the researchers was to determine λ_ε.

9. Conclusion

A variant of the tensor-nonlinear equations, which can become the main direction in the nonlinear theory of elasticity, is proposed for wide use. This concept leads to taking into account dilatancy and strain anisotropy, about which Novozhilov V.V. prudently expressed in his work. They were used to study the properties of different-module materials and show that this mathematical apparatus is suitable not only for describing second-order smallness effects but also for describing effects associated with changes in the material structure. The influence of dilatancy on all the characteristics of form change and bulk elasticity is revealed, since its development with proportional stress growth is the main cause of deformation anisotropy, both of transverse strain coefficients and of bulk elasticity yields (or modules), which are directly related to the changing elasticity parameter, which is a quantitative estimate of these changes. In tensile and near-tensile states, its values significantly exceed unity. This can be explained by the fact that, in the first direction, dilatancy, being transverse for the other directions, causes transverse strain coefficients with values exceeding the number 0.5. The assumption of dilatancy to elastic deformations is an unavoidable step to trace the behavior of all deformations along the three directions. The exact coincidence of the total bulk strain as the sum of its components in the direction of the principal stresses, or, as the sum of linear-elastic and dilatancy, indicates recognition of the fact that the apparatus of the proposed equations may be a major trend in nonlinear elasticity theory. Whatever concepts other elasticity theories may adhere to, taking into account the real values of transverse strain coefficients in tension and compression will implicitly lead to the consideration of dilatancy and, consequently, to the difference in the values of the bulk elasticity characteristics. The next stage in the development of the nonlinear theory of elasticity is the involvement of the apparatus of thermodynamics.

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