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**REM Working Paper 0221-2022**

March 2022

**REM – Research in Economics and Mathematics**

Rua Miguel Lúpi 20,  
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ISSN 2184-108X

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# Long-range connections, real-world networks and rates of diffusion

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## Abstract

Long range connections play an essential role in dynamical processes on networks, on the processing of information in biological networks, on the structure of social and economical networks and in the propagation of opinions and epidemics. Here we review the evidence for long range connections in real world networks and discuss the nature of the nonlocal diffusion arising from different distance-dependent laws. Particular attention is devoted to exponential and power laws.

## 1 Introduction

Long range connections play an important role in the dynamical processes on networks. For example, the existence and relevance of long-range connections in the brain has been studied [1] - [6] with diminished long-range functional connectivity being associated to cognitive disorders [7]. Long-range connections are also important for the structure of social and economic networks [8] [9] [10] as well as for the propagation of epidemics [11].

Dynamics on networks involving jumps over many links or cascades of many unit jumps, may lead to anomalous diffusion [12] - [16]. Likewise the existence of long range connections is expected to lead to anomalous diffusion effects. When the density of long range connections follows a power law, as a function of a suitably defined distance, the propagation of signals in the network behaves as the solution of a fractional diffusion equation [17] [18].

Because networks with power-law connections, leading to superdiffusion, display properties so very different from scale-free and hub dominated networks, it has been proposed to classify them as a new network class, **the fractional**

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**networks**<sup>1</sup>. However, it is to be expected that the nature of the diffusion and therefore the propagation of information in the network, may be different for other distance dependencies of the density of long range connections. This is an important issue because, for example in brain networks, existence of long-range connections between the specialized modules does not guarantee global integration of the cognitive functions. It is also necessary that the flow of information be sufficiently fast for the stimulus integration to be performed in a timely manner<sup>2</sup>.

In Section 2 of this paper we review the empirical evidence for long range connections in a few real-world networks. This includes both distance-dependent results obtained by other authors in several networks as well as our own analysis of some networks. A focus of our analysis is the extraction of the distance-dependent functional laws of the connections. A general result that is obtained in many networks is that short-distance and long-distance connections follows different laws. This is consistent with the modular nature of many networks, which have a high density of connections inside local modules and a much smaller number of long-range connections. This is typically the structure of brain networks. In these networks the long-range connections between the modules favor integration of perception but, on the other hand, they also imply a much larger biological cost. As we will find out, this two-laws behavior also appears in a transport network, again signaling a modular structure on the economic organization of society. In social networks the law of short- and long-range connections is more uniform. Probably it reflects the fact that short and long range connections have a similar cost.

Section 3 contains a detailed study of the nature of nonlocal diffusion, with particular emphasis on the comparison of the power law and exponential distance-dependence of the connections. It is clear that all real-world networks have a distance cut-off. Because of that, the distribution of the connections intensity has compact support and it is well-known [21] [22] that, for compact support diffusion kernels, the asymptotic behavior is identical to the one of the heat equation (normal diffusion). However, what is important in practice is not the behavior of the propagation of information for extremely large times, but for short and intermediate large times. We discuss in detail this question in Section 3 for power laws with a cut-off and find out that indeed, for intermediate large times, the solution behaves like an anomalous diffusion solution, very different from the asymptotics for extremely large times. Analytical estimates for small and large times are possible, but not for these intermediate times of practical interest. By numerical calculations we have nevertheless been able to parametrize and quantify the effective "fractionality" of the diffusion for intermediate times.

Power-law and exponential distance-dependence of the connections lead to very different diffusion behavior. However it is many times difficult to distinguish between these two-dependencies from the experimental data. To empha-

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<sup>1</sup>Networks with non-fractional couplings where nevertheless diffusion has fractional features, not to be confused with fractionally coupled networks [19] [20].

<sup>2</sup>This seems of particular relevance for the forward and backwards loops in the predictive coding mode of brain operation.

size this point we have displayed for all analyzed networks, both power-law and exponential fits. The reason for this difficulty is further analyzed in Section 4 (see Fig. 16). In Section 4 we also carry out a numerical experiment of propagation of a unit signal in power-law, exponential and nearest-neighbor networks. The smaller propagation times are, of course, for the power-law networks, but even the exponential network is much more efficient than the nearest-neighbor one.

The set of real-world networks that we were able to analyze is of course limited. This was the data that we were able to obtain either from the Web or kindly provided by the referenced authors. We encourage the readers that have access to other data to pursue our preliminary work. Interesting questions to explore are the nature of the functional law of connections in the human brain (both structural and functional) as compared to other mammals, as well as the nature of the "fractionality" index across species. Are some brain impairments, that have been identified, related to the nature of the distance-dependence law or just to the overall number of long-distance connections? Data on the more popular social networks and their distance-dependent links is harder to obtain. Interesting studies might be performed in that data when available, correlated also with the diffusion of trends and opinions. Some data on ecological and trophic networks was available, but not sufficient for robust results. Notice that in some of these networks the relevant notion of distance is not geographical distance.

## 2 Real-world networks: Analysis of empirical data

### 2.1 A human mobility network

The first network that was analyzed was the network of flights to and from airports in the New York region.

Data for the month of January 2019, for example, was collected from the database available in reference [23]. The file contains the following fields:

```
# Passengers (number of passengers),
# Distance (in miles),
# Origin_Airport_ID (unique identifier Airport_ID),
# Destination_Airport_ID,
# Month
```

From the data in these fields a weighted network of Airport\_IDs is defined. The Airport\_IDs are the nodes while the strength of the links is given by the number of passengers flying between the nodes (Origin\_Airport\_ID and Destination\_Airport\_ID, or vice-versa) during the time interval being considered (month), regardless of the flight direction. Figure 1 shows the network of flights to and from New York airports in January 2019. The network has 317 nodes (airports) and 641 weighted links, the strongest link being a Delta Airline flight

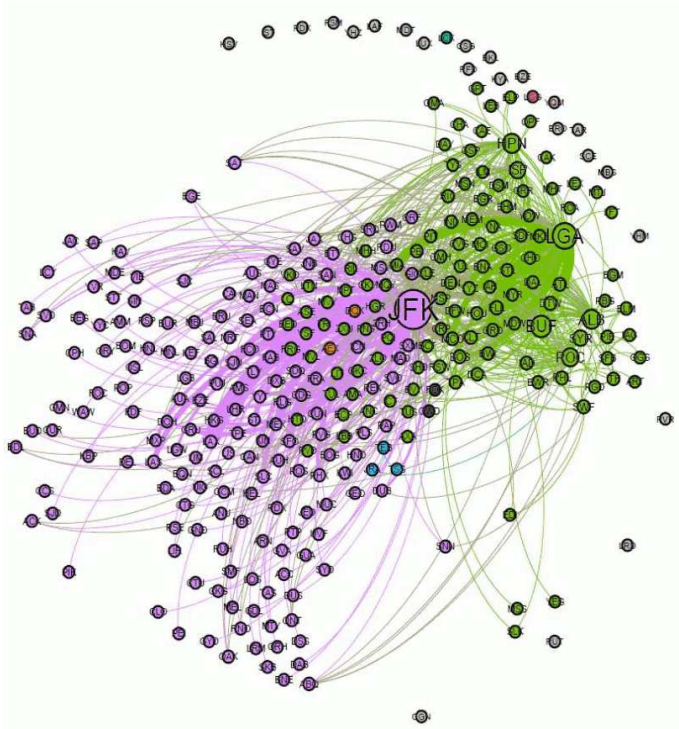


Figure 1: The network of flights to or from New York airports in January 2019

that links the airports of LaGuardia (LGA) and Atlanta (ATL). The largest distance (10201 miles) concerns the JFK airport in New York and the Melbourne Airport, in Australia. The average degree is 4 and the average weighted degree is 98319. The network diameter is 7 and the network modularity displays four different classes. The colors in Figure 1 characterize each one of the modularity classes, while the size of each node is proportional to its degree. The average clustering coefficient is 0.49 and the average path length is 2.4.

We have collected the same type of data for the full years of 2019 and 2020. The results are plotted in Figs 2 and 3

In these figures we also make polynomial fits corresponding in the left-hand plots to a power law  $w \sim d^{-\beta}$  where  $w$  is the number of passengers and  $d$  the inter-airports distance and to an exponential law  $w \sim \exp(-\lambda d)$  in the right-hand plots. Our conclusion is that the data is better described by a two-law structure separating short and the long range connections, with the long range connections obeying an approximate power law. According to the analysis in Section 3, this power law leads to anomalous diffusion, which has relevance for the fast propagation of trends and diseases.

Other authors have also emphasized the role of long-range connections in human mobility networks. For example Viana and Costa [24] analyzed their

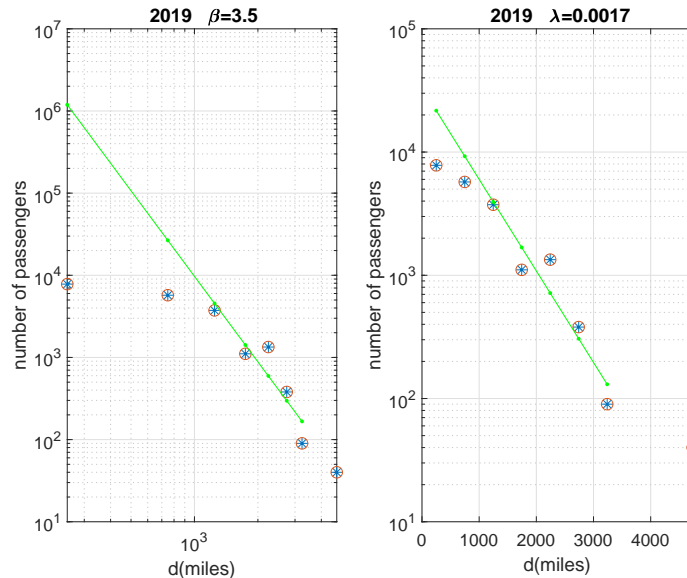


Figure 2: Distance-dependent distribution of the network links in the airports network for the year 2019.  $\beta$  is the exponent of the power law  $w \sim d^{-\beta}$  and  $\lambda$  the coefficient in the exponent of the exponential law  $w \sim \exp(-\lambda d)$ .

role in the London urban and US highway networks mostly in connection with the small world properties and travel velocity between nodes. The statistics of travel patterns and returns to the same locations using cell phones has also been studied [25]. The conclusion being that the individual travel patterns collapse into a single spatial probability distribution, it would be interesting to find out whether there are distinguishing features in short and long travel distances. . Another human mobility that has been analyzed by Riascos and Mateos [26] involved one billion taxi trips in New York City. The conclusion was that the probability of a trip to a site inside a circular region of radius  $R$  around the origin is approximately constant, whereas the probability to a long range trip outside this circle decays as a power law with an exponential cutoff. That is, a modular structure similar to what we have found for the airports network.

## 2.2 Brain networks

Network theory is a valuable tool to explore the complex nature of brain networks, the discovery of patterns of connections between cortical areas being the focus of most research works (see for example [27] [28] [29] [30]). Our analysis will concentrate on macaque and mouse brain networks.

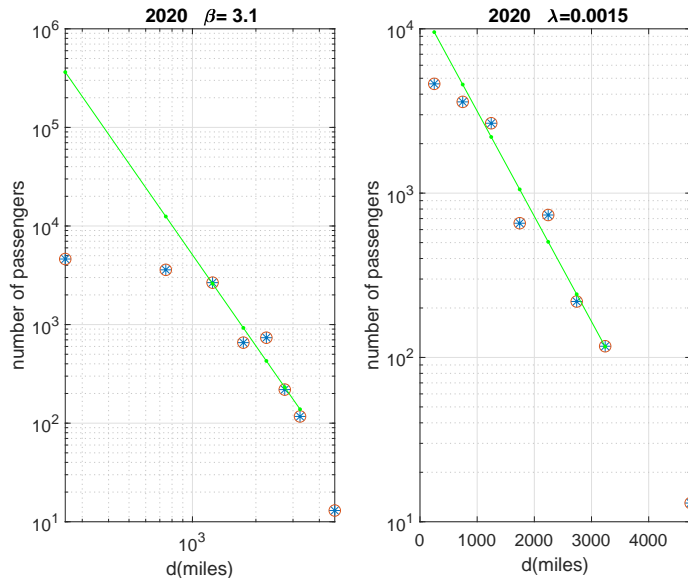


Figure 3: Distance-dependent distribution of the network links in the airports network for the year 2020.  $\beta$  is the exponent of the power law  $w \sim d^{-\beta}$  and  $\lambda$  the coefficient in the exponent of the exponential law  $w \sim \exp(-\lambda d)$ .

### 2.2.1 Macaque

The data used here was collected from Core-Net.org [27]. In this data the pathways linking cortical areas have been followed by retrograde tracing experiments using injections of fluorescent retrograde tracers<sup>3</sup>.

The file PNAS2013.xlsx downloaded from the data base has the following fields: 1. Monkey, 2. Source area, 3. Target area, 4. Neurons, 5. FLNe (fraction of labeled neurons), 6. Distance.

Source ( $j$ ) and target ( $i$ ) areas are labelled through retrograde labeling using fluorescent tracers [30]. This labeling method reveals all incoming connections  $j_i$  to an injected (target) area  $i$  by labeling the cell bodies of the neurons in source area  $j$  for which their axons connect to area  $i$ . The fraction of labeled neurons (FLNe), given by the ratio  $FLNe = \frac{Neurons}{TotalNeurons}$ , may be interpreted as the probability of a neuronal projection from Source area  $i$  to Target area  $j$  [31].

The fields Source and Target provide a weighted network of 91 brain areas. The Source and Target areas are the nodes while the intensity of the links corresponds to the number of Neurons connecting these areas. Figure 4 shows the network of the 91 areas in the PNAS2013.xlsx file, 39 macaques being sampled in 1989 instances of the database. Thus, this macaque network has 91 nodes,

<sup>3</sup>Localization of injection sites and labeled neurons was based on a new reference atlas that includes 91 cortical areas mapped to the left hemisphere of case M132 [28].



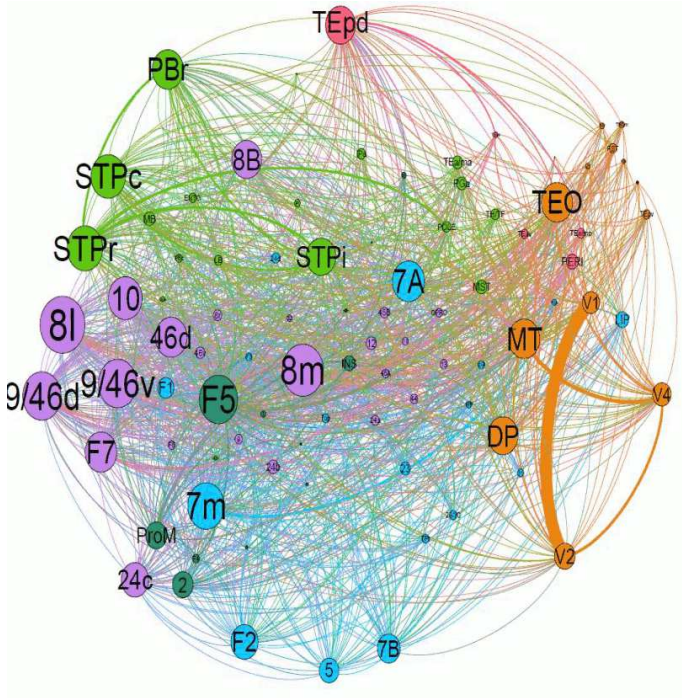


Figure 4: The network of 91 brain areas collected from 39 macaques

connected by 1401 links, the size of each node being proportional to its degree. The average degree is 30.7 and the average weighted degree is 869967.3. The network diameter is 3 and the network modularity displays six different classes, characterized by different colors in the figure. The average clustering coefficient is 0.74 and the average path length is 1.6.

The distance-dependent distribution of the network links is shown in Figure 5 both in a log-log and a semilog plot.

It seems that in this case the better fit is an exponential one  $w = \exp(-\lambda d)$ ,  $w$  being the links intensity (number of neurons) and  $\lambda = 0.16mm^{-1}$ .

Likewise, if instead of the Neuron field we focus on the Fraction of Labeled Neurons (FLNe) the result is quite similar, being in accordance with the work in [30], which by using data from macaque and mouse, infers the existence of a general organizational principle based on an exponential distance rule

$$FLNe(d) = c * \exp(-\lambda d) \quad (1)$$

### 2.2.2 Mouse

The database used here is obtained from reference [29], MouseDatabase.xlsx. The file MouseDatabase.xlsx contains, among others, the following fields:

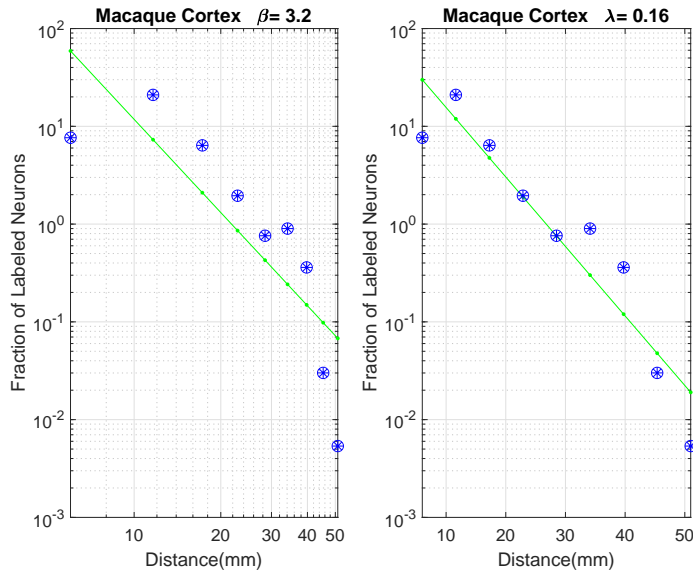


Figure 5: Distance-dependent distribution of the network links for the macaque cortex.  $\beta$  and  $\lambda$  have the same meaning as before.

1. Mouse, 2. Source area, 3. Target area, 4. FLNe (fraction of labeled neurons), 5. Neurons, 6. Distance. The fields Source and Target allow to define a network of 47 brain areas. The Source and Target areas are the nodes while the links correspond to the number of Neurons between the Source and Target areas, as in the macaque example studied before. Figure 6 shows the network of 47 areas in MouseDatabase.xlsx, where 27 mice were sampled in 1242 instances of the database. The mouse network has 47 nodes and 703 weighted links.

In this mouse example, the number of target areas is much smaller than in the macaque one. There are just seven (AM, AL, RL, P, ML, RDS and V1) target areas (against 43 source ones) and therefore these are the areas with the largest degrees, as Figure 6 shows. The network diameter is 2 and the network modularity displays three different classes characterized by different colors in Figure 6. The average clustering coefficient is 0.8 and the average path length is 1.35.

The distance-dependent distribution of the links is shown in Figure 7. This is a case where it is difficult to decide between an exponential and a power law. For an exponential law we would have  $w \sim \exp(-\lambda d)$  with  $\lambda = 3.9 \text{ mm}^{-1}$ . The average distance in our sample is  $\langle d \rangle_{\text{mouse}} = 0.78 \text{ mm}$ , smaller than the one reported in the samples analyzed by other authors (for example  $\langle d \rangle_{\text{mouse}} = 4.54 \text{ mm}$  in [29]). The occurrence of an exponential law for the macaque and mouse brain networks is in fact favored by most authors. Notice however that Knox et al. [32] in a high resolution study of the mouse connectome shows a power-law fit to the normalized connection density as a function of the distance.

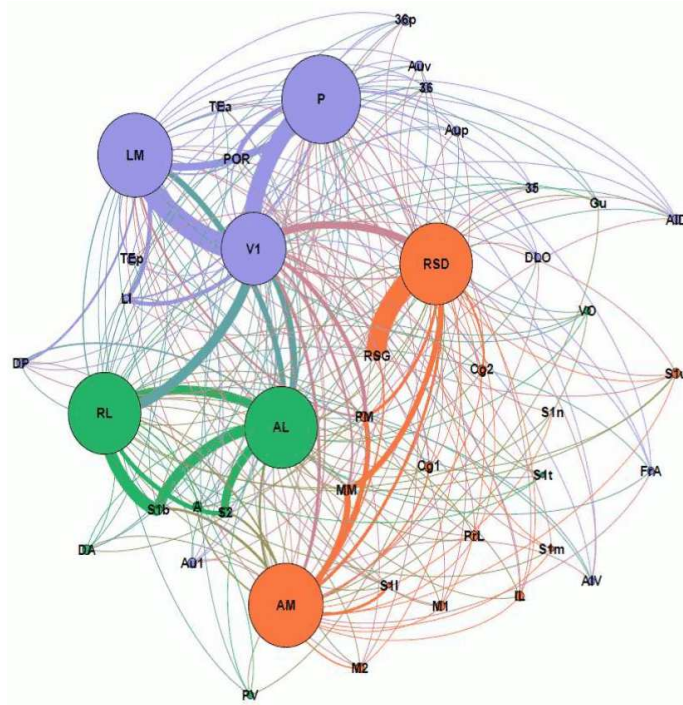


Figure 6: The network of neurons in 47 brain areas of 27 mice

The statistics of the data fit is however too poor to be able to decide. It would be interesting to compare the functional laws in the macaque and mouse networks with those of the human brain, both structural and functional, for which we have not yet had access to reliable data.

### 2.3 A social network

Here we analyze the Brightkite location-based online social network [33]. Brightkite is a location-based social networking service provider where users share their locations by checking-in. This friendship network was collected using users public API, and consists of 58,228 nodes and 214,078 edges. The network is originally directed but later transformed into a network with undirected edges when there is a friendship in both ways. Data include a total of 4,491,143 checkins of the network users over the period April 2008 - October 2010.

The first downloaded file (Loc-brightkite\_totalCheckins.txt) with information on the users location contains the fields: 1. User, 2. Latitude, 3. Longitude, 4. Location\_id, 5. Check-in time. The second Brightkite file (loc-brightkite\_edges.txt) is a friendship network of Brightkite users (58,228 nodes and 214,078 edges).

From the locations of each pair of Brightkite friends the corresponding dis-

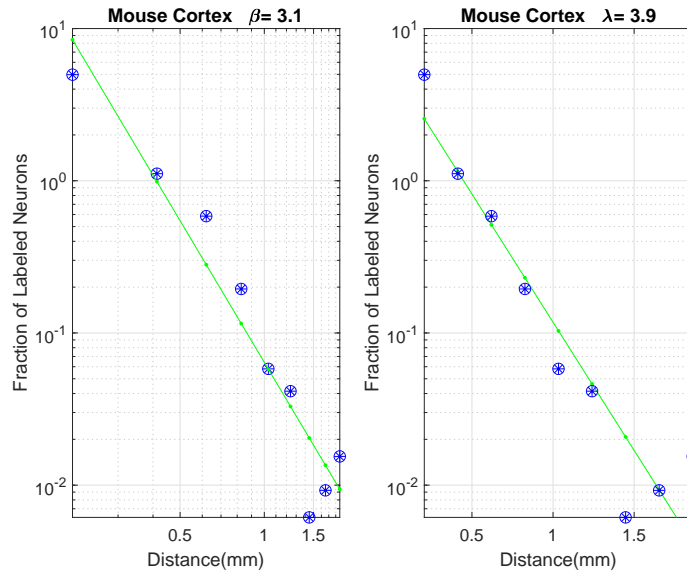


Figure 7: Distance-dependent distribution of the links in the mouse network.  $\beta$  and  $\lambda$  have the same meaning as before.

tance is computed. The characterization of the distance-dependent distribution of the network links (number of friends) is shown in Figure 8. Figure 8 shows a polynomial fits (in log-log and semilog plots) of the number of friends  $w_{i,j}$  and their distances  $d_{i,j}$ . The best fit corresponds to a power laws  $w_{i,j} = d_{i,j}^{-\beta}$  with  $\beta = 1.9$ . In this case one obtains a fairly uniform power law without any indication of a modular structure.

## 2.4 A fungi network

The network studied here is part of a large data set of 269 fungal networks available at [34] [35]. The data providers construct fungal networks by estimating cord conductances. In defining fungal networks, the nodes are located at hyphal tips, branch points, and anastomoses, while the edges represent cords. Structural networks are constructed by calculating edge weights based on how much nutrient traffic is predicted to occur along each edge.

Being location-based networks, in addition to the complete list of network links, the data set also includes the coordinates of each node. For the network present here, we used the sample identified as *Pp\_M\_Tokyo\_U\_N\_26h\_1*. This sample has 1357 nodes and 3716 undirect links. From the coordinates of each pair of linked nodes the corresponding distance is computed. The characterization of the distance-dependent distribution of the network links is shown in Figure 9.

This is again a situation where it is difficult to decide whether to infer an

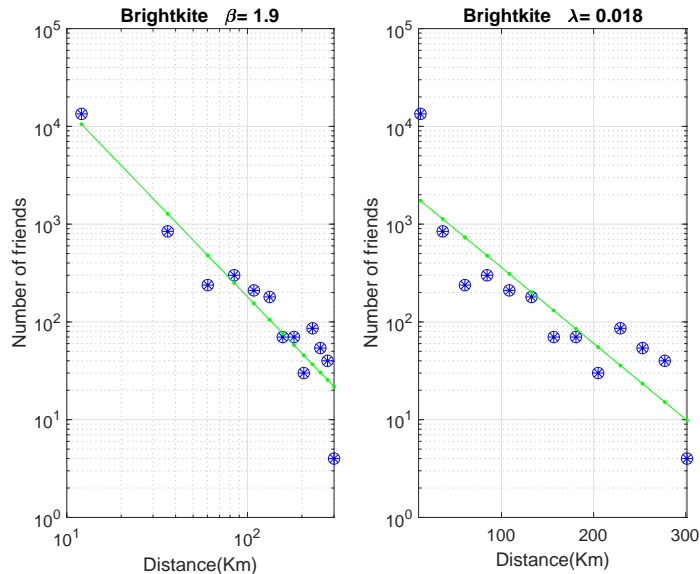


Figure 8: Friends vs. distance in the Brightkite network.  $\beta$  and  $\lambda$  have the same meaning as before.

exponential or a power-law.

### 3 Nonlocal diffusion and long range network connections

Short and long-range connections determine the global behavior of a network, both its emergent dynamical evolution and the spread of information (or pathologies) throughout the network. It is intuitive that long-range connections must play a role on the speed of the spread of information as well as on the context integration of the sensory inputs in brain networks. However it is not so obvious to infer how such phenomena may depend on the density and distance-dependence of the long-range connections.

Here we make an attempt to address these questions in the framework of the theory of nonlocal diffusion [21] [36] [37]. Let some time-dependent quantity (a field)  $\phi(t, x)$  stand for the density of individuals, quantity of information or activation in a  $d$ -dimensional space  $X$ . There are two related problems of nonlocal diffusion. If  $\phi(t, x)$  stands for the density of individuals or the degree of activation, the nonlocal diffusion equation is

$$\frac{\partial}{\partial t} \phi(t, x) = \int_X \rho(y, x) \phi(t, y) d^n y - \phi(t, x), \quad (2)$$

$\rho(y, x)$  being the probability density for a jump (or transmission) from  $y$  to  $x$

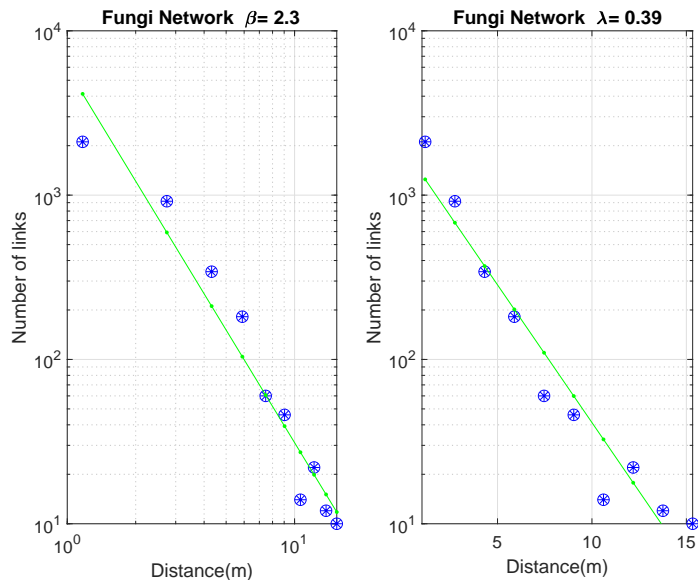


Figure 9: A fungi network.  $\beta$  and  $\lambda$  have the same meaning as before.

and the last term accounts for the rate of departure from  $x$  to other locations.

$$-\phi(t, x) = - \int_X \rho(x, y) \phi(t, x) d^n y \quad (3)$$

because  $\int_X \rho(x, y) d^n y = 1$ . Eq. (2) conserves the quantity  $Q = \int_X \phi(t, x) d^n x$ . In addition one might consider additional terms added to the right hand side of these equations to represent nonlocal internal interactions, localized sources or consumption terms.

However if  $\phi(t, x)$  stands for an information quantity or a disease, the last term might not make sense. Information does not decay in transmission to other nodes nor a disease is cured by infection of the neighbors. To simply suppress the last term in Eq. (2) to describe this situation does not make sense either because then  $Q$  grows exponentially. We will come back to this question on our numerical simulations of propagation of an impulse of information and for the time being our analysis will concern the Eq. (2).

In the continuous approximation, which would be a good approximation for a network with a very large number of nodes, the kernel  $\rho(y, x)$  may be considered to be proportional to the density of connections between nodes at the positions  $y$  and  $x$ . Here we will be concerned with the case where the kernel  $\rho(y, x)$  is a function of the distance  $|y - x|$  only

$$\rho(y, x) = \rho(|y - x|),$$

with the distance defined by a problem-adapted metric. Without loss of generality we will assume the parametrization of the network to be such that the

distances are Euclidean distances on that parametrization.

From our analysis of the real-world networks with long-range connections we have concluded that two important functional distance dependencies are the power law and the exponential law. It is therefore for these two laws that our calculations will be performed. As for dimensionality, two most important cases are  $d = 2$  and  $d = 3$ . However other higher dimensions may be important as well, when the network nodes are characterized by many parameters. Important examples are social networks, credit scoring networks, etc. Therefore we will derive general results for arbitrary (finite) dimensions. Our main interest is the asymptotic behavior of  $\phi(t, x)$  for long and intermediate times. For  $\rho(y, x) = \rho(y - x)$  the integral in (2) is a convolution and a Fourier transform treatment is appropriate

$$\frac{\partial}{\partial t} \tilde{\phi}(t, k) = \tilde{\rho}(k) \tilde{\phi}(t, k) - \tilde{\phi}(t, k) \quad (4)$$

with

$$\begin{aligned} \tilde{\phi}(t, k) &= \int_X e^{ik \cdot x} \phi(t, x) d^n x \\ \tilde{\rho}(k) &= \int_X e^{ik \cdot x} \rho(y - x) d^n x. \end{aligned} \quad (5)$$

From Eq. (4)

$$\begin{aligned} \tilde{\phi}(t, k) &= \tilde{\phi}(0, k) \exp(t(\tilde{\rho}(k) - 1)) \\ \phi(t, x) &= \frac{1}{(2\pi)^n} \int_{\tilde{X}} e^{-ik \cdot x} \tilde{\phi}(t, k) d^n k \end{aligned} \quad (6)$$

For definiteness we consider a Cauchy problem corresponding to the diffusion of a unit pulse at the origin at time zero. That is, we are considering what is called the fundamental solution to the equation. Once this is found, solutions for arbitrary smooth initial conditions are obtained by convolution with the fundamental solution. With,

$$\tilde{\phi}(0, k) = 1, \quad (7)$$

the solution is

$$\phi(t, x) = \frac{1}{(2\pi)^n} \int_{\tilde{X}} d^n k e^{-ik \cdot x} \exp(\tilde{\rho}(k) t) \quad (8)$$

with, for a kernel which only depends on the modulus of the distance ( $\rho(|y - x|)$ ), in dimension  $n$

$$\tilde{\rho}(k) = \tilde{\rho}(|k|) = (2\pi)^{n/2} \int dr r^{n-1} \rho(r) \frac{J_{\frac{n}{2}-1}(|k| r)}{(|k| r)^{\frac{n}{2}-1}}, \quad (9)$$

We will now particularize this solution for two types of kernels:

### 3.1 The power law case

Let the connection strength be proportional to  $\frac{c_1}{r^\beta}$  for most of the range of  $r$ . However, because this function is not normalizable, one truncates it with a  $G(r)$  function

$$G(r) = \begin{cases} 1 & \text{for } r_{\min} \leq r \leq r_{\max} \\ 0 & \text{otherwise} \end{cases}$$

$$\rho_1(r) = \frac{c_1}{r^\beta} G(r), \quad (10)$$

the normalization,  $\int d^n x \rho_1(r) = 1$ , implying

$$c_1 = \begin{cases} \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{n-\beta}{(r_{\max}^{n-\beta} - r_{\min}^{n-\beta})} & (n \neq \beta) \\ \frac{\Gamma(n/2)}{2\pi^{n/2}} \left(\log \frac{r_{\max}}{r_{\min}}\right)^{-1} & (n = \beta) \end{cases} \quad (11)$$

The asymptotic behavior for large times of the solution of Eq. (2) and its relation to the behavior of the Fourier transform  $\tilde{\rho}_1(|k|)$  of  $\rho_1(r)$  has been discussed by several authors in the past (see for example [21] [22]). Namely, the behavior of the solution for large times is controlled by the functional dependence of  $\tilde{\rho}_1(|k|)$  at small  $|k|$ . In particular, if for small  $|k|$

$$\tilde{\rho}_1(|k|) = 1 - A |k|^\alpha + o(|k|^\alpha) \quad (12)$$

then the large time asymptotic behavior of the solution is the same as for the solution  $v(t, x)$  of the fractional Laplacian equation  $\partial_t v(t, x) = -A(-\Delta)^{\alpha/2} v(t, x)$  with the same initial condition, namely

$$\lim_{t \rightarrow \infty} t^{n/\alpha} \max_x \{\phi(t, x) - v(t, x)\} = 0 \quad (13)$$

with an  $L^\infty$  rate of decay of the solution  $t^{-n/\alpha}$ . In particular for the  $\alpha = 2$  case, the asymptotic behavior is identical to the heat equation with a decay rate  $t^{-n/2}$  and the asymptotic profile is a Gaussian. Furthermore, refined asymptotics in terms of the derivatives of the fundamental solution, identical to those of the heat equation, were obtained [38]. What in principle would be of special importance for our network setting is also the fact that the same kind of estimate is obtained for nonlocal diffusion on a lattice [39]. All these are very interesting mathematical results, which however may not be very useful at relatively large, but finite, times. Let us compute the small  $|k|$  behavior of the power law Fourier transform  $\tilde{\rho}_1(|k|)$ ,

$$\begin{aligned} \tilde{\rho}_1(|k|) &= (2\pi)^{\frac{n}{2}} c_1 \int_{r_{\min}}^{r_{\max}} dr r^{n-1} \frac{J_{\frac{n}{2}-1}(|k|r)}{(|k|r)^{\frac{n}{2}-1}} \frac{1}{r^\beta} \\ &= 1 - A |k|^2 + \dots \end{aligned} \quad (14)$$

with

$$A = \begin{cases} \frac{n-\beta}{2n(n+2-\beta)} \frac{r_{\max}^{n+2-\beta} - r_{\min}^{n+2-\beta}}{r_{\max}^{n-\beta} - r_{\min}^{n-\beta}} & \text{for } n \neq \beta \\ \frac{1}{4n} \frac{r_{\max}^2 - r_{\min}^2}{\log r_{\max} - \log r_{\min}} & \text{for } n = \beta \end{cases} \quad (15)$$



According to the large time asymptotic results quoted above, the  $\alpha = 2$  behavior of the small  $|k|$  behavior of  $\tilde{\rho}_1(|k|)$ , might lead us to expect for this case a diffusion similar to the heat equation. This may indeed be true for extremely large times but not necessarily for the small or even relatively large times of practical importance. In any case the  $\alpha = 2$  was to be expected, of course, because the  $G(r)$  truncation makes  $\rho_1(r)$  a compact support function [22].

The limited usefulness of the (14) expansion is already apparent from the fact that whenever the power law range is large (large  $|r_{\max} - r_{\min}|$ ) the  $A$  coefficient becomes very large. Then  $\tilde{\rho}_1(|k|)$  decays with this rate only for a tiny fraction of  $|k|$  which contributes little to the inverse Fourier transform  $\phi(t, x)$  in (8). We illustrate this fact by numerically computing  $\tilde{\rho}_1(|k|)$  in the  $n = 2, \beta = 2$  case with  $r_{\min} = 1$  and  $r_{\max} = 200$  (Fig 10).

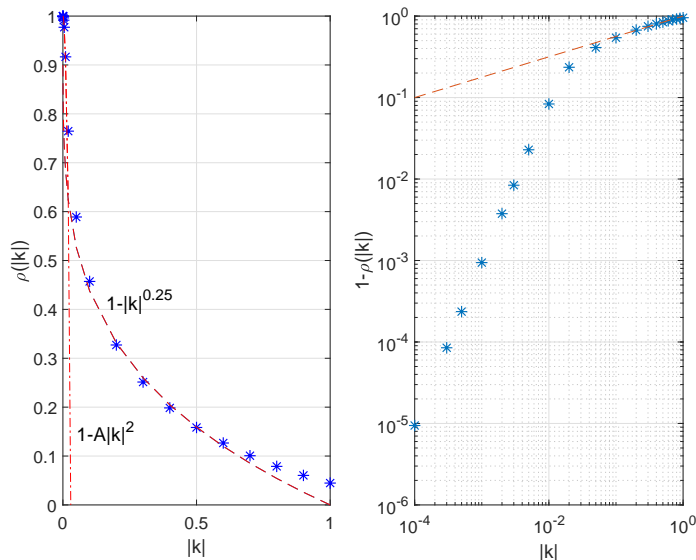


Figure 10:  $\tilde{\rho}(|k|)$  for the power law case with  $\beta = 2, n = 2, r_{\min} = 1, r_{\max} = 200$

One sees that the  $\alpha = 2$  scaling only occurs for very small  $|k|$ . This is put in evidence by plotting in the same figure the line  $1 - A|k|^2$ . At around  $|k| = 0.02$  there is a break for a quite different slope. We have numerically computed the slope  $\alpha$  after the break for dimensions  $n = 2$  to  $n = 4$  and several  $\beta$  values. The results are plotted in Fig 11 (left panel) which shows that they seem to follow a universal function of  $(n - \beta)$ . A rough fitting to a sigmoid function leads to  $\alpha(n - \beta) \sim 2/(1 + \exp(\gamma(n - \beta + c)))$  with  $\gamma = 1.5$  and  $c = 1.1$ .

In the right-hand side panel we have plotted the factor  $B$  in the  $\tilde{\rho}_1(|k|) \simeq 1 - B|k|^\alpha$  approximation with  $\alpha$  as in the left hand panel. The  $B$  coefficient has a weak dependence on  $n$ .

One sees that for large negative  $n - \beta$  the power exponent is consistent with normal diffusion, but there is a large range of  $n - \beta$  corresponding to anoma-

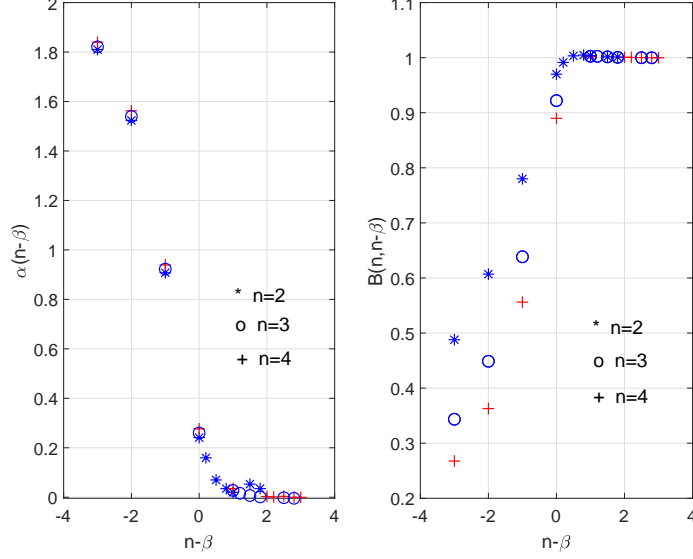


Figure 11: The power exponent of the function  $1 - \tilde{\rho}_1(|k|)$  after the break and the  $B$  coefficient in the  $\tilde{\rho}_1(|k|) \simeq 1 - B|k|^\alpha$  approximation

lous superdiffusion. This applies to intermediate times whereas, of course, for extremely large times the compact support nature of  $\rho(r)$  restores the slope to the one of normal diffusion ( $\alpha = 2$ ). As found above the dependence of the intermediate scaling factor is far more complex than the one guessed from comparison with the Grunwald-Letnikov representation of the fractional derivative [18]. In conclusion:

For power-law powers  $\beta$ , such that  $n - \beta \geq -1$  the nonlocal equation will display for intermediate times a fractional superdiffusive behavior with exponent that follows an universal law as represented in Fig. 10. In Fig. 10 we have also plotted the function  $1 - |k|^{\frac{1}{4}}$  to emphasize how this is a better first order approximation as compared to the asymptotic  $1 - A|k|^2$ .

The configuration space solution  $\phi_1(t, x)$  of (2) would be

$$\begin{aligned}
\phi_1(t, r) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty d|k| \frac{|k|^{\frac{n}{2}}}{r^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|k|r) e^{(\tilde{\rho}(|k|)-1)t} \\
&\simeq \frac{1}{(2\pi)^{\frac{n}{2}}} \left\{ \int_0^{|k|_b} d|k| \frac{|k|^{\frac{n}{2}}}{r^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|k|r) e^{-A|k|^2 t} + \int_{|k|_b}^\infty d|k| \frac{|k|^{\frac{n}{2}}}{r^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|k|r) e^{-B|k|^\alpha t} \right\} \\
&\simeq \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{r^n} \int_{r|k|_b}^\infty dz z^{\frac{n}{2}} J_{\frac{n}{2}-1}(z) \exp\left(-Bz^\alpha \frac{t}{r^\alpha}\right) \tag{16}
\end{aligned}$$

Therefore for times not extremely large, the breaking point  $|k|_b$  of the slope in

$\tilde{\rho}_1(|k|)$  being very small, one concludes that  $r^n \phi_1(t, r)$  is a function of  $\frac{t}{r^\alpha}$ ,

$$r^n \phi_1(t, r) \simeq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty dz z^{\frac{n}{2}} J_{\frac{n}{2}-1}(z) \exp\left(-Bz^\alpha \frac{t}{r^\alpha}\right) \quad (17)$$

This function is plotted in Figs. [12](#) [13](#) [14](#) for  $n = 2, 3$  and  $4$ . Notice that the integral in [\(17\)](#) is ill defined for  $t/r^\alpha = 0$ . It is a consequence of the initial condition of the fundamental solution being  $\phi_1(0, x) = \delta^n(x)$ <sup>4</sup>. Otherwise, as a function  $t/r^\alpha$ , it provides a complete description of the time evolution of the fundamental solution. One notices a similar behavior for all dimensions, whenever the same  $n - \beta$  is considered.

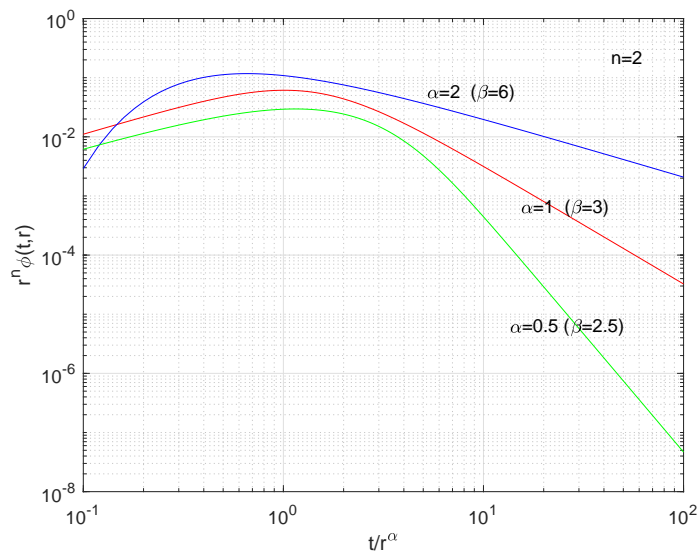


Figure 12: The  $r^n \phi(t, r)$  function in dimension 2 for the power law case

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<sup>4</sup>In contrast to the case of the heat equation, there is in general no regularizing effect of the singular initial condition in the nonlocal diffusion equation, the fundamental solution being  $e^{-t} \delta(x) + \phi(t, x)$ . [\[22\]](#). It is the smooth  $\phi(t, x)$  part of the solution that is of interest for the study of signal propagation in the networks.

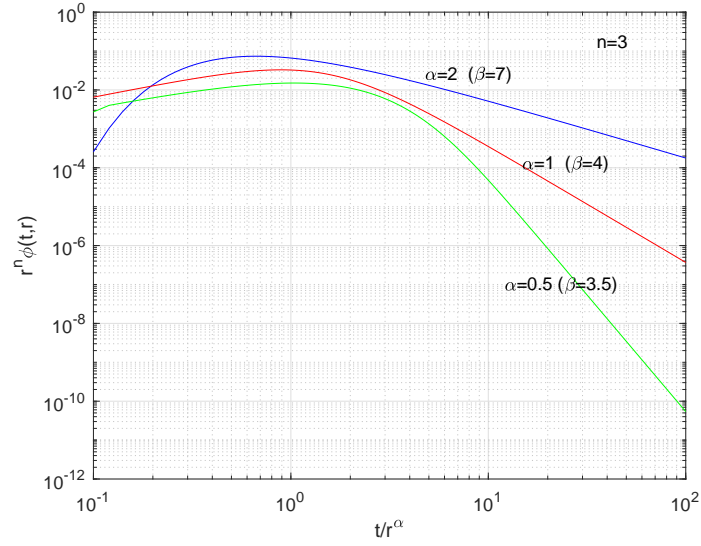


Figure 13: The  $r^n \phi(t, r)$  function in dimension 3 for the power law case

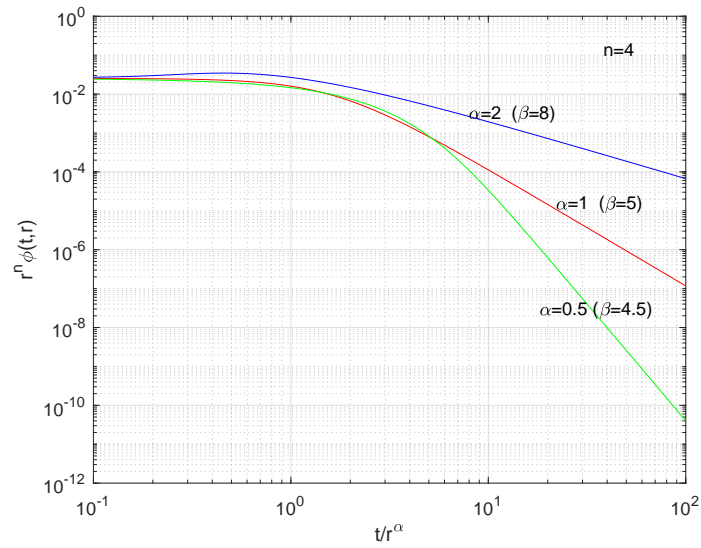


Figure 14: The  $r^n \phi(t, r)$  function in dimension 4 for the power law case

### 3.2 The exponential case

$$\rho_2(r) = c_2 e^{-\lambda r},$$

In this case because of the fast decay of the exponential we may, to a good approximation, consider an infinite network. Then the normalization,  $\int d^n x \rho_2(r) = 1$ , implies for  $n$  dimensions

$$c_2 = \frac{\Gamma\left(\frac{n}{2}\right) \lambda^n}{\Gamma(n) 2\pi^{\frac{n}{2}}}$$

and the Fourier transform (5) of  $\rho_2$  in dimension  $n$  is obtained in closed form

$$\tilde{\rho}_2(|k|) = \frac{\lambda^{n+1}}{\left(\lambda^2 + |k|^2\right)^{\frac{n+1}{2}}} = \frac{1}{\left(1 + \left(\frac{|k|}{\lambda}\right)^2\right)^{\frac{n+1}{2}}}.$$

and

$$\begin{aligned} \phi_2(t, r) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty d|k| \frac{|k|^{\frac{n}{2}}}{r^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|k|r) \exp\left(\frac{\lambda^{n+1}}{\left(\lambda^2 + |k|^2\right)^{\frac{n+1}{2}}} - 1\right) t \\ &= \frac{1}{(2\pi)^n} \frac{1}{r^n} \int_0^\infty dz z^{\frac{n}{2}} J_{\frac{n}{2}-1}(z) \exp\left(\frac{1}{\left(1 + (z/\lambda r)^2\right)^{\frac{n+1}{2}}} - 1\right) t \end{aligned}$$

$r^n \phi_2(t, r)$  being in this case a function of  $(t, \lambda r)$ . In Fig 15 we have plotted the  $|k|$  dependence of  $\tilde{\rho}_2(|k|)$  and  $1 - \tilde{\rho}_2(|k|)$ .

One sees that for a large  $|k|/\lambda$  region the slope of  $1 - \tilde{\rho}_2(|k|)$  is similar to the one of normal diffusion. Hence it is only for very small values of  $\lambda$  that one should expect anomalous diffusion effects. This is also clear from the size of the  $|k|^2$  coefficient in the Taylor expansion of  $\tilde{\rho}_2(|k|)$

$$\tilde{\rho}_2(|k|) = 1 - \frac{n+1}{2\lambda^2} |k|^2 + \dots$$

Therefore propagation of information on the network is expected to be much less efficient than in the power law case.

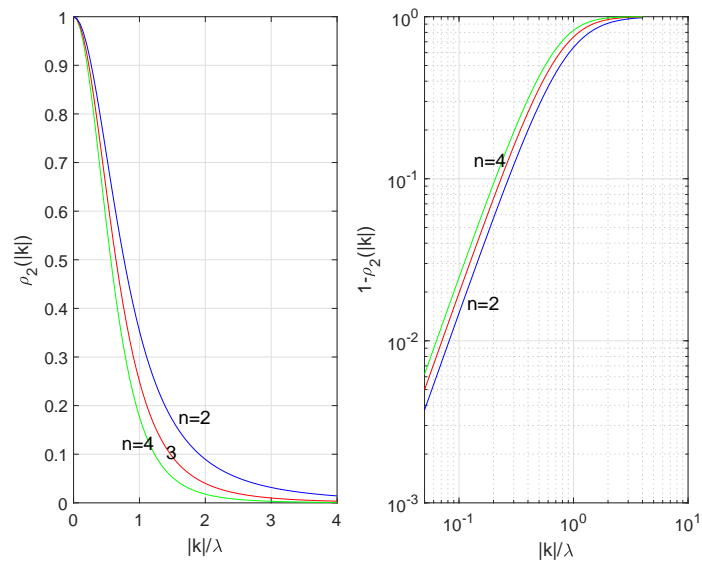


Figure 15: The Fourier kernel  $\rho_2(|k|)$  for the exponential case

### 3.3 Comparing the propagation of information in power-law and exponential networks

Here we compare by numerical simulation the propagation of information in power law and in exponential networks. We consider a network of 40000 agents placed on a  $200 \times 200$  two-dimensional lattice and establish among them networks with unit connection strengths distributed according to a distance-dependent law, either a power law  $p(d) = \frac{c_\alpha}{d^\alpha}$  or  $p(d) = c_e e^{-d}$ . These normalized probability functions are displayed in Fig. 16.

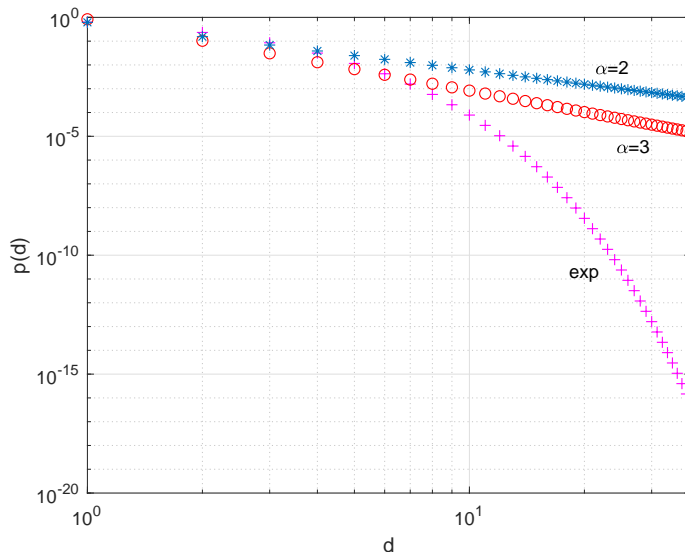


Figure 16: Connection probabilities for power law ( $\alpha = 2$  and 3) and exponential networks

One sees that up to distance  $\approx 10$  the connection probabilities are not very different, but for larger distances they very much differ. The distance that is used in the lattice is the taxi metric  $d_{AB} = |x_A - x_B| + |y_A - y_B|$ .

The propagation experiments are performed in the following way: One chooses at random two distant agents in the network, the "source"  $A$  and the "destination"  $B$ . At the initial time there is a unit pulse at  $A$  that this agent is going to transmit with unit intensity to all its neighbors (meaning the agents that are connected to it, regardless of the physical distance). At the next time step, the neighbors that received the pulse do the same operation and so on. We use a no-cycle condition, that is, each agent transmits the pulse to its neighbors only once, even if it receives any other pulse later on. In this sense we are not considering exactly the diffusion situation studied in Section 2 although the results are qualitatively the same. The process ends at a time when the destination  $B$  no longer receives any more pulses. This experiment has been repeated many times with very similar results every time. A typical result is shown in Fig. 17.

The intensities represent the number of pulses received at each time. We have also compared the speed of propagation with a nearest-neighbor network. All networks have the same number of connections (79600). The conclusion is that power law networks, with the same number of connections, are extremely more efficient in the propagation of information than the others. Although better than the nearest-neighbor one, the exponential network fares poorly.

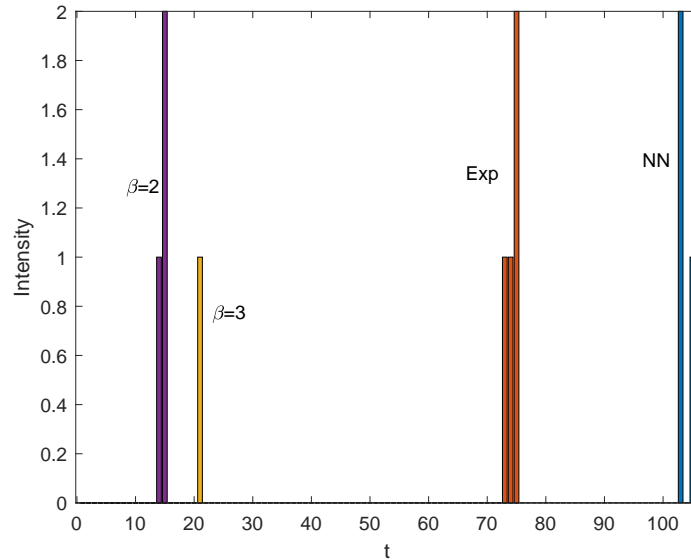


Figure 17: A typical result of the propagation of information experiment with power law ( $\alpha = 2$  and  $3$ ), exponential and nearest-neighbor networks

**Acknowledgement:** The authors acknowledge financial support from FCT - Fundação para a Ciência e Tecnologia (Portugal) through research grants UIDB/04561/2020 and UIDB/05069/2020.

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