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Estimating the failure rate of the log-logistic distribution by smooth adaptive and bias-correction methods

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ABSTRACT

The Log-logistic distribution has successfully earned attention in practical applications due to its good statistical properties. Because the traditional maximum likelihood estimators of the Log-logistic distribution parameters do not have an explicit form and are biased when the sample size is small. Therefore, the estimation and prediction of the failure rate is not well. In this study, we study the quality of the maximum likelihood, asymptotic maximum likelihood and bias-corrected maximum likelihood methods, and propose a smooth adaptive estimation method for estimating the Log-logistic distribution parameters. To reduce the bias of the asymptotic maximum likelihood and smooth adaptive estimators of the Log-logistic distribution parameters, the bias-corrected method is used to improve the asymptotic maximum likelihood and smooth adaptive estimation methods. Two new bias-corrected estimation methods are also proposed to obtain reliable estimates of the Log-logistic distribution parameters. An intensive Monte Carlo simulation study is conducted to evaluate the performance of these estimation methods. Simulation results show that the smooth adaptive and two new bias-corrected estimation methods are more competitive than other competitors. Finally, two real example is used for illustrating the applications of the smooth adaptive, CAML and CSA estimation methods.

1. Introduction

Failure rate is one of the main quantitative measures to describe the regular pattern of product's reliability. In this paper, the lifetime of product is considered to follow a Log-logistic distribution (LLD). The LLD has been widely used for modeling data in many areas, such as the manufacturing industry, economics and so on, see Kantam, Rao, and Sriram (2006), Wang, Wu, and Shu (2015), Seevali and Kiran (2015), Bennett (1983), Lu and Tsai (2009) and Francisco and Daniele (2015). Two advantages of the LLD were mentioned for modeling data. First, the cumulative distribution function (CDF) of the LLD has a simple form to make the statistical inference for censored data easier for implementation in a survival analysis, see Chen (2006). Second, the failure rate function of the LLD is not necessarily monotonic and this property makes the LLD can be widely applied for modeling data, see Lawles (1983) and Shakhatreh (2017). Because of the importance of the LLD in the field of survival and reliability analysis, many papers have extended LLD to obtain more useful data features, such as the beta log-logistic distribution (Lemonte, 2014), the extended log-logistic distribution (Lima & Cordeiro, 2017), the three-parameter log-logistic distribution (Shakhatreh, 2017).

When using the LLD for modeling data, it is necessary to obtain the estimates of the LLD parameters. The widely used parameter estimation methods, such as the ML, AML, generalized moment (GM) and generalized least square (GLS) estimation methods, could be unreliable due to these four estimation methods often result in unstable or bias parameter estimates no matter complete or censoring samples are used to implement parameter estimation when the sample size is small.

The ML estimation method is the most widely used parameter estimation method to obtain the ML estimates (MLEs) of the LLD parameters. Because no closed-formed solution of the MLEs can be found, numerical computation methods such as the Newton-Raphson or quasi-Newton methods are needed to obtain the MLEs of the LLD parameters. Some available optimization methods can be used to obtain the ML estimates of the LLD parameters, see Press, Fleming, Teukolsky, and Vetterling (1986) and Lange (1999). However, if the boundary of the solution space is wide or the initial solution of the parameter used in the numerical method is inaccurate enough, the obtained optimal solution

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could be a local optimal solution, see Hossain and Willan (2007). Some practitioners could use the GLS and GM estimation methods to replace the ML estimation methods to obtain the estimates of the LLD parameters due to the GLS and GM estimation methods can generate analytical solution forms for the estimates, see Ashkar and Mahdi (2006), Kantar (2015), Martin van Zyl (2017) and Reath, Dong, and Wang (2018).

Because it is difficult to obtain the approximate variance–covariance matrix of the GLS and GM estimators, the confidence interval (CI) inference for the LLD parameters could be a problem, see Kantar (2015) and Martin van Zyl (2017). Reath et al. (2018) has mentioned that the bias and mean square error (MSE) of the GM estimator could be large and this fact makes the GM estimation method less reliable. Chen (1997) and Chen (2006) have proposed interval inference methods for the shape and scale parameters of the LLD, respectively. AI-Shomrani et al. (2016) used the ML estimation method to establish approximate CIs of the LLD parameters.

The LLD can be transformed to a location and scale distribution, named logistic distribution (LD), by taking a logarithm transformation. Hossain and Willan (2007) studied the AML estimation method for the location and scale family and they applied the AML estimation method to the LD. Then, the close-formed solutions for the estimators of the LD parameters were proposed. Because the smooth adaptive (SA) estimation method proposed by Han and Hawkins (1994) and Shu, Hsu, and Han (2007) can also be used to obtain the closed-formed solution of the estimators of the population mean and population standard deviation in the location and scale distribution. Therefore, we propose using SA estimation method in this study to obtain reliable estimates of the LLD parameters. The AML and SA estimators can be obtained via using simple computation procedures, but the AML and SA estimators are biased estimators. Moreover, the consistency of the AML estimator could be a problem, see the discussion in Section 3.2. We are motivated to use bias-corrected method to reduce the impact of bias on the SA and AML estimation methods. The new proposed estimation methods are named the CSA and CAML estimation, respectively. The leading acronym "C" stands for bias-corrected method. Some good bias correction methods have been proposed by Hirose (1999), Zhang, Xie, and Tang (2006) and Reath et al. (2018), respectively. In addition, the sampling distributions of SA, CSA and CAML are difficult to obtain, the standard bootstrap (SB), percentile bootstrap (PB) and bias-corrected percentile bootstrap (CPB) methods are recommended in this study to obtain the bootstrap CIs (BCIs) of the LLD parameters.

The rest of this paper is organized as follows. In Section 2, some existing parameter estimation methods for the LLD are addressed. The proposed SA, CSA and CAML estimation methods for the LLD and the bias-corrected and bootstrap methods are studied in Section 3. In Section 4, the performance of the ML, CML, SA, CSA, AML and CAML estimation methods and their corresponding failure rate prediction performance are evaluated and compared via using Monte Carlo simulations based on the measures of bias and MSE. In Section 5, two examples are given to illustrate the proposed estimation method and failure rate prediction, respectively. Finally, some concluding remarks are given in Section 6.

Acronyms	
ML	maximum likelihood
AML	asymptotic maximum likelihood
SA	smooth adaptive
CML	bias-corrected maximum likelihood
CAML	bias-corrected asymptotic maximum likelihood
CSA	bias-corrected smooth adaptive
LLD	log-logistic distribution
LD	logistic distribution
SB	standard bootstrap
PB	percentile bootstrap
CPB	bias-corrected percentile bootstrap
CI	confidence interval
BCI	bootstrap confidence interval
CP	coverage probability
CDF	cumulative distribution function
PDF	probability density function
Notations	
n	sample size
$x_1, \cdots, x_n$	random sample drawn from the quality characteristic X LLD
$y_1, \cdots, y_n$	random sample drawn from the quality characteristic Y LD
z_γ	the γ^{th} quantile of the standard normal distribution
α	scale parameter of the log-logistic distribution
β	shape parameter of the log-logistic distribution
μ	location parameter of the logistic distribution
σ	scale parameter of the logistic distribution
$F(\cdot)$	CDF of the log-logistic distribution
$f(\cdot)$	PDF of the log-logistic distribution
$F_p(\cdot)$	CDF of the logistic distribution
$f_p(\cdot)$	PDF of the logistic distribution
$\Phi(\cdot)$	CDF of the standard normal distribution
$h(\cdot)$	failure rate function
$U_{MHi}(n)$	bias-corrector for parameters
l_j	coefficients of the $U_{MHi}(n)$, $j = 0, 1, 2, 3$
$\boldsymbol{\theta}$	a parameter vector of the log-logistic distribution, $\boldsymbol{\theta} = (\alpha, \beta)$
$\widehat{I}(\boldsymbol{\theta})$	fisher information matrix

2. The failure rate function of LLD and the ML, CML and AML estimation methods

Let $x_1, x_2, \cdots, x_n$ be a random sample taken from a LLD, which has the probability density function (PDF) and CDF that are defined by

$$f(x;\alpha,\beta) = \frac{(\beta/\alpha)(x/\alpha)^{\beta-1}}{\left[1+(x/\alpha)^{\beta}\right]^2}, \quad x > 0, \tag{1}$$

and

$$F(x;\alpha,\beta) = \frac{(x/\alpha)^{\beta}}{1+(x/\alpha)^{\beta}}, \quad x > 0, \tag{2}$$

respectively, where $\alpha > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. According to Elsayed (2012), the failure rate function of LLD at time t can be defined by

$$h(t;\alpha,\beta) = \frac{(\beta/\alpha)(t/\alpha)^{\beta-1}}{1+(t/\alpha)^{\beta}}, \quad t > 0. \tag{3}$$

We use the notation X LLD(α,β) to denote random variable X following a LLD with parameters α and β here and after. Let $Y = \ln(X)$, it can be shown that Y $F_p(\cdot;\mu,\sigma)$, where $F_p(\cdot;\mu,\sigma)$ is the CDF of the LD with location parameter $\mu = \log(\alpha)$ and scale parameter $\sigma = 1/\beta$. Three common parameter estimation methods are reviewed as follows:

2.1. The ML estimation method for LLD

Let $\boldsymbol{\theta} = (\alpha, \beta)$. According to the inference procedure of He, Chen, and Qian (2020), the ML estimator of $\boldsymbol{\theta}$, denoted by $\widehat{\boldsymbol{\theta}}^{ML} = \left(\widehat{\alpha}^{ML}, \widehat{\beta}^{ML}\right)$, are solutions of the following two likelihood equations,

$$n - 2\sum_{i=1}^{n} \frac{\left(\frac{x_i}{\alpha}\right)^{\beta}}{1+\left(\frac{x_i}{\alpha}\right)^{\beta}} = 0, \tag{4}$$

and

$$\frac{n}{\beta} + \sum_{i=1}^{n} ln\frac{x_i}{\alpha} - 2\sum_{i=1}^{n} \frac{\left(\frac{x_i}{\alpha}\right)^{\beta} ln\frac{x_i}{\alpha}}{1+\left(\frac{x_i}{\alpha}\right)^{\beta}} = 0. \tag{5}$$

Obviously, the closed-form solutions of $\widehat{\alpha}^{ML}$ and $\widehat{\beta}^{ML}$ do not exist. Numerical computation methods are needed to obtain $\widehat{\alpha}^{ML}$ and $\widehat{\beta}^{ML}$ by

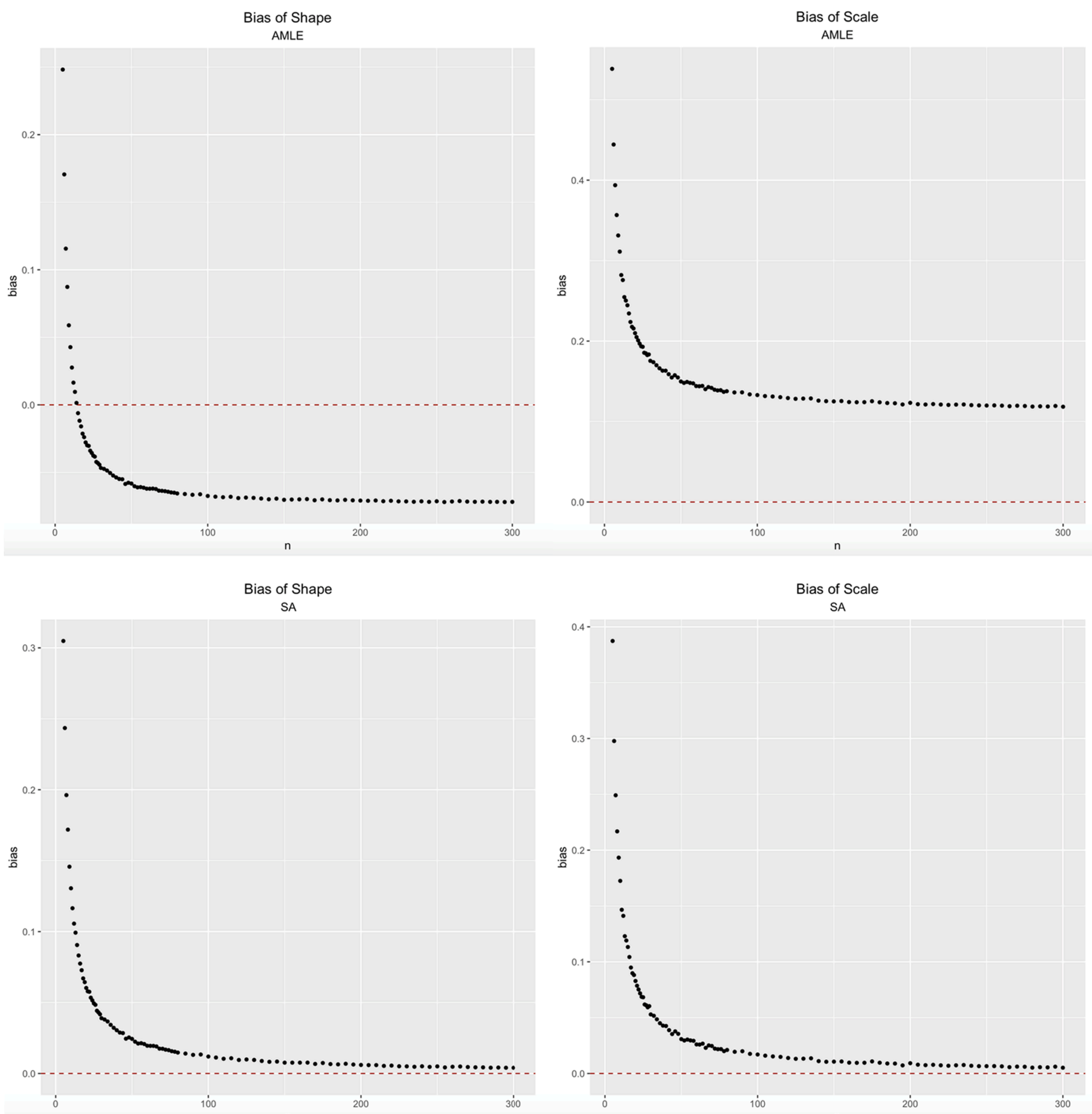

Fig. 1. The bias of AMLE and SA when $\beta = 1$ and $\alpha = 1$.

simultaneously solve Eqs. (4) and (5). AI-Shomrani et al. (2016) proposed an approximate interval estimation method to obtain the CI of α and β, denoted by $\mathrm{CI_{ML}}$. The $(1-\gamma)\times 100\%$ approximate CIs of α and β can be obtained by

$$\widehat{\alpha}^{ML} \pm z_{1-\gamma/2}\delta\left(\widehat{\alpha}^{ML}\right), \tag{6}$$

and

$$\widehat{\beta}^{ML} \pm z_{1-\gamma/2}\delta\left(\widehat{\beta}^{ML}\right), \tag{7}$$

respectively, where $z_{1-\gamma/2}$ is the $(1-\gamma/2)^{\text{th}}$ quantile of the standard normal distribution; that is $\Phi(z_{1-\gamma/2}) = 1-\gamma/2$ where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Moreover, AI-Shomrani et al. (2016) have shown that the asymptotic variances of $\widehat{\alpha}^{ML}$ and $\widehat{\beta}^{ML}$ can be presented by

$$\delta^2\left(\widehat{\alpha}^{ML}\right) = \frac{n\beta}{\alpha^2} - \frac{2\beta}{\alpha^2}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}}{1+\left(\frac{x_i}{\alpha}\right)^{\beta}} - \frac{2\beta^2}{\alpha^2}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}}{\left[1+\left(\frac{x_i}{\alpha}\right)^{\beta}\right]^2}, \tag{8}$$

and

$$\delta^2\left(\widehat{\beta}^{ML}\right) = -\frac{n}{\beta^2} - 2\sum_{i=1}^{n}\frac{\left(ln\frac{x_i}{\alpha}\right)^2\left(\frac{x_i}{\alpha}\right)^{\beta}}{\left[1+\left(\frac{x_i}{\alpha}\right)^{\beta}\right]^2}, \tag{9}$$

respectively.

Reath et al. (2018) proposed a CML estimation method to obtain the bias-corrected ML estimator of $\boldsymbol{\theta}$. Denote the obtained ML estimator of $\boldsymbol{\theta}$ by $\widehat{\boldsymbol{\theta}}^{CML}$. The $\widehat{\boldsymbol{\theta}}^{CML}$ can be presented by

$$\widehat{\boldsymbol{\theta}}^{CML} = \widehat{\boldsymbol{\theta}}^{ML} - \widehat{\boldsymbol{I}}(\boldsymbol{\theta})^{-1}\widehat{\boldsymbol{A}}\cdot vec\left(\widehat{\boldsymbol{I}}(\boldsymbol{\theta})^{-1}\right), \tag{10}$$

where $vec(V)$ denotes the vector of columns of matrix V,

$$\widehat{\boldsymbol{A}} = n\begin{bmatrix} \frac{\beta^2}{6\alpha^3} & \frac{\beta}{4\alpha^2} & \frac{-5\beta}{12\alpha^2} & 0 \\ \frac{\beta}{4\alpha^2} & 0 & 0 & \frac{1}{18\beta^3}\left(3+\frac{5\pi^2}{2}\right) \end{bmatrix}_{\alpha=\widehat{\alpha}^{ML},\beta=\widehat{\beta}^{ML}}, \tag{11}$$

and

$$\widehat{\boldsymbol{I}}(\boldsymbol{\theta}) = n\begin{bmatrix} \frac{\beta^2}{3\alpha^2} & 0 \\ 0 & \frac{n\left(3+\pi^2\right)}{9\beta^2} \end{bmatrix}_{\alpha=\widehat{\alpha}^{ML},\beta=\widehat{\beta}^{ML}}. \tag{12}$$

Hossain and Willan (2007) proposed an AML estimation procedure to obtain an explicit form of the estimator of the model parameter. Let $y_1,y_2,\cdots,y_n$ be a random sample taken from a LD with location parameter μ and scale parameter σ, denoted by $y_1,y_2,\cdots,y_n$ iidLD(μ,σ), in which the term of "iid" means all y's are independent and identically distributed, and $y_{(1)}\le y_{(2)}\le\cdots\le y_{(n)}$ are the order statistics obtained from the sample. The AML estimators of μ and σ can be obtain, respectively, by

$$\widehat{\mu}^{AML} = K - L\widehat{\sigma}^{AML} \tag{13}$$

and

$$\widehat{\sigma}^{AML} = \frac{-\lambda_1+\sqrt{\lambda_1^2+4n\lambda_2}}{2n}, \tag{14}$$

where $K = \frac{\sum_{i=1}^{n}B_iy_{(i)}}{\sum_{i=1}^{n}B_i}$, $L = \sum_{i=1}^{n}A_i/\sum_{i=1}^{n}B_i$, $\lambda_1 = \sum_{i=1}^{n}(y_{(i)}-K)A_i$, $\lambda_2 = \sum_{i=1}^{n}B_i(y_{(i)}-K)^2$, $A_i = (1+ln(q_i))/(1-ln(q_i)) - 2\,ln(-ln(q_i))ln(q_i)/(1-ln(q_i))^2$, $B_i = -2ln(q_i)/(1-ln(q_i))^2$, $q_i = 1-i/(n+1)$, $i = 1,2,\cdots,n$。 Taking anti-logrithm transformation to $y_i, i=1,2,\cdots,n$, the AML estimates of α and β can be obtained by

$$\widehat{\alpha}^{AML} = \exp\left(\widehat{\mu}^{AML}\right), \tag{15}$$

and

$$\widehat{\beta}^{AML} = \frac{1}{\widehat{\sigma}^{AML}}, \tag{16}$$

respectively.

3. The proposed SA, CSA and CAML estimation methods

3.1. The SA estimation method for LLD

Let $y_1,y_2,\cdots,y_n$ iidLD(μ,σ) and $y_{(1)}\le\cdots\le y_{(n)}$ denote the order statistics of $y_1,y_2,\cdots,y_n$. Following the estimation procedures proposed by Han and Hawkins (1994) and Hogg (1967), parameter μ can be estimated by

$$\widehat{\mu}^{SA} = \sum_{i=1}^{n}w_iy_{(i)}, \tag{17}$$

where $w_i = c_i/\sum_{j=1}^{n}c_j$, $c_i = \overline{U}_i(b) - \overline{L}_i(b)$, $\overline{U}_i(b)$ and $\overline{L}_i(b)$ are the average of the largest and smallest $[nb]$ observations in the sample with $y_{(i)}$ deleted, respectively; $[G]$ denotes the largest integers not greater than G. According to the recommendation of Han and Hawkins (1994), $b = 0.2$, is also considered in this paper. Then, the standard deviation of Y based on the estimator of $\widehat{\mu}^{SA}$ can be estimated by

$$SD(Y) = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(y_i-\widehat{\mu}^{SA})^2}. \tag{18}$$

Following the relationship between the $SD(Y)$ and parameter σ, we can obtain the estimator of σ by

$$\widehat{\sigma}^{SA} = \frac{\sqrt{3}\times SD(Y)}{\pi}. \tag{19}$$

Therefore, we can use $\widehat{\mu}^{SA}$ and $\widehat{\sigma}^{SA}$ to estimate μ and σ, respectively. Taking anti-logrithm transformation to $y_i, i=1,2,\cdots,n$, the SA estimators of α and β can be obtained and presented, respectively, by

$$\widehat{\alpha}^{SA} = \exp\left(\widehat{\mu}^{SA}\right), \tag{20}$$

and

$$\widehat{\beta}^{SA} = \frac{\pi}{\sqrt{3}\times SD(y)}. \tag{21}$$

3.2. The CSA and CAMLE estimation methods for LLD

Fig. 1 reports the bias of the AML estimates (AMLEs) and SA estimates of α and β based on 10,000 Monte Carlo simulation runs for sample size $n = 5, 10, 15, \ldots, 300$. In view of Fig. 1, we can see that the AMLE and SA are biased estimators for the LLD. The bias of the AML and SA estimators is declined as the sample size increases. Moreover, we also find that the AML estimator is an inconsistent estimator due to the AMLE cannot converge to its true value even the sample size is large. Therefore, these two bias-corrected method are used in this study to reduce the bias of the AML and SA estimators. The properties of ML and CML estimates vs. the sample size have been studied by Reath et al. (2018)

Let $x_1,x_2,\cdots,x_n$ iidLLD$(\boldsymbol{\theta})$ and denote the obtained estimate of $\boldsymbol{\theta}$ by $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha},\widehat{\beta})$, in which $\widehat{\boldsymbol{\theta}}$ can be obtained via using the SA or AML estimation method. Generate B_1 bootstrap samples, $x_{1,j}^*,x_{2,j}^*,\cdots,x_{n,j}^*, j=1,2,$

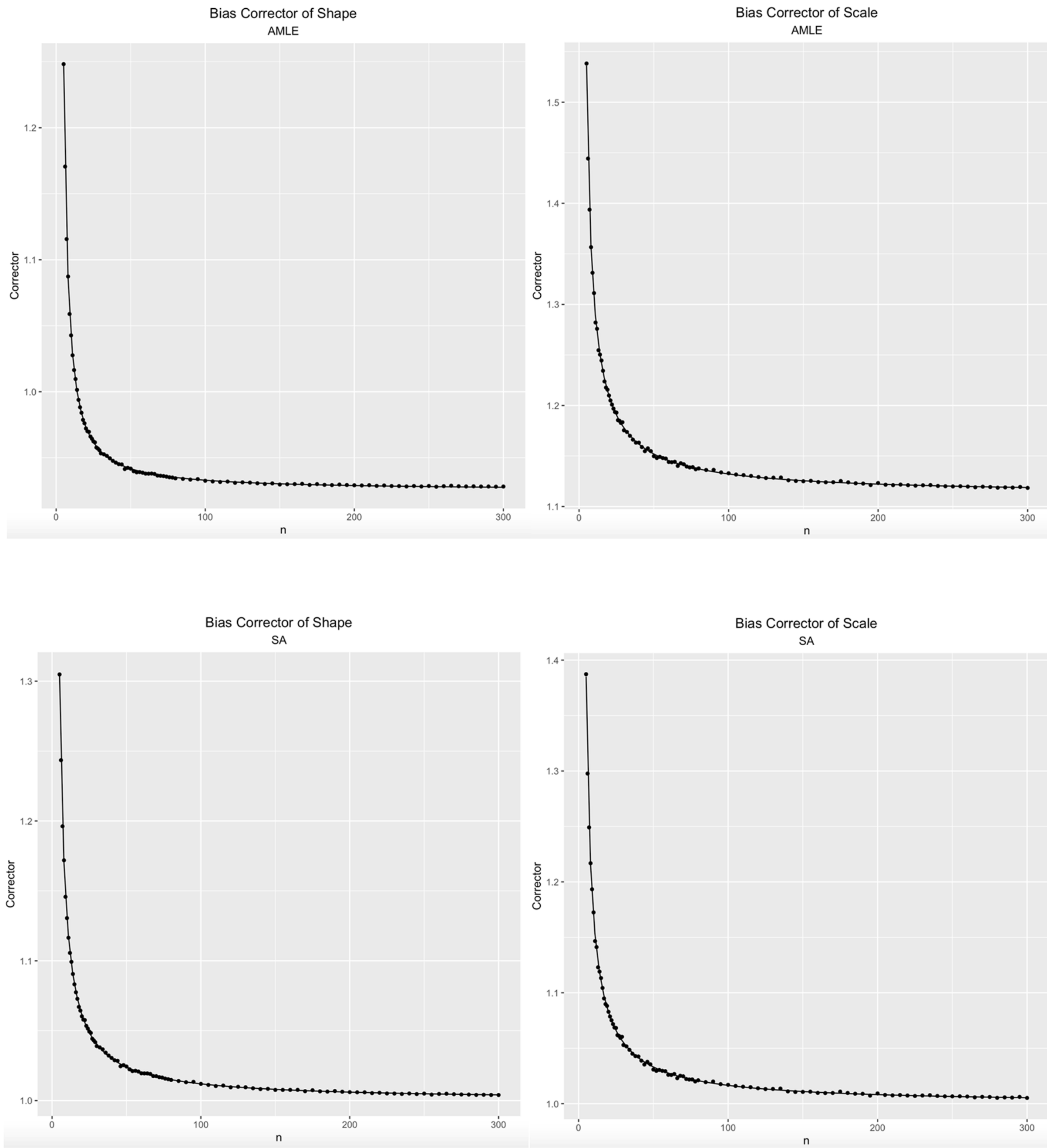

Fig. 2. The relationship between $U_{MH}(n)$ and n.

Table 1
The estimated values of l_i, l_1, l_2, l_3 and R^2 for $U_{MH1}(n)$ and $U_{MH2}(n)$ via using the AML or SA estimation methods.

AML estimation method									
$U_{MH1}(n)$ for α					$U_{MH2}(n)$ for β				
$\widehat{l}_0$	$\widehat{l}_1$	$\widehat{l}_2$	$\widehat{l}_3$	R^2	$\widehat{l}_0$	$\widehat{l}_1$	$\widehat{l}_2$	$\widehat{l}_3$	R^2
1.1120	2.0450	−2.6534	15.1618	0.9995	0.9251	0.7502	4.1109	1.0654	0.9998
SA estimation method									
$U_{MH1}(n)$ for α					$U_{MH2}(n)$ for β				
$\widehat{l}_0$	$\widehat{l}_1$	$\widehat{l}_2$	$\widehat{l}_3$	R^2	$\widehat{l}_0$	$\widehat{l}_1$	$\widehat{l}_2$	$\widehat{l}_3$	R^2
0.9997	1.6900	−1.2950	12.3930	0.9994	1.0002	1.1708	0.8938	4.4592	0.9998

$\cdots, B_1$, from the distribution of LLD$(\widehat{\boldsymbol{\theta}})$, whre B_1 is a large positive integer. For each bootstrap sample, $x_1^*, x_2^*, \cdots, x_n^*$, obtain the bootstrap estimate of $\boldsymbol{\theta}$ and denoted it by $\widehat{\boldsymbol{\theta}}^* = (\widehat{\alpha}^*, \widehat{\beta}^*)$ Let $\widehat{\boldsymbol{\alpha}}^* = (\widehat{\alpha}_1^*, \widehat{\alpha}_2^*, \cdots, \widehat{\alpha}_{B_1}^*)$ and $\widehat{\boldsymbol{\beta}}^* = (\widehat{\beta}_1^*, \widehat{\beta}_2^*, \cdots, \widehat{\beta}_{B_1}^*)$. Because B_1 is a big positive integer, it is reasonable to assume that $\mathrm{E}(\widehat{\alpha}^*) \cong \widehat{\alpha}$. Then we can obtain $\alpha/\widehat{\alpha} = \widehat{\alpha}/E(\widehat{\alpha}^*)$ and

$$\alpha = \frac{\widehat{\alpha}^2}{E(\widehat{\alpha}^*)} = \frac{\widehat{\alpha}}{U_{MH1}(n)}, \tag{22}$$

where $U_{MH1}(n) = \widehat{\alpha}/E(\widehat{\alpha}^*)$. Likewise, it can be shown that $\beta = \widehat{\beta}^2/E(\widehat{\beta}^*) = \widehat{\beta}/U_{MH2}(n)$ and $U_{MH2}(n) = \widehat{\beta}/E(\widehat{\beta}^*)$. Moreover, $E(\widehat{\boldsymbol{\theta}}^*)$ can be approximated by

$$E(\widehat{\boldsymbol{\theta}}^*) \cong \frac{1}{B_1}\sum_{j=1}^{B_1}\widehat{\boldsymbol{\theta}}_j^* = \left(\frac{1}{B_1}\sum_{j=1}^{B_1}\widehat{\alpha}_j^*, \frac{1}{B_1}\sum_{j=1}^{B_1}\widehat{\beta}_j^*\right) \tag{23}$$

When the CSA estimation method is used, the estimator of $\boldsymbol{\theta}$ can be expressed by

$$\boldsymbol{\theta}^{CSA} = (\alpha^{CSA}, \beta^{CSA}) = \left(\frac{\widehat{\alpha}^{SA}}{U_{MH1}(n)}, \frac{\widehat{\beta}^{SA}}{U_{MH2}(n)}\right) \tag{24}$$

with $U_{MH1}(n) = \widehat{\alpha}^{SA}/E(\widehat{\alpha}^*)$ and $U_{MH2}(n) = \widehat{\beta}^{SA}/E(\widehat{\beta}^*)$. When the CAML estimation method is used, the estimator of $\boldsymbol{\theta}$ can be expressed by

$$\boldsymbol{\theta}^{CAML} = (\alpha^{CAML}, \alpha^{CAML}) = \left(\frac{\widehat{\alpha}^{AML}}{U_{MH1}(n)}, \frac{\widehat{\beta}^{AML}}{U_{MH2}(n)}\right) \tag{25}$$

with $U_{MH1}(n) = \widehat{\alpha}^{AML}/E(\widehat{\alpha}^*)$ and$U_{MH2}(n) = \widehat{\beta}^{AML}/E(\widehat{\beta}^*)$.

The CSA and CAML estimation methods can help to reduce the bias of the SA and AML estimators. But the CSA and CAML estimation methods require more computation loading to obtain the CSA estimates and CAML estimates (CAMLEs) of the LLD parameters through using a bootstrap sampling procedure. Because the quality of the functions of $U_{MH1}(n)$ and $U_{MH2}(n)$ is highly dependent on the sample size, simulations are done for LLD$(\alpha = 1, \beta = 1)$ and $n = 5, 10, 15, \ldots, 300$ to study the relationship between the functions of $U_{MH1}(n)$ and $U_{MH2}(n)$ with the sample size. All simulation results are reported in Fig. 2. Fig. 2 indicates a cubic relationship between $U_{MH1}(n)$ (or$U_{MH2}(n)$) and the sample size. The reference cubic function can be expressed by

$$U_{MHi}(n) = l_0 + \frac{l_1}{n} + \frac{l_2}{n^2} + \frac{l_3}{n^3}, i = 1, 2. \tag{26}$$

All the coefficients of l_0, l_1, l_2 and l_3 can be estimated by using the least square estimation (LSE) method. Table 1 reports the estimated values of l_0, l_1, l_2 and l_3 for $U_{MH1}(n)$ and $U_{MH2}(n)$.

We can find that all the values of R^2 in Table 1 are closed to 1, which indicates a good model fitting by using a cubic function form to characterize $U_{MH1}(n)$ and $U_{MH2}(n)$ with the samle size. Let $\widehat{U}_{MHi}(n) = \widehat{l}_0 + \widehat{l}_1/n + \widehat{l}_2/n^2 + \widehat{l}_3/n^3$ for $i = 1, 2$. In practical applications, we can use $\widehat{U}_{MHi}(n)$ to replace the $U_{MHi}(n)$ in Eqs. (24) and (25) for$i = 1, 2$ to obtained the CSA or CAML estimates by

$$\widehat{\boldsymbol{\theta}}^{CSA} = (\widehat{\alpha}^{CSA}, \widehat{\beta}^{CSA}) = \left(\frac{\widehat{\alpha}^{SA}}{\widehat{U}_{MH1}(n)}, \frac{\widehat{\beta}^{SA}}{\widehat{U}_{MH2}(n)}\right), \tag{27}$$

and

$$\widehat{\boldsymbol{\theta}}^{CAML} = (\widehat{\alpha}^{CAML}, \widehat{\beta}^{CAML}) = \left(\frac{\widehat{\alpha}^{AML}}{\widehat{U}_{MH1}(n)}, \frac{\widehat{\beta}^{AML}}{\widehat{U}_{MH2}(n)}\right), \tag{28}$$

respectively. Please note that the simulation results in Table 1 can be used for the LLD with parameters $\alpha \neq 1$ or $\beta \neq 1$.

Table 2
The bias and MSE of shape parameter.β.

	Bias						MSE					
n	AMLE	MLE	CMLE	SA	CSA	CAMLE	AMLE	MLE	CMLE	SA	CSA	CAMLE
	$\beta = 2$	$\alpha = 1$										
5	0.461	0.731	0.686	0.568	−0.033	−0.028	2.098	2.835	2.795	2.374	1.204	1.211
8	0.174	0.411	0.381	0.342	0.003	0.003	0.669	0.947	0.930	0.862	0.545	0.543
12	0.039	0.252	0.232	0.216	0.003	0.005	0.319	0.436	0.429	0.423	0.308	0.307
20	−0.049	0.143	0.131	0.129	0.005	0.005	0.162	0.203	0.200	0.209	0.170	0.169
35	−0.101	0.076	0.068	0.068	−0.001	0.000	0.091	0.096	0.095	0.102	0.091	0.090
50	−0.116	0.053	0.048	0.048	0.000	0.000	0.070	0.066	0.065	0.071	0.066	0.064
75	−0.126	0.038	0.034	0.034	0.002	0.002	0.053	0.042	0.042	0.046	0.043	0.042
100	−0.134	0.027	0.024	0.024	0.000	0.000	0.045	0.030	0.030	0.033	0.032	0.031
	$\beta = 1.5$	$\alpha = 1$										
5	0.378	0.587	0.539	0.463	0.003	0.005	1.228	1.683	1.643	1.406	0.699	0.697
8	0.132	0.310	0.279	0.258	0.003	0.004	0.380	0.532	0.517	0.486	0.307	0.308
12	0.027	0.187	0.166	0.161	0.001	0.002	0.180	0.246	0.240	0.241	0.175	0.174
20	−0.046	0.099	0.086	0.087	−0.005	−0.005	0.091	0.112	0.109	0.114	0.094	0.094
35	−0.074	0.059	0.052	0.052	0.000	0.001	0.051	0.054	0.053	0.058	0.051	0.050
50	−0.089	0.039	0.034	0.035	−0.001	−0.001	0.040	0.036	0.036	0.040	0.037	0.036
75	−0.097	0.026	0.022	0.024	0.000	−0.001	0.030	0.023	0.023	0.025	0.024	0.023
100	−0.100	0.021	0.018	0.019	0.001	0.001	0.025	0.017	0.017	0.019	0.018	0.018
	$\beta = 1$	$\alpha = 1$										
5	0.246	0.384	0.336	0.302	−0.003	−0.001	0.523	0.719	0.686	0.597	0.297	0.297
8	0.082	0.201	0.170	0.166	−0.003	−0.003	0.166	0.233	0.221	0.212	0.135	0.135
12	0.022	0.129	0.108	0.111	0.004	0.005	0.082	0.113	0.107	0.109	0.079	0.079
20	−0.027	0.070	0.058	0.061	−0.001	0.000	0.040	0.050	0.048	0.051	0.042	0.041
35	−0.050	0.039	0.032	0.036	0.001	0.001	0.024	0.025	0.024	0.027	0.024	0.023
50	−0.058	0.027	0.022	0.024	0.000	0.000	0.017	0.016	0.016	0.017	0.016	0.016
75	−0.064	0.018	0.015	0.016	0.000	0.000	0.013	0.010	0.010	0.011	0.011	0.010
100	−0.068	0.012	0.009	0.011	−0.001	−0.002	0.011	0.007	0.007	0.008	0.008	0.007

Table 3
The bias and MSE of scale parameter.α.

n	Bias AMLE	MLE	CMLE	SA	CSA	CAMLE	MSE AMLE	MLE	CMLE	SA	CSA	CAMLE
	$\beta = 2$	$\alpha = 1$										
5	0.139	0.084	0.008	0.087	−0.215	−0.259	0.259	0.217	0.180	0.223	0.159	0.169
8	0.106	0.048	0.002	0.048	−0.137	−0.184	0.137	0.111	0.100	0.113	0.094	0.102
12	0.088	0.030	−0.001	0.030	−0.096	−0.145	0.088	0.070	0.065	0.072	0.064	0.070
20	0.076	0.017	−0.002	0.018	−0.060	−0.110	0.052	0.041	0.039	0.042	0.039	0.044
35	0.071	0.013	0.002	0.013	−0.033	−0.084	0.030	0.022	0.022	0.024	0.023	0.026
50	0.065	0.008	0.000	0.008	−0.024	−0.075	0.022	0.015	0.015	0.016	0.016	0.019
75	0.061	0.004	−0.001	0.004	−0.017	−0.069	0.015	0.010	0.010	0.011	0.011	0.013
100	0.059	0.003	−0.001	0.003	−0.013	−0.065	0.012	0.008	0.008	0.008	0.008	0.011
	$\beta = 1.5$	$\alpha = 1$										
5	0.233	0.153	0.011	0.156	−0.165	−0.197	0.583	0.465	0.334	0.479	0.264	0.263
8	0.171	0.088	0.003	0.091	−0.102	−0.136	0.294	0.227	0.184	0.234	0.164	0.163
12	0.139	0.057	0.000	0.058	−0.071	−0.105	0.179	0.136	0.119	0.139	0.110	0.110
20	0.117	0.035	0.001	0.037	−0.042	−0.077	0.104	0.076	0.070	0.080	0.069	0.067
35	0.096	0.018	−0.002	0.018	−0.028	−0.062	0.056	0.040	0.038	0.043	0.039	0.038
50	0.092	0.014	0.001	0.015	−0.018	−0.052	0.042	0.028	0.028	0.031	0.029	0.028
75	0.086	0.009	0.000	0.010	−0.012	−0.046	0.029	0.018	0.018	0.020	0.019	0.019
100	0.082	0.006	−0.001	0.006	−0.010	−0.045	0.022	0.014	0.013	0.015	0.014	0.014
	$\beta = 1$	$\alpha = 1$										
5	0.551	0.389	−0.004	0.396	0.008	0.009	2.888	2.070	1.023	2.151	1.040	1.096
8	0.353	0.205	−0.014	0.211	−0.004	−0.002	1.087	0.761	0.471	0.799	0.511	0.523
12	0.253	0.121	−0.014	0.121	−0.016	−0.016	0.526	0.374	0.280	0.379	0.281	0.285
20	0.204	0.075	−0.003	0.077	−0.005	−0.005	0.278	0.191	0.159	0.197	0.163	0.162
35	0.170	0.046	0.002	0.048	0.001	0.001	0.156	0.102	0.091	0.108	0.096	0.093
50	0.150	0.029	−0.001	0.031	−0.002	−0.002	0.104	0.065	0.060	0.069	0.064	0.061
75	0.137	0.019	−0.002	0.021	−0.001	−0.002	0.073	0.043	0.041	0.047	0.044	0.042
100	0.131	0.014	−0.001	0.015	−0.001	−0.001	0.056	0.031	0.030	0.034	0.032	0.030

3.3. Bootstrap CI

The sampling distributions of the AML, SA, CML, CAML and CSA estimators could be difficult to be obtained. In this study, three bootstrap methods proposed by Efron (1979) and Efron and Tibshirani (1986) are used to establish the CIs of α and β. The bootstrap CI (BCI) can be established according to the following steps:

Step 1: Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ be a random sample taken from LLD($\boldsymbol{\theta}$). Obtain the estimate of $\boldsymbol{\theta}$ based on $\boldsymbol{x}$ and denoted the obtained estimate by$\widehat{\boldsymbol{\theta}}$.

Step 2: Generate a bootstrap sample $\boldsymbol{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)$ from the distribution of LLD($\widehat{\boldsymbol{\theta}}$) and obtain the estimate of $\boldsymbol{\theta}$ based on $\boldsymbol{x}^*$,denote it by$\widehat{\boldsymbol{\theta}}^*$.

Step 3: Repeat Step 2 B_2 times and denote the bootstrap estimates by $\widehat{\boldsymbol{\theta}}_1^*, \widehat{\boldsymbol{\theta}}_2^*, \cdots, \widehat{\boldsymbol{\theta}}_{B_2}^*$, where B_2 is a large positive integer.

Step 4: The $(1-\gamma) \times 100\%$ BCIs of α and β can be obtained and denoted by

$$\left(\widehat{\alpha}_L^*, \widehat{\alpha}_U^*\right) \text{and} \left(\widehat{\beta}_L^*, \widehat{\beta}_U^*\right) \tag{29}$$

According to the procedures proposed by Efron (1979), three methods, the SB, PB and CPB methods, can be used to obtain the interval $(\widehat{\eta}_L^*, \widehat{\eta}_U^*)$, where η can be α or β. Based on the SB method, the BCI of η can be expressed by

$$\left(\widehat{\eta}_L^*, \widehat{\eta}_U^*\right) = \left(\widehat{\eta} - s^* z_{1-\gamma/2}, \widehat{\eta} + s^* z_{1-\gamma/2}\right) \tag{30}$$

where

$$s^* = \sqrt{\left(\frac{1}{B_2 - 1}\right) \sum_{i=1}^{B_2} \left[\widehat{\eta}_i^* - \left(\sum_{i=1}^{B_2} \widehat{\eta}_i^*\right) \Big/ B_2\right]^2} \tag{31}$$

Based on the PB method, the BCI of η can be expressed by

$$\left(\widehat{\eta}_L^*, \widehat{\eta}_U^*\right) = \left(\widehat{\eta}_{\gamma/2}^*, \widehat{\eta}_{1-\gamma/2}^*\right) \tag{32}$$

where $\widehat{\eta}_{\gamma/2}^*$ and $\widehat{\eta}_{1-\gamma/2}^*$ are the $(\gamma/2)^{\text{th}}$ and $(1-\gamma/2)^{\text{th}}$ empirical quantiles of the bootstrap sample $(\widehat{\eta}_1^*, \widehat{\eta}_2^*, \cdots, \widehat{\eta}_{B_2}^*)$. Based on the CPB method, the BCI can be expressed by

$$\left(\widehat{\eta}_L^*, \widehat{\eta}_U^*\right) = \left(\widehat{\eta}_{[P_L \times B_2]}^*, \widehat{\eta}_{[P_U \times B_2]}^*\right) \tag{33}$$

where $P_L = \Phi(2z_0 + z_{\gamma/2})$ and $P_U = \Phi(2z_0 + z_{1-\gamma/2})$, $p_0 = P(\widehat{\theta}^* \leq \widehat{\theta})$, $z_0 = \Phi^{-1}(p_0)$.

The CML, SA, CSA, AML or CAML estimation methods can be used to implement the above steps and the obtained BCIs are denoted by BCI_{CML}, BCI_{SA}, BCI_{CSA},BCI_{AML} and BCI_{CAML} in this study, respectively.

4. Monte Carlo simulation study

An extensive Monte Carlo simulation study is conducted in this section to evaluate the performance of using the ML, AML, SA, CML, CAML and CSA estimation methods to estimate the LLD parameters in terms of the quality measures of the bias and MSE and failure rate prediction performance. The coverage probabilities (CPs) of the $(1-\gamma) \times 100\%$ CI or BCIs of α and β are study.

Table 4
The values of CP for the LLD with $\beta = 2$ and $\alpha = 1$.

		BCI_{AML}		CI_{ML}		BCI_{CML}		BCI_{SA}		BCI_{CSA}		BCI_{CAML}	
n	Method	β	α	β	α	β	α	β	α	β	α	β	α
5	Eq.6–7			0.957	0.841								
	SB	1.000	0.919			0.811	0.846	0.999	0.879	0.943	0.611	0.947	0.526
	PB	0.882	0.896			0.907	0.862	0.847	0.879	0.941	0.690	0.941	0.614
	CPB	0.904	0.897			0.958	0.845	0.900	0.879	0.929	0.875	0.930	0.833
8	Eq.5–6			0.965	0.893								
	SB	0.985	0.947			0.866	0.910	0.993	0.917	0.946	0.717	0.949	0.599
	PB	0.929	0.916			0.923	0.921	0.878	0.910	0.956	0.782	0.950	0.677
	CPB	0.932	0.923			0.958	0.909	0.920	0.904	0.939	0.925	0.947	0.889
	Eq.6–7			0.952	0.898								
12	SB	0.956	0.956			0.875	0.924	0.973	0.918	0.946	0.753	0.945	0.606
	PB	0.946	0.924			0.934	0.927	0.905	0.912	0.947	0.812	0.946	0.677
	CPB	0.945	0.938			0.958	0.922	0.936	0.911	0.948	0.925	0.949	0.897
	Eq.6–7			0.953	0.931								
20	SB	0.917	0.966			0.911	0.922	0.966	0.939	0.947	0.826	0.952	0.657
	PB	0.943	0.928			0.939	0.919	0.919	0.942	0.949	0.865	0.950	0.720
	CPB	0.954	0.952			0.952	0.925	0.944	0.943	0.949	0.955	0.946	0.929
	Eq.6–7			0.949	0.938								
35	SB	0.845	0.940			0.927	0.939	0.962	0.946	0.947	0.876	0.945	0.646
	PB	0.885	0.896			0.943	0.941	0.934	0.941	0.951	0.903	0.939	0.697
	CPB	0.946	0.956			0.946	0.940	0.947	0.935	0.955	0.945	0.943	0.919
	Eq.6–7			0.949	0.938								
50	SB	0.753	0.918			0.931	0.939	0.959	0.941	0.949	0.904	0.951	0.674
	PB	0.798	0.875			0.932	0.941	0.947	0.937	0.951	0.919	0.954	0.709
	CPB	0.950	0.953			0.941	0.936	0.954	0.935	0.950	0.942	0.952	0.916
	Eq.6–7			0.955	0.937								
75	SB	0.682	0.886			0.935	0.958	0.953	0.943	0.948	0.909	0.947	0.640
	PB	0.722	0.832			0.947	0.956	0.947	0.941	0.950	0.924	0.947	0.681
	CPB	0.936	0.949			0.957	0.958	0.949	0.939	0.950	0.940	0.948	0.910
	Eq.6–7			0.951	0.936								
100	SB	0.562	0.858			0.943	0.942	0.954	0.944	0.954	0.917	0.952	0.597
	PB	0.605	0.809			0.951	0.943	0.946	0.941	0.952	0.926	0.950	0.631
	CPB	0.921	0.946			0.954	0.947	0.950	0.941	0.953	0.939	0.948	0.895

4.1. Point and interval estimation

Considering the sample size $n = 5, 8, 12, 20, 35, 50, 75, 100$ and $\gamma = 0.05$ for simulations. Morevoer, all the BCIs are obtained based on $B_2 = 10,000$ iteration runs. In the simulation study, data set with n observations are generated from the LLD$(\alpha = 1, \beta)$, where $\beta = 1, 1.5, 2$ via using the "rllogis()" function in the "STAR" package of R. Moreover, the "llogisMLE()" function of R was used for solving nonlinear equations of MLE.

Ten thousands iteration runs are used to evaluate the bias and MSEs of $\widehat{\boldsymbol{\theta}}^{ML}, \widehat{\boldsymbol{\theta}}^{AML}, \widehat{\boldsymbol{\theta}}^{SA}, \widehat{\boldsymbol{\theta}}^{CML}, \widehat{\boldsymbol{\theta}}^{CAML}$ and $\widehat{\boldsymbol{\theta}}^{CSA}$. The CPs of the LLD parameters, α and β are also evaluated based on $B = 10,000$ iteration runs. The bias, MSE and CP can be evaluated based on Equations (34), (35) and (36), respectively:

$$\text{bias} = \frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\eta}_i - \eta\right), \tag{34}$$

$$\text{MSE} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\eta}_i - \eta\right)^2}, \tag{35}$$

and

$$\text{CP} = \frac{1}{B}\sum_{i=1}^{B} I_i\{\eta | \eta \in (L_i, U_i)\}, \tag{36}$$

where η can be α or β; L_i and U_i are the lower and upper bounds of the CI or BCI in the i^{th} iteration run. Eqs. (6) and (7) are used to obtain the upper and lower bounds of the CI based on using ML estimation method and the other BCIs are obtained based on the proposed bootstrap methods in Section 3.3. All obtained bias and MSEs are reported in Tables 2 and 3. The obtained CPs of the CI and BCIs are reported in Tables 4–6.

In view of Tables 2 and 3, the following results are found:

1. In view of Tables 1–3, the SA, AMLE and MLE methods are competitive in terms of the bias and MSE for estimating the parameters α and β.
2. Bias-correction operation is helpful to improve the bias and MSE for the ML, SA and AMLE methods. The improvement is significant especially for the sample size is small.
3. The CSA estimation method outperforms the CML and CAML estimation methods if bias-correction is considered. It is noted that the CAML estimation method performs better than the CML estimation methods even the consistency of the AML estimator is a problem. After using the bias-corrected method, the CAML estimation method becomes competitive, too.

Table 5
The values of CP for the LLD with $\beta = 1.5$ and $\alpha = 1$.

		BCI_{AML}		CI_{ML}		BCI_{CML}		BCI_{SA}		BCI_{CSA}		BCI_{CAML}	
n	Method	β	α	β	α	β	α	β	α	β	α	β	α
	Eq.6–7			0.955	0.846								
5	SB	0.999	0.928			0.805	0.836	1.000	0.897	0.937	0.731	0.936	0.677
	PB	0.880	0.897			0.899	0.867	0.845	0.886	0.944	0.797	0.943	0.746
	CPB	0.910	0.898			0.954	0.836	0.902	0.884	0.938	0.919	0.936	0.908
	Eq.5–6			0.959	0.889								
8	SB	0.983	0.962			0.862	0.892	0.991	0.920	0.938	0.802	0.943	0.742
	PB	0.942	0.920			0.925	0.904	0.894	0.906	0.953	0.861	0.953	0.802
	CPB	0.943	0.929			0.963	0.892	0.930	0.905	0.949	0.937	0.952	0.932
	Eq.6–7			0.952	0.914								
12	SB	0.960	0.972			0.889	0.913	0.979	0.938	0.947	0.840	0.947	0.765
	PB	0.947	0.924			0.934	0.928	0.905	0.926	0.948	0.884	0.948	0.826
	CPB	0.944	0.945			0.957	0.905	0.931	0.925	0.947	0.946	0.943	0.945
	Eq.6–7			0.943	0.925								
20	SB	0.903	0.971			0.907	0.916	0.963	0.937	0.940	0.873	0.942	0.805
	PB	0.929	0.919			0.930	0.924	0.920	0.924	0.942	0.908	0.939	0.852
	CPB	0.944	0.947			0.947	0.912	0.940	0.926	0.944	0.936	0.945	0.933
	Eq.6–7			0.944	0.939								
35	SB	0.842	0.961			0.927	0.927	0.955	0.947	0.944	0.910	0.939	0.831
	PB	0.884	0.897			0.948	0.936	0.934	0.940	0.945	0.926	0.943	0.865
	CPB	0.948	0.955			0.960	0.930	0.943	0.942	0.947	0.950	0.950	0.951
	Eq.6–7			0.953	0.942								
50	SB	0.774	0.943			0.939	0.940	0.957	0.948	0.946	0.913	0.949	0.820
	PB	0.824	0.885			0.948	0.937	0.941	0.945	0.954	0.933	0.945	0.855
	CPB	0.946	0.964			0.947	0.940	0.950	0.947	0.952	0.953	0.946	0.947
	Eq.6–7			0.953	0.946								
75	SB	0.661	0.895			0.946	0.944	0.954	0.950	0.946	0.935	0.944	0.834
	PB	0.707	0.831			0.947	0.948	0.942	0.942	0.951	0.943	0.947	0.865
	CPB	0.941	0.952			0.955	0.945	0.946	0.943	0.950	0.948	0.951	0.945
	Eq.6–7			0.957	0.953								
100	SB	0.571	0.855			0.953	0.945	0.953	0.950	0.949	0.934	0.946	0.816
	PB	0.618	0.780			0.957	0.948	0.944	0.948	0.949	0.940	0.946	0.849
	CPB	0.918	0.956			0.958	0.944	0.947	0.947	0.948	0.947	0.949	0.950

4. The MSEs of all six estimation methods are closed to 0 as the sample size increases.

In view of Tables 4–6, we can find that that the BCI_{CAML} and BCI_{CSA} outperforms the other CI and BCIs due to their CPs are closer to the nominal values than other estimation methods in most cells. Overall, the CP is closer to the nominal value when the sample size increases. We also note that the bootstrap method of CPB is most competitive than the other bootstrap methods. Because the sampling distribution could be asymmetric, this could be the reason why the CPB method performs well. In this study, we recommend using the CPB method to obtain the BCI of α and β in the LLD.

4.2. Failure rate analysis under different parameter estimation methods

In this section, we evaluate the performance of predicting failure rate by using six estimation methods, including the methods of ML, AML, SA, CML, CAML and CSA estimation. Considering the sample size $n = 5, 8, 12, 20, 35, 50, 75, 100$, $\alpha = 1$, $\beta = 1, 1.5, 2$ and $t = 100$, 200, 500 min for simulations. The $bias^*$ and MSE^* can be evaluated based on Eqs. (37) and (38), respectively:

$$bias^* = \frac{1}{B}\sum_{i=1}^{B}\frac{(h(t;\hat{\alpha},\hat{\beta}) - h(t;\alpha,\beta))}{h(t;\alpha,\beta)}, \tag{37}$$

and

$$\mathrm{MSE}^* = \frac{1}{B}\sum_{i=1}^{B}\left[\frac{h(t;\hat{\alpha},\hat{\beta}) - h(t;\alpha,\beta)}{h(t;\alpha,\beta)}\right]^2. \tag{38}$$

The simulation results are listed in Tables 7–9. In view of Table 7–9, the following results are found:

1. The bias-correction methods outperforms the methods without bias-correction.
2. In most cases, with the increase of sample size, both $bias^*$ and MSE^* decline for all methods except for the $bias^*$ of the AML method.
3. The CAML and CSA methods are competitive and the CAML performs best in terms of the $bias^*$ andMSE^*.

5. Illustrative examples

Example 1. An example regarding the Scotland's annual maximum flood frequency series in m^3/s for specified periods (from 1952 to 1982) is used for illustration. The data set has been discussed by Acreman and Sinclair (1986) and Ahmad, Sinclair, and Werritty (1988). We report this data set in Table 10.

The LLD is used to characterize the data in Table 10. Moreover, all the ML, AML, SA, CML, CAML and CSA estimation methods are used to obtain the estimates of the LLD parameters. The 95% CI based on the ML estimation method and 95% BCI based on the AML, SA, CML, CAML and CSA estimation methods were obtained. All the estimation results are reported in Table 11. The p-value of the Kolmogorov-Smirnov test in Table 11 indicates that the LLD can well model this data set and we found the CSA estimation method is the best estimation method to obtain the estimates of the LLD parameters.

Example 2. This example is regarding to failure rate analysis. The data

Table 6
The values of CP for the LLD with $\beta = 1$ and $\alpha = 1$.

		BCI_{AML}		CI_{ML}		BCI_{CML}		BCI_{SA}		BCI_{CSA}		BCI_{CAML}	
n	Method	β	α	β	α	β	α	β	α	β	α	β	α
	Eq.6–7			0.965	0.843								
5	SB	0.999	0.944			0.801	0.850	0.999	0.920	0.934	0.851	0.931	0.842
	PB	0.899	0.905			0.824	0.883	0.868	0.896	0.954	0.885	0.953	0.872
	CPB	0.927	0.910			0.963	0.866	0.924	0.895	0.951	0.943	0.950	0.947
	Eq.5–6			0.952	0.881								
8	SB	0.983	0.965			0.862	0.857	0.991	0.930	0.939	0.873	0.933	0.865
	PB	0.936	0.928			0.912	0.896	0.888	0.914	0.949	0.905	0.948	0.891
	CPB	0.939	0.930			0.951	0.872	0.932	0.915	0.954	0.944	0.950	0.946
	Eq.6–7			0.949	0.894								
12	SB	0.956	0.971			0.896	0.889	0.975	0.937	0.941	0.882	0.942	0.873
	PB	0.943	0.930			0.932	0.926	0.899	0.926	0.943	0.918	0.940	0.906
	CPB	0.939	0.945			0.953	0.888	0.934	0.926	0.944	0.945	0.947	0.946
	Eq.6–7			0.951	0.909								
20	SB	0.901	0.980			0.919	0.921	0.966	0.934	0.941	0.896	0.939	0.891
	PB	0.928	0.920			0.942	0.934	0.928	0.928	0.949	0.924	0.947	0.919
	CPB	0.953	0.950			0.961	0.918	0.945	0.932	0.953	0.943	0.952	0.944
	Eq.6–7			0.953	0.934								
35	SB	0.838	0.977			0.924	0.938	0.962	0.954	0.942	0.924	0.947	0.921
	PB	0.885	0.893			0.938	0.938	0.935	0.940	0.944	0.937	0.949	0.930
	CPB	0.952	0.957			0.947	0.927	0.945	0.940	0.949	0.946	0.951	0.945
	Eq.6–7			0.954	0.936								
50	SB	0.779	0.976			0.945	0.945	0.964	0.943	0.955	0.926	0.953	0.918
	PB	0.821	0.886			0.951	0.952	0.949	0.939	0.954	0.934	0.950	0.934
	CPB	0.951	0.955			0.955	0.940	0.954	0.941	0.955	0.946	0.952	0.947
	Eq.6–7			0.956	0.930								
75	SB	0.688	0.933			0.940	0.933	0.953	0.937	0.949	0.921	0.953	0.920
	PB	0.728	0.839			0.943	0.941	0.941	0.931	0.948	0.929	0.950	0.925
	CPB	0.939	0.942			0.942	0.933	0.947	0.932	0.950	0.936	0.947	0.935
	Eq.6–7			0.948	0.933								
100	SB	0.562	0.894			0.949	0.949	0.953	0.941	0.945	0.925	0.944	0.929
	PB	0.611	0.782			0.952	0.954	0.944	0.938	0.946	0.935	0.944	0.936
	CPB	0.911	0.948			0.958	0.946	0.948	0.939	0.946	0.945	0.947	0.940

listed in Table 12 represent the failure times (in minutes) of a specific type of electrical insulation in an experiment, in which the insulation is subjected to a continuously increasing voltage stress. This data was firstly proposed by Lawles (1983), and discussed by Balakrishnan and Saleh (2012). The study of Balakrishnan and Saleh (2012) suggests that the failure times in Table 12 follow a LLD. In this case, we want to predict the failure rate at t = 200 min. The estimation results based on the SA, CSA and CAMLE methods and the predicted values of the failure rate function are given in Table 13. Based on the estimation results of Table 13, the failure rate at 200 min is about 1%.

6. Conclusions

Because the quality of the predicted failure rate depends on the estimates of the model parameters, it is necessary to find reliable parameter estimation methods for reliability analysis. In this study, we have studied the estimation quality of the ML, AML, SA, CML, CAML and CSA estimation methods to obtain reliable estimates of the LLD parameters. The estimation quality is evaluated in terms of the measures of bias and MSE. Moreover, the CI based on the ML estimation method and the BCI based on the AML, SA, CML, CAML and CSA estimation methods for the LLD are also studied via using the bootstrap methods of SB, PB and CPB. We focus on proposing estimation methods that can provide closed-form to obtain reliable estimates of the LLD parameters. Monte Carlo Simulations were conducted to assess the estimation performance of the proposed estimation methods. Based on the simulation results, we find that the SA, CSA and CAML estimation methods outperform the other competitors to obtain reliable estimates of the LLD parameters. The bootstrap method of CPB can generate the most closest CP to the nominal value than the SB and PB bootstrap methods. Moreover, the failure rate prediction performance of the CAML and CSA methods is better than other competitors.

A cubic function form is suggested in this study to reduce the computation loading to obtain the values of $U_{MH1}(n)$ and $U_{MH2}(n)$. The proposed computation procedure can provide a simple method to obtain an approximate values of $U_{MH1}(n)$ and $U_{MH2}(n)$ via using the proposed cubic function. The coefficients in the proposed cubic function were obtained and tabulated for practical use in this study. Two examples are used to demonstrate the proposed estimation method, the first example regarding the Scotland's annual maximum flood frequency series in m^3/s for specified periods in1952 to 1982 is used for illustrating the proposed inference method. In this example, we found that the CSA estimation method performs best among all competitors. The second example is used to evaluate the failure rate of electrical insulation when the failure time is 200 min. In this example, the predicted result based on proposed methods are about 1%.

The major difficulty in the parameter inference method is that the exact sampling distribution is difficult to be obtained. Other computation methods could be help to release the difficulty caused by the unknown exact sampling distribution of the estimators for the model parameters. How to extend the proposed methods to other widely used distribution is also important. For example, extend the proposed methods to the beta log-logistic distribution, the extended log-logistic distribution, the three-parameter log-logistic distribution. These two topics are still open and will be study in the future.

Table 7
The $bias^*$ and MSE^* of predicting Failure rate at $t = 100$.

	$bias^*$						MSE^*					
n	AMLE	MLE	CMLE	SA	CSA	CAMLE	AMLE	MLE	CMLE	SA	CSA	CAMLE
	$\beta = 2$	$\alpha = 1$										
5	0.2402	0.3774	0.3551	0.2961	−0.0081	−0.0068	0.5009	0.6847	0.6744	0.5716	0.2845	0.2850
8	0.0830	0.2010	0.1863	0.1669	−0.0022	−0.0021	0.1612	0.2262	0.2222	0.2072	0.1314	0.1311
12	0.0132	0.1189	0.1088	0.1028	−0.0036	−0.0034	0.0806	0.1088	0.1071	0.1057	0.0778	0.0777
20	−0.0287	0.0682	0.0620	0.0593	−0.0021	−0.0016	0.0420	0.0519	0.0513	0.0528	0.0438	0.0434
35	−0.0493	0.0387	0.0351	0.0353	0.0007	0.0010	0.0234	0.0249	0.0247	0.0264	0.0235	0.0232
50	−0.0589	0.0264	0.0239	0.0238	−0.0002	−0.0005	0.0178	0.0165	0.0164	0.0180	0.0167	0.0162
75	−0.0637	0.0183	0.0166	0.0173	0.0013	0.0006	0.0132	0.0104	0.0104	0.0115	0.0108	0.0105
100	−0.0668	0.0137	0.0124	0.0127	0.0007	0.0004	0.0113	0.0076	0.0076	0.0085	0.0082	0.0078
	$\beta = 1.5$	$\alpha = 1$										
5	0.2319	0.3709	−0.3737	0.2879	−0.0160	−0.0145	0.4847	0.6691	0.2612	0.5494	0.2756	0.2781
8	0.0858	0.2055	0.1847	0.1712	0.0006	0.0004	0.1702	0.2400	0.2330	0.2184	0.1390	0.1384
12	0.0216	0.1301	0.1159	0.1123	0.0045	0.0052	0.0835	0.1160	0.1128	0.1108	0.0806	0.0802
20	−0.0302	0.0666	0.0579	0.0586	−0.0031	−0.0025	0.0408	0.0499	0.0488	0.0511	0.0424	0.0419
35	−0.0486	0.0407	0.0357	0.0371	0.0023	0.0024	0.0236	0.0254	0.0250	0.0270	0.0240	0.0234
50	−0.0609	0.0248	0.0212	0.0220	−0.0021	−0.0019	0.0181	0.0164	0.0162	0.0178	0.0166	0.0161
75	−0.0645	0.0183	0.0160	0.0176	0.0015	0.0006	0.0136	0.0106	0.0105	0.0117	0.0111	0.0107
100	−0.0683	0.0126	0.0109	0.0116	−0.0005	−0.0005	0.0116	0.0077	0.0076	0.0085	0.0081	0.0079
	$\beta = 1$	$\alpha = 1$										
5	0.2426	0.3875	0.3424	0.3015	−0.0112	−0.0097	0.6242	0.8337	0.7418	0.7005	0.3623	0.3662
8	0.0741	0.1985	0.1604	0.1623	−0.0108	−0.0107	0.1682	0.2330	0.2287	0.2136	0.1390	0.1385
12	0.0098	0.1222	0.1025	0.1040	−0.0054	−0.0049	0.0840	0.1117	0.1059	0.1088	0.0809	0.0806
20	−0.0342	0.0677	0.0558	0.0593	−0.0040	−0.0036	0.0443	0.0526	0.0506	0.0540	0.0451	0.0449
35	−0.0570	0.0373	0.0304	0.0332	−0.0023	−0.0026	0.0267	0.0270	0.0263	0.0288	0.0260	0.0255
50	−0.0615	0.0301	0.0253	0.0265	0.0017	0.0014	0.0202	0.0183	0.0180	0.0199	0.0184	0.0181
75	−0.0697	0.0181	0.0149	0.0163	−0.0001	−0.0007	0.0151	0.0112	0.0111	0.0122	0.0116	0.0114
100	−0.0739	0.0116	0.0093	0.0106	−0.0018	−0.0022	0.0131	0.0083	0.0082	0.0092	0.0089	0.0086

Table 8
The $bias^*$ and MSE^* of predicting Failure rate at $t = 200$.

	$bias^*$						MSE^*					
n	AMLE	MLE	CMLE	SA	CSA	CAMLE	AMLE	MLE	CMLE	SA	CSA	CAMLE
	$\beta = 2$	$\alpha = 1$										
5	0.2575	0.3990	0.3768	0.3135	0.0056	0.0073	0.5714	0.7877	0.7771	0.6531	0.3258	0.3246
8	0.0820	0.1989	0.1842	0.1658	−0.0030	−0.0030	0.1621	0.2290	0.2251	0.2092	0.1330	0.1319
12	0.0174	0.1237	0.1136	0.1060	−0.0005	0.0007	0.0782	0.1071	0.1053	0.1034	0.0753	0.0753
20	−0.0277	0.0684	0.0622	0.0607	−0.0008	−0.0007	0.0409	0.0507	0.0501	0.0515	0.0424	0.0423
35	−0.0511	0.0378	0.0342	0.0339	−0.0005	−0.0010	0.0232	0.0246	0.0244	0.0262	0.0234	0.0228
50	−0.0578	0.0272	0.0247	0.0252	0.0012	0.0005	0.0176	0.0166	0.0165	0.0181	0.0166	0.0161
75	−0.0655	0.0170	0.0153	0.0155	−0.0005	−0.0013	0.0134	0.0102	0.0102	0.0112	0.0106	0.0104
100	−0.0668	0.0135	0.0122	0.0128	0.0008	0.0002	0.0113	0.0077	0.0077	0.0085	0.0082	0.0079
	$\beta = 1.5$	$\alpha = 1$										
5	0.2489	0.3865	−0.3628	0.3050	−0.0020	−0.0004	0.5597	0.7518	0.2735	0.6367	0.3201	0.3205
8	0.0839	0.2034	0.1825	0.1688	−0.0010	−0.0013	0.1630	0.2337	0.2268	0.2102	0.1333	0.1326
12	0.0152	0.1231	0.1088	0.1055	−0.0013	−0.0014	0.0806	0.1095	0.1065	0.1069	0.0784	0.0777
20	−0.0275	0.0691	0.0604	0.0615	−0.0002	−0.0002	0.0415	0.0510	0.0499	0.0529	0.0437	0.0429
35	−0.0537	0.0356	0.0306	0.0317	−0.0028	−0.0034	0.0240	0.0247	0.0244	0.0267	0.0241	0.0234
50	−0.0592	0.0262	0.0227	0.0236	−0.0005	−0.0006	0.0181	0.0168	0.0166	0.0181	0.0167	0.0164
75	−0.0653	0.0173	0.0149	0.0160	0.0000	−0.0008	0.0135	0.0104	0.0103	0.0114	0.0108	0.0105
100	−0.0686	0.0118	0.0101	0.0110	−0.0010	−0.0014	0.0113	0.0073	0.0073	0.0081	0.0078	0.0076
	$\beta = 1$	$\alpha = 1$										
5	0.2543	0.3964	0.3441	0.3131	−0.0001	0.0008	0.5906	0.7942	0.7696	0.6729	0.3411	0.3404
8	0.0861	0.2091	0.1792	0.1745	0.0013	0.0002	0.1738	0.2428	0.2293	0.2251	0.1439	0.1416
12	0.0141	0.1238	0.1037	0.1063	−0.0023	−0.0016	0.0834	0.1122	0.1066	0.1082	0.0798	0.0802
20	−0.0322	0.0670	0.0549	0.0595	−0.0031	−0.0032	0.0431	0.0517	0.0498	0.0533	0.0444	0.0440
35	−0.0545	0.0376	0.0307	0.0330	−0.0021	−0.0019	0.0254	0.0259	0.0253	0.0276	0.0248	0.0245
50	−0.0624	0.0264	0.0215	0.0232	−0.0012	−0.0015	0.0190	0.0170	0.0167	0.0185	0.0172	0.0168
75	−0.0678	0.0174	0.0142	0.0155	−0.0007	−0.0009	0.0142	0.0105	0.0104	0.0117	0.0111	0.0108
100	−0.0715	0.0116	0.0092	0.0104	−0.0018	−0.0017	0.0124	0.0079	0.0078	0.0088	0.0085	0.0082

Table 9
The *bias** and *MSE** of predicting Failure rate at $t = 500$.

n	bias* AMLE	bias* MLE	bias* CMLE	bias* SA	bias* CSA	bias* CAMLE	MSE* AMLE	MSE* MLE	MSE* CMLE	MSE* SA	MSE* CSA	MSE* CAMLE
	$\beta = 2$	$\alpha = 1$										
5	0.2435	0.3815	0.3591	0.3004	−0.0043	−0.0038	0.5558	0.7611	0.7508	0.6376	0.3213	0.3189
8	0.0825	0.2015	0.1868	0.1665	−0.0024	−0.0025	0.1612	0.2277	0.2236	0.2078	0.1318	0.1311
12	0.0135	0.1202	0.1101	0.1022	−0.0039	−0.0032	0.0767	0.1048	0.1031	0.1014	0.0743	0.0740
20	−0.0281	0.0683	0.0621	0.0598	−0.0016	−0.0012	0.0406	0.0500	0.0494	0.0514	0.0424	0.0420
35	−0.0493	0.0395	0.0359	0.0360	0.0015	0.0009	0.0229	0.0245	0.0243	0.0260	0.0231	0.0227
50	−0.0589	0.0255	0.0230	0.0233	−0.0007	−0.0007	0.0176	0.0162	0.0161	0.0175	0.0162	0.0159
75	−0.0646	0.0178	0.0161	0.0157	−0.0003	−0.0005	0.0130	0.0100	0.0099	0.0109	0.0104	0.0101
100	−0.0662	0.0141	0.0129	0.0130	0.0010	0.0009	0.0112	0.0077	0.0076	0.0085	0.0082	0.0079
	$\beta = 1.5$	$\alpha = 1$										
5	0.2461	0.3838	−0.3526	0.3020	−0.0036	−0.0021	0.5050	0.6994	0.2560	0.5722	0.2828	0.2859
8	0.0767	0.1948	0.1738	0.1606	−0.0077	−0.0079	0.1609	0.2240	0.2174	0.2066	0.1326	0.1318
12	0.0173	0.1245	0.1103	0.1059	−0.0007	0.0006	0.0803	0.1092	0.1062	0.1054	0.0770	0.0774
20	−0.0294	0.0668	0.0581	0.0599	−0.0016	−0.0024	0.0409	0.0503	0.0493	0.0515	0.0426	0.0422
35	−0.0511	0.0380	0.0330	0.0339	−0.0006	−0.0009	0.0241	0.0257	0.0254	0.0272	0.0243	0.0238
50	−0.0588	0.0264	0.0229	0.0234	−0.0006	−0.0005	0.0177	0.0165	0.0163	0.0178	0.0165	0.0161
75	−0.0644	0.0180	0.0156	0.0166	0.0006	−0.0001	0.0133	0.0103	0.0102	0.0114	0.0108	0.0104
100	−0.0674	0.0132	0.0114	0.0121	0.0002	−0.0003	0.0112	0.0075	0.0075	0.0083	0.0080	0.0077
	$\beta = 1$	$\alpha = 1$										
5	0.2571	0.3989	0.3590	0.3151	0.0034	0.0044	0.6042	0.8158	0.7595	0.6901	0.3494	0.3476
8	0.0759	0.1953	0.1686	0.1613	−0.0088	−0.0091	0.1647	0.2295	0.2290	0.2095	0.1354	0.1354
12	0.0169	0.1255	0.1052	0.1083	0.0004	0.0006	0.0832	0.1136	0.1081	0.1103	0.0810	0.0800
20	−0.0293	0.0684	0.0562	0.0608	−0.0013	−0.0013	0.0415	0.0505	0.0486	0.0519	0.0430	0.0427
35	−0.0517	0.0390	0.0320	0.0349	0.0001	−0.0003	0.0241	0.0252	0.0246	0.0268	0.0240	0.0235
50	−0.0599	0.0271	0.0222	0.0243	0.0000	−0.0003	0.0181	0.0167	0.0164	0.0180	0.0167	0.0162
75	−0.0651	0.0183	0.0151	0.0171	0.0009	0.0004	0.0140	0.0109	0.0108	0.0120	0.0114	0.0110
100	−0.0688	0.0130	0.0105	0.0120	−0.0001	−0.0005	0.0118	0.0077	0.0077	0.0087	0.0084	0.0080

Table 10
Annual maximum flood series in m^3/s for specified periods (1952–1982).

89.8	109.1	202.2	146.3	212.3	116.7	109.1
80.7	127.4	138.8	283.5	85.6	105.5	118
387.8	80.7	165.7	111.6	134.4	131.5	102
104.3	242.5	214.8	144.6	114.2	98.3	102.8
104.3	196.2	143.7				

Table 11
The estimation results based on the LLD for the example of annual maximum flood series.

Method	α	β	The p-value of K-S Test
AMLE	134.032 (116.661, 156.851)	4.120 (3.109, 6.036)	0.465
MLE	128.598 (112.233, 144.963)	4.815 (3.389, 6.240)	0.779
CMLE	128.330 (112.905, 147.198)	4.810 (3.569, 6.339)	0.801
SA	133.750 (115.240, 152.314)	4.770 (3.407, 6.454)	0.396
CSA	126.981 (115.893, 149.211)	4.591 (3.485, 6.600)	0.840
CAMLE	113.996 (112.794, 126.200)	4.321 (3.219, 6.055)	0.097

The numbers in brackets represent the lower and upper bounds of interval estimation of parameter.

Table 12
The failure times (in minutes).

12.3	21.8	24.4	28.6	43.2	46.9
70.7	75.3	95.5	98.1	138.6	151.9

Table 13
The estimation results and predicted values of the failure rate function.

Method	α	β	Failure rate prediction (time = 200)
SA	52.368	2.286	0.0102
CSA	45.988	2.066	0.0099
CAMLE	43.856	2.061	0.0099

CRediT authorship contribution statement

Xi Zheng: Writing - original draft, Writing - review & editing. **Jyun-You Chiang:** Conceptualization, Supervision, Funding acquisition. **Tzong-Ru Tsai:** Formal analysis, Funding acquisition. **Shuai Wang:** Methodology, Software.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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