Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 3465909, 29 pages
https://doi.org/10.1155/2018/3465909

Research Article

Model Selection Approaches for Predicting Future Order Statistics from Type II Censored Data

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Received 7 April 2018; Revised 4 July 2018; Accepted 14 August 2018; Published 8 October 2018

Academic Editor: Mohammed Nouari

This paper studies a discriminant problem of location-scale family in case of prediction from type II censored samples. Three model selection approaches and two types of predictors are, respectively, proposed to predict the future order statistics from censored data when the best underlying distribution is not clear with several candidates. Two members in the location-scale family, the normal distribution and smallest extreme value distribution, are used as candidates to illustrate the best model competition for the underlying distribution via using the proposed prediction methods. The performance of correct and incorrect selections under correct specification and misspecification is evaluated via using Monte Carlo simulations. Simulation results show that model misspecification has impact on the prediction precision and the proposed three model selection approaches perform well when more than one candidate distributions are competing for the best underlying distribution. Finally, the proposed approaches are applied to three data sets.

1. Introduction

For saving testing time and sample resource, censoring schemes often are considered to implement life tests. Type I censoring scheme and type II censoring scheme are two popular censoring schemes based on the criteria of test time censoring and failure number censoring. Plenty studies can be found for evaluating the reliability of lifetime components via using type I censoring test or type II censoring test. See examples like, [1–6] etc.

In this study, we mainly restrict our attention to using type II censoring scheme for predicting the censored sample for reliability evaluation when a discriminant problem is considered. In the type II censoring scheme, we consider an experiment where n identical components are placed in the test simultaneously. Assuming that r^{th} component fails, the experiment is terminated. Thus the last $(n-r)$ components are censored. In many engineering applications, censored data are not allowed for implementing statistical methods to obtain information. For example, if we like to conduct a factorial design or fractional factorial design based on the experimental design methods, most experimental design methods cannot be implemented with censored data. In such situation, a reliable procedure for predicting censored or unobserved observations is required. Moreover, if we can predict the unobserved observations and transform a censored data set into a complete data set, the parameter estimation problem becomes easy especially for dealing with the cases, which have no analytic solutions of the parameter estimators can be obtained. The purpose of predicting life length of the $s^{th}(r < s \leq n)$ item is equivalent to the life length of a (n-s+1)-out-of-n system that was made up of n identical components with independent life lengths. When $s = n$, it is better known as the parallel system. For this issue, various methods have been developed to predict the censored data. Kaminsky and Nelson [7] provided interval and point prediction of order statistics. Fertig et al. [8] provided Monte Carlo estimates of the distribution percentiles to construct prediction intervals for samples from a Weibull or smallest extreme value distribution (SEV). Kaminsky and Rhodin

[9] provided the maximum likelihood predictor (MLP) to predict the future order statistics and then estimate the unknown parameters. Wu et al. [10] proposed five new pivotal quantities to obtain prediction intervals of future order statistics from the Pareto distribution. Kundu and Raqab [11] describes the Bayesian inference and prediction of the two-parameter Weibull distribution. Panahi and Sayyareh [12] proposed parameter estimation and prediction of order statistics for the Burr type XII distribution. Some of these predictions are complex, or they need to construct complex statistical models. Therefore, these existing methods are not easy to apply.

In order to solve this problem, Raqab [13] modified the MLP method and proposed four modified MLPs (MMLPs) to predict the future order statistics for the normal distribution (ND). In order to simplify the estimation function, they considered four types of modification to approximate the terms of hazard rate and extended hazard rate functions form a ND, which has unknown mean and known standard deviation. Yang and Tong [14] used MMLP method to predict type II censored data from factorial experiments. They derived the simple explicit solutions for parameters for a ND, which has unknown mean and unknown standard deviation. Chiang [15] used another three MMLP procedures to predict type II censored data under the Weibull distribution. In his procedures, it is difficult to find the only root solution to the parameter estimation. However, the parameter estimation of MMLP method can be obtained via simple parameter explicit solution only in the ND. For other commonly used distributions, the likelihood equations of MMLP may be nonlinear and does not admit explicit solutions. Hence the parameter estimation of MMLP loses the advantage for other commonly used distributions.

Another important problem in life testing experiments is the model selection based on the existing sample. In practical applications, many statistical distributions are much alike, especially in censored data, and the underlying distribution of product quality characteristics is usually unknown. They may fit the data well in practical applications. However, their predictions may lead to a significant difference. Therefore, correctly identifying the underlying distribution is an important issue and it has long been studied. Dumonceaux and Antle [16] applied ratio of maximized likelihood (RML) to discriminating between the lognormal and Weibull distributions. Kundu and Manglick [17] proposed statistical methods to discriminate between the lognormal and gamma distributions. Kundu and Raqab [18] proposed a selection to discriminate between the generalized Rayleigh and lognormal distribution. Yu [19] provided a misspecification analysis method to discriminate between the ND and SEV for the design of experiment. Dey and Kundu [20] studied the discrimination problem between the lognormal and log-logistic distributions. Elsherpieny et al [21] considered the discrimination problem between the Weibull and log-logistic distributions. Ashour and Hashish [22] provided a numerical comparison study for using RML-procedure, S-procedure, and F-procedure in failure model discrimination. Pakyari [23] presented diagnostic tools based on the likelihood ratio test and the minimum Kolmogorov distance method to discriminate between the generalized exponential, geometric extreme exponential, and Weibull distributions. Elsherpieny et al. [24] provided a method to discriminate the gamma and log-logistic distributions based on progressive type II censored data. Although the inference methods in the aforementioned studies are valuable, the impacts of model misspecification on predicting the future order statistics have not been well studied.

Among the model discrimination problems, due to the well-developed theory and inferential procedures for the location-scale family of distributions, the model discrimination within the location-scale family of distributions is particularly important and it has received much attention. The main purpose of this paper is to address these issues and provide satisfactory estimators of parameters and predictors of future order statistics when the underlying distribution is unknown but it is a member in the location-scale family. Specifically, for lifetime analysis, the essence of this study is to predict the future order statistics for type II censored data when the underlying distribution is unknown but is a member of the location-scale family. The major contributions of this study for censored data prediction are presented in Figure 1.

The rest of this paper is organized as follows. Section 2 presents materials and methods. In this section, statistical methods to obtain approximate predictors for type II right censored variables are studied and two prediction methods are proposed to predict the type II right-censored variables based on the AMLEs. The ND and SEV are considered as the candidate distributions to compete the best distribution for obtaining the predictors of type II right-censored variables. In Section 3, we provide three algorithms to implement the three proposed model selection approaches to deal with the discrimination problem when obtaining the predictors of type II right-censored variables based on the proposed methods. An intensive simulation study is conducted in Section 4 to evaluate the performance of the proposed approaches. Then, three examples are used to demonstrate the applications of the proposed methodologies in Section 5. Some concluding remarks are provided in Section 6.

2. Methods for Approximate Predictors

2.1. Approximate Maximum Likelihood Estimation. Let Y_i denote the failure time of i^{th} item and $X_i = \log(Y_i)$, which follows a location-scale family, having the probability density function (PDF) and cumulative distribution function (CDF):

$$f(x;\mu,\sigma) = \frac{1}{\sigma} g\left(\frac{x-\mu}{\sigma}\right), \tag{1}$$

and

$$F(x;\mu,\sigma) = G\left(\frac{x-\mu}{\sigma}\right), \quad -\infty < \mu < \infty,\ \sigma > 0,\ -\infty < x < \infty, \tag{2}$$

respectively, where μ is location parameter and σ is scale parameter. $g(\cdot)$ and $G(\cdot)$ are the PDF and CDF of a member,

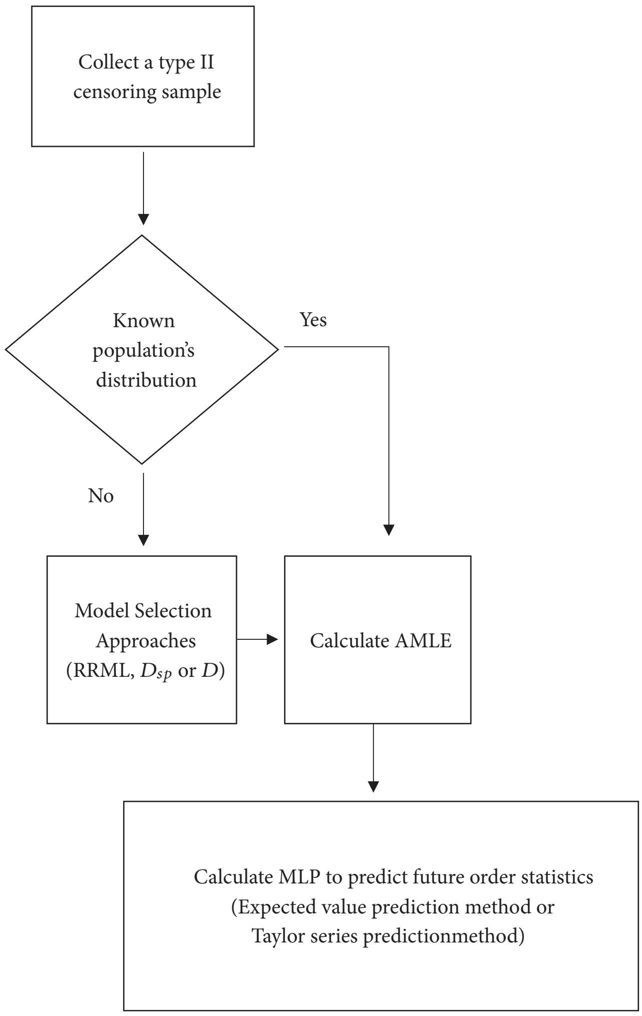

Figure 1: The flow chart of the major contribution of this study.

respectively, in the location-scale family. Denote the sample size by n, and denote type II censored sample with r failures by $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{r:n}$, which are the realizations of $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n}$, where $1 \leq r < s \leq n$. Our goal is to predict $x_{s:n}$ for $r < s \leq n$. Let $f(x) \equiv f(x;\mu,\sigma)$ and $F(x) \equiv F(x;\mu,\sigma)$ here and after to simplify the notations. Kaminsky and Rhodin [9] considered prediction of $x_{s:n}$ having observed $\mathbf{x} = (x_{1:n}, x_{2:n}, \ldots, x_{r:n})$, The predictive likelihood functions (PLF) of $X_{s:n}$, μ and σ is

$$L\left(X_{s:n},\mu,\sigma;\mathbf{x}\right) \equiv f\left(\mathbf{x}, X_{s:n};\mu,\sigma\right) = \frac{n!}{(s-r-1)!\,(n-s)!}\prod_{j=1}^{r} f\left(x_{j:n}\right) \cdot \left[F\left(X_{s:n}\right) - F\left(x_{r:n}\right)\right]^{s-r-1} f\left(X_{s:n}\right)\left[1 - F\left(X_{s:n}\right)\right]^{n-s}. \quad (3)$$

Please note that the capital notation $X_{s:n}$ in $F(X_{s:n})$ is unknown and can be predicted based on the sample $\mathbf{x}$. Based on the proposed method by Raqab [13], the PLF of $X_{s:n}$, μ and σ in (3) can be represented as a product of two likelihood functions, the PLF of μ and σ (i.e., which is denoted as L_1) and the PLF of $X_{s:n}$ (i.e., which is denoted as L_2). Both likelihood functions are presented, respectively, by

$$L_1\left(\mu,\sigma;\mathbf{x}\right) = \frac{n!}{(n-r)!}\prod_{j=1}^{r} f\left(x_{j:n}\right)\left[1 - F\left(x_{r:n}\right)\right]^{n-r}, \quad (4)$$

and

$$L_2\left(X_{s:n};\mu,\sigma,\mathbf{x}\right) = \frac{(n-r)!}{(s-r-1)!\,(n-s)!}\frac{\left[F\left(X_{s:n}\right) - F\left(x_{r:n}\right)\right]^{s-r-1}}{\left[1 - F\left(x_{r:n}\right)\right]^{n-r}} \times \left[1 - F\left(X_{s:n}\right)\right]^{n-s} f\left(X_{s:n}\right). \quad (5)$$

In practice, we can obtain the MLEs of μ and σ, denoted by $\widehat{\mu}$ and $\widehat{\sigma}$, respectively, through maximizing $L_1(\mu,\sigma;\mathbf{x})$ in (4). Then use $\widehat{\mu}$ and $\widehat{\sigma}$ to replace μ and σ as the plug-in parameters in (5) to predict $X_{s:n}$. Let $z_{j:n} = (x_{j:n} - \mu)/\sigma$ for $j = 1, \ldots, r$, $Z_{s:n} = (X_{s:n} - \mu)/\sigma$ for $s = r+1, \ldots, n$ and $\mathbf{z} = (z_{1:n}, z_{2:n}, \ldots, z_{r:n})$, then we can rewrite (4) and (5) by

$$L_1 \equiv L_1\left(\mu,\sigma;\mathbf{z}\right) = C_1\prod_{j=1}^{r}\sigma^{-1} f\left(z_{j:n}\right)\left[1 - F\left(z_{r:n}\right)\right]^{n-r} \quad (6)$$

and

$$L_2 \equiv L_2\left(Z_{s:n};\widehat{\mu},\widehat{\sigma},\mathbf{z}\right) = C_2\sigma^{-1} \cdot \frac{\left[F\left(Z_{s:n}\right) - F\left(z_{r:n}\right)\right]^{s-r-1}}{\left[1 - F\left(z_{r:n}\right)\right]^{n-r}}\left[1 - F\left(Z_{s:n}\right)\right]^{n-s} \cdot f\left(Z_{s:n}\right), \quad (7)$$

where $C_1 = n!/(n-r)!$ and $C_2 = (n-r)!/[(s-r-1)!(n-s)!]$. After straightforward computations, the MLEs of μ, σ and $Z_{s:n}$ respectively can be obtained as the solutions of

$$\frac{\partial \log\left(L_1\right)}{\partial \mu} = \frac{1}{\sigma}\left[\sum_{j=1}^{r}\Psi\left(z_{j:n}\right) + (n-r)\,h\left(z_{r:n}\right)\right] = 0 \quad (8)$$

$$\frac{\partial \log\left(L_1\right)}{\partial \sigma} = \frac{1}{\sigma}\left[-r + \sum_{j=1}^{r}\Psi\left(z_{j:n}\right) z_{j:n} + (n-r)\,h\left(z_{r:n}\right) z_{r:n}\right] = 0 \quad (9)$$

and

$$\frac{\partial \log\left(L_2\right)}{\partial Z_{s:n}} = (s-r-1)\,h_1\left(z_{r:n}, Z_{s:n}\right) - \Psi\left(Z_{s:n}\right) - (n-s)\,h\left(Z_{s:n}\right) = 0, \quad (10)$$

where

$$\Psi\left(Z_{j:n}\right) = -\frac{f'\left(Z_{j:n}\right)}{f\left(Z_{j:n}\right)}, \quad j = 1, \ldots, n, \text{ where } Z_{j:n} = z_{j:n} \text{ if } j \le r, \tag{11}$$

$$h\left(Z_{j:n}\right) = \frac{f\left(Z_{j:n}\right)}{1 - F\left(Z_{j:n}\right)}, \quad j = 1, \ldots, n, \text{ where } Z_{j:n} = z_{j:n} \text{ if } j \le r, \tag{12}$$

and

$$h_1\left(z_{r:n}, Z_{s:n}\right) = \frac{f\left(Z_{s:n}\right)}{F\left(Z_{s:n}\right) - F\left(z_{r:n}\right)}. \tag{13}$$

Because of no analytic presentation for $\hat{\mu}$ and $\hat{\sigma}$, one needs to use numerical gradient computation methods, for example, the Newton-Raphson method, for obtaining $\hat{\mu}$ and $\hat{\sigma}$ via by equating (8) and (9). To obtain proper initial solutions for implementing gradient computation methods, we consider using the approximate MLEs (AMLE) of μ and σ from Hossain and Willan [25] as their initial solutions in this study.

2.2. Approximate Maximum Likelihood Predictors. When we obtain the MLEs $\hat{\mu}$ and $\hat{\sigma}$, we can predict $X_{s:n}$ by using two approximation methods, the expected value prediction method and Taylor series prediction method. The resulting predictors of $X_{s:n}$ based on the expected prediction method is denoted by MLP_E, and the resulting predictors of $X_{s:n}$ based on the Taylor series prediction method is denoted by MLP_T. The two approximate methods mainly use two different methods to get the approximates of $h_1(z_{r:n}, Z_{s:n})$ and $h(Z_{s:n})$. Mehrotra and Nanda [26] proposed approximate maximum likelihood estimators for the ND and gamma distribution by replacing $h(x)$ and $xh(x)$ by their respective expected values and efficiencies compared to those for the best linear unbiased estimators for these distributions. Balakrishnan and Cohen [27] used the Taylor series expansion of $h(x)$ and $f(x)/F(x)$ at the points $F^{-1}(p_s)$ to obtain modified MLEs of the parameters of the ND and Rayleigh distribution, where $p_i = i/(n+1)$ for $i = 1, 2, \ldots, n$. The main point of their approach is that likelihood equations involve complicated terms and it is not possible to obtain an explicit form for MLE. So we follow their ideas and find an explicit form for the predictor of $X_{s:n}$.

Based on the expected value prediction method, replacing (μ, σ) with $(\hat{\mu}, \hat{\sigma})$, and replacing $h_1(z_{r:n}, Z_{s:n})$ and $h(Z_{s:n})$ by their respective expected values in (10). According to Raqab [13], the expected value of $f(Z_{j:n})$, $h_1(z_{r:n}, Z_{s:n})$ and $h(Z_{s:n})$ can be presented, respectively, by

$$E\left[f\left(Z_{j:n}\right)\right] = \frac{1}{n+1} \sum_{k=j+1}^{n+1} E\left[\Psi\left(Z_{k:n+1}\right)\right], \quad j \le n \text{ and } Z_{j:n} = z_{j:n} \text{ if } j \le r, \tag{14}$$

$$E\left[h\left(Z_{j:n}\right)\right] = \frac{1}{n-j} \sum_{k=j+1}^{n} E\left[\Psi\left(Z_{k:n}\right)\right], \quad j \le n-1 \text{ and } Z_{j:n} = z_{j:n} \text{ if } j \le r, \tag{15}$$

and

$$E\left[h_1\left(Z_{i:n}, Z_{j:n}\right)\right] = \frac{1}{j-i-1} \sum_{k=j}^{n} E\left[\Psi\left(Z_{k:n}\right)\right], \quad j - i \ge 2, \text{ and } Z_{j:n} = z_{j:n} \text{ if } j \le r. \tag{16}$$

Based on the Taylor series prediction method, replacing (μ, σ) with $(\hat{\mu}, \hat{\sigma})$ and replacing $h(Z_{s:n})$ and $h_1(Z_{r:n}, Z_{s:n})$ with their Taylor series approximations at points $F^{-1}(p_s)$ and $(F^{-1}(p_r), F^{-1}(p_s))$, respectively, in (10). In this study, we denote the MLP_E and MLP_T of $X_{s:n}$ under the candidate distribution M by $\hat{X}_{s:n}^{M,1}$ and $\hat{X}_{s:n}^{M,2}$, respectively.

There are many common distributions in location-scale family of distributions. The widely used members including the ND, SEV, logistic distribution, etc. It is impossible to list all inference formulas for predicting $X_{s:n}$ under all widely used members in the location-scale family. In this study, we use ND and SEV as candidates to illustrating the applications of the proposed methods. But the suggested algorithms in this study can be applied for the cases with more than two candidate members. The reason to select the ND and SEV as candidates is due to the fact that the Weibull distribution and lognormal distribution are two widely used distributions for life testing applications. The Weibull and lognormal distributions can be respectively transformed into the SEV and ND by taking log-transformation.

If the underlying distribution is normal, the PDF of normal distribution is given by

$$g(Z) = \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}. \tag{17}$$

Through using (17), we can obtain $\Psi(z) = -\phi'(z)/\phi(z) = z$. The MLEs of normal distribution parameters are denoted by $\hat{\mu}_N$ and $\hat{\sigma}_N$. Replacing μ and σ with $\hat{\mu}_N$ and $\hat{\sigma}_N$ in (6), we can represent (6) by

$$\hat{L}_N\left(\hat{\mu}_N, \hat{\sigma}_N\right) = C_1 \prod_{j=1}^{r} \hat{\sigma}_N^{-1} \phi\left(z_{j:n}\right) \left[1 - \Phi\left(z_{r:n}\right)\right]^{n-r}, \tag{18}$$

where $\Phi(\cdot)$ is the CDF of the standard ND. According to (15) and (16), $h_1(z_{r:n}, Z_{s:n})$ and $h(Z_{s:n})$ can be replaced with their respective expected values in (10). Equation (10) can be rewritten as

$$E\left(Z_{s:n}\right) - \hat{Z}_{s:n} = 0. \tag{19}$$

The values of $E(Z_{j:n})$ are available and have been tabulated by Teichroew [28]. Hence, MLP_E of $X_{s:n}$ for ND can be derived as

$$\hat{X}_{s:n}^{N,1} = \hat{\mu}_N + \hat{\sigma}_N E\left(Z_{s:n}\right). \tag{20}$$

Because $E(Z_{s:n}) \geq z_{r:n}$ is a necessary condition, we modify (20) by

$$\widehat{X}_{s:n}^{N,1} = \max\{\widehat{\mu}_N + \widehat{\sigma}_N E(Z_{s:n}), x_{r:n}\} \quad (21)$$

and use $\widehat{X}_{s:n}^{N,1}$ in (21) to protect $X_{s:n}$ for $r+1 \leq s \leq n$.

Based on the Taylor series prediction method, the functions $h(Z_{s:n})$ and $h_1(z_{r:n}, Z_{s:n})$ are expanded by using the Taylor series around points $F^{-1}(p_s)$ and $(F^{-1}(p_r), F^{-1}(p_s))$, respectively. According to Raqab [13], we can approximate $h(Z_{s:n})$ and $h_1(z_{r:n}, Z_{s:n})$ by

$$h(Z_{s:n}) = \frac{f(Z_{s:n})}{1 - F(Z_{s:n})} \approx \alpha + \beta Z_{s:n}, \quad (22)$$

and

$$h_1(z_{r:n}, Z_{s:n}) = \frac{f(Z_{s:n})}{F(Z_{s:n}) - F(z_{r:n})} \approx \gamma + \rho z_{r:n} - v_s Z_{s:n}. \quad (23)$$

The values of $\alpha, \beta, \gamma, \rho$ and v_s are given in Appendix A. Equation (10) can be rewritten by

$$(s-r-1)(\gamma + \rho z_{r:n} - v_s Z_{s:n}) - z_{s:n} - (n-s)(\alpha + \beta Z_{s:n}) = 0. \quad (24)$$

The MLP_T of $X_{s:n}$ can be obtained by

$$\widehat{X}_{s:n}^{N,2} = \max\left\{\frac{(s-r-1)\rho x_{r:n}}{(s-r-1)v_s + 1 + (n-s)\beta} + \left[1 - \frac{(s-r-1)\rho}{(s-r-1)v_s + 1 + (n-s)\beta}\right]\widehat{\mu}_N + \frac{(s-r-1)\gamma - (n-s)\alpha}{(s-r-1)v_s + 1 + (n-s)\beta}\widehat{\sigma}_N, x_{r:n}\right\}, \quad (25)$$

where $r+1 \leq s \leq n$.

If the underlying distribution is SEV, the PDF of the SEV is given by

$$g(z) = \phi_{sev}(z) = e^{z - e^z}. \quad (26)$$

Based on the expected value prediction method, $\Psi(z) = -\phi'_{sev}(z)/\phi_{sev}(z) = e^z - 1$. Using (8) and (9), the MLEs of μ and σ are denoted by $\widehat{\mu}_S$ and $\widehat{\sigma}_S$, respectively. Replacing μ and σ with $\widehat{\mu}_S$ and $\widehat{\sigma}_S$ in (6), (6) can be represented by

$$\widehat{L}_S(\widehat{\mu}_S, \widehat{\sigma}_S) = C_1 \prod_{j=1}^{r} \widehat{\sigma}_S^{-1} \phi_{sev}(z_{j:n}) [1 - \Phi_{sev}(z_{r:n})]^{n-r}, \quad (27)$$

where $\Phi_{sev}(z) = 1 - \exp[-\exp(z)]$ is the CDF of the standard SEV. Then $h_1(z_{r:n}, Z_{s:n})$ and $h(Z_{s:n})$ are replaced with their respective expected values in Eq. (10). Equation (10) can be rewritten as

$$(s-r-1)E[h_1(z_{r:n}, Z_{s:n})] - \left(e^{\widehat{Z}_{s:n}} - 1\right) - (n-s)E[h(Z_{s:n})] = 0. \quad (28)$$

The MLP_E of $X_{s:n}$ can be obtained as

$$\widehat{X}_{s:n}^{SEV,1} = \max\{\widehat{\mu}_S + \widehat{\sigma}_S \ln(E[\Psi(Z_{s:n})] + 1), x_{r:n}\} \quad (29)$$

for $r+1 \leq s \leq n$ and $E\Psi(Z_{s:n}) = E(e^{Z_{s:n}} - 1)$.

Based on the Taylor series prediction method, expanding $h(Z_{s:n})$ and $h_1(z_{r:n}, Z_{s:n})$ by using the Taylor series at the points $F^{-1}(p_s)$ and $(F^{-1}(p_r), F^{-1}(p_s))$, respectively. We obtain

$$h(Z_{s:n}) = \frac{f(Z_{s:n})}{1 - F(Z_{s:n})} \approx 1 - \alpha_s - \beta_s Z_{s:n}, \quad (30)$$

and

$$h_1(z_{r:n}, Z_{s:n}) = \frac{f(Z_{s:n})}{F(Z_{s:n}) - F(z_{r:n})} \approx \gamma_E + \rho_E z_{r:n} + v_E Z_{s:n}. \quad (31)$$

The values of $\alpha_s, \beta_s, \gamma_E, \rho_E$ and v_E are given in Appendix B. Equation (10) can be rewritten as

$$(s-r-1)(\gamma_E + \rho_E z_{r:n} + v_E Z_{s:n}) - e^{Z_{s:n}} - 1 - (n-s)(1 - \alpha_s - \beta_s Z_{s:n}) = 0 \quad (32)$$

The MLP_T of $X_{s:n}$ can be derived as

$$\widehat{X}_{s:n}^{SEV,2} = \max\left\{\frac{-(s-r-1)v_E x_{r:n}}{(s-r-1)\rho_E + \beta_s + (n-s)\beta_s} + \left[1 + \frac{(s-r-1)v_E}{(s-r-1)\rho_E + \beta_s + (n-s)\beta_s}\right]\widehat{\mu}_S - \frac{(s-r-1)\gamma_E + \alpha_s - (n-s) + (n-s)\alpha_s}{(s-r-1)\rho_E + \beta_s + (n-s)\beta_s}\widehat{\sigma}_S, x_{r:n}\right\}, \quad (33)$$

for $r+1 \leq s \leq n$.

3. Three Model Selection Approaches

When several candidate distributions are competing for the best underlying distribution and the users cannot identify which one distribution is the best, we suggest three approaches to discriminate the candidate distributions, the ratio of the maximized likelihood (RRML) approach, modification D_{SP} approach (shorted as D_{SP} approach), and modification D approach (shorted as the D approach), to obtain the predictor of $\widehat{X}_{s:n}$. It is noticed that the idea of the D_{SP} approach and D approach is based on goodness-of-fit test methods. All these three approaches can be implemented to obtain the predictor of $X_{s:n}$ via using Algorithms 1–3.

Algorithm 1 (the RRML approach).

Step 1. Collect a type II censored sample, which has size n and r observed failure times; we consider k candidate distributions.

Step 2. Obtain $(\widehat{\mu}_{M_i}, \widehat{\sigma}_{M_i})$ and $\widehat{L}_{M_i}(\widehat{\mu}_{M_i}, \widehat{\sigma}_{M_i})$ for the candidate distribution M_i, $i = 1, 2, \ldots, k$. Obtain $X_{s:n}$ under the candidate distribution M_i and label it by $\widehat{X}_{s:n}^{M_i,j}$ for $s = r+1,\ldots,n$, $i = 1,2,\ldots,k$ and $j = 1$ or 2.

Step 3. Let $\widehat{X}_{s:n}^{A1,j}$ denote the predicted value of $X_{s:n}$ for $j = 1$ or 2. Based on the method proposed by Dumonceaux and Antle [16], we can obtain $\widehat{X}_{s:n}^{A1,j}$, which can provide the largest maximum likelihood information by

$$\widehat{L}_{A1}\left(\widehat{\mu}_{A1},\widehat{\sigma}_{A1}\right) = \max\left\{\widehat{L}_{M_1}\left(\widehat{\mu}_{M_1},\widehat{\sigma}_{M_1}\right), \widehat{L}_{M_2}\left(\widehat{\mu}_{M_2},\widehat{\sigma}_{M_2}\right),\ldots,\widehat{L}_{M_k}\left(\widehat{\mu}_{M_k},\widehat{\sigma}_{M_k}\right)\right\}. \quad (34)$$

If the candidate distributions are ND and SEV, Steps 2 and 3 in Algorithm 1 can be reduced to Step 2' and Step 3' as the following, respectively:

Step 2'. Obtain $(\widehat{\mu}_N,\widehat{\sigma}_N)$, $(\widehat{\mu}_S,\widehat{\sigma}_S)$, $\widehat{L}_N(\widehat{\mu}_N,\widehat{\sigma}_N)$ and $\widehat{L}_S(\widehat{\mu}_S,\widehat{\sigma}_S)$. Obtain $X_{s:n}$ under the ND ($\widehat{X}_{s:n}^{N,j}$) and obtain $X_{s:n}$ under the SEV ($\widehat{X}_{s:n}^{SEV,j}$) for $s = r+1,\ldots,n$ and $j = 1$ or 2.

Step 3'. Let $\widehat{X}_{s:n}^{A1}$ denote the predicted value of $X_{s:n}$. Then

$$\widehat{X}_{s:n}^{A1} = \begin{cases} \widehat{X}_{s:n}^{N,j} & \text{if } \widehat{L}_N\left(\widehat{\mu}_N,\widehat{\sigma}_N\right) > \widehat{L}_S\left(\widehat{\mu}_S,\widehat{\sigma}_S\right) \\ \widehat{X}_{s:n}^{SEV,j}, & \text{otherwise.} \end{cases} \quad (35)$$

$$\text{for } s = r+1,\ldots,n \text{ and } j = 1 \text{ or } 2.$$

Algorithm 2 (the D_{SP} approach).

Step 1. Collect a type II censored sample, which has size n and r observed failure times.

Step 2. Obtain $(\widehat{\mu}_{M_i},\widehat{\sigma}_{M_i})$ for $i = 1,2,\ldots,k$, and then obtain $\widehat{X}_{s:n}^{M_i,j}$ for $s = r+1,\ldots,n$, $i = 1,2,\ldots,k$ and $j = 1$ or 2.

Step 3. Based on the method proposed by Castro-Kuriss et al. [29], the modification of D_{SP} with censored observations can be presented by

$$D_{SP}(\mu,\sigma) = \max_{1\le i\le r}\left\{\frac{2}{\pi}\left|\arcsin\left(\sqrt{\frac{i-0.5}{n}}\right) - \arcsin\left(\sqrt{U_{i:n}}\right)\right|\right\}, \quad (36)$$

where $U_{i:n} = G((x_{i:n}-\mu)/\sigma)$. The definition of $G(\bullet)$ is the same as that of (2), it represents the CDF of the assumed distribution in model selection. Evaluate the value of D_{SP} through using the candidate distribution M_i for $i = 1,2,\ldots,k$.

Step 4. Let $\widehat{X}_{s:n}^{A2,j}$ be the predicted value of $X_{s:n}$ for $j = 1$ or 2, then $\widehat{X}_{s:n}^{A2,j}$ can be obtained with the smallest $\widehat{D}_{SP}$. That is, $\widehat{X}_{s:n}^{A2,j}$ is the value corresponding to $\widehat{D}_{SP}^{A2}(\widehat{\mu}_{A2},\widehat{\sigma}_{A2})$, which is defined by

$$\widehat{D}_{SP}^{A2}\left(\widehat{\mu}_{A2},\widehat{\sigma}_{A2}\right) = \min\left\{\widehat{D}_{SP}\left(\widehat{\mu}_{M_1},\widehat{\sigma}_{M_1}\right), \widehat{D}_{SP}\left(\widehat{\mu}_{M_2},\widehat{\sigma}_{M_2}\right),\ldots,\widehat{D}_{SP}\left(\widehat{\mu}_{M_k},\widehat{\sigma}_{M_k}\right)\right\}. \quad (37)$$

If the candidate distributions are ND and SEV, Steps 2, 3, and 4 in Algorithm 2 can be reduced to Step 2' and Step 3' as the following, respectively:

Step 2'. Obtain $(\widehat{\mu}_N,\widehat{\sigma}_N)$ and $(\widehat{\mu}_S,\widehat{\sigma}_S)$. Obtain the $\widehat{X}_{s:n}^{N,j}$ under the ND and obtain the $\widehat{X}_{s:n}^{SEV,j}$ under the SEV for $s = r+1,\ldots,n$ and $j = 1$ or 2.

Step 3'. The modification of D_{SP} with censored observations can be presented by

$$D_{SP}(\mu,\sigma) = \max_{1\le i\le r}\left\{\frac{2}{\pi}\left|\arcsin\left(\sqrt{\frac{i-0.5}{n}}\right) - \arcsin\left(\sqrt{U_{i:n}}\right)\right|\right\}, \quad (38)$$

where $U_{i:n} = G((x_{i:n}-\mu)/\sigma)$. The definition of $G(\bullet)$ is the same as that of (2); it represents the CDF of the assumed distribution in model selection. Evaluate the values of D_{SP} through using the ND and SEV and denot them by $\widehat{D}_{SP}^{N}(\widehat{\mu}_N,\widehat{\sigma}_N)$ and $\widehat{D}_{SP}^{SEV}(\widehat{\mu}_S,\widehat{\sigma}_S)$, respectively.

Step 4'. Let $\widehat{X}_{s:n}^{A2,j}$ denote the predicted value of $X_{s:n}$, then $\widehat{X}_{s:n}^{A2,j}$ can be obtained by

$$\widehat{X}_{s:n}^{A2} = \begin{cases} \widehat{X}_{s:n}^{N,j}, & \text{if } \widehat{D}_{SP}^{N}\left(\widehat{\mu}_N,\widehat{\sigma}_N\right) < \widehat{D}_{SP}^{SEV}\left(\widehat{\mu}_S,\widehat{\sigma}_S\right) \\ \widehat{X}_{s:n}^{SEV,j}, & \text{if } \widehat{D}_{SP}^{N}\left(\widehat{\mu}_N,\widehat{\sigma}_N\right) \ge \widehat{D}_{SP}^{SEV}\left(\widehat{\mu}_S,\widehat{\sigma}_S\right) \end{cases} \quad (39)$$

$$\text{for } s = r+1,\ldots,n \text{ and } j = 1 \text{ or } 2.$$

Algorithm 3 (the D approach).

Step 1. Collect a type II censored sample, which has size n and r observed failure times.

Step 2. Obtain $(\widehat{\mu}_{M_i},\widehat{\sigma}_{M_i})$ for $i = 1,2,\ldots,k$, and then obtain $\widehat{X}_{s:n}^{M_i,j}$ for $s = r+1,\ldots,n$, $i = 1,2,\ldots,k$ and $j = 1$ or 2.

Step 3. Based on the method proposed by Castro-Kuriss et al. [29], the modification of $D(\mu,\sigma)$ with censored observations can be presented by

$$D(\mu,\sigma) = \max_{1\le i\le r}\left\{\frac{2}{\pi}\left|\sqrt{\frac{i-0.5}{n}} - U_{i:n}\right|\right\} + \frac{0.5}{n}, \quad (40)$$

where $U_{i:n} = G((x_{i:n}-\mu)/\sigma)$.

Step 4. Let $\widehat{X}_{s:n}^{A3,j}$ be the predicted value of $X_{s:n}$ for $j = 1$ or 2, then $\widehat{X}_{s:n}^{A3,j}$ can be obtained with the smallest $\widehat{D}(\widehat{\mu}_{M_i},\widehat{\sigma}_{M_i})$. That is, $\widehat{X}_{s:n}^{A3,j}$ is the value corresponding to $\widehat{D}^{A3}(\widehat{\mu}_{A3},\widehat{\sigma}_{A3})$, which is defined by

$$\widehat{D}^{A3}\left(\widehat{\mu}_{A3},\widehat{\sigma}_{A3}\right) = \min\left\{\widehat{D}\left(\widehat{\mu}_{M_1},\widehat{\sigma}_{M_1}\right),\widehat{D}\left(\widehat{\mu}_{M_2},\widehat{\sigma}_{M_2}\right), \ldots,\widehat{D}\left(\widehat{\mu}_{M_k},\widehat{\sigma}_{M_k}\right)\right\}. \quad (41)$$

If the candidate distributions are ND and SEV, Steps 2, 3, and 4 in Algorithm 3 can be reduced to Step 2' and Step 3' as the following, respectively:

Step 2'. Obtain $(\widehat{\mu}_N, \widehat{\sigma}_N)$ and $(\widehat{\mu}_S, \widehat{\sigma}_S)$. Obtain $\widehat{X}_{s:n}^{N,j}$ under the ND and obtain $\widehat{X}_{s:n}^{SEV,j}$ under the SEV for $s = r+1, \ldots, n$ and $j = 1$ or 2.

Step 3'. The modification of $D(\mu, \sigma)$ with censored observations can be presented by

$$D(\mu,\sigma) = \max_{1\le i\le r}\left\{\frac{2}{\pi}\left|\sqrt{\frac{i-0.5}{n}} - U_{i:n}\right|\right\} + \frac{0.5}{n}, \tag{42}$$

where $U_{i:n} = G((x_{i:n} - \mu)/\sigma)$. Evaluate the value of $D(\mu,\sigma)$ by using the ND and SEV and denote them by $\widehat{D}^N(\widehat{\mu}_N, \widehat{\sigma}_N)$ and $\widehat{D}^{SEV}(\widehat{\mu}_S, \widehat{\sigma}_S)$.

Step 4'. Let $\widehat{X}_{s:n}^{A3,j}$ denote the predicted value of $X_{s:n}$, then $\widehat{X}_{s:n}^{A3,j}$ can be obtained by

$$\widehat{X}_{s:n}^{A3,j} = \begin{cases}\widehat{X}_{s:n}^{N,j}, & \text{if } \widehat{D}^N(\widehat{\mu}_N,\widehat{\sigma}_N) < \widehat{D}^{SEV}(\widehat{\mu}_S,\widehat{\sigma}_S)\\ \widehat{X}_{s:n}^{SEV,j}, & \text{if } \widehat{D}^N(\widehat{\mu}_N,\widehat{\sigma}_N) \ge \widehat{D}^{SEV}(\widehat{\mu}_S,\widehat{\sigma}_S)\end{cases} \tag{43}$$

$$\text{for } s = r+1,\ldots,n \text{ and } j = 1 \text{ or } 2.$$

4. Monte Carlo Simulations

A Monte Carlo simulation study was conducted in this section, by using R language, to evaluate the performance of the proposed three approaches with two predicting methods. We consider the ND and SEV as the candidate distributions for competing the best lifetime model in the simulation study. The data sets of type II censoring sample, $x_{1:n}, \ldots, x_{r:n}$, used in the simulation were randomly generated from the ND and SEV with location parameter $\mu = 0$ and scale parameter $\sigma = 1$. Then, the s^{th} order statistic is predicted and denoted by $\widehat{X}_{s:n}$ for $s = r+1, r+2, \ldots, n$ for the sample sizes $n = 20, 30, 40, 50$ and 60. For the purpose of comparison, the values of the bias and mean square error (MSE) of $\widehat{X}_{s:n}$ are evaluated using $N = 10000$ Monte Carlo runs:

$$\text{bais} = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{X}_{s:n,i} - X_{s:n}\right) \tag{44}$$

and

$$\text{MSE} = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{X}_{s:n,i} - X_{s:n}\right)^2, \tag{45}$$

where $\widehat{X}_{s:n,i}$ is the predicted value of $X_{s:n}$ that is obtained in the i^{th} iteration of simulation for $i = 1, \ldots, N$. All simulation results are displayed in Tables 1 and 2 with the candidate distributions of ND and SEV. From Tables 1 and 2, we notice that the bias and MSE are large when the misspecification model is used. The impact of misspecification depends on the values of r and s. As n or r increases, the simulated bias and MSE are decreased. We also find that the MSE based on using the Taylor series prediction method is smaller than that based on using the expected values prediction method when the sample size is or larger than 30.

To evaluate the performance of the three proposed model selection approaches for MLP, Tables 3–5 report the simulation results for three model selection approaches from the ND. Tables 6–8 respectively report the simulation results for three model selection approaches from the SEV. The column "correct (%)" presented in Tables 3–8 is the correct model selection rate in all simulation runs. From Tables 3–8 we find that the three model selection approaches have good ability to identify the correct underlying distribution with a high probability. Moreover, the MSEs of these three approaches are close to those simulated MSEs of the cases by using the real underlying distribution. Overall, the correct model selection rates through using D_{SP} approach or D approach are higher than that of using the RRML approach when the sample size is smaller than 30. When the sample size grows to or over 30, the performance of the RRML approach is improved and the correct model selection rate of the RRML approach is higher than that are obtained by using the D_{SP} or D approach. To compare the performance of using two different MLPs, the MSEs of using the expected values prediction method are smaller than that using the Taylor series prediction method when the sample size is smaller than 30. The proposed approaches can perform well under large sample size cases.

5. Illustrative Examples

In this section, three numerical examples are presented to illustrate the proposed approaches in Sections 2–4.

5.1. Example 1. A test airplane component's failure time dataset provided in Mann and Fertig [30], in which 13 components were placed on test, and the test was terminated at the time of the 10^{th} failure. The failure times (in hours) of the 10 components that failed were

D_1: 0.22, 0.50, 0.88, 1.00, 1.32, 1.33, 1.54, 1.76, 2.50, 3.00.

Let Y_1 be the logs of the ten observations, i.e., $Y_1 = \ln(D_1)$. Figure 2 presents the histogram and the estimated PDFs of the ND and SEV. From Figure 2, we find a difficulty to fully decide the best distribution for lifetime fitting due to the fact that both candidate distributions can provide good fitting for this data set. In this example, we consider using D_{SP} approach to discriminate competing models and apply Taylor series prediction method to predicting the future order statistics, which are censored. The R source codes of Example 1 can be found in Appendix C and other designs can be obtained from the authors upon request.

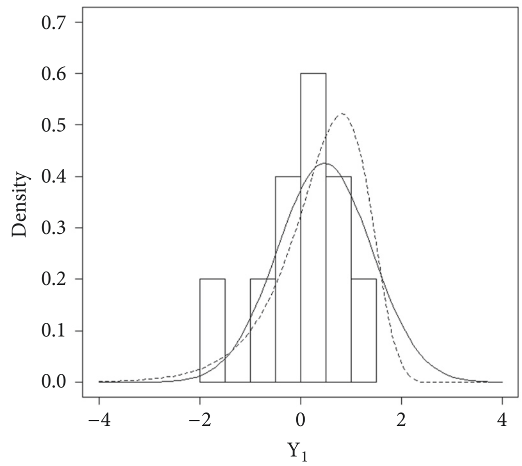

Figure 2: The histogram and the estimated probability density functions of airplane component's failure time in Example 1.

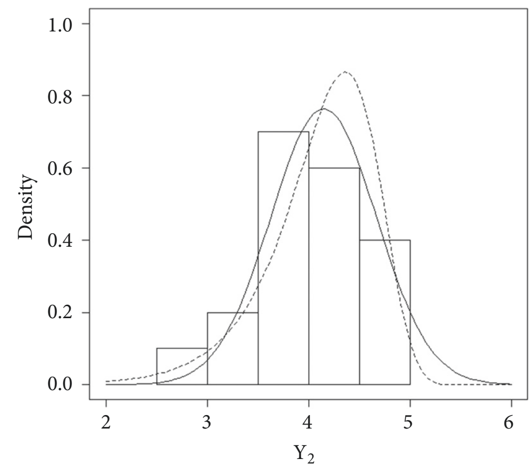

Figure 3: The histogram and the estimated probability density functions of tests on endurance of deep groove ball bearings in Example 2.

Through using Newton-Raphson algorithm, we obtained the MLEs of μ and σ as $(\widehat{\mu}_N, \widehat{\sigma}_N) = (0.479, 0.938)$ and $(\widehat{\mu}_S, \widehat{\sigma}_S) = (0.821, 0.705)$ for the ND and SEV, respectively.

The D_{SP} values via using ND and SEV are 0.223 and 0.212, respectively. Because the D_{SP} value obtained from the SEV is smaller than that obtained from the ND, we claim the best distribution of this data set is SEV. The Taylor series prediction for $(Y_{11:13}, Y_{12:13}, Y_{13:13})$ under the extreme value distribution with the censored sample can be obtained by $(\widehat{Y}^{A2,2}_{11:13}, \widehat{Y}^{A2,2}_{12:13}, \widehat{Y}^{A2,2}_{13:13}) = (1.098, 1.281, 1.567)$.

5.2. Example 2. In this example, we consider that the tests on endurance of deep groove ball bearings data, reported by Lieblein and Zelen [31] and further studied by Meeker and Escobar (1998), are used to illustrate the methodologies developed in this paper. The data are the numbers of million revolutions before failure for each of the 23 ball bearings in the life test. Meeker and Escobar [32] pointed out that this data (D_2) follows lognormal distribution or Weibull distribution. Hence $Y_2 = \ln(D_2)$ follows a ND or SEV. The data is given as follows:

> D_2: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

For more information about this carbon fiber breaking strength data set, one can be referred to Meeker and Escobar (1998). In this example, we assume that the censoring proportion is 0.8696 ($r = 20$, $n = 23$). Figure 3 presents the histogram and the estimated PDFs of ND and SEV based on the type II right-censored data set. From Figure 3, it is difficult to decide the best distribution from these two candidate distributions.

We consider using D approach in Example 2 for model selection and use expected values prediction method to predict the future order statistics, which are censored. The MLEs of μ and σ can be obtained via using Newton-Raphson algorithm, the resulting MLEs are $(\widehat{\mu}_N, \widehat{\sigma}_N) = (4.148, 0.524)$ and $(\widehat{\mu}_S, \widehat{\sigma}_S) = (4.369, 0.425)$ for the ND and SEV, respectively. The D values based on using the ND and SEV are 0.181 and 0.297, respectively. Because the D value obtained from ND is smaller than that obtained from SEV, we claim the best model is normal. The expected values prediction of $(Y_{21:23}, Y_{22:23}, Y_{23:23})$ via using ND are $(\widehat{Y}^{A3,1}_{21:23}, \widehat{Y}^{A3,1}_{22:23}, \widehat{Y}^{A3,1}_{23:23}) = (4.784, 4.922, 5.160)$. In addition, we compare our prediction results with the MMLP values that proposed by Yang and Tong (2006), in which the MMLP is $(\widehat{Y}_{21:23}, \widehat{Y}_{22:23}, \widehat{Y}_{23:23}) = (4.662, 4.936, 5.175)$. Our predicted results are close to that proposed by Yang and Tong [14] even we cannot initially assume which one of the ND or SEV is the best distribution.

5.3. Example 3. We consider the experiment on the pull-off performance for use in automotive engine components, reported by Byrne and Taguchi [33] and further studied by Yang and Tong [14], is used to illustrate the methodologies developed in this study. An experiment was conducted to find a method to maximize the pull-off force. Four control factors that could influence the assembly's pull-off force have been identified. Repeat 8 times for each run and record the pull-off force in pounds. Table 9 lists the four control factors with their levels and complete data of this experiment. In this example, we assume that the censoring proportion is 0.75 ($r = 6$, $n = 8$). Please note that censored data cannot support the practitioner to conduct experimental design methods. Predicting the unobserved data and using a pseudo-complete data set for conducting experimental design methods is required.

We consider using the RRML approach for model selection and use Taylor series prediction method to predict

Table 1: The corresponding bias and MSEs for different settings with model misspecification when true distribution is ND.

			Assumed Distribution							
			Normal distribution				Extreme Value distribution			
			$\widehat{X}_{s:n}^{N,1}$		$\widehat{X}_{s:n}^{N,2}$		$\widehat{X}_{s:n}^{SEV,1}$		$\widehat{X}_{s:n}^{SEV,2}$	
n	r	s	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	8	9	-0.1189	0.1295	-0.3430	0.2140	-0.1970	0.1406	-0.3430	0.2140
	7	8	-0.1193	0.0949	-0.2832	0.1509	-0.1597	0.0995	-0.2832	0.1509
	7	9	-0.1569	0.2159	-0.3308	0.2753	-0.3058	0.2619	-0.4007	0.3196
	6	7	-0.1206	0.0738	-0.2560	0.1170	-0.1402	0.0752	-0.2560	0.1170
	6	8	-0.3009	0.1652	-0.1570	0.2134	-0.3471	0.1947	-0.2592	0.2399
	6	9	-0.1964	0.3199	-0.2980	0.3498	-0.4147	0.4169	-0.4678	0.4566
	5	6	-0.1222	0.0747	-0.2448	0.1142	-0.1298	0.0749	-0.2448	0.1142
	5	7	-0.1690	0.1570	-0.2913	0.1968	-0.2452	0.1773	-0.3266	0.2152
	5	8	-0.2204	0.2661	-0.2979	0.2877	-0.3842	0.3310	-0.4304	0.3606
	5	9	-0.2702	0.4625	-0.3429	0.4841	-0.5557	0.6235	-0.5914	0.6567
20	16	18	-0.0675	0.0802	-0.2343	0.1178	-0.2074	0.1067	-0.2832	0.1414
	14	16	-0.0626	0.0514	-0.1964	0.0780	-0.1467	0.0622	-0.2208	0.0875
	14	18	-0.0802	0.1259	-0.1607	0.1372	-0.2995	0.1939	-0.3338	0.2143
	12	14	-0.0589	0.0411	-0.1753	0.0640	-0.1162	0.0477	-0.1903	0.0693
	12	16	-0.0863	0.0936	-0.1469	0.1019	-0.2381	0.1317	-0.2703	0.1467
	12	18	-0.1119	0.1857	-0.1694	0.1942	-0.4085	0.3102	-0.4333	0.3296
	10	12	-0.0664	0.0376	-0.1707	0.0579	-0.1075	0.0420	-0.1812	0.0614
	10	14	-0.0884	0.0818	-0.1385	0.0886	-0.2072	0.1084	-0.2392	0.1211
	10	16	-0.1151	0.1417	-0.1552	0.1474	-0.3376	0.2210	-0.3588	0.2343
	10	18	-0.1497	0.2628	-0.1960	0.2701	-0.5258	0.4627	-0.5460	0.4823
30	24	27	0.0295	0.0763	-0.1161	0.0584	-0.2544	0.1226	-0.2396	0.1020
	21	24	-0.0059	0.0500	-0.1078	0.0386	-0.1952	0.0725	-0.1819	0.0591
	21	27	-0.0238	0.0933	-0.0991	0.0827	-0.3317	0.1889	-0.2687	0.1366
	18	21	-0.0482	0.0343	-0.1087	0.0325	-0.1851	0.0574	-0.1700	0.0487
	18	24	-0.0748	0.0650	-0.1111	0.0605	-0.3107	0.1451	-0.2409	0.1030
	18	27	-0.0961	0.1203	-0.1328	0.1210	-0.4270	0.2816	-0.3717	0.2418
	15	18	-0.0654	0.0267	-0.1096	0.0310	-0.1354	0.0388	-0.1579	0.0430
	15	21	-0.0862	0.0540	-0.1052	0.0553	-0.2572	0.1096	-0.2331	0.0943
	15	24	-0.1138	0.0911	-0.1310	0.0941	-0.3836	0.2094	-0.3574	0.1924
	15	27	-0.1464	0.1578	-0.1722	0.1648	-0.5614	0.4196	-0.5463	0.4115
40	32	36	0.0653	0.0634	-0.0669	0.0404	-0.2696	0.1229	-0.2145	0.0840
	28	32	0.0564	0.0426	-0.0536	0.0225	-0.1961	0.0685	-0.1468	0.0418
	28	36	0.0622	0.0661	-0.0268	0.0495	-0.3224	0.1660	-0.2442	0.1073
	24	28	0.0503	0.0349	-0.0470	0.0173	-0.1673	0.0514	-0.1175	0.0294
	24	32	0.0567	0.0476	-0.0188	0.0307	-0.2464	0.1060	-0.1588	0.0554
	24	36	0.0679	0.0697	-0.0030	0.0560	-0.3202	0.1666	-0.2575	0.1199
	20	24	0.0402	0.0318	-0.0459	0.0152	-0.1629	0.0464	-0.1049	0.0244
	20	28	0.0388	0.0405	-0.0208	0.0261	-0.2236	0.0887	-0.1203	0.0394
	20	32	0.0498	0.0511	-0.0035	0.0402	-0.2503	0.1088	-0.1666	0.0637
	20	36	0.0548	0.0769	0.0015	0.0674	-0.3215	0.1688	-0.2658	0.1294

Table 1: Continued.

			Assumed Distribution							
			Normal distribution				Extreme Value distribution			
			$\widehat{X}_{s:n}^{N,1}$		$\widehat{X}_{s:n}^{N,2}$		$\widehat{X}_{s:n}^{SEV,1}$		$\widehat{X}_{s:n}^{SEV,2}$	
n	r	s	bias	MSE	bias	MSE	bias	MSE	bias	MSE
50	40	45	0.0660	0.0504	-0.0447	0.0310	-0.2813	0.1201	-0.1992	0.0700
	35	40	0.0564	0.0352	-0.0387	0.0175	-0.2043	0.0676	-0.1330	0.0342
	35	45	0.0671	0.0551	-0.0138	0.0407	-0.3223	0.1587	-0.2372	0.0970
	30	35	0.0506	0.0293	-0.0321	0.0131	-0.1742	0.0503	-0.1016	0.0228
	30	40	0.0580	0.0371	-0.0074	0.0239	-0.2479	0.0992	-0.1500	0.0467
	30	45	0.0676	0.0564	0.0076	0.0457	-0.3245	0.1580	-0.2547	0.1092
	25	30	0.0409	0.0260	-0.0313	0.0110	-0.1724	0.0446	-0.0903	0.0180
	25	35	0.0496	0.0313	-0.0065	0.0193	-0.2235	0.0822	-0.1108	0.0319
	25	40	0.0567	0.0393	0.0081	0.0299	-0.2530	0.1017	-0.1640	0.0559
	25	45	0.0702	0.0577	0.0221	0.0492	-0.3245	0.1573	-0.2646	0.1164
60	48	54	0.0639	0.0445	-0.0350	0.0266	-0.2901	0.1196	-0.1925	0.0630
	42	48	0.0554	0.0309	-0.0275	0.0144	-0.2101	0.0643	-0.1227	0.0285
	42	54	0.0644	0.0457	-0.0046	0.0341	-0.3264	0.1523	-0.2325	0.0886
	36	42	0.0500	0.0247	-0.0240	0.0106	-0.1803	0.0491	-0.0929	0.0186
	36	48	0.0573	0.0325	-0.0024	0.0209	-0.2505	0.0969	-0.1465	0.0429
	36	54	0.0678	0.0488	0.0148	0.0396	-0.3268	0.1542	-0.2526	0.1040
	30	36	0.0416	0.0215	-0.0226	0.0092	-0.1786	0.0466	-0.0798	0.0154
	30	42	0.0452	0.0260	-0.0030	0.0162	-0.2304	0.0786	-0.1082	0.0276
	30	48	0.0573	0.0333	0.0158	0.0257	-0.2532	0.0983	-0.1588	0.0512
	30	54	0.0674	0.0494	0.0233	0.0429	-0.3290	0.1568	-0.2668	0.1139

Table 2: The corresponding bias and MSEs for different settings with model misspecification when the true distribution is SEV.

			Assumed Distribution							
			Normal distribution				Extreme Value distribution			
			$\widehat{X}_{s:n}^{N,1}$		$\widehat{X}_{s:n}^{N,2}$		$\widehat{X}_{s:n}^{SEV,1}$		$\widehat{X}_{s:n}^{SEV,2}$	
n	r	s	bias	MSE	bias	MSE	bias	MSE	bias	MSE
10	8	9	0.1225	0.1563	-0.2965	0.1583	-0.0669	0.0868	-0.2969	0.1589
	7	8	0.0417	0.1109	-0.2685	0.1365	-0.0739	0.0793	-0.2692	0.1376
	7	9	0.1659	0.2742	-0.1233	0.1783	-0.1131	0.1742	-0.2539	0.2066
	6	7	-0.0155	0.0969	-0.2687	0.1355	-0.0864	0.0797	-0.2702	0.1364
	6	8	0.0710	0.2206	-0.1725	0.1852	-0.1287	0.1759	-0.2635	0.2118
	6	9	0.2075	0.4628	0.0304	0.3333	-0.1741	0.2954	-0.2571	0.3099
	5	6	-0.0532	0.1017	-0.2767	0.1497	-0.0913	0.0909	-0.278	0.1516
	5	7	-0.0037	0.2267	-0.2144	0.2142	-0.1548	0.202	-0.2848	0.2367
	5	8	0.0846	0.4086	-0.0544	0.3325	-0.2104	0.3158	-0.2858	0.3299
	5	9	0.2196	0.7279	0.0913	0.5969	-0.2717	0.4812	-0.3297	0.4914
20	16	18	0.3121	0.2071	-0.0726	0.0498	-0.0441	0.0543	-0.1745	0.0747
	14	16	0.2053	0.1345	-0.1028	0.0466	-0.0455	0.0431	-0.1658	0.0639
	14	18	0.3306	0.2389	0.1039	0.0949	-0.0702	0.0961	-0.1327	0.1031
	12	14	0.1424	0.1072	-0.1198	0.0481	-0.0445	0.0404	-0.1646	0.0605
	12	16	0.2234	0.1702	0.0440	0.0791	-0.0745	0.0868	-0.133	0.0925
	12	18	0.3685	0.3154	0.2098	0.1901	-0.0949	0.155	-0.1402	0.1594
	10	12	0.0967	0.0948	-0.1360	0.0561	-0.0496	0.0454	-0.1718	0.0667
	10	14	0.1624	0.1458	0.0156	0.0853	-0.0789	0.0949	-0.136	0.1015
	10	16	0.2667	0.2553	0.1467	0.1775	-0.1032	0.1629	-0.1417	0.1668
	10	18	0.4383	0.4968	0.3199	0.3774	-0.1335	0.2516	-0.1691	0.2546
30	24	27	0.5482	0.3543	0.0788	0.0330	0.0846	0.0522	-0.0552	0.0298
	21	24	0.4351	0.2472	0.0270	0.0212	0.0717	0.0477	-0.0573	0.0238
	21	27	0.5448	0.3562	0.2068	0.0821	0.082	0.0574	-0.0062	0.0386
	18	21	0.3743	0.1980	0.0003	0.0188	0.0594	0.0452	-0.0634	0.0228
	18	24	0.4215	0.2414	0.1135	0.0478	0.0588	0.0523	-0.0222	0.0343
	18	27	0.5250	0.3420	0.2612	0.1198	0.065	0.064	0.0012	0.0518
	15	18	0.3146	0.1777	-0.0232	0.0221	0.0308	0.0473	-0.0821	0.0289
	15	21	0.3257	0.1857	0.0575	0.0392	0.0069	0.0597	-0.0578	0.0447
	15	24	0.3688	0.2190	0.1409	0.0693	-0.0092	0.0766	-0.0588	0.0667
	15	27	0.4753	0.3279	0.2759	0.1478	-0.0185	0.1067	-0.0598	0.0998
40	32	36	0.5536	0.3486	0.1141	0.0342	0.0715	0.0397	-0.0358	0.0221
	28	32	0.4471	0.2346	0.0564	0.0183	0.0675	0.0352	-0.0346	0.0166
	28	36	0.5551	0.3485	0.2345	0.0847	0.0742	0.0422	0.0038	0.0289
	24	28	0.3923	0.1946	0.0293	0.0144	0.0611	0.0344	-0.0361	0.0148
	24	32	0.4471	0.2334	0.1434	0.0436	0.0666	0.0374	-0.0006	0.0234
	24	36	0.5584	0.3537	0.3026	0.1257	0.08	0.0454	0.0279	0.0349
	20	24	0.3732	0.1831	0.0142	0.0149	0.0575	0.0362	-0.0384	0.0163
	20	28	0.3929	0.1921	0.0932	0.0327	0.0588	0.0364	-0.0059	0.0229
	20	32	0.4455	0.2372	0.1958	0.0674	0.0678	0.0392	0.021	0.0289
	20	36	0.5526	0.3522	0.3374	0.1536	0.0762	0.0466	0.0362	0.0386

Table 2: Continued.

			Assumed Distribution							
			Normal distribution				Extreme Value distribution			
			$\widehat{X}_{s:n}^{N,1}$		$\widehat{X}_{s:n}^{N,2}$		$\widehat{X}_{s:n}^{SEV,1}$		$\widehat{X}_{s:n}^{SEV,2}$	
n	r	s	bias	MSE	bias	MSE	bias	MSE	bias	MSE
50	40	45	0.5605	0.3444	0.1386	0.0357	0.0686	0.0324	-0.0204	0.018
	35	40	0.4484	0.2291	0.0717	0.0174	0.0611	0.028	-0.0238	0.0127
	35	45	0.5600	0.3432	0.2491	0.0864	0.0698	0.0334	0.0086	0.0237
	30	35	0.3925	0.1859	0.0403	0.0127	0.0538	0.0282	-0.0265	0.0117
	30	40	0.4469	0.2289	0.1560	0.0426	0.0617	0.0304	0.0069	0.0189
	30	45	0.5594	0.3493	0.3134	0.1271	0.0722	0.0363	0.0277	0.0283
	25	30	0.3710	0.1736	0.0242	0.0125	0.0499	0.0287	-0.0276	0.0126
	25	35	0.3938	0.1869	0.1068	0.0296	0.0566	0.0303	0.0047	0.0184
	25	40	0.4441	0.2324	0.2015	0.0661	0.0612	0.0323	0.0217	0.0245
	25	45	0.5623	0.3468	0.3558	0.1555	0.0773	0.0377	0.0436	0.0318
60	48	54	0.5617	0.3438	0.1502	0.0374	0.0631	0.0267	-0.0151	0.0148
	42	48	0.4475	0.2247	0.0801	0.0170	0.0562	0.024	-0.0178	0.0107
	42	54	0.5629	0.3469	0.2619	0.0898	0.0666	0.0294	0.0139	0.0205
	36	42	0.3943	0.1802	0.0506	0.0119	0.0529	0.0232	-0.017	0.0097
	36	48	0.4478	0.2224	0.1630	0.0418	0.0586	0.0246	0.0103	0.0159
	36	54	0.5620	0.3444	0.3243	0.1275	0.0689	0.0312	0.0306	0.0246
	30	36	0.3690	0.1660	0.0312	0.0109	0.0448	0.0238	-0.0202	0.0102
	30	42	0.3904	0.1805	0.1103	0.0278	0.0503	0.0251	0.0059	0.0155
	30	48	0.4472	0.2239	0.2119	0.0634	0.0606	0.0266	0.0276	0.0204
	30	54	0.5610	0.3456	0.3606	0.1571	0.0701	0.0323	0.0405	0.0273

Table 3: The corresponding bias and MSEs for different settings of RML approach when the true distribution is ND.

			RML approach				
			$\widehat{X}_{s:n}^{A1,1}$		$\widehat{X}_{s:n}^{A1,2}$		
n	r	s	bias	MSE	bias	MSE	Correct (%)
10	8	9	-0.1796	0.1372	-0.3430	0.2140	0.6430
	7	8	-0.1580	0.1001	-0.2832	0.1509	0.6263
	7	9	-0.2459	0.2408	-0.3732	0.3023	0.6263
	6	7	-0.1466	0.0772	-0.2560	0.1170	0.5959
	6	8	-0.3316	0.1836	-0.2226	0.2313	0.5959
	6	9	-0.3150	0.3745	-0.3867	0.4098	0.5959
	5	6	-0.1399	0.0774	-0.2448	0.1142	0.5908
	5	7	-0.2206	0.1701	-0.3164	0.2098	0.5908
	5	8	-0.3144	0.3048	-0.3709	0.3303	0.5908
	5	9	-0.4189	0.5512	-0.4695	0.5794	0.5908
20	16	18	-0.1301	0.0875	-0.2542	0.1275	0.7195
	14	16	-0.1087	0.0556	-0.2093	0.0831	0.6955
	14	18	-0.1741	0.1534	-0.2281	0.1702	0.6955
	12	14	-0.0944	0.0443	-0.1850	0.0674	0.6657
	12	16	-0.1597	0.1112	-0.2008	0.1229	0.6657
	12	18	-0.2382	0.2419	-0.2764	0.2561	0.6657
	10	12	-0.0948	0.0404	-0.1787	0.0606	0.6477
	10	14	-0.1489	0.0949	-0.1849	0.1046	0.6477
	10	16	-0.2164	0.1806	-0.2432	0.1901	0.6477
	10	18	-0.3104	0.3566	-0.3419	0.3699	0.6477
30	24	27	0.0092	0.0718	-0.1224	0.0604	0.9508
	21	24	-0.0235	0.0472	-0.1130	0.0402	0.9345
	21	27	-0.0493	0.0950	-0.1135	0.0888	0.9345
	18	21	-0.0700	0.0348	-0.1172	0.0351	0.8478
	18	24	-0.1114	0.0752	-0.1357	0.0725	0.8478
	18	27	-0.1531	0.1561	-0.1788	0.1581	0.8478
	15	18	-0.0920	0.0295	-0.1234	0.0344	0.7104
	15	21	-0.1364	0.0676	-0.1451	0.0700	0.7104
	15	24	-0.1985	0.1293	-0.2061	0.1334	0.7104
	15	27	-0.2789	0.2534	-0.2945	0.2621	0.7104
40	32	36	0.0533	0.0603	-0.0708	0.0413	0.9717
	28	32	0.0476	0.0408	-0.0560	0.0229	0.9754
	28	36	0.0514	0.0651	-0.0324	0.0507	0.9754
	24	28	0.0445	0.0334	-0.0484	0.0175	0.9777
	24	32	0.0497	0.0464	-0.0218	0.0311	0.9777
	24	36	0.0593	0.0691	-0.0083	0.0572	0.9777
	20	24	0.0352	0.0308	-0.0469	0.0154	0.9817
	20	28	0.0331	0.0399	-0.0228	0.0263	0.9817
	20	32	0.0434	0.0508	-0.0069	0.0407	0.9817
	20	36	0.0467	0.0768	-0.0042	0.0683	0.9817

Table 3: Continued.

			RML approach				
			$\widehat{X}_{s:n}^{A1,1}$		$\widehat{X}_{s:n}^{A1,2}$		
n	r	s	bias	MSE	bias	MSE	Correct (%)
50	40	45	0.0589	0.0488	-0.0472	0.0313	0.9820
	35	40	0.0505	0.0341	-0.0403	0.0177	0.9847
	35	45	0.0595	0.0542	-0.0178	0.0411	0.9847
	30	35	0.0473	0.0286	-0.0329	0.0131	0.9876
	30	40	0.0539	0.0366	-0.0093	0.0241	0.9876
	30	45	0.0624	0.0559	0.0042	0.0463	0.9876
	25	30	0.0381	0.0257	-0.0319	0.0111	0.9878
	25	35	0.0466	0.0310	-0.0076	0.0194	0.9878
	25	40	0.0527	0.0389	0.0060	0.0300	0.9878
	25	45	0.0660	0.0577	0.0191	0.0497	0.9878
60	48	54	0.0586	0.0435	-0.0368	0.0270	0.9913
	42	48	0.0522	0.0300	-0.0284	0.0144	0.9919
	42	54	0.0611	0.0453	-0.0064	0.0345	0.9919
	36	42	0.0477	0.0243	-0.0245	0.0107	0.9933
	36	48	0.0543	0.0322	-0.0038	0.0210	0.9933
	36	54	0.0644	0.0487	0.0126	0.0400	0.9933
	30	36	0.0401	0.0212	-0.0229	0.0092	0.9928
	30	42	0.0433	0.0257	-0.0037	0.0163	0.9928
	30	48	0.0555	0.0333	0.0148	0.0258	0.9928
	30	54	0.0649	0.0492	0.0215	0.0429	0.9928

Table 4: The corresponding bias and MSEs for different settings of D_{sp} approach when the true distribution is ND.

			D_{sp} approach				
			$\widehat{X}_{s:n}^{A2,1}$		$\widehat{X}_{s:n}^{A2,2}$		
n	r	s	bias	MSE	bias	MSE	correct (%)
10	8	9	-0.1489	0.1300	-0.3430	0.2140	0.6694
	7	8	-0.1285	0.0932	-0.2832	0.1509	0.6654
	7	9	-0.2016	0.2273	-0.3517	0.2880	0.6654
	6	7	-0.1181	0.0716	-0.2560	0.1170	0.6616
	6	8	-0.3112	0.1713	-0.1809	0.2190	0.6616
	6	9	-0.2560	0.3466	-0.3461	0.3798	0.6616
	5	6	-0.1138	0.0725	-0.2448	0.1142	0.6305
	5	7	-0.1820	0.1603	-0.2965	0.1994	0.6305
	5	8	-0.2621	0.2818	-0.3339	0.3061	0.6305
	5	9	-0.3529	0.5053	-0.4170	0.5306	0.6305
20	16	18	-0.1159	0.0843	-0.2497	0.1252	0.7314
	14	16	-0.0892	0.0529	-0.2038	0.0810	0.7198
	14	18	-0.1456	0.1450	-0.2109	0.1610	0.7198
	12	14	-0.0748	0.0420	-0.1795	0.0655	0.6935
	12	16	-0.1308	0.1049	-0.1835	0.1159	0.6935
	12	18	-0.1989	0.2247	-0.2475	0.2369	0.6935
	10	12	-0.0741	0.0379	-0.1724	0.0584	0.6781
	10	14	-0.1193	0.0886	-0.1668	0.0976	0.6781
	10	16	-0.1791	0.1639	-0.2156	0.1721	0.6781
	10	18	-0.2580	0.3209	-0.2985	0.3318	0.6781
30	24	27	-0.0610	0.0618	-0.1422	0.0666	0.7082
	21	24	-0.0637	0.0390	-0.1211	0.0415	0.7488
	21	27	-0.0998	0.0901	-0.1355	0.0935	0.7488
	18	21	-0.0755	0.0314	-0.1174	0.0348	0.8027
	18	24	-0.1177	0.0693	-0.1369	0.0702	0.8027
	18	27	-0.1615	0.1472	-0.1826	0.1534	0.8027
	15	18	-0.0778	0.0277	-0.1184	0.0333	0.6998
	15	21	-0.1206	0.0629	-0.1365	0.0662	0.6998
	15	24	-0.1796	0.1204	-0.1935	0.1254	0.6998
	15	27	-0.2566	0.2319	-0.2774	0.2416	0.6998
40	32	36	-0.0416	0.0511	-0.1013	0.0493	0.7000
	28	32	-0.0287	0.0289	-0.0742	0.0259	0.6958
	28	36	-0.0467	0.0643	-0.0800	0.0629	0.6958
	24	28	-0.0225	0.0230	-0.0609	0.0193	0.7010
	24	32	-0.0278	0.0389	-0.0505	0.0353	0.7010
	24	36	-0.0390	0.0710	-0.0650	0.0702	0.7010
	20	24	-0.0207	0.0203	-0.0554	0.0163	0.7535
	20	28	-0.0232	0.0321	-0.0382	0.0277	0.7535
	20	32	-0.0218	0.0468	-0.0362	0.0445	0.7535
	20	36	-0.0346	0.0793	-0.0556	0.0784	0.7535

Table 4: Continued.

			D_{sp} approach				
			$\widehat{X}_{s:n}^{A2,1}$		$\widehat{X}_{s:n}^{A2,2}$		
n	r	s	bias	MSE	bias	MSE	correct (%)
50	40	45	-0.0314	0.0421	-0.0780	0.0383	0.7221
	35	40	-0.0244	0.0245	-0.0590	0.0203	0.7185
	35	45	-0.0376	0.0552	-0.0670	0.0528	0.7185
	30	35	-0.0171	0.0190	-0.0451	0.0145	0.7377
	30	40	-0.0193	0.0314	-0.0375	0.0278	0.7377
	30	45	-0.0303	0.0596	-0.0511	0.0583	0.7377
	25	30	-0.0166	0.0168	-0.0401	0.0118	0.7656
	25	35	-0.0112	0.0252	-0.0243	0.0210	0.7656
	25	40	-0.0116	0.0373	-0.0240	0.0342	0.7656
	25	45	-0.0179	0.0630	-0.0357	0.0611	0.7656
60	48	54	-0.0254	0.0383	-0.0664	0.0331	0.7520
	42	48	-0.0179	0.0219	-0.0463	0.0168	0.7444
	42	54	-0.0276	0.0485	-0.0524	0.0453	0.7444
	36	42	-0.0137	0.0164	-0.0364	0.0118	0.7635
	36	48	-0.0141	0.0283	-0.0306	0.0244	0.7635
	36	54	-0.0192	0.0537	-0.0383	0.0517	0.7635
	30	36	-0.0098	0.0146	-0.0304	0.0098	0.7864
	30	42	-0.0068	0.0217	-0.0184	0.0177	0.7864
	30	48	-0.0028	0.0321	-0.0131	0.0291	0.7864
	30	54	-0.0137	0.0565	-0.0307	0.0547	0.7864

Table 5: The corresponding bias and MSEs for different settings of D approach when the true distribution is ND.

			D approach				
			$\widehat{X}^{A3,1}_{s:n}$		$\widehat{X}^{A3,2}_{s:n}$		
n	r	s	bias	MSE	bias	MSE	correct (%)
10	8	9	-0.1491	0.1300	-0.3430	0.2140	0.6687
	7	8	-0.1286	0.0932	-0.2832	0.1509	0.6648
	7	9	-0.2016	0.2274	-0.3517	0.2881	0.6648
	6	7	-0.1181	0.0716	-0.2560	0.1170	0.6615
	6	8	-0.3113	0.1713	-0.1809	0.2190	0.6615
	6	9	-0.2560	0.3466	-0.3461	0.3798	0.6615
	5	6	-0.1138	0.0725	-0.2448	0.1142	0.6306
	5	7	-0.1820	0.1603	-0.2965	0.1994	0.6306
	5	8	-0.2621	0.2818	-0.3339	0.3061	0.6306
	5	9	-0.3529	0.5054	-0.4170	0.5307	0.6306
20	16	18	-0.1163	0.0844	-0.2498	0.1253	0.7293
	14	16	-0.0893	0.0529	-0.2039	0.0810	0.7190
	14	18	-0.1458	0.1451	-0.2109	0.1611	0.7190
	12	14	-0.0748	0.0420	-0.1795	0.0655	0.6933
	12	16	-0.1308	0.1049	-0.1835	0.1159	0.6933
	12	18	-0.1990	0.2248	-0.2476	0.2370	0.6933
	10	12	-0.0741	0.0379	-0.1724	0.0584	0.6781
	10	14	-0.1193	0.0886	-0.1668	0.0976	0.6781
	10	16	-0.1791	0.1639	-0.2156	0.1721	0.6781
	10	18	-0.2580	0.3209	-0.2984	0.3318	0.6781
30	24	27	-0.0640	0.0618	-0.1433	0.0669	0.7006
	21	24	-0.0646	0.0390	-0.1213	0.0416	0.7454
	21	27	-0.1009	0.0903	-0.1361	0.0937	0.7454
	18	21	-0.0757	0.0314	-0.1175	0.0348	0.8016
	18	24	-0.1178	0.0693	-0.1369	0.0702	0.8016
	18	27	-0.1620	0.1472	-0.1829	0.1534	0.8016
	15	18	-0.0778	0.0276	-0.1184	0.0333	0.6998
	15	21	-0.1206	0.0629	-0.1365	0.0662	0.6998
	15	24	-0.1796	0.1204	-0.1935	0.1254	0.6998
	15	27	-0.2565	0.2319	-0.2773	0.2416	0.6998
40	32	36	-0.0458	0.0514	-0.1028	0.0497	0.6912
	28	32	-0.0300	0.0290	-0.0745	0.0260	0.6916
	28	36	-0.0486	0.0642	-0.0811	0.0630	0.6916
	24	28	-0.0230	0.0230	-0.0610	0.0193	0.6990
	24	32	-0.0286	0.0389	-0.0508	0.0354	0.6990
	24	36	-0.0398	0.0711	-0.0655	0.0703	0.6990
	20	24	-0.0206	0.0203	-0.0554	0.0163	0.7536
	20	28	-0.0231	0.0321	-0.0382	0.0277	0.7536
	20	32	-0.0217	0.0468	-0.0362	0.0445	0.7536
	20	36	-0.0345	0.0792	-0.0554	0.0784	0.7536

Table 5: Continued.

			D approach				
			$\widehat{X}_{s:n}^{A3,1}$		$\widehat{X}_{s:n}^{A3,2}$		
n	r	s	bias	MSE	bias	MSE	correct (%)
50	40	45	-0.0349	0.0424	-0.0793	0.0387	0.7099
	35	40	-0.0259	0.0246	-0.0594	0.0204	0.7138
	35	45	-0.0397	0.0555	-0.0682	0.0531	0.7138
	30	35	-0.0179	0.0190	-0.0453	0.0146	0.7353
	30	40	-0.0199	0.0313	-0.0378	0.0278	0.7353
	30	45	-0.0310	0.0597	-0.0515	0.0584	0.7353
	25	30	-0.0165	0.0168	-0.0401	0.0118	0.7659
	25	35	-0.0111	0.0252	-0.0242	0.0210	0.7659
	25	40	-0.0115	0.0373	-0.0240	0.0342	0.7659
	25	45	-0.0179	0.0629	-0.0357	0.0610	0.7659
60	48	54	-0.0294	0.0387	-0.0679	0.0335	0.7393
	42	48	-0.0199	0.0220	-0.0469	0.0169	0.7384
	42	54	-0.0302	0.0487	-0.0538	0.0455	0.7384
	36	42	-0.0146	0.0164	-0.0366	0.0118	0.7617
	36	48	-0.0149	0.0283	-0.0309	0.0244	0.7617
	36	54	-0.0200	0.0539	-0.0388	0.0519	0.7617
	30	36	-0.0097	0.0146	-0.0304	0.0098	0.7867
	30	42	-0.0067	0.0217	-0.0184	0.0177	0.7867
	30	48	-0.0028	0.0320	-0.0130	0.0291	0.7867
	30	54	-0.0135	0.0565	-0.0306	0.0547	0.7867

Table 6: The corresponding bias and MSEs for different settings of RML approach when the true distribution is SEV.

			RML approach				
			$\widehat{X}_{s:n}^{A1,1}$		$\widehat{X}_{s:n}^{A1,2}$		
n	r	s	bias	MSE	bias	MSE	correct(%)
10	8	9	-0.0461	0.0865	-0.2969	0.1589	0.5938
	7	8	-0.0679	0.0798	-0.2692	0.1376	0.5587
	7	9	-0.0520	0.1779	-0.2258	0.1977	0.5587
	6	7	-0.0897	0.0814	-0.2702	0.1364	0.5445
	6	8	-0.0866	0.1746	-0.2454	0.2047	0.5445
	6	9	-0.0682	0.3140	-0.1716	0.3048	0.5445
	5	6	-0.1010	0.0935	-0.2780	0.1516	0.5242
	5	7	-0.1234	0.2004	-0.2715	0.2314	0.5242
	5	8	-0.1228	0.3226	-0.2119	0.3214	0.5242
	5	9	-0.1070	0.5253	-0.1835	0.5058	0.5242
20	16	18	-0.0089	0.0544	-0.1626	0.0713	0.7451
	14	16	-0.0222	0.0422	-0.1592	0.0618	0.7003
	14	18	-0.0012	0.1000	-0.0764	0.0988	0.7003
	12	14	-0.0272	0.0398	-0.1603	0.0592	0.6548
	12	16	-0.0183	0.0883	-0.0846	0.0882	0.6548
	12	18	0.0226	0.1753	-0.0332	0.1670	0.6548
	10	12	-0.0371	0.0447	-0.1692	0.0658	0.6058
	10	14	-0.0277	0.0946	-0.0895	0.0960	0.6058
	10	16	0.0028	0.1753	-0.0415	0.1707	0.6058
	10	18	0.0512	0.3056	0.0052	0.2909	0.6058
30	24	27	0.1143	0.0639	-0.0454	0.0299	0.9188
	21	24	0.0991	0.0541	-0.0495	0.0235	0.9066
	21	27	0.1171	0.0717	0.0121	0.0423	0.9066
	18	21	0.0776	0.0471	-0.0580	0.0223	0.9046
	18	24	0.0812	0.0569	-0.0110	0.0351	0.9046
	18	27	0.0978	0.0742	0.0241	0.0562	0.9046
	15	18	0.0360	0.0465	-0.0783	0.0281	0.8915
	15	21	0.0222	0.0569	-0.0453	0.0423	0.8915
	15	24	0.0181	0.0730	-0.0342	0.0626	0.8915
	15	27	0.0242	0.1017	-0.0200	0.0933	0.8915
40	32	36	0.0930	0.0472	-0.0282	0.0226	0.9441
	28	32	0.0869	0.0402	-0.0292	0.0167	0.9393
	28	36	0.0994	0.0542	0.0174	0.0325	0.9393
	24	28	0.0809	0.0391	-0.0312	0.0147	0.9346
	24	32	0.0896	0.0454	0.0096	0.0250	0.9346
	24	36	0.1109	0.0599	0.0478	0.0413	0.9346
	20	24	0.0760	0.0406	-0.0344	0.0161	0.9227
	20	28	0.0804	0.0423	0.0021	0.0239	0.9227
	20	32	0.0928	0.0485	0.0347	0.0322	0.9227
	20	36	0.1108	0.0631	0.0605	0.0476	0.9227

Table 6: Continued.

			RML approach				
			$\widehat{X}_{s:n}^{A1,1}$		$\widehat{X}_{s:n}^{A1,2}$		
n	r	s	bias	MSE	bias	MSE	correct(%)
50	40	45	0.0815	0.0378	-0.0157	0.0185	0.9673
	35	40	0.0735	0.0317	-0.0201	0.0128	0.9593
	35	45	0.0853	0.0408	0.0172	0.0262	0.9593
	30	35	0.0679	0.0311	-0.0230	0.0117	0.9557
	30	40	0.0771	0.0356	0.0139	0.0202	0.9557
	30	45	0.0929	0.0465	0.0415	0.0332	0.9557
	25	30	0.0629	0.0317	-0.0248	0.0126	0.9475
	25	35	0.0707	0.0343	0.0100	0.0190	0.9475
	25	40	0.0789	0.0389	0.0315	0.0269	0.9475
	25	45	0.0988	0.0506	0.0593	0.0391	0.9475
60	48	54	0.0718	0.0302	-0.0118	0.0154	0.9808
	42	48	0.0649	0.0266	-0.0152	0.0109	0.9731
	42	54	0.0774	0.0353	0.0201	0.0226	0.9731
	36	42	0.0618	0.0256	-0.0149	0.0098	0.9707
	36	48	0.0695	0.0285	0.0155	0.0169	0.9707
	36	54	0.0845	0.0384	0.0411	0.0281	0.9707
	30	36	0.0542	0.0256	-0.0183	0.0102	0.9633
	30	42	0.0606	0.0279	0.0099	0.0160	0.9633
	30	48	0.0727	0.0307	0.0344	0.0221	0.9633
	30	54	0.0855	0.0420	0.0518	0.0329	0.9633

Table 7: The corresponding bias and MSEs for different settings of D_{sp} approach when the true distribution is SEV.

			D_{sp} approach				
			$\widehat{X}_{s:n}^{A2,1}$		$\widehat{X}_{s:n}^{A2,2}$		
n	r	s	bias	MSE	bias	MSE	correct (%)
10	8	9	-0.0013	0.0904	-0.2969	0.1589	0.4340
	7	8	-0.0222	0.0817	-0.2692	0.1376	0.3635
	7	9	0.0392	0.2032	-0.1820	0.1869	0.3635
	6	7	-0.0418	0.0830	-0.2702	0.1364	0.3208
	6	8	0.0059	0.1959	-0.2014	0.1923	0.3208
	6	9	0.0879	0.3923	-0.0579	0.3199	0.3208
	5	6	-0.0539	0.0936	-0.2780	0.1516	0.3094
	5	7	-0.0312	0.2201	-0.2253	0.2182	0.3094
	5	8	0.0199	0.3925	-0.1053	0.3373	0.3094
	5	9	0.1007	0.6842	-0.0108	0.5837	0.3094
20	16	18	0.0221	0.0565	-0.1521	0.0687	0.6408
	14	16	0.0044	0.0437	-0.1515	0.0597	0.5695
	14	18	0.0503	0.1109	-0.0417	0.0991	0.5695
	12	14	0.0011	0.0405	-0.1527	0.0569	0.5011
	12	16	0.0354	0.0953	-0.0474	0.0858	0.5011
	12	18	0.1110	0.2044	0.0396	0.1780	0.5011
	10	12	-0.0017	0.0460	-0.1591	0.0628	0.4051
	10	14	0.0403	0.1047	-0.0403	0.0943	0.4051
	10	16	0.1041	0.2017	0.0431	0.1791	0.4051
	10	18	0.2156	0.3925	0.1515	0.3447	0.4051
30	24	27	0.1346	0.0585	-0.0364	0.0298	0.8704
	21	24	0.1191	0.0495	-0.0415	0.0230	0.8433
	21	27	0.1471	0.0743	0.0305	0.0463	0.8433
	18	21	0.1052	0.0451	-0.0491	0.0216	0.8042
	18	24	0.1168	0.0590	0.0069	0.0368	0.8042
	18	27	0.1424	0.0880	0.0548	0.0658	0.8042
	15	18	0.0703	0.0455	-0.0678	0.0267	0.7363
	15	21	0.0676	0.0589	-0.0249	0.0426	0.7363
	15	24	0.0689	0.0790	-0.0024	0.0649	0.7363
	15	27	0.0993	0.1233	0.0374	0.1067	0.7363
40	32	36	0.1130	0.0465	-0.0196	0.0234	0.9068
	28	32	0.1061	0.0375	-0.0221	0.0168	0.8795
	28	36	0.1258	0.0580	0.0337	0.0371	0.8795
	24	28	0.0985	0.0351	-0.0251	0.0147	0.8582
	24	32	0.1130	0.0446	0.0223	0.0272	0.8582
	24	36	0.1402	0.0694	0.0692	0.0507	0.8582
	20	24	0.0945	0.0358	-0.0285	0.0159	0.8438
	20	28	0.0994	0.0403	0.0112	0.0249	0.8438
	20	32	0.1191	0.0517	0.0519	0.0369	0.8438
	20	36	0.1413	0.0759	0.0852	0.0600	0.8438

Table 7: Continued.

			D_{sp} approach				
			$\widehat{X}_{s:n}^{A2,1}$		$\widehat{X}_{s:n}^{A2,2}$		
n	r	s	bias	MSE	bias	MSE	correct (%)
50	40	45	0.0996	0.0382	-0.0082	0.0194	0.9309
	35	40	0.0919	0.0304	-0.0137	0.0131	0.9069
	35	45	0.1094	0.0477	0.0318	0.0313	0.9069
	30	35	0.0854	0.0287	-0.0176	0.0118	0.8834
	30	40	0.1010	0.0368	0.0264	0.0223	0.8834
	30	45	0.1238	0.0571	0.0635	0.0422	0.8834
	25	30	0.0814	0.0283	-0.0198	0.0125	0.8760
	25	35	0.0925	0.0339	0.0199	0.0202	0.8760
	25	40	0.1022	0.0423	0.0465	0.0308	0.8760
	25	45	0.1344	0.0643	0.0870	0.0517	0.8760
60	48	54	0.0874	0.0319	-0.0054	0.0164	0.9491
	42	48	0.0804	0.0264	-0.0099	0.0114	0.9272
	42	54	0.1003	0.0408	0.0338	0.0267	0.9272
	36	42	0.0776	0.0242	-0.0102	0.0100	0.9116
	36	48	0.0912	0.0304	0.0266	0.0189	0.9116
	36	54	0.1110	0.0472	0.0601	0.0354	0.9116
	30	36	0.0725	0.0239	-0.0136	0.0103	0.8991
	30	42	0.0802	0.0285	0.0185	0.0172	0.8991
	30	48	0.0967	0.0358	0.0494	0.0262	0.8991
	30	54	0.1162	0.0420	0.0757	0.0430	0.8991

Table 8: The corresponding bias and MSE for different settings of D approach when the true distribution is SEV.

			D approach				
			$\widehat{X}_{s:n}^{A3,1}$		$\widehat{X}_{s:n}^{A3,2}$		
n	r	s	bias	MSE	bias	MSE	correct (%)
10	8	9	-0.0018	0.0902	-0.2969	0.1589	0.4362
	7	8	-0.0223	0.0817	-0.2692	0.1376	0.364
	7	9	0.0389	0.2031	-0.1821	0.1870	0.364
	6	7	-0.0419	0.0830	-0.2702	0.1364	0.321
	6	8	0.0058	0.1959	-0.2015	0.1923	0.321
	6	9	0.0877	0.3923	-0.0580	0.3199	0.321
	5	6	-0.0539	0.0936	-0.2780	0.1516	0.3093
	5	7	-0.0312	0.2201	-0.2253	0.2182	0.3093
	5	8	0.0199	0.3925	-0.1053	0.3373	0.3093
	5	9	0.1007	0.6842	-0.0108	0.5837	0.3093
20	16	18	0.0212	0.0564	-0.1524	0.0688	0.6441
	14	16	0.0041	0.0436	-0.1516	0.0597	0.5715
	14	18	0.0499	0.1106	-0.0419	0.0990	0.5715
	12	14	0.0011	0.0405	-0.1528	0.0569	0.5014
	12	16	0.0354	0.0954	-0.0474	0.0858	0.5014
	12	18	0.1110	0.2044	0.0395	0.1780	0.5014
	10	12	-0.0017	0.0460	-0.1591	0.0628	0.405
	10	14	0.0404	0.1047	-0.0403	0.0943	0.405
	10	16	0.1041	0.2017	0.0431	0.1791	0.405
	10	18	0.2157	0.3925	0.1516	0.3447	0.405
30	24	27	0.1310	0.0578	-0.0377	0.0298	0.8771
	21	24	0.1181	0.0493	-0.0418	0.0230	0.8474
	21	27	0.1452	0.0736	0.0295	0.0461	0.8474
	18	21	0.1046	0.0450	-0.0493	0.0216	0.8059
	18	24	0.1164	0.0589	0.0067	0.0368	0.8059
	18	27	0.1416	0.0876	0.0544	0.0656	0.8059
	15	18	0.0704	0.0455	-0.0677	0.0267	0.7359
	15	21	0.0678	0.0589	-0.0248	0.0426	0.7359
	15	24	0.0690	0.0790	-0.0023	0.0649	0.7359
	15	27	0.0994	0.1234	0.0374	0.1067	0.7359
40	32	36	0.1103	0.0454	-0.0206	0.0232	0.9126
	28	32	0.1041	0.0374	-0.0227	0.0168	0.8831
	28	36	0.1244	0.0575	0.0329	0.0369	0.8831
	24	28	0.0979	0.0350	-0.0252	0.0147	0.8598
	24	32	0.1125	0.0445	0.0221	0.0271	0.8598
	24	36	0.1393	0.0690	0.0687	0.0505	0.8598
	20	24	0.0946	0.0358	-0.0284	0.0159	0.8433
	20	28	0.0995	0.0403	0.0112	0.0249	0.8433
	20	32	0.1192	0.0518	0.0519	0.0369	0.8433
	20	36	0.1414	0.0759	0.0853	0.0601	0.8433

Table 8: Continued.

			D approach				
			$\widehat{X}_{s:n}^{A3,1}$		$\widehat{X}_{s:n}^{A3,2}$		
n	r	s	bias	MSE	bias	MSE	correct (%)
50	40	45	0.0973	0.0373	-0.0091	0.0192	0.9364
	35	40	0.0904	0.0301	-0.0142	0.0131	0.9113
	35	45	0.1074	0.0469	0.0306	0.0310	0.9113
	30	35	0.0848	0.0287	-0.0177	0.0118	0.8852
	30	40	0.1003	0.0367	0.0261	0.0222	0.8852
	30	45	0.1233	0.0568	0.0632	0.0421	0.8852
	25	30	0.0814	0.0283	-0.0197	0.0125	0.876
	25	35	0.0926	0.0339	0.0199	0.0202	0.876
	25	40	0.1022	0.0423	0.0465	0.0308	0.876
	25	45	0.1345	0.0643	0.0871	0.0517	0.876
60	48	54	0.0845	0.0310	-0.0065	0.0162	0.9554
	42	48	0.0794	0.0261	-0.0102	0.0113	0.9301
	42	54	0.0992	0.0402	0.0332	0.0265	0.9301
	36	42	0.0772	0.0241	-0.0103	0.0100	0.9127
	36	48	0.0905	0.0302	0.0263	0.0188	0.9127
	36	54	0.1099	0.0470	0.0595	0.0353	0.9127
	30	36	0.0725	0.0239	-0.0136	0.0103	0.899
	30	42	0.0803	0.0285	0.0186	0.0172	0.899
	30	48	0.0968	0.0358	0.0494	0.0262	0.899
	30	54	0.1163	0.0532	0.0758	0.0430	0.899

Table 9: Factors with levels of each factor and complete data in the experiments.

Run	Factor				Pull-off force for replicate							
	A	B	C	D	$x_{1:8}$	$x_{2:8}$	$x_{3:8}$	$x_{4:8}$	$x_{5:8}$	$x_{6:8}$	$x_{7:8}$	$x_{8:8}$
1	1	1	1	1	9.5	15.6	16.9	19.1	19.6	19.6	19.9	20
2	1	2	2	2	15	16.2	19.4	19.6	19.7	19.8	21.9	24.2
3	1	3	3	3	15.6	16.3	16.7	18.2	19.1	20.4	22.6	23.3
4	2	1	2	3	17.4	18.3	18.6	18.9	18.9	21	23.2	24.7
5	2	2	3	1	18.6	19.4	19.7	21.4	25.1	25.3	25.6	27.5
6	2	3	1	2	14.7	16.2	16.3	19.6	19.8	20	22.5	24.7
7	3	1	3	2	16.4	16.8	18.4	18.6	19.1	21.6	23.6	24.3
8	3	2	1	3	14.2	15.1	15.6	16.8	17.8	19.6	23.2	24.4
9	3	3	2	1	16.1	17.3	19.3	19.9	22.6	22.7	23.1	28.6

Note: Factor A is interference with Low (1), Medium (2) and High (3) levels.
Factor B is connector wall thickness with Thin (1), Medium (2) and Thick (3) levels.
Factor C is insertion depth with Shallow (1), Medium (2) and Deep (3) levels.
Factor D is Percent adhesive in connector pre-dip with Low (1), Medium (2) and High (3) levels.

Table 10: The pseudo-complete data and results of model selection.

Run	Pull-off force for replicate								Model selection
	$x_{1:8}$	$x_{2:8}$	$x_{3:8}$	$x_{4:8}$	$x_{5:8}$	$x_{6:8}$	$\widehat{X}_{7:8}^{A1,2}$	$\widehat{X}_{8:8}^{A1,2}$	
1	9.5	15.6	16.9	19.1	19.6	19.6	19.6	21.2	SEV
2	15	16.2	19.4	19.6	19.7	19.8	19.8	20.73	SEV
3	15.6	16.3	16.7	18.2	19.1	20.4	20.4	21.6	ND
4	17.4	18.3	18.6	18.9	18.9	21	21	21.8	ND
5	18.6	19.4	19.7	21.4	25.1	25.3	25.3	27.3	ND
6	14.7	16.2	16.3	19.6	19.8	20	20	21.1	SEV
7	16.4	16.8	18.4	18.6	19.1	21.6	21.6	22.8	ND
8	14.2	15.1	15.6	16.8	17.8	19.6	19.6	20.9	ND
9	16.1	17.3	19.3	19.9	22.6	22.7	22.7	24.5	ND

the future order statistics in this example. After combining the uncensored data and the predicted censored data, the pseudo-complete data are shown in Table 10.

6. Conclusions

It could be difficult to discriminate a best model sometimes from several candidate distributions. The sample size, estimation methods, and goodness-of-fit testing methods can affect the final results of model selection. In this study, we focus on providing reliable methods to obtain predicting values of censored data to reduce the impact of model misspecification. In this study, three model selection approaches are proposed for predicting the future order statistics from type II censored data, in which the quality characteristic is assumed to follow a location-scale family. The ND and SEV are considered as the candidate members in the location-scale distribution to compete the best underlying distribution. The ND can be the log transformation from the lognormal distribution and the SEV can be the log transformation from the Weibull distribution. Discrimination between lognormal and Weibull distributions is equivalent to the discrimination between ND and SEV. Hence, both ND and SEV are widely used for practical reliability applications.

Through any one of three proposed approaches, the robust predictions can be obtained even under model uncertainty. Three examples are used to illustrate the methodologies. Moreover, the performance of these three proposed approaches are evaluated through using Monte Carlo simulations. Numerical results show that the three proposed model selection approaches are robust and effective in obtaining good predicted values for the future order statistics, which are censored.

In comparing these three proposed approaches, we recommend using D_{SP} approach or D approach for model selection and use expected values prediction method to predict the future order statistics for small sample size cases, that is, the sample cases with a size n is less than 30. For large sample size cases (sample size n larger than 30), we recommend using RRML approach for model selection and use Taylor series prediction method to predict the future order statistics. Simulation results show that the proposed approaches are robust and can highly reduce the impact caused by model uncertainty. The proposed approaches can also work well if more than two candidate distribution are competing for the best distribution.

Other model selection methods from the current three proposed approaches could also be competitive. How to employ new model selection methods for the topic of type II censored data prediction can be studied in the future.

Appendix

A.

For the normal distribution case, the functions $h(Z_{s:n})$ and $h_1(z_{r:n}, Z_{s:n})$ can be expanded by using Taylor series at the points $F^{-1}(p_s)$ and $(F^{-1}(p_r), F^{-1}(p_s))$, respectively. We obtain

$$h(z_{s:n}) = \frac{f(Z_{s:n})}{1 - F(Z_{s:n})} \approx \alpha + \beta Z_{s:n} \quad \text{(A.1)}$$

and

$$\begin{aligned} h_1(z_{r:n}, Z_{s:n}) &= \frac{f(Z_{s:n})}{F(Z_{s:n}) - F(z_{r:n})} \\ &\approx \gamma + \rho z_{r:n} - \nu_s Z_{s:n}, \end{aligned} \quad \text{(A.2)}$$

in which the constants can be taken to be

$$\begin{aligned} \alpha &= \frac{f(\eta_s)\{(1+\eta_s^2)q_s - \eta_s f(\eta_s)\}}{q_s^2}, \\ \beta &= \frac{f(\eta_s)\{f(\eta_s) - q_s\eta_s\}}{q_s^2}, \\ \gamma &= \frac{f(\eta_s)\{(1+\eta_s^2)p_{sr} + \eta_s f(\eta_s) - \eta_r f(\eta_r)\}}{p_{sr}^2}, \\ \rho &= \frac{f(\eta_r)f(\eta_s)}{p_{sr}^2}, \\ \nu_s &= \frac{f(\eta_s)\{\eta_s p_{sr} + f(\eta_s)\}}{p_{sr}^2}, \end{aligned} \quad \text{(A.3)}$$

where $p_{ij} = p_i - p_j$, $p_i = i/(n+1)$ and $\eta_i = F^{-1}(p_i)$ for $i = 1, 2, \ldots, n$.

B.

For the smallest extreme value distribution case, the functions $h(Z_{s:n})$ and $h_1(z_{r:n}, Z_{s:n})$ can be expanded by using Taylor series at the points $F^{-1}(p_s)$ and $(F^{-1}(p_r), F^{-1}(p_s))$, respectively. We obtain

$$h(Z_{s:n}) = \frac{f(Z_{s:n})}{1 - F(Z_{s:n})} = 1 - \alpha_s - \beta_s Z_{s:n} \quad \text{(B.1)}$$

and

$$\begin{aligned} h_1(z_{r:n}, Z_{s:n}) &= \frac{f(Z_{s:n})}{F(Z_{s:n}) - F(Z_{s:n})} \\ &= \gamma_E + \rho_E z_{r:n} + \nu_E Z_{s:n}. \end{aligned} \quad \text{(B.2)}$$

The above constants can be taken to be

$$\begin{aligned} \alpha_s &= 1 + \ln(q_s) - \ln(q_s)\ln(-\ln q_s), \\ \beta_s &= \ln(q_s), \\ \gamma_E &= \frac{q_s \ln(q_s)\{q_{rs}[-1 + (1+\ln(q_s))\ln(-\ln(q_s))] + q_s\ln(q_s)\ln(-\ln(q_s)) - q_r\ln(q_r)\ln(-\ln(q_r))\}}{q_{rs}^2}, \\ \rho_E &= \frac{-q_s\ln(q_s)[(1+\ln(q_s))q_{rs} + q_s\ln(q_s)]}{q_{rs}^2}, \end{aligned} \quad \text{(B.3)}$$

and

$$\nu_E = \frac{q_s \ln(q_s) q_r \ln(q_r)}{q_{rs}^2}, \quad \text{(B.4)}$$

where $q_{ij} = q - q_j$.

C.

See Algorithm 1.

Data Availability

Data in examples of this study are cited from reference papers. We have put citation in each example and listed cited papers in references.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

```
rextreme=function(n,mu,sig){mu+sig*(log(-log(1-runif(n))))}
dextreme=function(x,mu,sig){(1/sig)*exp((x-mu)/sig-exp((x-mu)/sig))}
pextreme=function(x,mu,sig){1-exp(-exp((x-mu)/sig))}
n=13
r=10
data=log(c(0.22, 0.50, 0.88, 1.00, 1.32, 1.33, 1.54, 1.76, 2.50, 3.00))
data1=sort(data)[1:r]
Xr=data1[r]
# AMLE of normal
        pr=r/(n+1)
        qr=1-pr
        invpr=qnorm(pr,0,1)
        alpha=dnorm(invpr)*((1+invpr^2)*qr-invpr*dnorm(invpr))/(qr^2)
        beta=dnorm(invpr)*(dnorm(invpr)-invpr*qr)/(qr^2)
        A=sum(data1)+beta*(n-r)*Xr
        M=r+beta*(n-r)
        C=(n-r)*alpha
        D=A*C/M-C*Xr
        E=sum(data1^2)+(n-r)*beta*(Xr^2)-A^2/M
        sigma_hat=(-D+(D^2+4*r*E)^(1/2))/(2*r)
        u_hat=A/M+C*sigma_hat/M
## MLE of normal
L2=function(x){
         u=x[1]
         sigma=x[2]
         -(prod(dnorm(data1,u,sigma))*(1-pnorm(Xr,u,sigma))^(n-r))}
      Ans1=optim(c(u_hat,sigma_hat),L2)
      uN=Ans1$par[1]
      sigmaN=Ans1$par[2]
## AMLE of extreme value
      prE=r/(n+1)
      qrE=1-prE
      alphar=1+log(qrE)-log(qrE)*log(-log(qrE))
      betar=-log(qrE)
      SUMbeta=SUMbetaX=SUMalpha=SUMalphaX=SUMbetaX2=0
      for(h in 1:r){
      pi=h/(n+1)
      qi=1-pi
      alphai=1+log(qi)-log(qi)*log(-log(qi))
      betai=-log(qi)
      SUMbeta=SUMbeta+betai
      SUMbetaX=SUMbetaX+betai*data1[h]
      SUMalpha=SUMalpha+alphai
      SUMalphaX=SUMalphaX+alphai*data1[h]
      SUMbetaX2=SUMbetaX2+betai*((data1[h])^2)}
      M=SUMbeta+betar*(n-r)
      B=(SUMbetaX+(n-r)*betar*Xr)/M
      C=(SUMalpha-(n-r)*(1-alphar))/M
      D=-(n-r)*Xr+(n-r)*alphar*Xr+SUMalphaX-B*C*M
      E=(n-r)*betar*(Xr^2)+SUMbetaX2-M*(B^2)
      sigma_hat=(-D+(D^2+4*r*E)^(1/2))/(2*r)
      u_hat=B-C*sigma_hat
## MLE of extreme value
L9=function(x){
         u=x[1]
         sigma=x[2]
         -(prod(dextreme(data1,u,sigma))*(1-pextreme(Xr,u,sigma))^(n-r))}
      Ans9=optim(c(u_hat,sigma_hat),L9)
      uE=Ans9$par[1]
      sigmaE=Ans9$par[2]
## Model Selection Approaches
   Dsp1=Dsp2=D1=D2=array()
   for(j in 1:r){
L7=factorial(n)/(factorial(n-r))
```

Algorithm 1: Continued.

```
LN=L7*prod(dnorm(data1,uN,sigmaN))*(1-pnorm(Xr,uN,sigmaN))^(n-r)
LE=L7*prod(dextreme(data1,uE,sigmaE))*(1-pextreme(Xr,uE,sigmaE))^(n-r)
Dsp1[j]=(2/pi)*abs(asin(sqrt((j-0.5)/n))-asin(sqrt(pnorm(data1[j],uN,sigmaN))))
Dsp2[j]=(2/pi)*abs(asin(sqrt((j-0.5)/n))-asin(sqrt(pextreme(data1[j],uE,sigmaE))))
D1[j]=(2/pi)*abs(((j-0.5)/n)-pnorm(data1[j],uN,sigmaN))+0.5/n
D2[j]=(2/pi)*abs(((j-0.5)/n)-pextreme(data1[j],uE,sigmaE))+0.5/n}
DspN=max(Dsp1)
DspE=max(Dsp2)
#DN=max(D1)
#DE=max(D2)
## The Taylor series prediction
AMLP_E=function(r,n){
   pr=r/(n+1)
   qr=1-pr
   Xs2=array()
for(i in 1:(n-r)){
 s=r+i
  ps=s/(n+1)
 qs=1-ps
  alphas=1+log(qs)-log(qs)*log(-log(qs))
  betas=log(qs)
gamma1=qs*log(qs)*((qr-qs)*(-1+(1+log(qs))*log(-log(qs)))+qs*log(qs)*log(-log(qs))-
      qr*log(qr)*log(-log(qr)))/((qr-qs)^2)
rou1=qs*log(qs)*(-(1+log(qs))*(qr-qs)-qs*log(qs))/((qr-qs)^2)
      v1=qs*log(qs)*qr*log(qr)/((qr-qs)^2)
      A=s-r-1
      B=A*rou1+A*v1+betas+(n-s)*betas
      C=A*gamma1+alphas-(n-s)+(n-s)*alphas
      D=A*rou1+betas+(n-s)*betas
      Xs2[i]=-A*v1*data1[r]/D+uE*B/D-sigmaE*C/D}
      Xs2[which(Xs2<=data1[r])]=data1[r]
  Xs2}
AMLP_E(r,n)
```

Algorithm 1: R code of Example 1.

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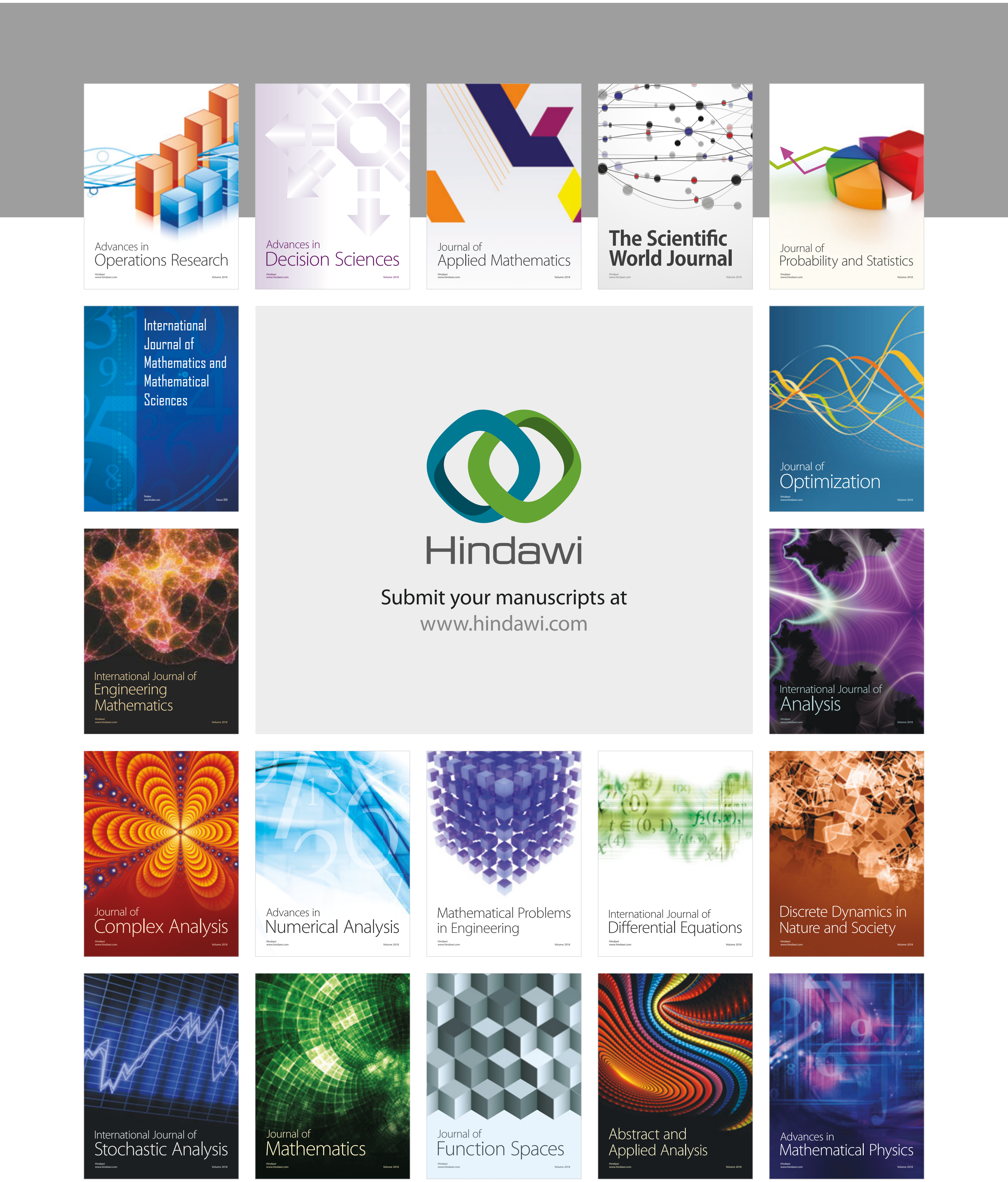
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