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Numerical Solution of a Two State Variable Contingent Claims Mortgage Valuation Model[ab]

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Abstract

Previously published work on mortgage component valuation has concentrated on the US market and is inapplicable to some of the mortgage arrangements outside of that market. We model UK repayment mortgages with capped Mortgage Insurance Guarantees (which affect both the equilibrium lending rates and the lender's residual exposure). A contingent claims framework is developed, with an explicit finite differences solution. Then the mortgage components are valued, assuming arrangement fees but no prepayment penalties, under various scenarios, and also under equilibrium conditions: The transformation of the original PDE, and the details of the finite difference solution are given, along with graphical sensitivities of the mortgage participants (including the options held or written by the borrower, the insurer, and the lender) to interest rates and house prices.

Introduction

Contingent claims analysis leads to the modelling of many derivative assets as partial differential equations (PDEs). A few of these models (usually those which use the simplest and strongest assumptions) allow for analytic or "closed-form" solutions. However, this is the exception, not the rule. The tendency to model increasingly sophisticated assets, and to relax the strongest assumptions in order to reduce the distance between models and reality, leads naturally to the development of valuation frameworks of enlarged complexity for which no closed-form solutions are available.

This paper presents a numerical procedure for the solution of a contingent claims valuation model aimed at valuing mortgage-related products. The theoretical cornerstone underlying this framework is the CIR model (Cox, Ingersoll and Ross, 1985a,b). The CIR (1985b) interest rate model is an attempt to overcome the strongest limitations of earlier term-structure models (e.g. Vasicek, 1977; Brennan and Schwartz, 1982). Under these earlier models, the functional form of the market price of risk and the stochastic processes governing the interest rate(s) are consistent with a general financial markets equilibrium. In contrast, CIR (1985b) is based on the comprehensive context of an inter-temporal capital asset pricing model (CIR 1985a). Departing from a set of strong assumptions about consumption preferences of the economic agents, it is possible to derive endogenously the stochastic process followed by the instantaneous risk-free interest rate. The resulting general valuation framework allows for the consideration of an arbitrary, but finite number of state-variables, each one representing a different source of risk. In their seminal paper, Litterman and Scheinkman (1991) investigated factors affecting bond yields. According to their results, the three main factors account for level, steepness and curvature. The first of these is normally able to explain 80%-90% or more of the variance. Given the complexity the numerical intensity associated with interest rate modeling, tractable models are obtained by limiting the number of state variables, commonly to one or two.

Mortgages are treated here as derivative assets whose prices depend on the evolution of the global economy, via the term structure of interest rates and house prices. Once the house price and the term structure are determined, the value of the mortgage is set through a process of arbitrage inference. All other factors that might exert some influence would only be taken into consideration through the market price of risk associated with each state variable. The first state variable, house price, is taken as a traded asset and so risk adjustment becomes unnecessary. The instantaneous spot interest rate is used as the state variable representing the term structure of interest rates.

The valuation procedure developed in this work considers two forms of endogenous termination prior to maturity: prepayment by the mortgage borrower and default. As the value of the mortgage is affected by the options to prepay and default in the future, it is necessary to use a numerical valuation procedure working progressively backwards in time, with the value of the assets in later periods feeding into the value of the same assets in earlier periods. This excludes the use of basic Monte Carlo valuation forwards in time, leaving tree and finite difference methods. Binomial (or trinomial) trees can be practically attractive for quick implementation for a single state variable but for two state variables, numerical solution of a PDE via a finite difference method is preferable. Amongst the alternative finite difference methods, the simplest is the explicit method. This requires care in the selection of parameters, making other methods appear more sophisticated and appealing. However, for practical purposes, with a financial valuation problem of this degree of complexity, we found the extra care needed in parameter selection trivial as a return measured in programming effort and speed of solution.

Although the general techniques of solution via finite differences are well known, first the problem must be set up in a suitable form. Despite their superficial similarity to problems in the Natural Sciences and Engineering, for which finite difference algorithms were originally developed, contingent claim valuation formulae differ in detail and the use of algorithms is not straightforward. The behaviour of solutions can be completely different and requires the development of specially designed adaptations. The solution to a specific problem can then be implemented using a convenient high level programming language (in this work we used a Fortran compiler supplied by the University of the Salford).

Valuation Framework

We model the spot interest rate, r(t), as a CIR mean-reverting square root process (Cox, Ingersoll and Ross, 1985b) and the house price , H(t), as a lognormal diffusion process (Merton, 1973). We represent these in equations (1) and (2). Stochastic elements are modelled by two standardised Wiener processes, $z_r(t)$ and $z_H(t)$ which are correlated as in equation (3). We have immediately dropped the time labels, in parentheses. We hope this will make later equations clear, so long as the reader remembers that both state variables and the Wiener processes are functions of time.

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz_r \qquad (1)$$

$$\frac{dH}{H} = (\mu - \delta)dt + \nu dz_H \qquad (2)$$

$$dz_r(t)\, dz_H(t) = \rho dt \qquad (3)$$

where:

- κ = speed of adjustment in the mean reverting process,
- θ = central location or long term mean of the short-term interest rate *r(t)* (steady state spot rate),
- σ = instantaneous standard deviation of the (interest rate) disturbance,
- z_r = standardised Wiener process,
- μ = instantaneous average rate of house price appreciation,
- δ = "dividend-type" per unit service flow provided by the house,
- v = instantaneous standard deviation of the house price,
- z_H = standardised Wiener process.
- ρ = instantaneous correlation coefficient between the Wiener processes.

Cox, Ingersoll and Ross (1985a) derived a general methodology for the valuation of contingent claims in an equilibrium framework. It is known from standard arguments in finance that the partial differential equation (PDE) for the valuation of any asset F(r,H,t), whose value is a function only of interest rate, r, house price, H, and time, t, takes the following form, again dropping labels in parentheses, for clarity (Cox, Ingersoll and Ross, 1985a,b; Epperson et al., 1985; Kau et al., 1992, 1993a)

$$\frac{1}{2}H^2v^2\frac{\partial^2F}{\partial H^2} + \rho H\sqrt{r}v\sigma\frac{\partial^2F}{\partial H\partial r} + \frac{1}{2}r\sigma^2\frac{\partial^2F}{\partial r^2} + \kappa(\theta-r)\frac{\partial F}{\partial r} + (r-\delta)H\frac{\partial F}{\partial H} + \frac{\partial F}{\partial t} - rF = 0 \qquad (4)$$

The PDE has several noteworthy characteristics:

1. A mixed derivative term
2. A "free boundary" (the mortgage borrower may terminate prior to maturity)
3. Variable coefficients

Components Of The Mortgage Value

Straightforward "repayment mortgages" are arranged such that the loan is repaid by a series equal annual payments on pre-determined, equally spaced dates. The monthly payments, MP, and the outstanding balance after each payment, O(i), are calculated using standard annuity formulae:

$$MP = \frac{(R/12)[1+R/12]^n O(0)}{[1+R/12]^n - 1} \qquad (5)$$

$$O(i) = \frac{[(1+R/12)^n - (1+R/12)^i]O(0)}{[1+R/12]^n - 1} \qquad (6)$$

where:

R = nominal annual interest rate with monthly compounding
O(0) = amount originally borrowed
n = the life of the mortgage in months
$\eta(i)$ = the time of the i^{th} month (i^{th} payment date)

The value of remaining future payments, A(r,t), promised to the lender is a function of the term structure of interest rates. We introduce notation which distinguishes value immediately before a payment is made by a negative superscript and value immediately afterwards by a positive superscript. At the end of the loan period, the value of the payment due, A^-, is equal to the final payment, MP, and so the terminal condition is:

$$A^-(r,t) = MP \quad \text{for } t = \eta(n) \tag{7}$$

The value of A at each of the other payment dates is subject to a similar condition:

$$A^-(r,t) = A^+(r,t) + MP \qquad \text{for } t = \eta(1),\ldots,\eta(n-1) \tag{8}$$

The mortgage value includes the value of A and other components (the borrower's options) and differs between borrower and lender in that the lender may also have insurance which is of no value to the borrower. We will discuss the mortgage value in terms of the negative of its value to the borrower, which we name $V_B(H,r,t)$. This consists not only of the present value of remaining future payments, A(r,t), promised to the lender but also the value of options implicit in the contract. These are the borrower's option to prepay, C(H,r,t), eliminating the debt early, and the option to default, D(H,r,t), reneging on the debt and turning over the house to the lender:

$$V_B(H,r,t) = A(r,t) - C(H,r,t) - D(H,r,t) \tag{9}$$

For valuation purposes, we assume that both options are legal and that either will be exercised, if it becomes financially rational to do so. If the house price exceeds the value of the remaining payments, a rational borrower will not default and so, clearly, the house price has a direct impact on the value of the default option. The situation is different in the case of prepayment, where the decision to pay the loan early would be affected by the evolution of the term structure of interest rates but not so obviously by the value of the underlying asset. However, the exercise of the default option terminates the loan, which implies automatically that the prepayment option expires worthless. Therefore, indirectly, the house price also affects prepayment. As a consequence of this interaction between both options to terminate the loan, they cannot be considered separately. The decision to default is not, therefore, triggered simply if the value of the remaining payments exceeds the house price but if it exceeds the house price plus the value of the joint option to terminate the mortgage:

$$A(r,t) > H + C(H,r,t) + D(H,r,t) \tag{10}$$

At termination, the borrower may either pay the required monthly amount, MP, or default. The value of the mortgage to the borrower immediately before a payment is made is the minimum of MP and the house value:

$$V_B^-(H,r,t) = \min[MP, H] \qquad \text{for } t = \eta(n) \qquad (11)$$

The default option, at mortgage maturity, will be worthless if the value of the house is greater than the final payment but otherwise equal to the difference between the two:

$$D^-(H,r,t) = \max[0,(MP - H)] \text{ for } t = \eta(n) \qquad (12)$$

Prepayment at this stage would have no meaning and the prepayment option value is zero:

$$C^-(H,r,t) = 0 \quad \text{for } t = \eta(n) \qquad (13)$$

At earlier payment dates, the value of the mortgage to the borrower immediately before payment is the greater of MP plus its value afterwards and the house price:

$$V_B^-(H,r,t) = \min[(V_B^+(H,r,t) + MP), H]$$
$$\text{for } t = \eta(1), \ldots \eta(n-1) \qquad (14)$$

We can now write conditions, immediately before payment falls due, for the borrower's option to prepay, C(H,r,t) and the option to default, D(H,r,t):

If	$V_B^-(H,r,t) = V_B^+(H,r,t) + MP$	(no default)	
then	$D^-(H,r,t) = D^+(H,r,t)$		(15)
and	$C^-(H,r,t) = C^+(H,r,t)$		(16)
If	$V_B^-(H,r,t) = H$	(default)	
then	$D^-(H,r,t) = A^-(r,t) - H$		(17)
and	$C^-(H,r,t) = 0$		(18)

Mortgage Indemnity Guarantees (MIG)

If the borrower prepays the mortgage, the total debt payment, TD(t), will include an early termination penalty. The amount is not standardised across UK mortgages and so we model it as a proportion, π, of the outstanding balance, O(i), plus accrued interest:

$$TD(t) = \{(1+\pi)[1 + R(t - \eta(i))]\}O(i)$$
$$\text{for } \eta(i) \le t = \eta(i+1) \qquad (19)$$

The lender may also benefit from a Mortgage Interest Guarantee (MIG). This is an insurance policy whose value depends on the mortgage contract and which benefits

only the lender. A financially rational borrower need not take this into account, hence our earlier definition of the mortgage value as (the negative of) the value to the borrower. The MIG, which depends on the mortgage, may then be valued in a separate step. The insurer agrees to pay a fraction of the total loss suffered by a mortgage lender on each loan included in a specific pool of mortgages. The precise terms for British MIG contracts varies from case to case but a common format, which we will use, is as follows. The insurer agrees to pay a fraction, γ, of the total loss, TD(t)-H, suffered by the lender but only up to a maximum indemnity, or "cap", of Γ. The loss will be considered to be the difference between the value of the borrower's total debt and the value of the house, $\{TD(t) - H\}$.
We assume the cap is .2 times H(0), based on a loan to value mortgage of .95 H(0), compared to a "normal" loan to value mortgage of .75 H(0). Also, we assume that γ=.8.

The combination of all these features gives a general terminal condition of the following form for the MIG at termination:

If $V_B^-(H,r,t) = MP$ for $t = \eta(n)$ (no default)
then $\Gamma^-(H,r,t) = 0$ (20)

If $V_B^-(H,r,t) = H$ for $t = \eta(n)$ (default)
then $\Gamma^-(H,r,t) = \min\{[\gamma(MP - H)], \Gamma\}$ (21)

On other Payment Dates:

If $V_B^-(H,r,t) = V_B^+(H,r,t) + MP$ for $t = \eta(n)$ (no default)
then $\Gamma^-(H,r,t) = \Gamma^+(H,r,t)$ (22)

If $V_B^-(H,r,t) = H$ for $t = \eta(n)$ (default)
then $\Gamma^-(H,r,t) = \min\{\{\gamma[TD^-(t) - H]\}, \Gamma\}$ (23)

Coinsurance

Coinsurance describes the potential loss not covered by the MIG and includes any loss above the cap. Its valuation can be relevant not only to the insurer, but also for the lender and for third party insurers eventually interested in selling coverage for this type of risk.

Letting CI(H,r,t) represent the value of coinsurance, the corresponding terminal conditions will be given by the following expressions:

Maturity of the Loan:

If $V_B^-(H,r,t) = MP$ for $t = \eta(n)$ (no default)
then $C\Gamma^-(H,r,t) = 0$ (24)

If $V_B^-(H,r,t) = H$ for $t = \eta(n)$ (default) then $CI^{-}(H,r,t) = \max\{[(1-\gamma)(MP - H)], [(MP - H) - \Gamma]\}$ (25)

On other Payment Dates:

If $V_B^{-}(H,r,t) = V_B^{+}(H,r,t) + MP$ for $t = \eta(n)$ (no default)

then $CI^{-}(H,r,t) = CI^{+}(H,r,t)$ (26)

If $V_B^{-}(H,r,t) = H$ for $t = \eta(n)$ (default)

then $CI^{-}(H,r,t) = \max\{\{(1-\gamma)\{TD^{-}[\eta(i)] - H\}\}, \{\{TD^{-}[\eta(i)] - H\} - \Gamma\}\}$ (27)

By definition, at any payment date the value of the coinsurance is equal to the difference between the value of the potential loss and the value of the insurance coverage. Therefore, in aggregate:

$CI^{-}(H,r,t) + I^{-}(H,r,t) = \{CI^{+}(H,r,t) + I^{+}(H,r,t)\}$ (no default) (28)

$CI^{-}(H,r,t) + I^{-}(H,r,t) = \{TD^{-}(t) - H\}$ (default) (29)

The No-Arbitrage Condition

Finally, the terms of the mortgage contract need to be set so as to avoid arbitrage and including the various fees and penalties applied to British mortgages. We now adjust our notation somewhat, to allow for these as variables. The equilibrium condition for the mortgage is:

$$V_B[H(0), r(0), t(0), R, \pi] - (1-\xi)L + I[H(0), r(0), t(0), R, \pi] = 0 \quad (31)$$

where:

$\dot{R}$ = contract rate; nominal annual interest rate paid by the borrower

π = early termination penalty

ξ = arrangement fee as a proportion of the amount lent

L = amount lent

I = value of the MIG

Although the MIG is not part of the contract with the borrower and benefits only the lender, it affects the equilibrium contract rate. In order to determine equilibrium for Equation (31), a contract rate was found using a secant iteration technique, in line with those described by Gerald and Wheatley (1994) and by Press et al. (1992). V_B, incorporates the joint option to terminate and we will consider the method used to solve the PDE and value its components in the next section.

The Finite Difference Methodology

The basic method of solution of partial differential equations by finite difference methods is well established and there are excellent textbook introductions in mathematics (e.g. Ames, 1992; Lapidus and Pinder, 1982; Morton and Myers, 1994) and in finance (e.g. Wilmott et al. 1993; Clewlow and Strickland, 1998). The underlying asset price, on which option values depend, versus time is approximated by a grid in which only small but finite changes in each dimension are considered. Terms in the partial differential equation are then approximated by linear slopes across the grid. With several underlying variables (in this work: H and r), this grid represents a multi-dimensional "state space". Knowing terminal conditions (such as when it is financially optimal to default on a mortgage) it is possible to work backwards in time, valuing the options at each point on the grid, until initial values are obtained at the start of the grid, at time zero.

The two most basic approaches to solution of PDEs via finite difference are explicit and implicit schemes. In an explicit scheme, the value of a single point on the grid is calculated from an odd number of points (usually three) with known values in the next time step. In implicit schemes, single points with known values are related to sets of points (again, usually three) whose values are unknown. The unknown values are then calculated by solution of simultaneous equations. Implicit methods have the advantage of not being so constrained in the size of time step required for a particular size of asset price step in the grid. They can also be improved so as to increase the rate at which errors decrease as the grid size is made finer (the best-known example of this is the Crank-Nicolson method where the error decreases with the square of the time step size rather than linearly as with plain implicit schemes).

In the only published work of comparable kind in real estate finance (Kau et al., 1992, 1993a, 1993b, 1995) an explicit finite difference method was employed. Complex problems in finance involving several dimensions can become so intricate and difficult to program that it may be considered preferable to use explicit methods. Dempster and Hutton (1995, 1996) reached this conclusion in two studies of cross-currency valuation functions. In the present work, there is a free boundary from an American option, several interconnected valuation functions and a stream of hundreds of European options. On balance, we concluded that an explicit finite difference approach would be most appropriate for its solution. Free boundaries can be attacked by boundary-tracking techniques in the programming (Crank, 1984), but we chose to transform the equations so as to convert to a fixed boundary problem, following a procedure similar to that proposed by Berger, Ciment and Rogers (1975). We demonstrate the transformation in the next section.

Transformation of the PDE

The state variables for house price, H, and interest rate, r, must be transformed to eliminate infinite boundary conditions which would otherwise be difficult to handle numerically. Arbitrage arguments require positive interest rates and, in a house market without major inefficiencies or transaction costs, similar arguments require positive house prices. Although in any particular practical application each state variable will remain finite, in terms of a mathematical solution we are left with the infinite domain $(0,\infty)(0,\infty)$. However, we can choose convenient transformations to map the infinite area $(0,\infty)(0,\infty)$ onto the unit square $(0,1)(0,1)$. These are shown in equations (32) and (33)

$$y = \frac{1}{1+\psi r} \tag{32}$$

$$x = \frac{1}{1+\omega H} \tag{33}$$

for ψ and $\omega > 0$

Of course, with such inverse relationships, the values used for the arbitrary factors ψ and ω considerably affect the density of points on the solution grid and were chosen so as to place the value ranges of most concern for the state variables around the centre of the grid. For example, giving ψ a value of 10 was convenient for interest rates around 0.1 (10%) p.a. House prices were normalized to an initial value of 1 and so it was convenient to set ω equal to 1.

Equation (4) is a backward parabolic PDE and we transform it into a forward equation (Wilmott et al., 1993) by reversing the temporal dimension, as in equation (34).

$$\tau = T - t \tag{34}$$

These transformations convert the original PDE, equation (4), for the valuation of any derivative asset F(r,H,t), whose value is a function only of interest rate, r, house price, H, and time, t, into its equivalent, W[r(y),H(x),t(τ)], whose value is a function of the new variables. We relegate details of the transformations to an appendix. Equation (35) shows the transformed PDE.

$$\frac{1}{2}H(x)^2 v^2 \omega^2 x^4 \frac{\partial^2 W}{\partial x^2} + \rho H(x)\sqrt{r(y)} v \sigma \psi \omega x^2 y^2 \frac{\partial^2 W}{\partial x \partial y}$$
$$+ \frac{1}{2} r(y) \sigma^2 \psi^2 y^4 \frac{\partial^2 W}{\partial y^2} + \left\{ H(x)^2 v^2 \omega^2 x^3 - \left[(r(y) - \delta) H(x) \omega x^2 \right] \right\} \frac{\partial W}{\partial x}$$
$$+ \left\{ r(y) \sigma^2 \psi^2 y^3 - \left[\kappa(\theta - r(y)) \psi y^2 \right] \right\} \frac{\partial W}{\partial y} - \frac{\partial W}{\partial \tau(t)} - r(y) W = 0 \tag{35}$$

We provide details of the finite difference representation and numerical solution in another appendix but in the following sections we will try to give some insight into the method. We will begin by considering the boundary conditions and then describe two-dimensional "slices" of house price versus interest rate moving backwards in time from the final, known, outcomes if the mortgage were to reach its full term.

Boundary Conditions

We now have the two state variables, house price, H, and interest rate, r, transformed and the third dimension time, t, (or its simple transform, τ), all of which together can be visualised as a three-dimensional lattice within a box. The two variables H and r may be visualised as a grid moving stepwise along the time axis as the finite difference calculations proceed.

In order to solve the problem, we need not only the finite difference approximations to first and second derivatives within the lattice but also known values or constraints within it. These are the boundary conditions. We will next describe the boundary conditions in terms of the original state variables, where financial insights are clearer but we will refer to the box for easy identification of the regions. The boundary conditions can be identified at the faces and edges of the box. We will consider the faces in turn, followed by the edges where they meet; for example, when r = 0 (and its transform is 1) the face is a plane of H versus time and when H = 0 (hence, its transform is 1) the face is a plane of r versus time.

House price is zero

When the house price is zero (H=0), the absolute value of the mortgage, owed by the borrower, cannot be less than the house price. The borrower's rational behaviour is to default. Consequently, the prepayment option is worthless. The value of the mortgage is then equal to zero, the house price. At this boundary, therefore, the default option is equal to the value of the remaining mortgage payments. These conditions are shown in equations (36)-(38).

$$C(0,r)=0 \qquad (36)$$

$$\mathrm{V_B}(0,\mathrm{r})=\mathrm{H}=0 \qquad (37)$$

$$\mathrm{D}(0,\mathrm{r})=\mathrm{A}(0,\mathrm{r}) \qquad (38)$$

The values of the mortgage interest guarantee, I, and the coinsurance, CI, are given by the degenerate form of equation (4), with H=0 and either I or CI substituted for the general variable F. For convenience, F(r,0,t) is replaced by F:

$$\frac{1}{2}\mathrm{r}\sigma^2\frac{\partial^2\mathrm{F}}{\partial\mathrm{r}^2}+\kappa(\theta-\mathrm{r})\frac{\partial\mathrm{F}}{\partial\mathrm{r}}+\frac{\partial\mathrm{F}}{\partial\mathrm{t}}-\mathrm{rF}=0 \qquad (39)$$

Interest rate is zero

When the interest rate is zero (r=0), there is no discounting. Given the CIR mean-reverting square root process in equation (1), the value of the interest rate in the next time step, s, is certain to be $\kappa\theta s$. Thus, the boundary condition for the value of future mortgage payments is given by equation (40).

$$A(0,t)=A(\kappa\theta s,t+s) \qquad (40)$$

Other asset values, I and CI, represented by F(r,H,t) in equation (4) are given by another degenerate form, equation (41), when the interest rate is zero. For convenience, F(0,H,t) is replaced by F. Note that derivative terms in r do not vanish unless also multiplied by r (contrast this with Kau et al., 1995).

$$\frac{1}{2}H^2v^2\frac{\partial^2 F}{\partial H^2}-H\delta\frac{\partial F}{\partial H}+\kappa\theta\frac{\partial F}{\partial r}+\frac{\partial F}{\partial t}=0 \tag{41}$$

House price becomes very large

As house price, H, tends to infinity, the value of the default option tends to zero. This corresponds to one face of the box where the transformed variable, x, is zero. We represent this in equation (42).

$$\lim_{H\to\infty} D(H,r)=0 \tag{42}$$

The prepayment option value, C, is determined by the same degenerate form of equation (4) as used for H=0, this time with F(r,H→∞,t) replaced by F. Since the value of the default option tends to zero, the mortgage value at this extreme is given by the difference in equation (43).

$$\lim_{H\to\infty} V_B(H,r)=A(r)-\lim_{H\to\infty} C(H,r) \tag{43}$$

Since there is no default, the mortgage interest guarantee and the coinsurance have no value:

$$\lim_{H\to\infty} I(H,r)=0 \tag{44}$$

$$\lim_{H\to\infty} CI(H,r)=0 \tag{45}$$

Interest rate becomes very large

In the limit of infinite interest rate, any expected future payment is worthless and so we can immediately write the following:

$$\lim_{r\to\infty} A(r)=0 \tag{46}$$

$$\lim_{r\to\infty} C(H,r)=0 \tag{47}$$

$$\lim_{r\to\infty} D(H,r)=0 \tag{48}$$

$$\lim_{r \to \infty} V_B(H,r) = 0 \tag{49}$$

$$\lim_{r \to \infty} I(H,r) = 0 \tag{50}$$

$$\lim_{r \to \infty} \mathrm{CI}(\mathrm{H},\mathrm{r}) = 0 \tag{51}$$

Both H and r have extreme values

Next, we consider the edges of the box where extreme values of H and r occur simultaneously; these are zero and infinity or, in transformed co-ordinates, 1 and 0. First, we will take r = 0 and consider the two extremes of H. As noted earlier, for an interest rate of zero the CIR model, equation (1), requires that the value of the interest rate in the next time step, s, is certain to be $\kappa\theta s$ and so we may write:.

$$F(0,0,t) = F(0,\kappa\theta s, t+s) \tag{52}$$

$$\lim_{H \to \infty} F(H,0,t) = \lim_{H \to \infty} F(H,\kappa\theta s, t+s) \tag{53}$$

Finally, when r tends to infinity (and its transformed co-ordinate is 0), the conditions on the rest of face H versus time are maintained, as given in equations (46)-(51).

The Free Boundary: The Prepayment Region

Prepayment can take place at any time (it is an "American" style option) and gives rise to a free boundary, on one side of which it is financially optimal for a borrower to exercise the option to prepay and on the other side of which it is not. We obtain a boundary condition by observing that at each moment in time the value of the total debt, TD, must be at least as high as the value of the mortgage :

$$V_B \leq TD \tag{54}$$

Consequently, the prepayment boundary has the "value matching" condition shown in equation (55).

$$V_B = TD \tag{55}$$

It is also necessary to observe a "high-order contact" or "smooth-pasting" condition requiring that both functions meet tangentially at the boundary (see Merton, 1973). Putting this in a different way, it is required that not only the values of the functions V_B and TD, but also their slopes, should match at the boundary. As for a repayment mortgage the derivatives involving TD are zero, we may write:

$$\frac{\partial V_B}{\partial H} = \frac{\partial TD}{\partial H} = 0 \qquad (56)$$

$$\frac{\partial V_B}{\partial r} = \frac{\partial TD}{\partial r} = 0 \qquad (57)$$

If prepayment occurs, the default option immediately becomes worthless. This also happens for the mortgage insurance products, MIG and coinsurance. At the prepayment boundary, slopes of functions must match and so we have the following conditions:

$$\frac{\partial D}{\partial H} = \frac{\partial D}{\partial r} = \frac{\partial I}{\partial H} = \frac{\partial I}{\partial r} = \frac{\partial CI}{\partial H} = \frac{\partial CI}{\partial r} = 0 \qquad (58)$$

An important aspect that it is necessary to mention is related to the interaction between the "normal" boundary conditions and the free boundary. Obviously, inside the prepayment region the valuation function obeys a different regime. Consequently, it is necessary to expand this regime to the boundary in order to assure the smoothness of the solution near the boundaries that "touch" the prepayment region.

Numerical Treatment of the Free Boundary

Free boundaries are difficult to treat numerically. As mentioned in the previous chapter, there are two approaches to deal with such features: boundary tracking methods or the use of transformations capable of reducing the original problem to a fixed boundary one, from which the free-boundary can be inferred afterwards. The solution adopted here is one of the latter type. Drawing on Berger, Cement and Rogers (1975) the problem is converted into a non-linear PDE with a fixed boundary. The valuation equation originally assumed the form:

$$\frac{\partial V_B}{\partial t} + LV_B = 0 \qquad \text{if } V_B < TD$$

$$V_B = TD \qquad \text{otherwise} \qquad (59)$$

where L is the second-order linear operator in equation (4).

Noting that when $V_B = TD$:

$$\frac{\partial V_B}{\partial t} + LV_B = \frac{\partial TD}{\partial t} + LTD = (1 + \pi)RO - rTD \qquad (60)$$

the valuation equation can be rewritten in the following form:

$$\frac{\partial V_B}{\partial t} + LV = 0 \qquad \text{if } VB < TD$$

$$\frac{\partial V_B}{\partial t} + LV = (1 + \pi)RO - rTD \qquad \text{if } VB = TD \qquad (61)$$

Where the problem is now defined for the entire (H,r) space.

The Default Region

Default is only rational both outside the prepayment region and on mortgage payment dates. Thus, the default boundary is fully specified by equations (12), (15) and (17). Next, we will describe the evolution of the mortgage, taking into account these boundary conditions.

Mortgage Terms and Component Values

Mortgages are not standardized products and their terms vary between lenders, with more than one product being offered by any single lender. It must be emphasized that the models designed to value US mortgages are inadequate for valuing British mortgage products and vice-versa. The reasons for this are twofold. In the first place, the amount of the arrangement fee ("points" in the US) differs significantly between the two countries. In the second place, the insurance coverage that is associated with both products is different. In the American case, authors such as Kau et al 1993a assume the coverage is the lower of the actual loss or a pre-defined percentage of the value of the debt. In the British case, the loss coverage is shared between the insurer and the lender. A common arrangement is coverage by the insurer of 80% of the actual loss, subject to a cap equal to 20% of the original house price for a LTV of 95% (based on the difference between the actual LTV ratio and an arbitrarily defined "normal" LTV ratio of 75%). Obviously, the values of the two products do not coincide. Both features have direct implications for the determination of the equilibrium contract rates (see equation 31). Consequently, even for two mortgages that coincide in every detail except those two, the contract rates would differ and so would the values of all the underlying assets. In addition, many British mortgages have early termination penalties that affect the exercise of the options by the borrower and consequently the equilibrium contract rates. Furthermore, insurance coverage constitutes another crucial factor in the determination of equilibrium combinations.

In order to present and discuss the numerical results provided by the mortgage valuation model presented a basic set of economic parameters was chosen. The choice was made mainly on the basis of the standard assumptions in the literature (see Buser and Hendershott, 1984; Dunn and McConnell, 1981a,b; Kau et al. 1993b, 1995; Leung and Sirmans, 1990; Stanton, 1995; Stanton and Wallace, 1995). Table 1 presents these values. Unless noted otherwise, this set of economic parameters has been applied.

Table 1. Base Parameters

ECONOMIC ENVIRONMENT	
. Spot interest rate, r(0)	10%
. Long term average of interest rate (steady state), θ	10%
. Speed of reversion, κ.	25%
. Interest rate volatility, σ	5%
. House service flow, δ	7.5%
. House price volatility, ν	5%
. Correlation coefficient, ρ	0
CONTRACT	
. Maturity, η	300 months
. Value of the house at origination, H	£ 100,000
. Arrangement fee, ξ	1%
. Prepayment penalty, π	0

Every parameter exercises a double influence on the value of the different mortgage related assets and, consequently, on the value of the mortgage itself. Any change in a parameter used to characterize the economic environment leads to a change in the equilibrium contract rate. Consequently, besides the direct implications that derive from the change of the parameters, the value of each of the mortgage-related assets is also influenced by the modification that takes place in the contract rate. This phenomenon has severe implications for the type of empirical work that can be done in the field. Complete repetition of a certain set economic conditions is improbable and consequently, we are faced with different equilibrium rates for similar contracts in different moments in time. A sound empirical test of the implications of changes in the economic environment in terms of the different components of a mortgage contract would consist of observing the evolution of the market value of a mortgage contract during its economic life. In other words, a study of this kind would imply the analysis of the resale market for old mortgages (see Kau et al., 1992, 1995).

Setting The Contract Rate: Arbitrage-Free Conditions At The Initiation Of A Mortgage

When the mortgage is first arranged, it must be structured such that neither the borrower nor the lender can make an immediate profit. The values of the state variables r(0) and H(0) are known and a contract rate, R, meeting the no-arbitrage condition must be found by an iterative process (in this work: a secant iteration technique set for a margin of error less than £10 for a £100,000 house). The mortgage contract clauses which influence the possibility of early termination by the borrower and, hence, the value of the mortgage to the lender, are the arrangement fee, ξ, the early termination penalty, π, and the Mortgage Indemnity Guarantee, I.

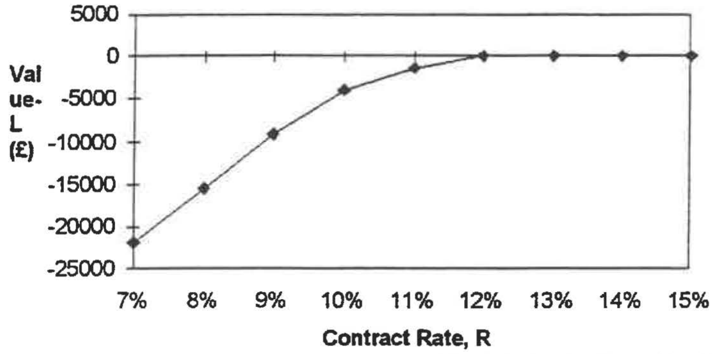
Neither the arrangement fee nor the mortgage indemnity guarantee affects the value of the other components of the loan (the value of the future payments and the options to prepay or default) and can be treated independently. Their effects on equilibrium coupon rates differ greatly. Inclusion of an arrangement fee simply adds to the value of the lender's position in the contract but the effects of the insurance component are more complex. Not only does its value evolve in different ways, according to the nature of the underlying contracts, but also it does not always change linearly within the same contract for different levels of the coupon rate.

In the most basic form of the repayment mortgage, without a MIG, arrangement fee or early termination fee, the equilibrium condition for setting up the mortgage becomes:

$$V_B[H(0),r(0),t(0),R]-L=0 \qquad (62)$$

For the contract to be viable, it is necessary that the value of the mortgage to the borrower, V_B, be equal the amount lent, L. As Kau et al. (1995) point out, for this to happen it is necessary that the prepayment region expands in such a way that (H(0), r(0)) becomes situated in the prepayment boundary (free-boundary), and immediate prepayment constitutes a possible optimal strategy for the borrower. Figure 1 illustrates the situation (note that the arrangement fee in Table 1 has been dropped).

Figure 1
Mortgage Value (Mortgage Contract Without Mortgage Indemnity Guarantee and Arrangement Fee)
Repayment Mortgage Without an Early Termination Penalty

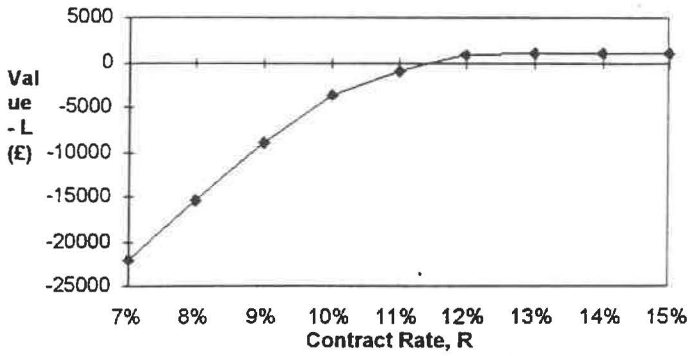


In this case, the borrower is indifferent between the alternatives of continuation and immediate repayment. Any increase in the contract rate that corresponds to this initial equilibrium situation generates a peculiar effect. It results in higher present value for future payments to be made by the borrower, but at the same time it also increases the value of the option of early repayment, C. As these are compensating effects, the borrower repays the loan immediately after taking it. Consequently, despite the possibility of finding contract rates capable of generating fair deals for both borrower and lender, no equilibrium exists, because those contract rates correspond to situations in which the mortgage is immediately terminated. The situation is resolved by inclusion of one or several of: arrangement fee, early termination penalty and MIG, which results in a single contract rate, as shown in Figure 6 for a prepayment penalty.

Figure 2

Mortgage Value (Mortgage Contract Without Mortgage Indemnity Guarantee and Arrangement Fee)

Repayment Mortgage With an Early Termination Penalty

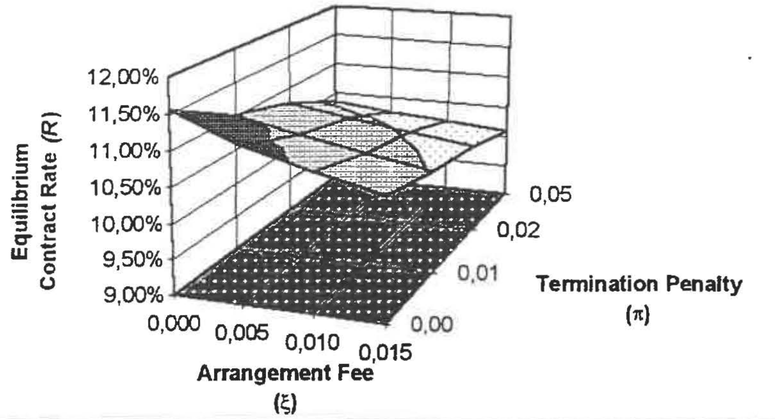


Each of the contractual features - arrangement fee, early termination penalty and MIG - generates a net benefit to the lender and lowers the equilibrium contract rate. The equilibrium combinations constitute a surface in (R,ξ,π) space. This is illustrated in Table 2 and Figure 3

Table 2: Trade-Off Between Arrangement Fee, Early Termination Penalty and Contract Rate

Prepayment	Arrangement Fee (ξ)			
Penalty (π)	0,000	0,005	0,010	0,015
0,00	11,57%	11,16%	10,92%	10,70%
0,01	11,02%	10,82%	10,64%	10,48%
0,02	10,75%	10,61%	10,46%	10,33%
0,05	10,34%	10,25%	10,16%	10,07%

Figure 3: Trade-Off Between Arrangement Fee, Early Termination Penalty and Contract Rate
(Repayment Mortgage)

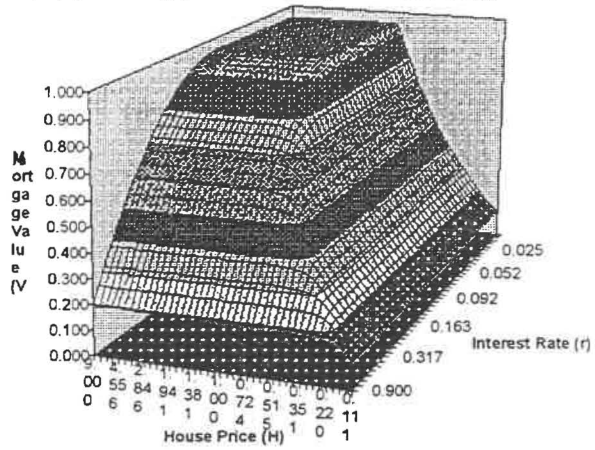


There is another situation in which no equilibrium exists: when the LTV ratio is unity so that L = H. A rational borrower is then indifferent between default and continuation. However, by delaying default, the borrower would benefit from the service flow of the house until the first payment was due and, consequently, there can be no equilibrium contract rate. Setting LTV ratios below unity is one way to reach equilibrium contract combinations (for a similar argument, see Kau et al., 1995).

Illustrative Results

Figure 4 shows the initial mortgage value, V_B, for a twenty five year mortgage specified by table 1, with the initial house price normalised to unity. Moving towards low levels of house price, default at the next payment date becomes increasingly likely, raising the value of the default option, D, and lowering the mortgage value. Moving towards higher levels, the prepayment option assumes greater significance. The values of both remaining future payments, A, and the prepayment option, C, vary inversely with interest rate, with opposite effects on V_B. Since C cannot be larger than A and is generally substantially smaller, the dominant effect on V_B is from A, reflected in the convex portion of Figure 4. The exception occurs when the house price is high and the interest rate is low, where there is a plateau in Figure 4.

Figure 4
Mortgage Value (V)
(Repayment Mortgage Without an Early Termination Penalty)



Figures 5 and 6 show the values of the default option, D, and the prepayment option, C. The default option is valuable throughout most of the state space for which the current house price is less than the initial price at which the mortgage was arranged and loses value on moving to higher interest rates. Since a mortgage in default cannot be prepaid, the prepayment option values in Figure 6 rise steeply on moving from low house prices closer to the initial price (unity).

Figure 5
Default Option (D)
Repayment Mortgage Without an Early Termination Penalty

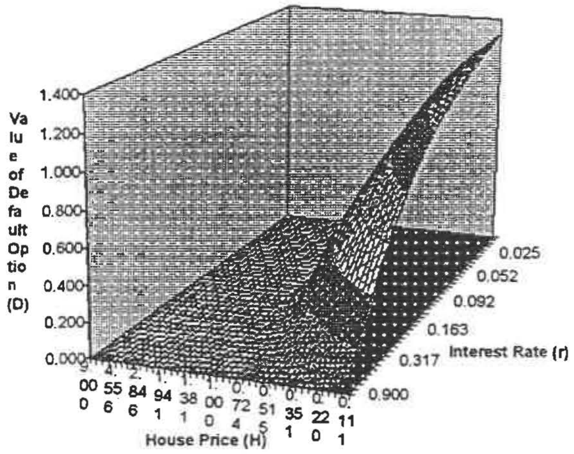
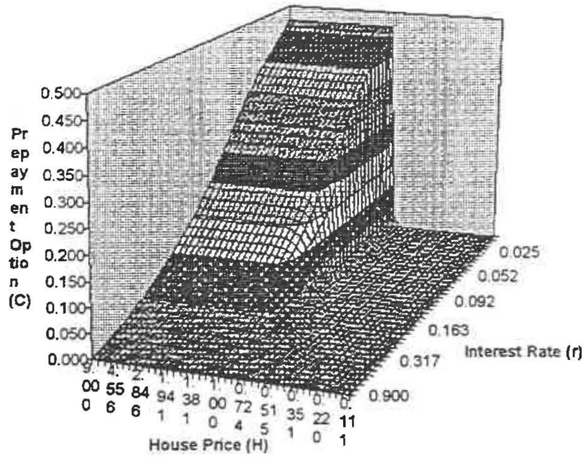


Figure 6
Value of Prepayment Option (C)
(Repayment Mortgage Without an Early Termination Penalty)



Figures 7 and 8 show the values of the mortgage indemnity guarantee, I, and the coinsurance, CI. Their relationship with the default option is apparent; for example, at high interest rate levels, house prices must fall greatly before the borrower will default, triggering the exercise of insurance. The coverage is capped and so Figure 7 shows a maximum being reached quite soon as the house price is reduced. Coinsurance covers the potential loss not covered by the mortgage interest guarantee, above the cap and Figure 8 shows its appearance in regions of (H,r) where I has reached the cap.

Figure 7
Value of Insurance Coverage (I)
(Repayment Mortgage Without an Early Termination Penalty)

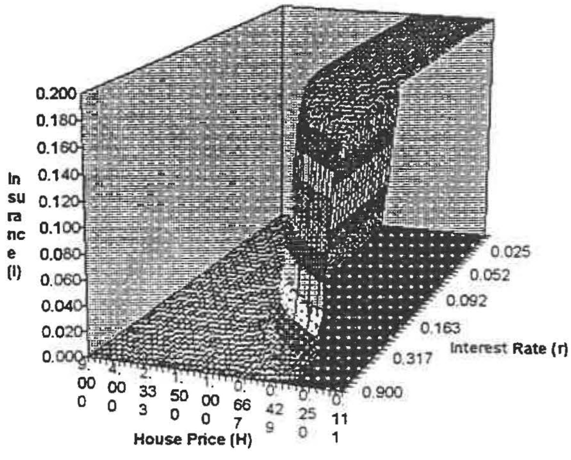
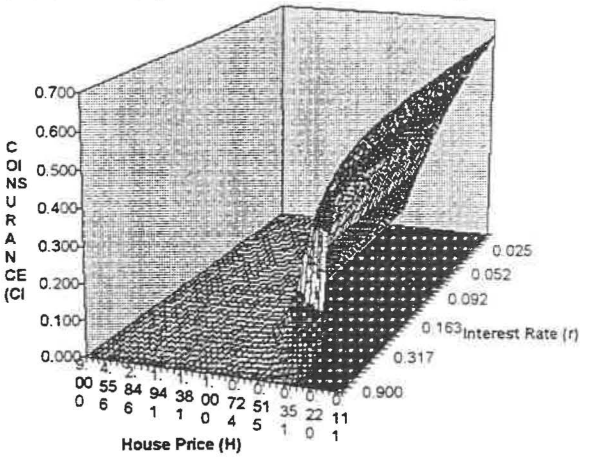


Figure 8
Value of Coinsurance (CI)
(Repayment Mortgage Without an Early Termination Penalty)



Note that the lender's exposure reaches a level of around 70% of the original house price, at the lower scale H=.11, for a L=.95 H(0) less the MIG cap of around .20 H(0). Inside the range of interest rates that can be considered historically common, the "delta" of the lender's exposure falls (eventually to −1) after house prices have collapsed to below 35% of the original house price of unity.

Conclusion

The theoretical framework developed in this work enables mortgage products to be valued which differ from those which can be handled by methods available in the literature for US products. Particular features are the early repayment penalties included in most UK fixed-rate mortgages, the mortgage indemnity guarantee (MIG) for a proportion of a lender's potential loss and the caps recently introduced into UK mortgage-related insurance for lending institutions. It also allows for valuation of coinsurance, the potential loss not covered by the MIG but covered by the lender.

No closed-form solutions are available for such complex contingent claims based on two stochastic factors. Closed-form solutions become less plausible in the presence of the special problems created by the free-boundary imposed by the American option to prepay the loan, and also by the need to cope with the idiosyncrasies dictated by the compound European option to default. As a consequence, the problem was solved numerically.

Our primary contribution is to model all of the mortgage components for the special factors of some British Mortgage Indemnity Guarantees using an explicit finite difference methodology. We show both equilibrium contract rates (in a "fair" environment) and the fair trade-offs between mortgage features such as arrangement fees, prepayment penalties and fixed contract rates. Then we show some aspects of the "at risk" elements (to interest rates and house prices) for all of the mortgage participants, including the borrower, the insurer and the lender. Extending this approach to viewing the full risk of each mortgage participant (to interest rate model parameters, to house price volatility, and to other realistic elements such as non-rational default and prepayment) are matters for future research.

APPENDIX A
Substitutions Needed To Transform The Original PDE

The transformation of the original PDE, with an infinite domain, into its equivalent in a unit square requires a series of substitutions which are given here. The original PDE, equation (4), for the valuation of any asset F(r,H,t), whose value is a function only of interest rate, r, house price, H, and time, t, is converted into its equivalent, W[r(y),H(x),t(τ)].

We begin with straightforward application of the chain rule, first in the r dimension:

$$\frac{\partial F}{\partial r} = \frac{\partial W}{\partial y}\frac{dy}{dr}$$

$$\frac{\partial^2 F}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial W}{\partial y}\frac{dy}{dr}\right)$$

$$\frac{\partial^2 F}{\partial r^2} = \frac{\partial^2 W}{\partial y^2}\left(\frac{dy}{dr}\right)^2 + \frac{d^2 y}{dr^2}\left(\frac{\partial W}{\partial y}\right)$$

Then in the H dimension:

$$\frac{\partial F}{\partial H} = \frac{\partial W}{\partial x}\frac{dx}{dH}$$

$$\frac{\partial^2 F}{\partial H^2} = \frac{\partial}{\partial H}\left(\frac{\partial W}{\partial x}\frac{dx}{dH}\right)$$

$$\frac{\partial^2 F}{\partial H^2} = \frac{\partial^2 W}{\partial x^2}\left(\frac{dx}{dH}\right)^2 + \frac{d^2 x}{dH^2}\left(\frac{\partial W}{\partial x}\right)$$

The treatment of the time derivative produces the following results:

$$\frac{\partial F}{\partial t} = \frac{\partial W}{\partial \tau}\frac{d\tau}{dt}$$

$$\frac{\partial F}{\partial t} = -\frac{\partial W}{\partial \tau}$$

Application of the chain rule to the mixed derivative leads to an apparently more complex expression:

$$\frac{\partial^2 F}{\partial H \partial r} = \frac{\partial}{\partial r}\left(\frac{\partial W}{\partial x}\frac{dx}{dH}\right)$$

$$\frac{\partial^2 F}{\partial H \partial r} = \frac{\partial^2 W}{\partial x \partial y} \frac{dy}{dr} \left(\frac{dx}{dH} \right) + \frac{\partial^2 x}{\partial H \partial r} \left(\frac{\partial W}{\partial x} \right)$$

The state variables r and H are independent of one another and hence the previous expression is simplified to:

$$\frac{\partial^2 F}{\partial H \partial r} = \frac{\partial^2 W}{\partial x \partial y} \frac{dy}{dr} \left(\frac{dx}{dH} \right)$$

Next, we need expressions for the derivatives of new state variables with respect to the old ones, first in the r dimension:

$$\frac{dy}{dr} = \frac{\partial}{\partial r} \left(\frac{1}{1 + \psi r} \right)$$

$$\frac{dy}{dr} = -\psi y^2$$

$$\frac{d^2 y}{dr^2} = \frac{d}{dr} \left(\frac{-\psi}{(1 + \psi r)^2} \right)$$

$$\frac{d^2 y}{dr^2} = 2\psi^2 y^3$$

Then in the H dimension:

$$\frac{dx}{dH} = \frac{d}{\partial H} \left(\frac{1}{1 + \omega H} \right)$$

$$\frac{dx}{dH} = -\omega x^2$$

$$\frac{d^2 x}{dH^2} = \frac{d}{dH} \left(\frac{-\omega}{(1 + \psi H)^2} \right)$$

$$\frac{d^2 x}{dH^2} = 2\omega^2 x^3$$

APPENDIX B
Details Of The Finite Difference Solution

In this appendix we present a finite difference representation of the transformed PDE on the lattice. We relate the parameters to positions in a three-dimensional box with sides representing house price, interest rate and time. Parameters and derivatives have calculated values only at the discrete grid points on the lattice (the "nodes"). Following a summary of notation, we begin by considering the interior nodes (the majority) and then consider boundary conditions on the sides and corners of the box.

Notation

The house price, H, and spot interest rate, r, were transformed into x and y respectively (equations 32 and 33). In the lattice, these are represented in the lengths of two dimensions of a unit cube, with the third dimension time, τ. The lengths are divided into J intervals for x, I intervals for y and N intervals for τ. Nodes on the lattice are then identified by lower case characters from j = 1 to J for x, i = 1, I for y and n = 1, N for τ. Clearly, the lattice spacings are 1/J for x, 1/I for y and 1/N for τ. It will be convenient later to refer to these spacings as h, l and s respectively. The positions of nodes are represented by x_j, y_i and τ_n where:

$$x_j = \frac{j}{J} = hj$$

$$y_i = \frac{i}{I} = li$$

$$\tau_n = \frac{n}{N} = sn$$

Following common practice (for example, see Morton and Mayers, 1994), we place time as a superscript and the other dimensions as subscripts when representing an approximation, U, of a function, W, at a node:

$$U^n_{i,j} \approx W(x_j, y_i, \tau_n)$$

Finite difference approximations are named forward, backward or central according to the direction of the approximation; for example, for derivatives with respect to x, increments in x correspond to successive integer values, j, and approximations between j and j+1 are termed "forward". Thus:

$$\frac{\partial W}{\partial x} \approx \frac{U^n_{i,j+1} - U^n_{i,j}}{h} \qquad \text{Forward difference}$$

$$\frac{\partial W}{\partial x} \approx \frac{U^n_{i,j} - U^n_{i,j-1}}{h} \qquad \text{Backward difference}$$

$$\frac{\partial W}{\partial x} \approx \frac{U^n_{i,j+1} - U^n_{i,j-1}}{2h} \qquad \text{Central difference}$$

A Finite Difference Representation Of The PDE

For the x and y dimensions we use the (more accurate) central difference approximations and for the temporal dimension, τ, we use a forward scheme in order to avoid known difficulties with stability (Wilmott et al., 1993, page 270). The transformed PDE can then be approximated in the finite difference lattice by making the following substitutions:

$$\frac{\partial W}{\partial x} \approx \frac{U^n_{i,j+1} - U^n_{i,j-1}}{2h}$$

$$\frac{\partial W}{\partial y} \approx \frac{U^n_{i+1,j} - U^n_{i-1,j}}{2l}$$

$$\frac{\partial W}{\partial \tau(t)} \approx \frac{U^{n+1}_{i,j} - U^n_{i,j}}{s}$$

$$\frac{\partial^2 W}{\partial x^2} \approx \frac{U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1}}{h^2}$$

$$\frac{\partial^2 W}{\partial x \partial y} \approx \frac{U^n_{i+1,j+1} - U^n_{i+1,j-1} - U^n_{i-1,j+1} + U^n_{i-1,j-1}}{4lh}$$

$$\frac{\partial^2 W}{\partial y^2} \approx \frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{l^2}$$

The finite difference equation is then:

$$\begin{aligned}
&\frac{1}{2}H(x)^2 v^2 \omega^2 x^4 \frac{U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1}}{h^2} \\
&+ \rho H(x)\sqrt{r(y)}v\sigma\psi\omega x^2 y^2 \frac{U^n_{i+1,j+1} - U^n_{i+1,j-1} - U^n_{i-1,j+1} + U^n_{i-1,j-1}}{4lh} \\
&+ \frac{1}{2}r(y)\sigma^2\psi^2 y^4 \frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{l^2} \\
&+ \left\{H(x)^2 v^2 \omega^2 x^3 - \left[(r(y) - \delta)H(x)\omega x^2\right]\right\} \frac{U^n_{i,j+1} - U^n_{i,j-1}}{2h} \\
&+ \left\{r(y)\sigma^2\psi^2 y^3 - \left[\kappa(\theta - r(y))\psi y^2\right]\right\} \frac{U^n_{i+1,j} - U^n_{i-1,j}}{2l} - \frac{U^{n+1}_{i,j} - U^n_{i,j}}{s} \\
&- r(y)U^n_{i,j} = 0
\end{aligned}$$

This equation is next rearranged to a form in which the value at a particular discrete time in the lattice is a function of its value in the previous time (recall that we work backwards in actual time, t, via the time parameter τ).

$$
\begin{aligned}
U_{i,j}^{n+1} = & \left\{1 - \left[H(x)^2 v^2 \omega^2 x^4 \left(\frac{s}{h^2}\right)\right] - \left[r(y)\sigma^2 \psi^2 y^4 \left(\frac{s}{l^2}\right)\right] - r(y)s\right\} U_{i,j}^n \\
& + \left\{\frac{1}{2} H(x)^2 v^2 \omega^2 x^4 \left(\frac{s}{h^2}\right)\right\}\left(U_{i,j+1}^n + U_{i,j-1}^n\right) \\
& + \left\{\left\{H(x)^2 v^2 \omega^2 x^3 - \left[(r(y) - \delta)H(x)\omega x^2\right]\right\}\frac{s}{2h}\right\}\left(U_{i,j+1}^n - U_{i,j-1}^n\right) \\
& + \left\{\frac{1}{2} r(y)\sigma^2 \psi^2 y^4 \left(\frac{s}{l^2}\right)\right\}\left(U_{i+1,j}^n + U_{i-1,j}^n\right) \\
& + \left\{\left\{r(y)\sigma^2 \psi^2 y^3 - \left[\kappa(\theta - r(y))\psi y^2\right]\right\}\frac{s}{2l}\right\}\left(U_{i+1,j}^n - U_{i-1,j}^n\right) \\
& + \rho H(x)\sqrt{r(y)}\, v\sigma\psi\omega x^2 y^2 \left(\frac{s}{4lh}\right)\left(U_{i+1,j+1}^n - U_{i+1,j-1}^n - U_{i-1,j+1}^n + U_{i-1,j-1}^n\right)
\end{aligned}
$$

The scheme is not quite finalised; in order to keep errors within bounds, it is necessary to guarantee that all the U^n coefficients are positive (the "maximum principle", Morton and Mayers, 1994). There are two ways to achieve this. Either the x and y step lengths can be reduced (substantially reducing the viability of the explicit finite difference algorithm) or changes must be made in the finite difference representation of the first derivatives, which we do in this work. The coefficients of the second derivative terms are always positive but this is not so for the first derivative terms. In our transformed valuation equation the coefficients are variable and the problem is even more acute, since the sign of the coefficients of the first derivative terms change across the lattice. This is dealt with by "upwind differencing". Central differences are used for second order derivatives but when the coefficient of a first derivative is positive, a forward difference approach is used and when negative a backward difference approach. This is readily handled in the Fortran code.

The solution involves an American-style option, exercisable at any time (the prepayment option) and a series of European-style options exercisable at each payment date (the default option). Working backwards, applying boundary conditions, involves a series of grids, one for each month. In this work, grids of 66 time steps per month were used, with I = 50 = J.

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