Instituto Superior de Economia e Gestão
UNIVERSIDADE TÉCNICA DE LISBOA

Departamento de Gestão

DOCUMENTO DE TRABALHO Nº 4/2007

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WORKING PAPERS

ISSN Nº 0874-8470

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WORKING PAPER N.4/2007

October 2007

Abstract

This chapter addresses the pricing of two popular portfolio credit derivatives: first-to-default swaps and collateralized debt obligations (CDOs). We use the recent model of Gaspar and Schmidt (2007) for the pricing of theses portfolio credit derivatives. This approach combines general quadratic models for term structures with shot-noise models and therefore naturally solves a number of important issues in credit portfolio risk. First, resulting pricing formulas are in closed form and therefore the model implementation is straightforward. Second, this class of models is able to incorporate well-known features of credit risky markets: realistic default correlations, default clustering and correlation between short-rate and credit spreads. Third, the recent turbulence in credit spreads caused by the U.S. subprime mortgage turmoil can be captured well.

Key Words: Credit Default Swaps (CDS), Collateralized Debt Obligations (CDO), First-to-default Swaps, Quadratic Term Structures, shot-noise

JEL Classification: C15,C12, G13, G33

Department of Management
Working paper Series
ISSNº 0874-8470

1 Introduction

The demand for investments with higher returns in areas other than the stock market has increased enormously over the last decade. Investing in credit markets, investors take on credit risk in exchange for an attractive yield, and as a result methodologies for pricing and hedging credit derivatives as well as for risk management of credit risky assets became very important. The efforts of the Basel Committee is just one of many examples which substantiate this.

In the last years the credit markets developed at a tremendous speed while at the same time the number of corporate defaults increased dramatically. It is therefore not surprising that the demand for credit derivatives is growing rapidly. Besides the liquidly traded single-instruments a number of portfolio products gained more and more attention recently. On one side there is the demand of investors for investment possibilities in diversified portfolios and the seek for new investment fields, while on the other side there is the difficulty to capture dependencies between defaults. Modelling and estimating default dependencies is still an area of ongoing research and the recent turbulence caused by the difficulties in the U.S. subprime mortgage markets confirm the necessity of developing suitable models.

The goal of this article is to propose a model for portfolio credit risk which is able to capture typical market effects, such as there are spread correlation and high default dependence leading to clustering of defaults and contagion effects. Moreover the model still remains tractable and leads to a large number of explicit pricing results.

The most liquid single-name credit risky instrument is a so-called credit default swap. In this contract a regularly paid, fixed spread is exchanged for a protection payment which covers the losses occurring at default of the underlying. For details, the reader may want to consider, for example, Schönbucher (2003), Lando (2004), McNeil, Frey, and Embrechts (2005) or the survey Schmidt and Stute (2004).

After a long success story of collateralized debt obligations (CDOs) in various forms, since four years there are a number of traded CDO-indices available which are traded at very high liquidity. The portfolio setup considered in this paper is a natural candidate for application in this area and we provide all necessary tools in this article. For more information on CDOs see Bluhm, Overbeck, and Wagner (2003).

For illustration purposes, we consider here the case of the iTraxx[3]. This indice has a number of different sub-indices and derivatives writen on iTraxx are the most liquid on the credit market. We summarize the indices and derivatives related to the iTraxx on Table 1. Figure 1, presents recent spreads on the iTraxx Europe w.r.t. different maturities and of the standardized tranches 3-6%,6-9%,9-12%,12-22%.

2 Portfolio Credit Derivatives

We start by introducing some notation required to deal with portfolio credit derivatives and then exactly define First-to-Default Swaps (FDS) and Collateralized debt Obligations (CDOs).

We will deal with a basket of securities of K different entities subject to default risk. Each entity (for instance a company) may default only once and its default time is denoted by τ^k.

[3] See www.itraxx.com

	BENCHMARK INDICES	
iTraxx Europe (125 investment grade entities)	iTraxx Europe HiVol (30 highest spread entities from iTraxx Europe)	iTraxx Crossover (50 sub-investment grade entitities)
	SECTOR INDICES	
Non-Financials (100 entities)	Financials Senior (25 entities)	Financials Sub (25 entities)
	DERIVATIVES	
Tranched iTraxx (5 standardized tranches: 0-3%,3-6%,6-9%,9-12%,12-22%)	iTraxx Options iTraxx Futures	First-to-Default Swaps (Baskets on: Autos, Consumer, Energy, Financials, Industrials, HiVol, Crossover, Diversified)

Table 1: iTraxx indices and most liquid Derivatives

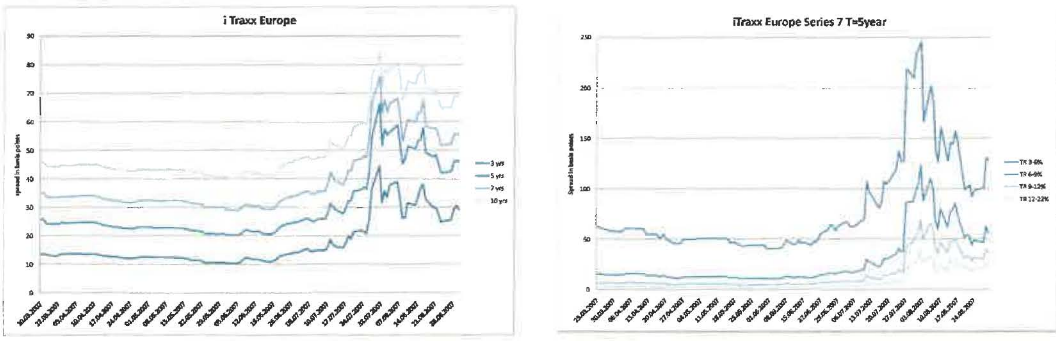


Figure 1: Spreads of the iTraxx Europe w.r.t. different maturities (left) and of the standardized tranches (right).

The counting process counting *all* defaults is denoted by $N_t := \sum_{k=1}^{K} \mathbf{1}_{\{\tau^k \leq t\}}$. If a default of entity k happens, we denote the loss quota by q^k. The notional in the basket associated with entity k is denoted by M^k, $k = 1, \ldots, K$.

It is market standard to name the two exchanging counterparties of a credit derivative *protection seller* and *protection buyer*. The protection seller offers a protection payment on specific default events and during some time span here denoted $[t_0, t_{N^*}]$, while the protection buyer pays a periodic fee in exchange. The fee payment dates are due in advance and tend to rely on a fixed tenure structure $t_0, t_1, \cdots, t_{N^*-1}$. As the protection buyer has only fixed payments, the payments due to him are also called *fixed leg*, while the payments of the protection seller are called *floating leg*.

2.1 First-to-Default Swaps

A *first-to-default swap* (FDS) is a contract which offers protection on the first default of a portfolio only. The FDS has an initiation date $t_0 < t_1$. If t_0 is in the future, the FDS is called *forward-starting FDS*. The FDS is characterized by the so-called *first-to-default spread* which is fixed at initiation of the contract.

- If the first default occurred in $(t_{n-1}, t_n]$, then the protection seller pays the default payment at t_n. Assuming name k defaulted first[4] ($\tau^{1:K} = \tau^k$), then the default payment is $M^k \cdot q^k$. If no default happens until t_{N^*} the protection seller pays nothing.
- The protection buyer pays the spread at all dates $t_1, \cdots, t_{N^*}$ until the maturity of the FDS or until the first default (whichever comes first).

The spread is chosen in such a way that entering the FDS is possible at zero cost. For the forward-starting FDS, such that its expected value at t_0 equals zero. Of course, after the spread is fixed, the value of the FDS changes. So the spread clearly depends on the current time, as agreements settled in different dates would originate different spreads. To emphasize this fact we write $s^{\mathrm{FD}}(t)$.

2.2 Collateralized Debt Obligations

A collateralized debt obligation (CDO) is a security backed by a pool of credits from various reference entities. The *asset side* of the CDO is formed by the credits themselves, while traded on the market are issued notes (typically swaps) on *tranches* of the CDO. These tranches have different seniorities, building the *liability side* of the CDO. There are different types of CDOs, depending on the type of the underlying credits. If the underlying are loans, bonds, mortgages, the CDO is named collateralized loan obligation, collateralized bond obligation or mortgage-backed security, respectively. This article treats only so-called *synthetic CDOs*, where the underlyings are credit default swaps (CDS). A particular kind of this type of CDO are those writen on the well know credit indices mentioned before. Swaps written on standardized tranches for these indices are among the most liquid portfolio credit derivatives.

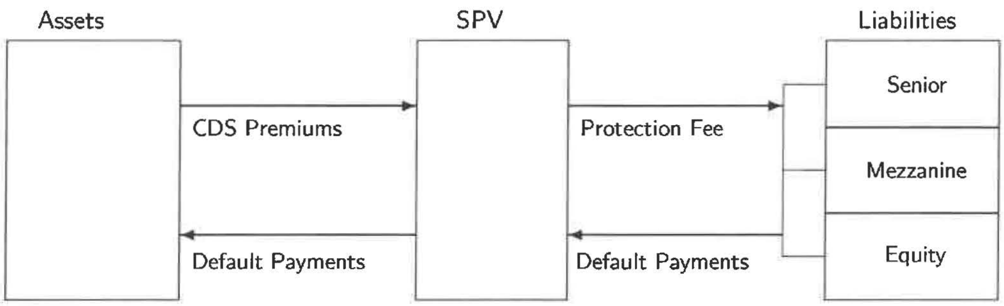
¿From now on we simply write CDO, while implicitly referring to synthetic CDOs. For further literature and other modelling approaches we refer to Bluhm and Overbeck (2003), Frey and Backhaus (2006) and Scherer (2007).

A CDO allocates interest income and principal repayments from a collateral pool of CDSs to a prioritized collection of CDO securities(tranches). While there are many variations, a standard prioritization scheme is simple subordination: *senior* CDO notes are paid before *mezzanine*; *equity* is paid with any residual cash-flow. The following picture clarifies the structure of a CDO.

In addition to the portfolio notation already introduced we need to introduce additional notation to describe the cash-flow of CDOs.

We consider a CDO with several tranches $i = 1, \cdots, I$. In the case where we have senior, mezzanine and equity tranches only we would simply take $I = 3$. The tranches are separated according to fixed barriers, the so-called attachment points, $b_0 < \cdots < b_I$. That is, b_1

[4] Throughout, we denote by $\tau^{1:K} < \cdots < \tau^{K:K}$ the *ordered default times*.



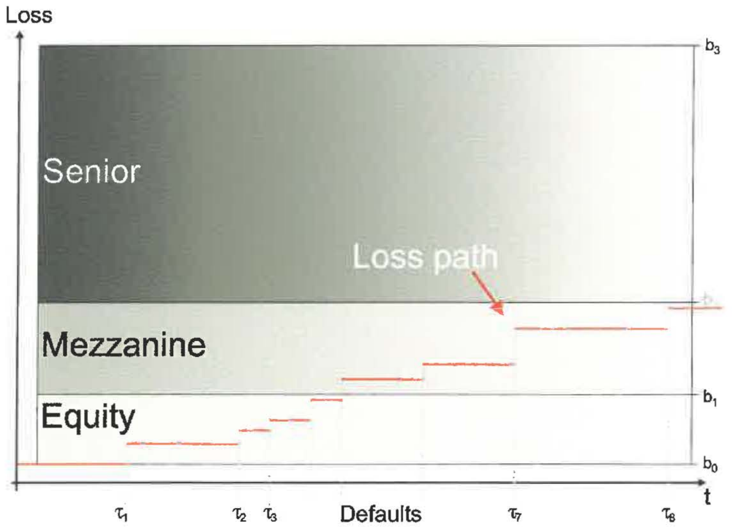


Figure 2: Tranches' losses in CDOs.

separates tranche 1 from tranche 2, b_2 separated the tranche 2 from tranche 3, and so on - compare Figure 2.

In general the *loss process* of the CDO, describing the reduction in face value of the whole underlying portfolio due to defaults, is given by $L(t) := \sum_{\tau^k \leq t} q^k M^k$. he *loss of tranche i* is, thus, given by

$$L^i(t) = \begin{cases} 0 & \text{if } \; L(t) < b^{i-1} \\ L(t) - b^{i-1} & \text{if } \; b^{i-1} \leq L(t) < b^i, \\ b^i - b^{i-1} & \text{if } \; L(t) \geq b^i \end{cases} \qquad i = 1, \ldots, I. \tag{1}$$

Figure 2 illustrates the CDO setup with a possible loss path affecting various tranches.

We now make standard normalizations and focus on the *pricing of tranches of CDOs*.

- The CDO offers notes on each tranche with par value 1. Recall that the attachment points of tranche i are b_{i-1} and b_i

- At each intermediate time $t_0, \dots, t_{N^*-1}$ the protection seller receives a coupon payment. The payment is on the remaining principal in the tranche, so that with fixed coupon S the payments due at t_n for tranche i are

$$S\,[t_n - t_{n-1}]\,\left(b^i - L^i(t_n)\right)^+$$

- In exchange, the protection seller covers in $(t_{n-1}, t_n]$ occurring losses at t_n, $n = 2, \cdots, N^*$. The protection payment, at t_n, for tranche i equals

$$L^i(t_n) - L^i(t_{n-1})$$

Note that coupon payments are exchanged at the beginning of each period, while the loss payments are due at the end of a period. Another critical point here is the reinvestment of the recovery payment. The default of an entity from the underlying pool leads to a non-payment of the future coupons. The recovery payment has to be re-invested at the current market level and possibly gets a lower coupon. We assume that these missing future coupons are already included in the recovery amount q^k. This means, that q^k is the actual recovery minus financing cost of the future coupons.

3 Model and Applications

This section considers the proposed model which is a combination of general quadratic term structures and shot-noise processes. First, the model is motivated and the precise setup is given. Then the prices of first-to-default swaps as well as CDO tranches are derived. Thereafter we discuss the important issue of obtaining realistic default correlations under the proposed model and finally we establish the link to credit indices.

3.1 Motivation and Setup

We assume that the default times are doubly stochastic random times. This setup is also referred to conditionally independent default times. For an introduction to this topic we refer to McNeil, Frey, and Embrechts (2005) or Bielecki and Rutkowski (2002). Given this, we model the default intensity of each firm as a linear combination of GQTS and shot noise processes.

Concretely, we consider an n-dimensional standard Brownian motion W and a state variable Z being the unique strong solution of

$$dZ_t = \alpha(t, Z_t)dt + \sigma(t, Z_t)dW_t.$$

Here $\alpha : \mathbb{R}_+ \times \mathbb{R}^m \mapsto \mathbb{R}^m$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^m \mapsto \mathbb{R}^{n\times n}$ are such that

$$\alpha(t, z) = d(t) + E(t)z \tag{2}$$

$$\sigma(t, z)\sigma^\top(t, z) = k_0(t) + \sum_{i=1}^{m} k_i(t)z_i + \sum_{i,j=1}^{m} z_i\, g_{ij}(t) z_j \tag{3}$$

with smooth functions $d : \mathbb{R}_+ \mapsto \mathbb{R}^m$, E, k_0, k_i and $g_{ij}, i, j = 1, \cdots, m$ map $\mathbb{R}_+$ to $\mathbb{R}^{m\times m}$.

The risk-free short rate $(r_t)_{t\geq 0}$ is given by

$$r(t, Z_t) = Z_t^\top Q(t) Z_t + g^\top(t) Z_t + f(t). \tag{4}$$

Q, g and f are smooth, mapping $\mathbb{R}_+$ to $\mathbb{R}^{m\times m}$, $\mathbb{R}^m$ and $\mathbb{R}$. $Q(t)$ is symmetricfor all t.

It is well-known that, under mild condition on the shape of the matrices k_i and g_ij (see Gaspar (2004) for further details on these conditions), the term structure of risk-free zero-coupon bond prices of exponential quadratic form

$$p(t, T) = \exp\left[A(t, T) + B^\top(t, T) Z_t + Z_t^\top C(t, T) Z_t\right] \tag{5}$$

where (A, B, C, f, g, Q) solves the basic ODE system defined in the appendix (Definition A.1).

We now extend the default setup in Gaspar and Schmidt (2007) to the portfolio case. We assume that the default time of firm k, τ^k, is a doubly stochastic random time. Its intensity $(\lambda_t^k)_{t\geq 0}$ satisfies

$$\lambda_t^k = \mu_t^k + \epsilon^k \mu_t^c. \tag{6}$$

The intensity of each firm depends on a firm specific term, μ^k, and a term common to all firms μ^c. All these expressions will have the same quadratic plus jump construction.

Specifically, we set $\mathbf{k} = \{1, \cdots, K\}$ and, for each $k \in \mathbf{k} \bigcup \{c\}$, we take

$$\mu_t^k = \eta_t^k + J_t^k \quad \text{with} \quad J_t^k = \sum_{\tilde{\tau}_j^k \leq t} Y_j^k h^k(t - \tilde{\tau}_j^i), \quad \eta_t^k = Z_t^\top \mathsf{Q}^k(t) Z_t + \mathsf{g}^k(t)^\top Z_t + \mathsf{f}^k(t) \tag{7}$$

where, $\tilde{N}^k$ are standard Poisson process with intensity l^k. We denote the jumping times of $\tilde{N}^k$ by $\tilde{\tau^k}_1, \tilde{\tau^k}_2, \ldots$.

Finally, we assume that the risk-free short rate r is independent of the firm specific intensity μ^k, but not necessarily of the common intensity μ^c.

Intuitively, the modelling of a quadratic component and a shot-noise component leads to the intensity being driven by a predictable component (the quadratic part) as well as by an unpredictable component (the jump part). We note that both η^k and J^k are assumed to be strictly positive. This assumption is needed because μ is supposed to be an intensity.

ϵ^k measures how sensitive is an entity to movements of the common factors. The higher ϵ_k is the bigger is the dependence of the common default risk driven by μ^c. For intuition take $\epsilon^i \equiv \epsilon$. Then, if μ^c jumps then suddenly the default risk of all the assets increase a lot and we will see numerous defaults. This can also be caused by a rise in the quadratic part to a high level, but then it is more or less predictable. The first effect causes some clustering similar to contagion effects, which means if one company defaults and others are closely related to this company, they are very likely to default also. The latter effect is more like a business cycle effect, so on bad days more companies default than on good days.

In the above setup, all necessary expressions for pricing relevant credit risky securities can be computed in closed-form. We delegate the necessary formulas to the appendix and refer to Gaspar and Schmidt (2007) for full proofs. At this point we only introduce a short-hand notation which will turn out extremely useful in the pricing formulas to follow.

For current time t, maturity T and $\theta \in \mathbb{R}$, we define[5]

$$S^k_\eta(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\eta^k_s ds}|\mathbf{F}^W_t\right] \qquad S^k_J(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta J^k_s ds}|\mathbf{F}^J_t\right] \tag{8}$$

$$\Gamma^k_\eta(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^k_T e^{-\int_t^T \theta\eta^k_s ds}|\mathbf{F}^W_t\right] \qquad \Gamma^k_J(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[\theta J^k_T e^{-\int_t^T \theta J^k_s ds}|\mathbf{F}^J_t\right] \tag{9}$$

$$\bar{S}^k_\eta(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta^k_s ds}|\mathbf{F}^W_t\right] \qquad \bar{\Gamma}^k(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^k_T e^{-\int_t^T r_s+\theta\eta^k_s ds}|\mathbf{F}^W_t\right] \tag{10}$$

Independence between the diffusion and jump components leads to

$$\begin{aligned}
S^k(\theta,t,T) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\mu^k_s ds}|\mathbf{F}^W_t\right] = S^k_\eta(\theta,t,T)\cdot S^k_J(\theta,t,T) && (11)\\
\bar{S}^k(\theta,t,T) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\mu^k_s ds}|\mathbf{F}^W_t\right] = \bar{S}^k_\eta(\theta,t,T)\cdot S^k_J(\theta,t,T) && (12)\\
\Gamma^k(\theta,t,T) &:= \mathbb{E}^{\mathbb{Q}}\left[\theta{\mu^k} e^{-\int_t^T \theta\mu^k_s ds}|\mathbf{F}^W_t\right]\\
&= \Gamma^k_\eta(\theta,t,T)S^k_J(\theta,t,T) + \Gamma^k_J(\theta,t,T)S^k_\eta(\theta,t,T) && (13)\\
\bar{\Gamma}^k(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta{\mu^k} e^{-\int_t^T r_s+\theta\mu^k_s ds}|\mathbf{F}^W_t\right]\\
&= \bar{\Gamma}^k_\eta(\theta,t,T)S^k_J(\theta,t,T) + \Gamma^k_J(\theta,t,T)\bar{S}^k_\eta(\theta,t,T). && (14)
\end{aligned}$$

If $\theta = 1$ we use the short-hand notation (t,T) instead of $(1,t,T)$, e.g. we write $S^k(t,T)$ for $S^k(1,t,T)$.

In the above short-hand notation we easily obtain, for each entity $k = 1,\cdots,K$ and on $\{\tau^k > t\}$

- implied survival probabilities

$$\mathbb{Q}^k_S(t,t_n) = \mathbb{Q}(\tau^k > T|\mathbf{F}_t) = S^k(t,t_n)\cdot S^c(\epsilon^k,t,t_n)$$

- prices of defaultable zero-coupon bonds

$$\bar{p}^k_0(t,t_n) = \mathbb{E}^{\mathbb{Q}}\Big(\exp(-\int_0^T r_u du)\mathbf{1}_{\{\tau^k>T\}}|\mathbf{F}_t\Big) = S^k(t,t_n)\cdot\bar{S}^c(\epsilon^k,t,t_n)$$

- prices of digitals (the price of a payoff of 1 u.c if the firm k defaults in $(t_{n-1},t_n]$)

$$e^{*k}(t,t_{n-1},t_n) = E^k(t,t_{n-1},t_n)\bar{p}^k_o(t,t_{n-1}) - \bar{p}^k_o(t,t_n) \tag{15}$$

  with E^k defined as in (26).

- $e^k(t,t_n) = \lim_{t_{n-1}\to t_n}\dfrac{1}{t_n-t_{n-1}}e^{*k}(t,t_{n-1},t_n) = \Gamma^k(t,t_n)\bar{S}^c(\epsilon^k,t,t_n)+\bar{\Gamma}^c(\epsilon^k,t,t_n)S^k(t,t_n).$

[5]Throughout, denote by $\mathbf{F}^X$ the natural filtration generated by a generic process X. We classify the market information according to the following filtrations: $\mathbf{F}^W$ the *information about the diffusion factors*; $\mathbf{F}^J$ the *information about the jump factors*; the filtration $\mathbf{H}_t := \sigma\left(\mathbf{1}_{\{\tau>s\}} : 0 \le s \le t\right)$, *information on the default state*; $\mathbf{F}_t := \mathbf{F}^W_t \vee \mathbf{F}^J_t = \sigma(Z_s, J_s : 0 \le s \le t)$, *information about all market factors* and $\mathbf{G}_t := \mathbf{F}_t \vee \mathbf{H}_t$, *the total information.*

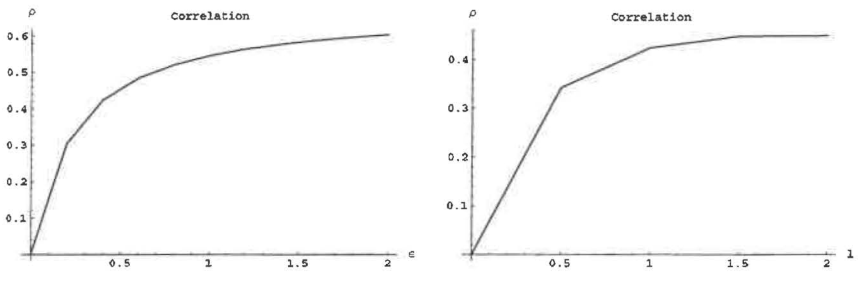


Figure 3: Default correlation in a concrete model.

3.2 Default correlation and Clustering

It is often argued that in the framework used here, where the default times are conditionally independent, the resulting default correlation is not high enough. However, already Duffie and Gârleanu (2001) showed that this is not the case. Especially through jumps or, more precisely, high peaks in the intensity a high default correlation is induced. However, in their formulation the authors had the sam parameter controlling the mean reversion speed of the diffusive as well as of the jump part. On the on hand big jumps we necessary to induce high default correlation but, on the other hand, this lead to unrealistic mean reversion specifications for the diffusion part.

In the framework presented above, this problem is solved, as the mean reversion speeds can be different.

The so-called default correlation is basically the correlation between the default indicators of two companies. Denote by $\mathbb{Q}_D^i$ the probability of company i defaulting in $(t, T]$ and by $\mathbb{Q}_D^{i,j}(t,T)$ the probability that companies i and j default in $(t, T]$. Using the building blocks we can easily get the default correlation in closed-form.

Lemma 3.1. *The default correlation of two different companies i and j is given by*

$$\rho^{i,j}(t,T) = \frac{S^i(t,T)S^j(t,T)\left[S^c(\epsilon^i+\epsilon^j,t,T) - S^c(\epsilon^i,t,T)S^c(\epsilon^j,t,T)\right]}{\sqrt{[1-S^i(t,T)S^c(\epsilon^i,t,T)]S^i(t,T)S^j(t,T)S^c(\epsilon^i,t,T)S^c(\epsilon^j,t,T)[1-S^j(t,T)S^c(\epsilon^j,t,T)]}}. \tag{16}$$

To illustrate the capability of the model to capture sufficiently high default correlation, we give a graph based on the concrete model supposed in Gaspar and Schmidt (2007), Section 5. We refer to this article for further details.

We in the remaining of the chapter we present the pricing results concerning portfolio credit derivatives under the quadratic/shot-noise model.

For notational simplicity we make the following *homogeneity assumption*[6]. However, the more general case is an immediate extension. Note that considering random, i.i.d. q^k leads to the same results when q is replaced by $\mathbb{E}(q^k)$.

Assumption 3.2. Assume that $q^k = q$, $\epsilon^k = \epsilon$ and $M^k = M$ for all $1 \leq k \leq K$. Moreover the tenor structure is equidistant, i.e. $t_j = j\Delta$, $1 \leq j \leq N^*$.

[6] All indices mentioned previously are homogeneous portfolios. Still, it is straightforward to extend our CDO results to the inhomogeneous case, but formulas get rather involved.

3.3 First-to-Default Swaps

The following results rely on the distribution of the first default time, which is the minimum of all default times. The main result on First-default-Swap is is Theorem 3.4. We start by computing the probability that the first default[7] occurs in $(t, T]$.

Lemma 3.3. *Consider a portfolio of K names and assume no default has occurred up to time t. Furthermore suppose Assumption 3.2 holds. Then, the survival probability of the first default is given by*

$$\mathbb{Q}\Big(\tau^{1:K} > T|\mathbf{F}_t\Big) = \mathbf{1}_{\{\tau^{1:K}>t\}}\, S^c(\epsilon K, t, T) \cdot \prod_{k=1}^{K} S^k(t, T).$$

Furthermore, the value of one unit of currency paid at T only if $\tau^{1:K} > T$ is given by

$$\bar{p}^{FtD}(t,T) := \mathbb{E}^Q\big(\exp(-\int_t^T r_u du)\mathbf{1}_{\{\tau^{1:K}>T\}}|\mathbf{F}_t\big) = \mathbf{1}_{\{T_1>t\}}\, \bar{S}^c(\epsilon K, t, T) \cdot \prod_{k=1}^{K} S^k(t, T).$$

With the above result at hand the spread of a First-to-Default Swap is easily derived:

Theorem 3.4. *Consider a portfolio of K names and assume no default has occurred up to time t. Then, the spread of the FDS is given by*

$$s^{FD}(t) = q\frac{\sum_{n=1}^{N^*} E^c(t, t_{n-1}, t_n)\bar{p}^{FtD}(t, t_{n-1}) - \bar{p}^{FtD}(t, t_n)}{\Delta \sum_{n=1}^{N^*} \bar{p}^{FtD}(t, t_n)}$$

with E^c defined as in (26).

3.4 Collateralized Debt Obligations

3.4.1 Portfolio loss distributions

We start by computing the *distribution of portfolios losses* under both the martingale measure and the T- forward measure. This will serve as building block for the pricing of CDOs.

Given our setup we can always conclude for the unconditional distribution of the loss function L. However, for pricing and risk management it is necessary to consider L after some time passed by, and we therefore will be interested in the *conditional* distribution of the loss function.

We note that, under Assumption 3.2, the loss process simplifies to $L(t) = qMN_t$.

To this it will be convenient to require the processes $(\lambda_t^k)_{t\geq 0}$, $k = 1, \dots, K$ to be Markovian. In Gaspar and Schmidt (2007) it is shown that this is the case whenever $h^k(t) = a^k \exp(-b^k t)$ for all $k = 1, \cdots, K$. Using the Markovian property, Theorem 3.5 gives us the conditional distribution of L. Before, however, to be able to handle defaulted and non-defaulted companies in a concise way, we need to introduce some more notation.

Denote by $\mathbf{S}_t$ the set which contains the indices of assets not defaulted until t, the "survivors":

$$\mathbf{S}_t := \{1 \leq k \leq K : \tau^k > t\}.$$

[7] Proof in the appendix B.

In the following proposition we will fix the number of defaults in the interval $(t, T]$ and then sum over all possible combinations of defaults. We write $\sum_{\mathbf{k}_n \in \mathbf{S}_t}$ for the sum over all sets $\mathbf{k}_n = \{k_1, \dots, k_n\}$ of size n with pairwise different elements and $k_1, \dots, k_n \in \mathbf{S}_t$. $\mathbf{k}_n$ represents the n companies which default in $(t, T]$.

Given $\mathbf{k}_n$, the companies *not* defaulting are denoted by

$$\mathbf{S}_t \backslash \mathbf{k}_n := \{1 \leq l \leq n : l \in \mathbf{S}_t, l \notin \mathbf{k}_n\}.$$

Furthermore, we write short $\{\tau^{\mathbf{k}_n} \in (t, T]\}$ for $\{\tau^{k_1} \in (t, T], \dots, \tau^{k_n} \in (t, T]\}$.

Theorem 3.5. *Suppose Assumption 3.2 holds. Suppose the function $h^k(x)$ in (7) are of the form $a_k e^{-b_k x}$, $k \in \mathbf{k} \bigcup \{c\}$. Then the conditional distribution of the portfolio losses is given by*

$$\mathbb{Q}(L_T \leq x | \mathbf{G}_t) = \mathbf{1}_{\{\tau^{\mathbf{S}_t} > t\}} \sum_{n=0}^{K-N_t} \Bigg\{ \mathbf{1}_{\{n \leq \frac{x}{qM} - N_t\}} \times$$

$$\times \sum_{\mathbf{k}_n \in \mathbf{S}_t} \Bigg[S^c(\epsilon(K - N_t - n), t, T) \Big(\prod_{k \in \mathbf{S}_t \backslash \mathbf{k}_n} S^k(t, T) \Big) - S^c(\epsilon(K - N_t), t, T) \Big(\prod_{k \in \mathbf{S}_t} S^k(t, T) \Big) \Bigg] \Bigg\}$$

where S^k and S^c are as previously defined.

Furthermore, if $t = 0$, the above expression gives the unconditional expectation and the functions $h^k(x)$ need not have any special form.

Corollary 3.6. *Denote by $\mathbb{Q}^T$ the T-forward measure. Under the assumptions of Theorem 3.5 we have that*

$$\mathbb{Q}^T \left(L_T \leq x \middle| \mathbf{G}_t \right) = \mathbf{1}_{\{\tau^{\mathbf{S}_t} > t\}} \frac{1}{p(t, T)} \sum_{n=0}^{K-N_t} \Bigg\{ \mathbf{1}_{\{n \leq \frac{x}{qM} - N_t\}} \times$$

$$\times \sum_{\mathbf{k}_n \in \mathbf{S}_t} \Bigg[\bar{S}^c(\epsilon(K - N_t - n), t, T) \Big(\prod_{k \in \mathbf{S}_t \backslash \mathbf{k}_n} S^k(t, T) \Big) - \bar{S}^c(\epsilon(K - N_t), t, T) \Big(\prod_{k \in \mathbf{S}_t} S^k(t, T) \Big) \Bigg] \Bigg\}$$

3.4.2 Link to Credit Indices

In this section we draw the link between CDOs and the to currently traded credit indices tranches. We give a pricing results for these tranche spreads using the quadratic/shot-noise model.

The iTraxx is effectively a portfolio of 125 single CDS. To guarantee liquidity, the portfolio is reorganized (the so-called series) semiannually by a voting scheme and entities whose rating fell below investment grade are removed in the new series. The aim of this procedure is to guarantee that the underlying portfolio stays in a certain class of credit worthiness. The recovery of each entity is fixed and assumed to be zero.

The mathematical setting for an credit index is as follows. W.l.o.g. we assume that the notional is 1. The credit index is on K names, each represented by a CDS with spread $s^i(t)$. Each names are in the same credit class, so that the homogeneity Assumption 3.2 will hold. Especially, the single names have equal weight.

The payment stream of the credit index is as follows. Recall that $N_t = \sum_{\tau^k \leq t}$ is the number of defaulted entities at time t.

- *Fixed leg:* The spread is paid on the remaining notional, i.e. at each time t_n of the tenor $t_1, \dots, t_{N^*-1}$ the payoff is

$$S\Delta \frac{K - N_{t_n}}{K}.$$

- *Defaulting leg:* We assume the payments of default protection occur at the end of the defaulting period, i.e. the payments of the floating or protection leg in the interval $(t_{n-1}, t_n]$ due at $t_n : 2 \le n \le N^*$ are

$$\sum_{\tau^k \in (t_{n-1}, t_n]} (1-q) = (1-q)(N_{t_n} - N_{t_{n-1}}).$$

  Typically, the recovery in the traded indices is set to zero, but for completeness we stay more general at this point.

Example 3.7. *Connection of index spread with underlying CDS spreads.* Before any default happens and if the recovery is paid as in the underlying CDS, it is clear that the payment streams of the index are equivalent to the payment streams of the portfolio of the equally weighted underlying CDS (with spread denoted by s_i) and so the spread of the index is simply

$$S_t = \frac{1}{K} \sum_{k=1}^{K} s^k(t).$$

Now, if a default happens, the situation gets more complicated. One entity is removed and the index still pays the spread S. However, the spread of the portfolio with equally weighted CDS, where now the defaulting entity is removed has a possibly different spread:

$$\frac{1}{K} \sum_{k=1}^{K} s^k(t) \mathbf{1}_{\{\tau^k > t\}}.$$

For example, if $K = 2$ and s^1 equals 100 and s^2 equals 200, both constant, we obtain for the index spread 150, but after default of name 1 the portfolio with equal weights pays the spread 100 while the index pays the spread 75. This may show that for pricing some more effort has to be done.

Summarizing we obtain the following result

Proposition 3.8. *The spread of a credit index on a pool of K entities where Assumption 3.2 holds, computes to*

$$S_t = \bar{q} K \frac{\sum_{n=2}^{N^*} \mathbf{1}_{\{t_n \ge t\}} \sum_{k=1}^{K} e^{*k}(t, t_{n-1}, t_n)}{\Delta \sum_{t_n \ge t}^{t_{N^*-1}} \sum_{k=1}^{K} \bar{p}_0^k(t, t_n)}.$$

3.4.3 Tranches on credit indices

Finally, we have to determine the pricing of *tranches* of credit indices. Investment in a tranche offers the possibility to separate between different credit qualities.

Recall that the overall nominal was assumed to be 1. A tranche refers to an interval $(b_1, b_2] \subset [0, 1]$. Investing (selling protection) in a tranche is again done by a swap where the following payments are exchanged

1. The investor receives at t_n, $1 \le n \le N^* - 1$ the payment

$$S\left(\mathbf{1}_{\{L_{t_n} \le b_1\}} + \frac{(b_2 - L_{t_n})^+}{b_2 - b_1}\mathbf{1}_{\{L_{t_n} > x_1\}}\right).$$

2. In turn, the investor has to cover eventual losses, i.e. pays at t_n, $2 \le n \le N^*$,

$$\left((b_2 - L_{t_{n-1}})^+ - (b_2 - L_{t_n})^+\right)\mathbf{1}_{\{L_{t_n} > b_1\}}.$$

It is cumbersome but not difficult to use Theorem 3.5 for pricing these expressions. However, under a quite common assumption in CDO analysis the pricing becomes rather straightforward.

Assumption 3.9. Assume that the risk-free rate of interest is independent[8] from the default intensities $\lambda^k, 1 \le k \le K$.

A special case is of course if the risk-free rate of interest is deterministic. If Assumption 3.9 holds, then the following result proved in Filipović, Overbeck, and Schmidt (2007), is the key tool for pricing.

Let us define

$$C(t, T, y) := \mathbb{E}^{\mathbb{Q}}\Big(\exp(-\int_t^T r_u du)\mathbf{1}_{\{L_T < y\}}|\mathbf{G}_t\Big). \tag{17}$$

Proposition 3.10. *Suppose Assumption 3.9 holds. The par-spread of tranche $(b_1, b_2]$ is given by*

$$S(t, b_1, b_2) = \frac{\int_{b_1}^{b_2}\Big(p(t,t_1)\mathbf{1}_{\{L_t<y\}} - C(t, t_{N^*}, y) + \sum_{n=2}^{N^*-1} C(t, t_n, y)\cdot\big(1 - \frac{p(t,t_{n+1})}{p(t,t_n)}\big)\Big)\,dy}{\sum_{n=1}^{N^*-1}\frac{1}{b_2-b_1}\int_{b_1}^{b_2} C(t, t_n, y)\,dy} \tag{18}$$

where, $p(t, T)$ denote the price at time t of a credit risk free zero-coupon bond price with maturity T and C is as in (22).

Note that under the additional assumption of vanishing interest rates, the spread equals

$$S(t, b_1, b_2) = \frac{\int_{b_1}^{b_2}\Big(\mathbf{1}_{\{L_t<y\}} - C(t, t_{N^*}, y)\Big)\,dy}{\sum_{n=1}^{N^*-1}\frac{1}{b_2-b_1}\int_{b_1}^{b_2} C(t, t_n, y)\,dy}.$$

Given the above Proposition, it remains to compute $C(t, T, y)$ that, under Assumption 3.9, becomes

$$C(t, T, y) = p(t, T)\mathbb{Q}^T(L_T < y|\mathbf{G}_t) = p(t, T)\mathbb{Q}(L_T < y|\mathbf{G}_t).$$

However, both these distributions had been computed in Theorem 3.5 and Corollary 3.6, so the spread of a tranche is available. Note also that, as L takes values in the finite set $\{0, qM, \dots, KqM\}$, the integral in (18) is simply a sum over at most $K + 1$ entries.

[8]In our setup, this means the Z components affecting the short rate of interest in (4) are different from those affecting the quadratic part of the intensities η^k in (7).

A Bulding Blocks in Closed-form

We start by defining three different types of ODE systems.

Definition A.1. (Basic ODE System) Denote $\mathbf{T} := \{(t,T) \in \mathbb{R}^2 : 0 \leq t \leq T\}$ and consider functions A, B and C on $\mathbf{T}$ with values in $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively. For functions ϕ_1 and ϕ_2, ϕ_3 on $\mathbb{R}^+$ with values in $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, respectively, we say that $(A, B, C, \phi_1, \phi_2, \phi_3)$ solves the *basic ODE system* if

$$\begin{aligned}\frac{\partial A}{\partial t} + d^\top(t)B + \frac{1}{2}B^\top k_0(t)B + \operatorname{tr}\{Ck_0(t)\} &= \phi_1(t)\\ \frac{\partial B}{\partial t} + E^\top(t)B + 2Cd(t) + \frac{1}{2}\tilde{B}^\top K(t)B + 2Ck_0(t)B &= \phi_2(t)\\ \frac{\partial C}{\partial t} + CE(t) + E^\top(t)C + 2Ck_0(t)C + \frac{1}{2}\tilde{B}^\top G(t)\tilde{B} &= \phi_3(t)\end{aligned}$$

subject to the boundary conditions $A(T,T) = 0, B(T,T) = 0, C(T,T) = 0$. A, B and C should always be evaluated at (t,T). E, d, k_0, are the functions from the above definitions (recall (2)-(3)) while

$$\tilde{B} := \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B \end{pmatrix}, \quad K(t) = \begin{pmatrix} k_1(t) \\ \vdots \\ k_m(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_{11}(t) & \cdots & g_{1m}(t) \\ \vdots & \ddots & \vdots \\ g_{m1}(t) & \cdots & g_{mm}(t) \end{pmatrix}, \tag{19}$$

where we have $\tilde{B}, K(t) \in \mathbb{R}^{m^2\times m}$ and $G(t) \in \mathbb{R}^{m^2\times m^2}$.

Definition A.2. (Interlinked ODE system) Consider smooth functions a, b, c, B, C on $\mathbf{T}$ with values in $\mathbb{R}$, $\mathbb{R}^m$, $\mathbb{R}^{m\times m}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$, and smooth functions ϕ_1, ϕ_2 and ϕ_3 on $\mathbb{R}_+$ with values in $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$ respectively. We say that $(a, b, c, B, C, \phi_1, \phi_2, \phi_3)$ solves the *interlinked ODE system* if it solves

$$\frac{\partial a}{\partial t} + d^\top(t)b + B^\top k_0(t)b + \operatorname{tr}\{ck_0(t)\} = 0 \tag{20}$$

$$\frac{\partial b}{\partial t} + E^\top(t)b + 2cd(t) + \frac{1}{2}\tilde{B}^\top k_0(t)b + 2ck_0(t)B + 2Ck_0(t)b = 0 \tag{21}$$

$$\frac{\partial c}{\partial t} + cE(t) + E^\top(t)c + 4Ck_0(t)c + \frac{1}{2}\tilde{B}^\top G(t)\tilde{b} = 0 \tag{22}$$

subject to the boundary conditions $a(T,T) = \phi_1(T)$, $b(T,T) = \phi_2(T)$, $c(T,T) = \phi_3(T)$. a, b, c and B, C should always be evaluated at (t,T). E, d, k_0, are the functions from (3) while $\tilde{B}, K \in \mathbb{R}^{m^2\times m}$ and $G \in \mathbb{R}^{m^2\times m^2}$ are as in (19).

Definition A.3. (Doubly-Interlinked ODE system) Denote $\mathbf{T} := \{(t, t_{n-1}, t_n) \in \mathbb{R}^3 : 0 \leq t \leq t_{n-1} \leq t_n\}$ and consider functions α, β, γ, on $\mathbf{T}$ with values in $\mathbb{R}$, $\mathbb{R}^m$, and $\mathbb{R}^{m\times m}$, respectively. For functions $\phi_1, \bar{\phi}_1$, $\phi_2, \bar{\phi}_2$, $\phi_3, \bar{\phi}_3$ on $\mathbb{R}^+$ with values in $\mathbb{R}$, $\mathbb{R}^m$, $\mathbb{R}^m$

and $\mathbb{R}^{m\times m}$, $\mathbb{R}^{m\times m}$ respectively, we say that $(\alpha, \beta, \gamma, \phi_1, \phi_2, \phi_3, \bar{p}hi_1, \bar{p}hi_2, \bar{p}hi_3)$ solves the *doubly-interlinked ODE system* if

$$\frac{\partial\alpha}{\partial t} + d^\top(t)\beta + \frac{1}{2}\beta^\top k_0(t)\beta + \operatorname{tr}\gamma k_0(t) + \beta^\top k_0(t)\bar{B} = 0 \tag{23}$$

$$\frac{\partial\beta}{\partial t} + E^\top(t)\beta + 2\gamma d(t) + \frac{1}{2}\tilde{\beta}^\top K(t)\beta + 2\gamma k_0(t)\beta + 2\bar{C}k_0(t)\beta + 2\gamma k_0(t)\bar{B} + \tilde{\beta}^\top K(t)\bar{B} = 0 \tag{24}$$

$$\frac{\partial\gamma}{\partial t} + \gamma E(t) + E^\top(t)\gamma + 2\gamma k_0(t)\gamma + \frac{1}{2}\tilde{\beta}^\top G(t)\tilde{\beta} + 4\bar{C}k_0(t)\gamma + \tilde{\bar{B}}^\top G(t)\tilde{\beta} = 0 \tag{25}$$

subject to the boundary conditions $\alpha(t_{n-1}, t_{n-1}, t_n) = A(t_{n-1}, t_n)$, $\beta(t_{n-1}, t_{n-1}, t_n) = B(t_{n-1}, t_n)$ and $\gamma(t_{n-1}, t_{n-1}, t_n) = C(t_{n-1}, t_n)$. Also $(A, B, C, \phi_1, \phi_2, \phi_3)$, $(\bar{A}, \bar{B}, \bar{C}, \bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ both solve the basic ODE system in (20)-(22). E, d, k_0, are the functions from (3), $\tilde{\beta}, K \in \mathbb{R}^{m^2\times m}$ and $\gamma \in \mathbb{R}^{m^2\times m^2}$. α, β, γ should be evaluated at (t, t_{n-1}, t_n) and A $\bar{A}, B, \bar{B}, C, \bar{C}$ at (t, t_{n-1}).

Theorem A.4. *Let $x = T - t$ and consider r as in (4), J and η as in (7) and $\theta \in \mathbb{R}$. For (ii) we also require existence of $D^k(\theta, x)$ and for (v) that D^k is bounded in some neighborhood of x.*

Then,

$$\begin{aligned}
&(i) && S^k_\eta(\theta, t, T) &&:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\eta^k_s ds} | \mathbf{F}^W_t\right] \\
&&&&&= \exp\left(\mathcal{A}^k(\theta, t, T) + \mathcal{B}^{k\top}(\theta, t, T)Z_t + Z_t^\top \mathcal{C}^k(\theta, t, T)Z_t\right) \\
&(ii) && S^k_J(\theta, t, T) &&:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta^k J_s ds} | \mathbf{F}^J_t\right] = \exp\left(\theta(\bar{J}^k_t - \bar{J}^k(t, T)) + lx[D^k(\theta, x) - 1]\right) \\
&(iii) && \bar{S}^k_\eta(\theta, t, T) &&:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s + \theta\eta^k_s ds} | \mathbf{F}^W_t\right] \\
&&&&&= \exp\left(\bar{A}^k(\theta, t, T) + \bar{B}^{k\top}(\theta, t, T)Z_t + Z^\top(t)\bar{C}^k(\theta, t, T)Z_t\right) \\
&(iv) && \Gamma^k_\eta(\theta, t, T) &&:= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^k_T e^{-\int_t^T \theta\eta^k_s ds} | \mathbf{F}^W_t\right] \\
&&&&&= \left(a^k(\theta, t, T) + b^{k\top}(\theta, t, T)Z_t + Z_t^\top c^k(\theta, t, T)Z_t\right) S_\eta(\theta, t, T) \\
&(v) && \Gamma^k_J(\theta, t, T) &&:= \mathbb{E}^{\mathbb{Q}}\left[\theta J^k_T e^{-\int_t^T \theta J^k_s ds} | \mathbf{F}^J_t\right] \\
&&&&&= S^k_J(\theta, t, T)\left\{\theta J^k(t, T) - l^k \cdot \left[D^k(\theta, x)(1 - x) - 1 + x\varphi^k_Y(\theta H^k(x))\right]\right\} \\
&(vi) && \bar{\Gamma}^k_\eta(\theta, t, T) &&:= \mathbb{E}^{\mathbb{Q}}\left[\theta\eta^k_T e^{-\int_t^T r_s + \theta\eta^k_s ds} | \mathbf{F}^W_t\right] \\
&&&&&= \left(\bar{a}^k(\theta, t, T) + \bar{b}^{k\top}(\theta, t, T)Z_t + Z_t^\top \bar{c}^k(\theta, t, T)Z_t\right) \cdot \bar{S}^k_\eta(\theta, t, T)
\end{aligned}$$

where $(\mathcal{A}^k, \mathcal{B}^k, \mathcal{C}^k, \theta\mathfrak{f}^k, \theta\mathfrak{g}^k, \theta\mathfrak{Q}^k)$ and $(\bar{A}^k, \bar{B}^k, \bar{C}^k, f + \theta\mathfrak{f}^k, g + \theta\mathfrak{g}^k, Q + \theta\mathfrak{Q}^k)$ solve the basic ODE system of Definition A.1, while $(a^k, b^k, c^k, \mathcal{B}^k, \mathcal{C}^k, \theta\mathfrak{f}^k, \theta\mathfrak{g}^k, \theta\mathfrak{Q}^k)$ and $(\bar{a}^k, \bar{b}^k, \bar{c}^k, \bar{B}^k, \bar{C}^k, \theta\mathfrak{f}^k, \theta\mathfrak{g}^k, \theta\mathfrak{Q}^k)$ solve the interlinked system of Definition (A.3).

Furthermore,

$$
\begin{aligned}
(vii)\quad & S^k(\theta,t,T) && := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\mu_s^k ds}|\mathbf{F}_t\right] = S_\eta^k(\theta,t,T)S_J^k(\theta,t,T)\\
(viii)\quad & \bar{S}^k(\theta,t,T) && := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\mu_s^k ds}|\mathbf{F}_t\right] = \bar{S}_\eta^k(\theta,t,T)S_J^k(\theta,t,T)\\
(ix)\quad & \Gamma^k(\theta,t,T) && := \mathbb{E}^{\mathbb{Q}}\left[\theta\mu_T^k e^{-\int_t^T \theta\mu_s^k ds}|\mathbf{F}_t\right] = \Gamma_\eta^k(\theta,t,T)S_J^k(\theta,t,T) + \Gamma_J^k(\theta,t,T)S_\eta^k(\theta,t,T)\\
(x)\quad & \bar{\Gamma}^k(\theta,t,T) && := \mathbb{E}^{\mathbb{Q}}\left[\theta\mu_T^k e^{-\int_t^T r_s+\theta\mu_s^k ds}|\mathbf{F}_t\right] = \bar{\Gamma}_\eta^k(\theta,t,T)S_J(\theta,t,T) + \Gamma_J^k(\theta,t,T)\bar{S}_\eta^k(\theta,t,T)
\end{aligned}
$$

We introduce one further handy notation.

Notation A.5. *For $k \in \mathbf{K} \cup \{c\}$ we define*

$$E^k(\theta,t,t_{n-1},t_n) := \exp\left\{\alpha^k(t,t_{n-1},t_n) + \beta^{k\top}(t,t_{n-1},t_n)Z_t + Z_t^\top\gamma^k(t,t_{n-1},t_n)Z_t\right\} \quad (26)$$

where $(\alpha^k,\beta^k,\gamma^k,f,g,Q,\theta\{,\theta\},\theta Q)$ solve the doubly interlinked ODE system in (23)-(25) with f,g,Q are as in 4 and $\mathsf{f},\mathsf{g},\mathsf{Q}$ as in (7). For $k=1,\cdots K$ $\theta=\epsilon$, while for $k=c$, $\theta=\epsilon K$.

B Proofs and Auxiliary Results

Proof of Lemma 3.1. By the definition of $\rho^{i,j}$ we have

$$\rho^{i,j}(t,T) = \frac{\mathbb{Q}_D^{i,j}(t,T) - \mathbb{Q}_D^i(t,T)\mathbb{Q}_D^j(t,T)}{\sqrt{\mathbb{Q}_D^i(t,T)[1-\mathbb{Q}_D^i(t,T)]\mathbb{Q}_D^j(t,T)[1-\mathbb{Q}_D^j(t,T)]}}.$$

where $\mathbb{Q}_D^{i,j}$ is the probability of joint default of the firms i,j until time T given that none has defaulted until t. We can easily get

$$
\begin{aligned}
\mathbb{Q}_D^{i,j}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^i<T\}}\mathbf{1}_{\{\tau^j<T\}}\middle|\,\mathbf{G}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(1-e^{-\int_t^T\lambda_s^i ds}\right)\left(1-e^{-\int_t^T\lambda_s^j ds}\right)\middle|\,\mathbf{F}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(1-e^{-\int_t^T\mu_s^i+\epsilon^i\mu_s^c ds}\right)\left(1-e^{-\int_t^T\mu_s^j+\epsilon^j\mu_s^c ds}\right)\middle|\,\mathbf{F}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[1-e^{-\int_t^T\mu_s^i+\epsilon^i\mu_s^c ds}-e^{-\int_t^T\mu_s^j+\epsilon^j\mu_s^c ds}+e^{-\int_t^T\mu_s^i+\mu_s^j+(\epsilon^i+\epsilon^j)\mu_s^c ds}\middle|\,\mathbf{F}_t\right]\\
&= 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T\mu_s^i+\mu_s^j+(\epsilon^i+\epsilon^j)\mu_s^c ds}\middle|\,\mathbf{F}_t\right].
\end{aligned}
$$

Using independence of μ^i, μ^j and μ^c and $\mathbb{Q}_S^k(t,T) = S^k(t,T)S^c(\epsilon^k,t,T)$ for $k=1,2$ we obtain

$$\mathbb{Q}_D^{i,j}(t,T) \;=\; 1--S^i(t,T)S^c(\epsilon^i,t,T) - S^j(t,T)S^j(\epsilon^k,t,T) + S^i(t,T)S^j(t,T)S^c(\epsilon^i+\epsilon^j,t,T).$$

the result follows from

$$
\begin{aligned}
\mathbb{Q}_D^i(t,T)\mathbb{Q}_D^j(t,T) &= (1-\mathbb{Q}_S^i(t,T))(1-\mathbb{Q}_S^j(t,T))\\
&= 1-\mathbb{Q}_S^i(t,T)-\mathbb{Q}_S^j(t,T)+\mathbb{Q}_S^i(t,T)\mathbb{Q}_S^j(t,T)
\end{aligned}
$$

and again substitution $\mathbb{Q}_S^i$ and Q_S^j by its short-hand notation. ■

Proof of Lemma 3.3. The result is trivial on $\{\tau_1 \leq t\}$, so we consider $\{\tau_1 > t\}$ from now on. Then, by definition $\mathbb{Q}_S^{\mathrm{FtD}}(t,T) = \mathbb{Q}(\tau_1 > T|\mathbf{G}_t)$. We start by conditioning on $\mu^c_{[t,T]} \vee \mathbf{G}_t$. Recall that the default time of name k is denoted by τ_k. Then,

$$\mathbb{Q}\left(\tau_1 > T|\mu^c_{[t,T]} \vee \mathbf{G}_t\right) = \mathbb{Q}\left(\min(\tau_1, \tau_2, \cdots, \tau_K) > T|\mu^c_{[t,T]} \vee \mathbf{G}_t\right)$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau_1>T,\tau_2>T,\cdots,\tau_K>T\}}|\mu^c_{[t,T]} \vee \mathbf{G}_t)\right]. \tag{27}$$

As $\tau_1, \ldots, \tau_K$ are independent, conditionally on $\mu^c_{[t,T]}$, we obtain

$$(27) = \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big[-\sum_{k=1}^{K}\int_t^T \lambda_s^k ds\Big]\Big|\mu^c_{[t,T]} \vee \mathbf{F}_t\Big)$$
$$= e^{-K\epsilon\int_t^T \mu_s^c ds} \cdot \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big[-\sum_{k=1}^{K}\int_t^T (J_s^k + \eta_s^k)ds\Big]\Big|\mu^c_{[t,T]} \vee \mathbf{F}_t\Big)$$

As $\eta^1, \ldots, \eta^{\bar{K}}, J^1, \ldots, J^K$ are mutually independent we obtain

$$(27) = e^{-\epsilon K\int_t^T \mu_s^c ds} \cdot \prod_{k=1}^{K} S^k(t,T).$$

It may be recalled that $S^k = S^k_\eta S^k_J$. Thus,

$$\mathbb{Q}(\tau_1 > T|\mathbf{G}_t) = \mathbb{E}^{\mathbb{Q}}\Big(e^{-\epsilon K\int_t^T \mu_s^c ds} \cdot \prod_{k=1}^{K} S^k(t,T)\Big|\mathbf{F}_t\Big)$$
$$= S^c(\epsilon\bar{K}, t, T) \cdot \prod_{k=1}^{K} S^k(t,T).$$

Using the same methodology with the fact that r is independent of μ^k for all $k \in \mathbf{k}$ but not of μ^c determines $\bar{p}^{\mathrm{FtD}}(t,T)$. ∎

Proof of Theorem 3.4. For ease of notation we write s^{FD} instead of $s^{\mathrm{FD}}(t)$. The value at time t of the fixed leg of the FDS follows from the results in the previous lemma:

$$\mathbb{E}^{\mathbb{Q}}\Big[\sum_{n=1}^{N^*} e^{-\int_t^{t_n} r_s\,ds}\, s^{\mathrm{FD}}\, \mathbf{1}_{\{\tau>t_n\}}\big|\mathbf{G}_t\Big] = s^{\mathrm{FD}}\sum_{n=1}^{N^*}(t_n - t_{n-1}) \cdot \bar{p}^{FtD}(t, T_n).$$

For the pricing of the floating leg we need to compute

$$\begin{aligned} e^{\mathrm{FD}*}(t, T_{n-1}, T_n) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau^{1:K}\in(t_n,t_{n-1}]\}}\Big|\mathbf{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau^{1:K}>t_{n-1}\}}\Big|\mathbf{G}_t\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau^{1:K}>t_n\}}\Big|\mathbf{G}_t\right], \end{aligned}$$

where the second expectation equals $\bar{p}^{\mathrm{FtD}}(t, t_n)$. Furthermore,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_n} r_s ds}\mathbf{1}_{\{\tau^{1:K}>t_{n-1}\}}\Big|\mathbf{G}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[p(t_{n-1}, t_n) \cdot e^{-\int_t^{t_{n-1}} r_s ds}\mathbf{1}_{\{\tau^{1:K}>t_{n-1}\}}\Big|\mathbf{F}_t\right]. \tag{28}$$

Following the steps from Lemma 3.3 we can deduce the following.[9]

$$(28) = \mathbb{E}^{\mathbb{Q}}\Big[p(t_{n-1},t_n)\cdot e^{-\int_t^{t_{n-1}}\left[r_s+\epsilon K\mu_s^c+\sum_{k=1}^K \mu_s^k\right]ds}\Big|\mathbf{F}_t\Big]$$

We write short $\tilde{\mathbf{F}}_{t,t_{N^*}}$ for $\mathbf{F}_t \vee \sigma(\mu_s^c, r_s : t \le s \le t_{N^*})$. Conditioning on $\tilde{\mathbf{F}}$ we obtain

$$(28) = \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\Big(e^{-\sum_{k=1}^K\int_t^{t_{n-1}}\mu_s^k\,ds}\Big|\tilde{\mathbf{F}}_{t,t_{N^*}}\Big)\cdot p(t_{n-1},t_n)e^{-\int_t^{t_{n-1}}\left(r_s+\epsilon K\mu_s^c\right)ds}\Big|\mathbf{F}_t\Big].$$

Let us consider the inner expectation more closely. We have that $\eta^1,\dots,\eta^K$ are independent of μ^c and r, so that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big(e^{-\sum_{k=1}^K\int_t^{t_{n-1}}\mu_s^k\,ds}\Big|\tilde{\mathbf{F}}_{t,t_{N^*}}\Big) &= \mathbb{E}^{\mathbb{Q}}\Big(e^{-\sum_{k=1}^K\int_t^{t_{n-1}}\eta_s^k\,ds}\Big|\mathbf{F}_t\Big)\cdot\mathbb{E}^{\mathbb{Q}}\Big(e^{-\int_t^{t_{n-1}}\sum_{k=1}^K J_s^k\,ds}\Big|\mathbf{F}_t\Big)\\ &= \prod_{k=1}^K S^k(t,t_{n-1}).\end{aligned}$$

It may be recalled that $S^k = S_\eta^k\cdot S_J^k$. It remains to compute

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big[p(t_{n-1},t_n)e^{-\int_t^{t_{n-1}}\left(r_s+\epsilon K\eta_s^c\right)ds}\Big|\mathbf{F}_t\Big] &= \bar{S}_\eta^c(\epsilon K,t,t_{n-1})\cdot\\ &\cdot\exp\Big(\alpha^c(t,t_{n-1},t_n)+\beta^{c\,\top}(t,t_{n-1},t_n)Z_t+Z_t^\top\gamma^c(t,t_{n-1},t_n)Z_t\Big).\end{aligned}$$

The remaining part with J^c is given by S_J^c such that by $\bar{S}^c=\bar{S}_\eta^c\cdot S_J^c$ expression (28) equals

$$e^{\alpha^c(t,t_n,t_{n-1})+\beta^{c\,\top}(t,t_n,t_{n-1})Z_t+Z_t^\top\gamma^c(t,t_n,t_{n-1})Z_t}\cdot\underbrace{\bar{S}^c(\epsilon\bar{K},t,t_{n-1})\cdot\prod_{k=1}^{\bar{K}}S^k(t,t_{n-1})}_{\bar{p}^{\mathrm{FtD}}(t,t_{n-1})},$$

where α,β,γ are as stated in (26). ■

Proof of Theorem 3.5. The conditional distribution of L is given by

$$\begin{aligned}\mathbb{Q}(L_T\le x|\mathbf{G}_t) &= \mathbb{Q}(L_T-L_t\le x-L_t|\mathbf{G}_t)=\mathbb{Q}\Big(\sum_{j=1}^{N_T-N_t}\xi_j\le x-L_t\big|\mathbf{G}_t\Big)\\ &=\mathbb{Q}\Big(\sum_{j=1}^{N_T-N_t}q^j\le\frac{x-L_t}{M}\big|\mathbf{G}_t\Big)=F_{q,N_T-N_t}\Big(\frac{x-L_t}{M}\Big).\end{aligned}$$

Recall that N is the counting process of *all* defaults. For the following, we first condition on μ^c. Then all individual defaults τ^k are independent and stem from independent Cox-processes with (also independent) intensities $(\lambda^k(t))_{t\ge0}$, $k=1,\dots,K$. Observe that N_T-N_t is not independent from N_t[10]. But, it is not difficult to compute the conditional

[9] Alternatively, in the conditionally independent approach the default intensity of the minimum of the default times is simply the sum over all intensities.

[10] For example, if all companies default before t, hence $N_t=K$ it follows that $N_T-N_t=0$.

distribution. However, in contrast to the unconditional distribution, we need to distinguish which company defaults.

Using the Markovianity of the processes μ^k we need to determine

$$\mathbb{Q}(N_T - N_t = k|\mathbf{S}_t, N_t, \mu^c_{[t,T]}, \mathbf{F}_t). \tag{29}$$

We write $\tilde{\mathbf{F}}_t := \sigma(\mathbf{S}_t, N_t, \mu^c_{[t,T]}, \mathbf{F}_t)$. In the above probability we will have k companies defaulting in $(t,T]$. Summing over all possible indices was denoted by $\sum_{\mathbf{k}_n \in \mathbf{S}_t}$. Then,

$$(29) = \sum_{\mathbf{k}_n \in \mathbf{S}_t} \mathbb{Q}(\tau^{\mathbf{k}_n} \in (t,T]|\tilde{\mathbf{F}}_t)\, \mathbb{Q}(\tau^{\mathbf{S}_t \setminus \mathbf{k}_n} > T|\tilde{\mathbf{F}}_t).$$

Note that the survival probability of asset k is given by

$$\begin{aligned}
\mathbb{Q}(\tau^k > T|\tilde{\mathbf{F}}_t) &= \mathbb{Q}(\tau^k > T|\mathbf{1}_{\{\tau^k > t\}}, \mu^c_{[t,T]}, \mathbf{F}_t) \\
&= \mathbf{1}_{\{\tau^k > t\}} \exp\Big(- \epsilon \int_t^T \mu^c_s\, ds \Big) \underbrace{\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(- \int_t^T \mu^k_s\, ds \Big) |\mathbf{F}_t \Big]}_{=S^k(t,T)}.
\end{aligned}$$

The expectation on the r.h.s. is of the exponential quadratic from as given by (vii) in Theorem A.4. In the Markovian case, we can simplify even further. Furthermore, since, conditionally on μ^c the defaults occur independently, we have

$$\mathbb{Q}(\tau^{\mathbf{k}_n} > T|\tilde{\mathbf{F}}_t) = \mathbf{1}_{\{\tau^{\mathbf{k}_n} > t\}} \exp\Big(- n\epsilon \int_t^T \mu^c_s\, ds \Big) \prod_{k \in \mathbf{k}_n} S^k(t,T).$$

On $\{\tau^k > t\}$ we also have that $\mathbb{Q}(\tau^k \in (t,T]|\tilde{\mathbf{F}}_t) = 1 - \mathbb{Q}(\tau^k > T|\tilde{\mathbf{F}}_t)$. Hence,

$$\begin{aligned}
&\mathbb{Q}(N_T - N_t = n|\tilde{\mathbf{F}}_t) = \\
&= \sum_{\mathbf{k}_n \in \mathbf{S}_t} \Big\{ 1 - e^{-n\epsilon \int_t^T \mu^c_s\, ds} \prod_{k \in \mathbf{k}_n} S^k(t,T) \Big\} \cdot e^{-(K-N_t-n)\epsilon \int_t^T \mu^c_s\, ds} \prod_{k \in \mathbf{S}_t \setminus \mathbf{k}_n} S^k(t,T) \\
&= \sum_{\mathbf{k}_n \in \mathbf{S}_t} \Big[e^{-(K-N_t-n) \int_t^T \mu^c_s\, ds} \prod_{k \in \mathbf{S}_t \setminus \mathbf{k}_n} S^k(t,T) - e^{-\epsilon(K-N_t) \int_t^T \mu^c_s\, ds} \prod_{k \in \mathbf{S}_t} S^k(t,T) \Big]
\end{aligned} \tag{30}$$

After we have done all calculation conditioned on μ^c we finally have to consider the unconditional expectation. This is, on $\{\tau^{\mathbf{S}_t} > t\}$,

$$\begin{aligned}
&\mathbb{Q}(N_T - N_t = n|\mathbf{S}_t, N_t, \mathbf{F}_t) \\
&= \sum_{\mathbf{k}_n \in \mathbf{S}_t} \Big[S^c(\epsilon(K - N_t - n), t, T) \prod_{k \in \mathbf{S}_t \setminus \mathbf{k}_n} S^k(t,T) - S^c(\epsilon(K - N_t), t, T) \prod_{k \in \mathbf{S}_t} S^k(t,T) \Big].
\end{aligned}$$

■

Proof of Corollary 3.6. Recall from (5) that $p(t,T) = \exp\left(A(t,T)+B^\top(t,T)Z_t+Z_t^\top C(t,T)Z_t\right)$. First, observe that

$$p(t,T)\mathbb{Q}^T\left(L_T \le x \big| \mathbf{G}_t\right) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\mathbf{1}_{\{L_T \le x\}} \big| \mathbf{G}_t\right).$$

We therefore just need to compute $\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\mathbf{1}_{\{L_T\le x\}}\big|\mathbf{G}_t\right)$.

To this, let $\tilde{\mathbf{G}}_t := \sigma(\mathbf{S}_t, N_t, \mu^c_{[t,T]}, \mathbf{G}_t)$ and recall r has common factors, i.e., conditional on μ^c it is known. We thus have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbf{1}_{\{L_T\le x\}}\Big|\mathbf{G}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{L_T\le x\}}\Big|\tilde{\mathbf{G}}_t\right]\Big|\mathbf{G}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{Q}\left(L_T \le x \big| \tilde{\mathbf{G}}_t\right)\Big|\mathbf{G}_t\right]
\end{aligned}$$

For the inner expectation we may use Equation (30) to obtain that the above equals

$$\mathbb{E}^{\mathbb{Q}}\Bigg\{e^{-\int_t^T r_s ds}\sum_{n=0}^{K-N_t} F_{q,n}\Big(\frac{x-L_t}{M}\Big)$$
$$\cdot\sum_{\mathbf{k}_n\in\mathbf{S}_t}\Bigg[e^{-\epsilon(K-N_t-n)\int_t^T\mu^c_s ds}\Bigg(\prod_{k\in\mathbf{S}_t\setminus\mathbf{k}_n} S^k(t,T)\Bigg) - e^{-\epsilon(K-N_t)\int_t^T\mu^c_s ds}\Bigg(\prod_{k\in\mathbf{S}_t}S^k(t,T)\Bigg)\Bigg]\Bigg|\mathbf{G}_t\Bigg\}.$$

Recalling the short-hand notation we get

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}e^{-\epsilon(K-N_t-n)\int_t^T\mu^c_s ds}\Big|\mathbf{G}_t\right) = \bar{S}^c(\epsilon(K-N_t-n),t,T) \qquad \blacksquare$$

Proof of Proposition 3.8. We start by determining the value of the index spread at a certain time t. The spread offered by the index is chosen, such that fixed and defaulting leg equal in value. We denote this spread by S_t. Using the above formulation, the value of the fixed leg at time t is

$$S_t\,\Delta K^{-1}\sum_{t_n\ge t}^{t_{N^*-1}}\frac{1}{K}\sum_{k=1}^K\mathbb{E}^{\mathbb{Q}}\Big(e^{-\int_t^{t_n} r_u du}\mathbf{1}_{\{\tau^k>t_n\}}\big|\mathbf{G}_t\Big),$$

where $\sum_{t_n\ge t}^{t_{N^*-1}}$ is, more precisely, the sum over all $t_n \in \{t_1,\dots,t_{N^*-1}\}$ with $t_n \ge t$. The last expectation is equal to $\bar{p}^k_0(t,t_n)$, the appropriate zero-recovery bond for the kth underlying. On the other side, the value of the floating leg equals

$$q\sum_{t_n\ge t}^{N^*}\mathbb{E}^{\mathbb{Q}}\Big(\sum_{\tau^j\in(t_{n-1},t_n]}e^{-\int_t^{t_n} r_u du}\big|\mathbf{G}_t\Big).$$

The expectation can be evaluated with the aid of e^* defined in Equation (15). Form there it may be recalled that the value of one unit of currency, paid at t_n, when name k defaults in $(t_{n-1},t_n]$ was named $e^{*k}(t,t_{n-1},t_n)$ and can be calculated in closed form. Thus, the value of the default leg is

$$q\sum_{t_n\geq t}\mathbb{E}^{\mathbb{Q}}\Big(\sum_{\tau^j\in(t_{n-1},t_n]} e^{-\int_t^{t_n} r_u\,du}\big|\mathbf{G}_t\Big) = q\sum_{t_n\geq t}\sum_{k=1}^{K}\mathbb{E}^{\mathbb{Q}}\Big(e^{-\int_t^{t_n} r_u\,du}\mathbf{1}_{\{\tau^k\in(t_{n-1},t_n]\}}\big|\mathbf{G}_t\Big)$$

$$= q\sum_{t_n\geq t}\sum_{k=1}^{K} e^{*k}(t,t_{n-1},t_n).$$

■

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