Long-range connections and mixed diffusion in fractional networks

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Abstract

Networks with long-range connections, obeying a distance-dependent power law of sufficiently small exponent, display superdiffusion, Lévy flights and robustness properties very different from the scale-free networks. It has been proposed that these networks, found both in society and in biology, be classified as a new structure, *the fractional networks*. Particular important examples are the social networks and the modular hierarchical brain networks where both short- and long-range connections are present. The anomalous superdiffusive and the mixed diffusion behavior of these networks is studied here as well as its relation to the nature and density of the long-range connections.

1 Introduction

The human brain contains up to 86 billion neurons connected by close to a million kilometers of axons and dendrites. Most of these connections ($\sim$80%) are short range on the order of a few hundred microns, the rest ($\sim$20%) being long-range myelinated fibers on the order of several centimeters. The insulating myelin sheath increases conduction velocity of the action potentials but at the cost of taking up more volume in the brain as well as rendering axons unable to synapse onto nearby neurons. That evolution has found profitable to accept this additional hardware cost, highlights the importance of long-range connections.

From a network point of view the brain has a modular and hierarchical structure [1] [2]. Each module is associated to a specialized function mediated by short-range connections whereas global integration, for higher cognition functions, relies on the long-range connections between modules.

The existence and importance of long-range connections in the brain has been much studied in recent years [3] [4] [5] [6] [7] [8], having been found that diminished long-range functional connectivity is associated to cognitive disorders [9]. Of course, by itself, existence of long-range connections between the specialized nodes does not guarantee global integration of the cognitive functions. It is also necessary that the flow of information be sufficiently fast for the stimulus integration to be performed in a timely manner. This seems of particular relevance for the forward and backwards loops in the predictive coding mode [10] [11] [12] [13] [14] [15] of brain operation. One may therefore ask what type of communication short- and long-range connections establish and whether it depends or not on the structure and density of the long-range connections.

The network modules in the brain are in fact repertoires of many neurons and, when dealing with the interactions of these intrinsic connectivity networks (ICN's), a continuous diffusion approximation might be a good modelling hypothesis. In another paper [16] the nature of the diffusion processes associated to short and long-range connections have been analyzed. In particular it was concluded that whereas for short-range connections information propagates as a normal diffusion, for long-range connections of a certain type, one has anomalous diffusion, sub- or super-diffusion depending on the power law distance-dependence of the connections. The interplay of short and long range connections and their role on the propagation of information might also be modelled by a Levy flight scenario.

Networks with long-range connections leading to superdiffusion display properties very different from scale-free and hub dominated networks. Distance dependent connections are also important for the structure of social and economic networks [17] [18] [19], for the evolution of epidemics [20] and for the small world properties of networks [21]. A particular important issue is the relation of the diffusion rates to the distance-dependence of the connections in real world networks [22].

Some authors have already studied dynamics on networks involving jumps over many links or cascades of many unit jumps, leading to anomalous diffusion (see for example [23] [24] [25] [26] [27]). What here and in the past [16] we wanted to emphasize is that, rather than imposing a multijump dynamics on a regular network, anomalous diffusion and other phenomena emerge naturally as structural properties in networks with long range connections. Hence these networks might be classified as a new network class, **the fractional networks**.

The term fractional networks has been used before in the literature (see for example [28] [29]) to denote networks where the coupling between the nodes follows a fractional differential equation. Therefore the fractional law is explicitly included on the dynamics and on the couplings. In contrast, we are here referring not to fractionally-coupled networks but to networks with non-fractional couplings, but which nevertheless display fractional dynamics features.

The central phenomena that is studied in this paper is the interplay of dynamical contributions that, in isolation, would lead either to normal or to anomalous diffusion. Modeling this by a diffusion equation, with both Laplacian anf fractional Laplacian terms, we find out that even a small amount of

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fractionality changes in a very distinct manner the nature of the dynamics, with much shorter propagation times. This being shown by explicitly displaying the solutions is then also confirmed by an asymptotic analysis. This is the subject of Section 3. Section 2 simply sets the notation and definition of fractionality. Finally in Section 4 we carry out a numerical experiment of propagation of a signal in networks with several power-law connection laws, comparing it with a similar propagation in a nearest-neighbor connected network. The importance of the power law connection law for long range connections is once more put into evidence.

2   Short vs. long range connections

Consider first a network with only nearest-neighbor connections. The Laplacian matrix is

$$L = G - A, \tag{1}$$

G being the degree matrix ($G_{ij} = \delta_{ij} \times$ number of connections of node i) and A the adjacency matrix ($A_{ij} = 1$ if i and j are connected, $A_{ij} = 0$ otherwise). Let $\psi(i)$ for each node i be the intensity of some function ψ across the network. For a node i connected along some coordinate to two other nearest neighbor nodes $i + 1$ and $i - 1$ the action of the Laplacian matrix on a vector leads to $-\psi(i-1) + 2\psi(i) - \psi(i+1)$, which is a discrete version of $-d^2$ (minus the second derivative). It is reasonable to think that ψ diffuses from i to j proportional to $\psi(i) - \psi(j)$ whenever i and j are connected. Then,

$$\frac{d\psi(i)}{dt} = -k \sum_j A_{ij} (\psi(i) - \psi(j)) = -k \left(\psi(i) \sum_j A_{ij} - \sum_j A_{ij} \psi(j) \right) \tag{2}$$

which in matrix form is

$$\frac{d\psi}{dt} + kL\psi = 0, \tag{3}$$

a heat-like equation. Therefore the Laplacian matrix controls the diffusion of quantities in the network and in the continuous approximation and for short-range connections the propagation of signals in the network may be represented by a normal diffusion equation

$$\frac{d\psi}{dt} = k\Delta\psi, \tag{4}$$

Δ being the Laplacian in the dimension of the space where the network is embedded.

However, for long-range connections the situation is different and we fall in the framework of nonlocal diffusion [30] [31], which is described by an equation

$$\frac{d\psi(x,t)}{dt} = \int p(x,y) \psi(y,t) d^n y - \psi(x,t) \tag{5}$$

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where $p(x, y)$ is the jumping probability density from y to x and the last term accounts for jumps from x to all other locations. For networks where the probability of establishment of a link to a distance $d = |x - y|$ is proportional to a power of the distance

$$p(x, y) = c |x - y|^{-\gamma}, \tag{6}$$

(c being a normalization constant such that $\sum_y p(x, y) = 1$) it is reasonable to assume that this is also the probability for the flow of information or activation between nodes. Then, in the continuous approximation, which we are assuming applies for the average field in a network with many nodes, the nature of the nonlocal diffusion is obtained by comparing the functional dependence of the probability density with the kernel of the the symmetrized Grünwald-Letnikov representation of the fractional derivative. This was done in [16] (see also the Appendix). The conclusion is that one obtains fractional diffusion of exponent $\beta = \gamma - 1$, $\beta = 2$ being normal diffusion and all $\beta < 2$ corresponding to superdiffusions,

$$\frac{d\psi}{dt} = -k (-\Delta)^{\frac{\beta}{2}} \psi. \tag{7}$$

Anomalous diffusion and other phenomena [16] emerge naturally as a structural property in long-range connection networks with distance dependence as in (6).

3 Mixed diffusion

In the case of networks characterized by a modular hierarchical structure one has both short and long range connections. This is the structure that occurs in brain networks and also in some social networks. Whereas in the networks studied in [16] the uniform scaling law of the connections leads to pure anomalous diffusion, here one faces a mixture of both normal and anomalous diffusion. This is the central phenomena that is studied in this paper with emphasis on the nature of the time scales of propagation of information. This is discussed in the framework of the continuous approximation to the network leading to a fractional differential equation, which, as stated before, is a reasonable approximation for very large networks. However it is also found that qualitatively similar results are obtained even for small discrete networks. This is illustrated in Section 3.

In the mixed case the diffusion equation will be

$$\frac{d\psi(x, t)}{dt} = \left(a\Delta - b(-\Delta)^{\frac{\beta}{2}} \right) \psi(x, t), \tag{8}$$

with $x \in \mathbb{R}^n$, n being the dimension of the embedding Euclidean space. Linear and nonlinear fractional diffusion studies are nowadays very rich mathematical fields both for functional (see for example [32]) and stochastic analysis [33]. Not so explored however is their dominant role in networks with a particular type (power law) of distance-dependence on the intensity of the connections. Here

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our concern is to estimate how information propagates in a network and, in particular, how that depends on the exponent in the distance-dependence of the connections. Of importance is also the interplay of short and long-range connections. For this purpose it suffices to consider the propagation of a single pulse in the linear equation (8).

With the Fourier transform

$$\widetilde{\psi}(k,t) = \int d^n x \psi(x,t) e^{-ik \cdot x}, \tag{9}$$

Eq.(8) becomes

$$\frac{d\widetilde{\psi}(k,t)}{dt} = \left(-a|k|^2 - b|k|^\beta\right)\widetilde{\psi}(k,t), \tag{10}$$

with solution

$$\widetilde{\psi}(k,t) = \widetilde{\psi}(k,0) e^{-t\left(a|k|^2 + b|k|^\beta\right)}. \tag{11}$$

$\widetilde{\psi}(k,0) = 1$ corresponds to $\psi(x,0) = \delta^{(n)}(x)$, that is, an initial localized disturbance at the origin. This is the situation of interest to study the propagation of information in the network. Computing the inverse Fourier transform one has the following exact solution in integral form,

$$\psi(x,t) = \frac{1}{(2\pi)^{n/2}} \int_0^\infty d|k| \, |k|^{n-1} e^{-t\left(a|k|^2 + b|k|^\beta\right)} \frac{J_{\frac{n}{2}-1}(|k||x|)}{(|k||x|)^{\frac{n}{2}-1}} \tag{12}$$

As in the purely fractional multidimensional solution [34] one notices the strong dependence on the dimension n.

Numerical evaluation of the exact solution (12) shows the remarkable difference in the speed of propagation of information between normal and mixed diffusion. For $n = 3$, Figures 1 and 2 compare the propagation of a delta signal at $(x = 0, t = 0)$ to distances $x = 10$ and 100 for normal and mixed diffusion. One sees that whereas for normal diffusion it takes a long time for the signal to be detected at a distance, for mixed diffusion the behavior is qualitatively very different.

Figures 3 and 4 show that this effect is obtained even with a very small amount of fractional diffusion.

Some of these effects may also be inferred by direct analytic estimates in Eq.(12). Let $n = 3$. Being interested in the large x behavior one sees that the term $J_{\frac{1}{2}}(|k||x|)$, having large sign fluctuations, a stationary phase estimate is possible. For $\beta = 2$ one obtains

$$\psi(x,t) \sim \frac{1}{2t} e^{-\frac{x^2}{4t}} \tag{13}$$

meaning that the signal only starts to be detected when $t \sim \frac{x^2}{4}$ (normal diffusion). On the other hand for $a, b \neq 0$ and $\beta = 1$

$$\psi(x,t) \sim \left(\frac{\cos\left(\frac{xb}{2a}\right)}{2at} + \frac{b\sin\left(\frac{xb}{2a}\right)}{2ax}\right) e^{-\frac{x^2 - t^2 b^2}{4at}}, \tag{14}$$
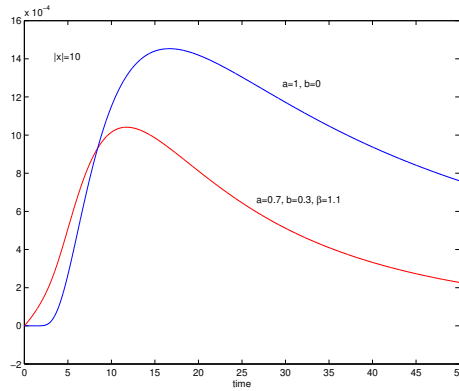
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Figure 1: Comparison of the propagation time of a delta signal at $(x = 0, t = 0)$ to a distance $x = 10$ for normal $(a = 1, b = 0)$ and mixed diffusion $(\beta = 1.1, a = 0.7, b = 0.3)$

implying $t \sim \frac{x}{b}$ That is, there is asymptotic ballistic motion for any $b \neq 0$. However, an effect that is not obvious from these asymptotic estimates, but is observed in the numerical solutions of Eq.(12), is the fact that even for small times the signal starts to be noticeable when $\beta < 2$ and $b \neq 0$.

Of course superdiffusion exists only if $\beta < 2$. For $\beta \geq 2$ the behavior would be practically indistinguishable from normal diffusion. This puts into evidence the fact that the mere existence of long-range connections does not guarantee the existence of fractional superdiffusion. That is, a sufficient small density decay of the long-range connections is required. This is an important hint to be taken into account on the relation of functional connectivity to brain cognitive disorders.
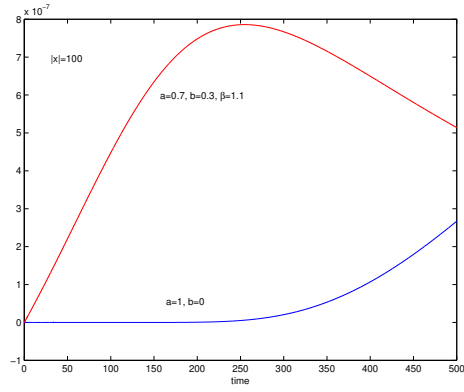
Figure 2: Comparison of the propagation time of a delta signal at $(x = 0, t = 0)$ to a distance $x = 100$ for normal $(a = 1, b = 0)$ and mixed diffusion $(\beta = 1.1, a = 0.7, b = 0.3)$

Figure 3: Comparison of the propagation time of a delta signal at $(x = 0, t = 0)$ to a distance $x = 10$ for normal $(a = 1, b = 0)$ and mixed diffusion $(\beta = 1.1, a = 0.9, b = 0.1)$
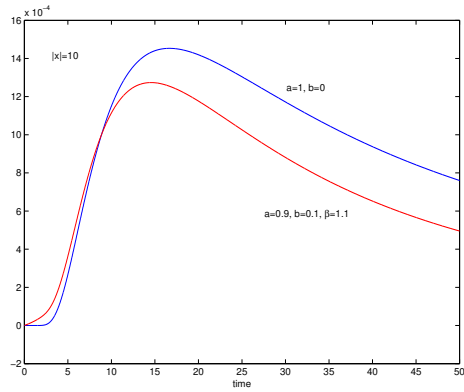
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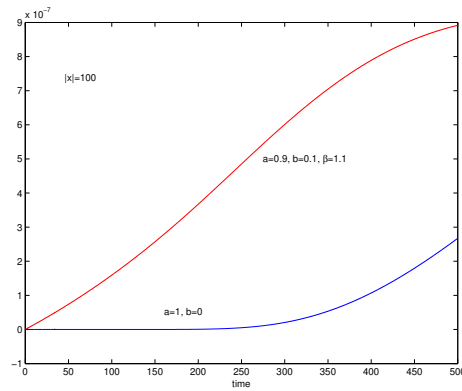Figure 4: Comparison of the propagation time of a delta signal at $(x = 0, t = 0)$ to a distance $x = 100$ for normal $(a = 1, b = 0)$ and mixed diffusion $(\beta = 1.1, a = 0.9, b = 0.1)$

4 Signal propagation in a discrete fractional network: Numerical results

So far we have discussed the diffusion behavior of fractional networks in the framework of the continuous approximation to the network. Here, by numerically simulating the propagation of a pulse of information in a discrete network, we show that the results are qualitatively consistent with those obtained from the continuous approximation modeled by the fractional differential equations.

We consider 40000 agents (nodes) placed in a two-dimensional 200x200 grid and establish connections among the nodes with a distance-dependent power-law distribution

$$p_{ij} \sim d^{-\gamma}. \tag{15}$$

Namely, we pick a node at random and establish a connection to another node at a distance d

$$d = \exp \left\{ \frac{\log \left(d_{\min}^{1-\gamma} - C\gamma y \right)}{1 - \gamma} \right\}, \tag{16}$$

y being a random number in the interval $[0, 1]$ and C a constant

$$C = \frac{\left(d_{\min}^{1-\gamma} - d_{\max}^{1-\gamma} \right)}{\gamma}. \tag{17}$$

In Fig.5 we show the pattern of connections, that is, the graphical representation of the adjacency matrix, for the networks with $\gamma = 2$, $\gamma = 3$ and also for a nearest-neighbor (NN) network. All networks have the same number of connections, meaning that the sparsity index of the first two networks is 9.95 10^{-5}. The topological parameters of the networks are listed in the table,

Network	Size	Avg. Degree	Avg. Path Length	Clustering	Assortiveness
$\gamma = 2$	39541	4.02	10.71	0.0667	0.022
$\gamma = 3$	39899	3.99	24.84	0.161	0.041
$\gamma = 4$	39982	3.98	56.08	0.225	0.083
NN	40000	3.98	133.33	0	0.66

For the $\gamma-$networks the parameters, as well as the signal propagation experiments, are performed for the largest connected component.

To study the signal propagation in the fractional network, we consider, at time zero, a unit pulse at one node (the source) and study how it propagates throughout the network until it reaches a distant node (the target). The source and target nodes are chosen among the most distant ones in the networks, that is, nodes near diagonally opposite corners, but not exactly at the corners to avoid boundary effects. It is also important to choose source and target nodes in the largest connected component of the networks, which we check using Tarjan's algorithm [35]. At each time step the pulse is transmitted to
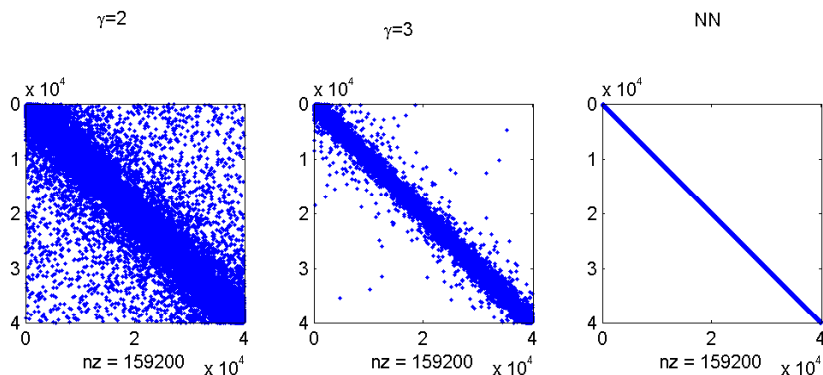
Figure 5: Connection patterns for network with $\gamma = 2$ and 3 and a nearest-neighbor (NN) one. All networks have the same total number of connections nz. The numbers in the axis are the node labels.

the neighbors of each activated node, with a "no information cycle" condition being imposed. That is, after a node transmits the pulse to its neighbors it no longer transmits the same pulse even if it receives it back through some cycle in the network. In the Fig.6 we show the results of a typical simulation for networks with $\gamma = 2, 3, 4$ and the nearest-neighbor one. As expected the speed of transmission diminishes with increasing γ. Not only is the signal transmitted much faster in the fractional network, but also its coherent nature is preserved, instead of being spread over a very large number of distinct times as it might occur in a sparse random network. Although for $\gamma \geq 3$ one might expect from the continuous approximation to have normal diffusion, one sees that for $\gamma = 3$ and 4 one has much faster propagation that in the nearest-neighbor network. It means that even a small number of long distance connections may speed-up the signal propagation.

In these numerical experiments we have studied the arrival of the emitted pulse rather than the establishment of the diffusion wave in the network[1]. It makes sense if one is concerned with the flow of news in the networks. In the continuous approximation it would correspond roughly to the arrival of the maximal intensity in the diffusion wave.

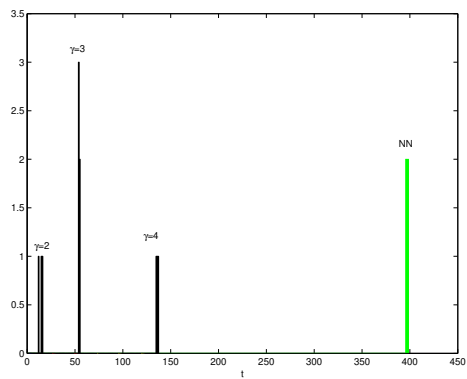
[1]Cf. the no information cycle condition.

Figure 6: Propagation of a unit pulse between two distant nodes for networks with exponents $\gamma = 2, 3, 4$ and a nearest-neighbor one. The label t is the number of time steps, different intensities (at the vertical axis) for the target node meaning that the signal may arrive simultaneously through different paths

5 Remarks and conclusions

1. As has been experimentally confirmed, existence of long-range connections between the brain intrinsic connectivity networks (ICN's) is critical for integration and interpretation of sensory stimuli and higher cognitive functions. One view of brain integration and consciousness [36] [37] is based on a percolation model. For percolation, that is, for the formation of a global cluster, it suffices that connections exist between the local clusters. However for the establishment of higher cognitive functions, and in particular in the predictive coding mode, it is necessary that the interaction between the ICN's be established at a sufficiently fast rate. Therefore the mere existence of long-range connections is not sufficient, it also necessary that they have, for example, a power-law dependence with $\gamma < 3$.

2. The additional hardware cost of myelinated long-range connections in the brain is compensated by the integration of information and higher cognitive functions. Another puzzling additional energetic cost is that, when tested with fMRI, the resting brain is in fact turbulent and restless [38]. There is a good reason for that, probably related to speed of reaction. With the operating time scales of individual neurons and their low average firing rate, pattern recognition by evolution towards an equilibrium fixed point or minimizing an energy function would be much too slow for practical living purposes. As has been conjectured, for example from the studies of the olfactory bulb [39], a much faster recognition is achieved by replacing the low-level chaos that exists in the absence of an external stimulus by, in the presence of a signal, a pattern of bursts with different intensities in different regions. A network of Bernoulli units [40] is a model confirmation of this conjecture.

3. Finally, as already discussed in [16], the robustness and controllability properties of the fractional networks are so very different from the scale-free networks that they deserve a detailed study. This is relevant not only for brain functions but also concern the uses and misuses of information flow in the social networks.

6 Appendix: Power-law long-range connections and fractional diffusion

For completeness we include here a short derivation of the relation between power-law long-range connections and fractional diffusion equations, already discussed in Ref.[16].

Let the probability of a link at distance d be proportional to a power of the distance

$$P_{ij} = cd_{ij}^{-\gamma} \qquad\qquad \text{with } \gamma \leq 3.$$

Consider now a block renormalized network N^* where each set of q nearby nodes in the original network N are mapped to a node of the N^* network. With the block renormalization, the power-law connection probability leads to

actual connection strengths in the renormalized network. In the N^* network the connections are

$$A_{ij}^* \simeq cq d_{ij}^{-\gamma},$$

with the Laplacian L^* and degree G^* matrices of the N^* network being

$$L^* \psi(i) = G_{ii}^* \psi(i) - cq \sum_{j \neq i} d_{ij}^{-\gamma} \psi(j).$$

Compare the distance dependence of the elements of the Laplacian matrix L^* along one of the coordinate axis with a discrete one-dimensional representation of a fractional derivative. The symmetrized Grünwald-Letnikov representation of the fractional derivative $(a < x < b)$ (see [41]) is

$$
\begin{aligned}
D^\beta \psi(x) &= \frac{1}{2} \lim_{h \to 0} \frac{1}{h} \left\{ \sum_{n=0}^{\left[\frac{x-a}{h}\right]} (-1)^n \begin{pmatrix} \beta \\ n \end{pmatrix} \psi(x - nh) \right. \\
&\quad \left. + \sum_{n=0}^{\left[\frac{b-x}{h}\right]} (-1)^n \begin{pmatrix} \beta \\ n \end{pmatrix} \psi(x + nh) \right\},
\end{aligned}
\tag{18}
$$

with coefficients

$$
\left| \begin{pmatrix} \beta \\ n \end{pmatrix} \right| = \frac{\Gamma(\beta+1) \left| \sin(\pi\beta) \right|}{\pi} \frac{\Gamma(n-\beta)}{\Gamma(n+1)} \underset{n \text{ large}}{\frown} \frac{\Gamma(\beta+1) \left| \sin(\pi\beta) \right|}{\pi} n^{-(\beta+1)}
\tag{19}
$$

and $sign \begin{pmatrix} \beta \\ n \end{pmatrix} = (-1)^{n+1}$.

Comparing (18-19) with the expression for $L^* \psi(i)$, the conclusion is that diffusion in the N^* network is fractional diffusion of exponent $\beta = \gamma - 1$. $\beta = 2$ would be normal diffusion, all $\beta < 2$ corresponding to superdiffusions. Notice however that the calculations here only lead to a rough estimation of the fractionality index of the networks. A better estimation requires a detailed study of the nonlocal diffusion equation at asymptotic and intermediate large times [22].

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