Two-Stage Maximum Likelihood Estimation Procedure for Parallel Constant-Stress Accelerated Degradation Tests

Cheng-Hsun Wu, Tzong-Ru Tsai, and Ming-Yung Lee

***Abstract*—The parallel constant-stress accelerated degradation test (PCSADT) is a popular method used to assess the reliability of highly reliable products in a timely manner. Although the maximum likelihood (ML) method is commonly utilized to estimate the PCSADT parameters, the explicit forms of the ML estimators, and their corresponding Fisher information matrix are usually difficult to obtain. In this article, we propose a two-stage ML (TSML) estimation procedure for a time-transformed model. In the proposed procedure, all the TSML estimators not only have explicit expressions but also possess consistency and asymptotic normality. Hence, this method is tractable for reliability engineers. Furthermore, the TSML estimators can provide constructive information about the unknown accelerated relationship law. The proposed method is also applied to analyze light-emitting diode data and compare the performance of our estimation procedures with the ML method via simulations.**

***Index Terms*—Accelerated relationship law, maximum likelihood estimation, parallel constant-stress accelerated degradation test, time-transformed model, two-stage estimation, Wiener process.**

Acronyms and Abbreviation

ADT	Accelerated degradation test.
CSADT	Constant-stress ADT.
PCSADT	Parallel constant-stress ADT.
SSADT	Step-stress ADT.
TSML	Two-stage maximum likelihood.
ML	Maximum likelihood.
MLE	Typical ML estimator/ ML estimate.
TSMLE	Two-stage MLE.
LED	Light emitting diode.
QC	Quality characteristic.

Manuscript received June 2, 2019; revised March 3, 2020; accepted May 13, 2020. Date of publication February 26, 2021; date of current version June 1, 2021. This work was supported by a research Grant from the Ministry of Science and Technology, Taiwan (MOST 105-2118-M-126-002) Associate Editor: Y. Deng. *(Corresponding author: Ming-Yung Lee.)*

Cheng-Hsun Wu is with the Department of Financial Engineering and Actuarial Mathematics, Soochow University, Taipei 215006, Taiwan (e-mail: chsunwu@scu.edu.tw).

Tzong-Ru Tsai is with the Department of Statistics, Tamkang University, New Taipei City 251301, Taiwan (e-mail: 078031@mail.tku.edu.tw).

Ming-Yung Lee is with the Department of Data Science and Big Data Analytics, Providence University, Taichung City 43301, Taiwan (e-mail: mylee@pu.edu.tw).

Color versions of one or more of the figures in this article are available online at https://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TR.2021.3053312

Notation

h $h+1$	is the group (stress level) number of the PCSADT.
k	k is the measurement number.
s_l	lth stress level for $l = 0, \ldots, h$.
t, t_j , Δt_j	Time, jth measurement times, and $\Delta t_j = t_j - t_{j-1}$ for $j = 01, \ldots, k$.
$W_i(t_j; s_l)$	Degradation of test unit i under s_l at time t_j , for $l = 0, \ldots, h, i = 1, \ldots, n_l$, $j = 1, \ldots, k$.
$B(t)$, $B_i(t_j)$	Standard Wiener process at time t and t_j for unit i.
ΔB_{lij}	Increment of two standard Wiener processes.
U_{lij}, $\tilde{U}_{lij}$	Increments and time-transformed increments of degradation measurements.
ξ, σ	Drift and diffusion parameters of standard Wiener process.
α	Termination time of PCSADT.
n_l, N	Sample size under s_l and total sample size.
$\beta(\theta; s_l), \beta_l$	Accelerated relationship laws under s_l.
θ	Accelerated parameter of accelerated relationship laws.
$\hat{\xi}, \hat{\sigma}^2$, $\hat{\xi}_l$, $\hat{\sigma}_l^2$	MLE of drift and diffusion parameters under CSADT with stress levels of s_0 and s_l.
$\hat{\xi}_{2S}, \hat{\sigma}^2_{2S}, \hat{\xi}_M, \hat{\sigma}^2_M$	TSMLE and MLE of drift and diffusion parameters under CSADT.
$\hat{\beta}_l$	Estimation of decay-acceleration value under s_l.
$\hat{\theta}_l$	MLE of θ under s_l.
a_l	Weight for TSMLE of θ under s_l.
$\hat{\theta}_M, \hat{\theta}_{2S}$	MLE and TSMLE of θ.
$\tilde{t}_l$, $\tilde{\alpha}_l$	Time-transformed jth measurement time and censoring time for units under s_l.

I. Introduction

In order to satisfy the stringent requirements and expectations of customers, contemporary products are designed with much longer life spans. Products with longer life spans make the lifetime information of reliable products more difficult to be collected at normal temperature and pressure. Therefore, reliability engineers adopt accelerated degradation test (ADT) methods to accelerate the quality degradation of reliable products for reliability assessment. More details on using ADT methods can be found in Meeker and Escobar [14].

In an ADT, the lifetime of a reliable product is closely related to a quality characteristic (QC). For instance, the lifetime of an alloy is defined as the observed duration until a crack in the alloy reaches a specified size. The crack is a QC of the lifetime of the alloy. Tseng *et al.* [24] presented a study of light-emitting diode (LED) lamps for contact image scanners according to the lifetime of LED, which depends on its QC, light intensity. Furthermore, Park and Padgett [20] adopted a generalized cumulative damage approach with a stochastic process to describe the accelerated degradation model.

The parallel constant-stress ADT (PCSADT), step-stress ADT (SSADT), and continuous-stress ADT (CSADT) are three popular ADT methods. Applications using these three methods can be found in the following references: Tang *et al.* [22], Nelson [17], [18], Liao and Tseng [11], Tseng *et al.* [26], Lim and Yum [13], Tsai *et al.* [25], Chiang *et al.* [1], Tsai *et al.* [24], Elsayed [4], and Lee *et al.* [12]. Collecting degradation information of reliable products is simpler through using the PCSDAT method than using the SSADT method. However, the PCSADT method requires more sample resources and less experiment time than the SSADT method. Fortunately, since the cost of reliable products has drastically decreased, using more sample resources from reliable products for ADT methods becomes affordable. Therefore, the PCSADT method has advantages over the SSADT method in some conditions. In this article, we focus on using the PCSADT methods in which the samples are classified into several groups according to the stress levels. This feature makes the statistical inference based on the PCSADT more tractable than other ADT methods.

Assume that a product has a critical QC whose (transformed) degradation sample path $\{W(t)|t \geq 0\}$, at normal stress s_0 follows a Wiener process:

$$W\ (t; s_0) =\ \xi t + \sigma B\left(t\right),\ \sigma > 0,\ t \geq 0$$

where $B(t)$ stands for the standard Brownian motion, ξ is the drift coefficient, and σ is the diffusion coefficient. The Wiener process is a crucial model for reliability and has successfully been applied, see Doksum and Normand [3], Whitmore and Schenkelberg [30], Liao and Elsayed [11], Tsai *et al.* [23], Tsai *et al.* [25], Tseng *et al.* [27], and Tseng and Wen [28].

Doksum and Hoyland [3] proposed a generalized degradation process, also called the time-transform model, based on stress level that satisfies the following stochastic process:

$$W\ (t; s) =\ \xi\tau\left(t; s\right) + \sigma B\left(\tau\left(t; s\right)\right),\ \sigma > 0,\ t \geq 0 \qquad (1)$$

where s is the stress level, $\tau\ (t; \mathrm{s}) =\ \beta(\theta; s)t$ as s is fixed, $t \geq 0$, and β is a nonnegative, continuous, and nondecreasing function with $\beta\ (\theta; s_0) =\ 1$. The function of β is also called accelerated relationship laws, in which θ is the accelerated parameter and the value of θ is related to the material characteristics. Based on the time-transformed model, the degradation process at high acceleration stress level can be represented by $W\ (ts_l) =\ W(\tilde{t}_l s_0)$, where $\tilde{t}_l =\ \beta(s_l)t$. Many exiting studies have been developed based on the time-transformed model assumption, such as Lee *et al.* [12], Tsai *et al.* [25], Liao and Elsayed [11], Tseng *et al.* [27], and Tseng and Wen [28].

Some accelerated relationship laws used for ADT can be found in Padgett and Tomlinson [19] and Meeker and Escobar [14]. Two most popular temperature accelerated relationship laws are the Arrhenius law and Eyring law.

Model (1) can be associated with the cumulative exposure (CE) model proposed by Nelson [16]. In the CE model, the rate of degradation only depends on the current stress. Denote $Y(t; s)$ as the performance model under stress s. Then, the degradation at time t can be defined as $W(t; \mathrm{s}) \cong Y(t; \mathrm{s}) - Y(0; \mathrm{s})$, and its expectation satisfies

$$E\ \left(W\left(t; s\right)\right) =\ \xi \int_0^t \beta\left(\theta; s\left(z\right)\right) dz\ =\ \xi\tau\left(t\right).$$

The derivative of $E(W(t; s))$ with respect to time t follows:

$$\frac{dE\left(W\left(t; s\right)\right)}{dt} = \frac{dE\left(Y\left(t; \mathrm{S}\right)\right)}{dt}\ =\ \xi\beta\left(\theta; s\left(t\right)\right).$$

The maximum likelihood (ML) method is the most popular estimation approach. Deriving the explicit forms of MLEs is usually an arduous task, so practitioners require the use of numerical methods to calculate the estimates. Because the MLEs and Fisher information matrix do not have explicit expressions, numerical methods with massive computation loading is needed to obtain the MLEs and the observed Fisher information matrix. This fact results in a difficulty to obtain an optimal ADT design via using the typical ML method. In this article, we propose a TSML inference procedure for the time-transformed Weiner model based on the PCSADT. The TSML inference procedure can provide explicit forms of the estimators and their asymptotic covariance matrix to help reliability engineers to obtain an optimal PCSADT design. The asymptotic properties of the TSML estimators, including consistency and asymptotic normality, are also clarified.

The remainder of this article is organized as follows. The degradation model and inference methods based on the typical ML method are introduced in Section II. The TSML inference procedure and its implementation are provided in Section III. The large sample properties and asymptotic normality of the proposed TSML estimators are analytically derived in Section IV. In Section V, we present an application of our method using the LED dataset. The performance of the proposed TSML inference procedure is compared with the typical ML method via using Monte Carlo simulations. All simulation results are reported in Section VI. Section VII concludes this article.

II. Degradation Model and Inference Methods

In this section, we discuss the degradation model and its statistical inference based on the typical ML method. Assume that a CSADT with n_0 test units is conducted under the normal stress s_0. The termination time of the CSADT is predetermined by α, and the measurement times for collecting degradation information are scheduled as $t_0(=0) < t_1 < t_2 < \cdots < t_k(=\alpha)$, where α is the termination time of the experiment. Denote the degradation paths by $W_i\ (t_j; s_0) = \xi t_j + \sigma B(t_j)$, for $j = 01, \ldots, k$, and $i = 1, \ldots, n_0$. The increments at t_j are evaluated by

$$\begin{aligned} U_{ij} &= W_i\ (t_j; s_0) - W_i\,(t_{j-1}; s_0) \\ &= \xi \Delta t_j + \sigma\left(B_i(t_j) - B_i\,(t_{j-1})\right) \end{aligned} \tag{2}$$

for $j = 1, \ldots, k,\ i = 1, \ldots, n_0$, where $\Delta\, t_j = t_j\ - t_{j-1}$ and $W_i\ (t_0; s_0) = W_i\ \ (0; s_0) = \ 0$. Then, U_{ij} is the normal distribution with mean $\xi \Delta t_j$ and variance $\sigma^2 \Delta t_j$. According to the independent increments of Brownian motion, the log-likelihood function can be denoted by

$$\ell\ = \sum_{i=1}^{n_0} \sum_{j=1}^{k} \left[-\frac{1}{2} \ln\left(\sigma^2\right) - \frac{1}{2} \ln\left(\Delta t_j\right) - \frac{\left(U_{ij} - \xi \Delta t_j\right)^2}{2\sigma^2 \Delta t_j} \right].$$

Then, maximizing ℓ gives the MLEs of the parameters as follows:

$$\hat{\xi} = \frac{\sum_{i=1}^{n_0} w_{ik}}{n_0 \alpha} = \frac{\sum_{i=1}^{n_0} w_i\,(\alpha)}{n_0 \alpha} \tag{3}$$

and

$$\hat{\sigma}^2 = \frac{1}{n_0 k} \sum_{i\,=\,1}^{n_0} \sum_{j\,=\,1}^{k} \frac{\left(U_{ij} - \hat{\xi} \Delta t_j\right)^2}{\Delta t_j} \tag{4}$$

where $w_{ik} = \sum_{j=1}^{k} U_{ij}\ = w_i\ (\alpha)$. The estimate of the drift parameter ξ in (3) is free of k but is dependent on sample size n_0 and the degradation value at termination time α.The large sample properties of the above MLEs (see Lehmann [9] for more details) are provided as follows:

1) $\hat{\xi}$ is a consistent estimator of ξ and is also the optimal linear unbiased estimator with weights $\Delta t_j / \sum_{j\,=\,1}^{k} \Delta t_j = \Delta t_j / \alpha$ for $j\ = \ 1, \ldots, k$.
2) The statistic $(U_{ij} - \xi \Delta t_j)/\sqrt{\Delta t_j}$ has an asymptotic normal distribution with mean 0 and variance σ^2.
3) $\hat{\sigma}^2$ is a consistent estimator of σ^2.

In a multi-group PCSADT, samples of products are divided into $h+1$ groups. We assume that the lth group, for $l\ =\ 12, \ldots, h$, contains n_l units tested under the stress $s_l(> s_0)$. Following the design in Doksum and Hoyland [3], we denote the degradation model as

$$W_i\ (t_j; s_l) = \xi_l\, t_j + \sigma B_i\left(\beta_l t_j\right),\ l\ =\ 0, \ldots, h\ i\ =\ 1, \ldots, n_l \tag{5}$$

$j\ =\ 01, \ldots, k$ where $\beta_l = \ \beta(\theta; s_l)$ and $\xi_l = \ \xi \beta_l$. Thus, the CSADT model in (2) only uses normal stress s_0 and is a special case of model (5) with $l\ =\ 0$. Model (5) is a Weiner process rescaled in time by (1) and describes that a higher stress level accompanies a higher expected accumulated degradation. Then, we denote the dataset as follows:

$$\begin{array}{lllll} \text{Under } s_0 & w_1(t_1; s_0) & w_1(t_2; s_0) & \ldots & w_1(\alpha; s_0) \\ & \ldots & \ldots & \ldots & \\ & w_{n_0}(t_1; s_0) & w_{n_0}(t_2; s_0) & & w_{n_0}(\alpha; s_0) \\ \text{Under } s_1 & w_1(t_1; s_1) & w_1(t_2; s_1) & \ldots & w_1(\alpha; s_1) \\ & \ldots & \ldots & & \ldots \\ & w_{n_l}(t_1; s_1) & w_{n_l}(t_2; s_1) & w_1(\alpha; s_1) & \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \text{Under } s_h & w_1(t_1; s_h) & w_1(t_2; s_h) & \ldots & w_1(\alpha; s_h) \\ & \ldots & \ldots & & \ldots \\ & w_{n_h}(t_1; s_h) & w_{n_h}(t_2; s_h) & & w_{n_h}(\alpha; s_h). \end{array}$$

Under the PCSADT experiment, we denote the increment of degradation measurement as

$$U_{lij} = W_i\ (t_j; s_l) - W_i\ (t_{j-1}; s_l) = \xi_l\ \Delta t_j + \sigma \Delta B_{lij} \tag{6}$$

for $l\ =\ 0, \ldots, h, i\ =\ 1, \ldots, n_l,\ j\ =\ 1, \ldots, k$, where $\Delta\, t_j = t_j - t_{j-1}$, and $\Delta\ B_{lij} = \ B_i(\beta_l t_j) - B_i(\beta_l t_{j-1})$. It is clear that U_{lij}'s are normally distributed with mean $\beta_l \xi \Delta t_j$ and variance $\beta_l \sigma^2 \Delta t_j$. On the basis of the independent increments property of Brownian motion, the log-likelihood function can be presented by

$$\begin{aligned} \ell\ = \sum_{l=0}^{h} \sum_{i=1}^{n_l} \sum_{j=1}^{k} &\left[-\frac{1}{2} \ln\left(\sigma^2\right) - \frac{1}{2} \ln\left(\beta_l \Delta t_j\right) \right. \\ &\left. - \frac{\left(U_{lij} - \beta_l \xi \Delta t_j\right)^2}{2\sigma^2 \beta_l \Delta t_j} \right] \end{aligned}$$

and its partial derivatives with respect to $\xi,\ \sigma^2$, and θ are

$$\frac{\partial \ell}{\partial \xi} = \sum_{l=0}^{h} \sum_{i=1}^{n_l} \sum_{j=1}^{k} \frac{\left(U_{lij} - \beta_l \xi \Delta t_j\right)}{\sigma^2},$$

$$\frac{\partial \ell}{\partial \sigma^2} = \sum_{l=0}^{h} \sum_{i=1}^{n_l} \sum_{j=1}^{k} \left[-\frac{1}{2\sigma^2} + \frac{\left(U_{lij} - \beta_l \xi \Delta t_j\right)^2}{2\sigma^4 \beta_l \Delta t_j} \right]$$

and

$$\frac{\partial \ell}{\partial \theta} = \sum_{l=0}^{h} \sum_{i=1}^{n_l} \sum_{j=1}^{k} \left[-\frac{\beta_l'}{2\beta_l} - \frac{U_{lij}^2 \beta_l'}{2\sigma^2 \beta_l^2 \Delta t_j} + \frac{\xi^2 \beta_l' \Delta t_j}{2\sigma^2} \right] \tag{7}$$

where $\beta_l' = \ \partial \beta_l / \partial \theta$. Then, solving the equations of $\partial \ell / \partial \xi = \ 0$ and $\partial \ell / \partial \sigma^2 = \ 0$, the typical MLEs of ξ and σ^2 are

$$\hat{\xi}_M = \frac{\sum_{l=0}^{h} \sum_{i=1}^{n_l} W_{lik}}{\sum_{j=0}^{h} n_l \hat{\beta}_{l,M} \alpha} = \frac{\sum_{l\,=\,0}^{h} \sum_{i\,=\,1}^{n_l} W_i\,(\alpha; s_l)}{\sum_{j\,=\,0}^{h} n_l \hat{\beta}_{l,M} \alpha} \tag{8}$$

and

$$\hat{\sigma}_M^2 = \frac{1}{Nk} \sum_{l\,=\,0}^{h} \sum_{i\,=\,1}^{n_l} \sum_{j\,=\,1}^{k} \frac{\left(U_{lij} - \hat{\xi}_M \hat{\beta}_{l,M} \Delta t_j\right)^2}{\hat{\beta}_{l,M} \Delta t_j} \tag{9}$$

where $W_{lik} = W_i\ (\alpha; s_l)\ = \sum_{j=1}^{k} U_{lij}$, $\hat{\beta}_{l,M} = \ \beta(\hat{\theta}_M; s_l)$ and $N\ = \sum_{l=0}^{h} n_l$. Note that the drift estimate $\hat{\xi}_M$ in (8) depends on the degradation value at termination time α and the numbers of measurement. However, the explicit forms of $\hat{\beta}_{l,M}$ and $\hat{\theta}_M$ in (8) and (9) remain uncertain.

The typical MLE of the accelerated parameter θ relies on a special relationship, $\beta_l' = c_l\ \beta_l$, for a constant $c_l \in \mathbb{R}$, and following such special relationship for the Arrhenius law:

$$\frac{\partial}{\partial\theta}\beta\ (\theta, s_l) = \frac{\partial}{\partial\theta}\ \beta_l = \delta_l\ \beta\ (\theta, s_l) = \delta_l\ \beta_l$$

where $\delta_l = 11605(s_0^{-1} - s_l^{-1})$. By substituting β_l' with $\delta_l\beta_l$ in (7) and setting $\partial\ell/\partial\theta = 0$

$$\sum_{l=0}^{h}\delta_l\left(\frac{\sum_{i=1}^{n_l}\sum_{j=1}^{k}U_{lij}^2/\Delta t_j}{\beta\,(\theta, s_l)} - \hat{\xi}_M^2\alpha n_l\beta\,(\theta, s_l) - k\hat{\sigma}_M^2 n_l\right) = 0. \tag{10}$$

Then, the MLE of θ denoted by $\hat{\theta}_M$ can be obtained by solving (10). It is also extremely difficult to determine the closed-form of $\hat{\theta}_M$. Therefore, the estimate of $\hat{\theta}_M$ should be calculated via using numerical methods.

III. Two-Stage Estimation Method

In this section, we propose a TSML inference procedure to derive the estimators of the lifetime distribution parameters. In the first stage, we demonstrate how to obtain the estimators of ξ, β_l, and the accelerated factor θ_l separately for each stress level s_l. The estimators of β_l and θ_l can be used to transform the original PCSADT data under stress s_l into degradation data under the normal use condition. In the second stage, we adopt the ML method to estimate the model parameters based on the transformed data. It is noteworthy that the TSML estimators are easily-interpretable and consistent, and all have closed-forms.

Another two-stage estimation procedure for the time-censored ADT was proposed by Lee *et al.* [12] In their design, the failure time data and degradation information are collected at the censored time. They used latent variable and failure time data to estimate the degradation value at the censored time, and then the pseudocomplete degradation data are used to estimate the ADT model parameters. In this article, the proposed TSML inference procedure uses a different data design from the one proposed by Lee *et al.* [12] The degradation datasets in this article are composed of true degradation values observed under the setup stresses. Moreover, the ADT parameters are estimated based on the true degradation datasets collected at some time points in the interval $[0, \alpha]$ for several stresses.

A. First Stage: Estimation of β_l

ξ_l in (5) can be regarded as a new drift parameter, and β_l can be considered a parameter of the degradation model under the stress s_l. Using (3) and (4), a comparison of the parameters in (5) and (2) can be used to represent the estimators of the drift parameter ξ_l and the diffusion parameter as follows:

$$\hat{\xi}_l = \frac{\sum_{i=1}^{n_l} W_{lik}}{n_l\alpha} = \frac{\sum_{i=1}^{n_l} W_i\,(\alpha; s_l)}{n_l\alpha} \tag{11}$$

and

$$\hat{\sigma}_l^2 = \frac{1}{n_l k}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij} - \hat{\xi}_l\Delta t_j\right)^2}{\hat{\beta}_l\Delta t_j} \tag{12}$$

for $l = 0, \ldots, h$. According to $\xi_l = \xi\beta_l$, the estimator of β_l can be represented by

$$\hat{\beta}_l = \frac{\hat{\xi}_l}{\hat{\xi}_0} = \frac{n_0}{n_l}\frac{\sum_{i=1}^{n_l} W_{lik}}{\sum_{i=1}^{n_0} W_{0ik}},\ \text{for}\ l = 1, \ldots, h. \tag{13}$$

Then, the estimator of θ at the stress s_l can be obtained through of using the functional form of $\beta(\cdot)$ by

$$\hat{\theta}_l = \beta^{-1}\left(\hat{\beta}_l\right)$$

for $l = 1, \ldots, h$. The Arrhenius law model is a commonly used stress model for ambient temperature in an ADT. Hu *et al.* [7] also used an ADT with the Arrhenius law model to study the reliability of LEDs. If the PCSADT cannot be carried out for the stress level s_0 (13) can be modified to estimate $\beta_l^* = \beta(\theta; s_1, s_l)$. The estimator of β_l^* satisfies

$$\hat{\beta}_l^* = \frac{\hat{\xi}_l}{\hat{\xi}_1} = \frac{n_1}{n_l}\frac{\sum_{i=1}^{n_l} w_{lik}}{\sum_{i=1}^{n_1} w_{1ik}},\ \text{for}\ l = 2, \ldots, h.$$

That is, $\hat{\beta}_l^*$ can be obtained based on the degradation information observed at the stress levels s_1 and s_l. In practice, although several explicit expressions of β are proposed, the best expression has not been determined. It is noteworthy that $\hat{\beta}_l^*$ is independent of the expression of β and provides an impersonal estimate for the value of β under stress s_l. Therefore, the explicit form of β can be explored using mathematical or statistical methods, for example, a regression analysis based on the dataset $\{(s_l,\ \hat{\beta}_l^*), l = 01, \ldots, h\}$. Our method can be extended to estimate the vector parameter θ in multiple stress factor cases; see Appendix B in Lee *et al.* [12] for an alternative method.

B. Second Stage: Estimation of Parameters ξ and σ Based on Transformed Data

In this section, we transform the data drawn based on (5) to the resulting data, which can be viewed as all of the data collected under s_0. The time t_j is changed to $\beta_l t_j$, and the transformed measurement time (called aged time) of the experiment can be denoted as

$$\tilde{t}_{lj} = \beta_l\, t_j, l = 1, \ldots, h,\ j = 1, \ldots, k. \tag{14}$$

The transformed termination times can be denoted as $\tilde{\alpha}_l = \beta_l\ \alpha$, for $l = 12,\ \ldots, h$, and the degradation model $W(t_j; s_l)$ can be rewritten as $W(\tilde{t}_{lj}; s_0)$. Note that $W\ (t_j; s_l) = W(\tilde{t}_{lj}; s_0)$. The transformed dataset of PCSADT under s_0 is listed in the following table:

$$\begin{array}{lllll}
\text{Under} & w_1(t_1; s_0) & w_1(t_2; s_0) & \ldots & w_1(\alpha; s_0) \\
s_0 & \ldots & \ldots & & \ldots \\
 & w_{n_0}(t_1; s_0) & w_{n_0}(t_2; s_0) & & w_{n_0}(\alpha; s_0) \\
\text{Under} & w_1(\tilde{t}_{11}; s_0) & w_1(\tilde{t}_{12}; s_0) & \ldots & w_1(\tilde{\alpha}_1; s_0) \\
s_1 & \ldots & \ldots & & \ldots \\
 & w_{n_l}(\tilde{t}_{11}; s_0) & w_{n_l}(\tilde{t}_{12}; s_0) & & w_{n_l}(\tilde{\alpha}_1; s_0) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\text{Under} & w_1(\tilde{t}_{h1}; s_0) & w_1(\tilde{t}_{h2}; s_0) & & w_1(\tilde{\alpha}_h; s_0) \\
s_h & \ldots & \ldots & \ldots & \ldots \\
 & w_{n_h}(\tilde{t}_{h1}; s_0) & w_{n_h}(\tilde{t}_{h2}; s_0) & & w_{n_h}(\tilde{\alpha}_h; s_0).
\end{array}$$

We omit s_0 and denote the modified model of (5) by

$$W_i\left(\tilde{t}_{lj}\right) = \xi \tilde{t}_{lj} + \sigma B_i\left(\tilde{t}_{lj}\right), l = 0, \ldots, h, i = 1, \ldots, n_l \quad j = 1, \ldots, k. \tag{15}$$

The increment of (15) follows:

$$\begin{aligned}\tilde{U}_{lij} &= W_i\left(\tilde{t}_{lj}\right) - W_i\left(\tilde{t}_{lj-1}\right) \\ &= \xi \Delta \tilde{t}_{lj} + \sigma\left(B_i\left(\tilde{t}_{lj}\right) - B_i\left(\tilde{t}_{l,j-1}\right)\right).\end{aligned}$$

Since $W_i\left(\tilde{t}_{lj}; s_0\right) = W_i\left(t_j; s_l\right)$, the abbreviations W_{lik} and U_{lij} are used to represent $W_i(\tilde{t}_{lj})$ and its increments $\tilde{U}_{lij}$. Therefore, U_{lij} follows a normal distribution with mean $\xi \Delta \tilde{t}_{lj}$ and variance $\sigma^2 \Delta \tilde{t}_{lj}$ for $l = 1, \ldots, h, i = 1, \ldots, n_l, j = 1, \ldots, k$. Note that Doksum and Hoyland proposed the time-transformed model and assumed that the acceleration stress not only affects the drift coefficient but also the diffusion coefficient. We present the log-likelihood function by

$$\ell = \sum_{l=0}^{h}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\left[-\frac{1}{2}\ln\left(\sigma^2\right) - \frac{1}{2}\ln\left(\Delta\tilde{t}_{lj}\right) - \frac{\left(U_{lij} - \xi\Delta\tilde{t}_{lj}\right)^2}{2\sigma^2\Delta\tilde{t}_{lj}}\right].$$

The first partial derivatives of ℓ with respect to ξ and σ^2 are as follows:

$$\begin{aligned}\frac{\partial \ell}{\partial \xi} &= \sum_{l=0}^{h}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{2\left(U_{lij} - \xi\Delta\tilde{t}_{lj}\right)\Delta\tilde{t}_{lj}}{2\sigma^2\Delta\tilde{t}_{lj}} \\ &= \sum_{l=0}^{h}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij} - \xi\Delta\tilde{t}_{lj}\right)}{\sigma^2}\end{aligned}$$

and

$$\frac{\partial \ell}{\partial \sigma^2} = \sum_{l=0}^{h}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\left[-\frac{1}{2\sigma^2} + \frac{\left(U_{lij} - \xi\Delta\tilde{t}_{lj}\right)^2}{2\sigma^4\Delta\tilde{t}_{lj}}\right].$$

By setting and solving $\partial\ell/\partial\xi = 0$ and $\partial\ell/\partial\sigma^2 = 0$, the TSMLEs of ξ and σ^2 follow (recall that $W_i(\tilde{t}_{lk}; s_0)$ and $W_i(t_k; s_l)$ have the same value and are denoted as W_{lik})

$$\hat{\xi}_{2\mathrm{S}} = \frac{\sum_{l=0}^{h}\sum_{i=1}^{n_l} W_{lik}}{\sum_{l=0}^{h} n_l\tilde{\alpha}_l} = \frac{\sum_{l=0}^{h}\sum_{i=1}^{n_l} W_i\left(\alpha; s_l\right)}{\sum_{l=0}^{h} n_l\tilde{\alpha}_l} \tag{16}$$

and

$$\hat{\sigma}^2_{2\mathrm{S}} = \frac{1}{Nk}\sum_{l=0}^{h}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij} - \hat{\xi}_{2\mathrm{S}}\Delta\tilde{t}_{lj}\right)^2}{\Delta\tilde{t}_{lj}}. \tag{17}$$

The parameter ξ_0 can be estimated by $\hat{\xi}_0$ in first-stage and $\hat{\xi}_{2S}$ in the second-stage. Via (13), it is clear that the two estimates of ξ_0 are the same.

Using the specific form of β, the parameter θ can be estimated directly as

$$\hat{\theta}_l = \beta^{-1}\left(\hat{\beta}_l\right) \tag{18}$$

according to the degradation data under stress s_l. From (13), $\hat{\xi}_0$ is the denominator of $\hat{\beta}_l$ and $\hat{\theta}_l$, $l = 1, \ldots, h$, are dependent. In this article, we denote the linear combination of $\hat{\theta}_l$ for $l = 1, \ldots, h$

$$\hat{\theta}_{2S} = \sum_{l=1}^{h} a_l\,\hat{\theta}_l$$

where a_l, $l = 12, \ldots, h$, are weights with $\sum_{l=1}^{m} a_l = 1$ as the TSMLE of θ. Suitable weights a_l, $l = 12, \ldots, h$ can be determined by minimizing the asymptotic variance of $\hat{\theta}_{2S}$, see Section IV.B. Finally, the unknown $\tilde{\alpha}_l$ and $\Delta\tilde{t}_{lj}$ are estimated using $\hat{\beta}_l\alpha$ and $\hat{\beta}_l\Delta t_{lj}$, respectively. Then, we have the proposed TSMLEs of ξ and σ^2 using all of the data.

IV. Asymptotic Properties

In this section, the aim is to derive the properties of consistency and asymptotic normality of the TSML estimators while the sample size tends to infinity but the number of the time point observations is fixed.

A. Consistency and Asymptotic Normality

Consistency is a fundamental property for evaluating the quality of the point estimator and ensuring that the estimator is approximate to the true parameter as the sample size increases. It is well known that the MLE of the variance σ^2 derived based on the n – sample drawn independently from $N(\theta_r, \sigma^2)$, $r = 12, \ldots, n$, is not consistent (see Lehmann [9, p. 524]). Despite the h ADT samples being drawn from different normal distributions, all of the TSML estimators are equipped with consistency.

First, we verify that all first-stage estimators are consistent in Lemma 1. Then, the consistency of the second-stage estimators $\hat{\theta}_{2S}$ is also given in Lemma 1. Moreover, the consistency of the second-stage estimator $\hat{\xi}_{2\mathrm{S}}$ and $\hat{\sigma}^2_{2\mathrm{S}}$ are reported in Proposition 1.

Lemma 1: Under the stress s_l, $\hat{\xi}_l$, $\hat{\beta}_l$, $\hat{\sigma}^2_l$, and $\hat{\theta}_l$ are consistent estimators of the parameters ξ_l, β_l, σ^2_l, and θ. Moreover, $\hat{\theta}_{2S} = \sum_{l=1}^{h} a_l\,\hat{\theta}_l$ is a consistent estimator of θ.

Proposition 1: The TSML estimators $\hat{\xi}_{2\mathrm{S}}$ and $\hat{\sigma}^2_{2\mathrm{S}}$ are consistent estimators of the parameters ξ and σ^2, respectively.

The proofs of Lemma 1 and Proposition 1 are included in Appendix A and B, respectively. The asymptotic normality of $\hat{\theta}_{2S}$ is discussed in Appendix C.

B. Selection of Coefficients a_l's

Selecting the weights $a_l,\ l = 12,\ldots,h$ in $\hat{\theta}_{2S}$ is important for specifying $\hat{\theta}_{2S}$. In this section, we discuss how to determine the weights $a_l,\ l = 12,\ldots,h$ by minimizing the asymptotic variance

$$\sigma^2_{\hat{\theta}_{2S}} = \left(\sum_{l=1}^{h} a_l \left(\beta^{-1}\right)'(\beta_l)\right)^2 \sum_{l\,=\,1}^{h}\left[\left(a_l\left(\beta^{-1}\right)'(\beta_l)\right)^2\left(\frac{\beta_l}{m_l}\right)\right] \tag{19}$$

subject to the equality constraint $\sum_{l\,=\,1}^{m} a_l = 1$. Since

$$\frac{\partial\sigma^2_{\hat{\theta}_{2S}}}{\partial a_l} = 2\left(\beta^{-1}\right)'(\beta_l)\sum_{l\,=\,1}^{m} a_l\left(\beta^{-1}\right)'(\beta_l) + 2\left[\left(\beta^{-1}\right)'(\beta_l)\right]^2\left(\frac{\beta_l}{m_l}\right)a_l$$

and

$$\frac{\partial\left(\sum_{l\,=\,1}^{m} a_l - 1\right)}{\partial a_l} = 1$$

for $l = 12,\ldots,h$, via using the Lagrange multipliers method, the following simultaneous linear equations are obtained:

$$2(\beta^{-1})'(\beta_l)\sum_{l=1}^{h} a_l(\beta^{-1})'(\beta_l) + 2\left[\left(\beta^{-1}\right)'(\beta_l)\right]^2\left(\frac{\beta_l}{m_l}\right)a_l - \lambda = 0, \text{for } l = 12,\ldots,h.$$

and

$$\sum_{l=1}^{m} a_l = 1$$

where λ is the Lagrange multiplier. Define $a_\lambda = (a_1, a_2,\ldots,a_h,\ \lambda)', \theta^{(\lambda)}_{i,j} = (\beta^{-1})'(\beta_i)(\beta^{-1})'(\beta_j)$ and

$$A_\lambda = \begin{pmatrix} 2\theta^{(\lambda)}_{11}\left(1+\frac{\beta_1}{m_1}\right) & 2\theta^{(\lambda)}_{12} & \cdots & 2\theta^{(\lambda)}_{1,m} & -1 \\ 2\theta^{(\lambda)}_{12} & 2\theta^{(\lambda)}_{22}\left(1+\frac{\beta_2}{m_2}\right) & \cdots & 2\theta^{(\lambda)}_{2,m} & -1 \\ \vdots & & \ddots & \vdots & \vdots \\ 2\theta^{(\lambda)}_{1,m} & 2\theta^{(\lambda)}_{2,m} & \cdots & 2\theta^{(\lambda)}_{m,m}\left(1+\frac{\beta_m}{m_m}\right) & -1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

where $(\beta^{-1})'$ is the derivative of the inverse function of β. Then, the solution of the simultaneous linear equations above is

$$a_\lambda = A_\lambda^{-1}(00,\ldots,01)'.$$

Since the Hessian matrix of $\sigma^2_{\hat{\theta}_{2S}}$ is a positive definite matrix, we determine that $\sigma^2_{\hat{\theta}_{2S}}$ is a convex function of $(a_1, a_2,\ldots,a_h)$ from (19). Therefore, the unique solution a_λ can be used to attain the global minimum of $\sigma^2_{\hat{\theta}_{2S}}$.

Note that the coefficients a_l can be easily obtained when the inverse of β can be expressed as an explicit form. If it is complicated to obtain the expression of the inverse function of β, the uniform-weighted estimator of θ is still an accurate estimator based on the consistency of $\hat{\theta}_{2S}$.

Example 1: For a three-level PSCADT with $h = 2$, the closed-form expressions of a_1 and a_2 can be carried out straightforwardly. Obviously, the Hessian matrix of $\sigma^2_{\hat{\theta}_{2S}}$ is a positive definite matrix. Define

$$\mathrm{A}_\lambda = \begin{pmatrix} 2\theta^{(\lambda)}_{11}\left(1+\frac{\beta_1}{m_1}\right) & 2\theta^{(\lambda)}_{12} & -1 \\ 2\theta^{(\lambda)}_{12} & 2\theta^{(\lambda)}_{22}\left(1+\frac{\beta_2}{m_2}\right) & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then, we can obtain

$$a_1 = \frac{\theta^{(\lambda)}_{22}\left(1+\frac{\beta_2}{m_2}\right) - \theta^{(\lambda)}_{12}}{\theta^{(\lambda)}_{11}\left(1+\frac{\beta_1}{m_1}\right) + \theta^{(\lambda)}_{22}\left(1+\frac{\beta_2}{m_2}\right) - 2\theta^{(\lambda)}_{12}}$$

and

$$a_2 = \frac{\theta^{(\lambda)}_{11}\left(1+\frac{\beta_1}{m_1}\right) - \theta^{(\lambda)}_{12}}{\theta^{(\lambda)}_{11}\left(1+\frac{\beta_1}{m_1}\right) + \theta^{(\lambda)}_{22}\left(1+\frac{\beta_2}{m_2}\right) - 2\theta^{(\lambda)}_{12}}$$

by $a_\lambda = A_\lambda^{-1}(00,1)'$. Following the Arrhenius law, we have $(\beta^{-1})'\ (\beta_l) = \ 1/(\delta_l\beta_l)$ for $l = 12$, where $\delta_l = 11605(s_0^{-1} - s_l^{-1})$. Therefore, the coefficients $a_l,\ l = 12$ have the following representations:

$$a_1 = \frac{\left(\frac{1}{\delta_2\beta_2}\right)^2\left(1+\frac{\beta_2}{m_2}\right) - \frac{1}{\delta_1\beta_1\delta_2\beta_2}}{\left(\frac{1}{\delta_1\beta_1}\right)^2\left(1+\frac{\beta_1}{m_1}\right) + \left(\frac{1}{\delta_2\beta_2}\right)^2\left(1+\frac{\beta_2}{m_2}\right) - \frac{2}{\delta_1\beta_1\delta_2\beta_2}}$$

and

$$a_2 = \frac{\left(\frac{1}{\delta_1\beta_1}\right)^2\left(1+\frac{\beta_1}{m_1}\right) - \frac{1}{\delta_1\beta_1\delta_2\beta_2}}{\left(\frac{1}{\delta_1\beta_1}\right)^2\left(1+\frac{\beta_1}{m_1}\right) + \left(\frac{1}{\delta_2\beta_2}\right)^2\left(1+\frac{\beta_2}{m_2}\right) - \frac{2}{\delta_1\beta_1\delta_2\beta_2}}.$$

Remark 1: Note that all of the commonly used accelerated relationship laws have the continuous derivative of the inverse function β^{-1}. However, the expressions of the optimal coefficient a_l involving unknown parameter β_l for $l = 12,\ldots,h$, we propose replacing β_l by the consistent estimator $\hat{\beta}_l$ to obtain the approximations of a_l for $l = 12,\ldots,h$. In addition, the accuracy of these approximations can be ensured by the consistency of the estimators $\hat{\beta}_l,\ l = 12,\ldots,h$ and the continuity of $(\beta^{-1})'$.

V. LED Example

In this section, we evaluate the performance of the proposed TSML estimators via using Monte Carlo simulations. A real-world application based on the proposed inference procedure is also given for illustration.

A. Simulation Study

Intensive simulations were conducted to study the performance of the proposed TSML estimation method. Referring the results of the LED example in the next section, we set $\xi = 0.001$ and $\sigma = 0.006$ for the Wiener process model in this simulation study. The life–stress relationship is set as the Arrhenius law

TABLE I
SIMULATION RESULTS FOR $\theta = 0.15, \xi = 0.001$ AND $\sigma = 0.006$.

	$\hat{\theta}_{2S}$	$\hat{\theta}_{M}$	$\hat{\xi}_{2S}$	$\hat{\xi}_{M}$	$\hat{\sigma}_{2S}$	$\hat{\sigma}_{M}$
			N=6×3	k=8		
AVE	0.150554	0.149356	0.001028	0.001010	0.006052	0.005976
SE	0.049539	0.023485	0.000269	0.000164	0.000740	0.000481
RMSE	0.049542	0.023494	0.000271	0.000164	0.000741	0.000482
			N=15×3	k=8		
AVE	0.150508	0.149275	0.001001	0.001004	0.005993	0.005994
SE	0.023471	0.015203	0.000140	0.000107	0.000370	0.000295
RMSE	0.023476	0.015220	0.000140	0.000107	0.000370	0.000295
			N=24×3	k=8		
AVE	0.149949	0.149793	0.001006	0.001005	0.005997	0.005990
SE	0.017282	0.011887	0.000110	0.000087	0.000300	0.000242
RMSE	0.017282	0.011889	0.000110	0.000087	0.000300	0.000243
			N=120×3	k=8		
AVE	0.149913	0.149797	0.001003	0.001003	0.006002	0.006002
SE	0.008069	0.005210	0.000050	0.000037	0.000125	0.000100
RMSE	0.008070	0.005214	0.000050	0.000037	0.000125	0.000100
			N=6×3	k=13		
AVE	0.150604	0.149484	0.001017	0.001011	0.006010	0.005987
SE	0.037495	0.020823	0.000227	0.000154	0.000552	0.000393
RMSE	0.037500	0.020830	0.000228	0.000155	0.000552	0.000394
			N=15×3,	k=13		
AVE	0.150025	0.150061	0.001008	0.001003	0.006011	0.005995
SE	0.023273	0.012966	0.000142	0.000099	0.000350	0.000242
RMSE	0.023273	0.012966	0.000143	0.000099	0.000350	0.000242
			N=24x3,	k=13		
AVE	0.150095	0.150304	0.001001	0.000997	0.006003	0.005991
SE	0.017714	0.010257	0.00011	0.000076	0.000259	0.000182
RMSE	0.017714	0.010261	0.00011	0.000076	0.000259	0.000183
			N=120×3,	k=13		
AVE	0.150034	0.149979	0.001000	0.001000	0.005999	0.005998
SE	0.007785	0.004807	0.000048	0.000036	0.000116	0.000085
RMSE	0.007785	0.004807	0.000048	0.000036	0.000116	0.000085
			N=6×3	k=52		
AVE	0.150361	0.149356	0.001019	0.001010	0.006029	0.006000
SE	0.037777	0.012418	0.000220	0.000122	0.000496	0.000216
RMSE	0.037779	0.012435	0.000221	0.000122	0.000497	0.000216
			N=15x3,	k=52		
AVE	0.149964	0.149786	0.001007	0.001002	0.006012	0.006002
SE	0.021007	0.008031	0.000136	0.000078	0.000280	0.000130
RMSE	0.021007	0.008034	0.000136	0.000078	0.000280	0.000130
			N=24×3,	k=52		
AVE	0.150183	0.149797	0.001002	0.001002	0.006004	0.006002
SE	0.016059	0.006385	0.000104	0.000061	0.000214	0.000106
RMSE	0.016060	0.006389	0.000104	0.000061	0.000214	0.000106
			N=120×3,	k=52		
AVE	0.149874	0.149879	0.001001	0.001001	0.006002	0.006001
SE	0.007048	0.002874	0.000046	0.000027	0.000096	0.000048
RMSE	0.007049	0.002876	0.000046	0.000027	0.000096	0.000048

model with $\theta = 0.15$. The stress levels $(s_0, s_1, s_2) = (25$ °C, 65 °C, 105 °C) are selected for the three-level PCSADT. Three measurement numbers, $k = 8$, 13, and 52, are adopted with the time intervals, which has length α/k. We set the termination time to $\alpha = 100$ and the sample sizes to $n = 6$, 15, 24, and 120 for all stress levels. All the designs are repeated 1000 times to obtain 1000 estimates of the model parameters. Then, the average estimate (AVE), the standard error (SE), and root-mean-square error (RMSE) are evaluated. These results are reported in Table I.

From Table I, we find that the AVE of $\tilde{\theta}_{2S}$ is closer to the true value $\theta = 0.15$ than $\tilde{\theta}_M$, except for the case of $N = 120 \times 3$ and $k = 52$. It is noted that the bias of $\tilde{\theta}_{2S}$ in most of the cells of Table I are smaller than that of $\tilde{\theta}_M$, but the difference of bias based on $\tilde{\theta}_{2S}$ and $\tilde{\theta}_M$ is insignificant. However, the SE of $\tilde{\theta}_{2S}$ is slightly greater than those of $\tilde{\theta}_M$. Their absolute difference decreases from 0.00055 to 0.000051 for the cases with $k = 6$, from 0.000361 to 0.002588 for the cases with $k = 13$ and decreases from 0.012976 to 0.004174 for the cases with $k = 52$, as the sample size increases. Furthermore, the SE of $\tilde{\theta}_{2S}$ is less sensitive than those of $\tilde{\theta}_M$ for the measurement member k. Since the SEs of $\tilde{\theta}_M$ and $\tilde{\theta}_{2S}$ are small, both estimators $\tilde{\theta}_{2S}$ and $\tilde{\theta}_M$ are competitive.

The AVEs of $\tilde{\xi}_{2S}$ and $\tilde{\xi}_M$ are extremely close to $\xi = 0.001$, and their SEs are extremely small. Although the SE of $\tilde{\xi}_M$ is slightly less than the SE of $\tilde{\xi}_{2S}$, $\tilde{\xi}_{2S}$ is less sensitive than $\tilde{\xi}_M$ for the measurement number k. For estimating σ, the AVEs of $\tilde{\sigma}_{2S}$ and $\tilde{\sigma}_M$ are also very close to $\sigma = 0.006$. However, $\tilde{\sigma}_M$ has a smaller SE and $\tilde{\sigma}_{2S}$ is less sensitive for the measurement numbers. The absolute difference of SEs between the MLEs and TSMLEs is very small, but the SE and RMSE of TSMLE are slightly bigger than the MLE. These two estimation methods are competitive. Due to the TSML estimators have explicit forms, the TSML method is more tractable than the ML method. In this article, we recommend the TSML method to engineers for implementing CSADTs.

B. LED Data Analysis

The LED is a critical part of a contact image sensor (CIS) module. The lifetime of the CIS module is highly dependent on the light intensity quality of the LEDs and the lifetime of an LED lamp is highly correlated with its light intensity (brightness). There is a fairly standard operating procedure for implementing an LED ADT (see Huang *et al.* [5], Wang [29], and U.S. Department of Energy [35] for more details). The main test equipment is a high-temperature aging degradation chamber. Note that IES LM-80-08 (IESNA Testing Procedures [33], which is an industry standard developed by the Illuminating Engineering Society of North America and sponsored by the U.S. Department of Energy[34], suggests using only one stress factor, namely, temperature, at three different levels when analyzing the lumen degradation and lifetime of LEDs.

Normally, the lumen degradation information of LED is collected under ADTs to evaluate the reliability of LEDs. Using a higher stress than the normal use condition in an ADT can accelerate the lumen degradation of LED and reduce the experimental time and testing cost. Temperature is a critical stress factor in this study. A lumen degradation dataset for a special type of LED was obtained from a leading LED manufacturer in Taiwan by using a three-level ($h = 2$) PCSADT with temperatures at $(s_0, s_1, s_2) = (25, 65, 105)$ in Celsius (°C) in which $n_0 = 15, n_1 = 18$, and $n_2 = 23$ LEDs were allocated for the PCSADT at the temperatures s_0, s_1, and s_2, respectively. The total measurement number is $k = 13$. The PCSADT was implemented for 2184 h and then terminated.

According to Yu and Tseng [31], [32], Tseng *et al.* [24], and Liao and Elsayed [10], a transformed degradation process can be modeled by a linear process in many engineering applications. In this example, we consider the transformed degradation path by $W_{li}\ (t) = -\ln(L_{li}(t^{0.6}))$. Therefore, the corresponding termination time of the PCSADT is also transformed as

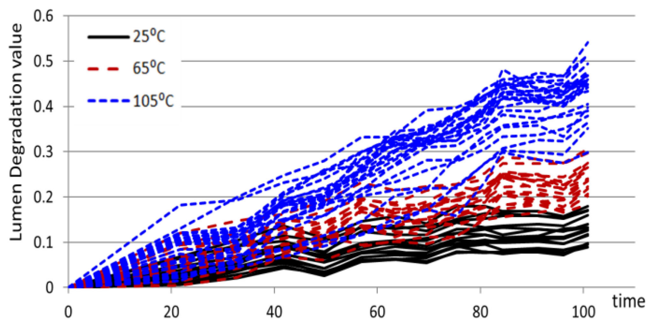

Fig. 1. Transformed Wiener process paths of LEDs.

$\alpha = 2148^{0.6} = 100.82$. The transformed Wiener process paths of LEDs over time are depicted in Fig. 1.

Consider the Arrhenius law model for the stress of temperature. Solving (10) based on (8) and (9) gives the MLEs of ξ, σ, and θ, which are listed as follows: $\hat{\xi}_M = 0.001189$, $\hat{\sigma}_M = 0.005870$, and $\hat{\theta}_M = 0.149048$. Following the TSML procedure, the estimate of θ is $\tilde{\theta}_{2S} = 0.146052$ and the representation of the estimate of the decay acceleration factor can be straightforwardly denoted as

$$\hat{\beta}_l = \beta\left(\hat{\theta}_{2S}; s_l\right) = e^{(0.146052/k_b)\left(\frac{1}{273.15+25} - \frac{1}{273.15+s_l}\right)}$$

for $l = 12$. The transformed times can be calculated by $\tilde{t}_{lj} = \hat{\beta}_l\, t_j$ for $l = 12$ and $j = 1, \ldots, k$, and the transformed termination times are $\tilde{\alpha}_l = \hat{\beta}_l\, \alpha$ for $l = 12$. Then, we obtain $\hat{\alpha}_1 = 196.15$ and $\hat{\alpha}_2 = 331.49$ at the stress levels $s_1 = 65\ ^\circ$C and $s_2 = 105\ ^\circ$C, respectively. Using (16) and (17), we obtain the TSMLEs of ξ and σ as $\hat{\xi}_{2S} = 0.001186$ and $\hat{\sigma}_{2S} = 0.005914$.

Comparing the results obtained from the ML and TSML methods, we have $|\hat{\xi}_M - \hat{\xi}_{2S}| = 3 \times 10^{-6}$, $|\hat{\sigma}_M - \hat{\sigma}_{2S}| = 4.4 \times 10^{-5}$, and $|\hat{\theta}_M - \hat{\theta}_{2S}| = 3 \times 10^{-3}$. These absolute differences indicate that the estimates using the two methods are close. However, the MLEs are lacking the explicit expressions and calculating the MLEs relies on a numerical method. By contrast, the TSML method can provide the explicit forms for estimating the parameters. Hence, our method can provide a more efficient and direct approach for estimating the parameters without using a numerical method.

In addition, evaluating the traditional MLE based on the numerical computation requires a suitable initial value. However, a suitable initial value is not always given, and as a result, the computation result is likely divergent or the local maximum cannot be attained. Therefore, in this section, we set the estimates of the TSMLEs as the initial values for evaluating the traditional MLEs. Moreover, the roots of (10) are difficult to obtain. Therefore, the explicit forms of the TSMLEs are not only a stable and trustworthy estimation but also can provide a suitable initial value for computing the traditional MLEs.

VI. Conclusion

To provide a tractable method to estimate the parameters of the Wiener process under a PCSADT for highly reliable products, a TSML estimation procedure was proposed in this article. The step-by-step estimation procedure was introduced, and the asymptotical properties of the proposed TSML estimators were analytically derived.

In the first stage of the proposed method, we obtained estimates of the acceleration factor for each stress level. These estimated acceleration factors were used to transform the original data collected from the PCSADT. In the second stage, the transformed data from the first stage were used to estimate the drift parameter and diffusion parameter of the Wiener process. Moreover, a linear combination estimator was proposed to estimate the accelerated parameter or the nuisance parameter. The proposed TSML estimators were close-formed and equipped with consistency. The asymptotical normality of all of the TSML estimators was also analytically derived. In addition, using the numerical method to evaluate the traditional MLE required a suitable initial value. However, unsuitable initial values often cause the computation results to diverge or fail to attain the local maxima. Contrasting the traditional MLE method, our proposed TSML method can provide a stable and trustworthy estimation. Furthermore, our method can provide a suitable initial value for evaluating the typical MLEs via the computation method. Hence, the TSML estimation procedure can be an auxiliary method for finding reliable MLEs. Moreover, the TSML estimation procedure is simple for implementation without working on an iteration process.

An intensive simulation study was conducted to evaluate the performance of the proposed estimation method. Simulation results show that the TSML estimators perform well for estimating the parameters. Since the typical ML estimators and their corresponding Fisher information matrix were not close-formed, constructing an optimal ADT plan relies on algorithms and complicated numerical methods. Our proposed method, which provides explicit forms, was more tractable and improved the PCSADT design efficiently. The LED example was presented to show the application of the proposed TSML estimation method.

One possible direction of future research is to extend our proposed TSML method to step-stress ADTs. The proposed estimation procedure can be applied to other popular stochastic models such as the gamma process and inverse Gaussian process. Based on these models, improving the PCSADT design is also an important issue that merits future study. Another possible direction of the future research is to extend our method to these popular models with taking the failure time into consideration.

Appendix A
Proof of Lemma 1

Since the MLE of the drift parameter of Brownian motion is consistent, it is clear that $\hat{\xi}_l$ is a consistent estimator of ξ_l. Then, by Slutsky's theorem, we obtain that $\hat{\beta}_l$ is a consistent estimator of β_l. Furthermore, since β is a monotonic and continuous function of θ, the consistency of $\hat{\theta}_l$ and $\hat{\theta}_{2S}$ can be clarified. $\hat{\sigma}_l^2$ can be rewritten as

$$\hat{\sigma}_l^2 = \frac{1}{n_l k} \sum_{i=1}^{n_l} \sum_{j=1}^{k} \frac{\left(U_{lij} - \xi_l \Delta t_j + \xi_l \Delta t_j - \hat{\xi}_l \Delta t_j\right)^2}{\hat{\beta}_l \Delta t_j}.$$

Hence, the consistency of $\hat{\sigma}_l^2$ can be verified directly via the following properties:

1)
$$\frac{1}{n_l k}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{(U_{lij}-\xi_l\Delta t_j)^2}{\hat{\beta}_l\Delta t_j}\xrightarrow{p}\sigma^2 \text{ as } n_l\to\infty \quad \text{(A1)}$$

2)
$$\frac{2}{n_l k}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{(U_{lij}-\xi_l\Delta t_j)\left(\xi_l\Delta t_i-\hat{\xi}_l\Delta t_j\right)}{\hat{\beta}_l\Delta t_j}\xrightarrow{p}0 \text{ as } n_l\to\infty \quad \text{(A2)}$$

3)
$$\frac{1}{n_l k}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(\xi_l\Delta t_j-\hat{\xi}_l\Delta t_j\right)^2}{\hat{\beta}_l\Delta t_j}\xrightarrow{p}\sigma^2 \text{ as } n_l\to\infty. \quad \text{(A3)}$$

Appendix B
Proof of Proposition 1

It is obvious that

$$\hat{\xi}_{2S}=\sum_{l=1}^{m}\left[\left(\frac{\sum_{i=1}^{n_l}\sum_{j=1}^{k}U_{lij}}{n_l\hat{\beta}_l\alpha}-\xi\right)\frac{n_l\hat{\beta}_l\alpha}{\sum_{l=1}^{m}n_l\hat{\beta}_l\alpha}\right]+\xi.$$

According to the following:

$$\frac{\sum_{i=1}^{n_l}\sum_{j=1}^{k}U_{lij}}{n_l\hat{\beta}_l\alpha}-\xi\xrightarrow{p}0 \text{ and } \frac{\sum_{l=1}^{m}\left|n_l\hat{\beta}_l\alpha\right|}{\left|\sum_{l=1}^{m}n_l\hat{\beta}_l\alpha\right|}\xrightarrow{p}1 \text{ as } n_l\to\infty$$

$\hat{\xi}_{2S}$ can be shown to be a consistent estimator of ξ using Slutsky's theorem.

Obviously, $\hat{\sigma}_{2S}^2$ can be rewritten as

$$\hat{\sigma}_{2S}^2=\left\{\frac{1}{Nk}\sum_{l=1}^{h}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij}-\hat{\xi}_l\Delta\tilde{t}_{lj}\right)^2}{\Delta\tilde{t}_{lj}}+\sum_{l=1}^{h}2\left(\hat{\xi}_l-\hat{\xi}_{2S}\right)\sum_{i=1}^{n_l}\sum_{j=1}^{k}\left(U_{lij}-\hat{\xi}_l\Delta\tilde{t}_{lj}\right)+\sum_{l=1}^{h}\left(\hat{\xi}_l-\hat{\xi}_{2S}\right)^2\sum_{i=1}^{n_l}\sum_{j=1}^{k}\Delta\tilde{t}_{lj}\right\}.$$

The following three properties:

1)
$$\frac{1}{Nk}\sum_{l=1}^{h}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij}-\hat{\xi}_l\Delta\tilde{t}_{lj}\right)^2}{\Delta\tilde{t}_{lj}}\xrightarrow{p}\sigma^2 \text{ as } n_l\to\infty,$$

2)
$$\frac{1}{Nk}\sum_{l=1}^{h}2\left(\hat{\xi}_l-\hat{\xi}_{2S}\right)\times\sum_{i=1}^{n_l}\sum_{j=1}^{k}\left(U_{lij}-\hat{\xi}_l\Delta\tilde{t}_{lj}\right)\xrightarrow{p}0 \text{ as } n_l\to\infty,$$

3)
$$\frac{1}{Nk}\sum_{l=1}^{h}\left(\hat{\xi}_l-\hat{\xi}_{2S}\right)^2\sum_{i=1}^{n_l}\sum_{j=1}^{k}\Delta\tilde{t}_{lj}\xrightarrow{p}0 \text{ as } n_l\to\infty,$$

can be verified by (A1)–(A3), Lemma 1 and the consistency of $\hat{\xi}_{2S}$. Therefore, we can verify that $\hat{\sigma}_{2S}^2$ is a consistent estimator of σ^2 using the above properties.

Appendix C
Asymptotic Normality

The asymptotic normality of the TSML estimators is discussed in this appendix. In practice, all sample sizes n_l, $l=01,\ldots,h$ are usually selected to be "nearly" equal in an ADT. Hence, it is reasonable to assume that $n_l/n_0\to m_l$ for some $m_l\in\mathbb{R}^+$ as $(n_0,n_l)\to(\infty,\infty)$, i.e., $n_l=O(n_0)$.

First, we discuss the sampling distribution of the first-stage estimators $\hat{\theta}_l$ and $\hat{\beta}_l$. Since W_{lij} has a normal distribution with mean $\alpha\xi_l$ and variance $\alpha\beta_l\sigma^2$, the MLE $\hat{\xi}_l$ follows a normal distribution with mean ξ_l and variance $\sigma^2\beta_l/(n\alpha)$ for $i=1,\ldots,n_l,l=1,\ldots,h$. Pham-Gia *et al.* [21] have established a closed-form expression for the general bivariate normal case by the Kummer hypergeometric function. $X\sim N(\mu_X,\sigma_X^2)$ and $Y\sim N(\mu_Y,\sigma_Y^2)$ are assumed to be two independent normally distributed random variables. The probability density function of the ratio variable $Z=X/Y$ is given by

$$P_Z\ (z)=\frac{b\,(z)\,c\,(z)}{d^3\,(z)}\frac{1}{\sqrt{2\pi\sigma_X\sigma_Y}}\left[2\Phi\left(\frac{b\,(z)}{d\,(z)}\right)-1\right]+\frac{1}{\pi\sigma_X\sigma_Y d^2\,(z)}\exp\left\{-\frac{1}{2}\left(\frac{\mu_X^2}{\sigma_X^2}+\frac{\mu_Y^2}{\sigma_Y^2}\right)\right\} \quad \text{(A4)}$$

where $b\ (z)=\frac{\mu_X}{\sigma_X^2}\ z+\frac{\mu_Y}{\sigma_Y^2}$, $c\ (z)=\exp\{\frac{b^2(z)}{2a^2(z)}-\frac{1}{2}(\frac{\mu_X^2}{\sigma_X^2}+\frac{\mu_Y^2}{\sigma_Y^2})\}$, $d\ (z)=\sqrt{\frac{1}{\sigma_X^2}z^2+\frac{1}{\sigma_Y^2}}$, and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Assuming that $\mu_Y>>\sigma_Y$ or $\mu_Y>5\sigma_Y$, Hayya *et al.* [6] used the Geary–Hinkley transformation to rewrite (A4) as

$$g_z(z,\mu_X,\sigma_X^2,\mu_Y,\sigma_Y^2)=\frac{\mu_X}{\sqrt{2\pi\sigma_Y^2}}\frac{\left(z+\frac{\mu_Y\sigma_X^2}{\mu_X\sigma_Y^2}\right)}{(z^2+\frac{\sigma_X^2}{\sigma_Y^2})^{\frac{3}{2}}}\exp\left\{-\frac{\left(z-\frac{\mu_X}{\mu_Y}\right)^2}{2\frac{\sigma_Y^2}{\mu_Y^2}\left(z^2+\frac{\sigma_X^2}{\sigma_Y^2}\right)}\right\} \quad \text{(A5)}$$

which is approximately a standard normal distribution.

Under the normal use condition s_0, $\hat{\xi}_0$ follows a normal distribution with mean ξ and variance $\sigma^2/(n\alpha)$. In an ADT, the assumption that $\sigma<\xi<0.1$ if α is large is reasonable. It can be shown that the ratio $\sqrt{n}\alpha\ \xi/\sigma>5$ if $10\le n\le 30$. The distribution of $\hat{\beta}_l=\hat{\xi}_l/\hat{\xi}_0$ can be expressed by

$$g_z\left(z,\xi,\frac{\sigma^2}{n_0\alpha},\xi_l,\frac{\beta_l\sigma^2}{n_1\alpha}\right)=\frac{\xi}{\sqrt{\frac{2\pi\beta_l\sigma^2}{n_1\alpha}}}\frac{\left(z+\frac{n_1}{n_0}\right)}{(z^2+\frac{n_1}{n_0\beta_l})^{\frac{3}{2}}}\times\exp\left\{-\frac{n_1\alpha\xi^2\left(z-\frac{1}{\beta_l}\right)^2}{2\sigma^2\left(z^2+\frac{n_1}{n_0\beta_l}\right)}\right\}$$

using (A5). Then, the distribution of $\hat{\theta}_l$ can be obtained as follows:

$$f_Z(z, \xi, \sigma, \beta_l, \beta(z)) = g_Z\left(\beta(z), \xi, \frac{\sigma^2}{n\alpha}, \xi_l, \frac{\beta_l \sigma^2}{n\alpha}\right) \times \left(\frac{\partial}{\partial z}\beta(z)\right)^{-1}.$$

However, the density function of $\hat{\theta}_l$ is extremely complicated, so it is difficult to derive the explicit form of the density function of $\hat{\theta}_{2S}$. The asymptotic normality can provide more elegant limiting distributions for the estimators. In the following proposition, we state the asymptotic normality of $\hat{\beta}_l$, and the proof is shown in Appendix D.

Proposition C.1: Let $n_l = O(n_0)$ for $l = 12, \ldots, h$. Then, $\sqrt{N}(\hat{\beta}_l - \beta_l) \xrightarrow{d} N(0,\ \sigma^2_{\hat{\beta}_l})$ as $n_0 \to \infty$, where $\sigma^2_{\hat{\beta}_l} = (\sum_{j=1}^{h} m_j)\ [\frac{\sigma^2}{\alpha\xi^2}(\beta_l^2 + \frac{\beta_l}{m_l})]$.

We proceed to clarify the asymptotic normality of $\hat{\theta}_{2S}$. Assume that the inverse of $\beta(\cdot)$ is differentiable. Define $g_\theta(\xi_0, \xi_1, \ldots, \xi_h) = \sum_{l=1}^{h} a_l \beta^{-1}(\xi_l/\xi_0)$. Then, the gradient of g_θ is

$$\nabla g_\theta = \left(-\sum_{l=1}^{h} a_l \frac{\xi_l}{\xi_0}\left[\beta^{-1}\left(\frac{\xi_l}{\xi_0}\right)\right]', \frac{a_1}{\xi_0}\left[\beta^{-1}\left(\frac{\xi_1}{\xi_0}\right)\right]', \ldots, \frac{a_h}{\xi_0}\left[\beta^{-1}\left(\frac{\xi_h}{\xi_0}\right)\right]'\right).$$

Let

$$\Sigma_\theta = \begin{pmatrix} \frac{1}{\alpha}\sigma^2 & 0 & \ldots & 0 \\ 0 & \frac{\sigma^2\beta_1}{\alpha m_1} & & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{\sigma^2\beta_h}{\alpha m_h} \end{pmatrix}.$$

Using the multivariate delta method and $n_l = O(n_0)$ for $l = 12, \ldots, h$, the following is obtained:

$$\sqrt{n_0}\left(\hat{\theta}_{2S} - \theta\right) \xrightarrow{d} N\left(0, \nabla g_\theta(u_{\xi,\beta})\,\Sigma_\theta[\nabla g_\theta(u_{\xi,\beta})]^T\right)$$

as $n_0 \to \infty$

where $u_{\xi,\beta} = (\xi, \beta_1\xi, \ldots, \beta_h\xi)$ and

$$\nabla g_\theta(u_{\xi,\beta}) \sum_\theta [\nabla g_\theta(u_{\xi,\beta})]^T = \frac{\sigma^2}{\alpha\xi^2}\left\{\left(\sum_{l=1}^{h} a_l (\beta^{-1})'(\beta_l)\right)^2 + \sum_{l=1}^{h}\left[a_l(\beta^{-1})'(\beta_l)\right]^2\left(\frac{\beta_l}{m_l}\right)\right\}.$$

Using the expression $\sqrt{N}\ (\hat{\theta}_{2S} - \theta) = \sqrt{\frac{N}{n_0}}\ \sqrt{n_0}(\hat{\theta}_{2S} - \theta)$ and Slutsky's theorem, we can verify that

$$\sqrt{N}\left(\hat{\theta}_{2S} - \theta\right) \xrightarrow{d} \sqrt{\sum_{l=1}^{h} m_l}\, N\left(0, \left(\nabla g_\theta(u_{\xi,\beta}) \sum_\theta [\nabla g_\theta(u_{\xi,\beta})]^T\right)\right)$$

as $n_0 \to \infty$. Therefore, we clarify the asymptotic normality of the second-stage estimator $\hat{\theta}_{2S}$ and summarize the result in Proposition C.2.

Proposition C.2: Assume that the inverse function of $\beta(\cdot)$ is differentiable and $n_l = O(n_0)$ for $l = 12, \ldots, h$. Then, $\sqrt{N}(\hat{\theta}_{2S} - \theta) \xrightarrow{d} N(0, \sigma^2_{\hat{\theta}_{2S}})$ as $n_0 \to \infty$, where

$$\sigma^2_{\hat{\theta}_{2S}} = \left(\sum_{l=1}^{h} m_l\right) \times (\nabla g_\theta(u_{\xi,\beta}) \sum_\theta [\nabla g_\theta(u_{\xi,\beta})]^T$$

and

$$[\nabla g_\theta(u_{\xi,\beta})] \sum_\theta [\nabla g_\theta(u_{\xi,\beta})]^T = \frac{\sigma^2}{\alpha\xi^2}\left\{\left(\sum_{l=1}^{h} a_l(\beta^{-1})'(\beta_l)\right)^2 + \sum_{l=1}^{h}\left[a_l(\beta^{-1})'(\beta_l)\right]^2\left(\frac{\beta_l}{m_l}\right)\right\}.$$

Before deriving the asymptotic normality of $\hat{\sigma}^2_{2S}$, we define

$$\mathrm{g}_{\sigma^2}(x_0, x_1, y_1, \ldots, x_h, y_h) = \left(\sum_{l=1}^{h} m_l\right)^{-1} \sum_{l=1}^{h}\left(\frac{m_l\beta_l y_l x_0}{x_l} - \sigma^2\right).$$

The gradient vector of g_{σ^2} is

$$\nabla \mathrm{g}_{\sigma^2} = \left(\sum_{l=1}^{h} m_l\right)^{-1}\left(\sum_{l=1}^{h}\frac{m_l\beta_l y_l}{x_l}, -\frac{m_1\beta_1 y_1}{x_1^2}, \frac{m_1\beta_1 x_0}{x_1}, \ldots, -\frac{m_h\beta_h y_h}{x_h^2}, \frac{m_h\beta_h x_0}{x_h}\right).$$

We verify the asymptotic normality of the second-stage estimators $\hat{\xi}_{2S}$ and $\hat{\sigma}^2_{2S}$ in the following propositions.

Proposition C.3. Let $n_l = O(n_0)$ for $l = 12, \ldots, h$. Then, $\sqrt{N}(\hat{\xi}_{2\mathrm{S}} - \xi) \xrightarrow{d} N(0, (\sum_{l=1}^{h} m_l)\sigma^2)$ as $n_0 \to \infty$.

Proof: See Appendix E.

Proposition C.4. Let $n_l = O(n_0)$ for $l = 12, \ldots, h$. Then,

$$\sqrt{N}\left(\hat{\sigma}^2_{2S} - \sigma^2\right) \xrightarrow{d} N\left(0, \left(\sum_{l=1}^{h} m_l\right)[\nabla g_{\sigma^2}(u_{\sigma^2})] \sum_{\sigma^2}[\nabla g_{\sigma^2}(u_{\sigma^2})]^T\right) \text{ as } n_0 \to \infty,$$

where

$$\left(\sum_{\ell=1}^{h} m_\ell\right)[\nabla g_{\sigma^2}(u_{\sigma^2})] \sum_{\sigma^2}[\nabla g_{\sigma^2}(u_{\sigma^2})]^T = \frac{\sigma^6}{\xi^2\alpha}\sum_{\ell=1}^{h} m_\ell + \frac{\sigma^6}{\xi^2\alpha}\left[\sum_{\ell=1}^{h}\frac{m_\ell}{\beta_\ell}\right]\left[\sum_{\ell=1}^{h} m_\ell\right]^{-1} + \frac{2\sigma^4}{k}.$$

Proof. See Appendix F.

APPENDIX D
PROOF OF PROPOSITION C.1

Define $\bar{X}_{n_l}^{(l)} = \sum_{i=1}^{n_l} \sum_{j=1}^{k} U_{lij} \ /(n_l \alpha)$ for $l = 01, \ldots, h$. Then, the following properties are obtained:

1)
$$\sqrt{n_l}\left(\bar{X}_{n_l}^{(l)} - \xi_l\right) \sim N\left(0, \sigma^2 \beta_l / \alpha\right).$$

2)
$$\sqrt{n_0}(\bar{X}_{n_0}^{(0)} - \xi) \sim N\left(0, \sigma^2/\alpha\right). \tag{A6}$$

3)
For each l, $\sqrt{n_0}\left(\bar{X}_{n_l}^{(l)} - \xi_l\right) \rightarrow N\left(0, \sigma^2 \beta_l / \left(\alpha m_l\right)\right)$

as $n_0 \rightarrow \infty$ and $n_l \rightarrow \infty$. (A7)

Furthermore, $\hat{\beta}_l$ can be rewritten as $\hat{\beta}_l = \bar{X}_{n_l}^{(l)} \ / \bar{X}_{n_0}^{(0)}$.

Let $\Sigma_\beta = \begin{pmatrix} \frac{\sigma^2}{\alpha} & 0 \\ 0 & \frac{\sigma^2 \beta_l}{\alpha m_l} \end{pmatrix}$. The gradient of $\beta_l = \xi_l \ / \xi_0$ is $\nabla \ \beta_l = (-\xi_l / \xi_0^2, 1/\xi_0)$.

By the multivariate delta method and the properties (A6) and (A7), we obtain

$$\sqrt{n_0}\left(\hat{\beta}_l - \beta_l\right) \xrightarrow{d} N\left(0, [\nabla \beta_l] \, \Sigma_\beta [\nabla \beta_l]^T\right) \text{ as } n_0 \rightarrow \infty.$$

Via Slutsky's theorem, we obtain

$$\sqrt{n_l}\left(\hat{\beta}_l - \beta_l\right) \xrightarrow{d} \sqrt{m_l} N \ \left(0, [\nabla \beta_l] \, \Sigma_\beta [\nabla \beta_l]^T\right)$$
$$= \ N\left(0, \frac{\sigma^2}{\alpha \xi^2}\left(m_l \beta_l^2 + \beta_l\right)\right)$$

as $n_0 \rightarrow \infty$, and

$$\sqrt{N}\left(\hat{\beta}_l - \beta_l\right) \xrightarrow{d} N\left(0, \left(\sum_{l=1}^{h} m_l\right)\left[\frac{\sigma^2}{\alpha \xi^2}\left(\beta_l^2 + \frac{\beta_l}{m_l}\right)\right]\right)$$

as $n_0 \rightarrow \infty$. Proposition C.1 is proved. □

APPENDIX E
PROOF OF PROPOSITION C.2

After a simple algebraic computation, we obtain $\hat{\xi}_{2S} - \xi = (\bar{X}_{n_0}^{(0)} - \xi)$ and the following property:

$$\sqrt{N}\left(\hat{\xi}_{2S} - \xi\right) \xrightarrow{d} \sqrt{\sum_{l=1}^{h} m_l} N\left(0, \frac{1}{\alpha}\sigma^2\right) \text{ as } n_0 \rightarrow \infty$$

can be verified using Slutsky's theorem. □

APPENDIX F
PROOF OF PROPOSITION C.3

It can be shown that

$$\hat{\sigma}_{2S}^2 - \ \sigma^2 = \frac{1}{Nk} \sum_{l=1}^{h} (n_l k)$$
$$\times \left[\frac{1}{n_l k} \sum_{i=1}^{n_l} \sum_{j=1}^{k} \frac{\left(U_{lij} - \ \hat{\xi}_{2S} \Delta t_j\right)^2}{\Delta \tilde{t}_{lj}} - \sigma^2\right]$$

and

$$\frac{1}{n_l k} \sum_{i\,=\,1}^{n_l} \sum_{j\,=\,1}^{k} \frac{\left(U_{lij} - \hat{\xi}_l \Delta t_j\right)^2}{\Delta \tilde{t}_{lj}}$$
$$= \frac{1}{n_l k} \sum_{i\,=\,1}^{n_l} \sum_{j\,=\,1}^{k} \frac{(U_{lij} - \xi_l \beta_l \Delta t_j)^2}{\hat{\beta}_l \Delta t_j} - \frac{\left(\bar{X}_{n_l}^{(l)} - \xi \beta_l\right)^2 \alpha}{\hat{\beta}_l k}. \tag{A8}$$

Based on Lemma 1, the second term of (A8) is $o_p = \ (n_0^{-1/2})$. This property can be clarified based on Lemma 1, (A5), and the fact that $\bar{\mathrm{X}}_{n_l}^{(l)}$ is a consistent estimator of $\xi \beta_l$. Therefore, we also have

$$\frac{\sqrt{\sum_{l=1}^{h} n_l}}{Nk} \sum_{l=1}^{h} (n_l k) \left[-\frac{\left(\bar{\mathrm{X}}_{n_l}^{(l)} - \xi \beta_l\right)^2 \alpha}{\tilde{\beta}_l k}\right] = \mathrm{o}_p \ \left(n_0^{-\frac{1}{2}}\right).$$

Define $\mathrm{X}_{n_l,i}^{(l)} = \frac{1}{\alpha} \sum_{j=1}^{k} U_{lij}$ and $\mathrm{Y}_{n_l,i}^{(l)} = \frac{\sigma^2}{k} \ \sum_{j=1}^{k} \frac{(U_{lij} - \xi_l \beta_l \Delta t_j)^2}{\sigma^2 \beta_l \Delta t_j}$ for $1 \leq i \leq n_l$ and $l = 12, \ldots, h$. Then, it is obvious that $\mathrm{E}\ (\mathrm{X}_{n_l,i}^{(l)}) = \ \xi \beta_l$, $\mathrm{Var}\ (\ \mathrm{X}_{n_l,i}^{(l)}) = \frac{\sigma^2 \beta_l}{\alpha}$, $\mathrm{E}\ (\mathrm{Y}_{n_l,i}^{(l)}) = \sigma^2$, $\mathrm{Var}\ (\ \mathrm{Y}_{n_l,i}^{(l)}) = \frac{2\sigma^4}{k}$ and $\mathrm{Cov}\ (\mathrm{X}_{n_l,i}^{(l)}, \mathrm{Y}_{n_l,i}^{(l)}) = 0$. Using the Central Limit Theorem and the multivariate Slutsky's theorem, we obtain, as n_0 approaches infinity

$$\sqrt{n_0}\left(\bar{\mathrm{X}}_{n_0}^{(0)} - \xi, \bar{\mathrm{X}}_{n_1}^{(1)} - \beta_1 \xi, \bar{\mathrm{Y}}_{n_1}^{(1)} - \sigma^2, \ldots,\right.$$
$$\left.\bar{\mathrm{X}}_{n_h}^{(h)} - \beta_h \xi, \bar{\mathrm{Y}}_{n_h}^{(h)} - \sigma^2\right) \rightarrow MN\left(0, \Sigma_{\sigma^2}\ \right)$$

where

$$\Sigma_{\sigma^2} = \begin{pmatrix} \frac{\sigma^2}{\alpha} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{\sigma^2 \beta_1}{m_1 \alpha} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{2\sigma^4}{m_1 k} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\sigma^2 \beta_m}{m_h \alpha} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{2\sigma^4}{m_h k} \end{pmatrix}.$$

Let

$$\mathrm{g}_{\sigma^2} \ (x_0, x_1, y_1, \ldots, x_h, y_h)$$
$$= \left(\sum_{l=1}^{h} m_l\right)^{-1} \sum_{l=1}^{h} \left(\frac{m_l \beta_l y_l x_0}{x_l} - \sigma^2\right).$$

Then, the gradient vector of g_{σ^2} is

$$\nabla \mathrm{g}_{\sigma^2} = \left(\sum_{l=1}^{h} m_l\right)^{-1}$$
$$\times \left(\sum_{l=1}^{h} \frac{m_l \beta_l y_l}{x_l}, -\frac{m_1 \beta_1 y_1}{x_1^2}, \frac{m_1 \beta_1 x_0}{x_1}, \ldots,\right.$$
$$\left.-\frac{m_h \beta_h y_h}{x_h^2}, \frac{m_h \beta_h x_0}{x_h}\right).$$

Using the multivariate delta method, we have

$$\sqrt{n_0}\left(\sum_{l=1}^{h} m_l\right)^{-1}$$
$$\sum_{l=1}^{h} m_l \left(\frac{1}{n_l k}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij}-\xi\beta_l\Delta t_j\right)^2}{\hat{\beta}_l\Delta t_j}-\sigma^2\right)$$
$$\rightarrow N\left(0, \nabla \mathrm{g}_{\sigma^2}\left(u_{\sigma^2}\right)\Sigma_{\sigma^2}\left[\nabla \mathrm{g}_{\sigma^2}\left(u_{\sigma^2}\right)\right]^T\right)$$

in distribution as $(n_0,\ldots,n_h)\rightarrow(\infty,\ldots,\infty)$, where $u_{\sigma^2}=(\xi,\beta_1\xi,\sigma^2,\beta_2\xi,\sigma^2,\ldots,\beta_h\xi,\sigma^2)$.

Therefore, via Slutsky's theorem, we obtain

$$\frac{\sqrt{n_0}}{Nk}\sum_{l=1}^{h}(n_l k)\left[\frac{1}{n_l k}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij}-\xi\beta_l\Delta t_j\right)^2}{\hat{\beta}_l\Delta t_j}-\sigma^2\right]$$
$$-\frac{\sqrt{n_0}}{\sum_{l=1}^{h} m_l}\sum_{l=1}^{h} m_l\left(\frac{1}{n_l k}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij}-\xi\beta_l\Delta t_j\right)^2}{\hat{\beta}_l\Delta t_j}-\sigma^2\right)$$
$$=\sum_{l=1}^{h}\left(\frac{n_l k}{\sum_{l=1}^{h} n_l k}-\frac{m_l}{\sum_{l=1}^{h} m_\ell}\right)\sqrt{n_0}$$
$$\times\left[\frac{1}{n_l k}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij}-\xi\beta_l\Delta t_j\right)^2}{\hat{\beta}_l\Delta t_j}-\sigma^2\right]\rightarrow 0$$

in distribution as $(n_0,\ldots,n_\mathrm{h})\rightarrow(\infty,\ldots,\infty)$. Consequently, we obtain

$$\frac{\sqrt{\sum_{l=1}^{h} n_l}}{Nk}\sum_{l=1}^{h}(n_l k)\left[\frac{1}{n_l k}\sum_{i=1}^{n_l}\sum_{j=1}^{k}\frac{\left(U_{lij}-\xi\beta_l\Delta t_j\right)^2}{\hat{\beta}_l\Delta t_j}-\sigma^2\right]$$
$$\xrightarrow{d} N\left(0,\left(\sum_{l=1}^{h} m_l\right)\nabla \mathrm{g}_{\sigma^2}\left(u_{\sigma^2}\right)\Sigma_{\sigma^2}\left[\nabla \mathrm{g}_{\sigma^2}\left(u_{\sigma^2}\right)\right]^T\right).$$

Thus, we can conclude that

$$\sqrt{\sum_{l=1}^{h} n_l}\left(\hat{\sigma}_{2S}^2-\sigma^2\right)$$
$$\rightarrow N\left(0,\left(\sum_{l=1}^{h} m_l\right)\nabla \mathrm{g}_{\sigma^2}\left(u_{\sigma^2}\right)\Sigma_{\sigma^2}\left[\nabla \mathrm{g}_{\sigma^2}\left(u_{\sigma^2}\right)\right]^T\right)$$

in distribution as $(n_0,\ldots,n_\mathrm{h})\rightarrow(\infty,\ldots,\infty)$, where

$$\left(\sum_{\ell=1}^{h} m_l\right)\nabla \mathrm{g}_{\sigma^2}\left(u_{\sigma^2}\right)\Sigma_{\sigma^2}\left[\nabla \mathrm{g}_{\sigma^2}\left(u_{\sigma^2}\right)\right]^T$$
$$=\frac{\sigma^6}{\xi^2\alpha}\sum_{l=1}^{h} m_l+\frac{\sigma^6}{\xi^2\alpha}\left[\sum_{l=1}^{h}\frac{m_l}{\beta_l}\right]\left[\sum_{l=1}^{h} m_l\right]^{-1}+\frac{2\sigma^4}{k}. \quad \square$$

ACKNOWLEDGMENT

The author would like to thank the Associate Editor and four anonymous referees for their many helpful suggestions, which improved this article.

REFERENCES

[1] J.-Y. Chiang, Y. L. Lio, and T.-R. Tsai, "Degradation tests using geometric Brownian motion process for lumen degradation data," *Qual. Rel. Eng. Int.*, vol. 31, no. 8, pp. 1797–1806, 2015.
[2] K. A. Doksum and S.-L. T. Normand, "Gaussian models for degradation processes, part I: Methods for the analysis of biomarker data," *Lifetime Data Anal.*, vol. 1, no. 2, pp. 131–144, 1995.
[3] K. A. Doksum and A. Hoyland, "Models for variable-stress accelerated life testing experiments based on Wiener processes and the inverse Gaussian distribution," *Technometrics*, vol. 34, no. 1, pp. 74–82, 1992.
[4] E. A. Elsayed, "Overview of reliability testing," *IEEE Trans. Rel.*, vol. 61, no. 2, pp. 282–291, Jun. 2012.
[5] J. Huang *et al.*, "Degradation modeling of mid-power white-light LEDs by using Wiener process," *Opt. Exp.*, vol. 23, no. 15, pp. 966–978, 2015.
[6] J. Hayya, D. Armstrong, and N. Gressis, "A note on the ratio of two normally distributed variables," *Manage. Sci.*, vol. 21, no. 11, pp. 1338–1341, 1975.
[7] C.-H. L. Hu, M.-Y. Lee, and J. Tang, "Optimal step-stress accelerated degradation test for Wiener degradation process under constraints," *Eur. J. Oper. Res.*, vol. 241, no. 2, pp. 412–421, 2015.
[8] Y. Komori, "Properties of the Weibull cumulative exposure model," *J. Appl. Statist.*, vol. 33, pp. 17–34, 2006.
[9] E. L. Lehmann, *Elements of Large-Sample Theory*. New York, NY, USA: Springer-Verlag, 1999.
[10] H. Liao and E. A. Elsayed, "Reliability inference for field conditions from accelerated degradation testing," *Nav. Res. Logistics*, vol. 53, no. 6, pp. 576–587, 2006.
[11] C. M. Liao and S. T. Tseng, "Optimal design for step-stress accelerated degradation tests," *IEEE Trans. Rel.*, vol. 55, no. 1, pp. 59–66, Mar. 2006.
[12] M.-L. Lee, C.-H. Hu, and J. Tang, "A two-stage latent variable estimation procedure for time-censored accelerated degradation tests," *IEEE Trans. Rel.*, vol. 66, no. 4, pp. 1266–1279, Dec. 2017.
[13] H. Lim and B.-J. Yum, "Optimal design of accelerated degradation tests based on Weiner process models," *J. Appl. Statist.*, vol. 38, no. 2, pp. 309–325, 2011.
[14] W. Q. Meeker and L. A. Escobar, *Statistical Methods for Reliability Data*. Hoboken, NJ, USA: Wiley, 1998.
[15] R. Miller and B. Nelson, "Optimal simple step stress plans for accelerating life testing," *IEEE Trans. Rel.*, vol. R-32, no. 1, pp. 59–65, Apr. 1983.
[16] W. Nelson, *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis.* Hoboken, NJ, USA: Wiley, 2004.
[17] W. Nelson, "A bibliography of accelerated test plans," *IEEE Trans. Rel.*, vol. 54, no. 2, pp. 194–197, Jun. 2005.
[18] W. Nelson, "A bibliography of accelerated test plans part II – references," *IEEE Trans. Rel.*, vol. 54, no. 3, pp. 370–373, Sep. 2005.
[19] W. J. Padgett and M. A. Tomlinson, "Inference from accelerated degradation and failure data based on Gaussian process models," *Lifetime Data Anal.*, vol. 10, no. 2, pp. 191–206, 2004.
[20] C. Park and W. J. Padgett, "Accelerated degradation models for failure based on geometric Brownian motion and gamma process," *Lifetime Data Anal.*, vol. 11, no. 4, pp. 511–527, 2005.
[21] T. Pham-Gia, T. N. Turkkan, and E. Marchand, "Density of the ratio of two normal random variables and applications," *Commun. Statist.—Theory Methods*, vol. 35, no. 9, pp. 1569–1591, 2006.
[22] L. C. Tang, G. Yang, and M. Xie, "Planning step-stress accelerated degradation tests," in *Proc. IEEE Annu. Rel. Maintainability Symp.*, 2004, pp. 287–292.
[23] T.-R. Tsai, C.-W. Lin, Y.-L. Sung, P.-T. Chou, C.-L. Chen, and Y. L. Lio, "Inference from lumen degradation data under Wiener diffusion process," *IEEE Trans. Rel.*, vol. 61, no. 3, pp. 710–718, Sep. 2012.
[24] T.-R. Tsai, W.-Y. Sung, Y.-L. Lio, S. I. Chang, and J.-C. Lu, "Optimal two-variable accelerated degradation test plan for gamma degradation process," *IEEE Trans. Rel.*, vol. 65, no. 1, pp. 459–468, Mar. 2016.
[25] C. C. Tsai, C. T. Lin, and N. Balakrishnan, "Optimal design for accelerated-stress acceptance test based on Wiener process," *IEEE Trans. Rel.*, vol. 64, no. 2, pp. 603–612, Jun. 2015.

[26] S. T. Tseng, N. Balakrishnan, and C. C. Tsai, "Optimal step-stress accelerated degradation test plan for gamma degradation processes," *IEEE Trans. Rel.*, vol. 58, no. 4, pp. 611–618, Dec. 2009.
[27] S. T. Tseng, J. Tang, and I. H. Ku, "Determination of optimal burn-in parameters and residual life for highly reliable products," *Nav. Res. Logistics*, vol. 50, no. 1, pp. 1–14, 2003.
[28] S. T. Tseng and Z. C. Wen, "Step-stress accelerated degradation analysis for highly reliable products," *J. Qual. Technol.*, vol. 32, pp. 209–216, 2000.
[29] J. Wang and X. Ma, "Study on LED device step-stress accelerated degradation test," *Proc. 14th Int. Conf. Electron. Packag. Technol*, 2013, pp. 1162–1165.
[30] G. A. Whitmore and F. Schenkelberg, "Modeling accelerated degradation data using Wiener diffusion with a time scale transformation," *Lifetime Data Anal.*, vol. 3, no. 1, pp. 27–45, 1997.
[31] H. F. Yu and S. T. Tseng, "Designing a degradation experiment," *Nav. Res. Logistics*, vol. 46, no. 6, pp. 689–706, 1999.
[32] H. F. Yu and S. T. Tseng, "Designing a screening experiment for highly reliability products," *Nav. Res. Logistics*, vol. 49, no. 5, pp. 514–526, 2002.
[33] IESNA Testing Procedures Committee, "LM-80-08: Measuring lumen maintenance of LED light source," Illuminating Engineering Society, 2020. [Online]. Available: https://www.energy.gov/eere/ssl/solid-state-lighting.
[34] U.S. Department of Energy, "LED Measurement Series: LED Luminaire Reliability," 2020. [Online]. Available: http:///www.eere.energy.gov
[35] U.S. Department of Energy, "L Prize® lumen maintenance testing of the Philips 60-watt replacement lamp L Prize entry" (prepared by Pacific Northwest National Laboratory), 2016. [Online]. Available: https://www.energy.gov/sites/prod/files/2018/06/f52/LPrize_60W-LumenMaintTesting_2016.pdf

Cheng-Hsun Wu received the Ph.D. degree in mathematics from National Central University, Taoyuan City, Taiwan, in 2009.

He is currently an Associate Professor with the Department of Financial Engineering and Actuarial Mathematics, Soochow University, Taipei City, Taiwan. His research interests include financial mathematics, statistical inference for stochastic processes, and financial engineering.

Tzong-Ru Tsai received the Ph.D. degree in statistics from National Chengchi University, Taipei, Taiwan, in 1996.

He is currently a Professor with the Department of Statistics, Tamkang University, Taipei. His major research interests include quality control and reliability analysis.

Ming-Yung Lee received the Ph.D degree in statistics from National Tsing Hua University, Hsinchu,Taiwan, in 2006.

He is currently an Associate Professor with the Department of Data Science and Big Data Analytics, Providence University, Taichung City, Taiwan. His research interests include clinical trial, epidemiology, and reliability analysis; specifically in accelerated degradation testing, analysis methods, and optimal designs.