ON ℓ_∞-GROTHENDIECK SUBSPACES

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ABSTRACT. A closed subspace S of ℓ_∞ is said to be a ℓ_∞-*Grothendieck subspace* if $c_0 \subset S$ (hence $\ell_\infty \subset S^{**}$) and every $\sigma(S^*, S)$-convergent sequence in S^* is $\sigma(S^*, \ell_\infty)$-convergent. Here we give examples of closed subspaces of ℓ_∞ containing c_0 which are or fail to be ℓ_∞-Grothendieck.

1. INTRODUCTION

The ℓ_∞-Grothendieck subspaces (defined in the abstract; see also Definition 2) naturally emerge when some versions of Schur's Lemma for bounded multiplier convergent series are sharpened (see, e.g., [15, 2, 1, 12]).

Apart from ℓ_∞, only one example of ℓ_∞-Grothendieck subspace is given in the literature (see [1, Remark 4.2]), using a result of [10]. This example is isomorphic to a $C(K)$ space with the Grothendieck property, and contains no subspaces isomorphic to ℓ_∞.

Here we prove that if X is a Grothendieck Banach space and M is a closed subspace of X with X/M separable then M is a Grothendieck space. As a consequence, we derive that a closed subspace S of ℓ_∞ containing c_0 is ℓ_∞-Grothendieck when the quotient ℓ_∞/S is separable, and using the fact that $L_q(0, 1)$ is isomorphic to a quotient of ℓ_∞ for $2 \leq q < \infty$, we prove the existence of an uncountable family of pairwise non-isomorphic ℓ_∞-Grothendieck subspaces. We also show that for each closed subspace Y of ℓ_∞ which is a Grothendieck space and contains a subspace isomorphic to c_0, there exists a ℓ_∞-Grothendieck subspace isomorphic to Y.

On the other hand, we show that a closed subspace S of ℓ_∞ containing c_0 is not ℓ_∞-Grothendieck when it is separable or, more generally, when the unit ball of S^* is weak*-sequentially compact.

2. PRELIMINARIES

Let X be a Banach space and let M be a subspace of X^{**} containing X. We say that a sequence (x_n^*) in X^* is $\sigma(X^*, M)$-convergent to x^* if $(\langle x^{**}, x_n^*\rangle)$ converges to $\langle x^{**}, x^*\rangle$ for every $x^{**} \in M$.

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A Banach space X *has weak* sequentially compact dual ball* (has W*SC dual ball, for short) if every sequence in the unit ball of X^* has a $\sigma(X^*, X)$-convergent subsequence. We refer to [6, Chapter XIII] for information about this property. The next result gives some examples of spaces of this kind.

Proposition 2.1. *A Banach space X has W*SC dual ball in the following cases:*

(1) *X is separable;*
(2) *X^* contains no copies of ℓ_1;*
(3) *X is isomorphic to the dual of a separable space containing no copies of ℓ_1.*

Proof. (1) is well-known [6]; (2) follows from Rosenthal's characterization of Banach spaces containing no copies of ℓ_1 and the fact that each weakly Cauchy sequence in X^* is weak*-convergent; and (3) is a consequence of [6, Theorem XIII.10]. □

A Banach space X is *Grothendieck* if every $\sigma(X^*, X)$-convergent sequence in X^* is $\sigma(X^*, X^{**})$-convergent.

Obviously reflexive Banach spaces are Grothendieck. Moreover, it is not difficult to show that Grothendieck spaces with W*SC dual ball are reflexive, it was proved in [9] that ℓ_∞ is a non-reflexive Grothendieck space (see [6, Theorem VII.15]), and the class of Grothendieck spaces satisfies the three-space property: If M is a closed subspace of a Banach space X and both M and X/M are Grothendieck, then so is X (see [8, Corollary 2.6]).

The following result collects some classical characterizations of Grothendieck spaces taken from [5, Chapter 5, Corollary 5].

Proposition 2.2. *For a Banach space X, the following assertions are equivalent:*

(1) *X is Grothendieck;*
(2) *every operator $T : X \to c_0$ is weakly compact;*
(3) *for each separable Banach space Y, every operator $T : X \to Y$ is weakly compact.*

It easily follows from Proposition 2.2 that quotients of a Grothendieck space are also Grothendieck.

3. Main results

The following notion extends the classical one of Grothendieck space.

Definition 1. *Let X be a Banach space and let M be a vector subspace of X^{**} containing X. We say that X is a M-*Grothendieck space *if every $\sigma(X^*, X)$-convergent sequence in X^* is $\sigma(X^*, M)$-convergent.*

Obviously, the Grothendieck spaces are the X^{**}-Grothendieck spaces. Moreover, since $\sigma(X^*, X)$-convergent sequences are bounded, the M-Grothendieck spaces coincide with the $\overline{M}$-Grothendieck spaces, where $\overline{M}$ is the closure of M. So we could always assume in Definition 1 that M is a closed subspace.

We are interested in a concrete case of Definition 1. Let S be a closed subspace of ℓ_∞ containing c_0, and let $j : c_0 \to S$ be the inclusion map. Then we can identify ℓ_∞ with a subspace $j^{**}(c_0^{**})$ of S^{**} containing S.

Definition 2. *Let S be a closed subspace of ℓ_∞. We say that S is a ℓ_∞-*Grothendieck subspace *if it contains c_0 and each $\sigma(S^*, S)$-convergent sequence in S^* is $\sigma(S^*, \ell_\infty)$-convergent.*

Clearly, if S is a closed subspace of ℓ_∞ that contains c_0 and S is a Grothendieck space, then S is a ℓ_∞-Grothendieck subspace.

The following result may be interesting on its own.

Proposition 3.1. *Let X be a Grothendieck Banach space. If M is a closed subspace of X and X/M is separable, then M is a Grothendieck space.*

Proof. Let $S : M \to c_0$ be an operator. Since the space c_0 is separably injective [3, Theorem 2.3] and the quotient X/M is separable, the operator S admits an extension $T : X \to c_0$ [3, Proposition 2.5], which is weakly compact by Proposition 2.2. Then S is weakly compact, and applying again Proposition 2.2 we conclude that M is Grothendieck. □

As a consequence of the previous result, we obtain that "big" subspaces are ℓ_∞-Grothendieck subspaces.

Corollary 3.2. *Let S be a closed subspace of ℓ_∞ containing c_0 such that ℓ_∞/S is separable. Then S is a ℓ_∞-Grothendieck subspace.*

Let us see that there exists an uncountable family of pairwise non-isomorphic ℓ_∞-Grothendieck subspaces.

Theorem 3.3. *Let $2 \leq p < \infty$.*

(1) *There exists a closed subspace N_p of ℓ_∞ containing c_0 such that the quotient ℓ_∞/N_p is isomorphic to $L_p(0,1)$. Hence N_p is a ℓ_∞-Grothendieck subspace.*
(2) *If $2 \leq r < \infty$, $p \neq r$, then the subspaces N_p and N_r are not isomorphic.*

Proof. (1) Recall that ℓ_∞ is isomorphic to $L_\infty(0,1)$, which is the dual of $L_1(0,1)$. Let q such that $1/p + 1/q = 1$, hence $1 < q \leq 2$.

By [14, Corollary 2.f.5], there exists a closed subspace M_q of $L_1(0,1)$ which is isometrically isomorphic to $L_q(0,1)$. Therefore, by duality,

$$M_q^* \equiv L_\infty(0,1)/M_q^\perp \equiv L_q(0,1)^* \equiv L_p(0,1).$$

Let $U : L_\infty(0,1) \to \ell_\infty$ be a bijective isomorphism. By taking $N_p = U(M_q^\perp)$ we guarantee that ℓ_∞/N_p is isomorphic to $L_p(0,1)$.

It remains to show that we can choose N_p containing c_0. This is a consequence of the fact that ℓ_∞/c_0 has a quotient isomorphic to ℓ_∞. So we can take as N_p the kernel of a composition of surjective operators like the following one:

$$\ell_\infty \to \ell_\infty/c_0 \to \ell_\infty \to L_p(0,1).$$

(2) Let $2 \leq p, r < \infty$, and assume that there exists a bijective isomorphism $T : N_p \to N_r$. Since both ℓ_∞/N_p and ℓ_∞/N_r are reflexive, by [13, Theorem 2.f.12] there exists an extension $\hat{T} : \ell_\infty \to \ell_\infty$ of T which is a Fredholm operator; i.e. the range $R(\hat{T})$ is closed and both the kernel $N(\hat{T})$ and $\ell_\infty/R(\hat{T})$ are finite dimensional. Then $\hat{T}$ induces a Fredholm operator $S : \ell_\infty/N_p \to \ell_\infty/N_r$, implying that $L_p(0,1)$ and $L_r(0,1)$ are isomorphic. Hence $p = r$, and the proof is done. □

Every quotient of $L_p(0,1)$ $(2 \leq p < \infty)$ is also a quotient of ℓ_∞, and we can assume as before that the kernel of the quotient map contains c_0, so it provides another example of ℓ_∞-Grothendieck subspace. In particular, we could have formulated Theorem 3.3 with ℓ_p instead of $L_p(0,1)$.

The next result provides additional examples.

Proposition 3.4. *Let Y be a closed subspace of ℓ_∞ which is a Grothendieck space and contains a subspace isomorphic to c_0. Then there exists a ℓ_∞-Grothendieck subspace isomorphic to Y.*

Proof. Let M be a closed subspace of Y isomorphic to c_0 and let $T : M \to c_0$ be a bijective isomorphism. Since both ℓ_∞/M and ℓ_∞/c_0 are non-reflexive, by [13, Theorem 2.f.12] there exists an extension $\hat{T} : \ell_\infty \to \ell_\infty$ of T which is a bijective isomorphism. Hence $\hat{T}(Y)$ is a ℓ_∞-Grothendieck subspace isomorphic to Y. □

A remarkable example of Grothendieck space obtained by Bourgain [4] is the space H^∞ of bounded analytic functions on the unit disc, which is not isomorphic to a $C(K)$ space, not even isomorphic to a $\mathcal{L}_\infty$-space. Moreover, it was proved in [7, Corollary 10] that the projective tensor product $\ell_\infty \widehat{\otimes}_\pi \ell_p$ is Grothendieck for $2 < p < \infty$.

Since both spaces H^∞ and $\ell_\infty \widehat{\otimes}_\pi \ell_p$ contain a subspace isomorphic to c_0 and they are isomorphic to dual spaces of separable spaces $(L_1/H_0^1)^*$ and $(\ell_1 \widehat{\otimes}_\varepsilon \ell_p^*)^*$, hence they embed in ℓ_∞, we get the following fact.

Corollary 3.5. *There exist ℓ_∞-Grothendieck subspaces which are isomorphic to H^∞ and $\ell_\infty \widehat{\otimes}_\pi \ell_p$ for $2 < p < \infty$.*

All known examples of ℓ_∞-Grothendieck subspace are Grothendieck spaces. So the following question arises:

Problem 1. *Is it possible to find an example of ℓ_∞-Grothendieck subspace which is not a Grothendieck space?*

To study this problem we would need a good characterization of ℓ_∞-Grothendieck subspaces, which we do not have yet.

Next we show that "small" subspaces are not ℓ_∞-Grothendieck subspaces.

Proposition 3.6. *Let S be a closed subspace of ℓ_∞ containing c_0. If S has W^*SC dual ball, then S is not a ℓ_∞-Grothendieck subspace.*

Proof. Let $j : c_0 \to S$ be the inclusion. Then $j^* : S^* \to c_0^*$ is surjective, and we can select a bounded sequence (x_n^*) in S^* such that $j^* x_n^* = e_n^*$ for each $n \in \mathbb{N}$, where (e_n^*) is the unit vector basis of $\ell_1 \equiv c_0^*$.

Since S has W^*SC dual ball, (x_n^*) has a $\sigma(S^*, S)$-convergent subsequence. Thus the proof is finished if we show that (x_n^*) has no $\sigma(S^*, \ell_\infty)$-convergent subsequence.

Indeed, let $(x_{n_k}^*)$ be a subsequence, and recall that $j^{**} : c_0^{**} \equiv \ell_\infty \to S^{**}$ is the inclusion. We take $z = (a_i) \in \ell_\infty$ with $a_i = 1$ for $i = n_{2k}$ $(k \in \mathbb{N})$ and $a_i = -1$ otherwise. Then

$$\langle j^{**} z, x_{n_k}^* \rangle = \langle z, j^* x_{n_k}^* \rangle = \langle z, e_{n_k}^* \rangle = (-1)^k,$$

hence $(x_{n_k}^*)$ is not $\sigma(S^*, \ell_\infty)$-convergent. □

Proposition 3.6 applies in the following cases:

(1) S is a separable closed subspace of ℓ_∞ containing c_0.

(2) $S = \overline{c_0 + M}$, where M is a non-separable subspace of ℓ_∞ containing no copies of ℓ_1 and isomorphic to a dual separable space. The space S has W*SC dual ball because M has W*SC dual ball (Proposition 2.1) and there is an injective operator with dense range $T : c_0 \times M \to S$; see [6, Chapter XIII]. For example, we can take M isomorphic to the dual of the James tree space JT [11].

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