UNIVERSIDADE DE LISBOA

ISEG

Pricing American Options by the Black-Scholes Equation with a Nonlinear Volatility Function

Yaser Faghan

Orientadores:

Prof. Doutor Maria do Rosário Grossinho
Prof. Doutor Daniel Ševčovič

Tese especialmente elaborada para obtenção do grau de Doutor em Matemática Aplicada à Economia e à Gestão

2021

UNIVERSIDADE DE LISBOA
ISEG

Pricing American Options by the Black-Scholes Equation with a Nonlinear Volatility Function

Yaser Faghan

Orientadores:

Prof. Doutor Maria do Rosário Grossinho
Prof. Doutor Daniel Ševčovič

Tese especialmente elaborada para obtenção do grau de Doutor em matemática aplicada à economia e à gestão

Júri:

Presidente:
Doutor Nuno João de Oliveira Valério - Professor Catedrático e Presidente do Conselho Científico - Instituto Superior de Economia e Gestão da Universidade de Lisboa

Vogais:
Doutor Maria do Rosário Grossinho - Professor Catedrática - Instituto Superior de Economia e Gestão da Universidade de Lisboa.
Doutor João Paulo Vicente Janela - Professor Associado - Instituto Superior de Economia e Gestão da Universidade de Lisboa.
Doutora Andrea Sofia Meireles Rodrigues - Lecturer - Department of Mathematics - University of York.
Doutora Eva Virgínia Araújo Morais - Professora Auxiliar - Escola de Ciências e Tecnologia e Universidade de Trás-os-Montes e Alto Douro.
Doutor Antonio Alberto Ferreira Santos - Professora Auxiliar - Faculdade de Economia de Universidade de Coimbra.

Instituição Financiadora: Cemapre

2021

Abstract

In this thesis we are concerned with the study of American-style options in presence of variable transactions costs. This leads to consider some generalized Black–Scholes equations with a nonlinear volatility function depending on the product of the underlying asset price and the second derivative of the option price. Mathematically, this involves the study of a free boundary problem for a nonlinear parabolic equation. The fully nonlinear character of the corresponding differential operator induces increased difficulties. By overcoming adequately those difficulties, we obtain qualitative and quantitative results regarding both types of American-style options, that is put and call options, as described next. Firstly, we investigate the qualitative and quantitative behaviour of a solution to the problem of pricing American style perpetual put options. We assume the option price is a solution to a stationary generalized Black–Scholes equation with a nonlinear volatility function. We prove existence and uniqueness of a solution to the free boundary problem. We derive a single implicit integral equation for the free boundary position and a closed form formula for the option price. It is a generalization of the well-known explicit closed form solution derived by Merton for the case of constant volatility. We also present results of numerical computations for the free boundary position, option price and their dependence on model parameters. Secondly, we analyse a nonlinear generalization of the Black–Scholes equation for pricing American-style call options, with nonlinear volatility. This model generalizes the well-known Leland model with constant transaction costs. Due to the fully nonlinear nature of the differential operator that appears in the model, the direct computation of the nonlinear complementarity problem becomes harder and unstable. Therefore, we propose a new approach to reformulate the nonlinear complementarity problem in terms of the new transformed variable for which the differential operator has the form of a quasilinear parabolic operator. We derive the nonlinear complementarity problem for the transformed variable in order to apply the Gamma transformation for American style options. We then solve the variational problem by means of the modified projected successive over relaxation (PSOR) for constructing an effective numerical scheme for discretization of the Gamma variational inequality. Finally, we present several computational examples of the nonlinear Black–Scholes equation for pricing American-style call options in the presence of variable transaction costs.

Keywords: American option pricing, nonlinear Black–Scholes equation, variable transaction costs, PSOR method.

Resumo

Esta dissertação incide sobre o estudo de opções americanas admitindo a existência de custos de transação variáveis. Tal estudo leva-nos a considerar equações de Black-Scholes generalizadas, com uma função de volatilidade não linear que depende do produto do preço do ativo subjacente e da segunda derivada do preço da opção, o que, do ponto de vista matemático, implica a análise de um problema de fronteira livre para uma equação parabólica não linear. O caráter não linear do operador diferencial correspondente gera dificuldades acrescidas. Contudo, um estudo adequado à condição de não linearidade permite-nos estabelecer resultados qualitativos e quantitativos sobre os dois tipos de opções americanas, mas precisamente, opções de venda e de compra, conforme descrito a seguir. Em primeiro lugar, investigamos o comportamento qualitativo e quantitativo de uma solução do problema de apreçamento de opções de venda perpétuas do tipo americano. Assumimos que o preço da opção é uma solução para uma equação de Black-Scholes generalizada estacionária com uma função de volatilidade não linear. Provamos existência e unicidade de uma solução do problema da fronteira livre. Derivamos uma equação integral implícita para o valor de fronteira livre e uma solução de forma fechada para o preço da opção. É uma generalização da conhecida solução de forma fechada explícita derivada por Merton para o caso de volatilidade constante. Também apresentamos resultados de cálculo numérico para o valor de fronteira livre, assim como para preço da opção e sua dependência dos parâmetros do modelo. Em segundo lugar, analisamos uma generalização não linear da equação de Black-Scholes para o apreçamento de opções de compra de tipo americano, com volatilidade não linear. Este modelo generaliza o conhecido modelo de Leland com custos de transação constantes. Devido à natureza totalmente não linear do operador diferencial que aparece no modelo, o cálculo direto do problema de complementaridade não linear torna-se mais difícil e instável. Portanto, propomos uma nova abordagem para reformular o problema de complementaridade não linear em termos de uma nova variável para a qual o operador diferencial tem a forma de um operador parabólico quase-linear. Derivamos o problema de complementaridade não linear para a variável transformada a fim de aplicar a transformação Gama para opções de tipo americano. Em seguida, resolvemos o problema variacional por meio do relaxamento projetado sucessivo modificado (PSOR) para construir um esquema numérico eficaz para discretização da desigualdade variacional Gama. Finalmente, apresentamos vários exemplos computacionais da equação não linear de Black-Scholes para apreçamento de opções de compra no tipo americano em presença de custos de transação variáveis.

Palavras-chave: Opção americana, equação de Black-Scholes não-linear, custos de transação, método PSOR.

Contents

1

Introduction

There are many types of financial markets where buyers and sellers participate in the trade of assets such as stocks, bonds, currencies and derivatives. These markets are typically using the transparent pricing, basic regulations on trading, costs and fees. While the market forces to determine the prices of securities in trading which is the important issue in the financial markets. A derivative is a contract between two or more parties whose value is based on an agreed-upon underlying financial asset, index or security. Examples of common derivatives are forwards, futures, swaps, and options. Derivatives can be used to either mitigate risk (hedging) or assume risk with the expectation of commensurate reward (speculation). Options are financial derivatives that give the buyer the right to buy or sell the underlying asset at a stated price within a specified period. The American option allows the holder to exercise their right at any time prior to the contract's expiration date while European options can only be exercised at the maturity date. Today, most securities traded on an exchange are American style of options. These contracts will specify at least four variables:

- **Underlying Asset**: common and preferred stock, commodities, interest rates.
- **Premium**: the price paid when an option is purchased.
- **Strike Price**: identifies the price at which the holder of the contract has a right to sell (put option) or buy (call option) the underlying asset.
- **Maturity Date**: also referred to as the expiry date; the option no longer has any value if not exercised before or at that date.

These financial instruments come in two basic forms:
Call Options: generally referred to as calls, this contract gives the holder the right to purchase the security at the strike price before the maturity date.
Put Options: also referred to as puts, this contract gives the holder the right to sell the security at the strike price before the maturity date.
Options provide their holder with certain rights, which are not obligations. For example, a call option gives the holder the right to purchase securities at the strike price. The holder is not required to complete this transaction. Investors can also short (sell) a call option, giving the buyer of the call option the right to purchase the asset at the strike price. The seller of the call option is compensated by the premium paid by the buyer, regardless if the buyer exercises their rights.
Options are extremely versatile securities that can be used in many different ways. Traders use options to speculate which is relatively risky practice while hedgers use options to reduce the risk of holding an asset. In terms of speculation, option buyers and writers have conflicting views regarding the outlook on the performance of an underlying security. For instance, the option writer will need to provide the underlying shares in the event that the stock's market price will exceed the strike. An option writer that sells a call option believes that the underlying stock's price will drop relative to the option's strike price during the life of the option, as that is how he or she will reap maximum profit. This is exactly the opposite outlook of the option buyer. The buyer believes that the underlying stock will rise, because if this happens, the buyer will be able to acquire the stock for a lower price and then sell it for a profit. One of the most commonly used methods of pricing stock options is the Black–Scholes method introduced by Merton, Black and Scholes [6]. They derived a pricing model by means of a solution to a certain PDE in 1973. The linear Black–Scholes equation with a constant volatility σ has been derived under several restrictive assumptions e.g., non-zero transaction costs, investors preferences, feedback and illiquid markets effects and risk from unprotected portfolio. Our work is to study the nonlinear Black–Scholes equation with a nonlinear volatility function arises from option pricing models that caused by the presence of transaction costs (see e.g., Hoggard, Whalley and Wilmott [21], Avellaneda and Paras [2]), feedback and illiquid market effects (Frey and Patie [15], Schönbucher and Wilmott [37]), imperfect replication and investor's preferences (Barles and Soner [4]), and risk from unprotected portfolio (Kratka [29], Ševčovič [42], Jandačka and Ševčovič [23]).

The main goals of our thesis can be summed up as follows:

- **Reviewing of some well-known nonlinear models**. We review

Blak–Scholes type of option pricing models with nonlinear volatility function. While utilizing more realistic variable transaction costs funcions concerning the amount of transactions, we followed the work of Žitňanská [45], and also the work of Ševčovič, Stehlíková and Mikula [43]. As proven, these models provide more accurate prices than the classical one in general by taking into account more realistic assumptions such as large investor's preferences, transaction costs, etc.

- **Pricing American Perpetual Put Options**. In the joint work of Grossinho, Faghan and Ševčovič [17, 18], we analyse American style of perpetual put options considering non-trivial transaction costs while trading in the financial stock market. We prove existence and uniqueness of a solution to the free boundary problem. We derive a single implicit equation for the free boundary position and the closed form formula for the option price.

- **Solving variational inequalities**. We analyze a nonlinear Black–Scholes equation for pricing American style of call option in which the volatility may depend on the underlying asset price and the Gamma of the option. We present that the generalized Black–Scholes equation can be transformed to the so-called Gamma equation in which we reformulate the free boundary problem (variational inequality) in terms of the solution to the Gamma equation. This result can be found in the joint work of Grossinho, Faghan and Ševčovič [19].

- **Numerical scheme and experiments.**. In this part we first propose efficient numerical discretization of the Gamma equation introduced by Jandačka and Ševčovič [23] based on the finite volume method and then use PSOR algoritm for solving variational inequalities for the nonlinear Black–Scholes equation.

2

Stochastic processes

In this chapter we present the Black–Scholes model for pricing financial options and review some basic concepts regarding to the mathematical framework. We need to model the stochastic behavior of underlying assets in order to deduce Black–Scholes equation. The Wiener process and its generalization the Brownian motion are fundamental tools for modelling stochastic evolution of asset prices that can be expressed using stochastic differential equation (SDE). We furthermore derive explicit formulae for pricing basic European options and finally discuss about the American plain vanilla options that can be exercised anytime before the expiration time T. This revision presents the basic framework that will be used in the next chapters. It is done in a very light way just to gather some notions and results that will underly the next chapters. Complete study can be found in Lamberton [33], Shreve [39], Privault [36], and Ševčovič, Stehlíková and Mikula [43].

2.1 Stochastic differential calculus

In this section we review some basic concepts of stochastic calculus.

Definition 2.1.1. *A continuous stochastic process X is a t-parametric system of random variables*

$$(X_t, t \in T) = (X_t(\omega), t \in T, \omega \in \Omega)$$

defined on some proper probability spaces $(\Omega, \mathcal{A}, P)$, with $T = [0, \infty)$.

Markov process is a particular class of stochastic process that use only the current value of a variable $\{X(s)\}$ for predicting the future values $X(t)$ for $\{t > s\}$. Markovian stochastic processes are the basic tool for explaining such a random evolution of the price of the asset. There is a wide range of well-known Markov processes, but the most widely used are the Wiener process and its generalization the Brownian motion.

Definition 2.1.2. *A continuous time stochastic process $\{X_t\}_{t\geq 0}$ in the probability space $(\Omega, \mathcal{A}, P)$ is called a Wiener process or (standard)Brownian motion, if*

- $X_0 = 0$ *almost surely.*
- *For any finite set of times $0 = t_0 < t_1 < \cdots < t_n$, the increments*

  $$X_{t_1},\ X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1},\ \ldots,\ X_{t_n} - X_{t_{n-1}}$$

  are independent.
- *For $s < t$, the increments $X_t - X_s$ has the Gaussian distribution $N(0, \sqrt{t-s})$.*
- *For all ω in a set of probability one, $X_t(\omega)$ has continuous trajectories of t.*

Definition 2.1.3. *Let $\mu \in R$ and $\sigma > 0$. A continuous-time stochastic process $\{X_t\}_{t\geq 0}$ is named a Brownian motion with drift μ and variance σ^2 on $[0, T)$ if*

- $X_0 = 0$ *almost surely.*
- *For $s < t$, the increments $X_t - X_s$ has the Gaussian distribution with mean $\mu(s-t)$ and variance $\sigma^2(s-t)$, $N(\mu(s-t), \sigma^2(s-t))$.*
- *For any finite set of times $0 = t_0 < t_1 < \cdots < t_n$, the increments*

  $$X_{t_1},\ X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1},\ \ldots,\ X_{t_n} - X_{t_{n-1}}$$

  are independent.
- *For all ω in a set of probability one, $\{X_t(\omega)\}$ is a continuous function of t.*

Remark 1. *Note that the equation $X_t = \mu t + \sigma W_t$ describes the relation between Brownian motion and Wiener process.*

Definition 2.1.4. *If $X(t), t \geq 0$ is a Brownian motion with parameters μ, σ and $y_0 \in R^+$, then the system of random variables*

$$Y(t) = y_0 e^{X(t)}, \quad t \geq 0 \tag{2.1}$$

is called a geometric Brownian motion.

We know that the Brownian path is not differentiable and also has unbounded variation. So the purpose is to manipulate these integrals that differ from ordinary calculus. A key tools in stochastic processes are the so–called Itô's integral and Itô's isometry.

Lemma 2.1.1. *(Itô's integral)*
Assume that $f(0,t) \to \mathbb{R}$ is a measurable function such that $\int_0^t f^2(\tau)\tau < \infty$. Then, there exists Itô's integral $\int_0^t f^2(\tau)\omega(\tau)$ such that the following properties hold:

$$\begin{aligned} E\left[\int_0^t f(\tau) d\omega(\tau)\right] &= 0, \\ E\left[\left(\int_0^t f(\tau)\ \omega(\tau)\right)^2\right] &= \int_0^t f^2(\tau)\tau. \end{aligned}$$

The last identity is called Itô's isometry.

Lemma 2.1.2. *(Itô's lemma)*
Assume that $\{X_s(\omega)\}$ is an Itô's process

$$dX_s(\omega) = a(s,\omega)ds + b(s,\omega)\ dW_s(\omega) \tag{2.2}$$

such that $f(s,x) \in C^2([0,\infty) \times \mathbb{R})$. Then

$$Z_s(\omega) = f(s, X_s(\omega)) \tag{2.3}$$

is again an Itô's process and

$$dZ_t = \frac{\partial f}{\partial t}(s, X_s)\ ds + \frac{\partial f}{\partial x}(s, X_s)\ X_s + \frac{1}{2} b^2 \frac{\partial f^2}{\partial x^2}(s, X_s)\ ds. \tag{2.4}$$

Itô' lemma provides a way to construct new SDE. It plays a very important role in pricing derivatives.

2.2 Linear Black–Scholes model

Fundamental work on option pricing theory was published by Black and Scholes and then by Merton. Although, this model is commonly used in option pricing theory in financial industry, but it does not efficiently reflect the financial market due to its restrictive assumptions. We list some of the Black–Scholes model assumptions:

- Continuous trading in the market with respect to time.
- No dividend payment on the underlying asset.
- Constant risk-free interest rate r with respect to time.
- Perfectly divisible shares are assumed.
- Zero transaction costs.
- Limited risk and no loss of short position.
- No risk-free arbitrage opportunities.

For modeling the dynamic of the asset price we need to utilize the stochastic differential equation (SDE) in order to present the geometric Brownian motion

$$dS = (\rho - q)Sdt + \sigma SdW, \tag{2.5}$$

where dS represent the change of asset price over the time interval of length dt, ρ is the evolution of underlying asset price, the positive q is an annualized dividend yield and σ is volatility. Now, we can deduce a SDE that describe the evolution of an arbitrary smooth function $V(S,t)$ of asset price and time. We can write a stochastic differential equation for the function $V(S,t)$ by utilizing Itô's lemma. Where the function $V(S,t)$ of the stochastic process S in (2.5) satisfies the following stochastic differential equation

$$dV = \left(\partial_t V + (\rho - q)S\partial_S V + \frac{1}{2}S^2\sigma^2\partial_S^2 V\right)dt + \sigma S\partial_S V dW \tag{2.6}$$

In this step we construct a portfolio involving underlying assets and one option on these assets considering self-financing concept. Assume that at time t, our portfolio has the amount of δ units assets with unit price S and one long position in the option with unit price V.
Thus, our portfolio value is

$$\Pi = V + \delta S \tag{2.7}$$

Then by using Itô's lemma (Lemma 2.1.2) and equations (2.5) and (2.6), we derive the equation in the form:

$$d\Pi = \left(\partial_t V + (\rho - q)S(\partial_S V + \delta) + \delta q S + \frac{1}{2}S^2\sigma^2\partial_S^2 V \right) dt + (\partial_S V + \delta)\sigma S dW \tag{2.8}$$

Now, by choosing the portfolio to be risk-free and take the parameter δ, i.e. number of assets in the portfolio, as follows:

$$-\partial_S V = \delta \tag{2.9}$$

and also by the no-arbitrage argument, the expected return of the portfolio is equal to the risk-free yield $r > 0$ of bonds, i.e.

$$d\Pi = r\Pi dt \tag{2.10}$$

Applying above assumptions and some algebraic calculations, one can derive the Black–Scholes PDE for an option price $V(S,t)$ as an function of underlying asset $S > 0$ and time $t \in [0,T]$.

Theorem 2.2.1. *(Black–Scholes Equation) Assuming that*

$$dX_t = rX_t dt \tag{2.11}$$

$$dS_t = S_t\alpha(t,S_t)dt + S_t\sigma(t,S_t)dW_t, \tag{2.12}$$

Then the unique pricing function $V(S_t,t)$, where V is a smooth function and consistent with no-arbitrage opportunities, is the solution of the folowing boundary value problem with $(t,S_t) \in [0,T] \times \mathbb{R}^+$,

$$V_t(s,t) + rsV_s(s,t) + \frac{1}{2}s^2\sigma^2(t,s)V_{ss}(s,t) - rV(s,t) = 0 \tag{2.13}$$

$$V(s,T) = \Phi(s). \tag{2.14}$$

To evaluate a contingent claim $\Psi = \Phi(S_T)$, we propose the following result which constructs the security price as the discounted value for presenting the expected payoff by means of Q martingle measure.

Theorem 2.2.2. *(Risk Neutral Valuation) The arbitrage free price of the contingent claim $\Phi(S(T))$ is given by $\Pi(t;\Phi) = V(t,S(t))$, where V is given by the formula*

$$V(t,s) = e^{-r(T-t)}E^Q_{t,s}\left[\Phi(S(T))\right],$$

where the Q-dynamics of S is

$$dS_t = S_t\alpha(t,S_t)dt + S_t\sigma(t,S_t)d\tilde{W}_t$$

with $\tilde{W}$ a Q-Wiener process.

Remark 2. *In the Black–Scholes world it is assumed that there are no arbitrage opportunities, i.e., any probable profit in the financial market leads to risk of loss. And the process of the price for the derivative asset with contingent claim* Ψ *has the form* $\Pi(t;\Psi) = V(S_t, t)$ *where* V *is assumed to be a smooth function.*
More details can be seen in [5].

In the next section we briefly review the European plain vanilla options.

2.3 European plain vanilla options

In the case of European put option on a stock S, at expiration time T, the pay-off function is:

$$V_T = (K - S_T)^+ = \max(0, K - S_T). \tag{2.15}$$

If the price of the asset S_T is greater than the strike price K then the option value is the difference $K - S_T$, thus we say that the option is in-the-money. In another word, if the strike price K is less than the price of the aseet S_T, then the option is out-of-the-money and is worthless. The same argument is true for the call option where the terminal pay-off for the call option is of the form:

$$V_T = (S_T - K)^+ = \max(0, S_T - K). \tag{2.16}$$

Where at expiry it values $S_T - K$ if S_t is greater than K, otherwise it is worthless.

For the European put case, the boundary conditions for $S = 0$ and $S \to \infty$, are

$$V(0,t) = Ke^{r(T-t)} \quad \text{and} \quad V(S \to \infty, t) = 0, \quad \forall t \in (0,T) \tag{2.17}$$

Respectively, The boundary conditions for the European call option are given by

$$V(0,t) = 0 \quad \text{and} \quad V(S \to \infty, t) = S, \quad \forall t \in (0,T) \tag{2.18}$$

We present in Fig. 2.1 the value of both call and put options by using the Black–Scholes PDE (2.13) with certain terminal conditions.
Here we give the following result which is well-known as the Black–Scholes formula (see e.g., Bjork [5]).

Proposition 1. *(Black–Scholes formula) The price of a European put (call) option with exercise price* K *and time to maturity* T *is given by* $V(t, S(t))$*, where*

European put: $V(t,s) = Ke^{-r(T-t)}N[-d_2(t,s)] - Se^{-q(T-t)}KN[-d_1(t,s)]$.

European call: $V(t,s) = Se^{-q(T-t)}N[d_1(t,s)] - Ke^{-r(T-t)}N[d_2(t,s)]$.

Here N is the cumulative distribution function (CDF) for the standardised normal distribution $N(0,1)$ and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

The function $N(x)$ is CDF

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-\alpha^2}{2}}\,d\alpha$$

Where $\{q \geq 0\}$ is the dividend yield. Rest of the parameters and constants must be familiar. The evolution of the solution in time, when time is passing up to expiration, for both put and call options are shown in Fig. 2.1(d) and Fig. 2.1(c), respectively.

In next section we give a brief introduction of the most basic American options.

2.4 American plain vanilla options

A contract that allows the holder to exercise their options to sell or buy the underlying derivaties at any time prior to the expiration date $t \in [0,T]$ is called American option. The real advantage of American contract over European contract is the flexibility that they offer. When you own this type of contract, it gives you the right to exercise earlier than the expiration date of the contract. But one may ask what is the the option premium at the time $t = 0$ of contracting. For the American call (put) option the challenge is to calculate the price $V^{ac}(S,t)$ of the American call option ($V^{ap}(S,t)$ for the put option), at the time $t \in [0,T]$. Thus, as compared to the European one, the relation between values of these two types of contracts gives an inequality presenting

$$V_t^{ac}(S,t) \geq V_t^{ec}(S,t), \quad V_t^{ap}(S,t) \geq V_t^{ep}(S,t), \tag{2.19}$$

at any time $t \in [0,T]$ and underlying asset price $S \geq 0$. Furthermore, we can describe the price of American call (put) option regarding to their prices at expiration time given by the pay-off diagram:

$$V_t^{ac}(S,t) \geq V_t^{ec}(S,T) = (S-K)^+, \tag{2.20}$$

$$V_t^{ap}(S,t) \geq V_t^{ep}(S,T) = (K-S)^+, \tag{2.21}$$

at any time $t \in [0,T]$ and underlying asset price $S \geq 0$. Thus, if the price of American call option before the expiration time is less than its pay-off then, by purchasing such an option and exercising immediately, one can receive underlying asset for the exercise price E and then sell it on the market to pocket the difference $S-K$. In this case the holder of the option earning some amounts without bearing any risk that may lead to an arbitrage opportunity. Due to the demand for these types of options the market price for such an option is higher or equal than its pay-off diagram. For the price of American call option with no dividend payment on the underlying asset ($q = 0$), one can write

$$\text{if} \quad q = 0 \quad \text{then} \quad V_t^{ap}(S,t) = V_t^{ep}(S,t), \text{for each} \quad S \geq 0, t \in [0,T]. \tag{2.22}$$

It means that it is better not to exercise the option prior to the expiration time. The condition is more sophisticated when the option paying dividends on the underlying asset ($q > 0$). In this case the price $V^{ec}(S,t)$ of the European call option ($V^{ep}(S,t)$ for the put option) intersects the pay-off diagram. Hence the price of the American call (put) option can be expressed as

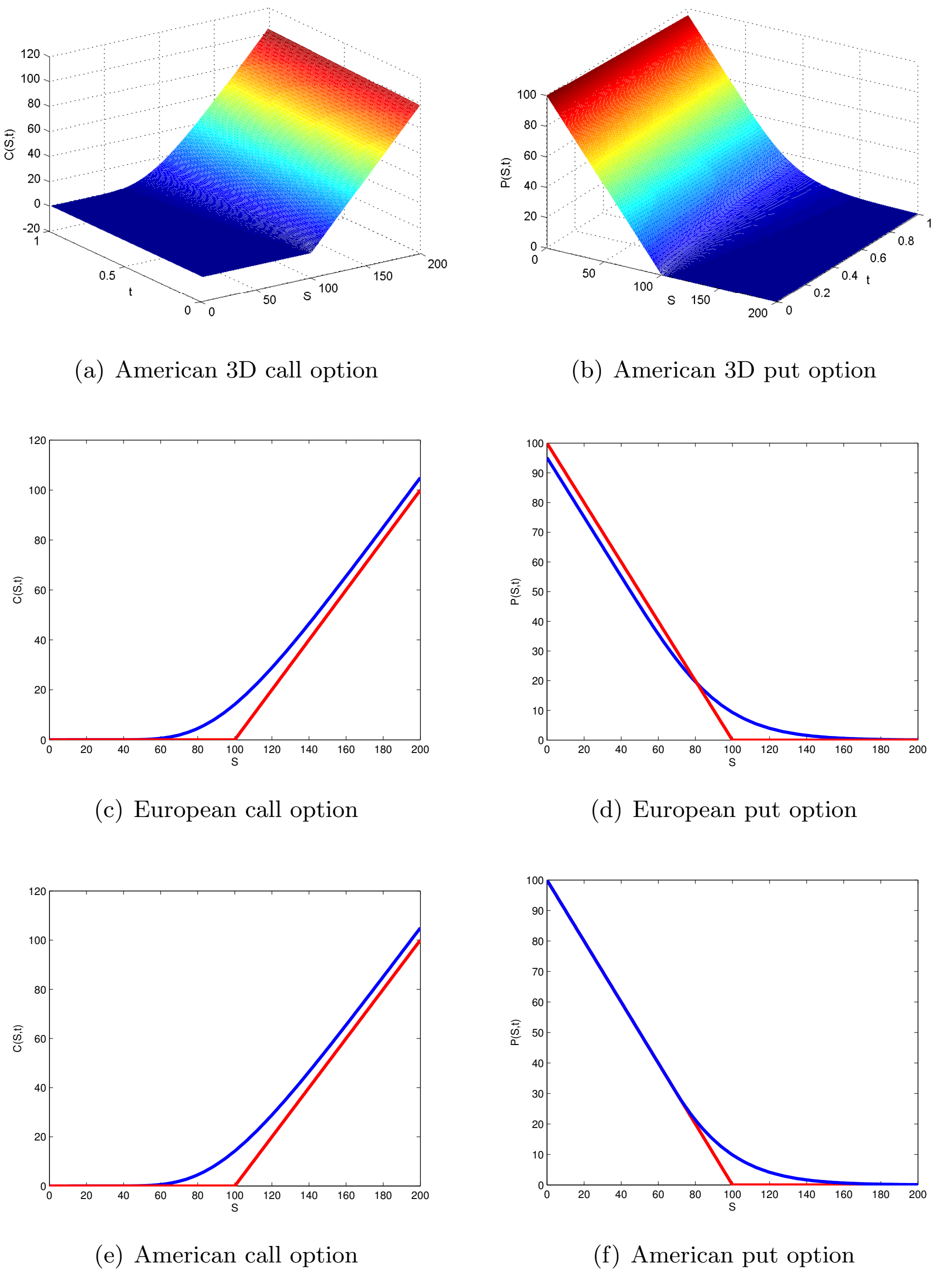
$$\text{if} \quad q > 0, r > 0 \quad V^{ac}(S,t) > V^{ec}(S,t), \quad \text{for each} \quad S > 0, t \in [0,T). \tag{2.23}$$

$$\text{if} \quad q \geq 0, r > 0 \quad V^{ap}(S,t) > V^{ep}(S,t), \quad \text{for each} \quad S \geq 0, t \in [0,T). \tag{2.24}$$

More details can be found in [43].

2.5 Chapter conclusions

In section 2.1 we present the basic concept of stochastic calculus such as random variables, stochastic processes and Itô' calculus which are the core mathematical tools for studying derivative pricing. In section 2.2 we give the brief connection between SDE and certain PDE, in particular deriving the Black–Scholes partial differential equation regarding to the model assumptions, by using self financing-portfolio argument and Itô' lemma. In sections 2.3 we discuss the most basic European options where its prices can be calculated as a solution of the Black–Scholes PDE and describe American plain vanilla option with dividend payment on the underlying asset in section 2.4 .



(a) American 3D call option

(b) American 3D put option

(c) European call option

(d) European put option

(e) American call option

(f) American put option

Figure 2.1: Plots of both American and European (put and call) options.

3 Models with transaction costs

In recent years the financial markets have become more complex that led to the development of more sophisticated mathematical models. The classical linear Black–Scholes equation was derived under several restrictive assumptions such as frictionless, liquid and complete markets, etc. Such assumptions have been relaxed in order to model the presence of transaction costs (see e.g., Leland [35], Hoggard, Whalley and Wilmott [21], Avellaneda and Paras [2]), feedback and illiquid market effects (Frey [14], Frey and Patie [15], Frey and Stremme [16], Schönbucher and Wilmott [37]), imperfect replication and investor's preferences (Barles and Soner [4]), risk from unprotected portfolio (Kratka [29], Jandačka and Ševčovič [23]). In this chapter we review some useful generalizations of the Black–Scholes equation where the diffusion coefficient in the nonlinear model is a function of the expression $H = S\partial_S^2 V$, of the derivative price and of the time $\tau = T - t$ to maturity.

Suppose that the dynamics of the underlying asset S is given by

$$dS = \mu S dt + \sigma S dw. \tag{3.1}$$

where μ is return on asset, σ is volatiliy of underlying asset and w is a standard Wiener process. Then, following the GBM and using self-financing portfolio argument, a PDE to model the price of the option $V(t, S)$ can be deduced

$$\partial_t V + rS\partial_S V + \frac{1}{2}S^2\sigma^2\partial_S^2 V - rV = 0 \tag{3.2}$$

Here r is the riskless interest rate and σ is the volatility.

We will focus our attention to the case when the diffusion coefficient σ depends on the product, $S\partial_S^2 V$, of the asset price S and the second derivative (Gamma) of the option price V, i.e.

$$\sigma = \sigma(S\partial_S^2 V)\,, \tag{3.3}$$

In the Leland model (generalized for more complex option strategies [21]) the volatility is given by

$$\sigma^2 = \sigma_0^2(1 + \mathrm{Le}\,\mathrm{sgn}(\partial_S^2 V))$$

where $\sigma_0 > 0$ is the constant historical volatility of the underlying asset price process and $\mathrm{Le} > 0$ is the so-called Leland number.
Another nonlinear Black–Scholes model has been derived by Frey model (see [14, 15, 16]). In this model the asset dynamics takes into account the presence of feedback effects due to a large trader choosing his/her stock-trading strategy (see also [37]). The diffusion coefficient σ is again non-constant:

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2\left(1 - \mu S\partial_S^2 V\right)^{-2}, \tag{3.4}$$

where $\sigma_0^2, \mu > 0$ are constants.
Another example of the Black–Scholes equation with a non-constant volatility is the so-called Risk Adjusted Pricing Methodology (RAPM) model proposed by Kratka [29] and revisited by Jandačka and Ševčovič [23]. In RAPM the purpose is to optimize the time-lag between consecutive portfolio adjustments in such way that the sum of the rate of transaction costs and the rate of a risk from unprotected portfolio is minimal. In this model, the volatility is as follows:

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2\left(1 + \mu(S\partial_S^2 V)^{\frac{1}{3}}\right). \tag{3.5}$$

By $\sigma_0 > 0$ we denoted the constant historical volatility of the asset price returns and $\mu = 3(C^2R/2\pi)^{\frac{1}{3}}$, where $C, R \geq 0$ are non-negative constants representing the transaction cost measure and the risk premium measure,

respectively (see [23] for more details). The non-constant transaction cost model has been generalized to more realistic transaction cost function by Ševčovič and Žitňanská in the recent paper [44].
In [3], Bakstein and Howison investigated a parametrized model for liquidity effects arising from the asset trading. In their model the volatility function is a quadratic function of the term $S\partial_S^2 V$:

$$\sigma(S\partial_S^2 V)^2 = \sigma_0^2 \Bigg(1 + \gamma^2(1-\alpha)^2 + 2\lambda S\partial_S^2 V + \lambda^2(1-\alpha)^2 \left(S\partial_S^2 V\right)^2 + 2\sqrt{\frac{2}{\pi}}\gamma \operatorname{sgn}\left(S\partial_S^2 V\right) + 2\sqrt{\frac{2}{\pi}}\lambda(1-\alpha)^2\gamma \left|S\partial_S^2 V\right| \Bigg). \quad (3.6)$$

The parameter λ corresponds to a market depth measure, i.e. it scales the slope of the average transaction price. The parameter γ models the relative bid–ask spreads and it is related to the Leland number through relation $2\gamma\sqrt{2/\pi} = \mathrm{Le}$. Finally, α transforms the average transaction price into the next quoted price, $0 \le \alpha \le 1$.
Notice that if additional model parameters (e.g., Le, $\mu, \kappa, \gamma, \lambda$) vanish, then all the aforementioned nonlinear models are consistent with the original Black–Scholes equation, i.e. $\sigma = \sigma_0$. Furthermore, for call or put options, the function V is convex in the S variable.
The above models have been developed in order to create better answers to the practical financial application, in particular, to avoid some practical drawbacks raised by the restrictive conditions under which the classical Black–Scholes model had been established. It is understandable that the $\sigma > 0$ could not be constant any more as assumed in the linear Black–Scholes model. Studying the nonlinear Black-Scholes PDE with a volatility function as in (3.3) enables us to consider non-zero transaction costs, market feedback and iliquid market effects, risk from investors preferences, etc. From the financial market point of view, these types of assumptions make the model more adequate to financial pratice. The rest of this chapter heavily followed by the works of Ševčovič and Žitňanská [44, 45].
Following our discussion, in the next part we review some models dealing with transaction costs in order to address nonlinear problems.

3.1 Leland model - proportional transaction costs

One of the most basic models dealing with transaction costs is Leland model [35] which suggests a market with proportional transaction costs. That is, considering $|\nu|$ the number of shares (it is positive if the agent buys or negative if the agent sells) and S the price of the asset at time t, the costs of the transaction of $|\nu|$ shares at time t is given by $kS|\nu|$. Where the constant $k > 0$ depends on the portion involved in the transaction.

Leland 's replication strategy is to use the common Black–Scholes formulae in periodical revisions of the portfolio, but with an appropriately enlarged volatility. This is a model widely accepted in the financial industry. The diffusion coefficient of the model is given

$$\sigma^2(1 - Le\ (H)) = \begin{cases} \sigma^2(1 - Le), & \text{if } H > 0, \\ \\ \sigma^2(Le + 1), & \text{if } H < 0, \end{cases} \tag{3.7}$$

where $H = S\partial_S^2 V$ and the Leland number is

$$Le = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}. \tag{3.8}$$

Here $C \equiv C_0$ is a positive constant transaction cost per unit dollar of transaction in the assets market. The constant volatility is called σ and the time leg between portfolio adjustment is Δt. When $(H) = 1$, then the diffusion coefficient becomes the same as in the original Black-Scholes equation with the adjusted constant volatility:

$$\sigma_{Le}^2 = \sigma^2(1 - Le).$$

Furthermore, in the case of European plain vanilla options we have $H > 0$. In order to specify the right sign of the diffusion coefficient for the plain vanilla options, we need to establish the lower and upper bounds of the Leland number $0 \leq Le < 1$. The solution of Leland backward parabolic partial differential equation with terminal condition, as discussed by Jandačka and Ševčovič [23], is an increasing function on the volatility σ. More precisely, the Leland model value is less than the one deduced from the Black–Scholes model when the Leland number is positive. This comes from the fact that the transaction costs are on the side of buyer or holder of the option. Therefore, the maximum price that a buyer can pay is the price deduced by the model which is called *bid* price. When the option is in short position in the self-financing portfolio, then the obtained price by the model is called *ask* price. This means that adjusted volatilities are:

$$\sigma^2(1 - Le) = \sigma_{bid}^2 \quad \text{and} \quad \sigma^2(Le + 1) = \sigma_{ask}^2,$$

And the price difference between ask and bid is:

$$V^{ask}(S, \tau, \sigma(1 + Le)^{\frac{1}{2}}) - V^{bid}(S, \tau, \sigma(1 - Le)^{\frac{1}{2}}).$$

In Fig. 3.1(a) we depicted the plot of the relevant function such that $C_0 = 0.02$.

3.2 Models with variable transaction costs

We want to review a new approach where the amount of transactions, $|\Delta\delta|$, has a general form of a decreasing function, i.e. $C = C(|\Delta\delta|)$, which contains and generalizes, Leland's model.
In the Black–Scholes model continuity of the rehedging of the portfolio is assumed to be continuous, but the problem is that hedging continuously leads to infinitely large costs. One of the simple but essential modification of the Black–Scholes model comprising transaction costs with discrete times possibility of portfolio rearrangement is precisely the Leland model. In the case of discrete time interval the whole transaction costs remain bounded.
Note that the model assumptions are generically the same as in the Black–Scholes model with some extensions. The main idea of the derivation of the option pricing model with transaction costs is the same as introduced by Leland [35]. We mention that the underlying stock is sold at lower price (bid price) S_{bid} and it is bought at a higher price (ask price) S_{ask}. By considering the case of small investor, the constant cost percentage of trading share is given

$$C_0 = 2\frac{S_{ask} - S_{bid}}{S_{bid} + S_{ask}} \quad \text{where} \quad S = \frac{(S_{bid} + S_{ask})}{2} \tag{3.9}$$

then the adjusted expression of the *bid* and *ask* price of the share are

$$S_{bid} = S(1 - \frac{C_0}{2}), \quad S_{ask} = S(\frac{C_0}{2} + 1). \tag{3.10}$$

We compute the additional cost as follows,

$$\frac{S_{ask} - S_{bid}}{2} |\Delta\delta| = \frac{S}{2} C_0 |\Delta\delta| \tag{3.11}$$

where we have the sales of $\Delta\delta < 0$ or buys of $\Delta\delta > 0$ shares at S price. As a result, we derive the Leland equation given

$$Le = \frac{C_0}{\sigma\sqrt{\Delta t}} \sqrt{\frac{2}{\pi}}. \tag{3.12}$$

where the cost C per one transaction is constant, i.e.

$$C \equiv C_0 = const. \tag{3.13}$$

In the case of large amounts of transactions, there is an advantage when the large investors can benefit from the more buying the less paying for every underlying stock on trading market. Assume that the price of the option is in the long position where keeping the option and then by trading underlying stocks strategy hedges the portfolio. And riskless bonds yield is equal to the expected return. If the option V is in long position and δ is the amount of assets S, by buying $\Delta\delta < 0$ or selling $\Delta\delta > 0$ short positioned stocks, then following the Black–Scholes equation we may establish portfolio Π including one option V and δ amount of stock shares S, i.e.

$$\Pi = \delta S + V. \tag{3.14}$$

Utilizing self-financing portfolio argument, the riskless interest rate at the time Δt has to be equal to the change of the expected value of the portfolio at the same time interval,

$$\Delta\Pi = r\Pi\Delta t, \tag{3.15}$$

where $\Delta\Pi_t = \Pi_{\Delta t+t} - \Pi_t$.
But practically replicating the portfolio, while trading assets, the portfolio value changes in transaction costs rate. Consequently, the portfolio value changes to:

$$\Delta\Pi = \Delta(\delta S + V) - \Delta TC \tag{3.16}$$

such that for the time interval of the length Δt, the transaction costs is ΔTC.
Knowing that the paid amount of the transaction costs is given by

$$\frac{S}{2}\,|\Delta\delta|\,C(|\Delta\delta|) = \Delta TC \tag{3.17}$$

Then the change of the portfolio in the time interval Δt, given

$$\delta\Delta S + \Delta V - \frac{S}{2}\,|\Delta\delta|\,C(|\Delta\delta|) = r\Pi\Delta t \tag{3.18}$$

while the portfolio adjustments are based on the δ-hedging strategy, i.e.

$$\delta = -\partial_S V. \tag{3.19}$$

Also remember that the dynamic of the underlying stock follows

$$\Delta S = \mu S\Delta t + \sigma S\Delta w, \tag{3.20}$$

Here the increment of the Wiener process is $\Delta w = w(\Delta t + t) - w(t)$. As the number of assets $\Delta\delta$ in the portfolio is a stochastic process, then utilizing Itô 's lemma on $-\partial_S V$, one can estimate $\Delta\delta$ as

$$\Delta\delta \approx -\sigma S\partial_S^2 V\Delta w, \tag{3.21}$$

where $|\Delta\delta| = \alpha\,|\Phi|$, with $\Phi \sim \mathcal{N}(0,1)$ and

$$\alpha := \sigma S\left|\partial_S^2 V\right|\sqrt{\Delta t}. \tag{3.22}$$

Here the stochastic term $C(|\Delta\delta|)\,|\Delta\delta|$ is assumed to be,

$$C(|\Delta\delta|)\,|\Delta\delta| \approx E[C(|\Delta\delta|)\,|\Delta\delta|].$$

Remark 3. *In the case of Leland model [35], $\Delta\delta$ is given by:*

$$|\Delta\delta| \approx \alpha E[|\Phi|] = \alpha\sqrt{\frac{2}{\pi}}.$$

where $C \equiv C_0 = const.$

we can give the definition of transaction costs measure r_{TC} utilizing the general transaction costs function. Where we apply the expectation operator E to relation (3.17) by using estimation (3.21) and equation (3.22), and some straightforward calculations.

Definition 3.2.1. *[45, Def 3.1.] The transaction cost measure, r_{TC}, can be expressed by using the expected value of the transaction costs:*

$$r_{TC} = \frac{1}{2}\frac{E[C(\alpha\,|\Phi|)\alpha\,|\Phi|]}{\Delta t} \tag{3.23}$$

where $\Delta\delta$ is the number of the traded assets each unit time interval Δt, C is the function of the transaction costs, $\Phi \sim \mathcal{N}(0,1)$ and α is defined in the equation (3.22).

By using the delta hedging strategy (3.19) and applying the self-financing portfolio argument (3.18), one may deduce the extended version of the Black–Scholes equation

$$\partial_t V + rS\partial_S V + \frac{1}{2}S^2\sigma^2\partial_S^2 V - r_{TC}V = 0 \tag{3.24}$$

where r_{TC} is introduced in definition 3.2.1.
We want to introduce the new function $\tilde{C}(\alpha)$ in order to simplify the notation.

Definition 3.2.2. *[45, Def 3.2.] Let $C = C(\xi)$, $C : \mathbb{R}^+ \to \mathbb{R}$, be a transaction cost function and consider*

$$\tilde{C}(\xi) = E[C(\xi\,|\Phi|)\,|\Phi|], \tag{3.25}$$

then, $\tilde{C}(\xi)$ is called the modified transaction costs function, where $\Phi \sim \mathcal{N}(0,1)$ and $\xi \geq 0$.

Remark 4. *We can apply equation (3.25) to the equation (3.23) in order to derive new expression for transaction cost measure, r_{TC}, given*

$$r_{TC} = \frac{\alpha}{2}\frac{\tilde{C}(\alpha)}{\Delta t} \tag{3.26}$$

Proposition 2. *Assume that $C : \mathbb{R}_0^+ \to \mathbb{R}$ is a measurable and bounded function of the transaction costs. Then the price of the option under variable* *transaction costs is the solution of the nonlinear Black–Scholes PDE given by*

$$\partial_t V + rS\partial_S V + \frac{1}{2}\hat{\sigma}^2(H)S^2\partial_S^2 V - rV = 0, \tag{3.27}$$

where the nonlinear diffusion coefficient $\hat{\sigma}^2(H)$ *is*

$$\hat{\sigma}^2(\tau, S, H) = \sigma^2\left(1 - \sqrt{\frac{2}{\pi}}\tilde{C}(\sigma S\,|H|\,\sqrt{\Delta t})\frac{(H)}{\sigma\sqrt{\Delta t}}\right). \tag{3.28}$$

In the next step we state some useful properties of the modified transaction costs function $\tilde{C}$ based on the work of Žitňanská [45].

Proposition 3. *Assume that the* $C(\xi)$ *is a measurable and bounded transaction costs function. Then its modification function* $\tilde{C}(\beta)$ *has the given properties:*

- $\tilde{C}(0) = C(0)$.
- *if* $\underline{C}_0 \leq C(\xi) \leq C_0$ *for all* $\xi \geq 0$, *then* $\underline{C}_0 \leq \tilde{C}(\xi) \leq C_0$ *for all* $\xi \geq 0$.
- *If* C *is increasing (decreasing) function,* $\tilde{C}(\xi)$ *is then increasing (decreasing) function.*
- *If* C *is variable concave (convex) function,* $\tilde{C}(\xi)$ *is then concave (convex) function.*
- *If* $C(\infty) = \lim_{\xi\to\infty} C(\xi)$, *then* $\tilde{C}(\infty) = C(\infty)$.
- *If* $C(\xi)$ *is* C^k *smooth function,* $\tilde{C}(\xi)$ *is then* C^k *smooth function.*

Proposition 4. *Suppose that the function* $C(\xi)$ *is a decreasing and measurable function of the transaction costs which is bounded for all positive* ξ. *Then,*

- *if* $C(0)$ *is positive, the function* $\xi \mapsto \frac{\tilde{C}(\xi)}{\xi}$ *is then strictly convex for positive* ξ.
- *For all positive* ξ *if* $\underline{C}_0 \leq C(\xi) \leq C_0$, *then*

$$2\underline{C}_0 - C_0 \leq \xi\tilde{C}'(\xi) + \tilde{C}(\xi).$$

At this point, we give some relevant examples of transaction costs functions. More details can be found in [45].

3.2.1 Non-increasing transaction costs function

Another interesting model is the one proposed by Amster, Averbuj and Mariani [1] where individual transaction costs can be treated as a non-increasing linear function. In this case the transaction cost decrease as the number of traded shares growing inversely. In their model the transaction cost function has the form

$$C(\xi) = C_0 - \kappa\xi, \tag{3.29}$$

where C_0, κ are positive constants depending on the individual investor and the volume of trading assets $\xi = |\Delta\delta|$ depends on the each time step of stocks hold in the portfolio. The mean value modification function of the model is given by:

$$\tilde{C}(\xi) = C_0\sqrt{\frac{2}{\pi}} - \kappa\xi \quad \text{where} \quad \xi \geq 0, \tag{3.30}$$

With this model the authors derived a nonlinear Black–Scholes PDE and worked on the stationary problem with relevant boundary conditions. Furthermore, the existence and uniqueness of the problem solution is proven by the upper (lower) solution arguments. In the real market $C(\xi)$ has to be non-negative, but the function (3.29) may attain negative value provided the transaction amount $|\Delta\delta| = \xi \geq C_0/\kappa$. This is a limitation which will be improved in the next section.

In Fig. 3.1(b) we depicted the plot of the relevant function when the parameter values are $\kappa = 0.4$ and $C_0 = 0.03$.

3.2.2 Piecewise decreasing transaction costs function

In financial markets, the transaction costs function should not reach the negative values. Therefore the previous model can be improved by using a piecewise linear decreasing function which has the capability of excluding the negative values. A piecewise linear decreasing transaction costs function can be described as:

$$C(\xi) = \begin{cases} C_0, & \text{if} \quad 0 \leq \xi < \xi_-, \\ C_0 - \kappa(\xi - \xi_-), & \text{if} \quad \xi_- \leq \xi \leq \xi_+, \\ \underline{C}_0, & \text{if} \quad \xi \geq \xi_+. \end{cases} \tag{3.31}$$

where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-)$.

Here $\xi_-, \kappa, C_0, \xi_+$ are given positive constants.

This transaction costs function seems to be more close to reality. It pays the amount C_0 for the small volume of traded assets. When the traded stocks volume is higher, there is a discount for that and finally when the trades are very large, it just pays a small constant $\underline{C}_0$.

The comparison between $\tilde{C}(\xi)$ and $C(\xi)$ can be seen in Fig. 3.1(c), with

$\kappa = 0.25, C_0 = 0.03$ and $\xi_+ = 0.12, \xi_- = 0.06$.
At this step, we give some results about the mean value modification function $\tilde{C}(\xi)$ related to the $C(\xi)$ function in (3.31).

Proposition 5. *Let C_0, κ be the given positive constants. Then, for the piecewise linear function (3.31), the modified mean value transaction costs function is of the form,*

$$\tilde{C}(\xi) = C_0\sqrt{\frac{2}{\pi}} - 2\kappa\xi \int_{\frac{\beta_-}{\xi}}^{\frac{\xi_+}{\xi}} \frac{e^{-u^2/2}}{\sqrt{2\pi}}u, \quad \textit{for} \quad \xi \geq 0. \tag{3.32}$$

Furthermore, there is a bound for this mean value transaction cost function $\tilde{C}(\xi)$.

Proposition 6. *The modification transaction costs function in (3.32) verifies*

$$C_0\sqrt{\frac{2}{\pi}} \leq \tilde{C}(\xi) \leq C_0\sqrt{\frac{2}{\pi}} \tag{3.33}$$

and

$$\lim_{\xi\to\infty} \tilde{C}(\xi) = \underline{C}_0\sqrt{\frac{2}{\pi}}, \tag{3.34}$$

where $\underline{C}_0 = C_0 - \kappa(\xi_+ - \xi_-)$.

In the last step, we discuss another realistc example of the transaction costs function introduced by Ševčovič and Žitňanská [44].

3.2.3 Exponentially decreasing transaction costs function

We can consider the following exponentially decreasing transaction costs function

$$C(\xi) = C_0 e^{\kappa\xi} \quad \text{for} \quad \xi \geq 0, \tag{3.35}$$

where κ, C_0 are given positive constants. It is easy to compute the modification transaction costs function by using (3.25):

$$\begin{aligned}
\tilde{C}(\xi) &= \sqrt{\frac{2}{\pi}}E[C(\xi\,|\Phi|)\,|\Phi|] = \int_0^\infty C(\xi u)ue^{-u^2/2}du \\
&= C_0\int_0^\infty e^{-\kappa\xi u}ue^{-u^2/2}du \\
&= C_0\left(\left[-e^{-\xi\kappa u}e^{-u^2/2}\right]_0^\infty - \int_0^\infty (e^{-\xi\kappa u})'(-e^{-u^2/2})du\right) \\
&= C_0\left(1 - \xi\kappa\int_0^\infty e^{-\xi\kappa u - u^2/2}du\right) = C_0\left(1 - \xi\kappa e^{\xi^2\kappa^2/2}\int_{\xi\kappa}^\infty e^{-t^2/2}dt\right) \\
&= C_0\Phi(-\sqrt{2}\xi\kappa),
\end{aligned}$$

Here

$$\Phi(u) = \frac{\sqrt{\pi}}{2} u e^{u^2/4}(1 + \text{erf}(u/2)) + 1$$

where the error function is $\text{erf}(u/2) = \frac{2}{\sqrt{\pi}} \int_0^{u/2} e^{-s^2} ds$.
In Fig. 3.1(d) we illustrated the $\tilde{C}(\xi), C(\xi)$ where the parameter values are $\kappa = 120$ and $C_0 = 0.03$.

In next part, we will follow the steps for modelling the risk arising from a volatile portfolio.

3.3 Modelling with risk

The nonlinearity of the model can arise not only from introducing transaction costs but also from applying risk due to a volatile portfolio. This model is the extended version of the RAMP model presented by Jandačka and Ševčovič in [23] based on the Kratka' s approach [29]. In financial markets, the portfolio is very volatile when it includes stocks and options. There is a direct relation between the exposure to risk and time-lag of portfolio adjustment. We require to measure the following risk which is proposed by Jandačka and Ševčovič [23] where the volatility of the portfolio changes. Assume that the replicating portfolio is $\Pi = \delta S + V$, then the volatility of the portfolio changes are described by the term $(\Delta\Pi/S)$. In other words, the risk measure r_{VP} from the unprotected portfolio can be expressed as:

$$r_{VP} = \frac{[\Delta\Pi/S]}{\Delta t} R. \tag{3.36}$$

It means, r_{VP} is proportional to the variance of the relative change of a portfolio per time interval Δt. A constant R is called the risk premium coefficient.
Then by utilizing Itô' s lemma to the $\Delta\Pi = \Delta V - \delta\Delta S$, we conclude

$$\Delta\Pi = \frac{1}{2}\sigma^2 S^2 \partial_S^2 V (\Delta W)^2 + \sigma S(\partial_S V + \delta)\Delta W + \phi, \tag{3.37}$$

where the deterministic term is $\phi = \partial_t V \Delta t + \mu S \Delta t(\partial_S V + \delta)$. Therefore

$$\Delta\Pi - E[\Delta\Pi] = \sigma S(\partial_S V + \delta)\Phi\sqrt{\Delta t} + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V(\Phi^2 - 1)\Delta t. \tag{3.38}$$

Here $E[\phi] = \phi$ is the lowest order Δt-term opproximation and $\Delta W = \Phi\sqrt{\Delta t}$ such that $\Phi \sim N(0,1)$, is a normally distributed random variable. Hence the variance can simply be calculated

$$\begin{aligned} [\Delta\Pi] &= E\left[(\Delta\Pi - E[\Delta\Pi])^2\right] \\ &= E\left[\left(\sigma S(\partial_S V + \delta)\Phi\sqrt{\Delta t} + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V(\Phi^2 - 1)\Delta t\right)^2\right]. \end{aligned}$$

By assuming the delta hedging $\delta = -\partial_S V$ for portfolio adjustment and applying the basic statistic rule $E[(\Phi^2 - 1)^2] = 2$ we deduce that the risk premium can be described as

$$r_{VP} = \frac{1}{2} R\sigma^4 (S\partial_S^2 V)^2 \Delta t. \tag{3.39}$$

where positive constant R describes the risk level of the risky portfolio.

In the next section we discuss the novel model including both non-trivial transaction costs and the risk from volatile portfolio which leads to a non-linear generalized Black–Scholes equation.

3.4 Modelling with risk and transaction costs

We want to present a nonlinear extended version of the Black–Scholes equation considering both the risky volatile portfolio and the transaction costs. If we combine the risk premium r_{VP} in equation (3.39) from the volatile portfolio to the problem, then we derive the more complicated model with the total risk measure r_R as:

$$r_R = r_{VP} + r_{TC} \tag{3.40}$$

where r_R follows the same assumption as both r_{VP} and r_{TC}.
We construct the derivation of the model by mimicking from the subsection (3.2). We start by the portfolio changes, of the form

$$\Delta\Pi = \Delta V + \delta\Delta S - r_R S \Delta t, \tag{3.41}$$

Here the total risk defined in equation (3.40) consists of transaction costs plus the risk level arising from the unprotected portfolio. From one side, the large rearranging time interval leads to the lower transaction costs. But from the other side, the portfolio is unprotected for a long time, so the trader is in danger.
The transaction cost measure due to the variable transaction costs is given by

$$r_{TC} = \frac{1}{2}\alpha \frac{\tilde{C}(\alpha)}{\Delta t} \quad \text{where} \quad \alpha = \sigma S \left|\partial_S^2 V\right| \sqrt{\Delta t}$$

and the risk measure from the volatile portfolio has the form

$$r_{VP} = \frac{1}{2} R\sigma^4 (H)^2 \Delta t \quad \text{where} \quad H = S\partial_S^2 V$$

By inserting r_R in (3.40) into the model we finally derive the new equation

$$\partial_t V + rS\partial_S V + \frac{1}{2}\hat{\sigma}^2(\Delta t, H) S^2 \partial_S^2 V - rV = 0, \tag{3.42}$$

where the nonlinear diffusion coefficient can be described by the equation

$$\hat{\sigma}^2(\Delta t, H) = \sigma^2 \left(1 - \tilde{C}(\sigma\,|H|\,\sqrt{\Delta t})\frac{(H)}{\sigma\sqrt{\Delta t}} - R\sigma^2 H \Delta t\right). \tag{3.43}$$

In the presented model we utilize the variable transaction costs with general mean value modification function $\tilde{C}$, namely with piecewise linear decreasing function introduced in subsection (3.2.2) or exponentially non-increasing function defined in subsection (3.2.3). Furthermore, the positive constant risk premium coefficient R is used to control the risk arises from the unprotected portfolio.

This new model includes some well-known models with different choices of R and $\tilde{C}$, respectively.

- **RAPM model with variable transaction costs with fixed time interval**
  Assuming a linearly non-increasing function from model proposed by Amster, Averbuj and Mariani, i.e.

  $$C(\xi) = C_0 - \kappa\xi \quad \text{where} \quad \xi = \Delta\delta,$$

  where the risk premium coefficient is not zero, $R \neq 0$, and the time-lag Δt is given. Then the volatility function can be expressed by

  $$\hat{\sigma}^2(\Delta t, H) = \sigma^2 \left(1 - \left(\frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}(H) + R\sigma^2 H \Delta t\right) + \kappa H\right).$$

  It is easy to check that the model includes both the volatility of Amster, Averbuj and Mariani model and the RAPM model with undetermined time-lag Δt.

- **RAPM model with optimal hedging time** The goal is to minimize the total risk $r_R = r_{VP} + r_{TC}$, which is equivalent to choose optimal hedging time interval. The transaction costs can be minimized by taking larger time interval Δt, but choosing larger time-lag Δt may lead to a higher risk exposure for the trader. Thus, we require to choose an optimal hedging time interval between the portfolio adjustment introduced by Jandačka and Ševčovič [23]. We first derive the transaction costs coefficient r_{TC}. Recalling self-financing portfolio argument and then utilizing Itô' s lemma and δ-hedging argument to the portfolio $\Pi = \delta S + V$, then the portfolio change can be presented as

  $$\Delta\Pi = \Delta V + \delta\Delta S - r_R S \Delta t.$$

Based on Leland' s approach [35], one can obtain the transaction costs coefficient r_{TC} as follows:

$$r_{TC} = \frac{C_0 \left|\Gamma \sigma S\right|}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}}.$$

Now, we can state the equation derived in (3.39) for the risk premium r_{VP} which has the form

$$r_{VP} = \frac{1}{2} R \sigma^4 (H)^2 \Delta t. \qquad (3.44)$$

with R as positive constant explaining the risk level of the unprotected portfolio. We have to find the solution of the problem of minimizing the total risk value r_R which is equal to minimize the value of function $g(\Delta t)$ according to Δt,

$$g(\Delta t) = r_{VP} + r_{TC} = \frac{1}{2} R \sigma^4 (H)^2 \Delta t + \frac{C_0 \left|H\right| \sigma}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}}.$$

The unique minimum attained solution of the function g at the time interval Δt is given by the formula

$$\Delta t_{opt} = \frac{A^2}{\sigma^2 \left|H\right|^{2/3}}, \quad \text{where} \quad A = \left(\frac{C_0}{R\sqrt{2\pi}} \right)^{1/3}. \qquad (3.45)$$

Inserting the relation in (3.45) into the function g, we obtain

$$g(\Delta t_{opt}) = B \sigma^2 \left|H\right|^{4/3} \quad \text{where} \quad B = \frac{3}{2} \left(\frac{C_0^2 R}{2\pi} \right)^{1/3}. \qquad (3.46)$$

Finally by applying $g(\Delta t_{opt}) = r_R(\Delta t_{opt})$ as the optimal value of total risk premium coefficient, we derive the generalized version of the Black–Scholes equation

$$\partial_t V + rS\partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V - rV - r_R S = 0, \qquad (3.47)$$

which can simply be expressed as

$$\partial_t V + rS\partial_S V + \frac{1}{2}\sigma^2 \left(\mu \left(H\right)^{1/3} + 1 \right) S^2 \partial_S^2 V - rV = 0, \qquad (3.48)$$

where $\mu = 3(\frac{\bar{C}^2 R}{2\pi})^{1/3}$.

Remark 5. *We state that the equation derived in (3.48) is a backward parabolic PDE if and only if the β function*

$$\beta(H) = \frac{\sigma^2}{2} H (\mu H^{1/3} + 1) \qquad (3.49)$$

is a non-decreasing function according to the $H = S\Gamma$.

Here the remark (5) is true if both H and μ are non-negative.

3.5 $\beta(H)$ functions for the nonlinear models

We want to give the relevant $\beta(H)$ function in the generalized nonlinear Black–Scholes equation, given by

$$\partial_t V + rS\partial_S V + S\beta(H) - rV = 0,$$

where the function $\beta(H)$ varies as the given model changes. General form of the $\beta(H)$ function can be written as

$$\beta(\tau, S, H) = \frac{1}{2}\hat{\sigma}^2(\tau, S, H)H. \tag{3.50}$$

- **Original Black–Scholes model**
  In the Black–Scholes model, the volatility is constant. Then $\hat{\sigma}^2(H) = \sigma^2$, so we can write the $\beta(H)$ function as:

  $$\beta(H) = \frac{\sigma^2}{2}H. \tag{3.51}$$

  where there is no transaction costs in the model.

- **Leland model**
  In this model transaction costs is included. The volatility function in the Leland model has the form

  $$\hat{\sigma}^2(H) = \begin{cases} \sigma^2(1 + Le), & \text{if} \quad H < 0, \\ \sigma^2(1 - Le), & \text{if} \quad H > 0, \end{cases}$$

  where the Leland number is $Le = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}$.
  Then the following β functions can be used for option pricing

  $$\beta_{ask}(H) = \frac{\sigma^2}{2}(1 - Le)H \quad \text{and} \quad \beta_{bid}(H) = \frac{\sigma^2}{2}(1 + Le)H$$

- **Model with non-increasing transaction costs**
  Here we present the function β for both *ask* and *bid* to derive the price of the option:

  $$\beta_{ask}(H) = \frac{\sigma^2}{2}H\,(Le - \kappa H + 1)$$

  and

  $$\beta_{bid}(H) = \frac{\sigma^2}{2}H\,(-Le + \kappa H + 1)$$

  where $Le = \frac{C_0}{\sigma\sqrt{\Delta t}}\sqrt{\frac{2}{\pi}}$ is the Leland number.

- **Liquidity model**
  Howison and Bakstein [3] developed a parameterised model related to the trading in an asset concerning liquidity effects. Where a combination of a trader's individual transaction cost and a price slippage impact describe the liquidity.

$$\beta(H) = \sigma^2\Bigg(1 + 2\lambda H + \bar{\gamma}^2(1-\alpha)^2 + 2\sqrt{\frac{2}{\pi}}\bar{\gamma}(H) + \lambda^2(1-\alpha)^2H^2 + 2\sqrt{\frac{2}{\pi}}\lambda(1-\alpha)^2\bar{\gamma}(H)\Bigg)H$$

  The market depth measure is shown by the parameter λ. Next, the average transaction price transmitted into the next quoted price is noted as α $(0 \leq \alpha \leq 1)$. Finally, the relative bid-ask price modeled by $\gamma = \sigma\bar{\gamma}\sqrt{\Delta t}$.

- **Risk adjustment pricing methodology model**
  We recall that in this model, the task was to find optimum value for the hedging time interval in order to minimize the risk premium coefficien r_R. The relevant β function is given

$$\beta_{ask}(H) = \frac{\sigma^2}{2}H(1 + \mu H^{1/3}),$$

  and

$$\beta_{bid}(H) = \frac{\sigma^2}{2}H(1 - \mu H^{1/3}),$$

  where $\mu = 3\left(\frac{C_0^2 R}{2\pi}\right)^{1/3}$.

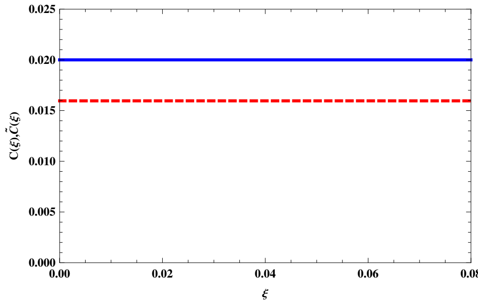
- **Model with risk and transaction cost for option pricing**
  In the novel model, we present the diffusion coefficient as in (3.43), then β function can be written:

$$\beta(H) = \frac{\sigma^2}{2}H\left(1 - \tilde{C}(\sigma\left|H\right|\sqrt{\Delta t})\frac{(H)}{\sigma\sqrt{\Delta t}} - R\sigma^2 H\Delta t\right)$$

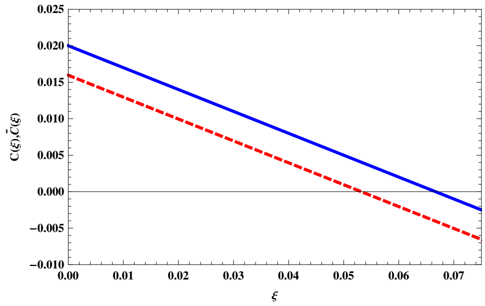
3.6 Chapter conclusions

In this chapter, our aim was to give a review of some nonlinear Black–Scholes models, in particular, models with variable transaction costs. We recall some well-known models such as Leland model [35], the Jumping volatility model proposed by Avellaneda, Levy and Paras [2], non-arbitrage liquidity model developed by Bakstein and Howison [3], Risk Adjusted Pricing Methodology model (RAPM) introduced by Kratka [29] and its generalization by

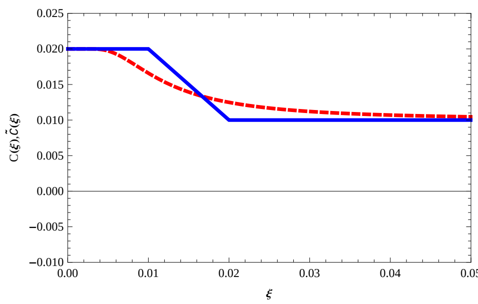
Jandačka and Ševčovič [23]. Furthermore, we focus on some more realistic examples of variable transaction costs function such as the exponentially decreasing function and the piecewise linear non-increasing function discovered by Ševčovič and Žitňanská [44]. The most important feature of these functions is actually to overcome the difficulty of possible negative transaction costs in some known introduced models. The core concept of chapter 3 will be applied to evaluate the price of American style of options with a nonlinear volatility functions in chapter 4 and 5.



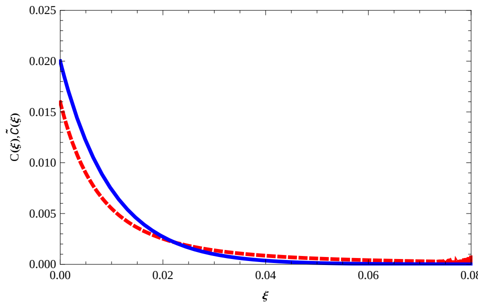
(a) Constant transaction costs.



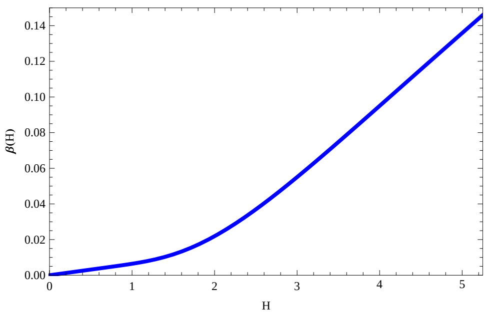
(b) Linearly decreasing costs.



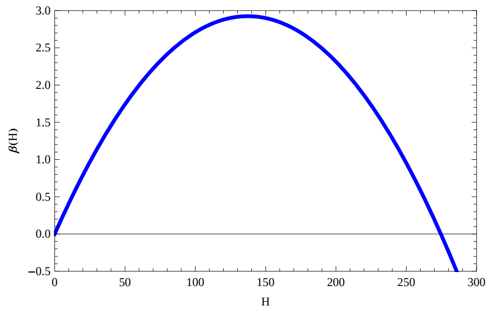
(c) Piecewise linear decreasing function.



(d) Exponential transaction costs.



(e) Function $\beta(H)$ related to the graph (c).



(f) Function $\beta(H)$ related to the graph (d).

Figure 3.1: [45, Fig 4.1] The graph of different types of transaction costs function C (blue line) with its modification $\tilde{C}$ (red line) and corresponding function $\beta(H)$ for the parameter values $R = 10$ and $\sigma = 0.3$.

4

Put option with transaction costs

In a stylized financial market, the price of a European option can be computed from a solution to the well-known Black–Scholes linear parabolic equation derived by Black and Scholes [6], and independently by Merton (cf. Kwok [31], Hull [22], and Dewynne, Howison and Wilmott [11]). A European call (put) option is the right but not obligation to purchase (sell) an underlying asset at the expiration price E at the expiration time T.
Despite the fact that a European option has fixed maturities, the owner of an American option has the right to exercise it at any moment prior to maturity. In fact, in contrast to European options, American style options can be exercised anytime in the temporal interval $[0, T]$ with the specified time of obligatory expiration at $t = T$.
This chapter concerns the qualitative and quantitative behavior of a solution to the problem of pricing American style perpetual put options, assuming

that the volatility may depend on the second derivative of the option price itself. This will lead to consider a nonlinear Black–Scholes model. The results will follow from the study of a nonlinear free boundary problem. We also present some numerical results. This is a contribution to the study of option pricing models with variable volatility. More precisely, such nonlinear Black–Scholes equation arises from option pricing models taking into account nontrivial transaction costs, market feedbacks and/or risk from a volatile (unprotected) portfolio. While the linear Black–Scholes equation with constant σ had been derived under several restrictive assumptions like frictionless, liquid and complete markets, etc.

In this chapter we mostly present the results from the joint work of Grossinho, Faghan and Ševčovič [17, 18].

4.1 Partial differential equation (PDE) approach

A mathematical model for pricing American put option leads to a free boundary problem involves a function $V = V(S,t)$ together with the early exercise boundary profile $S_f : [0,T] \to \mathbb{R}$ satisfying the following conditions:

1. V is a solution to the Black–Scholes partial differential equation:

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + rS\partial_S V - rV = 0 \tag{4.1}$$

   defined on the time dependent domain $S > S_f(t)$ where $0 < t < T$. Here σ is the volatility of the underlying asset price process and $r > 0$ is the interest rate of a zero-coupon bond. A solution $V = V(S,t)$ represents the price of an option if the price of the underlying asset is $S > 0$ at the time $t \in [0,T]$;

2. V satisfies the terminal pay-off condition:

$$V(S,T) = \max(E - S,\, 0); \tag{4.2}$$

3. and the boundary conditions for the put option:

$$V(S_f(t),t) = E - S_f(t), \quad \partial_S V(S_f(t),t) = -1, \quad V(+\infty,t) = 0, \tag{4.3}$$

   for $S = S_f(t)$ and $S = \infty$. Here $(x)_+ = \max(x,0)$ denotes the positive part of x.

If the diffusion coefficient $\sigma > 0$ in (4.1) is constant, then (4.1) is a classical linear Black–Scholes parabolic equation derived by Black and Scholes [6]. If we assume the volatility coefficient $\sigma > 0$ is a function of the solution V, then equation (4.1) with such a diffusion coefficient represents a nonlinear

generalization of the Black–Scholes equation. We recall that the American style of the put option has been investigated by many authors (c.f. Kwok [31] and references therein). Accurate analytic approximations of the free boundary position have been derived in Stamicar, Ševčovič and Chadam [40], Zhu [46], Lauko and Ševčovič [34], Evans, Kuske and Keller [12] dealing with analytic approximations on the whole time interval.
In this study we focus our attention to the case when the diffusion coefficient σ^2 may depend on the asset price S and the second derivative $\partial_S^2 V$ of the option price. More precisely, we assume that

$$\sigma = \sigma(S\partial_S^2 V)\,, \tag{4.4}$$

i.e. σ depends on the product $S\partial_S^2 V$ of the asset price S and the second derivative (Gamma) of the option price V. Recall that the nonlinear Black–Scholes equation (4.1) with the volatility σ having the form of (4.4) arises from option pricing models taking into account nontrivial transaction costs, market feedbacks and/or risk from a volatile (unprotected) portfolio. With this study we relax the conditions assumed for the derivation of the classical linear Black–Scholes equation with constant σ like e.g., frictionless, liquid and complete markets, etc. In the recent work [18] it has been investigated the case when the volatility function may depend on S and $\partial_S^2 V$ including other models proposed by Frey and Patie [15], Frey and Stremme [16]. However, for these models there is no single implicit equation for the free boundary position and numerical methods have to be adopted.

4.2 Perpetual American put option

Let us consider the problem of pricing the so-called perpetual put options. By definition, perpetual options are options with a very long maturity $T \to \infty$. Notice that both the option price and the early exercise boundary position depend on the remaining time $T-t$ to maturity only. Suppose that there exists a limit of the solution V and early exercise boundary position S_f for the maturity $T \to \infty$. Recently, stationary solutions to generalized Black–Scholes equation have been investigated [13, 20]. For an American style put option the limiting price $V = V(S) = \lim_{T-t\to\infty} V(S,t)$ and the limiting early exercise boundary position $\rho = \lim_{T-t\to\infty} S_f(t)$ of the perpetual put option is a solution to the stationary nonlinear Black-Scholes partial differential equation:

$$\frac{1}{2}\sigma(S\partial_S^2 V)^2 S^2 \partial_S^2 V + rS\partial_S V - rV = 0, \qquad S > \varrho, \tag{4.5}$$

and

$$V(\varrho) = E - \varrho, \quad \partial_S V(\varrho) = -1, \quad V(+\infty) = 0. \tag{4.6}$$

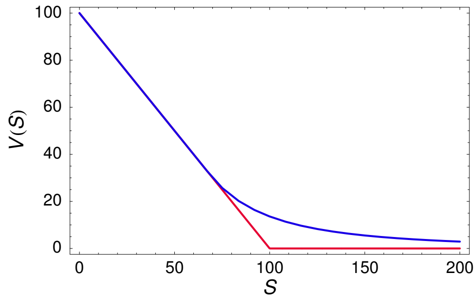


Figure 4.1: [17, Fig 1] A plot of the price $V(S)$ of a perpetual American put option and the pay-off diagram $\max(E-S,0)$ for the parameters: $E = 100, r = 0.1$ and constant volatility $\sigma_0 = 0.3$ and $\gamma = 2r/\sigma_0^2$.

Our purpose is to analyze the system of equations (4.5)–(4.6). In what follows, we will prove the existence and uniqueness of a solution pair $(V(\cdot), \varrho)$ to (4.5)–(4.6).
In the rest of this section, we will assume the volatility function

$$\mathbb{R}_0^+ \ni H \mapsto \sigma(H)^2 \in \mathbb{R}_0^+ \tag{4.7}$$

is non-decreasing, $\sigma(0) > 0$ and such that the function $H \mapsto \sigma(H)^2 H$ is C^1 smooth for $H \geq 0$. Under these assumptions there exists an increasing inverse function $\beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\frac{1}{2}\sigma(H)^2 H = u \qquad \text{iff} \qquad H = \beta(u). \tag{4.8}$$

which is an C^1 continuous and non-decreasing function such that $\beta(0) = 0$, and $\beta(u) > 0$ for $u > 0$. As $u = \frac{1}{2}\sigma(\beta(u))^2\beta(u) \geq \frac{1}{2}\sigma(0)^2\beta(u)$ we have

$$\beta(u) \leq M_1 u \quad \text{for all } u \geq 0, \tag{4.9}$$

where $M_1 = 2/\sigma(0)^2$. Moreover, for any $U_0 > 0$ there exists $M_0 > 0$ such that

$$\beta(u) \geq M_0 u \quad \text{for all } u \in [0, U_0]. \tag{4.10}$$

Notice that the transformation $H = S\partial_S^2 V$ is a useful tool when analyzing nonlinear generalizations of the Black–Scholes equations. For example, using this transformation the fully nonlinear Black–Scholes equation with a volatility function $\sigma = \sigma(S\partial_S^2 V)$ can be transformed into a quasilinear equation for the new variable H (more details can be checked in [23, 43]).

Remark 6. *Typically, the nonlinear volatility function $\sigma(H)$ is an increasing function satisfying the bounds:*

$$0 < \sigma_0^2 \leq \sigma(H)^2 \leq \sigma_0^2(1 + \mu H^a)$$

for some constants $\sigma_0 > 0$ and $\mu, a \geq 0$. Then it is easy to verify that, for any $U_0 > 0$ there are constants $M_0, M_1 > 0$ such that

$$M_0 u \leq \beta(u) \leq M_1 u \quad \text{for } 0 \leq u \leq U_0, \quad M_0 u^{\frac{1}{1+a}} \leq \beta(u) \leq M_1 u \quad \text{for } u \geq U_0. \tag{4.11}$$

The estimates (4.11) imply that the integral

$$\int_{U_0}^{\infty} \frac{\beta(u)}{u} du = +\infty.$$

In this section we will focus our attention on existence and uniqueness of a solution of the problem (4.5)–(4.6).

4.2.1 Existence and uniqueness of solution

Since β is the inverse function to $\frac{1}{2}\sigma(H)^2 H$ the pair $(V(\cdot), \varrho)$ is a solution to (4.5) if and only if

$$S\partial_S^2 V = \beta(rV/S - r\partial_S V).$$

Let us introduce the following transformation of variables

$$U(x) = r\frac{V(S)}{S} - r\partial_S V(S) = -rS\partial_S\left(\frac{V(S)}{S}\right), \qquad \text{where} \quad x = \ln S. \tag{4.12}$$

Since

$$\partial_x U(x) = \partial_S \left(rV(S)/S - r\partial_S V\right) \frac{dS}{dx} = -rS\partial_S^2 V(S) + rS\partial_S \left(\frac{V(S)}{S}\right)$$

the function $U(x)$ is a solution to the initial value problem

$$\partial_x U(x) = -U(x) - r\beta(U(x)), \qquad x > x_0 = \ln \varrho, \tag{4.13}$$

$$U(x_0) = \frac{rE}{\varrho}. \tag{4.14}$$

The latter initial condition easily follows from the smooth pasting conditions $V(\varrho) = E - \varrho$ and $\partial_S V(\varrho) = -1$. Equation (4.13) can be easily integrated. We have the following result:

Lemma 4.2.1. *[17, Lemma 1] The solution $U = U(x)$ to the initial value problem (4.13)-(4.14) is uniquely given by*

$$U(x) = G^{-1}(-x + x_0), \quad \text{for } x > x_0 = \ln \varrho,$$

where

$$G(U) = \int_{U(x_0)}^{U} \frac{1}{u + r\beta(u)} du. \tag{4.15}$$

The useful properties of the function G are summarized in the following lemma:

Lemma 4.2.2. *[17, Lemma 2] The function $G : \mathbb{R}_0^+ \to \mathbb{R}$ is nondecreasing and $G^{-1}(0) = rE/\varrho$. Suppose there exist constants $M_0, M_1, U_0 > 0$ such that*

- *$\beta(u) \leq M_1 u$ for all $u \geq U_0$. Then $G(+\infty) = +\infty$;*
- *$\beta(u) \geq M_0 u$ for all $u \leq U_0$. Then $G(0) = -\infty$,*

where $\beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the inverse function to the function $\mathbb{R}_0^+ \ni H \mapsto \sigma(H)^2 H \in \mathbb{R}_0^+$.

Henceforth we will assume that the inverse function β satisfies the growth assumptions from Lemma 4.2.2 and so $G(+\infty) = +\infty, G(0) = -\infty$ and, consequently, $G^{-1}(-\infty) = 0$.
Since

$$-rS\partial_S \left(\frac{V(S)}{S} \right) = U(\ln S) = G^{-1}(-\ln S + \ln \varrho)$$

we obtain, by taking into account the boundary condition $V(+\infty) = 0$ that the solution to equation (4.5) is given by

$$V(S) = \frac{S}{r} \int_S^\infty G^{-1} \left(-\ln \left(\frac{s}{\varrho} \right) \right) \frac{ds}{s}.$$

Using the substitution $u = G^{-1}(-\ln(s/\varrho))$ we have

$$\frac{ds}{s} = -G'(u)du = -\frac{1}{u + r\beta(u)} du.$$

As $G^{-1}(-\infty) = 0$ the expression for $V(S)$ can be simplified as follows:

$$V(S) = \frac{S}{r} \int_0^{G^{-1}(-\ln(S/\varrho))} \frac{u}{u + r\beta(u)} du. \tag{4.16}$$

We derive a single implicit equation for the free boundary position ϱ and the closed form formula for the option price. And also the first order expansion of the free boundary position with respect to the model parameter is also derived.

4.2.2 Equation for the free boundary position

Using the expression (4.16) we can deduce a single implicit integral equation for the free boundary position ϱ. Clearly, $V(\varrho) = E - \varrho$ if and only if

$$E - \varrho = \frac{\varrho}{r} \int_0^{G^{-1}(0)} \frac{u}{u + r\beta(u)} du. \tag{4.17}$$

As $G^{-1}(0) = \frac{rE}{\varrho}$ we obtain

$$\frac{rE}{\varrho} = r + \int_0^{\frac{rE}{\varrho}} \frac{u}{u + r\beta(u)} du = r + \frac{rE}{\varrho} - r \int_0^{\frac{rE}{\varrho}} \frac{\beta(u)}{u + r\beta(u)} du \qquad (4.18)$$

Therefore the free boundary position ϱ is a solution to the following implicit equation:

$$\int_0^{\frac{rE}{\varrho}} \frac{\beta(u)}{u + r\beta(u)} du = 1.$$

4.2.3 Main result

In this section we summarize the previous results and state the main result on existence and uniqueness of a solution to the perpetual American put option pricing problem (4.5)–(4.6).

Theorem 4.2.1. *[17, Theorem 1] Suppose that the volatility function $\sigma : \mathbb{R}_0^+ \to \mathbb{R}^+$ is non-decreasing, $\sigma(0) > 0$ and such that the function $H \mapsto \sigma(H)^2 H$ is C^1 smooth for $H \geq 0$. Then the perpetual American put option problem (4.5)–(4.6) has a unique solution $(V(\cdot), \varrho)$ where the free boundary position ϱ is a solution to the implicit equation*

$$\int_0^{\frac{rE}{\varrho}} \frac{\beta(u)}{u + r\beta(u)} du = 1, \qquad (4.19)$$

and the option price $V(S)$ is given by

$$V(S) = \frac{S}{r} \int_0^{G^{-1}(-\ln(S/\varrho))} \frac{u}{u + r\beta(u)} du, \qquad (4.20)$$

where β is the inverse function to the function $H \mapsto \frac{1}{2}\sigma(H)^2 H$.

Proof. According to results in Section 3.2 it suffices to prove that (4.19) has the unique solution ϱ. To this end, let us introduce the auxiliary function:

$$\phi(y) = \int_0^y \frac{\beta(u)}{u + r\beta(u)} du$$

we have $\phi'(y) > 0, \phi(0) = 0$. For a fixed $U_0 > 0$ we have $\beta(u) \geq \beta(U_0) > 0$ for $u \geq U_0$, and

$$\begin{aligned} \phi(+\infty) = \int_0^\infty \frac{\beta(u)}{u + r\beta(u)} du &\geq \int_{U_0}^\infty \frac{\beta(u)}{u + r\beta(u)} du \\ &\geq \frac{\beta(U_0)}{1 + rM_1} \int_{U_0}^\infty \frac{1}{u} du = +\infty. \end{aligned}$$

Hence equation (4.19) has the unique solution $\varrho > 0$. Clearly, $\varrho < E$ because the right hand side of (4.17) is positive.
Since ϱ is a solution to (4.18) we have $V(\varrho) = E - \varrho$. Moreover, as

$$U(x) = r\frac{V(S)}{S} - r\partial_S V(S), \qquad \text{where} \quad x = \ln S$$

(see (4.12)) we obtain, for $x_0 = \ln \varrho$,

$$\partial_S V(\varrho) = \frac{V(\varrho)}{\varrho} - \frac{U(x_0)}{r} = \frac{E-\varrho}{\varrho} - \frac{E}{\varrho} = -1.$$

Hence V is a solution to the perpetual American put option pricing problem (4.5)–(4.6), as claimed. □

Remark 7. *[17, Remark 1] In the case of a constant volatility function $\sigma(H) \equiv \sigma_0$ we have $\beta(u) = \frac{2}{\sigma_0^2}u$. It follows from equation (4.19) that*

$$\varrho = E\frac{\gamma}{1+\gamma}, \quad \textit{where } \gamma = \frac{2r}{\sigma_0^2},$$

and,

$$\begin{aligned} V(S) &= \frac{S}{r}\int_0^{G^{-1}(-\ln(S/\varrho))} \frac{u}{u+r\beta(u)}du \\ &= \frac{S}{r}\frac{1}{1+\gamma}G^{-1}(-\ln(S/\varrho)) \\ &= \frac{E}{1+\gamma}\left(\frac{S}{\varrho}\right)^{-\gamma} \end{aligned}$$

because $G(U) = \frac{1}{1+\gamma}\ln(U/U(x_0)), U(x_0) = rE/\varrho$, and so $G^{-1}(f) = \frac{rE}{\varrho}e^{(1+\gamma)f}$. Hence the solution is identical with Merton's explicit solution.

4.2.4 Sensitivity analysis

In this section we will investigate dependence of the free boundary position on model parameters. We consider the volatility function of the form:

$$\frac{1}{2}\sigma(H)^2 H = \frac{\sigma_0^2}{2}\left(1+\mu H^a\right)H + O(\mu^2) \quad \text{as } \mu \to 0.$$

Here $a \geq 0$ and $\mu \geq 0$ are specific model parameters. Our goal is to find the first order expansion of the free boundary position ϱ considered as a function of a parameter μ, i.e. $\varrho = \varrho(\mu)$.
First, we derive expression for the derivative $\partial_\mu \beta$ of the inverse function β. For $H = \beta(u;\mu)$ we have $u = \frac{1}{2}\sigma(\beta(u;\mu))^2\beta(u;\mu)$ and so

$$0 = \partial_\mu\left(\frac{\sigma_0^2}{2}\left(1+\mu H^a\right)H\right) = \frac{\sigma_0^2}{2}\left(1+\mu(a+1)\beta^a\right)\partial_\mu\beta + \frac{\sigma_0^2}{2}\beta^{a+1} + O(\mu)$$

For $\mu = 0$ we have $\beta(u;0) = \frac{2}{\sigma_0^2}u$. Therefore

$$\partial_\mu \beta(u;0) = -(\sigma_0^2/2)^{-(a+1)} u^{a+1}.$$

The first derivative of the free boundary position $\varrho = \varrho(\mu)$ can be deduced from the implicit equation (4.19). We have

$$\begin{aligned} 0 &= \frac{d}{d\mu}\int_0^{\frac{rE}{\varrho(\mu)}} \frac{\beta(u;\mu)}{u + r\beta(u;\mu)} du \\ &= \left.\frac{\beta(u;\mu)}{u + r\beta(u;\mu)}\right|_{u=\frac{rE}{\varrho(\mu)}} \left(-\frac{rE}{\varrho(\mu)^2}\partial_\mu \varrho(\mu)\right) + \int_0^{\frac{rE}{\varrho(\mu)}} \frac{u\partial_\mu \beta(u;\mu)}{(u + r\beta(u;\mu))^2} du. \end{aligned}$$

Since, for $\mu = 0$ we have $\varrho(0) = E\gamma/(1+\gamma)$ we conclude

$$\partial_\mu \varrho(0) = -\frac{E}{a+1}\gamma(1+\gamma)^{a-2}.$$

In summary we have shown the following result:

Theorem 4.2.2. *[17, Theorem 2] If the volatility function $\sigma(H)$ has the form $\frac{1}{2}\sigma(H)^2 H = \frac{\sigma_0^2}{2}(1 + \mu H^a) H + O(\mu^2)$ as $\mu \to 0$, where $\mu, a \geq 0$, then the free boundary position $\varrho = \varrho(\mu)$ of the perpetual American put option pricing problem has the asymptotic expansion:*

$$\varrho(\mu) = E\frac{\gamma}{1+\gamma} - \mu\frac{E}{a+1}\frac{\gamma}{(1+\gamma)^{2-a}} + O(\mu^2) \quad \text{as } \mu \to 0.$$

Remark 8. *[17, Remark 2] In the case $a = 0$ we have $\sigma(H)^2 = \sigma_0^2(1+\mu)$. It corresponds to the constant volatility model. Thus $\varrho(\mu) = E\frac{\gamma(\mu)}{1+\gamma(\mu)} = E\frac{1}{1+1/\gamma(\mu)}$ where $\gamma(\mu) = 2r/(\sigma_0^2(1+\mu))$. Hence*

$$\varrho(\mu) = E\frac{1}{1 + \frac{\sigma_0^2}{2r}(1+\mu)}, \quad \text{and,} \quad \partial_\mu \varrho(0) = -E\frac{\gamma}{(1+\gamma)^2},$$

as claimed by Theorem 4.2.2.

4.3 Extended version of the problem

In the previous section we provided the result for the models such as the RAPM model where the volatility function depends on $H = S\partial_S^2 V$ only, and it has the form:

$$\sigma(H)^2 = \sigma_0^2(1 + \lambda H^{\frac{1}{3}}) = \sigma_0^2(1 + \lambda(S\partial_S^2 V)^{\frac{1}{3}}), \tag{4.21}$$

where $\sigma_0 > 0$ is the constant historical volatility of the underlying asset and λ is a model parameter depending on the transaction cost rate and the unprotected portfolio risk exposure. But there are some models like Barles and Soner [4] in which investor's preferences are shown by an exponential utility function. In this model, the volatility function depends on $H = S\partial_S^2 V$ as well as S, and it has the following form:

$$\sigma(S,H)^2 = \sigma_0^2 \left(1 + \Psi(a^2 SH)\right) = \sigma_0^2 \left(1 + \Psi(a^2 S^2 \partial_S^2 V)\right), \tag{4.22}$$

where the function Ψ is the unique solution to the ODE: $\Psi'(x) = (\Psi(x) + 1)/(2\sqrt{x\Psi(x)} - x)$, $\Psi(0) = 0$ and $a \geq 0$ is a constant depending transaction costs and investor's risk aversion parameter (see [4] for more details). Notice that $\Psi(x) \geq 0$ for all $x \geq 0$ and it has the following asymptotic: $\Psi(x) = O(x^{\frac{1}{3}})$ for $x \to 0$ and $\Psi(x) = O(x)$ for $x \to \infty$. In this part we prove that under certain assumptions made on the volatility function the perpetual American put option problem (4.5)–(4.6) has the unique solution $(V(.), \varrho)$. where the volatility function is of the form

$$\sigma = \sigma(S,H) = \sigma(S, S\partial_S^2 V). \tag{4.23}$$

Throughout this section we will assume that the volatility function $\sigma = \sigma(S,H)$ fulfills the following assumption:
Assumption 1. *The volatility function $\sigma = \sigma(S,H)$ in (4.5) is assumed to be a C^1 smooth nondecreasing function in the $H > 0$ variable and $\sigma(S,H) \geq \sigma_0 > 0$ for any $S > 0$ and $H \geq 0$ where σ_0 is a positive constant.*
If we extend the volatility function $\sigma(S,H)$ by $\sigma(S,0)$ for negative values of H, i.e. $\sigma(S,H) = \sigma(S,0)$ for $H \leq 0$ then the function

$$\mathbb{R} \ni H \longmapsto \frac{1}{2}\sigma(S,H)^2 H \in \mathbb{R}$$

is strictly increasing and therefore there exists the unique inverse function $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{1}{2}\sigma(S,H)^2 H = w \qquad \text{if and only if} \qquad H = \beta(x,w), \quad \text{where } S = e^x. \tag{4.24}$$

Notice that the function β is a continuous and increasing function such that $\beta(0) = 0$.

Concerning the inverse function we have the following useful lemma:

Lemma 4.3.1. *[18, Lemma 1] Assume the volatility function $\sigma(S,H)$ satisfies Assumption 1. Then the inverse function β has the following properties:*

1. *$\beta(x,0) = 0$ and $\frac{\beta(x,w)}{w} \leq \frac{2}{\sigma_0^2}$ for all $x, w \in \mathbb{R}$;*

2. *$\beta'_w(x,w) \leq \frac{2}{\sigma_0^2}$ for all $x \in \mathbb{R}$ and $w > 0$.*

Proof. Clearly, $\beta(x,0)=0$. For $w>0$ we have $\beta(x,w)>0$ and $w=\frac{1}{2}\sigma(e^x,\beta(x,w))^2\beta(x,w)\geq\frac{\sigma_0^2}{2}\beta(x,w)$ and so $\frac{\beta(x,w)}{w}\leq\frac{2}{\sigma_0^2}$. If $w<0$ then $\beta(x,w)<0$ and we can proceed similarly as before.
Differentiating the equality $w=\frac{1}{2}\sigma(e^x,\beta(x,w))^2\beta(x,w)\geq\frac{\sigma_0^2}{2}\beta(x,w)$ with respect to $w>0$ yields:

$$1=\frac{1}{2}\sigma^2(e^x,\beta(x,w))\beta'_w(x,w)+\partial_H\left(\frac{1}{2}\sigma(e^x,H)^2\right)H\geq\frac{1}{2}\sigma_0^2\beta'_w(x,w)$$

for $H=\beta(x,w)>0$ and the proof of the second statement of Lemma follows.

□

The key step how to solve the perpetual American put option problem (4.5)–(4.6) consists in introduction of the following variable:

$$W(x)=\frac{r}{S}\left(V(S)-S\partial_S V(S)\right)\quad\text{where}\quad S=e^x. \tag{4.25}$$

Lemma 4.3.2. *[18, Lemma 2] Let $x^0\in\mathbb{R}$ be given. The function $V(S)$ is a solution to equation (4.5) for $S>\varrho=e^{x_0}$ satisfying the boundary condition:*

$$V(S)-S\partial_S V(S)=E,\quad at\ \ S=\varrho,$$

if and only if the transformed function $W(x)$ is a solution to the initial value problem for the ODE:

$$\begin{aligned}&\partial_x W(x)=-W(x)-r\beta(x,W(x)),\quad x>x_0,\\&W(x_0)=rEe^{-x_0}.\end{aligned} \tag{4.26}$$

Proof. As $\partial_x=S\partial_S$ we obtain

$$\begin{aligned}\partial_x W(x)&=rS\partial_S(S^{-1}V(S)-\partial_S V(S))\\&=rSS^{-1}\partial_S V(S)-rS^{-1}V(S)-rS\partial_S^2V(S)\\&=-W(x)-rS\partial_S^2V(S)=-W(x)-r\beta(x,W(x)),\end{aligned}$$

because $\beta(x,W(x))=H\equiv S\partial_S^2V(S)$ if and only if $\frac{1}{2}\sigma(S,H)^2H=W(x)$ and V solves (4.5), i.e.

$$\frac{1}{2}\sigma(S,H)^2H+\frac{r}{S}\left(S\partial_S V(S)-V(S)\right)=0.$$

Finally, $W(x_0)=\frac{r}{S}\left(V(S)-S\partial_S V(S)\right)=rEe^{-x_0}$ where $S=\varrho=e^{x_0}$, as claimed. □

Notice the equivalence of conditions:

$$V(S) - S\partial_S V(S) = E \text{ and } V(S) = E - S \quad \Longleftrightarrow \quad \partial_S V(S) = -1 \quad (4.27)$$

and V(S)=E-S.

Concerning the solution W of the ODE (4.26) we have the following auxiliary result:

Lemma 4.3.3. *[18, Lemma 3] Assume $x^0 \in \mathbb{R}$. Let $W = W_{x_0}(x)$ be the unique solution to the ODE (4.26) for $x \in \mathbb{R}$ satisfying the boundary condition $W(x_0) = rEe^{-x_0}$ at the initial point x_0. Then*

1. *$W_{x_0}(x) > 0$ for any $x \in \mathbb{R}$,*
2. *the function $x_0 \mapsto W_{x_0}(x)$ is increasing in the x_0 variable for any $x \in \mathbb{R}$,*
3. *if the volatility function depends on $H = S\partial_S^2 V$ only, i.e. $\sigma = \sigma(H)$, then*

$$W_{x_0}(x) = F^{-1}(x_0 - x) \quad \text{where} \quad F(W) = \int_{W_0}^{W} \frac{1}{w + r\beta(w)} dw \quad \text{and}$$

$$W_0 = W(x_0) = rEe^{-x_0}.$$

Proof. According to Lemma 4.3.1 we have $\beta(x, w)/w \le 2/\sigma_0^2$ for any $x \in \mathbb{R}$ and $w \neq 0$. Hence

$$\partial_x |\ln(W(x)| = -\left(1 + r\frac{\beta(x, W(x))}{W(x)}\right) \ge -(1 + \gamma)$$

where $\gamma = 2r/\sigma_0^2$. Therefore

$$|W(x)| \ge |W(x_0)| e^{-(1+\gamma)(x - x_0)} > 0,$$

and this is why the function $W(x)$ does not change the sign. As $W(x_0) = rEe^{-x_0} > 0$ we have $W_{x_0}(x) > 0$ as well.

The solution $W_{x_0}(x)$ to the ODE (4.26) can be expressed in the form

$$\begin{aligned} W_{x_0}(x) &= W_{x_0}(x_0) - \int_{x_0}^{x} (W_{x_0}(\xi) + r\beta(\xi, W_{x_0}(\xi)))d\xi \\ &= rEe^{-x_0} - \int_{x_0}^{x} (W_{x_0}(\xi) + r\beta(\xi, W_{x_0}(\xi)))\, d\xi. \end{aligned}$$

Let us introduce the auxiliary function

$$y(x) = \partial_{x_0} W_{x_0}(x).$$

Then

$$\begin{aligned} y(x) &= -rEe^{-x_0} + W_{x_0}(x_0) + r\beta(x_0, W_{x_0}(x_0)) \\ &- \int_{x_0}^{x} \left(1 + r\beta'_w(\xi, W_{x_0}(\xi))\right) y(\xi)d\xi \\ &= r\beta(x_0, W_{x_0}(x_0)) - \int_{x_0}^{x} \left(1 + r\beta'_w(\xi, W_{x_0}(\xi))\right) y(\xi)d\xi. \end{aligned}$$

Hence y is a solution to the ODE:

$$\begin{aligned} &\partial_x y(x) = -\left(1 + r\beta'_w(x, W_{x_0}(x))\right) y(x), \quad x \in \mathbb{R}, \\ &y(x_0) = r\beta(x_0, rEe^{-x_0}) > 0. \end{aligned} \tag{4.28}$$

With regard to Lemma 4.3.1 we have $\beta'_w(x, W_{x_0}(x)) \leq 2/\sigma_0^2$. Therefore the function y is a solution to the differential inequality:

$$\partial_x y(x) \geq -(1+\gamma)y(x), \quad x \in \mathbb{R},$$

where $\gamma = 2r/\sigma_0^2$. As a consequence we obtain

$$|y(x)| \geq |y(x_0)|e^{-(1+\gamma)(x-x_0)} > 0 \tag{4.29}$$

and this is why the function $y(x)$ does not change the sign. Therefore $\partial_{x_0} W_{x_0}(x) = y(x) > 0$ and the proof of the statement 2) follows.
Finally, if $\sigma = \sigma(H)$ we have $\beta = \beta(w)$ and so

$$\partial_x F(W(x)) = \frac{1}{W(x) + r\beta(W(x))} \partial_x W(x) = -1.$$

Hence $F(W(x)) = F(W(x_0)) - (x - x_0) = x_0 - x$ and the statement 3) follows.

□

Lemma 4.3.4. *[18, Lemma 4] Under Assumption 1, there exists the unique root $x_0 \in \mathbb{R}$ of the implicit equation*

$$\int_{x_0}^{\infty} \beta(x, W_{x_0}(x))dx = 1. \tag{4.30}$$

Proof. Denote $\phi(x_0) = \int_{x_0}^{\infty} \beta(x, W_{x_0}(x))dx$. Then $\phi(\infty) = 0$ and

$$\phi'(x_0) = -\beta(x_0, W_{x_0}(x_0)) + \int_{x_0}^{\infty} \beta'_w(x, W_{x_0}(x))y(x)dx$$

where $y(x) = \partial_{x_0} W_{x_0}(x)$ is the solution to (4.28).
That is

$$\partial_x y(x) = -\left(1 + r\beta'_w(x, W_{x_0}(x))\right) y(x)$$

and
$$y(x_0) = r\beta(x_0, W_{x_0}(x_0)) = r\beta(x_0, rEe^{-x_0})$$
Therefore
$$\begin{aligned}\phi'(x_0) &= -\beta(x_0, W_{x_0}(x_0)) - \frac{1}{r}\int_{x_0}^{\infty} \partial_x y(x) + y(x)dx \\ &= -\frac{1}{r}y(\infty) - \frac{1}{r}\int_{x_0}^{\infty} y(x)dx \le -\frac{1}{r}\int_{x_0}^{\infty} y(x)dx.\end{aligned}$$
As $y(x) = \partial_{x_0} W_{x_0}(x) \ge y(x_0)e^{-(1+\gamma)(x-x_0)}$ we have
$$\phi'(x_0) \le -\frac{1}{r}\frac{y(x_0)}{1+\gamma} = -\frac{\beta(x_0, W_{x_0}(x_0))}{1+\gamma}.$$
It means that the function ϕ is strictly decreasing. Since
$$\frac{1}{2}\sigma(e^{x_0}, \beta(x_0, W_{x_0}(x_0)))^2\beta(x_0, W_{x_0}(x_0)) = W_{x_0}(x_0) = rEe^{-x_0} \to +\infty$$
as $x_0 \to -\infty$. We have $\lim_{x_0\to-\infty}\beta(x_0, W_{x_0}(x_0)) = \infty$ and therefore $\lim_{x_0\to-\infty}\phi'(x_0) = -\infty$. Therefore $\phi(-\infty) = \infty$. In summary, there exists the unique root x_0 of the equation $\phi(x_0) = 1$, as claimed.

□

Now we are in a position to state our main result on unique solvability of the perpetual American put option problem (4.5)–(4.6).

Theorem 4.3.1. *[18, Theorem 1] Assume the volatility function σ satisfies Assumption 1. Then there exists the unique solution $(V(.), \varrho)$ to the perpetual American put option problem (4.5)–(4.6). The function $V(S)$ is given by*
$$V(S) = \frac{S}{r}\int_{\ln S}^{\infty} W_{x_0}(x)dx, \quad \text{for } S \ge \varrho = e^{x_0},$$
where $W_{x_0}(x)$ is the solution to the ODE (4.26) and x_0 is the unique root of the implicit equation (4.30).

Proof. Differentiating the above expression for $V(S)$ we obtain
$$\begin{aligned}\partial_S V(S) &= \frac{1}{r}\int_{\ln S}^{\infty} W_{x_0}(x)dx - \frac{1}{r}W_{x_0}(\ln S) \\ S\partial_S^2 V(S) &= -\frac{1}{r}\left(W_{x_0}(x) + \partial_x W_{x_0}(x)\right) = \beta(x, W_{x_0}(x)),\end{aligned}$$
where $x = \ln S$. Hence, assume that
$$A = \frac{1}{2}\sigma(S, S\partial_S^2 V)^2 S^2\partial_S^2 V + rS\partial_S V - rV,$$

then one can write

$$A = S\left(\frac{1}{2}\sigma(e^x, \beta(x, W_{x_0}(x)))^2\beta(x, W_{x_0}(x)) - W_{x_0}(x)\right) = 0,$$

i.e. $V(S)$ is the solution to (4.5) for $S > \varrho = e^{x_0}$. Furthermore,

$$\begin{aligned}
[V(S) - S\partial_S V(S)]_{S=\varrho} &= V(\varrho) - \frac{\varrho}{r}\int_{\ln\varrho}^{\infty} W_{x_0}(x)dx + \frac{\varrho}{r}W_{x_0}(\ln\varrho) \\
&= E\varrho e^{-\ln\varrho} \\
&= E,
\end{aligned}$$

and,

$$\begin{aligned}
V(\varrho) &= \frac{\varrho}{r}\int_{\ln\varrho}^{\infty} W_{x_0}(x)dx = \frac{\varrho}{r}\int_{\ln\varrho}^{\infty} -\partial_x W_{x_0}(x) - r\beta(x, W_{x_0}(x))dx \\
&= \frac{\varrho}{r}W_{x_0}(\ln\varrho) - \varrho\int_{\ln\varrho}^{\infty}\beta(x, W_{x_0}(x))dx = E - \varrho
\end{aligned}$$

because x_0 is the unique solution to (4.30). With regard to the equivalence (4.27) we have $\partial_S V(S) = -1$ at $S = \varrho$. In summary, $(V(.), \varrho)$ is the unique solution to the perpetual American put option problem (4.5)–(4.6). □

Remark 9. *In the case the volatility function depends on $H = S\partial_S^2 V$ only, i.e. $\sigma = \sigma(H)$, then equation (4.30) can be simplified by introducing the change of variables $w = W_{x_0}(x)$. Indeed, $\beta = \beta(w)$ and $dw = \partial_x W_{x_0}(x)dx = -(W_{x_0}(x) + r\beta(W_{x_0}(x)))dx$. Therefore*

$$\int_{x_0}^{\infty}\beta(W_{x_0}(x))dx = -\int_{W_{x_0}(x_0)}^{0}\frac{\beta(w)}{w + r\beta(w)}dw = \int_0^{\frac{rE}{\varrho}}\frac{\beta(w)}{w + r\beta(w)}dw.$$

Equation (4.30) can be rewritten in the following form

$$\int_0^{\frac{rE}{\varrho}}\frac{\beta(w)}{w + r\beta(w)}dw = 1. \qquad (4.31)$$

This is the condition for the free boundary position ϱ recently derived by the authors in [18].

4.3.1 Comparison principle and Merton's solutions

In this part our aim is to derive sub– and super–solutions to the perpetual American put option pricing problem.
Let $\gamma > 0$ be positive constant. By V_γ we will denote the explicit Merton solution, i.e.

$$V_\gamma(S) = \begin{cases} \frac{E}{1+\gamma}\left(\frac{S}{\varrho_\gamma}\right)^{-\gamma}, & S > \varrho_\gamma, \\ E - S, & 0 < S \le \varrho_\gamma, \end{cases} \tag{4.32}$$

where

$$\varrho_\gamma = E\frac{\gamma}{1+\gamma}. \tag{4.33}$$

It means that the pair $(V_\gamma(\cdot), \varrho_\gamma)$ is the explicit Merton solution corresponding to the constant volatility $\sigma_0^2 = 2r/\gamma$ (see [39]). Then, for the transformed function $U_\gamma(x)$ we have

$$U_\gamma(x) = -rS\partial_S\left(\frac{V_\gamma(S)}{S}\right) = rE\varrho_\gamma^\gamma e^{-(1+\gamma)x}, \quad \text{for } x = \ln S > x_{0\gamma} = \ln \varrho_\gamma.$$

Clearly,

$$\partial_x U_\gamma + U_\gamma + r\beta(U_\gamma) = r\beta(U_\gamma) - \gamma U_\gamma. \tag{4.34}$$

Next we will construct a super–solution to the solution U of the equation $\partial_x U_\gamma = -U_\gamma - r\beta(U_\gamma)$ by means of the Merton solution U_γ where $\gamma = \gamma^+$ is the unique root of the equation

$$\gamma^+\sigma(1+\gamma^+)^2 = 2r. \tag{4.35}$$

Since

$$U_{\gamma^+}(x) \le U_{\gamma^+}(x_{0\gamma}) = \frac{rE}{\gamma^+} = r\frac{1+\gamma^+}{\gamma^+}$$

we obtain

$$\frac{1}{2}\sigma((\gamma^+/r)U_{\gamma^+}(x))^2\frac{\gamma^+}{r}U_{\gamma^+}(x) \le \frac{1}{2}\sigma(1+\gamma^+)^2\frac{\gamma^+}{r}U_{\gamma^+}(x).$$

By taking the inverse function β we finally obtain

$$\frac{\gamma^+}{r}U_{\gamma^+}(x) \le \beta(U_{\gamma^+}(x)).$$

With regard to (4.34) we conclude that

$$\partial_x U_{\gamma^+}(x) \ge -U_{\gamma^+}(x) - r\beta(U_{\gamma^+}(x)) \quad \text{for } x > x_{0\gamma^+} = \ln \varrho_{\gamma^+} \tag{4.36}$$

Similarly, we will construct the Merton sub–solution U_{γ^-} satisfying the opposite differential inequality. Let γ^- be given by

$$\gamma^-\sigma(0)^2 = 2r, \tag{4.37}$$

i.e. $\gamma^- = 2r/\sigma(0)^2$. Then

$$U_{\gamma^-} = \frac{1}{2}\sigma(0)^2\frac{\gamma^-}{r}U_{\gamma^-} \le \frac{1}{2}\sigma((\gamma^-/r)U_{\gamma^-})^2\frac{\gamma^-}{r}U_{\gamma^-}$$

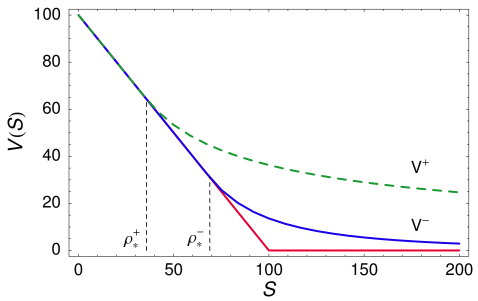


Figure 4.2: A plot of Merton solutions V^+, V^-, and the pay-off diagram $\max(E-S,0)$ corresponding to constant volatilities $\sigma^+=0.6$ and $\sigma^-=0.3$ for model parameters: $E=100, r=0.1$.

and so, by taking the inverse function β we obtain $\beta(U_{\gamma^-})\le \frac{\gamma^-}{r}U_{\gamma^-}$. Then, from (4.34) we conclude that

$$\partial_x U_{\gamma^-}(x)\le -U_{\gamma^-}(x)-r\beta(U_{\gamma^-}(x)) \quad \text{for } x>x_{0\gamma^-}=\ln\varrho_{\gamma^-}. \tag{4.38}$$

In Fig. 4.2 we plot Merton's solutions $V^\pm(\cdot)$ corresponding to $\gamma^+=0.555$ ($\sigma^+\doteq 0.6$) and $\gamma^-=2.222$ ($\sigma^-\doteq 0.3$) where $(\sigma^\pm)^2=2r/\gamma^\pm$.
In what follows, we will prove the inequalities

$$\varrho_{\gamma^+}\le\varrho\le\varrho_{\gamma^-}, \tag{4.39}$$

where ϱ is the free boundary position for the nonlinear perpetual American put option pricing problem (4.5)–(4.6).
Denote

$$\beta^-(u)=\frac{\gamma^-}{r}u$$

the inverse function to the function $H\mapsto\frac{1}{2}\sigma(0)^2H$. As $\frac{1}{2}\sigma(0)^2H\le\frac{1}{2}\sigma(H)^2H$ we have $\beta(u)\le\beta^-(u)$ for any $u\ge 0$. Since

$$\int_0^{\frac{rE}{\varrho}}\frac{\beta(u)}{u+r\beta(u)}du=1=\int_0^{\frac{rE}{\varrho_{\gamma^-}}}\frac{\beta^-(u)}{u+r\beta^-(u)}du\ge\int_0^{\frac{rE}{\varrho_{\gamma^-}}}\frac{\beta(u)}{u+r\beta(u)}du$$

we conclude the inequality $\varrho\le\varrho_{\gamma^-}$. On the other hand, let

$$\beta^+(u)=\frac{\gamma^+}{r}u$$

be the inverse function to the function $H\mapsto\frac{1}{2}\sigma(1+\gamma^+)^2H$. Then, for $u\le rE/\varrho_{\gamma^+}$ we have

$$H=\beta(u)\le\beta(rE/\varrho_{\gamma^+})=\beta(\frac{1}{2}\sigma(1+\gamma^+)^2(1+\gamma^+))=1+\gamma^+.$$

Therefore, for $u \le rE/\varrho_{\gamma^+}$ we have $\beta(u) \ge \beta^+(u)$ and arguing similarly as before we obtain the estimate $\varrho_{\gamma^+} \le \varrho$ and so the inequalities (4.39) follows. For initial conditions we have $U_{\gamma^\pm}(x_{0\gamma^\pm}) = \frac{rE}{\varrho_{\gamma^\pm}}, U(x_0) = \frac{rE}{\varrho}$ and so

$$U_{\gamma^-}(x_{0\gamma^-}) \le U(x_0) \le U_{\gamma^+}(x_{0\gamma^+}).$$

Using the comparison principle for solutions of ordinary differential inequalities we have $U_{\gamma^-}(x) \le U(x) \le U_{\gamma^+}(x)$. Taking into account the explicit form of the function $V(S)$ from Theorem 4.2.1 (see (4.20)) we conclude the following result:

Theorem 4.3.2. *[17, Theorem 3] Let $(V(\cdot), \varrho)$ be the solution to the perpetual American pricing problem (4.5)–(4.6). Then*

$$V_{\gamma^-}(S) \le V(S) \le V_{\gamma^+}(S) \quad \textit{for any} \quad S \ge 0,$$

and,

$$\varrho_{\gamma^+} \le \varrho \le \varrho_{\gamma^-}$$

where $V_{\gamma^\pm}, \varrho_{\gamma^\pm}$ are explicit Merton's solutions where $\gamma^\pm$ are given by (4.35) and (4.37).

A graphical illustration of the comparison principle is shown in Fig. 4.6.

4.4 Numerical results

In this section we propose a simple and efficient numerical scheme for constructing a solution to the perpetual put option problem (4.5)–(4.6). Using transformation $H = \beta(u)$, i.e. $u = \frac{1}{2}\sigma(H)^2 H$ and $du = \frac{1}{2}\partial_H(\sigma(H)^2 H)dH$ we can rewrite the equation for the free boundary position (see (4.19)) as follows:

$$\int_0^{\beta(rE/\varrho)} \frac{H\frac{1}{2}\partial_H(\sigma(H)^2 H)}{\frac{1}{2}\sigma(H)^2 H + rH} dH = 1. \tag{4.40}$$

Similarly, the option price (4.19) can be rewritten in terms of the H variable as follows:

$$V(S) = \frac{S}{r} \int_0^{\beta(G^{-1}(-\ln(S/\varrho)))} \frac{\frac{1}{2}\sigma(H)^2 H \frac{1}{2}\partial_H(\sigma(H)^2 H)}{\frac{1}{2}\sigma(H)^2 H + rH} dH. \tag{4.41}$$

With this transformation we can reduce computational complexity in the case when the inverse function β is not given by a closed form formula.
We present the results of numerical calculation based on the two different nonlinear models, given

- Frey model (see [14, 15, 16]) which is a nonlinear Black–Scholes model. In this model the asset dynamics takes into account the presence of feedback effects due to a large trader choosing his/her stock-trading strategy (see also [37]). The diffusion coefficient σ is non-constant:

$$\sigma^2(S\partial_S^2 V) = \sigma_0^2 \left(1 - \mu S\partial_S^2 V\right)^{-2}, \tag{4.42}$$

where σ_0^2 and μ are given positive constants.

$$\beta(H) = \frac{1}{2}\sigma_0^2 \left(1 - \mu S\partial_S^2 V\right)^{-2} H$$

The range of the parameter μ is therefore limited to satisfy the strict inequality $1 - \mu H = 1 - \mu S\partial_S^2 V(S) > 0$. However, using the identity

$$\frac{1}{1-\mu H} = 1 + \sum_{n=1}^{\infty} \mu^n H^n.$$

we can approximate the Frey volatility function as follows:

$$\sigma(H)^2 = \sigma_0^2 \left(1 + \sum_{n=1}^{N} \mu^n H^n\right)^2, \tag{4.43}$$

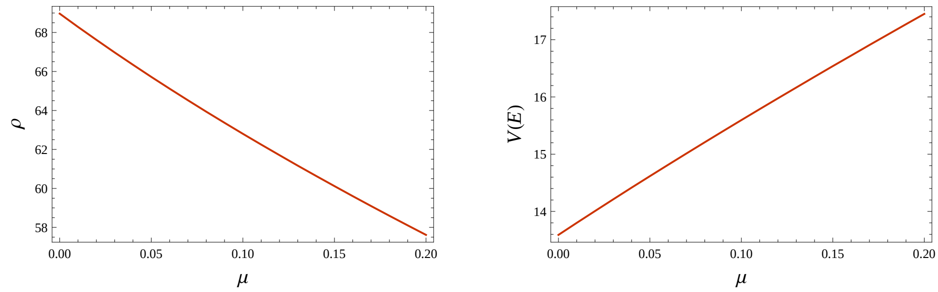
where N is sufficiently large. Interestingly, a similar power series expansion of $\sigma(H)^2$ can be found in the generalized Black–Scholes model proposed by Cetin, Jarrow and Protter [9].

- Risk Adjusted Pricing Methodology model (RAPM) proposed by Kratka [29] and revisited by Jandačka and Ševčovič [23]. In this model, the volatility is non-constant:

$$\sigma(S\partial_S^2)^2 = \sigma_0^2 \left(1 + \mu (S\partial_S^2 V)^{\frac{1}{3}}\right). \tag{4.44}$$

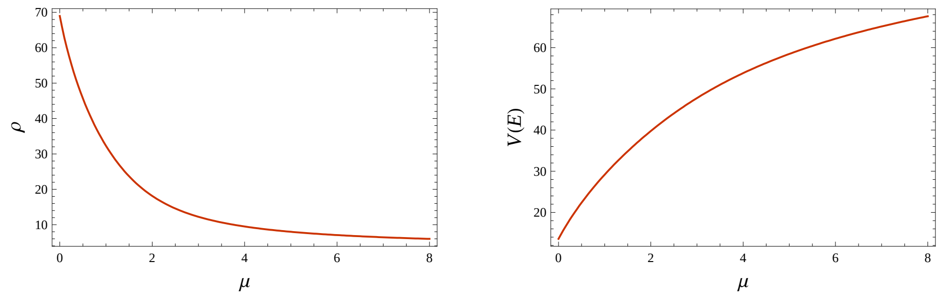
By $\sigma_0 > 0$ we denoted the constant historical volatility of the asset price returns and $\mu = 3(C^2R/2\pi)^{\frac{1}{3}}$, where $C, R \geq 0$ are nonnegative constants representing the transaction cost measure and the risk premium measure, respectively. (see [23] for more details).

Results of numerical calculation for the Frey model (4.42) and the RAPM model (4.44) are summarized in Tables 4.1 and 4.3. We show the position of the free boundary ϱ and the perpetual option value V evaluated at the exercise price $S = E$. The results are computed for various values of the parameter μ for the Frey model and the RAPM model. Other model parameter were chosen as: $E = 100, r = 0.1$ and $\sigma_0 = 0.3$.
In computations shown in Fig. 4.4 and Tab. 4.2 we present results of the free boundary position and the perpetual American put option price $V(E)$



a) b)

Figure 4.3: [17, Fig 3] A plot of dependence of the free boundary position ϱ a) and the perpetual American put option price $V(E)$ b) on the model parameter μ for the Frey model (4.42).



a) b)

Figure 4.4: [17, Fig 4] A plot of dependence of the free boundary position ϱ a) and the perpetual American put option price $V(E)$ b) on the model parameter μ for the modified Frey model (4.43).

Table 4.1: [17, Tab 1] The free boundary position $\varrho = \varrho(\mu)$ and the option price $V(S)$ evaluated at $S = E$ for various values of the model parameter $\mu \geq 0$ for the Frey model (4.42).

μ	0.00	0.01	0.05	0.10	0.15	0.20	0.22
ϱ	68.9655	68.2852	65.7246	62.8036	60.1175	57.6177	56.6627
$V(E)$	13.5909	13.8005	14.6167	15.5961	16.5389	17.4510	17.8083

Table 4.2: [17, Tab 2] The free boundary position ϱ and the option price $V(S)$ evaluated at $S = E$ for various values of the model parameter $\mu \geq 0$ for the modified Frey model.

μ	0.00	0.10	0.50	1.00	2.00	4.00	8.00
ϱ	68.9655	62.8037	45.3007	31.0862	16.3126	8.3818	5.4556
$V(E)$	13.5909	15.5961	22.4529	29.5719	41.0654	56.1777	70.2259

Table 4.3: [17, Tab 3] The free boundary position ϱ and the option price $V(S)$ evaluated at $S = E$ for various values of the model parameter $\mu \geq 0$ for the RAPM model.

μ	0.00	0.10	0.50	1.00	2.00	4.00	8.00
ϱ	68.9655	66.7331	59.6973	53.3234	44.5408	34.0899	23.6125
$V(E)$	13.5909	14.5761	17.9398	21.3434	26.6857	34.3393	44.1774

for $N = 10$ and larger interval of parameter values $\mu \in [0, 8]$. Note that the results for small values $\mu \leq 0.1$ computed from the original Frey volatility (4.42) and (4.43) are very close to each other.
In our next computational example we consider the Risk adjusted pricing methodology model (RAPM). In computations shown in Fig. 4.5, a) and Tab. 4.3 we present results of the free boundary position and the perpetual American put option price $V(E)$ for the RAPM model (see Fig. 4.5, b)). We also show comparison of the free boundary position $\varrho = \varrho(\mu)$ and its linear approximation derived in Theorem 4.2.2 (see Fig. 4.5, c)).
In the last examples shown in Fig. 4.6 we present comparison of the option price $V(S)$ and the free boundary position ϱ for the Frey model (left) and the Risk adjusted pricing methodology model (right) with closed form explicit Merton's solutions corresponding to the constant volatility.

4.5 Chapter conclusions

This chapter involves the main theoretical results of the thesis i.e. the existence and uniqueness of a solution to the price of American perpetual put option under nonlinear volatility function. Where the volatility function depends on $H = S\partial_S^2 V$ only, as the equation (4.20) in section 4.2.3. The first order expansion of the free boundary position with respect to the model parameter is also derived in equation (4.19). In section 4.3, we also extend

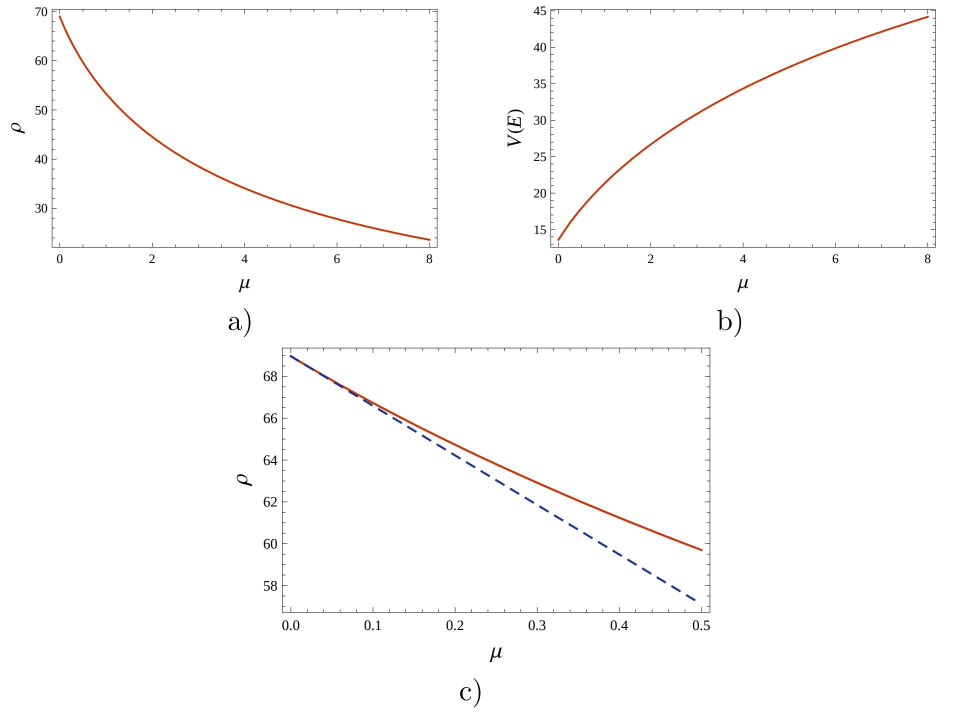


Figure 4.5: [17, Fig 5] A plot of dependence of the free boundary position ϱ a) and the perpetual American put option price $V(E)$ b) on the model parameter μ for the RAPM model (4.44). The comparison of the free boundary position and its linear approximation c).

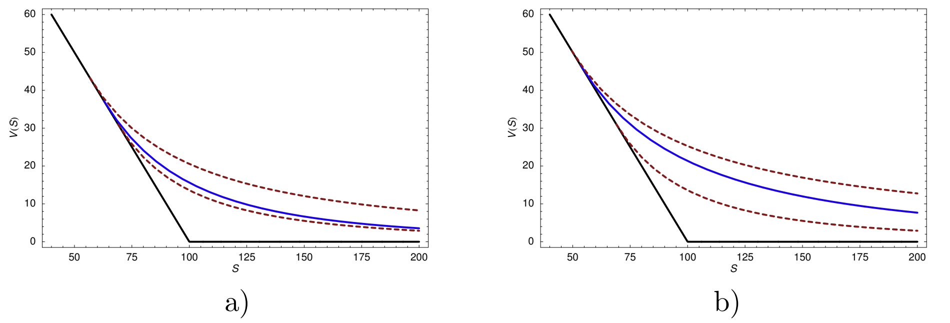


Figure 4.6: [17, Fig 6] The solid curve represents a graph of a perpetual American put option $S \mapsto V(S)$ for the Frey model a) with $\mu = 0.1$ and the RAPM model b) with $\mu = 1$. Sub- and super- solutions $V^- = V_{\gamma^-}$ and $V^+ = V_{\gamma^+}$ are depicted by dashed curves, $V^+ < V^-$. The model parameters: $E = 100, r = 0.1$ and $\sigma_0 = 0.3$.

our results in section 4.2.3 to other models like the well-known Barles and Soner [4] model where the volatility function depends on $H = S\partial_S^2 V$ as well as S. Then, we construct suitable sub– and supper–solutions based on Merton's explicit solutions with constant volatility in section 4.3.1. Finally, we present a numerical approximation scheme and computational results of the free boundary position, option price and their dependence on the model parameter.

5

Call option with transaction costs

The nonlinearity of the original Black–Scholes model can also arise from the feedback and illiquid market effects due to large traders choosing given stock-trading strategies (Schönbucher and Wilmott [38], Frey and Patie [15], Frey and Stremme [16]), imperfect replication and investor preferences (Barles and Soner [4]), risk from unprotected portfolio (Kratka [29], Jandačka and Ševčovič [23]). In this chapter we consider a new nonlinear model recently derived by Ševčovič and Žitňanská [44] for pricing call or put options in the presence of variable transaction costs. The model generalizes the well-known Leland model with constant transaction costs (c.f. [21, 35]) and Amster model [1] with linearly decreasing transaction costs. It leads to the generalized Black–Scholes equation with the nonlinear volatility function $\hat{\sigma}$ which depends on the product $H = S\partial_S^2 V$ of the underlying asset price S and the second derivative (Gamma) of the option price V:

$$\partial_t V + \frac{1}{2}\hat{\sigma}(S\partial_S^2 V)^2 S^2 \partial_S^2 V + (r-q)S\partial_S V - rV = 0, \quad V(T,S) = (S-E)^+, \tag{5.1}$$

where $r, q \geq 0$ are the interest rate and the dividend yield, respectively. The price $V(t, S)$ of a call option is given by a solution to the nonlinear parabolic equation (5.1) depending on the underlying stock price $S > 0$ at the time $t \in [0, T]$, where $T > 0$ is the time of maturity and $E > 0$ is the exercise price.

Our goal is to study American style call option which, as known, leads to a free boundary problem. Their prices can be computed by means of the generalized Black-Scholes equation with the nonlinear volatility function (5.1). If the volatility function is constant then it is well known that American options can be priced by means of a solution to a linear complementarity problem (cf. Kwok [31]). Similarly, for the nonlinear volatility model, one can construct a nonlinear complementarity problem involving the variational inequality for the left-hand side of (5.1) and the inequality $V(t, S) \geq (S - E)^+$. However, due to the fully nonlinear nature of the differential operator in (5.1), the direct computation of the nonlinear complementarity problem becomes harder and unstable. Therefore, we propose an alternative approach and reformulate the nonlinear complementarity problem in terms of the new transformed variable H for which the differential operator has the form of a quasilinear parabolic operator appearing in the left-hand side of (5.3). In order to apply the Gamma transformation for American style options we derive the nonlinear complementarity problem for the transformed variable H and we solve the variational problem by means of the modified projected successive over relaxation method (cf. Kwok [31]). Using this method we compute American style call option prices for the Black–Scholes nonlinear model for pricing call options in the presence of variable transaction costs.

We mention that most of the used results in this chapter can be found in the joint work of Grossinho, Faghan and Ševčovič [19].

5.1 Nonlinear complementarity problem (NLCP)

We recall the discussion in chapter 2 where $C : \mathbb{R}_0^+ \to \mathbb{R}$ is a measurable and bounded transaction costs function then the price of the option based on the variable transaction costs is given by the solution of the nonlinear Black–Scholes PDE (5.1) (for more details we refer to the work of Ševčovič and Žitňanská [44]). Where the nonlinear diffusion coefficient $\hat{\sigma}^2(H)$ is

$$\hat{\sigma}^2(\tau, S, H) = \sigma^2 \left(1 - \sqrt{\frac{2}{\pi}} \tilde{C}(\sigma S \,|H|\, \sqrt{\Delta t}) \frac{(H)}{\sigma \sqrt{\Delta t}} \right). \tag{5.2}$$

For European style call options various numerical methods for solving the fully nonlinear parabolic equation (5.1) were proposed and analyzed by Ďuriš, Tan and Ševčovič in [41]. Also, Ševčovič and Jandačka [42] and Žitňanská [44] investigated a new transformation technique (referred to as the Gamma transformation). They showed that the fully nonlinear parabolic equation (5.1) can be transformed to a quasilinear parabolic equation

$$\partial_\tau H - \partial_u^2 \beta(H) - \partial_u \beta(H) - (r-q)\partial_u H + qH = 0, \qquad (5.3)$$
$$\text{where} \quad \beta(H) = \frac{1}{2}\hat{\sigma}(H)^2 H,$$

of a porous-media type for the transformed quantity $H(\tau, u) = S\partial_S^2 V(t,S)$ where $\tau = T - t$, $u = \ln(S/E)$. The advantage of solving the quasilinear parabolic equation in the divergent form (5.3) compared to the fully nonlinear equation (5.1) is twofold. Firstly, from the analytical point of view, the theory of existence, uniqueness of solutions to quasilinear parabolic equation of the form (5.1) is well developed and understood. Using the general theory of quasilinear parabolic equations due to the work in Ladyženskaya, Solonnikov and Uralaceva [32], the existence of Hölder smooth solutions to (5.3) has been shown in the work of Ševčovič and Žitňanská [44]. Secondly, the quasilinear parabolic equations in the divergent form can be numerically approximated by means of the finite volume method (cf. LeVeque [35]). Furthermore, we propose the semi-implicit approximation scheme fits into a class of methods which have been shown to be of the second order of convergence (see e.g. Kilianová and Ševčovič [24]). In a series of papers [25, 26, 27, 28] Koleva and Vulkov investigated the transformed Gamma equation (5.3) for pricing European style of call and put options. They also derived the second order positivity preserving numerical scheme for solving (5.1) and (5.3). In the context of European style options the transformation method to the Gamma equation was proposed and analyzed by Jandačka and Ševčovič [23]. If we consider the generalized nonlinear Black-Scholes equation (5.3) for the European style of an option, then making the change of variables $u = \ln(S)$ and $\tau = T - t$ and computing the second derivative of equation (5.1) with respect to u, we derive the so-called Gamma equation (5.3).

Lemma 5.1.1. *Let us consider the call option with the pay-off diagram $V(T,S) = (S-E)^+$. Then the function $H(\tau,u) = S\partial_S^2 V(t,S)$ where $u = \ln(\frac{S}{E})$ and $\tau = T - t$ is a solution to (5.3) subject to the Dirac initial condition $H(0,x) = \delta(x)$ if and only if*

$$V(t,S) = \int_{-\infty}^{+\infty} (S - Ee^u)^+ H(\tau,u)du$$

is a solution to (5.1).

More details of derivation of the Gamma equation, existence and uniqueness of classical Hölder smooth solutions can be found in[44].

In this section we investigate the transformation method of a free boundary problem arising in pricing American style of options by means of a solution to the so-called Gamma variational inequality. It is well-known that pricing an American call option on an underlying stock paying continuous dividend yield $q > 0$ leads to a free boundary problem. In addition to a function $V(t,S)$, we need to find the early exercise boundary function $S_f(t)$ with respect to time $t \in [0,T]$. Furthermore, we note that the function $S_f(t)$ has the following properties:

- If $S_f(t) > S$ for $t \in [0,T]$ then $V(t,S) > (S-E)^+$.
- If $S_f(t) \leq S$ for $t \in [0,T]$ then $V(t,S) = (S-E)^+$.

In the last decades many authors analyzed the free boundary position function S_f. Stamicar, Ševčovič and Chadam [40] derived accurate approximation to the early exercise position for times t close to expiry T for the Black–Scholes model with constant volatility (see also [12, 34, 46]). The method has been generalized for the nonlinear Black-Scholes model by Ševčovič [42]. Following Kwok [31] (see also [43]) the free boundary problem for pricing the American call option consists in finding a function $V(t,S)$ and the early exercise boundary function S_f such that V solves the Black–Scholes PDE (1) on a time depending domain: $\{(t,S) : 0 < S < S_f(t)\}$ and $V(t,S_f(t)) = S_f(t) - E$, and $\partial_S V(t,S_f(t)) = 1$. Alternatively, a C_1 smooth function V is a solution to the free boundary problem for (5.1) if and only if it is a solution to the nonlinear variational inequality

$$\begin{aligned} &\partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV \leq 0, \qquad V(t,S) \geq g(S), \\ &\big(\partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV\big) \times (V-g) = 0, \end{aligned} \tag{5.4}$$

for any $S > 0$ and $t \in [0,T]$ where $g(S) \equiv (S-E)^+$.

5.1.1 Gamma transformation of the variational inequality

In this section we present a transformation technique how to transform the nonlinear complementarity problem (5.4) for the function $V(t,S)$ into the so-called Gamma variational inequality involving the transformed function $H(\tau,x)$. We need two auxiliary lemmas.

Lemma 5.1.2. *[19, Lemma 3.2] Let $V(t,S)$ be a given function. Let $u = \ln(\frac{S}{E})$, $\tau = T - t$ and define the function $Y(\tau,u)$*

$$Y(\tau,u) = \partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV.$$

Then

$$-\partial_\tau H + \partial_u \beta(H) + \partial_u^2 \beta(H) + (r-q)\partial_u H - qH = \frac{1}{E} e^{-u}[\partial_u^2 Y - \partial_u Y],$$

where $H(\tau,u) = S\partial_S^2 V(t,S)$.

Proof. By differentiating the function Y with respect to the variable u and using the fact $\partial_u = S\partial_S$, we obtain

$$\partial_u Y = \partial_t(S\partial_S V) + S(\beta + \partial_u\beta) + (r-q)SH - qS\partial_S V \quad \text{where} \quad S = Ee^u.$$

Furthermore, since

$$\begin{aligned} \partial_u^2 Y &= \partial_t(S\partial_S V + S^2\partial_S^2 V) \\ &\quad +(r-q)S(H + \partial_u H) + S(\beta + \partial_u\beta) + S(\partial_x^2\beta + \partial_u\beta) \\ &\quad -qS\partial_S V - qH, \end{aligned}$$

then

$$\partial_u^2 Y - \partial_u Y = Ee^u\Psi[H], \tag{5.5}$$

where $\Psi[H] := -\partial_\tau H + \partial_u\beta(H) + \partial_u^2\beta(H) + (r-q)\partial_u H - qH$.

□

Remark 10. *For the particular case $Y = 0$, we conclude that the function $V(t,S)$ is a solution to the European style call option satisfying the nonlinear Black-Scholes equation (5.1) if and only if the function $H(\tau,u)$ is a solution to the so-called Gamma equation*

$$-\partial_\tau H + \partial_u\beta(H) + \partial_u^2\beta(H) + (r-q)\partial_u H - qH = 0,$$

subject to the initial condition $H(x,0) = \delta(x)$, where δ is the Dirac function (cf. [42, 44]).

Lemma 5.1.3. *[19, Lemma 3.3] If we assume the asymptotic behavior*

$$\lim_{u\to-\infty} Y(\tau,u) = 0 \quad \text{and} \quad \lim_{u\to-\infty} e^{-u}\partial_u Y(\tau,u) = 0,$$

then we have

$$\begin{aligned} \int_{-\infty}^{+\infty} (S - Ee^u)^+\Psi[H](\tau,u)du &= Y(\tau,u)|_{u=\ln(S/E)} \\ &\equiv \partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV. \end{aligned}$$

Proof. Using Lemma 2 and equation (5.5) we can express the term $\int_{-\infty}^{+\infty}(S - Ee^u)^+\Psi[H](\tau,u)du$ as follows:

$$\int_{-\infty}^{+\infty}(S - Ee^u)^+\frac{1}{E}e^{-u}[\partial_u^2 Y - \partial_u Y]du =$$

$$\frac{1}{E}\int_{-\infty}^{\ln(S/E)}(Se^{-u} - E)[\partial_u^2 Y - \partial_u Y]du =$$

$$\frac{1}{E}\int_{-\infty}^{\ln(S/E)}\left[Se^{-u}\partial_u Y - (Se^{-u} - E)\partial_u Y du\right] + \underbrace{[(Se^{-u} - E)\partial_u Y]_{-\infty}^{\ln(S/E)}}_{0} =$$

$$\frac{1}{E}\int_{-\infty}^{+\infty} E\partial_u Y du = Y(\tau,u)|_{u=\ln(S/E)} =$$

$$\partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV.$$

□

Theorem 5.1.1. *[19, Theorem 3.4] The function $V(t,S)$ is a solution to the nonlinear complementarity problem (NLCP):*

$$V(t,S) \geq g(S) \quad \text{and} \quad \partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV \leq 0$$

$$\left(\partial_t V + (r-q)S\partial_S V + S\beta(S\partial_S^2 V) - rV\right) \times (V-g) = 0$$

for any $S > 0$ and $t \in [0,T]$ where $g(S) \equiv (S-E)^+$ if and only if the following Gamma variational inequality and complementarity constraint:

$$\int_{-\infty}^{+\infty} (S - Ee^u)^+ [-\Psi[H](\tau,u)] du \geq 0, \tag{5.6}$$

$$\int_{-\infty}^{+\infty} (S - Ee^u)^+ H(\tau,u) du \geq g(S), \tag{5.7}$$

$$\int_{-\infty}^{+\infty} (S - Ee^u)^+ \Psi[H](\tau,u)) du \times f(S,H) = 0. \tag{5.8}$$

where $f(S,H) = \int_{-\infty}^{+\infty} (S-Ee^u)^+ H(\tau,u)du - g(S)$. It hold for any $S \geq 0$ and $\tau \in [0,T]$.

Proof. It directly follows by applying Lemma 5.1.2 and Lemma 5.1.3. □

Remark 11. *[19, Remark 3.5] For calculating $V(T,S)$ in Theorem 5.1.1 we use the fact that $H(0,u) = \bar{H}(u), u \in \mathbb{R}$, where $\bar{H}(u) := \delta(u)$ is the Dirac delta function such that*

$$\int_{-\infty}^{+\infty} \delta(u) du = 1, \qquad \int_{-\infty}^{+\infty} \delta(u-u_0)\phi(u) du = \phi(u_0),$$

for any continuous function ϕ.
In what follows, we will approximate the initial Dirac δ-function can be approximated as follows:

$$H(x,0) \approx f(d)/(\hat{\sigma}\sqrt{\tau^*}),$$

where $0 < \tau^* \ll 1$ is a sufficiently small parameter, $f(d)$ is the PDF function of the normal distribution, that is: $f(d) = e^{-d^2/2}/\sqrt{2\pi}$ and $d = \left(x + (r-q-\sigma^2/2)\tau^*\right)/\sigma\sqrt{\tau^*}$. Note that this approximation follows from observation that for a solution of the linear Black–Scholes equation with a constant volatility $\sigma > 0$ at the time $T - \tau^*$ close to expiry T the value $H^{lin}(x,\tau^*) = S\partial_S^2 V^{lin}(S, T-\tau^*)$ is given by $H^{lin}(x,\tau^*) = f(d)/(\hat{\sigma}\sqrt{\tau^*})$. Moreover, $H^{lin}(.,\tau^*) \to \delta(.)$ as $\tau^* \to 0$ in the sense of distributions.

5.2 Solving the Gamma variational inequality

According to Theorem 5.1.1, the American call option problem can be rewritten in terms of the function $H(\tau, u)$ in the form of the Gamma variational inequality (5.6)–(5.7) with the complementarity constraint (5.8).
In order to apply the Projected Successive Over Relaxation method (PSOR) (c.f. Kwok [31]) to the variational inequalities (5.6)–(5.7), we need first to discretize the nonlinear operator Ψ:

$$-\Psi[H] \equiv \partial_\tau H - (r-q)\partial_u H - \partial_u \beta(H) - \partial_u^2 \beta(H) + qH. \tag{5.9}$$

In the next, we follow the paper by Ševčovič and Žitňanská [44] in order to derive an efficient numerical scheme for solving the Gamma variational inequality for a general form of the function $\beta(H)$ including the special case of the variable transaction costs model.

5.2.1 Numerical scheme

The proposed numerical discretization is based on the finite volume method. Assume that the spatial interval belongs to $u \in (-L, L)$ for sufficiently large $L > 0$ where the time interval $[0, T]$ is uniformly divided with a time step $k = \frac{T}{m}$ into discrete points $\tau_j = jk$ for $j = 1, 2, \cdots, m$. Furthermore, we divide the spatial interval $[-L, L]$ into a uniform mesh of discrete points $u_i = ih$ where $i = -n, \cdots, n$ with a spatial step $h = \frac{L}{n}$.
The discretization of the operator $\Psi[H]$ leads to a tridiagonal matrix multiplied by the vector $H^j = (H^j_{-n+1}, \cdots, H^j_{n-1}) \in \mathbb{R}^{2n-1}$. More precisely, the vector $\Psi[H]^j$ at the time level τ_j is given by $\Psi[H]^j = -(A^j H^j - d^j)$ where the $(2n-1)\times(2n-1)$ matrix A^j has the form

$$A^j = \begin{pmatrix} b^j_{-n+1} & c^j_{-n+1} & 0 & \cdots & 0 \\ a^j_{-n+2} & b^j_{-n+2} & c^j_{-n+2} & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \cdots & a^j_{n-2} & b^j_{n-2} & c^j_{n-2} \\ 0 & \cdots & 0 & a^j_{n-1} & b^j_{n-1} \end{pmatrix} \tag{5.10}$$

with coefficients of the form:

$$\begin{aligned} a^j_i &= -\frac{k}{h^2}\beta'(H^{j-1}_{i-1}) + \frac{k}{2h}(r-q), \\ c^j_i &= -\frac{k}{h^2}\beta'(H^{j-1}_{i}) - \frac{k}{2h}(r-q), \\ b^j_i &= (1+kq) - (a^j_i + c^j_i), \end{aligned}$$

and

$$d^j_i = H^{j-1}_i + \frac{k}{h}\left(\beta(H^{j-1}_i) - \beta(H^{j-1}_{i-1})\right).$$

Finally, using numerical integration the variational inequality (5.6)–(5.7) can be discretized as follows:

$$V(S, T-\tau_j) = h\sum_{i=-n}^{n}(S - Ee^{u_i})^+ H_i^j, \quad j = 1, 2, \cdots, m. \tag{5.11}$$

Then, the full space-time discretized version of the problem (5.6)–(5.7) is given by

$$h\sum_{i=-n}^{n}(S - Ee^{u_i})^+ \left[(A^j H^j)_i - d_i^j\right] \geq 0, \tag{5.12}$$

$$h\sum_{i=-n}^{n}(S - Ee^{u_i})^+ H_i^j \geq g(S) \equiv (S-E)^+. \tag{5.13}$$

Let us assume that

$$P_{li} = h[\max(S_l - Ee^{u_i}, 0)] = h[\max(Ee^{v_l} - Ee^{u_i}, 0)] \tag{5.14}$$

where

$$v_l = \frac{u_{l+1} + u_{l-1}}{2}, \quad \text{for } l = -n, \cdots, n.$$

Remark 12. *The matrix $P = (P_{li})$ is invertible.*

5.2.2 Applying the PSOR method

In this section, our aim is to solve the problem (5.12)–(5.13) by means of the PSOR method. Then, according to (5.14), we can rewrite (5.12)–(5.13) for the American call option in this form

$$\begin{aligned} PAH &\geq Pd \\ PH &\geq g \\ (PAH - Pd)_i(PH - g)_i &= 0, \quad \text{for all } i, \end{aligned}$$

where $A = A^j$, $g_i = (S_i - E)^+$ and $H = H^j$.
This nonlinear complementarity problem can be solved by the PSOR algorithm, given by the following iterative scheme:

1. for $k = 0$ set $v^{j,k} = v^{j-1}$,

2. until $k \leq k_{max}$ repeat:

$$\begin{aligned} t_i^{j,k+1} &= \frac{1}{\tilde{A}_{ii}}\left(-\sum_{l<i}\tilde{A}_{il}v_l^{j,k+1} - \sum_{l>i}\tilde{A}_{il}v_l^{j,k} + \tilde{d}_i^j\right), \\ v_i^{j,k+1} &= \max\left\{v_i^{j,k} + \omega(t_i^{j,k+1} - v_i^{j,k}),\, g_i\right\}, \end{aligned}$$

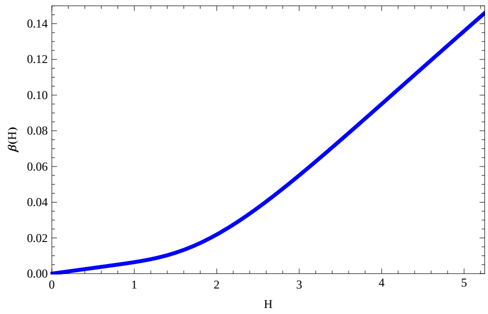


Figure 5.1: [19, Fig 2] A graph of the function $\beta(H)$ related to the piecewise linear decreasing transaction costs function (see [23]).

3. set $v^j = v^{j,k+1}$,

for $i = -n, \cdots, n$ and $j = 1, 2, \cdots, m$, where $v^j = PH^j$, $\tilde{d^j} = Pd^j$ and $\tilde{A} = PA^jP^{-1}$. Here $\omega \in [1, 2]$ is a relaxation parameter which can be tuned in order to speed up convergence process.
Finally, using the value $H^j = P^{-1}v^j$ and equation (5.11), we can evaluate the option price.
The full description of new PSOR method for calculating American call option can be seen in algorithm 1.

5.2.3 Numerical results

In this section, we focus our attention on numerical experiments for computing American style call option prices based on the nonlinear Black–Scholes equation that includes a piecewise linear decreasing transaction costs function. In Fig. 5.1, we show the corresponding function $\beta(H)$ given by

$$\beta(H) = \frac{\sigma^2}{2}\left(1 - \sqrt{\frac{2}{\pi}}\tilde{C}(\sigma|H|\sqrt{\Delta t})\frac{\mathrm{sgn}(H)}{\sigma\sqrt{\Delta t}}\right)H,$$

where $\tilde{C}$ is the mean value modification of the transaction costs function.
The parameters $C_0, \kappa, \xi_\pm, \Delta t$ characterizing the nonlinear piecewise linear variable transaction cost function and other model parameters are given in Table 5.1. Here Δt is the time interval between two consecutive portfolio rearrangements, the maturity time T, the historical volatility σ, the dividend yield q, the strike price E and r is the risk free interest rate. The small parameter $0 < \tau^* \ll 1$ represents the smoothing parameter for approximation of the Dirac δ function.
For the given numerical parameters in Table 5.1, we computed option values V_{vtc} for several underlying asset prices S. The prices were calculated by numerical solutions for both Bid and Ask option prices in Table 5.2. The

Table 5.1: [19, Tab 1] Model and numerical parameter values for calculation of numerical experiments.

Model parameters	Numerical parameters
$C_0 = 0.02$	m=200, 800
$\kappa = 0.3, \xi_- = 0.05\xi_+ = 0.1$	n=250, 500
$\Delta t = 1/261$	h=0.01
$\sigma = 0.3$	$\tau^* = 0.005$
$r = 0.011, q = 0.008$	$k = T/m$
$T = 1, E = 50$	$L = 2.5$

Bid price $V_{Bid_{vtc}}$ is compared to the price V_{BinMin} computed by means of the binomial tree method (cf. [31]) with the lower volatility $\hat{\sigma}^2_{min} = \sigma^2(1 - C_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}})$, whereas the upper bound price V_{BinMax} corresponds to the solution with the higher constant volatility $\hat{\sigma}^2_{max} = \sigma^2(1 - \underline{C_0}\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}})$. Similarly, as well as for the Ask price $V_{Ask_{vtc}}$ the lower bound V_{BinMin} corresponds to the solution of the binomial tree method with the lower volatility $\hat{\sigma}^2_{min} = \sigma^2(1 + \underline{C_0}\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}})$, whereas the upper bound V_{BinMax} corresponds to the solution with the higher constant volatility $\hat{\sigma}^2_{max} = \sigma^2(1 + C_0\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}})$.

Remark 13. *In the case of a European style option, it can be shown analytically by using the parabolic comparison principle that*

$$V_{\sigma_{min}}(S,t) \leq V_{vtc}(t,S) \leq V_{\sigma_{max}}(t,S), \quad S > 0, t \in [0,T].$$

For more details we refer to the work in [44]. For the case of American style options, these inequalities can be observed in Table 5.2.

In Table 5.3, we present a comparison of results obtained by our method based on the Gamma equation in which we considered constant volatilities σ_{min} and σ_{max} and those obtained by the well-known method based on binomial trees for American style call options (cf. [31]). The difference in the prices is in the order of the mesh size $h = L/n$.

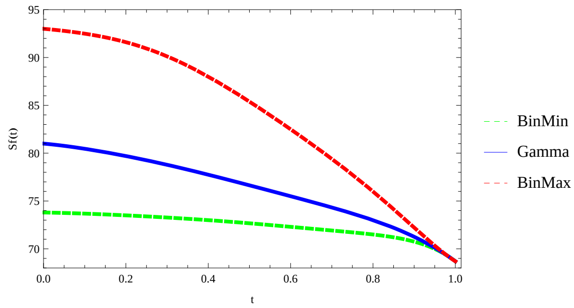
In Fig. 5.2 we present the free boundary function $S_f(t)$ obtained by our method with variable transaction costs function for bid option value compared to the binomial trees with $\sigma_{min}, \sigma_{max}$ in which parameter values are given by $E = 50, \sigma = 0.3, r = 0.011, q = 0.008, T = 1$. In Fig. 5.3 we plot the graphs of the solutions $V_{vtc}(t,S)$ at $t = 0$ for both bid and ask prices. We also plot the prices obtained by the binomial tree method with the constant lower volatility σ_{min} and the higher volatility σ_{max}, respectively.

Table 5.2: [19, Tab 2] Bid (top table) and Ask (bottom table) American call option prices $V_{Bid_{vtc}}$ and $V_{Ask_{vtc}}$ obtained from the numerical solution of the nonlinear model with variable transaction costs for different meshes. Comparison to the option prices V_{BinMin} and V_{BinMax} computed by means of binomial trees for constant volatilities σ_{min} and σ_{max}.

	$n = 250, m = 200$				$n = 500, m = 800$		
S	V_{BinMin}	$V_{Bid_{vtc}}$	V_{BinMax}	S	V_{BinMin}	$V_{Bid_{vtc}}$	V_{BinMax}
40	0.0320	0.0513	1.3405	40	1.4511	1.6594	2.8670
42	0.1075	0.3252	1.8846	42	2.0137	2.3869	3.6039
44	0.2901	0.8232	2.5527	44	2.6979	3.2309	4.4371
46	0.6535	1.5097	3.3483	46	3.5064	4.1868	5.3645
48	1.2675	2.3859	4.2711	48	4.4382	5.2488	6.3833
50	2.1740	3.4244	5.3175	50	5.4897	6.4133	7.4889
52	3.3738	4.6126	6.4817	52	6.6553	7.6764	8.6772
54	4.8304	5.9521	7.7555	54	7.9270	9.0342	9.9423
56	6.4862	7.4377	9.1295	56	9.2959	10.4824	11.2798
58	8.2809	9.0643	10.5943	58	10.7532	12.0179	12.6832
60	10.1635	10.8273	12.1397	60	12.2892	13.6385	14.1481

	$n = 250, m = 200$				$n = 500, m = 800$		
S	V_{BinMin}	$V_{Ask_{vtc}}$	V_{BinMax}	S	V_{BinMin}	$V_{Ask_{vtc}}$	V_{BinMax}
40	1.4511	1.6594	2.8670	40	1.4420	1.6692	2.8519
42	2.0137	2.3869	3.6039	42	2.0027	2.3945	3.5870
44	2.6979	3.2309	4.4371	44	2.6851	3.2412	4.4187
46	3.5064	4.1868	5.3645	46	3.4922	4.2134	5.3450
48	4.4382	5.2488	6.3833	48	4.4231	5.2601	6.3627
50	5.4897	6.4133	7.4889	50	5.4742	6.4300	7.4678
52	6.6553	7.6764	8.6772	52	6.6395	7.6922	8.6557
54	7.9270	9.0342	9.9423	54	7.9115	9.2167	9.9211
56	9.2959	10.4824	11.2798	56	9.2812	11.0264	11.2586
58	10.7532	12.0179	12.6832	58	10.7393	12.2017	12.6628
60	12.2892	13.6385	14.1481	60	12.2763	13.6505	14.1283

Figure 5.2: [19, Fig 3] An early exercise boundary function $S_f(t), t \in [0, T]$, computed for the model with variable transaction costs (dashed line Gamma) and comparison with early exercise boundary computed by means of binomial trees with constant volatilities σ_{min} (bottom curve) and σ_{max} (top curve).



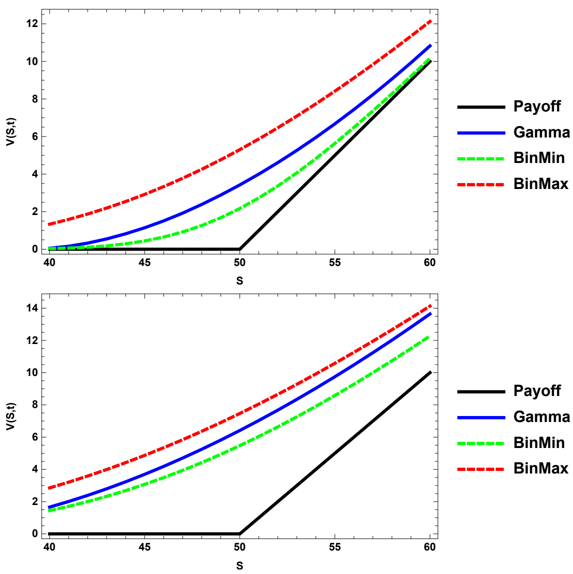


Figure 5.3: [19, Fig 4] The American Bid (top) and Ask (bottom) call option price $V(t, S)$ at $t = 0$ computed by means of the nonlinear Black-Scholes model with variable transaction costs with the mesh size $n = 500, m = 800$ in comparison to solutions $V_{\sigma_{min}}, V_{\sigma_{max}}$ calculated by the binomial trees with constant volatilities σ_{min} and σ_{max}.

Algorithm 1 New PSOR algorithm to compute American call option

Initialization:

$\sigma, \quad r, \quad q, \quad \tau^*, \quad T, \quad X, \quad \omega, \quad \gamma, \quad h;$

$C_0, \quad \kappa, \quad \xi_-, \quad \xi_+, \quad N, \quad M_{iter};$

$\alpha = \frac{r-q}{\sigma^2} - \frac{1}{2}, \quad \beta = \frac{r+q}{2} + \frac{\sigma^2}{8} + \frac{(r-q)^2}{2\sigma^2}, \quad k = \frac{T}{2N-1};$

$H(x) = \frac{e^{-((x+(r-q-\sigma^2/2)\tau^*)/(\sigma\sqrt{\tau^*})^2/2}}{(\sigma\sqrt{\tau^*}\sqrt{2\pi})}.$

Computing beta function:

Define $\tilde{C}(\xi) \leftarrow$ Modified variable transaction costs function

$\beta(H) = \frac{\sigma^2}{2}\left(1 + \frac{\tilde{C}\left(\sigma|H|\sqrt{\Delta t}\right)}{\sigma\sqrt{\Delta t}}\right)H.$

$\beta_H(H) \leftarrow$Derivative of β with respect to H

$\beta_x(H) \leftarrow$Derivative of β with respect to x

$g(x,\tau) = e^{\alpha x + \beta\tau}\max(e^x - 1, 0) \leftarrow$Payoff function

$u^{(0)} = (g(x_0,0), g(x_1,0), ..., g(x_{2N-1},0));$

$P = h\max[X(e^{\tau_j} - e^{x_i}), 0];$

MAIN LOOP

for j=1,...,2N-1 **do**

$\bar{A}$=Tridiag(a,b,c) where:

$a_i^j = -\frac{k}{h^2}\beta_H(H_{i-1}^{j-1}, x_{i-1}, \tau_{j-1}) + \frac{k}{2h}r;$

$c_i^j = -\frac{k}{h^2}\beta_H(H_i^{j-1}, x_i, \tau_{j-1}) - \frac{k}{2h}r;$

$b_i^j = 1 - (a_i^j + b_i^j).$

$d_i^j = H_i^{j-1} + \frac{k}{h}(\beta(H_i^{j-1}, x_i, \tau_{j-1}) - \beta(H_{i-1}^{j-1}, x_{i-1}, \tau_{j-1}))$

$+\beta_x(H_i^{j-1}, x_i, \tau_{j-1}) - \beta_x(H_{i-1}^{j-1}, x_{i-1}, \tau_{j-1}))).$

$gvec = P^{-1} * g(x, \tau_j).$

  for p=1,...,M_{iter} **do**

    for s=1,...,$2N-1$ **do**

$unew_s^j = (1-\omega)u_s^{j-1} + (\omega/\bar{A}_{(s,s)})(d_i^j + u_s^{j-1} - \bar{A}_{s,s-1}u_{s-1}^{j-1} - \bar{A}_{s,s+1}u_{s+1}^{j-1});$

$u^j = \max(unew^j, gvec(s)).$

    end for

  end for

$H_{sol} = u^j.$

end for

final result:

$V(S) = h\sum_{j=1}^{2N-1}\max(S - Xe^{x_j}, 0)H_{sol}(j)$ =0

Table 5.3: [19, Tab 3] Ask call option values $V_{Ask_{vtc}}$ of the numerical solution of the model under constant volatilities $\sigma = \sigma_{min}$ (left) and $\sigma = \sigma_{max}$ (right) and comparison to the prices computed by the Binomial tree method (with $n = 100$ and $n = 200$ nodes, respectively.

	$\sigma = \sigma_{min}$					$\sigma = \sigma_{max}$			
	$n = 250, m = 200$		$n = 500, m = 800$			$n = 250, m = 200$		$n = 500, m = 800$	
S	$V_{Ask_{vtc}}$	V_{BinMin}	$V_{Ask_{vtc}}$	V_{BinMin}	S	$V_{Ask_{vtc}}$	V_{BinMax}	$V_{Ask_{vtc}}$	V_{BinMax}
40	1.4737	1.4511	1.4634	1.4420	40	2.8827	2.8670	2.8663	2.8519
42	2.2417	2.0137	2.110	2.002	42	3.6273	3.6039	3.5923	3.5870
44	2.7156	2.6979	2.7025	2.6851	44	4.4618	4.4371	4.4067	4.4187
46	3.5287	3.5064	3.5193	3.4922	46	5.3945	5.3645	5.3561	5.3450
48	4.4572	4.4382	4.4498	4.4231	48	6.4095	6.3833	6.3515	6.3627
50	5.5019	5.4897	5.4996	5.4742	50	7.5002	7.4889	7.4710	7.4678
52	6.6993	6.6553	6.6684	6.6395	52	8.7049	8.6772	8.6682	8.6557
54	7.9537	7.9270	7.9350	7.9115	54	9.9765	9.9423	9.9326	9.9211
56	9.3367	9.2959	9.3145	9.2812	56	11.3071	11.2798	11.2742	11.2586
58	10.8015	10.7532	10.7683	10.7393	58	12.7103	12.6832	12.6790	12.6628
60	12.3369	12.2892	12.3189	12.2763	60	14.1640	14.1481	14.1374	14.1283

5.3 Chapter conclusions

This chapter brings the main contribution in the form of a solution to the problem of pricing American call options dealing with nonlinear Black–Scholes equation with variable transaction costs function for the trading underlying assets. The mathematical model is constructed by the fully nonlinear parabolic equation when the nonlinear diffusion coefficient may depends on the second derivative of theoption price. Furthermore, section 5.1 devoted to transform the nonlinear complementarity problem into the so called Gamma equation based on PSOR approach [19]. Finally, section 5.2 is fully described a finite volume discretization of the complementarity problem and its solution obtained by means of the projected super over relaxation (PSOR) approach following the work in [44] and then expressed results of various numerical experiments for pricing American style of call options, the early exercise boundary position and comparison with models with constant volatility terms.

6

Conclusions

In this dissertation we analysed recent topics on pricing American style options and established qualitative and quantitative results in presence of transaction costs. We considered a nonlinear generalization of the classical Black–Scholes model in which the diffusion coefficient is no longer constant but it depends on the product of the underlying asset price and the second derivative of the option price. Mathematically, we had to investigate a free boundary problem for a fully nonlinear parabolic equation. After having reviewed, in the chapters 2 and 3, some basic concepts and results of stochastic processes and of the theory of models with variable transaction costs (see [45]), we adressed the study of American-style options in presence of variable transactions costs, considering put and call types. In Chapter 4, we investigated the qualitative and quantitative behaviour of a solution to the problem of pricing American style perpetual put options which was based on the study of a stationary generalized Black–Scholes equation with a nonlinear volatility function. We proved existence and uniqueness of a solution to the associated free boundary problem. More precisely, we derived a single implicit integral equation for the free boundary position and a closed form formula for the option price, which is a generalization of the well-known explicit closed form solution derived by Merton [39] for the case of constant volatility. We also presented results of numerical computations for the free boundary position and option price, including their dependence on model parameters (as can be seen in Grossinho, Faghan and Sevcovic [17, 18]). In Chapter 5, we were concerned with pricing American-style call options, with nonlinear volatility, which generalizes the well-known Leland model with constant transaction costs (c.f. [21, 35]) and the Amster model

[1] with linearly decreasing transaction costs.. We analysed a corresponding nonlinear generalization of the Black–Scholes equation. Since the direct computation of the nonlinear complementarity problem became harder and unstable due to the fully nonlinear nature of the differential operator that appears in the model, we proposed a new approach to reformulate the nonlinear complementarity problem in terms of the new transformed variable. With that, the differential operator acquired the form of a quasilinear parabolic operator. We derived the nonlinear complementarity problem for the transformed variable in order to apply the Gamma transformation for American style options. We then solve the variational problem by means of the modified projected successive over relaxation (PSOR) (cf. Kwok [31]), the main numerical approach used in this thesis, for constructing an effective numerical scheme for discretization of the Gamma variational inequality. We illustrated our study by presenting several computational examples of the nonlinear Black–Scholes equation for pricing American-style call options in the presence of variable transaction costs (as can be seen in Grossinho, Faghan and Sevcovic [19]). As we saw, studying american options in presence of transaction costs led to nonlinear models of the Black–Scholes type, for which is in general difficult to find an explicit solution. However, in Chapter 4 we derived a single implicit integral equation for the free boundary position and a closed form formula for the option price of an American perpetual put option. As an extension of this result, we can consider using the method of lines to study the general problem, once we have already important information for the stationary problem. Both analytical and numerical results are worth being investigated. Generically, for problems of the types considered in this dissertation, other numerical schemes could be considered. The work of Casabán, Jódar and Pintos [8], Cheng-hu and Zhou [10], and Kútik and Mikula [30], for other schemes, as well as from Bordag and Frey [7], for some some explicit solutions for special type of nonlinear models, can be good references to support the extension of the results of this dissertation. Considering other types of financial derivatives, as for example exotic options, can also be a future fruitful line of research. We believe that the original results we present in this dissertation not only contribute positively to the recent investigation that aims to overcame the drawbacks that appear when modelling financial markets by the classical Black–Scholes model but also are worth being considered for future investigation.

Acknowledgements

First of all, I want to thank my PhD supervisors Prof. Maria do Rosario Grossinho and Prof. Daniel Sevcovic for the patience they had with me, for the useful discussions and the many emails we have exchanged in recent years. I would like to express my thanks to Prof. Fernando Goncalves, who gave me the opportunity to start this doctoral program together with the European network STRIKE `http://www.itn-strike.eu/` and FCT scolarship. Last, but not least, I want to thank my family and my friends Gilson Silva, Nicola Cantarutti and my lovely landlord Natalia de Carvaliho in Lisbon, for their never-ending support. This research was supported by the European Union in the FP7-PEOPLE-2012-ITN project STRIKE - Novel Methods in Computational Finance (304617), and by CEMAPRE MULTI/00491, financed by FCT/MEC through Portuguese national funds.

This was remove recently! (it is for Appendix part)

Bibliography

[1] P. Amster, C. Averbuj, M. Mariani, and D. Rial. A black–scholes option pricing model with transaction costs. *J. Math. Anal. Appl.*, 303:688–695, 2005.

[2] M. Avellaneda and A. Paras. Dynamic hedging portfolios for derivative securities in the presence of large transaction costs. *Applied Mathematical Finance*, 1:165–193, 1994.

[3] D. Bakstein and S. Howison. A non–arbitrage liquidity model with observable parameters. *Working paper*, 2004.

[4] G. Barles and H. Soner. Option pricing with transaction costs and a nonlinear black–scholes equation. *Finance and Stochastics*, 2:369–397, 1998.

[5] T. Bjork. Arbitrage theory in continuous time. *Oxford Finance; New York.*, 2004.

[6] F. Black and M. Scholes. The pricing of options and corporate liabilities. *J. Political Economy*, 81:637–654, 1973.

[7] L. Bordag and R. Frey. Pricing options in illiquid markets: Symmetry reductions and exact solutions. *Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing*, page 83–109, 2008.

[8] M. Casabán, R. Company, L. Jódar, and J. Pintos. Numerical analysis and computing of a non-arbitrage liquidity model with observable parameters for derivatives. *Computers an Mathematics with Applications*, 61(32):1951–1956, 2011.

[9] U. Cetin, R. Jarrow, and P. Protter. Liquidity risk and arbitrage pricing theory. *Finance and Stochastics*, 8(3):311–441, 2004.

[10] N. Cheng-hu and S. Zhou. Numerical solution of non-arbitrage liquidity model based on uncertain volatility. *Journal of East China Normal University (Natural Science) 1*, 2012.

[11] J. Dewynne, S. Howison, J. Rupf, and P. Wilmott. Some mathematical results in the pricing of american options. *Euro. J. Appl. Math.*, 4:381–398, 1993.

[12] J. Evans, R. Kuske, and J. Keller. American options on assets with dividends near expiry. *Mathematical Finance*, 12(3):219–237, 2002.

[13] F. Fabiao and M. R. Grossinho. Positive solutions of a dirichlet problem for a stationary nonlinear black-scholes equation. *Nonlinear Analysis: Theory, Methods and Applications*, 71(10):4624–4631, 2009.

[14] R. Frey. Perfect option hedging for a large trader. *Finance and Stochastics*, 2:115–142, 1998.

[15] R. Frey and P. Patie. Risk management for derivatives in illiquid markets: A simulation study. *Advances in Finance and Stochastics*, pages 137–159, 2002.

[16] R. Frey and A. Stremme. Market volatility and feedback effects from dynamic hedging. *Mathematical Finance*, 4:351–374, 1997.

[17] M. R. Grossinho, Y. Faghan, and D. Ševčovič. Analytical and numerical results for american style of perpetual put options through transformation into nonlinear stationary black-scholes equations. *Novel Methods in Computational Finance, Springer International Publishing*, 25(3):129–142, 2017.

[18] M. R. Grossinho, Y. Faghan, and D. Ševčovič. Pricing perpetual put options by the black-scholes equation with a nonlinear volatility function. *Asia-Pacific Financial Markets*, 24(4):291–308, 2017.

[19] M. R. Grossinho, Y. Faghan, and D. Ševčovič. Pricing american call options using the black-scholes equation with a nonlinear volatility function. *Journal of Computational Finance*, 23(4):291–308, 2020.

[20] M. R. Grossinho and E. Morais. A note on a stationary problem for a black-scholes equation with transaction costs. *International Journal of Pure and Applied Mathematics*, 51:557–565, 2009.

[21] T. Hoggard, A. Whalley, and P. Wilmott. Hedging option portfolios in the presence of transaction costs. *Advances in Futures and Options Research*, 7:21–35, 1994.

[22] J. Hull. Options, futures and other derivative securities. *Prentice Hall*, 1989.

[23] M. Jandačka and D. Ševčovič. On the risk adjusted pricing methodology based valuation of vanilla options and explanation of the volatility smile. *Journal of Applied Mathematics*, 3:235–258, 2005.

[24] S. Kilianová and D. Ševčovič. A transformation method for solving the hamilton-jacobi-bellman equation for a constrained dynamic stochastic optimal allocation problem. *ANZIAM Journal*, 55:14–38, 2013.

[25] M. Koleva. Efficient numerical method for solving cauchy problem for the gamma equation. *AIP Conference Proceedings*, 1410:120–127, 2011.

[26] M. Koleva and L. Vulkov. A second-order positivity preserving numerical method for gamma equation. *Applied Mathematics and Computation*, 220:722–734, 2013.

[27] M. Koleva and L. Vulkov. On splitting-based numerical methods for nonlinear models of european options. *International Journal of Computer Mathematics*, 93(5):781–796, 2016.

[28] M. Koleva and L. Vulkov. Computation of delta greek for non-linear models in mathematical finance. *Numerical Analysis and Its Applications*, 10187:430–438, 2017.

[29] M. Kratka. No mystery behind the smile. *Risk*, 9:67–71, 1998.

[30] P. Kútik and K. Mikula. Finite volume schemes for solving nonlinear partial differential equations in financial mathematics. *In Proceedings of the Sixth International Conference on Finite Volumes in Complex Applications, Prague*, 4(9):643–561, 2011.

[31] Y. Kwok. Mathematical models of financial derivatives. *Springer-Verlag*, 1998.

[32] O. Ladyženskaya, V. Solonnikov, and N. Uralaceva. Linear and quasilinear equations of parabolic type. *American Mathematical Society, Providence*, 23, 1968.

[33] D. Lamberton and B. Lapeyre. Introduction to stochastic calculus applied to finance. *Chapman and Hall, UK*, 1996.

[34] M. Lauko and D. Ševčovič. Comparison of numerical and analytical approximations of the early exercise boundary of american put options. *ANZIAM journal*, 51:430–448, 2011.

[35] H. Leland. Option pricing and replication with transaction costs. *Journal of Finance*, 40:1283–1301, 1985.

[36] N. Privault. An elementary introduction to stochastic interest rate modeling advanced series on statistical science and applied probability. *Springer*, 2012.

[37] P. Schönbucher and P. Wilmott. The feedback-effect of hedging in illiquid markets. *SIAM Journal of Applied Mathematics*, 61:232–272, 2000.

[38] P. Schönbucher and P. Wilmott. The feedback-effect of hedging in illiquid markets. *SIAM Journal of Applied Mathematics*, 61:232–272, 2000.

[39] S. Shreve. Stochastic calculus for finance ii, continuous-time models. *Springer*, 2003.

[40] R. Stamicar, D. Ševčovič, and J. Chadam. The early exercise boundary for the american put near expiry: numerical approximation. *Canad. Appl. Math. Quarterly*, 7:427–444, 1999.

[41] K. Ďuriš, S.-H. Tan, C.-H. Lai, and D. Ševčovič. Comparison of the analytical approximation formula and newton's method for solving a class of nonlinear black-scholes parabolic equations. *Int. J. Appl. Theor. Finance*, 16(1):35–50, 2016.

[42] D. Ševčovič. An iterative algorithm for evaluating approximations to the optimal exercise boundary for a nonlinear black-scholes equation. *Canad. Appl. Math. Quarterly*, 15:77–97, 2007.

[43] D. Ševčovič, B. Stehlíková, and K. Mikula. Analytical and numerical methods for pricing financial derivatives. *Nova Science Publishers, Inc., Hauppauge*, 2011.

[44] D. Ševčovič and M. Žitňanská. Analysis of the nonlinear option pricing model under variable transaction costs. *Asia-Pacific Financial Markets*, 23(2):153–174, 2016.

[45] M. Žitňanská. Qualitative and quantitative analysis of black-scholes type models of pricing derivatives on assets with general function of volatility. *PhD thesis, Comenius University, Faculty of Mathematics*, 2014.

[46] S.-P. Zhu. A new analytical approximation formula for the optimal exercise boundary of american put options. *Int. J. Theor. Appl. Finance*, 9:1141–1177, 2006.