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Characterizing the effect of income distribution on the voluntary provision of public goods

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Characterizing the effect of income distribution on the voluntary provision of public goods*

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Abstract

The impact of income distribution changes on the provision of public goods has attracted a great deal of attention in various theoretical, empirical, and experimental fields. Applying a notion based on stochastic order, we consider how the distribution of income affects the amount of voluntarily provided public goods. We show that increasing convex order characterizes the total supply of public goods when the preferences of households are identical. Even if there is heterogeneity among households, it is still possible to describe how the distribution of income increases the voluntary supply of public goods by using a modified notion of increasing convex order. We can readily confirm the properties shown here by comparing reverse generalized Lorenz curves or a modified version.

key words: income distribution, public goods, increasing convex order
JEL code: D31, H41

1 Introduction

The voluntary provision of public goods by households plays a key role in sustaining the lives of people at various levels of society. Charitable gifts and donations are typical examples of contributions to public goods. Patronage funding has stimulated art and culture, and individual members of local communities spend considerable time on activities intended to provide common intangible resources such as safety and amenity. Individual actions to reduce the emissions of pollutants and greenhouse gases are an example of the private provision of global public goods. As is well known, the voluntary provision of public goods leads to an undersupply in the non-cooperative equilibrium. Thus, establishing the determinants of the voluntary provision of public goods has

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attracted attention not only as a theoretical issue but also as a policy matter.

Income is a significant factor affecting the total supply of public goods. Bergstrom et al. (1986), in their seminal paper, suggest that an increase in income inequality increases the aggregate provision of public goods in the Nash equilibrium. Although many experimental and empirical studies have examined the relationship between income inequality and public goods supply, their results and conclusions do not necessarily coincide. For example, Chan et al. (1996) reported a positive correlation between income inequality and public goods supply based on their experimental results, while other authors suggest a negative correlation (e.g., Cherry et al., 2005; Anderson et al., 2008) or no relationship (Chan et al., 1999; Buckley and Croson, 2006). A number of empirical studies have focused on the relationship between environmental quality and income inequality. For example, Ravallion et al. (2000) and Hübler (2017) found a positive correlation between carbon emissions and income inequality. In contrast, Baek and Gweisah (2012), and Kasuga and Takaya (2017) found that income equality is beneficial to environmental quality.

In this paper, we consider the income distribution characterizing the total supply of voluntary provided public goods. Despite their extensive analysis of Bergstrom et al. (1986), and considering the results from the experimental and empirical analyses described above, there remains a clear need to carefully investigate the effects of income distribution on the total supply of voluntarily provided public goods. Bergstrom et al. (1986) mainly considered the redistribution of income within the set of contributors or between the contributors and the non-contributors of public goods, focusing on relative inequality under the condition of constant total income. If we consider a situation in which overall income is variable, it is impossible to capture the characteristics of the distribution only by the transfer of income. In empirical analyses, aggregate indices such as the Gini coefficient have been used as explanatory variables to capture the level of income inequality. However, considering the variation in overall income and the heterogeneity of preference, we need to compare the distributions themselves rather than aggregate indices such as Gini coefficients.

To characterize the distribution of income, we use the notion of stochastic order introduced into the analysis of equity and social welfare (e.g., Atkinson, 1970). The results of this paper are quite simple: Increasing convex order characterizes the total supply of public goods when the preferences of households having a quasi-concave utility function with normality for both private and public goods are the same. The distribution that is larger in terms of increasing convex order leads to more public goods in the Nash equilibrium. Furthermore, even if there is heterogeneity among households, it is possible to characterize the distribution of income to increase the voluntary supply of public goods by using a modified version of increasing convex order. Usefully, reverse generalized Lorenz curves corresponding to the distributions for the purpose of comparison provide simple graphical representations of such partial ordering.

Our results contrast with those taken from the theory of social welfare and income inequality.

For two distributions with the same mean, social welfare as characterized by the Schur concavity is greater in the distribution that dominates the other in terms of Lorenz curves (Atkinson, 1970). In contrast, in the Lorenz-dominated distribution, the total supply of public goods in the Nash equilibrium is larger. In this sense, it is correct to say that an increase in inequality increases the total supply of public goods. Furthermore, when total incomes differ, the generalized Lorenz dominance criterion describes welfare superiority (Shorrocks, 1983). On the other hand, the total supply of public goods is greater in the distribution that dominates the other in terms of the reverse generalized Lorenz curve derived from the inverse of the integrated survival function.

To our knowledge, few studies have sought to characterize the relationship between income distribution and the supply of public goods by using the distribution function. Andreoni (1988) analyzed the relationship between group size and the voluntary supply of public goods with the aid of the distribution function, taking into account differences in household preferences and income. Although our analytical framework is based on Andreoni (1988), our focus is on how the distribution of income affects the aggregate supply of public goods, while Andreoni (1988) considered the set of contributors when the number of individuals increases. To characterize the income distribution, we refer to the procedure proposed in Bourguignon (1989) in which he established a dominance criterion for ordering the distribution of income with different family sizes in terms of welfare. Fleurbaey et al. (2003) developed a simple formula to test the theorem in Bourguignon (1989). We apply their procedures in our analysis of the aggregate supply of public goods with necessary modifications. This makes it easy to compare our results on income distribution with the literature on welfare dominance (e.g., Shorrocks, 1983; Atkinson and Bourguignon, 1987; Jenkins and Lambert, 1993; Fleurbaey et al., 2003).

The remainder of the paper is organized as follows: In the next section, we present our analytical framework; in Section 3, we consider the income distribution in a society consisting of identical households; in Section 4, we study the case in which the households are heterogeneous; in Section 5, we focus on various demographics; in Section 6, we offer concluding remarks and suggest future directions.

2 Analytical Framework

The model employed here is a conventional one of privately provided public goods according to, e.g., Bergstrom et al. (1986) and Andreoni (1988). Consider a society consisting of n households. Let $\mathcal{N} := \{1,2,\ldots,n\}$ be the set of households. We denote the exogenous income of household i as $w_i \in \mathcal{D} := [0, v] \subset \mathbb{R}_+$, where $v < \infty$ is the maximum conceivable income. The distribution of income is represented by a vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. The set of income distribution is represented by

$$\mathcal{L} = \{\mathbf{w} = (w_1, w_2, \dots, w_n): w_i \in \mathcal{D}, i \in \mathcal{N}\}.$$

We assume that there is one private good and one public good. Let x_i and g_i be household i's consumption of private goods and its contribution to public goods, respectively. Assume that the price of the private good is constant at 1 and that one unit of private goods can transform into one unit of public good. The total provision of the public good is $g = \sum_{i=1}^{n} g_i$. Utility is represented by a continuous and strictly quasi-concave function $u_i = u_i(x_i, g)$ for $i \in \mathcal{N}$, where x_i denotes household i's consumption of the private goods. We assume that both the public good and the private good are normal. We denote by $\mathcal{U}$ the set of utility function satisfying quasi-concavity and normality.

Under the Nash conjecture, each household decides its contribution to the public good by solving the following maximization problem:

$$\max_{x_i, g} u_i(x_i, g),$$

subject to

$$x_i + g = w_i + g_{-i},$$
$$g \geq g_{-i},$$

where $g_{-i} = g - g_i$ is the aggregate contributions of the society other than household i. Solving this yields a continuous demand function for the public good: $g = \max\{f_i(w_i + g_{-i}), g_{-i}\}$ for $i \in \mathcal{N}$.

If the inequality constraint is not binding, the contribution by household i will be $g_i = f_i(w_i + g_{-i}) - g_{-i}$. From the normality condition, we have $f_i'(w_i + g_{-i}) \in (0,1), \forall i \in \mathcal{N}$ which implies that f_i is increasing. Under these conditions, there exists a unique Nash equilibrium (e.g., Bergstrom et al., 1986). Furthermore, inverting $g = f_i(w_i + g_{-i})$ yields $f_i^{-1}(g) = w_i + g_{-i}$. Because $g_{-i} = g$ holds for the non-contributor to the public good, $w_i^*(g) = f_i^{-1}(g) - g$ denotes the critical income that distinguishes the contributors from the non-contributors: $w_i^*(g) \geq w_i \Leftrightarrow g_i = 0$ and $w_i^*(g) < w_i \Leftrightarrow g_i > 0$. From the normality condition, $w_i^*(g)$ is strictly increasing function: $w_i^{*\prime}(g) \in (0, \infty)$. For example, given a utility function such as $u_i = x_i^{\beta/(1+\beta)} g^{1/(1+\beta)}$, we can represent the critical income as $w_i^*(g) = \beta g$.

Thus, for given $\mathbf{w} \in \mathcal{L}$, the total provision of the public good can be implicitly given as

$$g = \sum_{i=1}^{n} \left(w_i - w_i^*(g)\right)^+, \tag{1}$$

where $(a)^+ = \max\{a, 0\}$. Solving (1), we obtain the total provision of the public goods in the Nash equilibrium as $g = g(\mathbf{w})$. It should be noted that, for given g, $g_i(w_i) = \left(w_i - w_i^*(g)\right)^+$ is an increasing (non-decreasing) convex function in w_i[1]. In (1), $w_i^*(g)$ can be regarded as the needs for private goods; that is, household i, which is a contributor, spends $w_i^*(g)$ for private goods.

If all households are contributors to the public goods, then the total provision of public goods is $g = \sum_{i=1}^n w_i - \sum_{i=1}^n w_i^*(g)$. In such a situation, the neutrality theorem of public goods by Warr (1983) is valid. In general, we can summarize the relationship between the total supply of public goods and the distribution of income as follows.

Lemma 1. Consider two distributions $\mathbf{w}^X, \mathbf{w}^Y \in \mathcal{L}$. Let $g^J := g(\mathbf{w}^J)$ for $J \in \{X, Y\}$ be the equilibrium provision of the public good under the distribution of $\mathbf{w}^J$. If and only if the following conditions are met, $g^X \geq g^Y$ holds.

$$\sum_{i=1}^n \left(w_i^X - w_i^*(g^J)\right)^+ \geq \sum_{i=1}^n \left(w_i^Y - w_i^*(g^J)\right)^+, \qquad \text{for } J \in \{X, Y\}. \tag{2}$$

Proof. Suppose that $g^X \geq g^Y$ holds. From (1) and the increasingness of $w_i^*(g)$, we have

$$\sum_{i=1}^n \left(w_i^X - w_i^*(g^Y)\right)^+ \geq \sum_{i=1}^n \left(w_i^X - w_i^*(g^X)\right)^+ \geq \sum_{i=1}^n \left(w_i^Y - w_i^*(g^Y)\right)^+ \geq \sum_{i=1}^n \left(w_i^Y - w_i^*(g^X)\right)^+.$$

Thus, the condition holds. Next, suppose that $g^Y > g^X$. From (1) and the increasingness of $w_i^*(g)$, we have

$$\sum_{i=1}^n \left(w_i^Y - w_i^*(g^X)\right)^+ \geq \sum_{i=1}^n \left(w_i^Y - w_i^*(g^Y)\right)^+ > \sum_{i=1}^n \left(w_i^X - w_i^*(g^X)\right)^+ \geq \sum_{i=1}^n \left(w_i^X - w_i^*(g^Y)\right)^+,$$

which implies that $\sum_{i=1}^n \left(w_i^X - w_i^*(g^J)\right)^+ < \sum_{i=1}^n \left(w_i^Y - w_i^*(g^J)\right)^+$ for $J \in \{X, Y\}$. ■

In the analysis that follows, we compare two distributions, based on the theory of stochastic dominance. We can think of distribution $\mathbf{w}^X \in \mathcal{L}$ as a random variable X taking values w_i^X with probability $P\{X = w_i^X\} = 1/n$ for $i \in \mathcal{N}$. In the case of $\mathbf{w}^Y \in \mathcal{L}$, random variable Y is similarly defined. $F^J(t): \mathbb{R} \to [0,1]$ for $J \in \{\mathrm{X}, \mathrm{Y}\}$ denotes the distribution function corresponding to

[1] Hereafter, the term "increasing" includes both non-decreasing and strictly increasing.

$\mathbf{w}^J \in \mathcal{L}$. Distribution function F^J is non-decreasing and right-continuous. The arithmetic mean of distribution $\mathbf{w}^J \in \mathcal{L}$ is denoted by $\mu^J := \sum_{i=1}^{n} w_i^J / n$.

Based on the distribution function, we employ the survival function to characterize the aggregate quantity of privately provided public goods. We denote by $\bar{F}^J(t) = 1 - F^J(t)$ for $t \in \mathbb{R}$ and $J \in \{X, Y\}$ the survival function of F^J. The following expressions are well known in the theory of stochastic dominance (e.g., Shaked and Shanthikumar, 1994).

$$\bar{F}^{J(2)}(t) := \int_t^{\infty} \bar{F}^J(z)dz = \frac{1}{n}\sum_{i=1}^{n}\left(w_i^J - t\right)^+, \quad t \in \mathbb{R} \text{ and } J \in \{X, Y\}. \tag{3}$$

where $\bar{F}^{J(2)}$ is sometimes referred to as the integrated survival function (e.g., Müller and Stoyan, 2002), which is decreasing convex in t. Since our concern is the distribution $\mathbf{w}^J \in \mathcal{L}$, $\bar{F}^{J(2)}(0) = \mu(\mathbf{w}^J)$ and $\bar{F}^{J(2)}(t) = 0$ for $t \geq v$ hold. For a given distribution with finite mean μ^J, the following expression connects the integrated survival function with the integrated distribution function, $F^{J(2)}(t) := \int_{-\infty}^{t} F^J(x)dx$.

$$\bar{F}^{J(2)}(t) = F^{J(2)}(t) - (t - \mu^J), \qquad t \in \mathbb{R} \text{ and } J \in \{X, Y\}. \tag{4}$$

Let

$$\bar{H}^{(2)}(t) := \bar{F}^{X(2)}(t) - \bar{F}^{Y(2)}(t) \text{ for } t \in \mathbb{R}, \tag{5}$$

be the difference in the integrated survival functions. Given two distributions $\mathbf{w}^X, \mathbf{w}^Y \in \mathcal{L}$, $\mathbf{w}^Y$ is smaller than $\mathbf{w}^X$ in terms of the increasing convex order if $\bar{H}^{(2)}(t) \geq 0$ holds $\forall t \in \mathbb{R}$. Furthermore, if $\bar{H}^{(2)}$ is non-increasing $\forall t \in \mathbb{R}$, $\mathbf{w}^Y$ is smaller than $\mathbf{w}^X$ with respect to the usual stochastic order (e.g., Müller and Stoyan, 2002).

3 Identical Agents

First, we consider a situation in which all households' preferences are identical: $w_i^*(g) = w^*(g)$, $\forall i \in \mathcal{N}$. From (1), we can represent the total provision of public goods by $g^J = \sum_{i=1}^{n}\left(w_i^J - w^*(g^J)\right)^+$ for $J \in \{X, Y\}$. We immediately obtain the following result.

Proposition 1 Suppose that all household preferences are identical. Consider two distributions $\mathbf{w}^X$, $\mathbf{w}^Y \in \mathcal{L}$ such that g^X and g^Y are the total provision of public goods in the Nash equilibrium, respectively. The following two conditions are equivalent.

i). $\overline{H}^{(2)}(t) \geq 0$ holds $\forall t \in \mathbb{R}$,

ii) $g^X \geq g^Y$ holds $\forall u \in \mathcal{U}$.

Proof. Suppose that i) holds. For $\mathbf{w}^J$, $J \in \{X, Y\}$, let g^J and $w^*(g^J)$ be the total provision of the public good and the critical income in the Nash equilibrium, respectively. Condition i) implies that

$$\sum_{i=1}^{n} \left(w_i^X - w^*(g^J)\right)^+ \geq \sum_{i=1}^{n} \left(w_i^Y - w^*(g^J)\right)^+, \; J \in \{X, Y\}, \tag{6}$$

holds. From Lemma 1, we obtain $g^X \geq g^Y$. Next, suppose that $\overline{H}^{(2)}(t) < 0$ for some $t \in \mathbb{R}$. In such a case we can find $\hat{t} \in (0, v)$ such that $\sum_{i=1}^{n}\left(w_i^X - \hat{t}\right)^+ < \sum_{i=1}^{n}\left(w_i^Y - \hat{t}\right)^+$ holds[2]. Considering a utility function $u_i = x_i^{\beta/(1+\beta)} g^{1/(1+\beta)}$ for $i \in \mathcal{N}$ and $\beta \in (0, \infty)$, we can confirm that $w^*(g) = \beta g$ and $u_i \in \mathcal{U}$. Noting that $\sum_{i=1}^{n}\left(w_i^Y - \hat{t}\right)^+$ is strictly positive, when we set $\beta = \hat{t} / \sum_{i=1}^{n}\left(w_i^Y - \hat{t}\right)^+$, the following relations hold in the Nash equilibrium:

$$g^Y = \sum_{i=1}^{n}\left(w_i^Y - \beta g^Y\right)^+ = \sum_{i=1}^{n}\left(w_i^Y - \hat{t}\right)^+ > \sum_{i=1}^{n}\left(w_i^X - \hat{t}\right)^+ = \sum_{i=1}^{n}\left(w_i^X - \beta g^Y\right)^+. \tag{7}$$

Thus, from Lemma 1, we obtain $g^X < g^Y$. ■

Proposition 1 states that the increasing convex order characterizes the relationship between the quantity of public goods supply and income distribution. In Proposition 1, we do not restrict the total amount of income in the two distributions. If the total incomes, and, therefore, the mean incomes, are the same for $\mathbf{w}^X$ and $\mathbf{w}^Y$, $\overline{H}^{(2)}(t) \geq 0$ also implies $H^{(2)}(t) := \int_{-\infty}^{t}[F^X(z) - F^Y(z)]dz \leq 0, \forall t \in \mathbb{R}$. That is, if $\sum_{i=1}^{n} w_i^X = \sum_{i=1}^{n} w_i^Y$ holds, then $g^X \geq g^Y$ is equivalent to that $\mathbf{w}^X$ second-order stochastically dominates $\mathbf{w}^Y$. In this situation, we can predict the total supply of public goods by comparing the Lorenz curves (e.g., Iritani and Kuga, 1983).

When the total incomes differ, the reverse generalized Lorenz (RGL) curve characterizes the total supply of the public goods[3]. Let

[2] Since $\left(w_i^J - t\right)^+ \geq 0$, $\sum_{i=1}^{n}\left(w_i^X - t\right)^+ < \sum_{i=1}^{n}\left(w_i^Y - t\right)^+$ implies that t must be smaller than $\max\{w_i^Y : i \in \mathcal{N}\}$. Furthermore, $\max\{w_i^Y : i \in \mathcal{N}\}$ must be strictly positive. Thus, if $\sum_{i=1}^{n}\left(w_i^X - t\right)^+ < \sum_{i=1}^{n}\left(w_i^Y - t\right)^+$ holds for $t \leq 0$, this inequality still hold for $t \in (0, \mathrm{w}_0]$, where $w_0 = \min\{w_i^X, w_i^Y : w_i^J > 0, J = X, Y \text{ and } i \in \mathcal{N}\}$.

[3] The term "reverse generalized Lorenz curve" can be found in Bazen and Moyes (2012). We can also define the RGL curve as the integral of the inverse survival function.

$$F^{(-1)}(p) \coloneqq \inf\{x: F(x) \geq p\}, \qquad \text{for } p \in (0,1], \tag{8}$$

be the inverse distribution function of $F(x)$. We define the RGL curve $\overline{GL}^J(\theta)$ as follows:

$$\overline{GL}^J(\theta) := \mu^J - F^{J(-2)}(1-\theta), \qquad \text{for } \theta \in [0,1] \text{ and } J \in \{X,Y\}, \tag{9}$$

where

$$F^{J(-2)}(p) \coloneqq \int_0^p F^{J(-1)}(x)dx, \qquad \text{for } p \in (0,1] \text{ and } J \in \{X,Y\},$$

and $F^{J(-2)}(0) \coloneqq 0$ denote the second quantile function and $\mu^J = \mu(\mathbf{w}^J)$. For later discussion, we set $F^{(-2)}(p) = \infty$ for $p \notin [0,1]$. In order to connect the previous result to the RGL curves, the following property from Ogryczak and Ruszczyński (2002) is useful.

Lemma 2 (Theorem 3.1, Ogryczak and Ruszczyński, 2002). For the distribution F^J with finite mean, the following two expressions hold.

$$F^{J(-2)}(p) = \sup_t\{tp - F^{J(2)}(t)\}, \tag{10a}$$

$$F^{J(2)}(t) = \sup_p\{tp - F^{J(-2)}(p)\}. \tag{10b}$$

Proof. See Ogryczak and Ruszczyński (2002). ■

By using Lemma 2, we can characterize the income distribution in terms of the RGL curves.

Proposition 2 Consider two distributions $\mathbf{w}^X, \mathbf{w}^Y \in \mathcal{L}$. The following two conditions are equivalent.

i). $\overline{H}^{(2)}(t) \geq 0$ holds $\forall t \in \mathbb{R}$,

ii) $\overline{GL}^X(\theta) \geq \overline{GL}^Y(\theta), \forall \theta \in [0,1]$.

Proof. First, suppose that condition i) holds. From (9) and Lemma 2, we obtain

$$\overline{GL}^X(\theta) - \overline{GL}^Y(\theta) = \mu^X - \mu^Y - \sup_t\{(1-\theta)t - F^{X(2)}(t)\} + \sup_t\{(1-\theta)t - F^{Y(2)}(t)\}. \tag{11}$$

For a given $\theta \in [0,1]$, we can choose t^* such that $(1-\theta)t - F^{X(2)}(t)$ attains the supremum at $t = t^*$. Inserting t^* into (11) and considering (4), we obtain

$$\begin{aligned} \overline{GL}^X(\theta) - \overline{GL}^Y(\theta) &= \mu^X - \mu^Y - (1-\theta)t^* + F^{X(2)}(t^*) + \sup_t\{(1-\theta)t - F^{Y(2)}(t)\} \\ &\geq \mu^X - \mu^Y + F^{X(2)}(t^*) - F^{Y(2)}(t^*) \\ &= \overline{H}^{(2)}(t^*) \geq 0. \end{aligned} \tag{12}$$

Conversely, suppose that condition ii) holds. From Lemma 2, we obtain

$$\overline{H}^{(2)}(t) = \mu^X - \mu^Y + \sup_p\{tp - F^{X(-2)}(p)\} - \sup_p\{tp - F^{Y(-2)}(p)\}. \tag{13}$$

For a given $t \in \mathcal{D}$, we can choose p^* such that $tp - F^{Y(-2)}(p)$ attains the supremum at p^*. Inserting p^* into (13), we obtain

$$\begin{aligned} \overline{H}^{(2)}(t) &= \mu^X - \mu^Y + \sup_p\{pt - F^{X(-2)}(p)\} - \left(p^*t - F^{Y(-2)}(p^*)\right) \\ &\geq \left(\mu^X - F^{X(-2)}(p^*)\right) - \left(\mu^Y - F^{Y(-2)}(p^*)\right) \\ &= \overline{GL}^X(1-p^*) - \overline{GL}^Y(1-p^*) \geq 0. \end{aligned} \tag{14}$$

Thus, the claim is proved. ■

Note that a relative index of income inequality such as the Gini coefficient does not fully characterize the total supply of public goods in the Nash equilibrium even though the Lorentz curves for comparison do not intersect. Furthermore, the generalized Lorenz (GL) curve proposed by Shorrocks (1983) also does not coincide with the comparison based on the RGL curves. Let $GL^J(\theta) := \int_0^\theta F^{J(-1)}(x)dx$ be the GL function corresponding to F^J. Inserting this into (9), we obtain $\overline{GL}^J(\theta) := \mu^J - GL^J(1-\theta)$. Thus, $\overline{GL}^X(\theta) \geq \overline{GL}^Y(\theta)$ $\forall\theta \in [0,1]$ does not necessarily imply $GL^X(1-\theta) \geq GL^Y(1-\theta)$ $\forall\theta \in [0,1]$.

4 Heterogeneous Agents

We now consider a situation in which the households are partitioned into $K(>1)$ types. Let $\mathcal{K} = \{1, \dots, K\}$ be the set of household types. Each household is classified according to its needs for the private goods: $w_i^*(g)$. We assume that $\infty > w_1^*(g) \geq w_2^*(g) \geq \dots, \geq w_K^*(g) > 0$ for $g \in (0, \infty)$. That is, we focus on the set of utility profile such that

$$\mathcal{U}_{\mathcal{K}} \coloneqq \{(u_1, \dots, u_K): u_k \in \mathcal{U}, k \in \mathcal{K}, \infty > w_1^*(g) \geq w_2^*(g) \geq \dots, \geq w_K^*(g) > 0, g \in (0, \infty)\}.$$

The number of households of type $k \in \mathcal{K}$ is n_k. Thus, $n = \sum_{k=1}^{K} n_k$. In this section, we compare distributions with identical demographic composition, denoting the set of households belonging to type $k \in \mathcal{K}$ as $\mathcal{N}_k \coloneqq \{1, \dots, n_k\}$.

The income of households of type k is distributed in the range $\mathcal{D}_k := [0, v_k] \subset \mathbb{R}_+$. We denote $\bar{v} \coloneqq \max\{v_1, \dots, v_K\}$. As in the previous section, we consider the distributions $\mathbf{w}_{\mathcal{K}} := [\mathbf{w}_1, \dots, \mathbf{w}_K]$ where $\mathbf{w}_k := (w_{k,1}, \dots, w_{k,n_k})$. We represent the set of income distribution consisting of the subset partitioned by the type of household as

$$\mathcal{L}_{\mathcal{K}} \coloneqq \left\{[\mathbf{w}_1, \dots, \mathbf{w}_K]: w_{k,i} \in \mathcal{D}_k, i \in \mathcal{N}_k, k \in \mathcal{K}\right\}.$$

For $\mathbf{w}_{\mathcal{K}}^J \in \mathcal{L}_{\mathcal{K}}$, total provision of the public goods in the Nash equilibrium can be written as

$$g^J = \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left(w_{k,i}^J - w_k^*(g^J)\right)^+. \tag{15}$$

Consider two distributions $\mathbf{w}_{\mathcal{K}}^X, \mathbf{w}_{\mathcal{K}}^Y \in \mathcal{L}_{\mathcal{K}}$. Let X_k be a random variable consisting of possible value $w_{k,1}^X$ with probability $1/n_k$. Y_k is similarly defined. Let $F_{\mathcal{K}}^J = (q_1, \dots, q_K, F_1^J, \dots, F_K^J)$ be the profile of distribution $J \in \{X, Y\}$, where F_k^J, which is non-decreasing and right-continuous, is the distribution function corresponding to $\mathbf{w}_k^J$ and $q_k \coloneqq n_k/n$ is the proportion of type k household. We denote by $\bar{F}_k^J(t) = 1 - F_k^J(t)$ for $J \in \{X, Y\}$ the survival functions of F_k^J for $k \in \mathcal{K}$. Furthermore, the integrated survival functions for type k households are

$$\bar{F}_k^{J(2)}(t_k) := \int_{t_k}^{\infty} \bar{F}_k^J(z)dz = \frac{1}{n_k} \sum_{i=1}^{n_k} \left(w_{k,i}^J - t_k\right)^+, \quad \text{for } t_k \in \mathbb{R}. \tag{16}$$

and the difference of the integrated survival functions is

$$\bar{H}_k^{(2)}(t) \coloneqq \bar{F}_k^{X(2)}(t_k) - \bar{F}_k^{Y(2)}(t_k), \qquad \text{for } t_k \in \mathbb{R} \text{ and } k \in \mathcal{K}. \tag{17}$$

From (15) - (17), and Lemma1, we have the following result.

Proposition 3. Consider two distributions $\mathbf{w}_{\mathcal{K}}^X, \mathbf{w}_{\mathcal{K}}^Y \in \mathcal{L}_{\mathcal{K}}$ such that $g^X = g(\mathbf{w}_{\mathcal{K}}^X)$ and $g^Y = g(\mathbf{w}_{\mathcal{K}}^Y)$ are the total provision of public goods in the Nash equilibrium. The following two conditions are equivalent.

i). $\sum_{k=1}^{K} q_k \bar{H}_k^{(2)}(t_k) \geq 0$ holds $\forall \infty > t_1 \geq t_2 \geq, \dots, \geq t_K > -\infty$,

ii) $g^X \geq g^Y$ holds $\forall (u_1, \dots, u_K) \in \mathcal{U}_{\mathcal{K}}$.

Proof. Suppose that i) is satisfied. From the definition of $\bar{H}_k^{(2)}(t_k)$ and (16), we obtain

$$\sum_{k=1}^{K} q_k \bar{H}_k^{(2)}(t_k) = \frac{1}{n} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left[\left(w_{k,i}^X - t_k\right)^+ - \left(w_{k,i}^Y - t_k\right)^+ \right] \geq 0. \tag{18}$$

$\forall \infty > t_1 \geq t_2 \geq, \dots, \geq t_K > -\infty$. Thus, for all $\infty > w_1^*(g^J) \geq w_2^*(g^J) \geq, \dots, \geq w_K^*(g^J) > 0$, condition i) also holds. From Lemma 1, this implies $g^X \geq g^Y$. Next, suppose that $\sum_{k=1}^{K} q_k \bar{H}_k^{(2)}(t_k) < 0$ holds for some $\infty > \hat{t}_1 \geq \hat{t}_2 \geq, \dots, \geq \hat{t}_K > -\infty$. In this case, we can find a utility function such that $g^X < g^Y$ holds as in Proposition 1.[4] Therefore, necessity is verified. ∎

Although condition i) in Proposition 3 is a simple application of the increasing convex order, it is difficult to confirm its establishment. Instead, we can show the condition that is equivalent to condition i). This assertion is a straightforward application of Bourguignon (1989) and Fleurbaey et al. (2003). First, for $t \in [0, \infty)$, we define the following function.

$$\bar{Z}_K(t_{K-1}) = \min_{t_K \leq t_{K-1}} \left\{ q_K \bar{H}_K^{(2)}(t_K) \right\}, \tag{19a}$$

$$\bar{Z}_k(t_{k-1}) = \min_{t_k \leq t_{k-1}} \left\{ q_k \bar{H}_k^{(2)}(t_k) + \bar{Z}_{k+1}(t_k) \right\}, \qquad \text{for } k \in \mathcal{K} \backslash \{1, K\}. \tag{19b}$$

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[4] Noting that $\sum_{k=1}^{K} q_k \bar{H}_k^{(2)}(t_k) < 0$ implies $\sum_{k=1}^{K} \sum_{i=1}^{n_k} \left(w_{k,i}^Y - t_k\right)^+ > 0$, we can find a utility function such that $u_i^k = x_i^{\beta_k/(1+\beta_k)} g^{1/(1+\beta_k)}$, where $\beta_k = \hat{t}_k / \sum_{k=1}^{K} \sum_{i=1}^{n_k} \left(w_{k,i}^Y - \hat{t}_k\right)^+$.

By the definition $\bar{Z}_k(t)$ is a decreasing and $\bar{Z}_k(0) = \sum_{r=k}^{K} q_k(\mu_k^X - \mu_k^Y)$. Furthermore, when $\bar{Z}_k(t) \geq 0\ \ \forall t \in [0, \bar{v}]$, $\bar{Z}_k(\bar{v}) = 0$ holds.

We obtain the following result.

Proposition 4 Consider two distributions $\mathbf{w}_{\mathcal{K}}^X, \mathbf{w}_{\mathcal{K}}^Y \in \mathcal{L}_{\mathcal{K}}$ such that $g^X = g(\mathbf{w}_{\mathcal{K}}^X)$ and $g^Y = g(\mathbf{w}_{\mathcal{K}}^Y)$ are the total provision of public goods in the Nash equilibrium. The following two conditions are equivalent.

i) $\sum_{k=1}^{K} q_k \bar{H}_k^{(2)}(t_k) \geq 0$ holds $\forall \infty > t_1 \geq t_2 \geq, \dots, \geq t_K > -\infty$,

ii) $q_1 \bar{H}_1^{(2)}(t_1) + \bar{Z}_2(t_1) \geq 0,\ \forall t_1 \in \mathbb{R}$.

Proof. First, suppose that condition i) holds. For arbitrary $t_1^* \in \mathbb{R}$, let $t_k^* := \arg\min_{t_k \leq t_{k-1}^*}\{q_k \bar{H}_k^{(2)}(t_k) + \bar{Z}_{k+1}(t_k)\}$ for $k = 2, \dots, K-1$ and $t_K^* := \arg\min_{t_K \leq t_{K-1}^*}\{q_K \bar{H}_K^{(2)}(t_K)\}$. Inserting t_k^* into condition ii), we obtain

$$q_1 \bar{H}_1^{(2)}(t_1^*) + \bar{Z}_2(t_1^*) = \sum_{k=1}^{K} q_k \bar{H}_k^{(2)}(t_k^*) \geq 0. \tag{20}$$

Thus, condition i) implies condition ii). Conversely, suppose that $\sum_{k=1}^{K} q_k \bar{H}_k^{(2)}(\hat{t}_k) < 0$ holds for some $\infty > \hat{t}_1 \geq \hat{t}_2 \geq, \dots, \geq \hat{t}_K > -\infty$. From the definition of $Z_k(t_{k-1})$ for k, we have

$$\begin{aligned}
\sum_{k=1}^{K} q_k \bar{H}_k^{(2)}(\hat{t}_k) &= \sum_{k=1}^{K-1} q_k \bar{H}_k^{(2)}(\hat{t}_k) + q_K \bar{H}_K^{(2)}(\hat{t}_K) \\
&\geq \sum_{k=1}^{K-1} q_k \bar{H}_k^{(2)}(\hat{t}_k) + \min_{t_K \leq \hat{t}_{K-1}} \left\{ q_K \bar{H}_K^{(2)}(t_K) \right\} \\
&= \sum_{k=1}^{K-2} q_k \bar{H}_k^{(2)}(\hat{t}_k) + q_{K-1} \bar{H}_{K-1}^{(2)}(\hat{t}_{K-1}) + \min_{t_K \leq \hat{t}_{K-1}} \left\{ q_K \bar{H}_K^{(2)}(t_K) \right\} \\
&\geq \sum_{k=1}^{K-2} q_k \bar{H}_k^{(2)}(\hat{t}_k) + \min_{t_{K-1} \leq \hat{t}_{K-2}} \left\{ q_{K-1} \bar{H}_{K-1}^{(2)}(t_{K-1}) + \bar{Z}_K(t_{K-1}) \right\} \\
&= \sum_{k=1}^{K-3} q_k \bar{H}_k^{(2)}(\hat{t}_k) + q_{K-2} \bar{H}_{K-2}^{(2)}(\hat{t}_{K-2}) + \bar{Z}_{K-1}(\hat{t}_{K-2}) \\
&\geq \sum_{k=1}^{K-3} q_k \bar{H}_k^{(2)}(\hat{t}_k) + \min_{t_{K-2} \leq \hat{t}_{K-3}} \left\{ q_{K-2} \bar{H}_{K-2}^{(2)}(t_{K-2}) + \bar{Z}_{K-1}(t_{K-2}) \right\} \\
&\geq, \dots, \geq q_1 \bar{H}_1^{(2)}(\hat{t}_1) + \bar{Z}_2(\hat{t}_1).
\end{aligned} \tag{21}$$

Thus, $\sum_{k=1}^{K} q_k \overline{H}_k^{(2)}(\hat{t}_k) < 0$ for some $\infty > \hat{t}_1 \geq \hat{t}_2 \geq, \dots, \geq \hat{t}_K > -\infty$ implies that there exists $\hat{t}_1$ such that $q_1 \overline{H}_1^{(2)}(\hat{t}_1) + \bar{Z}_2(\hat{t}_1) < 0$ holds. ■

From Proposition 4, we can easily test whether the provision of public goods is increased by the change in the income distribution. Note that the LHS of (20) can be rewritten as

$$\begin{aligned} q_1 \overline{H}_1^{(2)}(t_1) + \bar{Z}_2(t_1) & \\ &= q_1 \overline{H}_1^{(2)}(t_1) \\ &+ \min_{t_2 \leq t_1} \left\{ q_2 \overline{H}_2^{(2)}(t_2) + \min_{t_3 \leq t_2} \left\{ q_3 \overline{H}_3^{(2)}(t_3) +, \dots, \min_{t_K \leq t_{K-1}} \left\{ q_K \overline{H}_K^{(2)}(t_K) \right\} \right\} \right\}. \end{aligned} \tag{22}$$

From (22), we can confirm that if and only if $q_k \overline{H}_k^{(2)}(t) + \bar{Z}_{k+1}(t) \geq 0,\ \forall k \in \mathcal{K} \backslash \{K\}$ and $\bar{Z}_K(t) \geq 0$ hold $\forall t \in [0, \bar{v}]$, then condition ii) of Proposition 4 is met. That is, the sequential testing of the sign of $\bar{Z}_k(t)$ starting from $k = K$ enables us to predict the change in the supply of public goods.

We can also reformulate the results stated in Proposition 4 in terms of the RGL curves. First, we define

$$\tilde{F}_k^{X(2)}(t) \coloneqq \frac{1}{\tilde{q}_k} \sum_{r=k}^{K} q_r \bar{F}_r^{X(2)}(t), \qquad \text{for } k \in \mathcal{K}, \tag{23}$$

where $\tilde{q}_k = \sum_{r=k}^{K} q_r$. Furthermore, if $\bar{Z}_{k+1}(\bar{v}) \geq 0$ holds, we define

$$\tilde{F}_k^{Y(2)}(t) \coloneqq \frac{1}{\tilde{q}_k} \left[q_k \bar{F}_k^{Y(2)}(t) + \tilde{q}_{k+1} \tilde{F}_{k+1}^{X(2)}(t) - \bar{Z}_{k+1}(t) \right], \text{for } k \in \mathcal{K} \backslash \{K\}. \tag{24}$$

and $\tilde{F}_K^{Y(2)}(t) \coloneqq \bar{F}_K^{Y(2)}(t)$. In (23) and (24), $\tilde{F}_k^{X(2)}(t)$ represents a new integrated survival function that is simply a convex combination of the survival functions $\bar{F}_r^{X(2)}$ for $r \in \{k, \dots, K\}$. On the other hand, $\tilde{F}_k^{Y(2)}(t)$ is a convex combination of $\bar{F}_k^{Y(2)}$ and $\tilde{F}_{k+1}^{X(2)}(t) - (1/\tilde{q}_{k+1}) \bar{Z}_{k+1}(t)$ that separates $\tilde{F}_{k+1}^{X(2)}(t)$ and $(1/\tilde{q}_{k+1}) \sum_{r=k+1}^{K} q_r \bar{F}_r^{Y(2)}(t)$ in the sense of

$$\tilde{F}_{k+1}^{X(2)}(t) \geq \tilde{F}_{k+1}^{X(2)}(t) - \left(\frac{1}{\tilde{q}_{k+1}} \right) \bar{Z}_{k+1}(t) \geq \frac{1}{\tilde{q}_{k+1}} \sum_{r=k+1}^{K} q_r \bar{F}_r^{Y(2)}(t), \tag{25}$$

under the condition of $\bar{Z}_{k+1}(\bar{v}) \geq 0$. In such a situation, by the definition of $\bar{Z}_{k+1}(t)$, $\tilde{F}_{k+1}^{X(2)}(t) - (1/\tilde{q}_{k+1}) \bar{Z}_{k+1}(t)$ is decreasing and convex in t. Furthermore, $\tilde{F}_{k+1}^{X(2)}(0) - (1/\tilde{q}_{k+1}) \bar{Z}_{k+1}(0) =$

$\tilde{\mu}^{Y}_{k+1}$ and $\tilde{F}^{X(2)}_{k+1}(\bar{v}) - (1/\tilde{q}_{k+1})\bar{Z}_{k+1}(\bar{v}) = 0$ hold. Thus, $\tilde{F}^{Y(2)}_{k}(t)$ also can be regarded as an integrated survival function.

By using (23) and (24), if $\bar{Z}_{k+1}(\bar{v}) \geq 0$, we obtain

$$q_k \bar{H}^{(2)}_{k}(t) + \bar{Z}_{k+1}(t) = \tilde{q}_k \left[\tilde{F}^{X(2)}_{k}(t) - \tilde{F}^{Y(2)}_{k}(t)\right], \qquad k \in \mathcal{K}\backslash\{K\}. \tag{26}$$

For $k = K$, we have $q_K \bar{H}^{(2)}_{K}(t) = \tilde{q}_K[\tilde{F}^{X(2)}_{K}(t) - \tilde{F}^{Y(2)}_{K}(t)]$. Thus, $\bar{Z}_k(t) \geq 0 \;\; \forall t \in [0, \bar{v}]$ implies $\tilde{F}^{X(2)}_{k}(t) \geq \tilde{F}^{Y(2)}_{k}(t) \;\; \forall t \in [0, \bar{v}]$. From Lemma 2, for $k \in \mathcal{K}$, we can consider the RGL curves associated with $\tilde{F}^{J}_{k}(t)$ as follows:

$$\widetilde{GL}^{J}_{k}(\theta) := \int_0^{\theta} \tilde{F}^{J(-1)}_{k}(p)dp, \qquad \text{for } \theta \in [0,1) \text{ and } J \in \{X, Y\}, \tag{27}$$

where $\tilde{F}^{J(-1)}_{k}(p)$ is an inverse survival function. Taking account of the relationship between the inverse survival function and the inverse distribution function, we can state a sequential dominance condition in terms of the RGL curves[5].

Proposition 5. Suppose that $\bar{Z}_r(t) \geq 0$ holds for $r \in \{k+1, \dots, K\} \subseteq \mathcal{K}$. The following two conditions are equivalent:

i) $q_k \bar{H}^{(2)}_{k}(t_k) + \bar{Z}_{k+1}(t_k) \geq 0, \;\; \forall t_k \in [0, \bar{v}]$,

ii) $\widetilde{GL}^{X}_{k}(\theta) \geq \widetilde{GL}^{Y}_{k}(\theta), \;\; \forall \theta \in [0,1]$.

Proof. From (26), condition i) implies $\tilde{F}^{X(2)}_{k}(t) - \tilde{F}^{Y(2)}_{k}(t) \geq 0$. Hence, similar procedures as in Proposition 2 can be applied to $\tilde{F}^{X(2)}_{k}$ and $\tilde{F}^{Y(2)}_{k}$. ■

Proposition 5 states that the sequential comparisons of the RGL curves based on the modified income characterize the aggregate quantity of privately provided public goods under the income distributions with heterogeneous agents. It is not so difficult to modify the income distribution according to (27).

The procedure to confirm the establishment of Proposition 5 is straightforward. First, we

[5] Let $G(x)$ and $\bar{G}(x)$ be a distribution function and its survival function, respecctively. The inverse survival function $\bar{G}^{-1}(p)$ is related to the inverse distribution function such that $\bar{G}^{-1}(p) = G^{-1}(1-p)$ for $p \in [0,1)$. Thus, we can apply Lemma 2 to (27) because $\int_0^p \bar{G}^{-1}(z)dz = \int_{1-p}^1 G^{-1}(z)dz = \mu - \int_0^{1-p} G^{-1}(z)dz$ holds.

compare two RGL curves obtained from the integrated survival functions $\bar{F}_K^{X(2)} = \tilde{F}_K^{X(2)}$ and $\bar{F}_K^{Y(2)} = \tilde{F}_K^{Y(2)}$. If $\widetilde{GL}_K^X(\theta) < \widetilde{GL}_K^Y(\theta)$, $\exists\theta \in [0,1]$, then we terminate the comparison and conclude that Proposition 5 fails. Otherwise, we construct the survival functions $\tilde{F}_{K-1}^{X(2)}$ and $\tilde{F}_{K-1}^{Y(2)}$ according to (23) and (24), and compare the RGL curves for $K-1$. Repeating this step up to $k=1$, we obtain insights for the total provision of public goods.

The following example illustrates the procedure.

Example 1. Let us suppose that a society consists of three types of households: $\mathcal{K} = \{1\ 2\ 3\}$. Consider two distributions $\mathbf{w}_{\mathcal{K}}^X = [\mathbf{w}_1^X, \mathbf{w}_2^X, \mathbf{w}_3^X]$ and $\mathbf{w}_{\mathcal{K}}^Y = [\mathbf{w}_1^Y, \mathbf{w}_2^Y, \mathbf{w}_3^Y]$ in both of which households are partitioned into 3 groups as follows:

$$\mathbf{w}_1^X = (50\ 20\ 10),\ \ \mathbf{w}_2^X = (35\ 10),\ \ \mathbf{w}_3^X = (40\ 30\ 5),$$

$$\mathbf{w}_1^Y = (45\ 30\ 15),\ \ \mathbf{w}_2^Y = (25\ 25),\ \ \mathbf{w}_3^Y = (40\ 10\ 10).$$

First, we compare $\mathbf{w}_3^X$ and $\mathbf{w}_3^Y$. Since $\bar{Z}_3(v) = 0$, we can confirm $\bar{Z}_3(t) \geq 0\ \ \forall t \in [0, v]$. Next, we compare the distributions combining types 2 and 3 households. Once $\tilde{F}_2^{X(2)}$ and $\tilde{F}_2^{Y(2)}$ are obtained, we can construct the corresponding survival functions $\tilde{F}_2^X$ and $\tilde{F}_2^Y$ (e.g., Müller 1996), based on which we modify $\mathbf{w}_3^Y$ as $\mathbf{w}_3^{Y\prime} = (40\ 15\ 5)$. Thus, in the second sequence, we compare the distributions

$$\tilde{\mathbf{w}}_2^X \coloneqq [\mathbf{w}_2^X, \mathbf{w}_3^X] = (35\ 10\ 40\ 30\ 5),$$

$$\tilde{\mathbf{w}}_2^Y \coloneqq \left[\mathbf{w}_2^Y, \mathbf{w}_3^{Y\prime}\right] = (25\ 25\ 40\ 15\ 5).$$

We can confirm that $\bar{Z}_2(t) \geq 0\ \ \forall t \in [0, v]$, and obtain $\tilde{F}_1^{X(2)}$ and $\tilde{F}_1^{Y(2)}$ according to (23) and (24). As a result, we modify $\tilde{\mathbf{w}}_2^Y$ as $\tilde{\mathbf{w}}_2^{Y\prime} := (27.5\ 27.5\ 40\ 10\ 5)$. In the third sequence, the following two distributions,

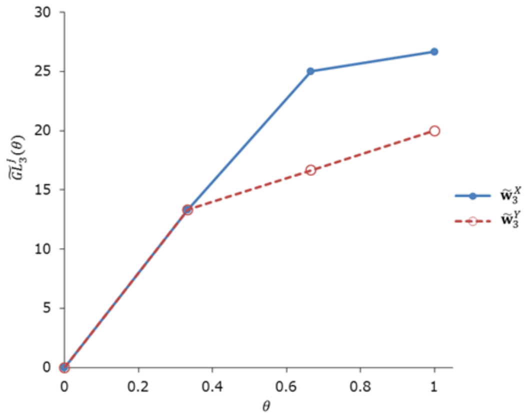
$$\tilde{\mathbf{w}}_1^X := [\mathbf{w}_1^X, \tilde{\mathbf{w}}_2^X] = (50\ 20\ 10\ 35\ 10\ 40\ 30\ 5),$$

$$\tilde{\mathbf{w}}_1^Y := \left[\mathbf{w}_1^Y, \tilde{\mathbf{w}}_2^{Y\prime}\right] = (45\ 30\ 15\ 27.5\ 27.5\ 40\ 10\ 5),$$

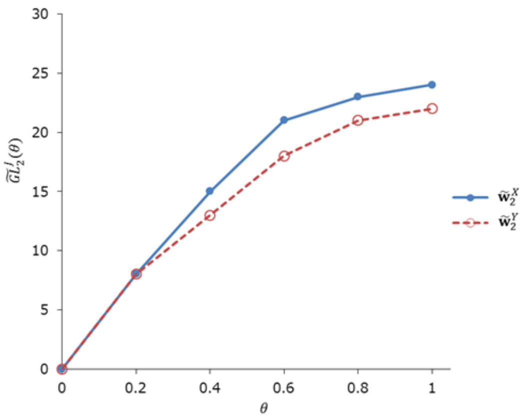
should be compared. According to Corollary 1, by drawing and comparing RGL curves for 3 pairs of distribution, that is $\{\mathbf{w}_3^X, \mathbf{w}_3^Y\}$, $\{\tilde{\mathbf{w}}_2^X, \tilde{\mathbf{w}}_2^Y\}$ and $\{\tilde{\mathbf{w}}_1^X, \tilde{\mathbf{w}}_1^Y\}$, we can test the dominance condition: Fig. 1a-c represent the comparisons of the RGL curves in each sequence. From Fig 1, we can confirm X

dominates Y in the sense of condition ii) in Proposition 5. In this example, if we replace $\mathbf{w}_1^Y$ to $\mathbf{w}_{11}^Y \coloneqq (45\ 40\ 5)$, the condition i) in Proposition 3 is not satisfied. Indeed, for $\tilde{\mathbf{w}}_1^X$ and $\tilde{\mathbf{w}}_{11}^Y :=$ $(45\ 40\ 5\ 27.5\ 27.5\ 40\ 10\ 5)$, we have $q_1 \bar{H}_1^{(2)}(15) + \bar{Z}_2(15) = -0.625$. In this case, consider the utility functions such as $u_i^k = x_i^{\beta_k/(1+\beta_k)} g^{1/(1+\beta_k)}$ for $k = 1,2,3$ with $\beta_1 = 1/24$, $\beta_2 = 1/12$, and $\beta_3 = 1/6$. Solving (1) for the current values, we have $g^X = 116.58$ and $g^Y = 120$.

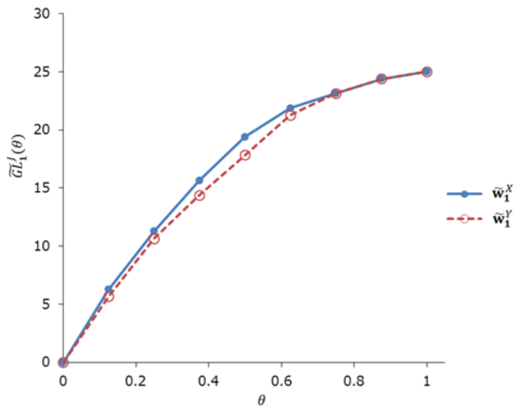
Fig 1 Sequential comparisons of the RGL curves in Example 1.



a



b



c

5 Difference in Demographics

Thus far, we have focused on the comparison of income distributions whose demographics are the same. If we intend to compare intertemporal or interregional differences in income distributions, the demographics of the societies involved in the comparison will differ in the usual ways.

We can easily extend the previous results to the case of different demographics. We begin with identical preferences. Consider two situations X and Y. There are n^X and n^Y households in X and Y, respectively. The distributions of income are $\mathbf{w}^X = (w_1^X, \dots, w_{n^X}^X)$ and $\mathbf{w}^Y = (w_1^Y, \dots, w_{n^Y}^Y)$, where $n^X \neq n^Y$ and $w_i^J \in [0, v]$. We can state the following Lemma.

Lemma 3. The following two inequalities are equivalent.

$$\sum_{i=1}^{n^X}\left(w_i^X - t\right)^+ - \sum_{i=1}^{n^Y}\left(w_i^Y - t\right)^+ \geq 0, \forall t \in \mathbb{R}, \tag{28}$$

$$\sum_{i=1}^{n^X}\left(w_i^X - t\right)^+ - \sum_{i=1}^{n^Y}\left(w_i^Y - t\right)^+ - (n^X - n^Y)(-t)^+ \geq 0, \forall t \in \mathbb{R}. \tag{29}$$

Proof. For $t \in [0, \infty)$, (29) is reduced to (28). For $t \in (-\infty, 0)$, (29) is $\sum_{i=1}^{n^X} w_i^X - \sum_{i=1}^{n^Y} w_i^Y = \sum_{i=1}^{n^X}\left(w_i^X - 0\right)^+ - \sum_{i=1}^{n^Y}\left(w_i^Y - 0\right)^+$ because $w_i^J \in [0, v]$ for $J \in \{X, Y\}$. ∎

In Lemma 3, we add the virtual households whose incomes are zero to the original distribution with fewer households. Without loss of generality, suppose that $n^X < n^Y$. In such a case, Lemma 3 compares the distributions $\mathbf{w}^{X^*} := [\mathbf{w}^X, \mathbf{0}_{\Delta n}]$ with $\mathbf{w}^Y$ where $\mathbf{0}_{\Delta n} := (0, \ldots, 0)$ is an $n^Y - n^X$-dimensional vector whose entries are all zero. Based on $\mathbf{w}^{X^*}$ and $\mathbf{w}^Y$, we can consider the survival functions and apply the procedures in Section 2.

In the case of heterogeneous agents, a similar argument is valid. As in the previous section, we assume that there are K types of households. We modify the original distribution of type $k \in \mathcal{K} = \{1, \ldots, K\}$ households as follows: for $I, J \in \{X, Y\}, I \neq J$,

$$\mathbf{w}_k^{J^*} := \begin{cases} \left[\mathbf{w}_k^J, \mathbf{0}_{\Delta n_k}\right] & if\ n_k^I - n_k^J > 0, \\ \mathbf{w}_k^J & if\ n_k^I - n_k^J \leq 0. \end{cases}$$

where $\mathbf{0}_{\Delta n_k}$ is an $\left|n_k^X - n_k^Y\right|$-dimensional zero vector. Thus, $\mathbf{w}_k^{J^*}$ is a vector consisting of $n_k^* = \max\{n_k^X, n_k^Y\}$ entries. Applying the procedures described in the previous section to the distributions, $\mathbf{w}_{\mathcal{K}}^{X^*} := \left[\mathbf{w}_1^{X^*}, \ldots, \mathbf{w}_K^{X^*}\right]$ and $\mathbf{w}_{\mathcal{K}}^{Y^*} := \left[\mathbf{w}_1^{Y^*}, \ldots, \mathbf{w}_K^{Y^*}\right]$, we can obtain insights into the aggregate supply of privately provided public goods.

6 Remarks

In this paper, we considered the relationship between the voluntary provision of public goods and the distribution of income using the notion of stochastic order. With minimal assumptions about the utility function, we characterized the change in income distribution that increases the voluntary supply of public goods. Furthermore, we showed that the RGL curve is a useful tool for identifying such a change in the supply of public goods. The visualization proposed here is a reverse version of the increasing concave order used for characterizing the relationship between income distribution and social welfare.

We showed that in a society consisting of homogeneous households, increasing convex order characterizes the total amount of the voluntary provided public goods. The existing literature argues that a rise in income inequality has a positive impact on the total supply of public goods. Indeed, in comparing two societies with the same total income, a rise in income inequality in terms of the Lorentz dominance criterion leads to an increase in the total supply of public goods. However, when the total incomes are different in the societies being compared, the change in the total provision of public goods cannot be predicted by the ordinal and generalized Lorentz curves; rather, the dominance relation of the RGL curves predicts the change in the total amount of public goods supply. This relationship is a mirror image of Shorrocks' theorem in the theory of income distribution.

On the other hand, when the society consists of several types of households, a mirror image of the dominance criterion proposed by Bourguignon (1989) characterizes the amount of public goods supply. That is, sequentially comparing the modified RGL curves leads to insight into the total provision of public goods. In the literature of the income distribution, the Sequential Generalized Lorenz criterion proposed by Atkinson and Bourguignon (1987) is well known. However, the dominance criterion proposed in this paper is not its mirror image. Because $(w - w_k^*)^+ - (w - w_{k-1}^*)^+$ is neither a convex nor concave function. Therefore, we need to modify the distribution of income for the sequential comparison of the RGL curves. We also showed that the comparison is possible even if the group composition is different.

The focus of this paper is not on the social welfare effects of a change in income distribution in the presence of voluntarily provided public goods. Increasing the supply of public goods starting from the Nash equilibrium potentially improves social welfare. However, if we take account of social welfare in a strict sense, then we must simultaneously consider changes in the public goods supply and the consumption of private goods. Further study will be needed to characterize the relationship between the distribution of income and social welfare in the presence of voluntarily provided public goods.

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stochastic order, we consider how the distribution of income affects the amount of voluntarily provided public goods.

Increasing convex order characterizes the total supply of public goods when the preferences of households are identical.

Even if there is heterogeneity among households, it is still possible to describe how the distribution of income increases the voluntary supply of public goods by using a modified notion of increasing convex order.

We can readily confirm the properties shown here by comparing reverse generalized Lorenz curves or a modified version.