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One-Stage Multiple Comparisons with the Control for Exponential Median Lifetimes under Heteroscedasticity

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Abstract: When the additional sample for the second stage may not be available, one-stage multiple comparisons for exponential median lifetimes with the control under heteroscedasticity including one-sided and two-sided confidence intervals are proposed in this paper since the median is a more robust measure of central tendency compared to the mean. These intervals can be used to identify treatment populations that are better than the control or worse than the control in terms of median lifetimes in agriculture, stock market, pharmaceutical industries. Tables of critical values are obtained for practical use. An example of comparing the survival days for four categories of lung cancer in a standard chemotherapeutic agent is given to demonstrate the proposed procedures.

Keywords: one-stage procedure; two-stage procedure; multiple comparison procedures with the control

1. Introduction

In the lifetime test problems, the lifetime of some products follows an exponential distribution (see Lawless [1]). As the lifetime of products possesses a two-parameter exponential distribution, this research focuses on the development of multiple comparison procedures with the control population in terms of median lifetime. We consider k (≥ 2) independent populations $\pi_1, \cdots, \pi_k$, where π_i possesses a two-parameter exponential distribution denoted by $E(\theta_i, \sigma_i)$, $i = 1, \cdots, k$. The kth population is regarded as the control population and the first $k-1$ populations are regarded as the treatment populations. The location parameters $\theta_1, \cdots, \theta_k$ are unknown and usually called the guaranteed time in reliability analysis. The unequal and unknown scale parameters $\sigma_1, \cdots, \sigma_k$ are regarded as the mean lifetime minus their location parameters $\theta_1, \cdots, \theta_k$ since the mean lifetime for the ith population is $\mu_i = \theta_i + \sigma_i$, $i = 1, \ldots, k$. Regarding the multiple comparisons with the control population, Ng et al. [2] proposed a procedure in terms of the location parameter under the assumption of equal scale parameters. Under heteroscedasticity (unequal scale parameters), Lam and Ng [3] developed a design-oriented two-stage multiple comparison procedure in terms of location parameters. For the problem of multiple comparisons with the average under heteroscedasticity, Wu and Wu [4] investigated the two-stage procedures in terms of exponential location parameters. However, it may happen that the experimenters are not able to collect the additional sample for the second stage for the two-stage procedure. Because of this reason, Wu et al. [5] propose one-stage multiple comparisons with the average instead. Wu [6] proposes an one-stage multiple comparisons with the control for exponential distributions in terms of mean lifetimes under heteroscedasticity. Instead of doing the multiple comparisons with the control based on mean lifetimes, the median lifetimes should be considered since the median lifetimes are more robust for measuring the central tendency of the exponential lifetime distributions than mean lifetimes. Therefore, we consider the

multiple comparisons with the control in terms of median lifetimes instead of mean lifetimes in this study. The median lifetime for the ith population is obtained as $\delta_{0.5,i} = \theta_i + \ln 2\ \sigma_i$, $i = 1, \cdots, k$. The one-sided and two-sided confidence interval of the ith median lifetime deviated from the control median lifetime denoted by $\delta_{0.5,i} - \delta_{0.5,k}$, $i = 1, \cdots, k-1$ are proposed in the next section. All critical values are computed and listed in a table for the application of users. In Section 3, an example of comparing the survival days for four categories of lung cancer is considered for the illustrative aims to illustrate the implementation of proposed methods. Finally, our conclusions are summarized in Section 4.

2. One-Stage Multiple Comparisons with the Control for Exponential Median Lifetimes under Heteroscedasticity

The exponential distribution is primarily used in reliability applications and this distribution is used to model data with a constant failure rate, see for example, Johnson et al. [7]. The probability density function (pdf) and cumulative distribution function (cdf) for the ith exponentially distributed population are defined as

$$f(x) = \frac{1}{\sigma_i} \exp\left(-\frac{x-\theta_i}{\sigma_i}\right),\ x \geq \theta_i, \theta_i > 0, \sigma_i > 0,$$

and

$$F(x) = 1 - \exp\left(-\frac{x-\theta_i}{\sigma_i}\right), x \geq \theta_i, \theta_i > 0, \sigma_i > 0, i = 1, \cdots, k.$$

Location parameters $\theta_1, \cdots, \theta_k$ and scale parameters $\sigma_1, \cdots, \sigma_k$ are unknown and possibly unequal. The survival function for the ith exponentially distributed population is $S(x) = \exp\left(-\frac{x-\theta_i}{\sigma_i}\right)$. The qth quantile of the ith exponential distribution denoted by $\eta_{q,i}$ can be obtained by solving $F(x) = 1 - \exp\left(-\frac{x-\theta_i}{\sigma_i}\right) = q$ and results in $\delta_{q,i} = \theta_i - \ln(1-q)\ \sigma_i$, $i = 1, \cdots, k$. That is there are at least $1-q$ percentage of products having lifetime longer than $\delta_{q,i}$ units. Let $q = 0.5$ and then the median lifetimes can be obtained as $\delta_{0.5,i} = \theta_i + \ln 2\ \sigma_i$, $i = 1, \cdots, k$. In other words, there are at least 50% of products having lifetime longer than $\delta_{0.5,i}$ units.

Lam and Ng [3] proposed two-stage multiple comparison procedures with the control. But for some reasons like lacking budget or encountering experimental difficulties, it is possible that the experimenters cannot collect the additional sample for the second stage. In this case, one-stage procedures should be considered instead. Therefore, we propose one-stage multiple comparison procedures for exponential median lifetimes with the control as follows:

Take a random sample of size $m\ (\geq 2)$ from π_i denoted by $X_{i1}, \cdots, X_{im}$ for the one-stage procedures. Let $Y_i = \min(X_{i1}, \cdots, X_{im})$ and $S_i = \sum_{j=1}^{m} (X_{ij} - Y_i)/(m-1)$ and let

$$c^* = \max_{i=1,\cdots,k} \frac{S_i}{m}. \tag{1}$$

It is well-known that the complete sufficient statistics for (θ_i, σ_i) are (Y_i, S_i). From Roussas [8], the following three distributional results are observed.

(1) $Q_i = 2(m-1)S_i/\sigma_i =, i = 1, \ldots, k$ has a chi-square distribution with $2m-2$ degrees of freedom (df) denoted by χ^2_{2m-2}.

(2) $E_i = m(Y_i - \theta_i)/\sigma_i$ has a standard exponential distribution denoted by $Exp(1)$.

(3) E_i and Q_i are independent.

Using the distribution results of (1) and (2), we can find the uniformly minimum variance unbiased estimator (UMVUE) for (θ_i, σ_i) as $(Y_i - S_i/m, S_i)$. Furthermore, the UMVUE for the ith median lifetime $\delta_{0.5,i} = \theta_i + \ln 2\ \sigma_i$ is $Y_i - S_i/m + \ln 2 S_i = Y_i + (m \ln 2 - 1)S_i/m$. Then we find the UMVUE for the ith median lifetime deviated from the control median lifetime denoted by $\delta_{0.5,i} - \delta_{0.5,k}$ as $Y_i + (m \ln 2 - 1)S_i/m - Y_k - (m \ln 2 - 1)S_k/m$. Based on this estimator, we are going to propose

the simultaneous confidence intervals for $\delta_{0.5,i} - \delta_{0.5,k} = \theta_i + \ln 2\ \sigma_i - \theta_k + \ln 2\ \sigma_k$, $i = 1, \cdots, k-1$ in Theorem 1.

For the ith population, we consider the pivotal quantity

$$G_i = \frac{\delta_{0.5,i} - Y_i - (m \ln 2 - 1) S_i/m}{S_i/m} = \frac{\theta_i + \ln 2\ \sigma_i - Y_i - (m \ln 2 - 1) S_i/m}{S_i/m},$$
$$= \frac{-(m \ln 2 - 1)(S_i/\sigma_i) + m \ln 2 - m(Y_i - \theta_i)/\sigma_i}{S_i/\sigma_i}, i = 1, \cdots, k.$$

Based on these pivotal quantities, we propose the one-stage multiple comparison procedures with the control in terms of median lifetimes in Theorem 1.

Theorem 1. *For a given $0 < P^* < 1$, we can find the upper confidence bounds, lower confidence bounds, and two-sided confidence intervals for $\delta_{0.5,i} - \delta_{0.5,k}$, $i = 1, \ldots, k-1$ as follows:*

(a) If s_U^* is the $100P^{th}$ percentile of the distribution of U, where $U = \max(-G_k, G_i, G_i - G_k, i = 1, \cdots, k-1)$, then the simultaneous P^* upper confidence bounds for $\delta_{0.5,i} - \delta_{0.5,k}$ are $Y_i + (m \ln 2 - 1) S_i/m - Y_k - (m \ln 2 - 1) S_k/m + c^* s_U^*$, $i = 1, \cdots, k-1$.

(b) If s_L^* is the $100P^{th}$ percentile of the distribution of L, where $L = \max(-G_i, G_k, G_k - G_i, i = 1, \cdots, k-1)$, then the simultaneous P^* lower confidence bounds for $\delta_{0.5,i} - \delta_{0.5,k}$ are $Y_i + (m \ln 2 - 1) S_i/m - Y_k - (m \ln 2 - 1) S_k/m - c^* s_L^*$, $i = 1, \cdots, k-1$.

(c) If s_t^* is the $100P^{th}$ percentile of the distribution of T, where $T = \max(|G_i|, |G_k|, |G_k - G_i|, i = 1, \cdots, k-1)$, then the simultaneous P^* two-sided confidence intervals for $\delta_{0.5,i} - \delta_{0.5,k}$ are

$$(Y_i + (m \ln 2 - 1) S_i/m - Y_k - (m \ln 2 - 1) S_k/m \pm c^* s_t^*),\ i = 1, \cdots, k-1.$$

The technique we use to prove the above Theorem is the following Lemma given in Lam [9,10]:

Lemma 1. *Suppose X and Y are two random variables, a and b are two positive constants, then $[aX \geq bY - d\max(a, b)] \supseteq [X \geq -d,\ Y \leq d$ and $X \geq Y - d]$.*

Proof of Theorem 1.

For (a), we have

$$\begin{aligned}
&\mathrm{P}(\delta_{0.5,i} - \delta_{0.5,k} \leq Y_i + (m \ln 2 - 1) S_i/m - Y_k - (m \ln 2 - 1) S_k/m + c^* s_U^*,\ i = 1, \cdots, k-1)\\
&= \mathrm{P}(\theta_i + \ln 2\sigma_i - \theta_k - \ln 2\sigma_k \leq Y_i + (m \ln 2 - 1) S_i/m - Y_k - (m \ln 2 - 1) S_k/m + c^* s_U^*,\ i = 1, \cdots, k-1)\\
&= \mathrm{P}\Big(-(m \ln 2 - 1) S_i/m + \ln 2\sigma_i + \theta_i - Y_i \leq -(m \ln 2 - 1) S_k/m + \ln 2\sigma_k + \theta_k - Y_k + c^* s_U^*,\ i = 1, \cdots, k-1\Big)\\
&= \mathrm{P}\Big(\tfrac{S_i}{m}\tfrac{\sigma_i}{S_i}\tfrac{m(-(m \ln 2 - 1) S_i/m + \ln 2\sigma_i + \theta_i - Y_i)}{\sigma_i} \leq \tfrac{S_k}{m}\tfrac{\sigma_k}{S_k}\tfrac{m(-(m \ln 2 - 1) S_k/m + \ln 2\sigma_k + \theta_k - Y_k)}{\sigma_k}\\
&+ c^* s_U^*,\ i = 1, \cdots, k-1\Big)\\
&= \mathrm{P}\Big(\tfrac{S_k}{m} G_k \geq \tfrac{S_i}{m} G_i - c^* s_U^*,\ i = 1, \cdots, k-1\Big)\\
&\geq E_{S_1, \cdots, S_k} P\Big(\tfrac{S_k}{m} G_k \geq \tfrac{S_i}{m} G_i - \max(\tfrac{S_i}{m}, \tfrac{S_k}{m})\ s_U^*,\ i = 1, \cdots, k-1\Big)\\
&\geq \mathrm{P}(G_k \geq -s_U^*,\ G_i \leq s_U^*,\ G_k \geq G_i - s_U^*,\ i = 1, \cdots, k-1) \text{(By using Lemma1)}\\
&= \mathrm{P}(-G_k \leq s_U^*,\ G_i \leq s_U^*,\ G_i - G_k \leq s_U^*,\ i = 1, \cdots, k-1)\\
&= \mathrm{P}(\max(-G_k, G_i, G_i - G_k,\ i = 1, \cdots, k-1) \leq s_U^*) = P^*
\end{aligned}$$

It is clear that s_U^* represents the $100P^{th}$ percentile of the distribution of U and thus the proof is completed.

For (b), we have

$$
\begin{aligned}
&\mathrm{P}(\delta_{0.5,i}-\delta_{0.5,k}\geq Y_{\mathrm{i}}+(m\ln 2-1)S_i/m-Y_k-(m\ln 2-1)S_k/m-c^*s_L^*,\ i=1,\cdots,k-1)\\
&=\mathrm{P}(\theta_i+\ln 2\sigma_i-\theta_k-\ln 2\sigma_k\geq Y_{\mathrm{i}}+(m\ln 2-1)S_i/m-Y_k-(m\ln 2-1)S_k/m-c^*s_L^*,\ i=1,\cdots,k-1)\\
&=\mathrm{P}\Big(\tfrac{S_i}{m}\tfrac{\sigma_i}{S_i}\tfrac{m(-(m\ln 2-1)S_i/m+\ln 2\sigma_i+\theta_i-Y_i)}{\sigma_i}\geq\tfrac{S_k}{m}\tfrac{\sigma_k}{S_k}\tfrac{m(-(m\ln 2-1)S_k/m+\ln 2\sigma_k+\theta_k-Y_k)}{\sigma_k}\\
&-c^*s_L^*,\ i=1,\cdots,k-1\Big)\\
&=\mathrm{P}\Big(\tfrac{S_i}{m}G_i\geq\tfrac{S_k}{m}G_k-c^*s_L^*,\ i=1,\cdots,k-1\Big)\\
&\geq E_{S_1,\cdots,S_k}P\Big(\tfrac{S_i}{m}G_i\geq\tfrac{S_k}{m}G_k-\max(\tfrac{S_i}{m},\tfrac{S_k}{m})s_L^*,\ i=1,\cdots,k-1\Big)\\
&\geq\mathrm{P}(G_i\geq-s_L^*,\ G_k\leq s_L^*,\ G_i\geq G_k-s_L^*,\ i=1,\cdots,k-1)(\text{By using Lemma 1})\\
&\geq\mathrm{P}(-G_i\leq s_L^*,\ G_k\leq s_L^*,\ G_k-G_i\leq s_L^*,\ i=1,\cdots,k-1)\\
&=\mathrm{P}(\max(-G_i,G_k,G_k-G_i,i=1,\cdots,k-1)\leq s_L^*)=P^*
\end{aligned}
$$

It is clear that s_L^* represents the $100P^{th}$ percentile of the distribution of L and thus the proof is completed.

For (c), combining (a) and (b), we have

$$
\begin{aligned}
&\mathrm{P}(Y_{\mathrm{i}}+(m\ln 2-1)S_i/m-Y_k-(m\ln 2-1)S_k/m-c^*s_t^*\leq\delta_{0.5,i}-\delta_{0.5,k}\\
&\leq Y_{\mathrm{i}}+(m\ln 2-1)S_i/m-Y_k-(m\ln 2-1)S_k/m+c^*s_t^*,\ i=1,\cdots,k-1)\\
&=E_{S_1,\cdots,S_k}P(-G_i\leq s_t^*,\ G_k\leq s_t^*,\ G_k-G_i\leq s_t^*\cap\\
&\qquad -G_k\leq s_t^*,\ G_i\leq s_t^*,\ G_i-G_k\leq s_t^*,\ i=1,\cdots,k-1)\\
&\geq\mathrm{P}(\max(\left|G_i\right|,\left|G_k\right|,\left|G_k-G_i\right|,i=1,\cdots,k-1)\leq s_t^*,\ i=1,\cdots,k-1)=P^*.
\end{aligned}
$$

It is clear that s_t^* represents the $100P^{th}$ percentile of the distribution of T and thus the proof is completed. □

When the lifetime of products follows a two-parameter exponential distribution, this theorem can be used to find the upper confidence bounds and the lower confidence bounds for the parameters of $\delta_{0.5,i}-\delta_{0.5,k}$, $i=1,\ldots,k-1$, where $\delta_{0.5,k}$ represents the median lifetime of the control population. This theorem can also be used to find the two-sided simultaneous confidence intervals for parameters $\delta_{0.5,i}-\delta_{0.5,k}$, $i=1,\ldots,k-1$. Based on these estimations, experimenters can identify better-than-the-control, worse-than-the-control, and not-much-different-from-the-control treatment populations in terms of median lifetimes. The real-life example to demonstrate the application of this theorem is given in Section 3.

It is very difficult to derive the p.d.f. or c.d.f. for U, L, and T. Using the above three distributional results (1)~(3), we observe that

$G_i=\frac{-(m\ln 2-1)(\nu S_i/\sigma_i)+m\nu\ln 2-m\nu(Y_i-\theta_i)/\sigma_i}{\nu S_i/\sigma_i}\sim-(m\ln 2-1)+\frac{\nu(m\ln 2-Exp(1))}{\chi^2_{2m-2}}$, $i=1,\cdots,k$, $\nu=2\ m-2$.

If we can generate independent random variables G_i, $i=1,\cdots,k$, then we can find the empirical distribution of U, L, and T and the critical values s_U^*, s_L^*, and s_t^* are the empirical $100P^{th}$ percentiles of the distributions of U, L, and T, through Monte-Carlo simulation methods.

The steps to find the critical values of s_U^*, s_L^*, and s_t^* in theorem 1 are enumerated as follows:

Step 1: Generate k independent random variables E_i~$Exp(1)$ and another k independent random variables Q_i~χ^2_{2m-2} and then obtain the k independent random variables $G_i=-(m\ln 2-1)+\frac{\nu(m\ln 2-E_i)}{Q_i}$, $i=1,\ldots,k$.

Step 2: Compute $U=\max(-G_k,G_i,G_i-G_k,i=1,\cdots,k-1)$;

$L=\max(-G_i,G_k,G_k-G_i,i=1,\cdots,k-1)$ and

$T=\max(\left|G_i\right|,\left|G_k\right|,\left|G_k-G_i\right|,i=1,\cdots,k-1)$, $i=1,\cdots,k$.

Step 3: Repeat Steps 1,2 for 100,000 times. After sorting, we have $U_{(1)}\leq\ldots\leq U_{(100,000)}$; $L_{(1)}\leq\ldots\leq L_{(100,000)}$; $T_{(1)}\leq\ldots\leq T_{(100,000)}$.

Step 4: The critical values are obtained as $s_U^* = U_{([100{,}000*P^*]+1)}$; $s_L^* = L_{([100{,}000*P^*]+1)}$; $s_t^* = T_{([100{,}000*P^*]+1)}$, where $[x]$ is the largest integer less than or equal to x.

Remark: *For unequal initial sample sizes denoted as* $m_i, i = 1, \cdots, k$, *Theorem 1 can be modified by replacing* m *by* $m_i, i = 1, \cdots, k$.

For the practical use of application, we find the critical values s_U^*, s_L^*, and s_t^* by using the above algorithm under k = 3,4, ... ,10, m = 2,3, ... , 10,15,20,25,30 and P^* = 0.90, 0.95 and 0.975. The critical values are listed in the following table. From part (c) of Theorem 1, we observe that the length of the two-sided confidence intervals for $\delta_{0.5,i} - \delta_{0.5,k}$ is $L_1 = 2c^* s_t^*$. The larger the critical values, the larger the confidence length when c* is fixed. From Table 1, we observe that the critical value s_t^* increases when P^* increases for fixed k and m or when k increases for fixed P^* and m. Therefore, the confidence length L_1 increases when P^* increases for fixed k and m or when k increases for fixed P^* and m.

Table 1. Approximate critical values of s_U^*, s_L^*, and s_t^*.

k	m	$P^* = 0.90$			$P^* = 0.95$			$P^* = 0.975$		
		s_U^*	s_L^*	s_t^*	s_U^*	s_L^*	s_t^*	s_U^*	s_L^*	s_t^*
3	2	14.78	11.28	26.26	30.40	23.10	53.50	61.52	46.58	107.45
	3	7.31	5.71	9.92	11.16	8.79	14.78	16.55	13.00	21.59
	4	6.32	5.03	8.00	8.89	7.16	10.94	12.03	9.78	14.56
	5	6.12	4.94	7.52	8.28	6.77	9.85	10.74	8.87	12.60
	6	6.12	5.00	7.40	8.08	6.72	9.48	10.27	8.61	11.82
	7	6.22	5.14	7.45	8.07	6.81	9.39	10.08	8.62	11.52
	8	6.36	5.32	7.57	8.14	6.97	9.43	10.06	8.71	11.42
	9	6.52	5.47	7.72	8.30	7.12	9.53	10.13	8.86	11.46
	10	6.70	5.67	7.90	8.44	7.31	9.66	10.24	9.01	11.53
	15	7.55	6.53	8.83	9.33	8.29	10.60	11.11	10.02	12.37
	20	8.36	7.38	9.77	10.25	9.26	11.59	12.05	11.06	13.37
	25	9.12	8.13	10.62	11.08	10.12	12.52	12.97	12.01	14.36
	30	9.81	8.83	11.43	11.90	10.95	13.44	13.87	12.95	15.34
4	2	21.05	13.94	34.96	42.87	28.48	70.77	85.82	57.02	142.45
	3	9.17	6.33	11.72	13.87	9.58	17.36	20.37	14.06	25.14
	4	7.64	5.44	9.14	10.56	7.63	12.37	14.10	10.29	16.34
	5	7.24	5.27	8.44	9.58	7.13	10.93	12.29	9.27	13.86
	6	7.18	5.33	8.26	9.28	7.07	10.45	11.63	8.99	12.93
	7	7.23	5.47	8.27	9.20	7.14	10.30	11.34	8.95	12.51
	8	7.34	5.65	8.35	9.24	7.30	10.27	11.25	9.05	12.34
	9	7.48	5.84	8.49	9.32	7.49	10.36	11.23	9.22	12.32
	10	7.64	6.03	8.66	9.46	7.69	10.49	11.32	9.40	12.38
	15	8.53	6.98	9.60	10.33	8.72	11.38	12.11	10.45	13.19
	20	9.40	7.87	10.59	11.28	9.75	12.43	13.09	11.57	14.23
	25	10.20	8.68	11.49	12.16	10.68	13.38	14.01	12.58	15.22
	30	10.95	9.45	12.35	13.00	11.56	14.32	14.93	13.55	16.22
5	2	27.21	16.50	43.64	55.78	33.61	88.55	112.91	66.81	178.20
	3	10.75	6.88	13.27	16.09	10.35	19.53	23.52	15.07	28.26
	4	8.69	5.77	10.08	11.86	7.99	13.49	15.80	10.67	17.71
	5	8.12	5.56	9.21	10.65	7.43	11.84	13.60	9.59	14.91
	6	7.96	5.61	8.94	10.19	7.36	11.23	12.67	9.30	13.80
	7	7.97	5.72	8.88	10.02	7.40	10.97	12.26	9.20	13.30
	8	8.07	5.87	8.94	10.02	7.53	10.94	12.12	9.30	13.07
	9	8.19	6.08	9.07	10.07	7.75	10.98	12.06	9.48	13.00
	10	8.37	6.28	9.25	10.22	7.95	11.10	12.12	9.67	13.04
	15	9.24	7.28	10.19	11.05	9.05	11.98	12.85	10.78	13.78
	20	10.13	8.22	11.17	12.01	10.08	13.01	13.84	11.89	14.82
	25	10.96	9.10	12.12	12.92	11.09	14.02	14.78	12.99	15.84
	30	11.74	9.88	12.99	13.78	11.98	14.97	15.71	13.98	16.85

Table 1. *Cont.*

k	m	$P^* = 0.90$			$P^* = 0.95$			$P^* = 0.975$		
		s_U^*	s_L^*	s_t^*	s_U^*	s_L^*	s_t^*	s_U^*	s_L^*	s_t^*
6	2	33.14	19.15	51.76	67.64	38.55	105.07	136.19	76.71	211.14
	3	12.17	7.38	14.66	18.15	10.98	21.55	26.47	15.86	31.09
	4	9.58	6.06	10.89	13.00	8.32	14.57	17.26	11.08	19.08
	5	8.85	5.78	9.83	11.51	7.68	12.63	14.61	9.84	15.86
	6	8.62	5.78	9.49	10.94	7.55	11.87	13.50	9.50	14.54
	7	8.57	5.93	9.40	10.69	7.60	11.55	13.00	9.43	13.92
	8	8.65	6.09	9.44	10.64	7.77	11.46	12.79	9.53	13.66
	9	8.76	6.27	9.54	10.69	7.93	11.50	12.75	9.68	13.58
	10	8.90	6.47	9.69	10.78	8.14	11.58	12.74	9.84	13.56
	15	9.78	7.52	10.64	11.61	9.27	12.47	13.44	10.99	14.28
	20	10.70	8.48	11.65	12.58	10.36	13.49	14.41	12.16	15.30
	25	11.57	9.38	12.63	13.52	11.37	14.53	15.40	13.28	16.37
	30	12.35	10.21	13.50	14.36	12.32	15.46	16.28	14.31	17.35
7	2	39.31	21.78	60.49	80.06	44.01	122.99	162.06	87.46	249.19
	3	13.47	7.82	15.93	20.01	11.55	23.42	29.24	16.63	33.86
	4	10.35	6.31	11.61	13.97	8.63	15.46	18.48	11.43	20.25
	5	9.46	5.99	10.39	12.24	7.91	13.25	15.45	10.09	16.59
	6	9.16	5.96	9.95	11.53	7.72	12.39	14.20	9.67	15.10
	7	9.08	6.09	9.83	11.24	7.79	12.02	13.61	9.62	14.41
	8	9.13	6.24	9.86	11.18	7.92	11.93	13.38	9.69	14.16
	9	9.23	6.43	9.94	11.20	8.09	11.93	13.27	9.81	13.99
	10	9.36	6.63	10.08	11.28	8.29	12.01	13.27	10.02	14.02
	15	10.24	7.70	11.02	12.08	9.46	12.85	13.91	11.18	14.67
	20	11.14	8.70	12.02	13.02	10.55	13.87	14.84	12.38	15.67
	25	12.01	9.63	13.00	13.96	11.61	14.89	15.82	13.50	16.73
	30	12.84	10.47	13.90	14.85	12.56	15.86	16.78	14.52	17.75
8	2	45.50	24.27	68.97	92.79	48.74	140.29	186.20	97.45	282.34
	3	14.60	8.22	17.11	21.63	12.12	25.03	31.42	17.34	35.96
	4	11.08	6.52	12.28	14.92	8.87	16.30	19.68	11.69	21.34
	5	10.02	6.15	10.88	12.92	8.08	13.85	16.28	10.25	17.30
	6	9.63	6.12	10.37	12.07	7.88	12.88	14.87	9.84	15.69
	7	9.53	6.23	10.22	11.77	7.92	12.49	14.20	9.74	14.95
	8	9.55	6.38	10.21	11.64	8.05	12.31	13.87	9.81	14.57
	9	9.65	6.57	10.31	11.67	8.23	12.34	13.80	9.95	14.46
	10	9.75	6.77	10.41	11.69	8.44	12.35	13.70	10.15	14.40
	15	10.62	7.84	11.35	12.47	9.61	13.19	14.34	11.37	15.04
	20	11.54	8.88	12.35	13.41	10.75	14.20	15.26	12.52	16.03
	25	12.43	9.81	13.33	14.36	11.79	15.23	16.22	13.67	17.06
	30	13.27	10.68	14.27	15.28	12.78	16.24	17.19	14.76	18.12
9	2	51.55	26.67	77.45	105.29	53.72	158.46	213.80	108.06	319.87
	3	15.70	8.60	18.23	23.24	12.62	26.66	33.73	18.05	38.33
	4	11.70	6.74	12.85	15.76	9.12	17.08	20.76	11.97	22.33
	5	10.51	6.29	11.32	13.48	8.22	14.37	16.94	10.42	17.92
	6	10.08	6.25	10.76	12.60	8.00	13.32	15.42	9.94	16.18
	7	9.93	6.34	10.57	12.19	8.03	12.86	14.66	9.86	15.37
	8	9.92	6.51	10.55	12.05	8.18	12.68	14.30	9.96	14.96
	9	9.97	6.68	10.59	12.01	8.33	12.62	14.15	10.06	14.78
	10	10.10	6.89	10.73	12.08	8.54	12.70	14.12	10.28	14.75
	15	10.95	7.98	11.64	12.82	9.73	13.48	14.68	11.48	15.34
	20	11.88	9.01	12.65	13.76	10.88	14.49	15.59	12.68	16.30
	25	12.77	9.98	13.63	14.71	11.96	15.53	16.58	13.86	17.39
	30	13.63	10.85	14.57	15.64	12.95	16.52	17.54	14.92	18.41

Table 1. *Cont.*

k	m	$P^* = 0.90$			$P^* = 0.95$			$P^* = 0.975$		
		s_U^*	s_L^*	s_t^*	s_U^*	s_L^*	s_t^*	s_U^*	s_L^*	s_t^*
10	2	57.40	29.34	86.01	117.14	59.19	175.47	237.09	118.10	352.95
	3	16.70	8.95	19.24	24.59	13.04	28.02	35.59	18.63	40.30
	4	12.28	6.92	13.39	16.46	9.34	17.79	21.66	12.24	23.22
	5	10.94	6.45	11.72	14.00	8.40	14.87	17.58	10.61	18.54
	6	10.45	6.35	11.11	13.02	8.13	13.73	15.93	10.09	16.68
	7	10.27	6.43	10.88	12.59	8.14	13.20	15.10	9.96	15.78
	8	10.23	6.60	10.81	12.38	8.27	12.99	14.69	10.05	15.31
	9	10.33	6.79	10.90	12.41	8.45	12.98	14.55	10.18	15.14
	10	10.43	6.98	11.01	12.41	8.65	13.00	14.51	10.37	15.09
	15	11.23	8.09	11.88	13.11	9.85	13.73	14.99	11.59	15.62
	20	12.17	9.14	12.88	14.04	10.99	14.74	15.89	12.79	16.57
	25	13.08	10.10	13.89	15.02	12.08	15.78	16.88	13.95	17.65
	30	13.95	11.02	14.84	15.94	13.09	16.79	17.83	15.06	18.66

3. Example

Referring to Maurya et al. [11], the example of survival days of patients with inoperable lung cancer who were subjected to a standard chemotherapeutic agent is used to illustrate our proposed multiple comparison procedures with the control in Theorem 1. The patients are divided into four categories based on the histological type of their tumor: squamous, small, adeno, and large. The data are a part of a larger data set collected by the Veterans Administrative Lung Cancer Study Group in the United States. The survival days of 9 patients for four kinds of lung cancer are listed in Table 2:

Table 2. Survival times for four categories of lung cancer.

	Category	m	Survival Times							
1	Squamous	9	72	10	81	110	100	42	8	25	11
2	Small	9	30	13	23	16	21	18	20	27	31
3	Adeno	9	8	92	35	117	132	12	162	3	95
4	Large	9	177	162	553	200	156	182	143	105	103

Maurya et al. [11] had indicated that the data in the four categories may be assumed to be drawn from the two-parameter exponential distributions $E(\theta_i, \sigma_i)$, $i = 1, \cdots, k$. We regard the first category as the control population.

The required statistics and critical values for $P^* = 0.90,\ 0.95$ and 0.975 are summarized in Table 3.

Table 3. The required statistics and critical values.

Statistics	Category 1	Category 2	Category 3	Category 4
Y_i	8	13	3	103
S_i	48.375	10.250	78.265	106.750
c^*	11.861			
$Y_i - Y_1$		5	−5	95
$(m \ln 2 - 1)(S_i - S_1)/m$		−22.190	17.397	33.976
$Y_i - Y_1 + (m \ln 2 - 1)(S_i - S_1)/m$		−17.190	12.397	128.976
P^*	s_U^*	s_L^*	s_t^*	
0.900	7.48	5.84	8.49	
0.950	9.32	7.49	10.36	
0.975	11.23	9.22	12.32	

The upper confidence bounds and the lower confidence bounds for $\delta_{0.5,i} - \delta_{0.5,1}$, $i = 2,3,4$ under confidence coefficients 0.90, 0.95, and 0.975 are listed in Table 4 using parts (a) and (b) in Theorem 1. Since all upper bounds are positive, no categories are selected in a subset of all treatment populations which are worse than the control population (Category 1 lung cancer). Since only Category 4 has positive lower bound for all confidence coefficients, we conclude that only Category 4 is selected in a subset of all treatment populations which are better than the control population with the probability of correct selection being at least 0.90, 0.95, and 0.975.

Table 4. The 90%, 95%, and 97.5% upper bounds (the number before comma) and lower bounds (the number after comma) for three categories compared with the control category (Category 1).

	$Y_i - Y_4 + (m\ln 2 - 1)(S_i - S_4)/m + c^* s_U^*$, $Y_i - Y_4 + (m\ln 2 - 1)(S_i - S_4)/m - c^* s_L^*$		
	90%	**95%**	**97.5%**
$\delta_{0.5,2} - \delta_{0.5,1}$	71.530, −86.458	93.355, −106.029	116.009, −126.548
$\delta_{0.5,3} - \delta_{0.5,1}$	101.117, −56.871	122.942, −76.442	145.596, −96.961
$\delta_{0.5,4} - \delta_{0.5,1}$	217.696, 59.708	239.521, 40.137	262.175, 19.618

The two-sided confidence intervals for $\delta_{0.5,i} - \delta_{0.5,1}$, $i = 2,3,4$ with confidence coefficients 0.90, 0.95, and 0.975 are computed using part (c) in Theorem 1 and the results are listed in Table 5. For confidence levels of 0.90 and 0.95, only the confidence interval for Category 4 does not contain zero and the lower limit is positive. We conclude that only Category 4 has median survival days better than Category 1. For confidence level of 0.975, no categories are identified to have median survival days greater than Category 1 in terms of median survival days.

Table 5. The 90%, 95% and 97.5% two-sided confidence intervals for three categories compared with the control category (Category 1).

	$Y_i - Y_4 + (m\ln 2 - 1)(S_i - S_4)/m + c^* s_t^*$, $Y_i - Y_4 + (m\ln 2 - 1)(S_i - S_4)/m - c^* s_t^*$		
	90%	**95%**	**97.5%**
$\delta_{0.5,2} - \delta_{0.5,1}$	−117.89,83.51	−140.070, 105.690	−163.318, 128.937
$\delta_{0.5,3} - \delta_{0.5,1}$	−88.303, 113.097	−110.483, 135.277	−133.730, 158.525
$\delta_{0.5,4} - \delta_{0.5,1}$	28.276, 229.676	6.096, 251.856	−17.151, 275.104

4. Conclusions

We analyze the impact of confidence levels P^* and number of population k on the confidence length in this paper. Instead of doing multiple comparisons with the control for exponential mean lifetimes, we propose multiple comparison procedures with the control in terms of median lifetimes since the measurement of median lifetimes are more robust than mean lifetimes for the measurement of central tendency for exponential lifetime distribution. For the illustrative aim, we give a real life example to illustrate how to find the upper bounds, lower bounds, and two-sided confidence intervals for our parameters related to median lifetimes.

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