Chapter

Obtaining of a Constitutive Models of Laminate Composite Materials

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Abstract

The study of the mechanical behavior of composite materials has acquired great importance due to the innumerable number of applications in new technological developments. As a result, many theories and analytical models have been developed with which its mechanical behavior is predicted; these models require knowledge of elastic properties. This work describes a basic theoretical framework, based on linear elasticity theory and classical lamination theory, to generate constitutive models of laminated materials made up of orthotropic layers. Thus, the models of three orthotropic laminated composite materials made up of layers of epoxy resin reinforced with fiberglass were also obtained. Finally, by means of experimental axial load tests, the constants of the orthotropic layers were determined.

Keywords: composite materials, elasticity theory, orthotropic materials, experimental methods, Sheet theory

1. Introduction

One of today's engineering needs is to develop new materials capable of improving the common materials that exist today (such as metals), in weight, wear resistance, corrosion resistance, high strength and stability at high temperatures, among others [1, 2]. The properties are improved through the use of reinforcements with fibers or particles in polymers, metals and ceramics, among others, giving rise to composite materials. The uses of composite materials can be found in the automotive industry, in the wind, aerospace and military industries, in civil applications, among others [3, 4]. The mechanical behavior of CM in tension, bending, torsion, etc., have been studied for decades [5–7]. For example, Sun [8] used glass fiber reinforced polyester resin to improve mechanical properties such as tensile strength, flexural strength, and Young's Modulus for single and multiple fibers. Acosta [9] developed a novel method to determine the stresses in torsion problems of laminated trimetallic and bimetallic composite bars, for which experimental and numerical analysis were carried out.

On the other hand, a necessary task for engineering applications is obtaining the mechanical properties of composite materials such as Young's Modulus (E), Rigidity Modulus (G), Yield Stress and Maximum Stress at traction, among others. In this regard, various authors have developed various numerical models and experimental techniques (photo-acoustic, ultrasound, Moiré interferometry, electrical extensometry, etc.) which have been applied to the design of composite materials [10, 11]. To obtain the effective properties of composite materials with different configurations, the authors Acosta et al. [12], developed an analytical constitutive model that is used for the mechanical analysis of intralaminar and global stresses in laminated composite materials with isotropic plies subject to axial load and to determine the elastic constants (E, v, and G)) of each of its components, using the method of electrical extensometry.

Of the laminated composite structures, the most widely used are those formed by layers of orthotropic materials. The design and mechanical analysis of laminated composite material structures involves a large number of variables (fiber orientation, layer thickness and stacking sequence, material densities, topological design, etc.) [13]. Of the laminated composite structures, the most widely used are those formed by orthotropic layers.

The study of the mechanical behavior of laminate composite materials is of great importance for engineering, so it is necessary to have a theoretical framework for its analysis, both globally and locally. In this work, the conceptual and analytical models foundations of the theory of linear elasticity and of the classical theory of laminated composite materials are presented for the theoretical and experimental approach of models that predict the mechanical behavior and allow obtaining its effective mechanical properties of a multi directionally reinforced laminated composite by orthotropic layers reinforced with longitudinal fibers [14–17]. With the models, the real properties obtained imply the effects of the existing defects in the interfaces between the layers (glue, gluing defects, layer fusion, etc.), which should considerably improve the efficiency in stress analysis.

2. Theory of elasticity

Stresses are internal forces that occur in bodies as a result of applying forces on their boundaries. If an imaginary cut is made in a body and if the internal distribution of forces on the cut surface is analyzed, then the stresses can be obtained as follows:

$$\sigma_{ij} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1}$$

where i and j are the Cartesian components x, y or z. For this case, i corresponds to the normal to the imaginary plane analyzed and j is associated with the direction of the force ΔF applied on the element of area ΔA. The state of stresses at a point can be represented graphically, as shown in **Figure 1**.

The stresses that act normally to the surface are called normal stresses $(\sigma_x, \sigma_y, \sigma_z)$, while those forces that act tangent to the surface are known as shearing stresses $(\tau_{xy}, \tau_{xz}, \tau_{yz})$. The complete stress analysis in a body implies determining the state of stresses in each of the points that make it up, and a partial analysis, in one or a set of points sufficient to solve a particular problem. The stress model is continuous and linear, from the mathematical point of view, which implies working with neighborhoods of infinitesimal points. By applying static equilibrium, the following field or equilibrium equations are obtained [14]:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0 \tag{2}$$

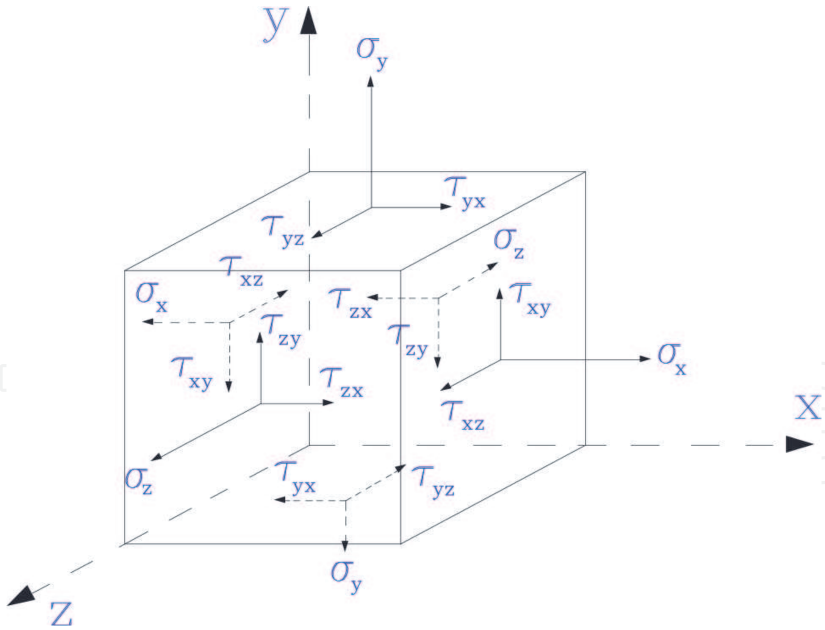

Figure 1.
State of stresses on a point.

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0$$

On the other hand, a strain is defined as the relative displacement between the internal points of a body. If we consider a change in length in a straight-line segment in the x, y and z axes, we have normal or longitudinal strain ε_x, ε_y y ε_z. In the same way, if we have a transformation between the angles of two straight lines, the shear, or angular, strains γ_{xy}, γ_{xz}, y γ_{yz}, are obtained in the xy, xz and yz planes, respectively. The following expression represents the strain–displacement equations for normal and shear strains:

$$\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{3}$$

Or, explicitly:

$$\varepsilon_x = \frac{\partial u}{\partial x}; \varepsilon_y = \frac{\partial v}{\partial y}; \varepsilon_z = \frac{\partial w}{\partial z}; \tag{4}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + ; \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

The strain model described in expressions (Eq. (3)) is considered linear and continuous, which implies a model of infinitesimal strains. According to (Dally), the linear equations between stresses and strains give rise to the constitutive model. These equations are determined according to the following expression:

$$\sigma_i = C_{ij}\varepsilon_j; i, j = 1, 2, 3, \ldots, 6 \tag{5}$$

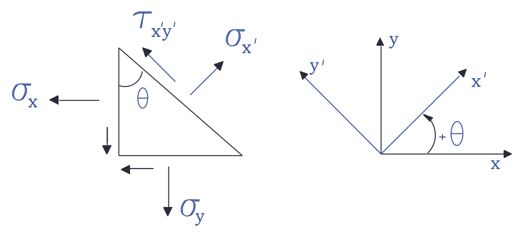

Figure 2.
Transformation of stresses at a point with state of plane stresses.

where σ_i are the stress components, C_{ij} is the stiffness tensor and ε_j are the strain components. The explicit form of expression (Eq. (5)) is known as Hooke's Generalized Law. This is:

$$\begin{aligned}
\sigma_x &= C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z + C_{14}\gamma_{xy} + C_{15}\gamma_{yz} + C_{16}\gamma_{zx} \\
\sigma_y &= C_{21}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\varepsilon_z + C_{24}\gamma_{xy} + C_{25}\gamma_{yz} + C_{26}\gamma_{zx} \\
\sigma_z &= C_{31}\varepsilon_x + C_{32}\varepsilon_y + C_{33}\varepsilon_z + C_{34}\gamma_{xy} + C_{35}\gamma_{yz} + C_{36}\gamma_{zx} \\
\tau_{xy} &= C_{41}\varepsilon_x + C_{42}\varepsilon_y + C_{43}\varepsilon_z + C_{44}\gamma_{xy} + C_{45}\gamma_{yz} + C_{46}\gamma_{zx} \\
\tau_{yz} &= C_{51}\varepsilon_x + C_{52}\varepsilon_y + C_{53}\varepsilon_z + C_{54}\gamma_{xy} + C_{55}\gamma_{yz} + C_{56}\gamma_{zx} \\
\tau_{xz} &= C_{61}\varepsilon_x + C_{62}\varepsilon_y + C_{63}\varepsilon_z + C_{64}\gamma_{xy} + C_{65}\gamma_{yz} + C_{66}\gamma_{zx}
\end{aligned} \tag{6}$$

Here, $C_{11}, C_{12}, \ldots C_{66}$ (C_{ij} for $i,j = 1, 2, \ldots, 6$), are called the stiffness constants of the material and are independent of the stress values or the strain values. The following expression shows the inverse form between strains and stresses:

$$\varepsilon_i = S_{ij}\sigma_j, i,j = 1, 2, 3, \ldots, 6 \tag{7}$$

Here, S_{ij} is known as the compliance tensor. According to Durelli [14] and when considering the strain energy in the analysis, the following expressions are fulfilled:

$$C_{ij} = C_{ji}; S_{ij} = S_{ji} \tag{8}$$

It is worth mentioning that the constants of the constitutive models can be put as a function of the so-called engineering constants (Young's modulus (E), Poisson's ratios (ν) and shear modulus (G)), which can be obtained through tests of pure tension and shear. According to Durelli [14] the model of the Theory of Linear Elasticity that governs the bodies´ mechanical analysis, is composed of a system of 15 partial differential equations and 15 unknowns (σ_{ij}, $\varepsilon_{ij} y u_i$). On the other hand, for practical purposes, it is necessary to characterize the state of stress at a body's point on an arbitrary plane. The analytical equations that govern the state of stress at a body's point that make it possible to linearly transform the stress components, refer to a reference coordinate system and find the stress components regarding any other system. **Figure 2** shows a graphic example of a state of plane stresses.

In the case of the strain transformation laws, a similar process is carried out.

3. Mechanical analysis in orthotropic materials

An orthotropic material is one in which the values of its elastic properties are different for each orientation, referred to three coordinate axes, each perpendicular to another (see **Figure 3**). Examples of orthotropic materials are: wood, unidirectionally materials reinforced with fiberglass, carbón, Kevlar, among others. In orthotropic materials, the stiffness changes depending on the orientation of the

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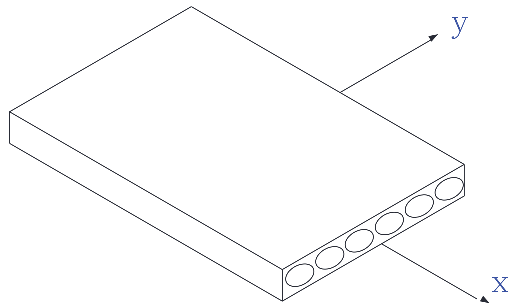

Figure 3.
Axes of symmetry of a plane orthotropic material.

fibers. To determine the stress or strain components in any direction, it is necessary to know the states of stress or strain at a point and apply the analytical transformation equations. The stress–strain relations and the stress and strain transformation equations are the basis for the construction of constitutive models to study the stresses and to determine the effective mechanical properties of laminated composite materials as a function of the orientation and direction [16].

To clarify the analysis, the *xy* system will be used when it coincides with the axes of symmetry of the properties of the unidirectional material and the system of 12 will be used for any coordinate system outside of it (see **Figure 4**).

The equations that govern the transformation of plane stresses, in a unidirectionally reinforced laminate, allow the obtention of the value of the stresses in the xy system once the stresses in the 12 system are known (see **Figures 5** and **6**).

To know the relations between the stresses of the *xy* system regarding the stresses in the 12 system, a cut perpendicular to the fibers is analyzed and the director cosines, m = cos θ and n = sin θ, are used (see **Figure 6**). These equations are expressed as follows:

$$\begin{aligned} &\sigma_x = \sigma_1 m^2 + \sigma_2 n^2 + \sigma_6 m n_6; \sigma_y = \sigma_1 n^2 + \sigma_2 m^2 - 2\sigma_6 mn; \\ &\sigma_s = -\sigma_1 mn + \sigma_2 mn + \sigma_6 [m^2 - n^2] \end{aligned} \tag{9}$$

Here, σ_x and σ_y are the normal stresses and σ_s corresponds to the shear stress in the xy system; σ_1 and σ_2 are the normal stresses and σ_6 is the shear stress in the 12 system.

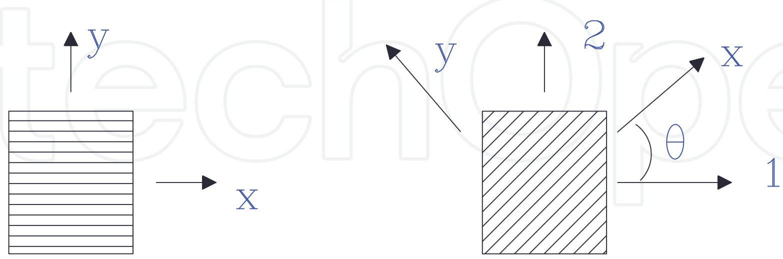

Figure 4.
System xy *in the direction of the fibers and system* 12 *in a different orientation.*

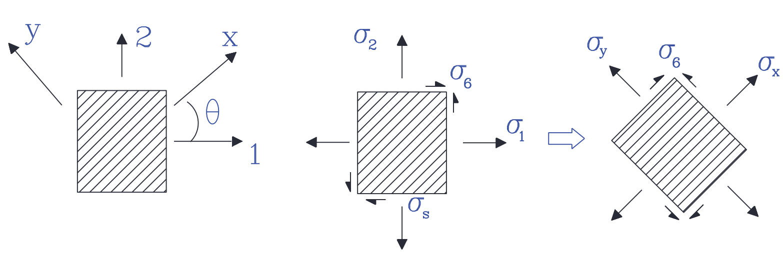

Figure 5.
Transformation of stress.

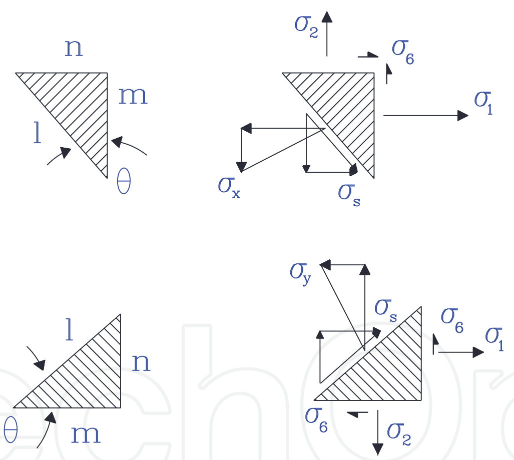

Figure 6.
Stress components in system xy *as a function of stress components in system 12.*

The strain transformation equations, between the xy Cartesian strains and the strains in system 12, in terms of the director cosines are shown in equations (Eq. (10)).

$$\begin{aligned} \varepsilon_x &= \varepsilon_1 m^2 + \varepsilon_2 n^2 + \varepsilon_6 m n_6; \varepsilon_y = \varepsilon_1 n^2 + \varepsilon_2 m^2 - \varepsilon_6 mn \\ \varepsilon_s &= -\varepsilon_1 mn + \varepsilon_2 mn + \varepsilon_6 [m^2 - n^2] \end{aligned} \tag{10}$$

Here, ε_x and ε_y are the normal strains and ε_s corresponds to the shear strain in the xy system; ε_1 and ε_2 are the normal strains and ε_6 is the shear strain in the 12 system. The linear relations, between stresses and strains in a plane system for an orthotropic composite, are obtained by means of Hooke's generalized law. These relations are as follows:

$$\begin{aligned} \sigma_x &= C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z; \sigma_y = C_{21}\varepsilon_x + C_{22}\varepsilon_y + C_{23}\varepsilon_z; \\ \sigma_z &= C_{31}\varepsilon_x + C_{32}\varepsilon_y + C_{33}\varepsilon_z \; \tau_{xy} = C_{44}\gamma_{xy}; \tau_{yz} = C_{55}\gamma_{yz}; \tau_{xz} = C_{66}\gamma_{zx} \end{aligned} \tag{11}$$

It is worth mentioning that the state of stress at all points is considered plane stress because it is assumed that the distribution of strains is homogeneous through the thickness of the orthotropic composite. The planes of symmetry correspond to the longitudinal direction of the fibers and the transverse direction, respectively. The composite material and its symmetry planes are shown in **Figure 3**. The material stiffness coefficients are obtained from the development of the following simple axial tests (see **Figure 7**): 1) Tension test in the longitudinal direction of the fibers, 2) Tension test in the cross-fiber direction and 3) Pure shear test.

For a uniaxial state of stress, we have the equations $\varepsilon_x = S_{11}\sigma_x$ and $\varepsilon_y = S_{21}\sigma_y$. By Hooke's law, the stress in the direction of the applied load P is: $\sigma = \frac{P}{A}$, if and only if it is assumed that the P force is applied uniformly to a cross section and the change of the latter for any P value is negligible [14]. The equations between the stress σ and the strain Ɛ (in the direction of σ) within the elastic-linear range is: $\sigma = \varepsilon E$, where E is the Young's modulus of the material. Note that if $S_{11} = \frac{1}{E_x}$, then

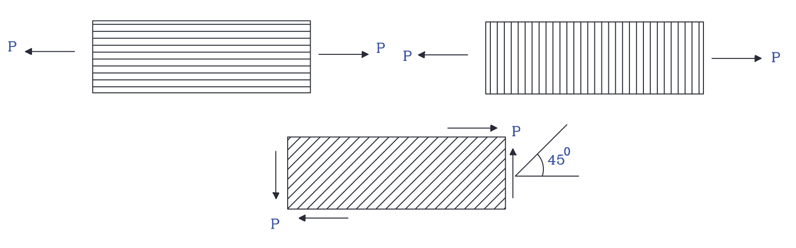

Figure 7.
Tests, tension in the direction of the fibers, tension in the direction transverse to the fibers and pure shear at 45°.

$$\varepsilon_x = \frac{\sigma_x}{E_x}; \varepsilon_y = -\frac{\upsilon_x \sigma_x}{E_x} \tag{12}$$

Here, $\nu_x = -\frac{\varepsilon_y}{\varepsilon_x}$ is the longitudinal Poisson's ratio and ε_x and ε_y are the longitudinal and transverse deformations, respectively.

By following a similar process performed in the previous test (see **Figure 1**.4), the following equations are obtained:

$$\varepsilon_x = -\frac{\nu_y \sigma_y}{E_y}; \varepsilon_y = -\frac{\sigma_y}{E_y} \tag{13}$$

where $\nu_y = -\frac{\varepsilon_x}{\varepsilon_y}$ is the transverse Poisson's ratio, E_y is the transverse Young's Modulus and ε_y and ε_x are the longitudinal and transverse strains, respectively. In a pure shear test (see **Figure 7**), the constitutive relation between shear stress and shear strain is as follows:

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \tag{14}$$

Here, τ_{xy} is the shear stress, γ_{xy} is the shear strain, and G_{xy} is the shear modulus of the material. As result of the three tests, the following constitutive relations are obtained:

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_y \sigma_y}{E_y}; \varepsilon_y = -\frac{\nu_x \sigma_x}{E_x} - \frac{\sigma_y}{E_y}; \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \tag{15}$$

The relations that define the stiffness constants as a function of the engineering constants are as follows:

$$Q_{xx} = \frac{E_x}{1 - \upsilon_x \upsilon_y}; Q_{yy} = \frac{E_y}{1 - \upsilon_y \upsilon_x}; Q_{yx} = \frac{\upsilon_x E_y}{1 - \upsilon_x \upsilon_y}; Q_{xy} = \frac{\upsilon_y E_x}{1 - \upsilon_x \upsilon_y}; Q_{ss} = E_{ss} \tag{16}$$

The relations between compliance constants and engineering constants are:

$$S_{xx} = \frac{1}{E_x}; S_{yy} = \frac{1}{E_y}; S_{yx} = \frac{\upsilon_y}{E_y}; S_{xy} = \frac{\upsilon_x}{E_x} Q_{ss} = \frac{1}{G_{xy}} \tag{17}$$

By symmetry of the stiffness tensor and the compliance tensor, we have: $\frac{\upsilon_x}{E_x} = \frac{\upsilon_y}{E_y}$. This relation reduces the number of independent constants, from five to four, in the plane problem. To determine the stresses in terms of the strains (for a reference system), different from the material's symmetry axes, system 12 (see **figure 8**). Is necessary to apply a strain transformation process to change to the symmetry axes (system xy), configuration (a) to (b) and determine the state of stresses with constitutive model, configuration (b) to (c) to subsequently apply a stress transformation and come at the stresses in axes 12 (configuration (c) to (d)). To transform configuration (a) to (d), knowing the stiffness constants in the specific orientation and the value of the strain components in same direction, are the following steps:

Step 1) The step from configuration (a) to (b) is obtained by making a positive strain transformation and using equations (Eq. (10)), that is:

$$\begin{aligned} \varepsilon_{\mathrm{x}} &= m^2\varepsilon_1 + n^2\varepsilon_2 + mn\varepsilon_6; \varepsilon_{\mathrm{y}} = n^2\varepsilon_1 + m^2\varepsilon_2 - mn\varepsilon_6; \\ \varepsilon_{\mathrm{s}} &= -2\mathrm{mn}\varepsilon_1 + 2\mathrm{mn}\varepsilon_2 + \left[m^2 - n^2\right]\varepsilon_6 \end{aligned} \tag{18}$$

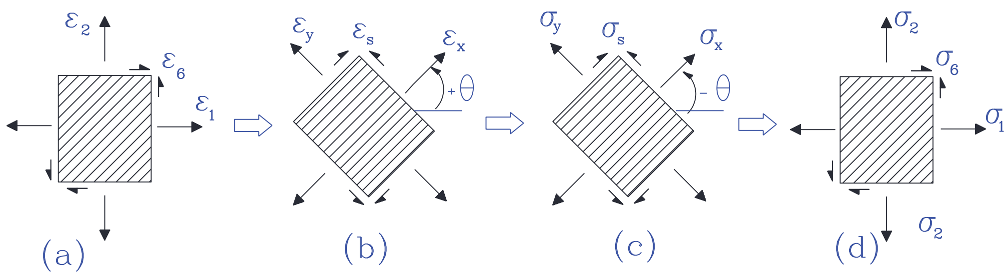

Figure 8.
Transformation process, from the state of strains to the state of stresses, in a coordinate system 12. Strain transformation process, (a) to (b), getting stresses with constitutive model, (b) to (c) and finally arrive at the state of stresses in axes 12, (c) to (d).

Step 2) To go from configuration (b) to configuration (c), the stress–strain relations are used in the material's symmetry axes. These relations are as follows:

$$\begin{aligned} \sigma_x &= Q_{xx}[m^2\varepsilon_1 + n^2\varepsilon_2 + mn\varepsilon_6] + Q_{xy}[n^2\varepsilon_1 + m^2\varepsilon_2 - mn\varepsilon_6]; \\ \sigma_y &= Q_{yx}[m^2\varepsilon_1 + n^2\varepsilon_2 + mn\varepsilon_6] + Q_{yy}[n^2\varepsilon_1 + m^2\varepsilon_2 - mn\varepsilon_6]; \\ \sigma_{ss} &= Q_{ss}[-2m\varepsilon_1 + 2mn\varepsilon_2 + [m^2 - n^2]\varepsilon_6] \end{aligned} \tag{19}$$

Step 3) To go from configuration c) to d), the stress transformation equations with negative angle of rotation are used. This is:

$$\begin{aligned} \sigma_1 = m^2&\left[Q_{xx}\left[m^2\varepsilon_1 + n^2\varepsilon_2 + mn\varepsilon_6\right] + Q_{xy}\left[n^2\varepsilon_1 + m^2\varepsilon_2 - mn\varepsilon_6\right]\right] \\ &+ n^2\left[Q_{yx}\left[m^2\varepsilon_1 + n^2\varepsilon_2 + mn\varepsilon_6\right] + Q_{yy}\left[n^2\varepsilon_1 + m^2\varepsilon_2 - mn\varepsilon_6\right]\right] \\ &- 2mn\left[Q_{ss}\left[-2mn\varepsilon_1 + 2mn\varepsilon_2 + \left[m^2 - n^2\right]\varepsilon_6\right]\right] \\ \sigma_2 = n^2&\left[Q_{xx}\left[m^2\varepsilon_1 + n^2\varepsilon_2 + mn\varepsilon_6\right] + Q_{xy}\left[n^2\varepsilon_1 + m^2\varepsilon_2 - mn\varepsilon_6\right]\right] \\ &+ m^2\left[Q_{yx}\left[m^2\varepsilon_1 + n^2\varepsilon_2 + mn\varepsilon_6\right] + Q_{yy}\left[n^2\varepsilon_1 + m^2\varepsilon_2 - mn\varepsilon_6\right]\right] \\ &+ 2mn\left[Q_{ss}\left[-2mn\epsilon_1 + 2mn\varepsilon_2 + \left[m^2 - n^2\right]\epsilon_6\right]\right] \\ \sigma_6 = mn&\left[Q_{xx}\left[m^2\varepsilon_1 + n^2\varepsilon_2 + mn\varepsilon_6\right] + Q_{xy}\left[n^2\varepsilon_1 + m^2\varepsilon_2 - mn\varepsilon_6\right]\right] \\ &- mn\left[Q_{yx}\left[m^2\varepsilon_1 + n^2\varepsilon_2 + mn\varepsilon_6\right] + Q_{yy}\left[n^2\varepsilon_1 + m^2\varepsilon_2 - mn\varepsilon_6\right]\right] \\ &+ \left(m^2 - n^2\right)\left[Q_{ss}\left[-2mn\varepsilon_1 + 2mn\varepsilon_2 + \left[m^2 - n^2\right]\varepsilon_6\right]\right] \end{aligned} \tag{20}$$

Considering the strains and the symmetry of the stiffness tensor, the constitutive relations can be obtained in an arbitrary orientation, system 12 (see **Figure 9**), to a direct transformation from the state of strain to the state of stress. What would be:

$$\begin{aligned} &\sigma_1 = Q_{11}\varepsilon_1 + Q_{12}\varepsilon_2 + Q_{16}\varepsilon_6; \sigma_2 = Q_{21}\varepsilon_1 + Q_{22}\varepsilon_2 + Q_{26}\varepsilon_6; \\ &\sigma_6 = Q_{61}\varepsilon_1 + Q_{62}\varepsilon_2 + Q_{66}\varepsilon_6 \end{aligned} \tag{21}$$

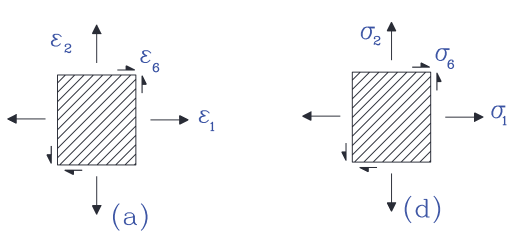

Figure 9.
Direct transformation from the state of strain to the state of stress in a system 12, getting the state of stresses in terms of the state of strains, with constitutive relations. Configuration a) to d).

The stiffness constants in an arbitrary orientation are defined as follows:

$$\begin{aligned}
Q_{11} &= m^4 Q_{xx} + 2m^2n^2 Q_{xy} + n^4 Q_{yy} - 4m^2n^2 Q_{ss} \\
Q_{12} &= m^2n^2 Q_{xx} + (m^4 + n^4) Q_{xy} + m^2n^2 Q_{yy} - 4m^2n^2 Q_{ss} \\
Q_{16} &= m^3 n Q_{xx} + (mn^3 + m^3 n) Q_{xy} - mn^3 Q_{yy} + 4[mn^3 - m^3 n] Q_{ss} \\
Q_{22} &= n^4 Q_{xx} + 2m^2n^2 Q_{xy} + m^4 Q_{yy} + 4m^2n^2 Q_{ss} \\
Q_{26} &= mn^3 Q_{xx} + (m^3 n - mn^3) Q_{xy} - m^3 n Q_{yy} + 2[m^3 n - mn^3] Q_{ss} \\
Q_{66} &= m^2\, n^2 Q_{xx} - 2m^2n^2 Q_{xy} + m^2n^2 Q_{yy} + [m^2 - n^2]^2 Q_{ss}
\end{aligned} \tag{22}$$

The obtained results are of great importance due to these are the equations that define the variation of the constants as a function of the orientation. On the other hand, the constitutive relations and the compliance constants, in a different orientation from the axis of symmetry of the material, are obtained by applying a similar process to the one in the previous section. But now we start from the known state of stresses and is required to know the state of the strains. To process it, need to apply a stresses transformation process to change to the symmetry axes (system xy), configuration (a) to (b), determine the state of strains with constitutive model, configuration (b) to (c) to subsequently apply a strain transformation and get the strains in axes 12, configuration (c) to (d), see **Figure 10**. The relations between the constants of compliance, on the lines of symmetry and outside of them, are:

$$\varepsilon_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{16}\sigma_6;\, \varepsilon_2 = S_{21}\sigma_1 + S_{22}\sigma_2 + S_{26}\sigma_6;\, \varepsilon_6 = S_{61}\sigma_1 + S_{62}\sigma_2 + S_{66}\sigma_6 \tag{23}$$

The relations between the compliance constants, corresponding to the lines symmetry and outside of them, are:

$$\begin{aligned}
S_{11} &= m^4 S_{xx} + 2m^2n^2 S_{xy} + n^4 S_{yy} + m^2n^2 S_{ss} \\
S_{12} &= m^2n^2 S_{xx} + (m^4 + n^4) S_{xy} + m^2n^4 S_{yy} - m^2n^2 S_{ss} \\
S_{16} &= 2m^3 n S_{xx} + 2(mn^3 - m^3 n) S_{xy} - 2mn^3 S_{yy} - mn(m^2 - n^2) S_{ss} \\
S_{22} &= n^4 S_{xx} + 2m^2n^2 S_{xy} + m^4 S_{yy} + m^2n^2 S_{ss} \\
S_{26} &= 2m^3 n S_{xx} + 2(mn^3 - m^3 n) S_{xy} - 2mn^3 S_{yy} - mn(m^2 - n^2) S_{ss} \\
S_{66} &= 4m^2n^2 S_{xx} - 8m^2n^2 S_{xy} + 4m^2n^2 S_{yy} + (m^2 - n^2)^2 S_{ss}
\end{aligned} \tag{24}$$

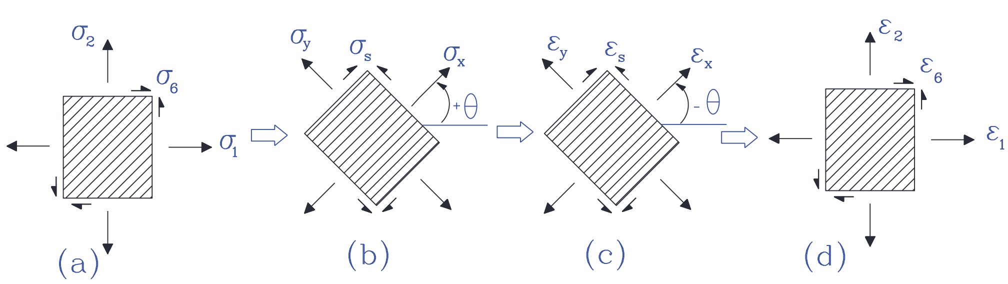

Figure 10.
Transformation process from the state of stresses to the state of strains in a coordinate system 12. Stresses transformation process, (a) to (b), getting strains with constitutive model, (b) to (c), and to finally obtain the state of strains in axes 12, (c) to (d).

Equations (Eq. (22)) can be put in terms of trigonometric identities. This is:

$$\begin{aligned}
Q_{11} &= \frac{1}{8}\left(3Q_{xx} + 2Q_{xy} + 3Q_{yy} + 4Q_{ss}\right) + \frac{1}{8}\left(4Q_{xx} - 4Q_{yy}\right)\cos 2\theta \\
&\quad + \frac{1}{8}\left(Q_{xx} - 2Q_{xy} + Q_{yy} - 4Q_{ss}\right)\cos 4\theta \\
Q_{12} &= \frac{1}{8}\left(Q_{xx} + 6Q_{xy} + Q_{yy} - 4Q_{ss}\right) - \frac{1}{8}\left(Q_{xx} - 2Q_{xy} + Q_{yy} - 4Q_{ss}\right)\cos 4\theta \\
Q_{16} &= \frac{1}{8}\left(2Q_{xx} - 2Q_{yy}\right) sen 2\theta + \frac{1}{8}\left(Q_{xx} + Q_{yy} - 2Q_{xy} - Q_{ss}\right) sen 4\theta \\
Q_{22} &= \frac{1}{8}\left(3Q_{xx} + 2Q_{xy} + 3Q_{yy} + 4Q_{ss}\right) - \frac{1}{8}\left(4Q_{xx} - 4Q_{yy}\right)\cos 2\theta \\
&\quad + \frac{1}{8}\left(Q_{xx} - 2Q_{xy} + Q_{yy} - 4Q_{ss}\right)\cos 4\theta \\
Q_{26} &= \frac{1}{2}\left(Q_{xx} - Q_{yy}\right) sen 2\theta - \frac{1}{8}\left(Q_{xx} - 2Q_{xy} + Q_{yy} - 4Q_{ss}\right) sen 4\theta \\
Q_{66} &= \frac{1}{8}\left(Q_{xx} - 2Q_{xy} + Q_{yy} - 4Q_{ss}\right) - \frac{1}{8}\left(Q_{xx} + Q_{yy} - 2Q_{xy} - Q_{ss}\right) sen 4\theta
\end{aligned} \tag{25}$$

If and only if the following relations are satisfied:

$$\begin{aligned}
&m^4 = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta); m^3 n = \frac{1}{8}(2sen 2\theta + sen 4\theta); m^2 n^2 = \frac{1}{8}(1 - \cos 4\theta); \\
&mn^3 = \frac{1}{8}(2sen 2\theta - sen 4\theta); n^4 = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)
\end{aligned} \tag{26}$$

By defining the following relations:

$$\begin{aligned}
&U_1 = \frac{1}{8}\left(3Q_{xx} + 2Q_{xy} + 3Q_{yy} + 4Q_{ss}\right); U_2 = \frac{1}{2}\left(Q_{xx} - Q_{yy}\right); \\
&U_3 = \frac{1}{8}\left(Q_{xx} - 2Q_{xy} + Q_{yy} - 4Q_{ss}\right); U_4 = \frac{1}{8}\left(Q_{xx} + 6Q_{xy} + Q_{yy} - 4Q_{ss}\right); \\
&U_5 = \frac{1}{8}\left(Q_{xx} - 2Q_{xy} + Q_{yy} - 4Q_{ss}\right)
\end{aligned} \tag{27}$$

And, when ordering terms, the following relatios are obtained:

$$\begin{aligned}
&Q_{11} = U_1 + U_2\cos 2\theta + U_3\cos 4\theta; Q_{12} = U_4 - U_3\cos 4\theta; Q_{16} = \frac{1}{2}U_2 sen 2\theta + U_3 sen 4\theta \\
&Q_{22} = U_1 - U_2\cos 2\theta + U_3\cos 4\theta; Q_{26} = \frac{1}{2}U_2 sen 2\theta - U_3 sen 4\theta; Q_{66} = U_5 - U_3\cos 4\theta
\end{aligned} \tag{28}$$

4. Laminate composite materials theory

A laminate is a set of plies or ply groups that have different orientations from their main axes [16]. The classical laminate theory assumes, in the mechanical model, the following [16, 17]: the laminate is symmetric; the behavior of the plies and the laminate complies with Hooke's law; each ply is considered orthotropic; the union between plies is perfect and thin; the functions of the displacements and strains are considered continuous through the interface; the laminate is homogeneous, elastic and linear; and the ply thicknesses are constant, thin and homogeneous throughout the laminate.

For the study of global stresses, the following aspects are assumed: the model is linear [14]; and the state of stress is homogeneous throughout the laminate. Thus, edge effects on the laminate can be ignored, allowing the problem to be about plane stresses. For the stress analysis at the local level, it is assumed that the problem for each of the plies is biaxial of stresses, and that the normal stresses have an average constant distribution through the thickness of the plies (see **Figure 11**). On a global level at a symmetric laminate: it is made up of homogeneous plies; the union between plies is perfect; and the laminate's composite thickness is homogeneous. Finally, when considering a state of homogeneous strain in the laminate and in the plies, the intralaminar stresses τ_{yz} can be ignored, so that all the points in the laminate and locally in the plies present a state of plane stresses.

A laminate composite material can be defined by means of a code [16]. **Figure 12** shows a diagram of a symmetrical laminate. Its code is $\left[45^0_3/90^0_1/0^0_2\right]_S$ and it is interpreted as follows: if the analysis is started from z = −h / 2, we have three plies oriented at 45^0, followed by a ply with orientation 90^0, and finally two plies at 0^0. The subscript S means that the laminate is symmetric and that from the central axis up, the sequence is in reverse order. If instead of S there were the subscript T, this would mean that the code would be written in full, that is: $\left[45^0_3/90^0_1/0^0_2/0^0_2/90^0_1/45^0_3\right]_T$. If the laminate were not symmetric, we would have a code like the following: $\left[45^0_3/90^0_1/0^0_2/45^0_3/90^0_1/0^0_2\right]_T$.

4.1 Mechanics of symmetric laminate

In the study of symmetric laminates, the strains in the *xy* plane are constant throughout the lamina if and only if its thickness is small compared to the length

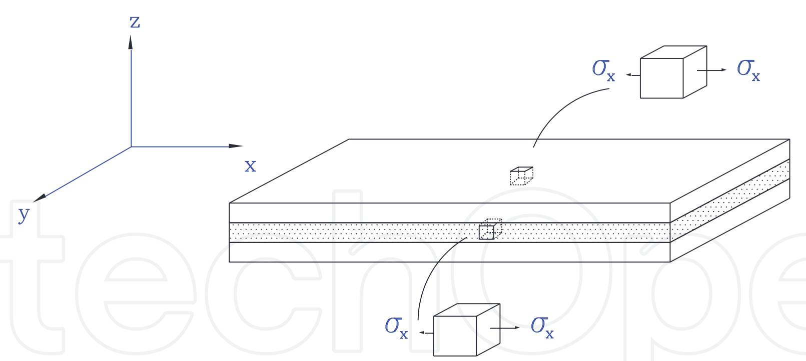

Figure 11.
Intralaminar stress state for a three-ply laminate.

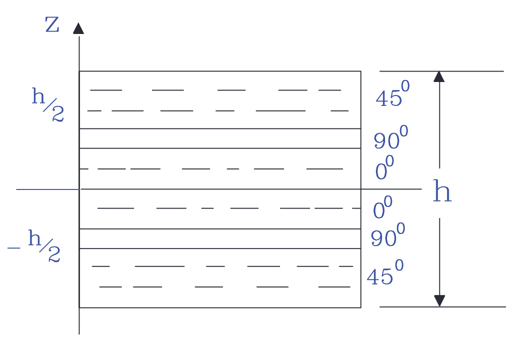

Figure 12.
Sequence of a symmetric laminate.

and width. Therefore, $\varepsilon_1(z) = \varepsilon_1^0$, $\varepsilon_2(z) = \varepsilon_2^0$ and $\varepsilon_{12}(z) = \varepsilon_{12}^0$. The exponent (0) means that the strains as a function of z are constant. It is worth mentioning that the stress distribution is not constant and varies from plies or ply group to plies. If a global analysis of the laminate is carried out, the constitutive relations are obtained based on the properties and orientation in each ply group [16]. For this study it is necessary to start from the concept of average effort, (see **Figure 13**). This is:

$$\overline{\sigma}_1 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 dz; \overline{\sigma}_2 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_2 dz; \overline{\sigma}_6 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_6 dz \tag{29}$$

On the other hand, the stresses are defined as a function of the stiffness constants in any direction. The stress–strain relations are as follow:

$$\begin{aligned} &\sigma_1 = Q_{11}\varepsilon_1 + Q_{12}\varepsilon_2 + Q_{16}\varepsilon_6; \sigma_2 = Q_{21}\varepsilon_1 + Q_{22}\varepsilon_2 + Q_{26}\varepsilon_6; \\ &\sigma_6 = Q_{61}\varepsilon_1 + Q_{62}\varepsilon_2 + Q_{66}\varepsilon_6 \end{aligned} \tag{30}$$

If the strains are constant, then the average stresses are expressed as follows:

$$\begin{aligned} &\overline{\sigma}_1 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(Q_{11}\varepsilon_1^0 + Q_{12}\varepsilon_2^0 + Q_{13}\varepsilon_6^0\right) dz; \overline{\sigma}_2 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(Q_{21}\varepsilon_1^0 + Q_{22}\varepsilon_2^0 + Q_{23}\varepsilon_6^0\right) dz \\ &\overline{\sigma}_6 = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(Q_{61}\varepsilon_1^0 + Q_{62}\varepsilon_2^0 + Q_{66}\varepsilon_6^0\right) dz \end{aligned} \tag{31}$$

Considering that the constants Q vary from ply to ply, the average stresses take thefollowing form:

$$\begin{aligned} &\overline{\sigma}_1 = \frac{1}{h}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} dz \varepsilon_1^0 + \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{12} dz \varepsilon_2^0 + \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13} dz \varepsilon_6^0\right]; \\ &\overline{\sigma}_2 = \frac{1}{h}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{21} dz \varepsilon_1^0 + \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{22} dz \varepsilon_2^0 + \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{23} dz \varepsilon_6^0\right] \\ &\overline{\sigma}_6 = \frac{1}{h}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{61} dz \varepsilon_1^0 + \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{62} dz \varepsilon_2^0 + \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{66} dz \varepsilon_6^0\right] \end{aligned} \tag{32}$$

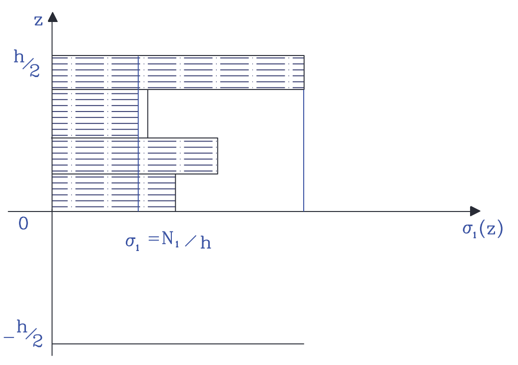

Figure 13.
Representation of mean stresses in a multidirectional lamina.

If:

$$A_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}dz; A_{21} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{21}dz; A_{22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{22}dz; A_{61} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{61}dz;$$
$$A_{62} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{62}dz; A_{66} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{66}dz \tag{33}$$

And: $A_{21} = A_{12}$; $A_{61} = A_{16}$ y $A_{62} = A_{26}$ (The equivalent modulus tensor A_{ij} is symmetric), the average stresses are rewritten as follows:

$$\overline{\sigma}_1 = \frac{1}{h}\left[A_{11}\varepsilon_1^0 + A_{21}\varepsilon_2^0 + A_{61}\varepsilon_6^0\right]; \overline{\sigma}_2 = \frac{1}{h}\left[A_{21}\varepsilon_1^0 + A_{22}\varepsilon_2^0 + A_{62}\varepsilon_6^0\right];$$
$$\overline{\sigma}_3 = \frac{1}{h}\left[A_{61}\varepsilon_1^0 + A_{62}\varepsilon_2^0 + A_{63}\varepsilon_6^0\right] \tag{34}$$

These last equations are known as the effective or global constitutive relations, where the equivalent modulus of a multidirectional laminate is the arithmetic average of the individual modulus of stiffness outside their axis of symmetry of the plies or ply groups. The units of the Qs are Pa *(or N / m2)* and the As are in $Pa \cdot m$ (or N / m). A stress resultant (*N's*) can be defined, with units of force per unit of length or force per unit of thickness h. This is:

$$N_1 = h\overline{\sigma}_1; N_2 = h\overline{\sigma}_2; N_6 = h\overline{\sigma}_6 \tag{35}$$

Or, equivalently:

$$N_1 = A_{11}\varepsilon_1^0 + A_{21}\varepsilon_2^0 + A_{61}\varepsilon_6^0; N_2 = A_{21}\varepsilon_1^0 + A_{22}\varepsilon_2^0 + A_{63}\varepsilon_6^0;$$
$$N_3 = A_{61}\varepsilon_1^0 + A_{62}\varepsilon_2^0 + A_{66}\varepsilon_6^0 \tag{36}$$

These equations relate the resultant stresses to the strains. To know the stress in each ply or ply group from the strains and global stresses, it is shown schematically in **Figure 14**. First, determine the average strains in terms of the average stresses,

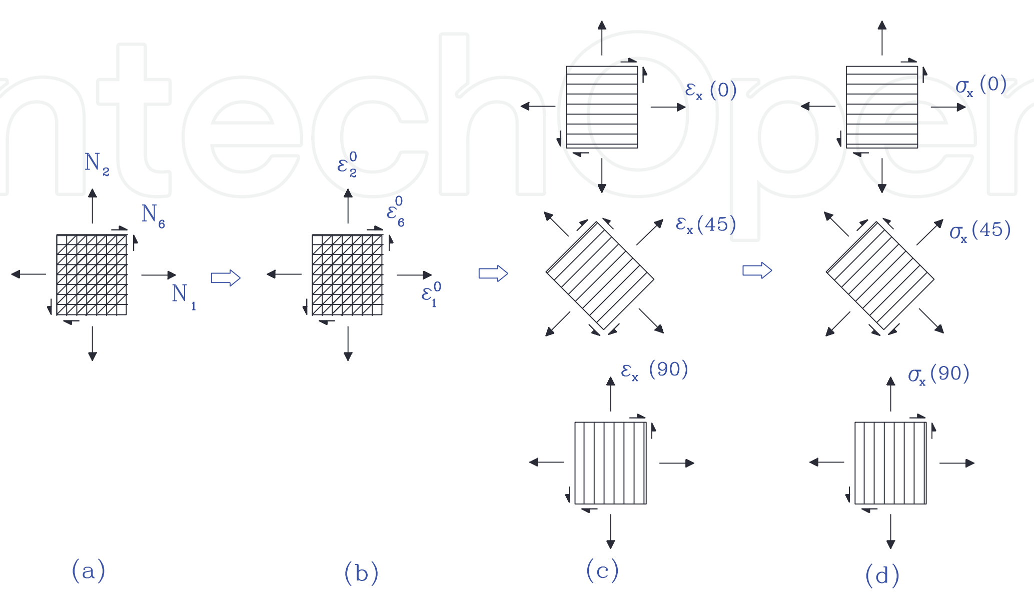

Figure 14.
Process to obtain the state of stress in each ply group from the average state of stress, determining the average strains in terms of the average stresses, (a) to (b), getting the strains for the symmetry axes (system xy), configuration (b) to (c) and finally, determine the state of stress of each ply or plies group, (c) to (d).

for a reference system 12 and apply an effective or global constitutive relations, configuration (a) to (b). Second, getting the strains for the symmetry axes (system xy) of each orthotropic plies or orthotropic ply groups with different orientation, configuration (b) to (c). Finally, apply the constitutive relations to determine the state of stress of each ply or ply group.

Equivalent modulus can also be expressed in terms of the multi-angle equations, that is:

$$\begin{aligned}
A_{11} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} (U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta) dz \\
&= U_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} dz + U_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 2\theta dz + U_3 \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 4\theta dz \\
A_{21} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{21} dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} (U_4 - U_3 \cos 4\theta) dz = U_4 \int_{-\frac{h}{2}}^{\frac{h}{2}} dz - U_3 \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 4\theta dz \\
A_{22} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{22} dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} (U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta) dz \\
&= U_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} dz - U_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 2\theta dz + U_3 \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 4\theta dz \\
A_{61} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{31} dz = \frac{1}{2} U_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} sen 2\theta dz + U_3 sen 4\theta) dz = \frac{1}{2} U_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} sen 2\theta + U_3 \int_{-\frac{h}{2}}^{\frac{h}{2}} sen 4\theta dz \\
A_{62} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{32} dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{1}{2} U_2 sen 2\theta - U_3 sen 4\theta \right) dz = \frac{1}{2} U_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} sen 2\theta - U_3 \int_{-\frac{h}{2}}^{\frac{h}{2}} sen 4\theta dz \\
A_{66} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{33} dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} (U_5 - U_3 \cos 4\theta) dz = U_5 \int_{-\frac{h}{2}}^{\frac{h}{2}} dz - U_3 \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 4\theta dz
\end{aligned} \tag{37}$$

As the Us have no variation with respect to the z axis, they are considered constant. If:

$$V_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 2\theta dz; V_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 4\theta dz; V_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} sen 2\theta dz; V_4 = \int_{-\frac{h}{2}}^{\frac{h}{2}} sen 4\theta dz \tag{38}$$

Then:

$$\begin{aligned}
&A_{11} = U_1 h + U_2 V_1 + U_3 V_2; A_{21} = U_4 h - U_3 V_2; \\
&A_{22} = U_1 h - U_2 V_1 + U_3 V_2; A_{61} = \frac{1}{2} U_2 V_3 + U_3 V_4 \\
&A_{62} = \frac{1}{2} U_2 V_3 - U_3 V_4; A_{66} = U_5 h - U_3 V_2
\end{aligned} \tag{39}$$

The V values now depend on the orientation of the plies or ply groups in the multidirectional laminate (see **Figure 15**). When normalizing the equations' Vs in terms of the thickness of the laminate so that the values are dimensionless, the following expressions are obtained:

$$V_1^* = \frac{V_1}{h} = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 2\theta dz;\ V_2^* = \frac{V_2}{h} = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \cos 4\theta dz;$$
$$V_3^* = \frac{V_3}{h} = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} sen 2\theta dz;\ V_4^* = \frac{V_4}{h} = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} sen 4\theta dz \tag{40}$$

If the plies or ply groups that make up the laminate have the same orientation and the same material, the expressions described above take the following form:

$$V_1^* = \frac{1}{h}\sum_{i=1}^{h} \cos 2\theta_i [z_i - z_{i-1}] = \frac{1}{h}\sum_{i=1}^{h} \cos 2\theta_i h_i;$$
$$V_2^* = \frac{1}{h}\sum_{i=1}^{h} \cos 4\theta_i [z_i - z_{i-1}] = \frac{1}{h}\sum_{i=1}^{h} \cos 4h_i$$
$$V_3^* = \frac{1}{h}\sum_{i=1}^{h} sen 2\theta_i [z_i - z_{i-1}] = \frac{1}{h}\sum_{i=1}^{h} sen 2\theta_i h_i;$$
$$V_4^* = \frac{1}{h}\sum_{i=1}^{h} sen 4\theta_i [z_i - z_{i-1}] = \frac{1}{h}\sum_{i=1}^{h} sen 4\theta_i h_i \tag{41}$$

Where h_i is the thickness of i-th ply group and starts from $\frac{h}{2}$, as in **Figure 15**.

The volumetric fraction can be expressed as follows: $v_i = \frac{h_i}{h}$ and if each i represents an orientation, then Equations (Eq. (41)) are as follows:

$$V_1^* = \sum_{i=1}^{h} \cos 2\theta_i v_i = v_1 \cos 2\theta_1 + v_2 \cos 2\theta_2 + v_3 \cos 2\theta_3 + \ \dots$$
$$V_2^* = \sum_{i=1}^{h} \cos 4\theta_i v_i = v_1 \cos 4\theta_1 + v_2 \cos 4\theta_2 + v_3 \cos 4\theta_3 + \ \dots$$
$$V_3^* = \sum_{i=1}^{h} sen 2\theta_i v_i = v_1 sen 2\theta_1 + v_2 sen 2\theta_2 + v_3 sen \theta_3 + \ \dots$$
$$V_4^* = \sum_{i=1}^{h} sen 4\theta_i v_i = v_1 sen 4\theta_1 + v_2 sen 4\theta_2 + v_3 sen 4\theta_3 + \ \dots \tag{42}$$

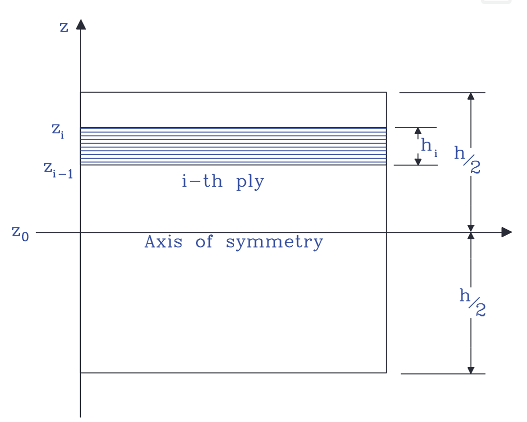

Figure 15.
Graphic representation of n *ply groups in a multidirectional symmetric laminate.*

The sum of the volume fractions fulfills the condition $v_1 + v_2 + v_3 + \ldots = 1$ and the limits of $V^*{'}s$ are $-1 \leq V^*{'}s \geq 1$. By considering the above, the effective or global mechanical properties can be determined, if the orientation and fraction volume of each ply group is known.

5. Theoretical approach to obtain the constitutive models

This section presents the algebraic development to generate the constitutive mathematical model of a laminate composite material taking into account the properties of the constituent orthotropic plies. The laminate to be modeled is made up of longitudinal plies of fiberglass reinforced with epoxy resin, oriented orthogonally. The mechanical engineering properties that characterize an orthotropic laminate are five: 1) Two Young's moduli (*Ex* and *Ey*, two Poisson's ratios ν_{xy} and ν_{yx}, and 2) a shear modulus (*E6 or Gxy*). **Figure 16** shows an orthotropic lamina.

The constitutive relations for a lamina in terms of the engineering constants of each of the layers are obtained by considering equations (Eq. (16)) and (Eq. (27)), that is:

$$U_1 = \frac{1}{8}\left[\frac{3E_x}{1-\nu_x\nu_y} + \frac{3E_y}{1-\nu_x\nu_y} + \frac{2\nu_y E_x}{1-\nu_x\nu_y} + 4E_{ss}\right] = \frac{1}{8}\left[\frac{3E_x + 3E_y + 2\nu_y E_x}{1-\nu_x\nu_y} + 4E_{ss}\right];$$

$$U_2 = \frac{1}{2}\left[\frac{E_x - E_y}{1-\nu_x\nu_y}\right]; U_3 = \frac{1}{8}\left[\frac{E_y + E_x - 2\nu_y E_x}{1-\nu_x\nu_y} - 4E_{ss}\right];$$

$$U_4 = \frac{1}{8}\left[\frac{E_y + E_x + 6\nu_y E_x}{1-\nu_x\nu_y} - 4E_{ss}\right]; U_5 = \frac{1}{8}\left[\frac{E_y + E_x - 2\nu_y E_x}{1-\nu_x\nu_y} + 4E_{ss}\right] \tag{43}$$

By considering equations (Eq. (40)) and (Eq. (42)), the relations for any multidirectional lamina are obtained. This is:

$$\begin{aligned} V_1 &= (\nu_1 cos\ 2\theta_1 + \nu_2 cos\ 2\theta_2 + \nu_3 cos\ 2\theta_3 + \ldots)h; \\ V_2 &= (\nu_1 cos\ 4\theta_1 + \nu_2 cos\ 4\theta_2 + \nu_3 cos\ 4\theta_3 + \ldots)h; \\ V_3 &= (\nu_1 sen\ 2\theta_1 + \nu_2 sen\ 2\theta_2 + \nu_3 sen\ 2\theta_3 + \ldots)h; \\ V_4 &= (\nu_1 sen\ 4\theta_1 + \nu_2 sen\ 4\theta_2 + \nu_3 sen\ 4\theta_3 + \ldots)h \end{aligned} \tag{44}$$

From the relations of (Eq. (43)), the following expressions are obtained:

$$A_{11} = U_1 h + U_2 V_1 + U_3 V_2; A_{21} = U_4 h - U_3 V_2; A_{22} = U_1 h - U_2 V_1 + U_3 V_2$$
$$A_{61} = \frac{1}{2} U_2 V_3 + U_3 V_4; A_{62} = \frac{1}{2} U_2 V_3 - U_3 V_4; A_{66} = U_5 h - U_3 V_2 \tag{45}$$

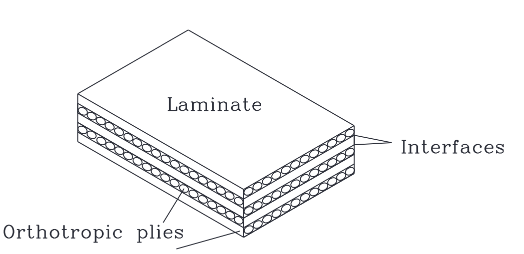

Figure 16.
Orthotropic laminate.

The average values of the modules for "n" plies or "n" ply groups are obtained by making equations (Eq. (45)) explicit. This is:

$$
\begin{aligned}
A_{11} &= \frac{1}{8}\left[\frac{3E_x + 3E_y + 2\nu_y E_x}{1-\nu_x\nu_y} + 4E_{ss}\right]h \\
&+ \frac{1}{2}\left[\frac{E_x - E_y}{1-\nu_x\nu_y}\right](\nu_1 cos\ 2\theta_1 + \nu_2 cos\ 2\theta_2 + \nu_3 cos\ 2\theta_3 + \ \dots)h \\
&+ \frac{1}{8}\left[\frac{E_y + E_x - 2\nu_y E_x}{1-\nu_x\nu_y} - 4E_{ss}\right](\nu_1 cos\ 4\theta_1 + \nu_2 cos\ 4\theta_2 + \nu_3 cos\ 4\theta_3 + \ \dots)h \\
A_{21} &= \frac{1}{8}\left[\frac{E_y + E_x + 6\nu_y E_x}{1-\nu_x\nu_y} - 4E_{ss}\right]h \\
&- \frac{1}{8}\left[\frac{E_y + E_x - 2\nu_y E_x}{1-\nu_x\nu_y} - 4E_{ss}\right](\nu_1 cos\ 4\theta_1 + \nu_2 cos\ 4\theta_2 + \nu_3 cos\ 4\theta_3 + \ \dots)h \\
A_{22} &= \frac{1}{8}\left[\frac{3E_x + 3E_y + 2\nu_y E_x}{1-\nu_x\nu_y} + 4E_{ss}\right]h \\
&- \frac{1}{2}\left[\frac{E_x - E_y}{1-\nu_x\nu_y}\right](\nu_1 cos\ 2\theta_1 + \nu_2 cos\ 2\theta_2 + \nu_3 cos\ 2\theta_3 + \ \dots)h \\
&+ \frac{1}{8}\left[\frac{E_y + E_x - 2\nu_y E_x}{1-\nu_x\nu_y} - 4E_{ss}\right](\nu_1 cos\ 4\theta_1 + \nu_2 cos\ 4\theta_2 + \nu_3 cos\ 4\theta_3 + \ \dots)h \\
A_{61} &= \frac{1}{4}\left[\frac{E_x - E_y}{1-\nu_x\nu_y}\right](\nu_1 sen\ 2\theta_1 + \nu_2 sen\ 2\theta_2 + \nu_3 sen\ 2\theta_3 + \ \dots)h \\
&+ \frac{1}{8}\left[\frac{E_y + E_x - 2\nu_y E_x}{1-\nu_x\nu_y} - 4E_{ss}\right](\nu_1 sen\ 4\theta_1 + \nu_2 sen\ 4\theta_2 + \nu_3 sen\ 4\theta_3 + \ \dots)h \\
A_{62} &= \frac{1}{4}\left[\frac{E_x - E_y}{1-\nu_x\nu_y}\right](\nu_1 sen\ 2\theta_1 + \nu_2 sen\ 2\theta_2 + \nu_3 sen\ 2\theta_3 + \ \dots)h \\
&- \frac{1}{8}\left[\frac{E_y + E_x - 2\nu_y E_x}{1-\nu_x\nu_y} - 4E_{ss}\right](\nu_1 sen\ 4\theta_1 + \nu_2 sen\ 4\theta_2 + \nu_3 sen\ 4\theta_3 + \ \dots)h \\
A_{66} &= \frac{1}{8}\left[\frac{E_y + E_x - 2\nu_y E_x}{1-\nu_x\nu_y} + 4E_{ss}\right]h \\
&- \frac{1}{8}\left[\frac{E_y + E_x - 2\nu_y E_x}{1-\nu_x\nu_y} - 4E_{ss}\right](\nu_1 cos\ 4\theta_1 + \nu_2 cos\ 4\theta_2 + \nu_3 cos\ 4\theta_3 + \ \dots)h
\end{aligned}
\tag{46}
$$

By substituting equations (Eq. (46)) in equations (Eq. (36)), the following relations are obtained:

$$
\begin{aligned}
&N_1 = A_{11}\varepsilon_1^0 + A_{21}\varepsilon_2^0 + A_{61}\varepsilon_6^0; N_2 = A_{21}\varepsilon_1^0 + A_{22}\varepsilon_2^0 + A_{62}\varepsilon_6^0; \\
&N_6 = A_{61}\varepsilon_1^0 + A_{62}\varepsilon_2^0 + A_{66}\varepsilon_6^0
\end{aligned}
\tag{47}
$$

And, when considering the expression (Eq. (35)) the following relations are obtained:

$$\bar{\sigma}_1 = \frac{h}{N_1}; \bar{\sigma}_2 = \frac{h}{N_2}; \bar{\sigma}_6 = \frac{h}{N_6} \tag{48}$$

If the volume fractions and the orientations of each ply group are known, equations (Eq. (48)) represent the constitutive relations for any multidirectional laminate. These equations contain global or effective properties A's and the average stresses and strains.

5.1 Constitutive equations for laminate

The constitutive model for a configuration laminate $\left[0^0/90^0/0^0/90^0/0^0\right]_T$ is obtained from equations (Eq. (47)) and (Eq. (48)) by considering $\theta = 0^0$ and $\theta = 90^0$ with volume fractions $v_1 = \frac{3}{5}$ and $v_2 = \frac{2}{5}$. This is:

$$\bar{\sigma}_1 = \frac{1}{5}\left[\frac{3E_x + 2E_y}{1-\nu_x\nu_y}\right]\varepsilon_1^0 + \left[\frac{\nu_y E_x}{1-\nu_x\nu_y}\right]\varepsilon_2^0; \bar{\sigma}_2 = \left[\frac{\nu_y E_x}{1-\nu_x\nu_y}\right]\varepsilon_1^0 + \frac{1}{5}\left[\frac{2E_x + 3E_y}{1-\nu_x\nu_y}\right]\varepsilon_2^0; \bar{\sigma}_6 = E_{ss}\varepsilon_6^0 \tag{49}$$

For the case of the laminate $\left[90^0/0^0/90^0/0^0/90^0\right]_T$ the constitutive model is obtained considering $\theta = 90^0$ and $\theta = 0^0$ with volumetric fractions $v_1 = \frac{2}{5}$ y $v_2 = \frac{3}{5}$. This is:

$$\bar{\sigma}_1 = \frac{1}{5}\left[\frac{2E_x + 3E_y}{1-\nu_x\nu_y}\right]\varepsilon_1^0 + \left[\frac{\nu_y E_x}{1-\nu_x\nu_y}\right]\varepsilon_2^0; \bar{\sigma}_2 = \left[\frac{\nu_y E_x}{1-\nu_x\nu_y}\right]\varepsilon_1^0 + \frac{1}{5}\left[\frac{3E_x + 2E_y}{1-\nu_x\nu_y}\right]\varepsilon_2^0; \bar{\sigma}_6 = E_{ss}\varepsilon_6^0 \tag{50}$$

Finally, the constitutive model for the laminate $\left[0^0/90^0/0^0\right]_T$ is generated considering that $\theta = 0^0$ and $\theta = 90^0$ with volume fractions $v_1 = \frac{2}{5}$ and $v_2 = \frac{3}{5}$. The model is as follows:

$$\bar{\sigma}_1 = \frac{1}{3}\left[\frac{2E_x + E_y}{1-\nu_x\nu_y}\right]\varepsilon_1^0 + \left[\frac{\nu_y E_x}{1-\nu_x\nu_y}\right]\varepsilon_2^0; \bar{\sigma}_2 = \left[\frac{\nu_y E_x}{1-\nu_x\nu_y}\right]\varepsilon_1^0 + \frac{1}{3}\left[\frac{2E_x + E_y}{1-\nu_x\nu_y}\right]\varepsilon_2^0; \bar{\sigma}_6 = E_{ss}\varepsilon_6^0 \tag{51}$$

6. Experimental obtaining of the elastic engineering properties to ply

In order to obtain the engineering properties of laminated plies, the experimental electrical extensometry method is used, which is supported by Perry [18]. It is worth mentioning that, due to the arrangement of plies in the proposed composite laminate, the number of constitutive analytical equations are three and the number of elastic constants of orthotropic plies are five, so five- and three-ply laminate are used. A CEA-O6-240UZ-120 strain gage is selected for the study (see **Figure 17**).

For the experimental tests, an INSTRON Universal Testing Machine was used. The engineering properties E_x, E_y, ν_x and ν_y were determined. Several tests were carried out and, in the solution, the equations of two models were combined due to the number of constants to be determined.

Figure 18 shows the tests as well as the location of the installed strain gages, two for each deformation in a full Wheatstone bridge configuration to eliminate signals outside the required measurement (deflections, temperature changes, etc.). Tests were performed with laminates $\left[0^0/90^0/0^0/90^0/0^0\right]_T$ and $\left[90^0/0^0/90^0/0^0/90^0\right]_T$ for characterization and $\left[0^0/90^0/0^0\right]_T$ for evaluation of results. The measurement

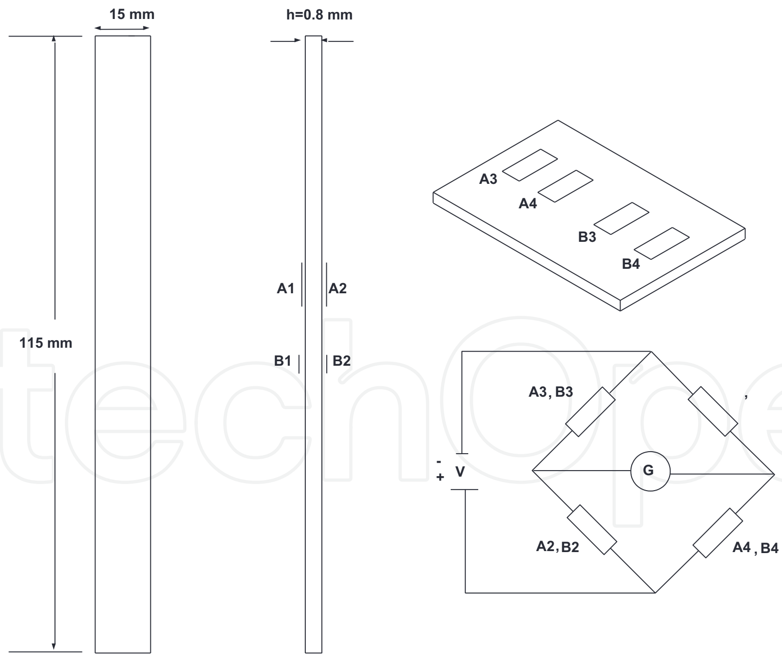

Figure 17.
Installation of strain gages, active and dummy.

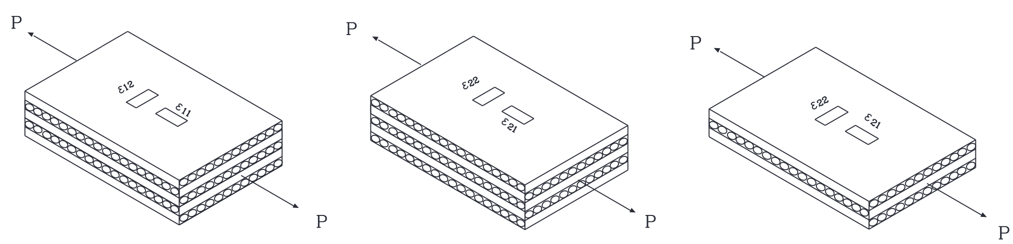

Figure 18.
Tension test system with test specimens 3–2, 2–3 and 2–1 and installation of strain gages.

of the strains was carried out with electrical resistance strain gages on a theoretically homogenized surface, and the values obtained are average and not punctual. From the readings provided by the meter, the effective properties for both a laminate and plies are calculated.

Combining test specimens 3–2 and 2–3 and from equations (Eq. (49)) and (Eq. (50)), the following equations are obtained to determine the constants E_x, E_y, ν_x and ν_y:

$$
\begin{aligned}
&v_x = \frac{\varepsilon_{12}^0 \varepsilon_{22}^0}{\left(2\varepsilon_{12}^0 \varepsilon_{21}^0 - 3\varepsilon_{11}^0 \varepsilon_{22}^0\right)_;} \;\; v_y = \frac{\varepsilon_{12}^0 \varepsilon_{22}^0}{\left(2\varepsilon_{11}^0 \varepsilon_{22}^0 - 3\varepsilon_{12}^0 \varepsilon_{21}^0\right)} \\
&E_x = \frac{5\overline{\sigma}_{11} v_x \left(1 - v_x v_y\right)}{3v_x \varepsilon_{11}^0 + 2v_y \varepsilon_{11}^0 + 5v_x v_y \varepsilon_{12;}^0} \;\; E_y = \frac{5\overline{\sigma}_{11} v_y \left(1 - v_x v_y\right)}{\left(3v_x \varepsilon_{11}^0 + 2v_y \varepsilon_{11}^0 + 5v_x v_y \varepsilon_{12}^0\right)}
\end{aligned}
\tag{52}
$$

Here, $\overline{\sigma}_{11}$, $\overline{\sigma}_{12,}$ ε_{11}^0, ε_{12}^0 are the global average stress and strain values, respectively, in directions 1 and 2 for the laminate $\left[0^0/90^0/0^0/90^0/0^0\right]_T$, and $\overline{\sigma}_{21}$, $\overline{\sigma}_{22,}$ ε_{21}^0, ε_{22}^0 are the average stress and strain values in directions 1 and 2 for the laminate$\left[90^0/0^0/9\ 0^0/0^0/90^0\right]_T$. In the tests, $\overline{\sigma}_{12} = \overline{\sigma}_{22} = 0$.

6.1 Analysis of results

The analysis of the data complied with the symmetry of the stiffness and compliance tensor, the identity $\frac{v_x}{E_x} = \frac{v_y}{E_y}$, for calculated values of a ply. For the results of

Engineering Constants Plies	Magnitude
E_x	53.05 GPa
E_y	23.3 GPa
υ_x	0.233
υ_y	0.1

Table 1.
Properties of the average experimental engineering constants.

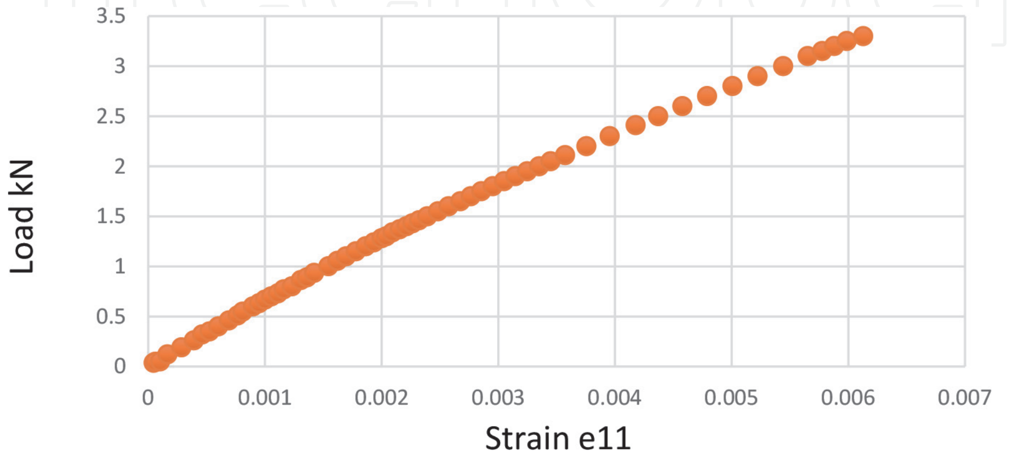

Figure 19.
Graph of laminate 3–2 under tension.

the effective mechanical properties obtained experimentally E_x, E_y, ν_x and ν_y of the plies that make up the laminate composite material, see **Table 1**. **Figure 19** shows a representative test on laminate 3–2.

The consistency of the tests was carried out by comparing the results obtained for tests 3–2 and 2–1, showing very proximate values.

7. Conclusions

In this chapter, a theoretical framework based on the theory of linear elasticity and the classical theory of composite laminates was established for the analysis of axial load problems by analyzing concepts and establishing scopes and restrictions of the models applied by the theories. The necessary bases were established for the general obtention of the composite laminates' constitutive models made up of orthotropic plies or orthotropic ply groups with different orientations. With the established models, it is possible: a) with the known stiffness constants, to calculate the state of plane stresses at a point from the state of deformations and b) with the known conformity constants, to calculate the state of strains from the state of stresses. For ease of use, both the stiffness constants and the compliance constants were made explicit in terms of the engineering constants. This allows an analysis of overall or average stresses in the laminates under axial load. To show the efficiency of the developments presented, the constitutive models of three orthotropic composite laminates were obtained. Furthermore, by performing simple stress tests on the laminates and measuring the state of strain with strain gages, the engineering constants of the plies were determined.

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