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The Pedagogy of Children's Mathematics in Number: Teacher Perspectives

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Abstract

This thesis evidenced changing pedagogies that support children's mathematical graphics. The study was in two parts: the first part was divided into two group case studies involving seven Reception teachers and eight Nursery School teachers in English state funded early years' settings; the second part focused on two single case studies drawn from the original Reception teacher case study group. This research was a qualitative, longitudinal study within an interpretative, socio-cultural paradigm. The data was gathered from questionnaires, interviews, written reflections of the participants' classroom practice and focus groups.

A main finding of the study was that the two case study Reception teachers during the years of the study shifted from a constructivist model to a child-centred /socio-cultural model of pedagogy. This thesis supports the view that pedagogies which support children to use their own ways of representing *their* mathematics are child orientated. Detailed shifts in pedagogy were also reported outlining the ways that the teachers moved to child-orientated socio-cultural practices. It was noted that these changes in practice were not easy if you are situated in a culture and belief system that is almost solely adult focused. Schools that advocated pedagogic relationships that enabled both the agency of the teacher and the child were vital for the teachers to experiment and trial their ideas in supporting children's mathematical graphics.

Two other important findings of this study were the importance of teachers' understanding and skills in child-led pretend play. Secondly, it surfaced that the mathematical pedagogies of the Nursery School and Reception teachers were fundamentally different and this discontinuity between the two teaching practices could be confusing for children. This study proposes a shift is needed in moving away from solely adult perspectives in early mathematics teaching, to authentically listening and noticing children's own mathematics and their mathematical cultural knowledge.

Key words: children's mathematical graphics, teacher perspectives, English Nursery Schools, Reception classes, child-orientated practices.

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My daughters, Mhairi-Zoel and Sovay, encouraged me to keep going and were, as always, by my side. I give special thanks to my husband, Tom, who helped with the references.

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Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's *Regulations and Code of Practice for Research Degree Programmes* and that it has not been submitted for any other academic award.

Except where indicated by specific reference in the text, the work is the candidate's own work.

Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

Signed: Elizabeth Carruthers Date:12th July

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Glossary

Early Years' Teaching School - Outstanding Nursery Schools designated by the UK government to support the early years sector in their community. They are involved in research and teachers training.

Copy recording - Children think through a mathematical problem, perhaps by using practical equipment, and then copy on paper what they have worked out practically. It also refers to copying teacher's calculations from the board.

Mastery teaching - Mastery learning is an Asian based style of mathematics learning which the English government has supported and financed in schools (Drury, 2018).

Unifix cubes - Small plastic cubes which can be connected on one side. Unifix Cubes were invented by Charles Tacey in 1953 to replace counting beads. The Tacey family was involved in the production of Froebel and Montessori educational materials in Europe and brought Unifix Cubes to the US through Didax Inc. Unfix cubes are versatile and easy to handle. They can be used to make patterns, record data, count, model fractions and compute.

Nursery Schools - English state funded nursery education settings with a head teacher and deputy head teacher and qualified teachers. They are part of a historical nursery tradition in England founded by the sisters, Rachel and Margaret Mc Millan, who were pioneers of nursery education and health care for children in poverty in the 1920's.

Numicon - A system of flat plastic shapes with holes. Each shape represents a number from one to 10 and each number has its own colour. Numicon was recommended by the Williams Review of Mathematics in England in 2008.

Multi-link - Small interlocking coloured cubes used in schools for counting and mathematical problem solving.

Reggio Emilia Pre-schools - Internationally famous pre-schools in the Reggio Emilia area of Italy. They are renowned for children's access to express themselves in multiple ways but mostly through art based materials.

Specialist Leader of Education - These are local teachers designated as part of the UK's Government's Teaching School initiative. They are usually specialists in a curriculum area. They support other schools and teachers.

Early Years Foundation Stage - The birth to five years phase of English education and care.

Reception Class - The last age band (4-5 years) in the Foundation Stage phase.

Funds of Knowledge - A term used by Moll et al. 1991 which means the cultural knowledge that children acquire from their everyday living experiences.

Key Stage One – The five to seven years phase of the English school system.

Chapter 1: Introduction

1.1 Introduction

I have been particularly interested in two areas of education over the past thirty years, children's mathematics and teacher agency. The power I saw in teacher's own experimentation and classroom research began when I was part of a teacher group. This was a small group of teachers who were dismayed by the imposition of the, then, new English National Curriculum (Department of Education and Science, 1989) which we felt stifled teachers' freedoms to develop their own thinking about how children learn and what pedagogy was the most effective. We met after school, meeting in our homes and schools. We were self-motivated and the agenda was always open, unlike the imposed professional development some of us were used to in our school situations. We were all particularly interested in improving our mathematics teaching and shared ideas and literature we had read on the subject. We came from different schools and taught different age groups, ranging from nursery to year 2 (3-7 years old). In our monthly meetings we sometimes invited researchers to have discussions with us. One of those researchers, who generously gave their time to share and discuss their research, was Martin Hughes. This thesis is borne out of the work of Martin Hughes (1986) who closely studied children and their families and his research influenced my inquiry into children's own mathematical representations. This thesis is also grounded in my history, as a teacher listening to, debating and having exciting conversations about young children's mathematics with other teachers and the insightful Nursery School teachers in the school where I was Headteacher for twelve years. I found that these teachers had their own strong voices which can change every day mathematical practice in classrooms to positively support children's mathematics.

1.1.2 The research problem

My research area is the pedagogy of children's mathematical graphics. It is about teachers' perspectives on their mathematics teaching and how they can teach in a way that uncovers and supports children's own representations of their mathematics. This builds on my earlier work and co-research with Worthington on children's mathematical graphics (Carruthers & Worthington, 2005). Children's mathematical graphics "is a term first used by Carruthers and Worthington (2005) and is used to describe children's (2-7 year-olds) own ways of representing their mathematical thinking and this can be through any kind of signs, drawings scribble-like, marks, mathematical notation and writing" (Carruthers, 2015, p. 1). Children's mathematical signs and representations are described in a range of terms including inscriptions, notations, symbolic tools, models, schematisations, visual signs, and children's mathematical graphics (Carruthers & Worthington, 2006.)

The Carruthers & Worthington (2006) research uncovered that children choose to use a range of graphics including their own invented mathematical symbols and standard symbols when they were working out a mathematical problem (for a more detailed account see Appendix 1). This research was rooted in Hughes' (1986) work on children and number where he observed that children of four years of age used their own non-standard symbols and each child represented quantities in different ways. "The noteworthy aspect of this research was that children were using their own thinking; they were not told what to write or given a model to copy" (Carruthers, 2015, p. 1).

In England, young children's written mathematics is often described as *recording* (Tucker, 2014). This is an activity that children will do after they have finished, usually, a practical mathematics session with a teacher. The teacher will ask children to record their work which could be, for example, copying what they have already done or copying what the teacher has put on the board. This is different from, but often confused with, children's own mathematical graphics (Carruthers & Worthington, 2006) where children use their graphics to aid their mathematical thinking which could be for a play situation or to work out their own mathematics problems or those set by a teacher. Key features of children's mathematical graphics include:

- children's own choice of ways to represent their mathematical thinking;
- children's own meanings, both independent and co-constructed;
- children's own graphics (e.g. drawings, writing, symbols, marks);
- children's own layouts and organisation of their mathematics;
  children challenge themselves as they explore their mathematical thinking and meanings, communicate their ideas and solve problems.
  (Carruthers & Worthington, 2010)

The data from my original research with Worthington (Carruthers & Worthington, 2005) was collected from our time as classroom teachers in the Foundation Stage and Key Stage One of the English school system. I wrote,

> When we first presented our findings at a research conference (Carruthers & Worthington, 2003) one of the professors in the audience, Bert van Oers from the Free University, Amsterdam, asked us how we got such rich examples of children's own mathematics. We could not give him a considered answer as we were so focused on the meanings of the children's graphics and not on the pedagogy that supported the children's mathematical representations. However, his question was an important and obvious one (Carruthers, 2015, p. 1).

The research on children's mathematical graphics mainly centres on exploring the graphics and children's meanings (Carruthers & Worthington, 2005; Worthington & van Oers, 2016). There is a link between the children's mathematical graphics and the pedagogy (Carruthers & Worthington, 2011), however, there has been insufficient research that directly addresses pedagogies that recognise and foster children's mathematical graphics. I, therefore, emphasise,

> There is a need for teachers to understand the pedagogical strategies that might be employed to uncover children's mathematical graphics and develop children's mathematical thinking. Knowing that children can use their own methods, symbols and notation to help them solve mathematical problems is not enough. If we only know but do not support or value this, then it will become hidden and the knowledge we have about young children's own mathematics will remain static. Therefore, the focus of this study is finding out how teachers can support children's own mathematical representations in Nursery School and Reception classes in the Early Years Foundation Stage (DfE, 2012) (Carruthers, 2015 p. 2).

It is important that Reception teachers are involved because my own reflections, on my years of studying children's representations of their mathematical thinking are, Reception is a year where teachers are unsure of how to introduce the formal symbols of mathematics (Gifford, 1997). Therefore, my premise is, if, given the opportunity, children can produce many ways of using their graphics to solve mathematical problems in Reception. The Reception year could be a transition period, where children come from informal and intuitive mathematical representation to later using and understanding standard ways of symbolic mathematical representation. The Foundation (pre-school and Reception) and Year One phase of schooling may be an optimum transition period for children to develop their mathematical understanding for more formal standard mathematics. This is why I feel this study might be crucial in understanding the pedagogical aspects that promote children's mathematical graphics in Reception classes to continue in Year One. Therefore, the engagement with Reception teachers is vital for this study.

1.1.3 Focus on children's mathematical graphics through the area of number

To focus the study, I am selecting one area of mathematics education and, therefore, I plan to centre on number and how teachers support children's emerging understanding of number, which includes counting and calculation. Gifford (2005) observed that teachers find it difficult introducing standard algorithms (Ofsted, 2011; DCSF, 2008). However, in our research (Carruthers & Worthington, 2006) we discovered young children had many ways of representing their mathematical thinking in number and the range is explained in our taxonomy (Appendix 2). It will be informative if this present research reveals pedagogical strategies that illuminate how teachers respond to children's mathematical graphics.

1.1.4 Background context

The participants in this study are fifteen early years teachers working in English Primary schools and Nursery Schools. In England official recognition that young children could use their own ways in writing their mathematics was first recognised in the Curriculum Guidance for the Foundation Stage (DfEE, 2000) where children's early number representations acknowledged Hughes' (1986) research and advised early years practitioners to ask children to, "put something on paper" when they were engaged in mathematical learning (DfEE, 2000, P. 71). The guidance gave an example of Hughes' (1986) tins game (4.2.2) which he used in his original research with children under five. There was also some reference to supporting children's written methods in the National Numeracy Strategy (QCA, 1999).

In the Williams Review (2008) of mathematics teaching in early years' settings, a strong case was made for early years' professionals not to miss opportunities for children to experiment with early mathematical marks. Our taxonomy (Carruthers & Worthington, 2006) was re-published and highlighted within this review and one of the major recommendations was to publish a set of materials on mathematical mark-making and children's mathematical development. Two documents for teachers and early years' practitioners followed. One firstly on mark-making (DCSF, 2008) and secondly on developing children's mathematical thinking which included mathematical graphics (DCSF, 2009). However, although these two booklets were published by the government there was no official professional development to accompany these materials and therefore it was very much left to teachers' own understanding. An issue with the Williams Review is that it only emphasised children in the Foundation Stage using their own graphics and it did not mention the development into Key Stage One; although that was redressed, to some extent, in the materials following the report in the document above (DCSF, 2009). I believe if we only see children in the very early stages of school life experimenting and devising their own ways of thinking through mathematical problems, with their own graphics to aid them, then there is a disconnection in mathematics education between the Early Years Foundation Stage, Key Stage One and Key Stage Two. There is also increasingly a recognition in research findings that children's own ways to represent their mathematical thinking will help their mathematical understanding (Pascal, Bertram & Rouse 2019; Anantharajan, 2020; Worthington & van Oers, 2016).

Within this study, therefore, an aim is to look more closely at the pedagogy of children's mathematical graphics through early years' teachers' eyes, seeing their barriers and ways forward to support children's mathematical graphics. As Stenhouse (1975) reflects, it is teachers who will make a difference to education. It is the teachers who are firmly centred in the real classrooms of experience and are constantly faced with the everyday dilemmas of the teaching world. However, in England there are major obstructions to teachers' freedoms to address, in their own ways, the pedagogy of children's mathematics because of the effect of neoliberal policies. Neoliberalism promotes agendas like competition, heavy accountability and micromanagement within the workforce and in schools. Therefore, there is a heavy emphasis on government structured curriculum and testing agendas which can make teachers less trusted and dampen their creativity and enthusiasm (Rogers, Davigo & Doan, 2020). There is also the pull of government initiatives like mastery learning.[1]Another obstacle is that the school starting age is much younger in England and the UK than in most other European countries and therefore young children and teachers can be faced with formal schooling curriculums. This means that there is less opportunity for teachers to experiment with new teaching approaches outside government controlled agendas. I expand on this in the next

[1] Mastery learning is an Asian based style of mathematics learning which the English government has supported and financed (Drury, 2018).

section.

1.1.5 The political agenda of performativity

One of the current stresses of the English education system is the government drive on accountability and performance concerning testing of academic skills. As Ball states, from the early 1990's there has been an increasing shift in education to "a culture and a mode of regulation measured and judged against a set of standards and is sweeping the education sector; it is constant and ongoing" (Ball, 2016, p. 36). Professional identities and personal theories and values may be undermined by the performativity agenda. Tensions must exist in organisations, especially when the values are different. Performativity can skew teacher identity and resilience (Ball, 2016) because there may be a danger of teachers becoming enslaved to the educational system through direct government agendas.

Rogers, Dovigo and Doan (2020) reflect on the English government's policy of austerity, cost-effectiveness and the education of nursery age children. This neoliberal context has created perceptions of quality within a performative agenda and there is a "stress on control and performativity hinders the development of the holistic child and undermines the professional identity of teachers" (Rogers, Dovigo & Doan, 2020, p. 809). Perhaps in the UK the Reception classes have the most pressure for the children in their class to achieve. Rose and Rogers' (2012) research demonstrates, quite starkly, the presence of severe pressures on Reception class teachers to meet the regulatory demands of a standards-based educational context, despite recent major developments towards a play-based framework for children under five years of age in England. This demand has been exasperated recently with the publication of Ofsted's *Bold Beginnings* (2017) in which the recommendations were that the EYFS should be streamlined, keeping it more in line with the increasing demands of year one and their narrow focus on mathematics and literacy. There are more pressures on government literacy education targets than numeracy but this focus on targets has moved the agenda away from child-led learning and play orientated teaching and learning. Moving away from child-centred practices has given early years national groups serious reservations about the lack of research behind this move (TACTYC, 2017). The new changes to the EYFS (DfE, 2021) has also made early years groups question the source of these changes. Although the early years' sector was consulted on the changes, the finalised document, including the mathematics section, was strongly opposed by the early years sector and their views were mostly ignored. This intense opposition to the new EYFS has, for the first time, boldly urged on early years' groups to join together to write their own document for the early years' workforce (Early Education, 2021) disregarding the government document. This is increasingly a turbulent time for the early years' sector and for early mathematics education. I now explain the thesis structure.

1.2 Thesis Structure

This thesis is presented in 10 chapters; chapters 2, 3, and 4 are a review of the literature of the research pertaining to the subject of the thesis. Pedagogy is a main thread of the thesis and chapter 2 defines pedagogy and pertinent research in this area including; a discussion of the main theorists; adult-child interactions, the debates surrounding the pedagogy of play and practical mathematics. Chapter 3 has a strong emphasis on the rights of children to be listened to and if children are listened to, how are they listened to? I also discuss the many ways children communicate and the importance of valuing all children's methods of communicating. Chapter 4 focuses on early mathematics education including mathematical play, research on children's mathematical representations and underlines teacher studies that focus on children's mathematical graphics. In chapter 5, I state my ontological and epistemological stance arguing a socio-cultural historical position. I put forward that mathematics is a fallible, human creation, embedded in socio-cultural practices and I adopt a socio-cultural perspective on teaching and also the pedagogy of mathematics. I then explain the methodology of my research where I describe how I am using case study research within an interpretative socio-cultural paradigm. Chapters 6 to 9 are the analysis of the data, firstly of the Nursery School and Reception teacher group studies and then of the two Reception teachers' case studies. Finally, in chapter 10, I present the main findings, recommendations, implications and conclusion of the thesis.

As this thesis is mainly about pedagogy therefore in the next chapter I present research and theories about pedagogy as the first chapter of the literature review.

Chapter 2: Pedagogical Perspectives

My research encompasses mathematics pedagogy focusing on early years' teachers' perceptions and therefore to begin the review of the literature this chapter focuses on pedagogy. It is in six sections. In Section 1, I define pedagogy and discuss social pedagogy and political contexts. Section 2 critically analyses play pedagogies and Section 3 debates changing perspectives on pedagogy from Piaget to Vygotsky. In Section 4, differing views on Vygotsky's idea of the Zone of Proximal Development and his two levels of concept formation are considered. The issues surrounding practical mathematics teaching and understanding abstract concepts in mathematics are underlined in Section 5. Finally, in Section 6 I reflect on some of the research theories regarding adult-child teaching interactions.

2.1 Defining Pedagogy, Highlighting Democratic Pedagogies and the Importance of Political Contexts

In this section, as well as defining the word pedagogy from different early childhood standpoints, I am drawing on the literature that focuses on democratic pedagogies particularly social pedagogy which fits, to some extent, with the socio-cultural stance of this thesis.

Siraj-Blatchford et al., (2002) describe pedagogy as the art, science or craft of teaching. The word is derived from the ancient Greek word for teacher of young children and is used in European early childhood practice (Rinaldi, 2006). Siraj-Blatchford et al., (2002) debate the nature of teaching in early years and argue that in "early years any adequate conception of educative practice must be wide enough to include the provision of learning environments for play and exploration" (p. 2). They also believe it is about instruction, although not necessarily direct instruction, and initiating and sustaining learning processes. The Department for Education (2012) in England states that it is an interactive process between teacher and learner and the educational environment and the aim is to achieve educational goals.

The definition of early years' pedagogy also varies between and in countries. Some European definitions seem to be rooted in early childhood and care. These include skills-based teaching and learning and a holistic style including families and community. In Australia, it is less centralised and varies from state to state (Raban & Kilderry, 2017). In Italy, early childhood education involves the scientific, political and philosophical (Mantovani, 2007). In most European countries early childhood pedagogy encompasses play and countries, for example, Finland and Norway, start formal schooling at seven years of age, at least two years after the United Kingdom. The age difference between countries in school starting age makes a difference to the understanding of the data and comparison of data between different countries, as the term pre-school can be anything from age three to seven years unless children's ages are specified.

Moss and Petrie (2019) highlight the importance of social pedagogy and its potential in education. They state that a predominant feature of social pedagogy is that it is holistic both practically and theoretically, seeking to understand the person as a whole. The pedagogues, as practitioners, see themselves as people in a relationship with the child as a whole person, supporting the child's overall development (Boddy et al., 2005). Moss and Petrie (2019) commenting on the current English education system put forward that social pedagogy can empower school-based education away from "its current dominant narrow, technical and highly instrumental incarnation, to acquire instead a broader, more holistic understanding, in which the school plays an important role but as only one locus for education" (p. 394). In this social pedagogical perspective, teachers are reflective practitioners encouraged to continually evaluate their practice applying theoretical and self-knowledge. They critically analyse a situation and respond to the particular circumstances involved. "They do not follow predetermined procedures, to be applied irrespective of context or circumstance" (Moss & Petrie 2019, p. 399).

Social Pedagogy has similarities to the ethos of the Early Years Foundation Framework in England (DfE, 2008, 2012, 2014). This framework claims to offer a holistic curriculum that includes personal, social and emotional development, communication and language and physical education as basic priorities for early years' learning. However, Moss (2007) argues that it comes across as a manual for technicians because it creates no "democratic space" (p. 10) and gives little opportunity for teachers and practitioners to engage their thinking in the curricula they present to children. He goes on to contrast this with the Nordic countries who, he says, give their teachers broad principles, values and goals to work with but leave much open to their own interpretation as they are trusted individuals.

Democracy is a key focus of social pedagogy and has links to the work of Dewey (1916) and Rousseau (Damrosch, 2007) who both emphasised the importance of democracy in education where children's views and interests are central in determining educational environments. Dewey particularly advocated a democratic society of informed and engaged enquirers which upheld the intellectual freedom of teachers. For Dewey, schools are only one way of educating children and, he argues, "compared with other agencies, a relatively superficial means". This view of schools as limiting education is echoed by Dhalberg, Moss and Pence (2006) who see that pedagogy is in society and is intricate:

> Pedagogical work is embedded in life and the work we live in. It is not some decontextualized abstraction that can be readily measured and categorised. Instead making meaning, deepening understanding, or attempting to make judgements, will be a struggle, full of 'contradictions and ambiguities' (p. 110).

Moss stresses that democracy is vital as an important principle of citizenship. It is a way in which children and adults can be involved in matters which affect them which include challenging oppression and "injustices that arise from the unrestrained exercise of power" (Moss, 2007, p. 2).

Moss's views on power and oppression is an argument that Freire (2000) puts forward when he inserted questions of power, culture and oppression in schooling. Children in some education cultures, to a varying degree, understand that "to achieve some satisfaction they must adapt to the precepts which have been set from above. One of these precepts is not to think" (P. 36). Giroux (2004) also asserts that pedagogy is a moral and political practice, and Simon (1987) observed that,

> any discussion on pedagogy must begin with a discussion of educational practice as a form of cultural politics, as a particular way in which a sense of identity, place, worth, and above all value is informed by practices which organize knowledge and meaning (p. 372).

Neoliberalism has taken over some political education agendas, and in England over the last forty years, as already stated in 1.1.4, pedagogy has been increasingly guided by government interference and teacher agency has been heavily reduced, resulting in restricted pedagogies (Smith, Tesar & Myers, 2016). Giroux recommends that pedagogy needs to be reclaimed and there needs to be "a critical subversion of power itself" (Giroux, 2004, p. 33).

Moss (2007) concludes that "democracy creates the possibility for diversity to flourish. By so doing, it offers the best environment for the production of new thinking and new practice" (p. 2). It is this new thinking and practice that is vital to this thesis as children's own mathematical inscriptions, writings or graphics are not a widely understood or practised pedagogy (Worthington, Dobber & van Oers, 2019). Unravelling the pedagogy needs freedom of thought for the teachers to experiment, and for some to work outside the boundaries of accepted school practice and perhaps against government agendas. Strauss, Ziv and Sten (2002) highlight the many decisions the classroom teacher has to make and that "teaching is an extraordinarily complex enterprise" (p. 1476). Early years' teachers also have to consider their pedagogical stance on play, and this adds to the many intricacies involved in everyday pedagogy in early years' classrooms. Therefore, to delve further, in the next section I discuss play and pedagogies.

2.2 Play Pedagogy

Early years' concepts of teaching and learning are a heavily contested, sensitive area (Pramling et al. 2019) and play itself, although mentioned in every version of the English Early Years Foundation Stage[2] documents since 1996 (SCAA, 1996), is interpreted in many different ways (Rogers & Evans, 2008). Therefore, in this section I underline current debates in play, firstly outlining Vygotskian views on play.

Vygotsky (1978) understood play as paramount, particularly socio-dramatic play (Hedges, 2010), or imaginative play which van Oers (1996) described as pretend play. This, in Vygotsky's (1978) view, "gives opportunities for the development of everyday concepts and provides a channel between spontaneous and scientific (academic) concepts (p. 238)" (This is explored further in 2.4.1). Vygotsky (1978) saw imaginative and symbolic play as a medium through which children can represent the meanings of their individuality in everyday experiences. This concurs with the research on children's funds of knowledge,[3] and play as a place where children can own and act out their knowledge (Gonzalez, 2005). Holzman (1997) and van der Veer and Valsiner (1991) state play has a learning capacity only when the play environment has the capacity to challenge children to span their Zone of Proximal

[2] The Early Years Foundation Stage in England is from birth to five years.
[3] The term means the knowledge that children accumulate from their home, community and their everyday lives.

Development (This will be explored further in section 2.4). Challenging children in play demands "social interaction where the preschool teacher plays an active role, challenging the child and encouraging him or her to create new meanings and understandings" (van der Veer & Valsiner, 1991, p. 61).

For the Western early years' professional community, play is deemed central to children's learning (Moyles, 2014) but the pedagogy and understanding of play are intricate (Wood, 2010). Play in Western education communities includes physical play, play with blocks, playing with toys and malleable play (Bilton, 2018). How much the adults are involved in children's play is a highly debated issue. Fleer (2019) believes that many teachers and practitioners advocate that the emphasis should be on setting up the environment and then the children can play freely with the selected resources. Bruce (1991) who puts forward a concept of free-flow play agrees with a mostly non-interventionist approach letting the children wallow in their own play worlds. She cautions that if adults take part in children's play, they must be sensitive to the child's agenda and not dominate and that children choose to play, they cannot be made to play. Brostrom (2016) believes that Bruce's concept of free-flow play comes from a Froebelian perspective but argues this is not in line with Froebel's original concept "where play can also be adult-directed with a learning perspective […] for example, a system of gifts and occupations (play gifts) were incorporated in purposeful play"[4] (p. 3). However, the term "purposeful play" which is also used in the EYFS Statutory Framework (2017, p. 9) in England, is also problematic. It gives the impression that play without careful planning and expected aims such as children's spontaneous play without adult-intended outcomes is not acceptable. As Wood (2010) states, "the emphasis on 'purposeful play' carries the opposite assumption that without pedagogical framing, play would be purposeless" (p. 18). Wood goes on to say free play, with few restrictions, has always been a problem in educational settings "as it seems to conflict with the set pedagogies of policy frameworks" (p. 18).

Play, therefore, can be seen as unpredictable and the outcomes cannot be planned for. Adults who promote mostly adult-led learning can find the unpredictability of play, chaotic, difficult to understand and sometimes bewildering (Rose & Rogers, 2012). In England, with pressures from a goal-orientated curriculum and assessment-led education system, confusion arises when

[4] Froebel occupations are materials given to the children to practise a skill, e.g. clay, paper folding, and weaving. Froebel gifts are for exploring, e.g. different sized blocks and beads.

teachers in Reception classes, for example, plan role play with expected outcomes (Rogers, 2010). Adams (2007) exemplifies this challenge for teachers as she described a possible situation of a Reception teacher who transformed the roleplay area into a well-resourced shop (including cash till, receipts, real money, notepads, pencils and food) to give children a vehicle to incorporate shopping into their play. The mathematical intent was to develop children's understanding of numeracy. However, instead of using the resources as the teacher had planned, one child selected a bunch of bananas from the shop area, cuddled it into a blanket, walked out of the shop to the book corner, chose a story and then read her *baby* a story. As she finished the story, she gently "put the bananas to bed" (p. 246). The teacher's immediate response was to remove the bananas and suggest the child "goes shopping" instead of observing the mathematical learning possibilities that the child's play afforded (Adams, 2007, p. 246). Adams discussed the pressure teachers are under and the many decisions they have to make in a busy early childhood environment with little time for reflective practice.

Rogers (2010) also found dismissal of children's play themes in Reception classes, stating that the dominant adult curriculum and goal-orientated culture takes over. She suggests what is left is "the myth of a play-based curriculum grounded in an illusion of choice" (p. 154). Hedges (2010, p. 20) reflects that "the adage of learning through play has never sat comfortably alongside the notion of teaching through play". One of the questions that early years' pedagogy highlights, in the bananas' scenario, is, how informed is the teacher about children's play and development. Rogers (2010) argues that "the relationship between play on the one hand and pedagogy, on the other, is under-theorised and under-developed in early childhood education" (p. 154).

Fleer (2019) concurring with Rogers' views that play and pedagogy has not been sufficiently researched argues for "new understandings of the relations between play and learning as a synthesis" and, in so doing, claims a new pedagogical theoretical concept of "play-responsive teaching" which she explains means:

> the teachers' participation in children's play requires a high level of responsivity to the children's perspectives. Their role is not just to shadow children and to simply observe them and contribute very little to their play. What is argued is that teachers need to find the ways to introduce the seeds of new directions and give

> new possibilities for the children's play – but without compromising children's agency and play narrative (2019, p. 31).

What Fleer suggests needs highly skilled teachers. It is what Goouch (2010) describes as she observed the play pedagogy of two Nursery School teachers, "Adults and children were 'multi-voicing' [but were also…] directing the play according to the socially constructed narrative of home and experiences" (p. 130). Goouch (2010) goes on to say that these "remarkable" (p. 10) teachers were "both insiders and outsiders of the play-action, inside and outside of reality and fantasy" (p. 130).

In summary it has been highlighted that play holds possibilities for teaching and learning but not automatically; thought-provoking material and considerate, collaborative adult-child interactions need to be prioritised (Brostrom, 2017). There is a move to shift practice in play pedagogy from a child-centred orientation characterised by free play, and provide more space for adult interaction (Fleer, 2019) but, as Fisher (2016) states, not adult interference. However, for this to happen teachers need to be more in tune with children's own play themes and understand their funds of knowledge (Gonzalez, 2005) to be with their learning processes. Hedges (2010) argues that teachers need to be more critically aware of children's backgrounds and home learning knowledge, including their current interests and enquiries, to plan an effective play curriculum.

Children's play is also relevant to mathematics pedagogical practice and also has relevance to children's own mathematical representations which can occur in play (Carruthers & Worthington, 2011; Carruthers & Butcher, 2013; Worthington & van Oers, 2016; Papandreou & Tsiouli, 2020). It has been stated that play pedagogy, in general, is underdeveloped and similarly research in play and mathematics is embryonic. This thesis may contribute to the area of mathematics play pedagogy by involving teachers and their understanding of mathematical play and will be discussed further in chapter four of this study. I now critically discuss some seminal pedagogical perspectives highlighting the work of Piaget and Vygotsky.

2.3 Changing Perspectives on Pedagogy; Piagetian and Vygotskian views

Vygotsky and Piaget have been key theorists in learning, influencing educational thinking in the past seventy years. One of the main aspects of the Carruthers and Worthington's (2005; 2006; 2011) research is children's mathematical graphics (see 1.2) are explained as cultural tools (Vygotsky, 1978) which children use to make meaning of their mathematics. The children use their own graphics to communicate their mathematics and to solve mathematical problems (Carruthers & Worthington, 2011). Therefore, to uncover the children's mathematical graphics, the pedagogy needs to be democratic where children are given opportunities to make their meanings and the teacher gives children that freedom to explore their own thinking. The teacher has to recognise the cultural knowledge that children bring to their learning. Therefore, Vygotsky's (1978) theory on socio-cultural learning and related pedagogy is paramount in this thesis which looks at Vygotsky's work as a foundation, to reflect on the understanding of the child as a learner and the role of the teacher as a mediator in the child's learning.

Rogoff (1990) believes that Vygotsky's socio-cultural historical theory came from an understanding that individual intellectual development arises from the social and cultural situation in which the child is embedded. Vygotsky termed this cultural-historical; it is also known as socio-cultural. It means that the child learns from everything around them including their previous experiences, their community and their families. The teacher is only one influence in this learning environment.

2.3.1 Piaget

Piaget (1964) strongly influenced early childhood educators and play theorists and the English Infant School model of the 1960s and 70s was heavily based on Piagetian research (Clemson & Clemson, 1994). Piaget predominately focused on the child as a lone scientist working with materials and making their sense of the world. The application of Piagetian theory in practice was *discovery learning*. It was a non-interventionist approach to play and learning for young children at least up to the age of seven (Dickson, Brown & Gibson, 1984). When Vygotsky's work began to be translated, Piaget's research was questioned and comparisons were made (Sutton-Smith, 1966; Matusov & Hayes, 2000). The Piagetian approach was questioned by early childhood researchers; for example, Bennett, Wood and Rogers (l997) argue that in their study of Reception teachers that this Piagetian view limited the teachers' role in supporting the children's learning and, as a result, learning opportunities were missed.

Piaget, however, is important, especially in the field of early childhood, as he was one of the first researchers to analyse young children's cognitive development and how they learn (Matusov & Hayes, 2000). His theories also influenced the way researchers and educators thought about young children. Rather than simply viewing them as smaller versions of adults, early childhood educators began to acknowledge that the way children think may be fundamentally different from the way adults think (McMahon & Rose, 2019). Piaget acknowledged young children's capabilities, but his stage theory seemed to restrict the scope of their learning. The translation of his work into teaching practice brought in the concept of readiness and the premise that children need to be ready for the next stage of their development. His work on the conservation of numbers (Piaget, 1952) influenced mathematics curricula. Interpretations of this included a teaching focus on sorting, sets and matching activities until children were ready for direct number experiences (Clemson & Clemson, 1994; Williams, 1996). This has since been questioned in early mathematics teaching (Aubrey, 1994; Williams, 1996). Interpretation of Piaget's work also stressed practical mathematics teaching until children were ready for more abstract concepts. Teaching children to write mathematical operations was discouraged as it was thought that they would not understand abstract mathematical standard notation (Gifford, 1997). Certainly, this understanding has relevance to Hughes' work (1986), where he found that children even up to nine years old, when given the option, would prefer not to use standard mathematical algorithms.

In Piaget's pre-operational stage (2 to 7 years) children begin to use symbols and representations, for example, in imaginary play (Piaget, 1954). This was interpreted as meaning that children should have access to play situations up to the age of seven, with which many early years' specialists and researchers would agree (Wood, 2014; Goouch, 2010; Rose & Rogers, 2012; Moyles, 2014). However, Piaget did not sufficiently emphasise the part society and culture played in children's learning and critics argue that children are not in a pre-state of learning (Hall, 1987). Children have already been exposed to knowledge of many concepts outside of school and, although their understanding may be partial (Athey, 1990), it is part of their growing knowledge, their developing schemas[5] (Arnold, 2003). This knowledge needs to be recognised by teachers as part of classroom enquiry and thinking (Gonzalez, 2005).

2.3.2 Vygotsky

There has been a shift away from the Piagetian stage theory of readiness to learn concepts and regarding the child as an isolated thinker. Research has moved to studying how the child's thoughts are being influenced by their communities and social worlds. This has now heavily

[5] Schemas are children's patterns of thinking. The term originated from Piaget's work (1954) and was developed by Athey (1990).

shaped many early childhood education scholars further understanding of young children and how they learn (Rogoff, 1990; Fleer, 2010; Jordan, 2009; Hedges & Cullen, 2012). Rogoff (1990, p.52) argues that,

> For Vygotsky children's cognitive development must be understood not only as taking place with social support in interaction with others, but also as involving the development of skill with socio-historically developed tools that mediate intellectual activity. Thus, individual development of higher mental processes cannot be understood without considering the social roots of both the tools for thinking that children are learning to use and the social interactions that guide children in their use.

The social interactions Rogoff is underlining above are the adults or community that children are involved in and this includes teachers. The socio-historically developed tools are what the child uses to gain and use knowledge such as forms of language and writing and communications systems which include mathematics. Vygotsky's cultural-historical theory, therefore, has implications for the role of the adult, in the child's learning, and this is a major move away from the Piagetian non-intervention position. Vygotsky sees the teacher as a mediator connecting with the child's social world. Therefore, it is important that teachers know children's backgrounds, life out of school and their home learning perspectives for this is the knowledge base that the children draw upon to make connections with school learning. Scrimsher and Tudge (2003) emphasise this and believe that, from a Vygotskian perspective, the child's culture is highly important and so effective teaching requires teachers knowing about the child's background to understand the context of the child's world. For example, Jordan refers to teachers responding to children's "funds of knowledge" (2010, p. 98). This means teachers knowing the children's home and community experiences and making links between their experiences and the offered curriculum. Jordan (2010) goes on to explain,

> Teachers' responses to what they notice children engaged with also depend on their learning more about those interests in which children themselves could already be well informed, so that they, the children, will be able to co-engage in developing ever greater complexity and breadth of understanding about the topic area (p. 99).

Cullen and Jordan (2004) also emphasise the importance of relationships in and beyond the school for the child's intellectual growth. Ma (2012) stresses "knowledge construction as social in nature and cultural in origin is a dynamic process of learning and development that mediates and is mediated by children's social and cultural experiences" (p. 453).

Fisher (2016), in her project with early years' teachers and practitioners again highlights the importance of knowing the child's home and community experiences as it gives context to the exchanges between teacher and child. These 'funds of knowledge' can also be mathematical (Papandreou & Tsiouli, 2020). Research into young children's community experiences, outside the home, found that children were noticing and using mathematical knowledge in explicit and implicit ways. These mathematical experiences can affect the mathematical knowledge that they bring to the classroom (Macdonald & Carmichael, 2018). The teacher, therefore, has a pivotal role in valuing and building on these relationships to co-construct the child's learning.

2.4 The Zone of Proximal Development and Concept Formation

The *Zone of Proximal Development* is a term used by Vygotsky. Daniels (2007) explains this term by observing that the individual learner learns from a more knowledgeable other who can be an interested adult or a more mature child. Scrimsher and Tudge's (2003) view is that the Zone of Proximal Development "is not some clear-cut space that exists independently of the process of joint activity itself" (i.e., between the child and knowledgeable other in collaboration) (p.300). They add, this space is not, as some authors state that the teacher's role is to identify the space "between what the child currently knows and what the teacher can help him to know" (p. 300). Instead, the ZPD is developed in co-operation between teacher-child or peer interaction as a joint emergent understanding.

Wood, Bruner and Ross (1976) and Brown and Ferrara (1985) explain that scaffolding (this term is explained in 2.6.1) in the ZPD is about the adult or mature child carefully controlling the task by making it easier for the child to perform it until they can do the whole task independently. Scrimsher and Tudge (2003) argue that although simplifying the task might be successful, it is far too teacher-dominated and controlled and this, they conclude, is not what Vygotsky meant.

Brown, Metz and Campione's (1996) work on social learning communities sees the classroom as a busy hive where interlocking and overlapping *Zones* of Proximal Development are being supported. This support is not by one adult, but by the whole classroom learning community.

This is in opposition to the traditional model of direct teaching where one skill is taught in isolation. Therefore, the ZPD can be seen as broader and more complex than some original interpretations. The teacher is only part of the learning influence.

2.4.1 Concept formation

Another vital part of Vygotskian pedagogy is knowing how children learn concepts. Fleer (2010) explains that Vygotsky was particularly interested in two types of concept formation. The first level is what he termed the everyday concept formation which is through interacting directly with the world and developing an intuitive understanding of how the world works by, for example, going shopping or planting seeds, but without understanding the scientific concepts behind these interactions. Level 2 concept formation is the scientific or academic level: for example, knowing about the metamorphosis of a butterfly; and that a current of electricity is a steady flow of electrons. The first concept formation, the everyday, is different as it develops in the child's knowledge "in the process of living and which were assisted with his social environment" (Vygotsky, 1978, p. 130). They "are formed aside from any process specially at mastering them" (p. 130) whereas the scientific/academic concepts formation is where the child switches to a "new specifically cognitive relationship to the world" (p. 130) where a science/academic system of concepts takes the precedent, especially in school learning. Fleer (2010), referring to Vygotsky's work, argues that when scientific or academic concepts are introduced in isolation away from the young child's everyday life then they become disembedded. The two, every day and academic concepts are dialectically connected which means the teacher and child are thinking together about the academic, linked to the everyday aspect of being involved in the process. Vygotsky (1987) was clear that direct instruction of concepts was impossible; the transmission mode of teaching was not fruitful in supporting the child's understanding of concepts.

> The teacher who attempts to use this approach (direct instruction) achieves nothing but a mindless learning of words, an empty verbalism that stimulates or imitates the presence of concepts in the child. Under these conditions, the child learns not the concept but the word and this word is taken over by the child through memory rather than thought. Such knowledge turns out to be inadequate in any meaningful application (Vygotsky, 1987 p. 170).

For example, a teacher teaching about horses, to children who have never seen or known anything about horses, simply shows them the word horse and says, “it has four legs”. The children then recite the word horse several times. In this example, the teacher has not fully engaged the children in the concept of what a horse is. This learning is disembedded and the children cannot make connections to what they already know, for example, about farms or animals. Donaldson (1985) discusses this, “as presented in abrupt isolation and presented, to begin with at least, by some other person (the teacher) whose purposes are obscure” (p. 122). It would seem Donaldson is also implying the children are confused and do not see the sense or aim in this delivery of information by the teacher.

Hedegaard and Chaiklin (2005) proposed a teaching concept of “double move”, meaning that the teacher remembers Vygotsky’s two levels: everyday concepts and scientific concepts. For example, by wiping the table after a cooking session, the teacher keeps in mind scientific concepts and everyday concepts. The teacher supporting the child wiping the table, therefore, might use language such as area, corners and perimeter and demonstrate them in the context of table-wiping. This is different from the disembedded model discussed above and the non-interference model proposed by Piaget (1952) in which children are supposed to learn by themselves in the environment. McNaughton (2003) argues that the Piagetian environment was set up to create a learning atmosphere with carefully selected resources for the children to select, engage and problem solve by themselves. The environment in early childhood is a major part of the pedagogy (Fleer, 2019; Rose & Rogers, 2012; Fisher, 2016). However, from a Vygotskian perspective, the adult’s role is active and this has implications for mathematics teaching and children’s mathematical graphics. Children in a well-resourced environment will choose to write their mathematics (Worthington & van Oers, 2016; Carruthers, 2012) and the teacher will support their mathematical thinking in this environment. The resources that teachers select for the environment dictate how children might represent their mathematics. It is common in early years’ mathematics teaching to provide only practical materials in teaching mathematics (Anatharajan, 2020). The issue is, if children are only presented with practical materials and standard mathematical resources, for example, cubes or beads then they may not be able to use graphical ways of representing their mathematics. In the next section I further discuss practical mathematics teaching and contentions surrounding abstract and concrete concepts.

2.5 Practical Mathematics Teaching

The recurring problem in the teaching of mathematics seems to be how do we teach so that children understand the abstract (Askew, 2001). For example, the algorithms in calculations are abstract and difficult concepts for young children to understand (Hughes, 1986; Thompson, 2008). There is still a Piagetian influence on much of the teaching practice for young children and his theory seems to imply that young children needed to handle and manoeuvre objects, to develop visualization, reasoning and understanding which are often termed concrete experiences (Piaget, 1952). Working with concrete experiences has been a trend in early mathematics education since the 1960s, along with worksheets (Pound, 2006). Many early childhood writers emphasise teaching through practical mathematics (Threfall, 1992; Griffiths, Back & Gifford, 2017) where children count objects or have tins of food to calculate in shop play. Early years' teaching of mathematics relies heavily and sometimes solely on using practical resources as representations to aid children's mathematical thinking (Pound, 2006). Hughes (1986), Askew (2001), Carruthers and Worthington (2006) argue that it is quite a shift in thinking from manipulatives to abstract symbols and may add to children's lack of understanding standard calculation notation and other mathematical concepts at a higher level. Gravemeijer (1991) states that the peril in working with manipulatives is that it does not give children the preparation of working without manipulatives and thinking about abstract conceptual understanding. Van den Brink (1993) believes that abstraction is not linear and can be seen as children's mathematisation,[6] activities in which children move backwards and forwards between the real world and the world of symbols. Coles and Sinclair (2019) question the assumption of starting with the concrete when teaching mathematics. They highlight the work of Gattengo (1974) and of Davydov (1990), explaining "that there is no universal pattern of learners reaching the abstract as a culmination of experiences that begin with concrete examples, and/or in which mathematical symbols start off having direct and absolute concrete referents" (p. 5). Therefore, whilst it is well documented that manipulatives have a positive effect on children's mathematical learning (Griffiths, Back & Gifford, 2017) there are increasing debates contesting that when children use practical materials they are not necessarily making the connection between the concrete and abstract.

[6] Freudenthal (1973) states that mathematization is organising or structuring subject matter by mathematical means.

Through studying children using their own forms of written mathematics to calculate their problems, Carruthers and Worthington (2006) argue that children are abstracting for themselves and they may more easily understand the mathematics because they are making their own connections. There is no sharp change from their methods to more abstract forms of mathematics; instead, they weave in and out of both, only taking on as much as they understand (Carruthers & Worthington, 2005). There are similarities between this and the 'double move' concept described by Hedegard and Chaklin (2005). Exposing children to the scientific (abstract) concepts of mathematics, but at the same time, they are using what they already know and also incorporating some of these more abstract concepts. For example, some children draw hands to signify take-away but at the same time use standard ways to write digits (Carruthers & Worthington, 2006). The teacher has kept in mind the importance of modelling standard methods of calculation but at the same time is encouraging and supporting the children's own methods.

In summary, pedagogical approaches to mathematics have to consider the materials and opportunities that are afforded to give children access to both practical activities and abstract thinking. Teachers also might have to think about the interface between concrete and abstract thinking. Whatever resource or approach teachers use to teach mathematics, or any subject, it is the teachers' skills that are vital to successful teaching, in this is the importance of child-teacher exchanges (Stephens, 2010) which is the focus of the next section.

2.6 Adult-child Pedagogical Interactions

How teachers interact with young children to support their learning and development has always been a prominent part of the pedagogical debate (Rose & Rogers, 2012). In this section, I discuss theories of teacher-child pedagogical relationships.

2.6.1 Scaffolding

Section 2.4 looked at Vygotsky's writings in terms of mediation, concept development and the ZPD, touching on the teacher supporting children's learning. However, Vygotsky's writings are translated and they rely on the interpreter's understanding of the language nuances and conceptions (John-Steiner, 1985). Also, Vygotsky's work on teacher mediation is not extended. For example, Moll (1990) points out that Vygotsky never made it completely clear what kind of support the more knowledgeable others give to the learner. He only mentions collaboration and direction, assisting children through demonstration and leading questions and introducing the initial element of the task's solution. However, other researchers have added to Vygotsky's concept of the ZPD and explained the role of teaching as *scaffolding* (Stone, 1998). Scaffolding, as defined by Wood, Bruner and Ross (1976) is when the adult assists a child in a sensitive way that gradually enables a child or novice to solve a problem, carry out a task, or achieve a goal that they cannot do alone. The child eventually masters the task and carries it out unassisted. The important part of the teaching is that the adult supports the child in the bits of the task they cannot do and encourages them in the parts they can do.

The issue is how is this best done and how much support should the adult give? Newman, Griffin and Cole (1989) questioned the part of the adult as the scaffolder and explained that this is not a one-way scaffold but that it is negotiated between adult and learner. Cole and Griffin (1984) also queried the scaffolding part of the teacher in that the child's creativity might be underused. Brown and Palinscar (1989), working with older children, developed a system called reciprocal teaching where teachers and their students engage in collaborative dialogue about a text. The role of the teacher rotated and involved all the students. Brown and Palinscar (1989) suggested that this promoted *expert scaffolding* in co-operation with teachers and more able peers in an activity that they would not understand if they worked on it alone. In early years' teaching, this might have links with sustained shared thinking (Siraj-Blatchford et al., 2002) and can be applied to any subject, including mathematics. Again, there was stress on the collaboration between teacher and child and thinking together in democratic practice.

Daniels (2007) was concerned that the term scaffolding, absent of a clear set of theoretical explanations, can be sculpted and formed into any teaching and learning model. Goouch (2010), studying two English Nursery School teachers' practices, questioned the use of the term scaffolding as sometimes limiting the scope of children's learning. She argues that there is a much deeper relational way of understanding teaching practice with young children. She believes that terms and labels to describe scaffolding are pedagogical pegs (p. 131) that legitimise teaching in an overly accountable education system. These pegs could be described as teaching methods. Stenhouse (1975) states that *teaching methods* is a term that suggests teachers are to be trained in a variety of skills. However, both Stenhouse (1975) and Drummond (2003) see that teachers need to have a vision for themselves and principles which they apply in their teaching and this links with the previous discussion on democracy and social pedagogy (2.1).

Scaffolding, therefore, has a variety of interpretations and explanations which can sometimes appear adult-dominated with less emphasis on the child's views and cultures. There is much debate on how much the child controls their learning and between who leads in the teacher-child relationship and some views are polarised (Fleer, 2019). Some scholars believe that play belongs to the child and therefore the term adult-led and play are conflicting (Fisher, 2016). Rose and Rogers (2012) also identify this problem, they argue,

> By viewing all activities and exchanges as a process of initiation that immediately becomes an interconnected negotiation, rather than an act of being led or directed by either the child or the adult, we can envisage the adult-child relationship as one that involves interchangeable processes of give-and-take and mutual co-construction. (2012, p.9)

Rose and Rogers, therefore, view the relationship between adult and child in teaching and learning as constructing shared understandings; and the confusion around who leads is not as important as who initiates ideas and possible learning situations. They consider the terms child-initiated and adult-initiated as a better way of conveying the sense of equal and shared relationships between teacher and child.

2.6.2 Sustained shared thinking

In the extensive research report, *Researching Effective Pedagogy in the Early years in England*, (Siraj-Blatchford et al. 2002) the findings revealed that one of the most effective pedagogical practices is when a teacher or one or more children are involved in sustained shared thinking (2002, p. 10) by engaging in a conversation about a shared interest. The adult shows genuine interest, encourages thinking and clarifies and extends ideas. Fleer (2010) highlights this as one of the most important pedagogical discoveries in early years in the last thirty years. However, the Siraj-Blatchford et al. research also expressed that this happened infrequently in early years settings and mostly in Ofsted judged outstanding settings. Another similar approach to teaching and learning is co-construction (Jordan, 2009) where there is collaborative dialogue and the adult and child think together in a "mutual bridge of meaning" (Stephen 2010, p. 64). In a study by Jordan (2009), teachers reported that children were more empowered when interactions were co-constructed compared to scaffolding which was more direct and adult-led. Ma (2013) writes that "the developing mind can benefit from interpersonal relationships through which learning becomes shared and integrated, rather than an individual, solitary activity" (p. 453). His study emphasised that the collaboration in connecting mind and culture, between adult and child, can produce a new level of understanding as opposed to simple knowledge transmission.

2.6.3 Obuchenie

Scrimsher and Tudge (2003), like Fleer (2010), discuss the important relationship from a Vygotskian root of the interplay between teacher and child. They describe the Russian word obuchenie as meaning teaching and learning together and therefore teaching and learning are not seen as separate. This explanation is also supported by Bodrova and Leong (1996) and by van der Veer and Valsiner (1991). This view of teaching and learning as connected is in opposition to the teacher-as-giver-of-knowledge, directly instructing pupils. Scrimsher and Tudge (2003) explain the importance of the word obuchenie;

> The more accurate interpretation of the word (obuchenie) as teaching and learning connotes highly interactive relations involving all participants in creative activity and growth (p. 298).

Based on Vygotsky's cultural-historical view of learning, Rogoff (1990) describes a metaphor of apprenticeship in which "active novices advance their skills and understanding through participation with more skilled partners in culturally organised activities" (p. 39). She talks about the workings of apprenticeship in cognitive development as *guided participation* (ibid) in skilled cultural activity and concludes that children and their social partners, especially their caregivers, are interdependent rather than dependent. She studied families and communities around the world and saw a variety of patterns in guided participation. She comments on these joint socialisation roles of the child in the family and community, concluding that they are much more common than the direct instructional didactic mode of teaching that has been the focus of much research. Tizard and Hughes' seminal study (1984) uncovered that working class (a phrase used in the study) mothers in the home were engaged in intellectually challenging conversations with their four-year-old children, including, to the researchers' surprise, mathematical enquiries. It is this awareness of the mathematical knowledge that all children bring to school and that many children are already participating in mathematical enquiry, before school entry, that is important in this study.

2.7 Conclusion

Much of the literature on pedagogy focuses on democratic pedagogical practices. For early years, the debate is moving towards the adults having an active but sensitive role in children's play but the research on play pedagogy is underdeveloped. The literature reported in this chapter, especially the stance on social pedagogy, knowing children and the rich knowledge they bring to school is of crucial value to teachers who employ democratic practices. For mathematics teaching in the early years, there seems not to be a sufficient move away from Piagetian stage theory. The traditional approach to early mathematics teaching using practical equipment as a precursor to abstract concepts has been questioned. Teacher-child pedagogical interactions that promote a reciprocal understanding are dominant in the literature but there is debate on how this could be accomplished. From a Vygotskian perspective, the part the teacher or knowledgeable other plays in the ZPD is much debated as is the term *scaffolding*. More enlightened views include Scrimsher and Trudge's perspective (2003). They explain that individual factors, for example, children's interests and background histories, should be interwoven with the teacher's motivations and together in democratic practice they make not a zone but Zones of Proximal Development. This is far removed from many writers' conceptions where they see only the child as being in the ZPD (Daniels, 2007). This view could be seen as lessening the child's agency and sense of self-empowerment. Changing perspectives on pedagogy are moving towards an appreciation of the child's view,

> the understanding that the child is not the material from which an adult sculpts his or her future is becoming clearer. Accordingly, an understanding of child development and teaching practice is not understood as a linear process; it is becoming more and more multi-dimensional and builds on the mutual interaction and communication between teacher and child (Veraksa & Sheridan 2018, p. 4).

Therefore, the teacher is an equal partner in the pedagogical relationship and often the child takes the lead, projecting an interest to be discussed and extended. The interest comes from the child's world of home, school and community and the power lies in the social and cultural collaboration between the teacher and the child. This is similar to the concept of obuchenie (2.6.3) where there is no distinct line between teaching and learning, both child and adult are teaching and learning together. The importance of this democratic pedagogical relationship also has relevance to mathematics teaching as children's mathematical graphics are rooted in liberal classrooms where the teacher places a great emphasis on children's mathematical meanings (Papandreou & Tisolou, 2020; Carruthers, Coles & Rose, 2020; Worthington, Carruthers & Hattingh, 2020).

If effective pedagogy claims a more democratic tone, then one aspect of this is listening to children views. This is explored in the next chapter.

Chapter 3: Listening to Children

This chapter begins with defining listening within educational contexts and the rights of children to be listened to. Next, I argue, historically, early childhood scholars and pioneers have always purported a listening stance. I go on to discuss how children are viewed; the image of the child and how this might impact on, if, and how, we listen to children. I consider to what extent are teachers listening to children's mathematics and look at ways of listening to children's mathematics, including the importance of a listening culture. Finally, I return to the theme of children's rights and their right to be listened to, in any ways they can communicate and that make sense to them.

3.1 Introduction

Within this chapter, I am defining listening, in the broadest sense, as a metaphor for hearing and acknowledging children's own behaviours and perspectives. Within an early years' pedagogical stance listening to children also includes tuning into their feelings, ideas and thinking. Active and affective listening has been part of an early years' ethos, way of working and espoused practice, across many Western European early childhood communities, for example, in the Scandinavian countries and parts of Italy (Rinaldi, 2006; Dhalberg, Moss & Pence, 2006; Bath, 2013). Although, an ethos of listening to children is also recognised, to some extent, within English government curriculum frameworks for the Foundation Stage, the latest revised framework has reduced the focus on child perspectives (DfE, 2020). In this new framework mathematics is given as a set of skills to be learned and this points to the current issue, in England, that listening has become counter-culture in an early childhood education and care system determined by measurement (Clarke, 2020).

Listening and documenting observations of young children have been an evolving pedagogical theory for over 100 years (Isaacs, 1970; Carr, 2001; Rinaldi, 2006). The earliest pioneers of early childhood education, for example, Froebel and Steiner, based their understanding and ideas on listening and watching children in play and therefore play is proposed as a major component of early childhood education, in many Western countries. There are also related discourses on listening, including listening as participation (Theobald, Danby & Ailwood, 2011; Bae, 2010) and children's agency (Caiman & Lundergard, 2014). Lenz Taguchi (2010), a Swedish researcher, influenced by the Reggio Emilia pre-schools and the work of Barad (2007) has extended the concept of a listening pedagogy as an intra-active pedagogy. Therefore, research on listening to young children is an ever growing and challenging field.

In contrast to the general early years' aspect on listening, there is not a substantial amount of literature that is specifically around listening in early years' mathematics teaching and this may be due to the lack of research in general in early mathematics, certainly compared to literacy and language (Littleton & Mercer, 2013). However, an emerging body of mathematical research related to older children is focusing on a listening pedagogy and within this, there is a heavy emphasis on the rights of learners (Kalinec-Craig, 2017). A definition of listening, defined by these mathematics researchers, is a humanistic approach to listening that includes a broader perspective than classroom discourses that only include the teacher's focused agenda. Within this broader definition of listening Davis (1997) declares learning as a social process and listening is one of participating, interpreting, interrogating and transforming. Building on the writings of Davis, more recent literature in mathematics teaching has included the rights of the mathematics learner to be authentically[7] listened to without negative judgement of their mathematical ideas (Kalinec-Craig, 2017; Hintz, Tyson & English, 2018). Hintz, Tyson and English's (2018) work highlight that listening is defined by equitable mathematics discourses. Listening to children's mathematics, therefore, can be seen as an ethical issue. Freire (2000) talks about listening as a form of tolerance and for Freire it is a political discourse as well as an ethical duty. Davis (2014) discusses listening as an ethical relationship and respect for other opinions and differences, "listening as continual openness to the not-yet-known" (p.11). Therefore, there are many perspectives on listening within and beyond early childhood pedagogies and I draw upon these lines of thought as I pursue this pedagogical stance of listening as the main theme of this chapter.

3.2 The Rights of the Child

The importance of listening to young children was presented in the United Nations' Convention on the Rights of the Child,

[7] Authentic listening occurs when you listen at a level beyond hearing. You respond to the speaker giving every opportunity to allow him/her to complete their thinking, allowing for silences (Hancock & Mc Donald, 2015)

> The child has the right to freedom of expression. This right shall include freedom to seek, receive and impart ideas of all kinds, regardless of frontiers, either orally, or in writing, or in print or in the form of art, or through any other media of the child's choice (UNICEF 2009, Article 13).

The United Kingdom Children's Act (1989) also emphasised that children's views be taken into consideration. This legislation was to do with their rights in the broader sense of children's worlds and not necessarily focused on curriculum or schools. However, as Lancaster (2003) highlighted as a result of this declaration children were increasingly seen as "competent to express their views" (p. 2). Both nationally and internationally the rights of children to be listened to, not only at the human level of decisions about themselves, but educationally, is an evolving paradigm. If it is understood children have the right to be listened to then to what extent is this adhered to in early childhood? The next section discusses listening as part of early childhood practice.

3.3 Listening in Early Childhood

A listening pedagogy can be traced back to Dewey (1916) who, from a philosophical point, suggested that schooling should be democratic and children should be part of the decision making process. In England, in the early twentieth century, Isaacs, through her time teaching at the Malting House School in Oxford (Isaacs, 1970) was a pioneer of looking at young children's perspectives. She based her teaching on observing what children did in their play and how they went about tasks, rather than presume what they needed and then directly teach a set of skills. Isaac's observational notes on children could be said to be a form of listening to children. This observational note taking approach has extended in recent years to the documentation of young children's actions on the world, in the renowned Reggio Emilia nurseries in Italy (Rinaldi, 2006) and Carr's (2001) *Learning Stories* in New Zealand. The Reggio Emilia nurseries offer what they call visual listening (Rinaldi, 2006). They propose that there are many ways to listen to children, and, importantly, children are born with an innate ability to listen to others as they are social beings (Rinaldi, 2006). Written observations and documentation have now become standard practice in Western early childhood settings. However, to what extent are these observational notes based on listening and what kinds of listening? In England, observational notes have sometimes drawn teachers into a tick list assessment mode and the listening has been restricted to whether a child can achieve a pre-set objective or not (Bradbury & Robert-Holmes, 2018). This seems far removed from the articulated and compelling vision of the listening proposed by the Reggio community of educators who focus on children's meanings and not the objectives of a set government curriculum (Rinaldi, 2006).

Although visual listening can be mis-interpreted and ill-used it has had a positive impact. As Oliveira-Formosinho (2007) reflects, from a Western European perspective, there is now a move away from a solitary pedagogy of direct *doing to children* but entering, instead, in dialogue with children which includes their affective learning, families and society. Pedagogies have therefore become more individualised and "a means to deconstruct a ready to wear single size curriculum" (Oliveira-Formosinho, 2007 p. 20). Individualising has made teachers and those who work with children, aware of children's needs, and, in turn they promote the child's autonomy through honest listening. When observing children, the early years' teacher does not have an agenda to skew the notes into the direction she/he wants it to go. The teacher is open to authentic listening and "creating space and time for the child's exploration and communication, through documenting it" (Formosinho & Oliveira-Formosinho, 2012, p. 593). There is perhaps a slowing down of the hectic schedules imposed on even pre-school children so that there is a chance of children's agendas surfacing. Freedom and space to listen to children links to social pedagogies, which was discussed in the last chapter (2.1), where children are listened to in a way that is not decontextualized from the rest of society. The culture of the children are brought within educational agendas, as an active force (Dhalberg & Moss, 2005).

Attitudes towards young children have increasingly embraced some aspects of children's views and Piaget (1952) was one of the first Western known researchers to study child development. His ways of listening to and observing children were through mostly set up clinical tasks, which have advantages but the disadvantage is that because they were not open-ended observations they did not see variant patterns of thought over the years of his study. They produced a set response from the children he studied, because of the limited nature of the tasks. For example, the closed tasks he used produced evidence that children at a certain age could not conserve or categorise correctly (Piaget, 1954). Piaget's findings, it can be argued, did not depict a broad picture of the child; and researchers since have questioned these Piagetian tasks (Donaldson, 1985; Genovese, 2003).

In contrast, later work by Athey (1990) using open observations of children, in self-chosen activity, including play, revealed what she called *illuminative research* (p.17) on children. Her research uncovered children's developing schematic thinking revealing that children had their own agendas and threads of thinking that they pursued. These schematic patterns varied from child to child and included a range of mathematical concepts. She concluded children could not be categorised as having deficiencies as in Piaget's tasks where children were seen to be understanding conservation of number or not. Instead, through close observation, her research revealed young children independently pursued a range of self-initiated mathematical inquiries. Her conclusion was not either they had a concept or not, as in Piaget's studies, but, importantly, she acknowledged young children as competent and that they were coming to know certain mathematical concepts; they had partial knowledge. Athey's research was particularly rigorous as she made detailed observations over three years in a nursery setting. Through listening and observing children's behaviours, over a lengthy period, rather than single clinical tasks, her research uncovered a variety of children's cognitive patterns of thought. Athey (1990) reflected, when researching young children in play it was not always easy to see the children's behaviours, as play is messy and chaotic at times, and not as neat as clinical studies might be.

Although listening may be understood as part of early years' practice, Bath (2013) cautions that it is in peril of becoming a part of early childhood teaching that is already well known, and, therefore, not in need of further deliberation. She reflects that nothing is ever absolute and finished and listening to children requires continual discussion and rethinking. I think that this is particularly true for the area of mathematics teaching in the early years. This may be because it is under researched and much of the research is from adult-centred perspectives (Dockett & Goff, 2005). It may also have to do with the nature of mathematics (5.1.1) which can be seen as a precise exact science influencing some teachers "lines of mathematical instruction" (Mholo & Schaffer, 2012, p. 4).

However, how adults listen to young children may depend on how they perceive them, in the next section I discuss how adults view young children.

3.4 How Young Children are Viewed

Fleer (2010) puts forward the premise, how we view children determines how we teach them. Athey's (1990) work, as explained above (3.3), emphasises that children are competent to pursue their own lines of inquiry and much of the literature is in accord with this, highlighting that young children have much to bring to the learning process. It is well documented they are curious self-motivated individuals who are able to analyse and hypothesise (Vygotsky 1978; Nutbrown, 2006). For example, Wells' (1986) seminal study illuminates young children as powerful meaning makers, trying to make sense of their world. Wells reflects that the child has an expert role in their learning, and this learning thrives when those around listen to their ideas and lines of thought.

However, Colliver (2017) puts forward that many European cultures see young children as cute, vulnerable or immature and she stresses that if we want to listen to children then perhaps there needs to be a shift to viewing children as strong, knowledgeable individuals and to take their actions seriously. Langsted (1994) views children as "experts in their own lives" (p.29) and they are being, rather than becoming. This shift to children's agency discusses young children as life theorists and active agents in their world (Hedges, 2012). Hedges (2012) puts forward the concept of children's *working theories* that represent the tentative evolving ideas and understandings formulated by children as they engage with others to think, ponder, wonder, learn and make sense of the world, in order to participate more effectively within it (p.144).

Within the premise that children are active agents in their own worlds (Hedges, 2012; Dhalberg, Moss & Pence, 2006) research *on* children has also moved more to research *with* children. From a view that childhood is a socially constructed phenomenon and children are socially active within this (Mayall, 2002; Smith, 2007), promotes the right, not only for children's views to be heard and taken seriously, but there is a responsibility for adults to listen and to respond to those views. Research studies with children, as active participants, have seen children play a critical role and this redefines our structures and conventions about childhood (Clark & Moss, 2001; Harcourt & Einarsdottir, 2011). Researching with children is an ethical stance and socially just (Millikan, 2003). A pedagogy of actively listening to young children and socially inclusive relationships can be pivotal in improving children's life chances and education, including their learning of mathematics (Coley, Lombardi & Simms, 2015).

In summary, how children are viewed across the field of education is increasingly towards a positive and strong image of the child. This means children's ideas and ways of being can seriously be taken into consideration; their contribution to school experiences counts. However, to what extent is this true in mathematics education; do teachers listen to children's mathematics? I consider teachers responses to children's mathematics in the next section.

3.5 Teachers Listening to Children's Mathematics

Brown (2015) points to how we all see things differently, and, for example, in mathematics, although we might all see a sphere, we see it from varying perspectives "every interpretation is partial embedded in an interpreter's ultimately contingent subjective position" (p.85). Are teachers able to accept this diversity of mathematical thinking especially in the more embryonic thinking of young children? Are teachers as Brown (p. 86) puts forward able to conceive mathematics from the point of view of the learner?

Gattengo (2010), working in the United States in the mid-twentieth century observed that in schools, often mathematical knowledge is something that is passed on to the children, not something they own. He provides the example of children reciting 2+3=5 not because they are certain of this but because the teacher told them. He further explains "They have not been allowed to use the basis of surety that exists in their perception. They have not been allowed to believe in their sense of truth" (Gattengo 2010, p. 19). Gattengo's scenario exemplifies that children can receive a mathematics education without personal engagement, for example, exploring and testing their mathematical theories. Similar concerns exist today, as Brown (2015) more recently reflects that mathematics in schools can be reduced by restricting the variety of the more personal mathematical insights of learners. Mholo and Schafer (2012), in their study of mathematical pedagogy, concluded that the teachers ignored the learners mathematical thinking by enforcing their "formalised constructions" (p. 1) and this is consistent with other critiques of Western mathematics education (Boaler & Greeno, 2000; Nasir, Hand & Taylor, 2008). Kazemi and Franke (2004) worked alongside teachers to look at how children worked out their own strategies in mathematics problems. The teachers uncovered a range of written strategies the children used, when given the opportunity to do so. Similarly, in Aanatharajan's (2020) study the teachers pointed out that the children were also able to apply more sophisticated reasoning in mathematics than they had previously realised. In both studies the teachers commented on how, time to listen to children and reflect on their written work was crucial to revealing children's mathematical thinking.

Authentically listening to young children's behaviours and perspectives may require familiarisation with children's individual backgrounds and cultures (Colliver, 2017). Colliver suggests teachers working every day with children have an advantage of familiarity with the children over, for example, researchers visiting classrooms. Knowing the children and their lives puts a context to what they say and do, therefore, really taking note of what they say can be more easily accomplished. However, do teachers have the capacity to listen; meaning are they afforded the opportunities, for example, in space and time to encourage and acknowledge the various ways that children can communicate? Within educational structures Bradbury (2013) discusses how teachers hold power over student subjects, determining what is taught and how students exhibit adequate learning. As Drummond emphasises (2003) these structures can influence a more authoritarian style of teaching, classrooms can impede teachers' facility to accumulate the broader child-led evidence of children's learning, by not noticing aspects of this learning. This may result in more emphasis on adult-led provision. In educational settings where there are more democratic and socially inclusive relationships between children and teachers then authentic listening can more easily occur (Lancaster, 2003).

Colliver (2017) challenges us when he says; when children are listened to, to what extent are they being truly understood? Adults can make their own adult assumptions about what they hear from children's experiences and miss out on the child meanings. Paley (2007), a teacher all her life, describes the difficulties of knowing and understanding young children, "So often I drift around on the edge of their knowing without finding a place to land" (p. 131). Young children's mathematical thinking is usually further away from the standard norms of adult expectations than older children's mathematics (Carruthers & Worthington, 2006). Seeing, listening and noticing the distinct aspects of young children's mathematical thinking may be more difficult to understand or recognise than that of older children.

Central to children being heard is the adults *hearing* (Moss & Clark, 2006). When teachers listen to children they are not necessarily hearing or responding to the children's perspectives. They may be still focusing on school objectives and driving the children towards that, in some kind of illusion of responding to the children's thoughts or trajectories (Rose & Rogers, 2012). Paley (2007) narrates how listening to children in her classroom offered a new means of being with children, "The rules of teaching had now changed; I now wanted to hear the answers I could not myself invent" (p. 125). She said she realised that she had been in the "wrong forest" (p. 125). I interpret this to mean she was listening to children from a very different perspective. Her work outlines that often she saw children as she wanted them to be. She was obliged to reflect further and reconsider her frame of understanding young children and see the children as they are. Similarly, in mathematics teaching, Haniford and Tyson (2018) found it a struggle to develop a space for listening to their children's mathematics and found that they, as well as the children, were dependent on text books. Lahman (2008) recommends that "if you want to uncover what children know then abandon your agenda, let them drive the activity so that you can fully understand the child" (p. 26). However, I am not sure that it is possible to fully understand children and abandoning agendas may be difficult in a culture that expects explicit objectives to be taught.

Therefore, although listening to children's ways of understanding and representing mathematics seems difficult for teachers, it is vital for supporting children's growing knowledge and uncovering what they know. Mholo and Scahfer (2012) state "There needs to be a personal engagement with mathematics so that it becomes an integral part of the learner's personal identity" (p. 3). If children become part of the process of mathematical thinking, within classrooms, then they might develop personal mathematical knowledge which may help them understand and remember. The next section continues the theme of listening to children's mathematics but with a focus on ways of listening.

3.6 Ways of Listening to Children's Mathematics

Research suggests that listening to student thinking is important in mathematics. For example, research on Cognitively Guided Instruction (CGI) demonstrates that teachers who participate in professional learning, focused on student thinking, develop greater understanding of children's mathematical strategies, and this has a positive impact on their instruction, and, in turn, on their students' problem-solving ability and confidence (Carpenter et al., 2014). However, what are the most effective ways of listening to children's mathematics? I now focus on the literature that highlights the kind of listening that supports children's mathematical thinking.

Although Carpenter and Fennema as early as 1992, in their analysis of effective mathematics teaching behaviours, emphasised that listening to pupils was a critical factor they did not expand on this, it was merely an aside. Davis (1997), however, expanding on Carpenter and Fennemas' research and through an enactivist lens examined pedagogical listening as a possible dynamic teacher-learner communication within a mathematics classroom. He proposed three kinds of listening (evaluative, interpretive and hermeneutic) as "both an effective means of interpreting classroom phenomena and as a useful starting place for transforming mathematics teaching practice" (p. 357). The first two modes of listening evaluative and interpretive generally involved the teacher constantly assessing the pupils and listening for the right answers to ensure coverage of a set of pre-determined skills. However, hermeneutic listening involved the teacher abandoning the pre-set objectives and becoming a participant in the investigation of the mathematics. It is a pedagogical situation where the teacher becomes part of the community of learners and is prepared to analyse the assumptions and prejudices that surround her perceptions and actions.

Using Davis' research on hermeneutic listening Mholo and Schaffer's (2012) looked at situations where the teachers, in the study, were responding to learners whose contribution was unexpected. The researchers were looking for a hermeneutic orientation to listening. They found that, for the most part, teachers did not acknowledge contributions from learners that did not fit in to the teachers' lines of mathematical construction. The researchers reported that the teachers did not probe further the learners' meanings, some dismissed the response and drew the learner into their line of thinking. This study suggests that teachers are less democratic in their listening orientation. Previous studies have found similar disconnections between the teacher and the learner's lines of mathematical inquiry. (Brodie, 2010; Ball & Forzani, 2010). As Brodie (2010) expresses a vital aspect of teaching mathematics is to support learners to express their mathematical thinking and reasoning however embryonic or defective that might be. This means tuning into the learner's mathematics. Ball and Forzani (2010) point out, this may be difficult if there is a great difference between the learners and their teachers in cultural background, language and experience as "the less precisely attuned the teaching is likely to be" (p.41). Tuning in to the children's mathematics is what Belenky et al. (1986) term "connected knowing" (p. 32) where the teacher has an uncritical empathy and understanding of the learner. Teaching from the perspective of the learner may require the teacher to consider a listening culture in his/her classroom.

Kalinec-Craig's (2017) work on the rights of the learner and promoting a listening culture in mathematics draws attention to five rights that she outlined the learner must have (Rights of Learning Framework). The first two are powerful within a mathematics learning environment that is listening to children (1) the right to be confused and (2) the right to claim a mistake. They seem to sit together and when actualised could change the culture of a classroom where previous mathematics learning was seeking right answers and that the emergent process of learning was not recognised. For young children the world is new and they are seekers of knowledge, within their world, and that includes building mathematical knowledge. It can be argued mathematics is about problem solving (van Oers, 2001; Hoyles, 2019) and if so then the young child's world is about new experiences and new problems and enquiries, a constant finding out. Within this premise learning is not always neat and tidy but messy and chaotic at times (Athey, 1990). The listening teacher is not only aware of but supporting and responding to these complex confusions and states of disequilibrium (Vygotsky, 1978). It is, perhaps, our struggle to comprehend that gives us that understanding, eventually, and that battle can make that understanding stronger. This could enhance the child's mathematical cognitive development as they may build up their pathway in mathematical thinking. For young children it is often through play and exploring as these can be powerful experiences that challenge "the boundaries and assumptions of their own understandings about mathematics" (Kalinec-Craig, 2017, p. 5).

One of the possible barriers to effective listening to children's mathematics is how the teacher or supporting adult deals with what is called mathematical errors, where children are seen to make a mistake. Mathematical errors can be seen as part of the learning process (Van de Walle 2004; Bray, 2013) and this can also be a discussion point for young children's learning (Gifford, 2004). However, from an early childhood perspective young children's mathematics might not be described as mistakes. They do not make errors, instead, they have working theories (Hedges 2012); they are coming to know the world. It can be argued, therefore that there is no place for teachers to identify errors but only acknowledging what children know now (Athey, 1990; Brostrom, 2006). For example, when young children start counting they often repeat numbers or miss out numbers. They use the numbers they know and put them to a counting situation. They slowly build up their repertoire of number names and refine counting skills through the experience of their individual culture (Carruthers, 1997; Sarama & Clements, 2009). Haniford and Tyson (2018) stated that the classroom should be a place where students share emerging thinking and not just "finished thoughts" (p. 339). In Athey's (1990) seminal work on observing and listening to children she advised teachers to throw out all negatives as then you will see what children can do. Brostrom (2006) agrees we need to view "children from an additive perspective rather than deficit perspective" (p. 228). Recognising "children's competencies can help adults reflect on the limitations of their (the adults) understanding of children's lives; to listen to children more, rather than assume we already know the answers" (Clark & Moss, 2005, p. 6). Johnsen-Hoines (2004) uses the term "the authoritarian nature of the mathematics" (p. 1) and she says this may restrict us listening to children's own mathematics and only see mathematics as right or wrong. She acknowledges that a mathematics curriculum can be imposed on children and even although, within her study, they claimed to use children's ideas and thinking as the basis of their approach they later reflected "we realised we did not do what we claimed to do" (p. 1). Instead, they based the mathematics on what they called traditional school mathematics of text books and curriculum. They then moved to abandon the formal mathematics and standard representations of mathematics, encouraging children's own ways. It is only when they accepted the diversity of children's ways of representing their mathematics, which could have been seen in traditional mathematics teaching as mistakes, that they uncovered children's mathematics. They concluded, just as the languages of indigenous people need to be protected so to do children's own informal mathematics.

In summary, although it has been recognised that listening to and understanding children's mathematics is not straightforward, it is what will bring their mathematics to the fore and be recognised. This is what this thesis is centring on; teachers' awareness, actively listening and responding to children's mathematics. One aspect that is vitally important to listening is to understand that communication is not just linguistic. The broader sense of listening, which was defined in the first section (3.1), is discussed in the next section underlining the many ways that children communicate their mathematical thinking.

3.7 The Many Ways Children Communicate *their* Mathematics

The fourth right of Kalinec-Craig's Frame of Rights (2017) is heavily influenced by her experience of working with second language learners and immigrants to the United States. The fourth right is "You have the right to do and represent only what makes sense to you" (p. 6). This statement is pivotal in educators thinking about children's mathematical representations; mathematical representations must make sense to children. This sense making is what Gattengo (2010) was emphasising when he discusses children's unquestionable acceptance of the written standard calculations the education system imposes on them, even although they do not understand them (see 3.5).

Kalinec-Craig explains there is no one way to express one's mathematical thinking. Culture and mathematics are interwoven because of our everyday lived experiences (Barta, Eglash & Barkley, 2014). Therefore, she recognises the multiple ways that children can represent their mathematics. I think her position as a bilingual teacher may have given her an insight into learners who perhaps could not express themselves in standard English or American cultural ways. This has parallels to listening to young children's mathematics as they too, do not always communicate their mathematics in standard ways (Hughes, 1986; Carruthers & Worthington, 2005; Anantharajan, 2020). As Hughes (1986) stressed, in his research, the difficulties that young children had with the standard abstract symbolism of mathematics was like learning a foreign language. A main premise of this study is educators need to understand the often informal and non-standard ways children represent their mathematics and this may be like a written foreign language to teachers.

Pressure is often put on children to express themselves linguistically (Anghileri, 2006). Anning (1999) highlighted the signal children receive from school teachers are that formal ways of representing literacy and numeracy are valued and children's drawings are merely recreational, therefore, underestimating the intellectual potential of the children's graphics. Worryingly, Anning finally concluded, in school, children's "capabilities, in using alternative modes of representation, as tools for learning, wither away" (1999, p. 1). Similar to Anning's drawing study, research in the last twenty years expresses the need to listen to the many ways that children communicate. (Kress, 1997; Anning & Ring, 2004; Matthews, 2003; Flewitt, 2005b; Rinaldi, 2006; Carruthers & Worthington, 2005). As Carpenter et al. (2014) express teachers can learn an immense amount about children's mathematical knowledge through their multiple mathematical representations.

In conclusion, the literature shows that children are capable of pursuing their own ways of thinking mathematically, however, there is doubt that mathematics' pedagogies prioritise authentic listening to children, taking into consideration their views and understandings, from their perspective. A review of 46 countries highlighted that policy makers and researchers around the world attribute a sparsity or no consideration of young children's perspectives (Powell et al., 2011). Added to this, Colliver (2017) emphasised research that did listen to children did not always sufficiently understand them, as it is not always easy to hone in on young children's meanings. In England, although government documents are open to interpretation, increasingly there has been a narrowing of the curriculum for the early years (TACTYC, 2017) which is again under review (DfE, 2020). Early childhood professional groups have particularly argued that the new proposals are not child-orientated, instead, there is a trajectory towards adult-centred perspectives (TACTYC, 2017). This, therefore, will impact teachers' pursuit in listening to children's perspectives and ways of being, including their mathematics.

Building on this chapter's argument that it is essential for teachers and those in education to listen to children and their mathematics, in the next chapter, I review the literature on; young children's mathematical play; children's own representations of their mathematical thinking; and teacher studies that focus on supporting and listening to children's mathematical graphics.

Chapter 4: Young Children's Mathematics with a Focus on Their Ways of Representing Mathematics.

This chapter is written in three sections. The first section focuses on early mathematics education, including mathematical play. The second discusses research on young children's representations of their mathematical thinking. The final and third section highlights teacher studies that focus on children's mathematical graphics.

4.1 Early Mathematics Education, Including Mathematical Play.

In this section a critical debate regarding current issues in early mathematics education will be discussed. I include the importance of play and the confusions around play and mathematical learning, foregrounding pretend play as a possible useful springboard in children's mathematical learning. Next, I highlight the many ways children can communicate mathematics; and I finally point to the position that young children's own written mathematics is less well researched, compared to spoken language and using practical resources for mathematical teaching.

4.1.2 Introduction

There is an immense amount of literature in early mathematics education on children's understanding of number, with a considerable focus on counting (Baroody, Lai, & Mix, 2006; Fuson & Hall, 1982; Gelman & Gallistel, 1978; Slaughter et al., 2011; Kazemi et al., 2016; Carpenter et al., 2017). For example, Fuson (1988) presented an extensive account of children's counting errors stating that, by the age of 3, children exhibited organised and mature counting structures. Gelman and Gallistel's (1978) seminal work found counting principles that generally applied when children counted. However, many of the studies mainly used clinical tasks and although they were significant in our understanding of children and number the research design did not allow for children's unrestricted engagement with mathematics in more democratic environments. Children's number knowledge in every day, free and social activities seem less well researched.

There is general agreement that young children are just as competent in mathematics as in other areas of learning (Gelman & Gallistell, 1978; Dockett & Goff, 2013). Young children develop an informal mathematics as they go about their daily lives at home and in their communities.

They attend to factors in their environment, for example, using money in shopping, measuring in cooking, age in birthday celebrations; and this includes aspects of division, subtraction, multiplying shape, space and patterning, without direct instruction (Ginsberg, 2008; Dockett & Goff, 2013). Ginsberg (2008) refers to young children's engagement with the mathematics they use in their homes and community as the *everyday mathematics* of young children (p. 3). Gelman (2000) says; young children learn mathematics *on the fly* (p. 26), they have an instinct to learn regardless of whether they are with adults or not. For some time, it has been noted that teachers and especially pre-school practitioners are sometimes unaware of the mathematics that young children engage in, especially in play (Munn, & Schaffer, 1993; Ginsberg, 2008; Papandreou & Tsiouli, 2020). In many ways the mathematics that children learn informally, from their everyday experiences, is hidden knowledge, schools do not recognise it (Hughes, 1986, Aubrey, 1997; Papandreou & Tsiouli, 2020). Children are learning mathematics from their every-day cultural experiences and van Oers terms this children's early enculturation[8] into mathematics (2010). The emphasis on children's cultural mathematical knowledge is recognised by the research on funds of knowledge (Moll et al., 1992; Gonzalez, 2005) and cultural learning (Rogoff, 1990).

Within the debate on teaching young children mathematics, there have been ongoing tensions that directly teaching or alluding to any kind of prescriptive teaching of mathematics to young children is inappropriate (Ginsburg, 2008; Sarama & Clements, 2009; Gifford, 2004). Similarly, the questions of the place of formal notation and standard written mathematics have puzzled teachers in English Reception classes; they are not clear when, and how to teach children formal addition and subtraction (Gifford, 2004). Correspondingly, in a study in Northern Ireland, teachers stated that they were uneasy about introducing formal symbols in the first class of school (Moffat & Eaton, 2018). When to introduce formal symbols has not been clearly addressed in English government official guidance for the Early Years Foundation Stage (DfES, 2014). In England, a Reception class study (Adams, Drummond & Moyles, 2004) found that mathematics, other than direct number work teaching, was seriously lacking from the child's school environment. This study, although on a smaller scale, was repeated 10 years

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[8] Enculturation is the process by which people learn the requirements of their surrounding culture and acquire values and behaviours appropriate or necessary in that culture (van Oers, 2010) .

later, with similar results, that this dearth of mathematics within the broader classroom environment was still a concern (Moyles & Worthington, 2013).

Is this scarcity of mathematics in the broader Foundation Stage environment also in play? In the next section, I present research on mathematical play that addresses this issue and other concerns to do with play and mathematics.

4.1.3 Play and mathematics

Vygotsky recognised the importance of play and young children's developing understanding of mathematics "Children have their own preschool arithmetic, which only myopic psychologists could ignore" (Vygotsky, 1978, p. 84). There may be a whole aspect of children's mathematical learning that we are not listening to if we do not take play seriously. I have observed play almost daily and was part of the debates and discussions with colleagues about play when I worked as a Headteacher of a Nursery School and as teacher in a Nursery class, Reception class and a Year 2 class. I have seen and taken part in rich mathematical play where children are leading and learning and the teachers are learning about them (Worthington, Carruthers & Hattingh, 2020; Carruthers & Butcher, 2015). However, studies and research on child-led play in mathematics are rare (Papandreou & Tsiouli, 2020; Worthington & van Oers, 2016) and this may be to do with issues that surround mathematical education researchers' understanding of play, which I am going to discuss in the following paragraphs.

Few studies have concentrated on mathematical play and the studies that do are not always in child-led activity (van Oers,1996; Vogt et al., 2018), which brings somewhat of a conundrum to the research area of play and mathematics. If we are listening to children's play in research then a prevailing argument is that it should be children who have the opportunity to lead (Moyles, 2014; Bruce, 2011; Fisher, 2016). Although, Ginsberg, in researching mathematical play in pre-school classrooms in the United States agrees that children do acquire a significant amount of mathematics in their own play, he says it is not enough, "it does not usually help children to mathematize which means to interpret their experiences in mathematical form and understand the relations between the two" (Ginsberg, 2008, p. 7).

Child-centred advocates (Dodge, Colker & Herman, 2002; Copley, Jones & Dighe, 2007) declare that in children's spontaneous play teachers can seize upon *teachable moments*[9] to support and extend children's mathematical thinking. Ginsberg (2006) questions the extent of the pre-school teachers' mathematical knowledge and maintains they would need an immense amount of support, to recognise and make these teachable moments a reality. Teachers need to seize these teachable moments when the chance occurs but Helenius (2018) says the difficulty is, how many opportunities will there be of such play episodes happening? Another issue she puts forward is, what kind of interaction happens in teachable moments? Teachable moments could very easily turn into direct teacher-child interaction and this may dampen the spontaneous play and mathematical thinking of the child. Ginsberg et al.'s (2008) research concurs with some of the difficulties outlined above and concluded teachable moments are not sufficiently reliable to provide high quality mathematics education to children. Instead, Ginsburg advises that pre-school teachers focus on playful teaching.[10] He presents an example of a teacher directly teaching children numbers up to 100 and letting the children play with the numbers afterwards. He says, this was presented in a playful manner. However, playful teaching and children's own free play are different (Moyles, 2014; Walsh et al., 2017) as Bodrova and Leong (2015) state, "adding playful elements to a lesson will not turn it into play" (p. 386).

Ginsberg's seeming doubt about children's spontaneous play being adequate to provide sufficient mathematical input for young children, appears similar to many mathematical researchers who are seeking answers to a more child-appropriate mathematics play pedagogy (Williams, 2014; Gifford, 2005; Perry & Dockett, 2007). I put forward, playful teaching, which could be described as an enjoyable mathematics teacher-directed activity, could become a red herring; the diversion of moving away from spontaneous play to centring on playful teaching could draw teacher and mathematical researchers focus away from children's spontaneous mathematical play and the many learning opportunities within this. If there is a pedagogical shift away from developing spontaneous play to playful activities in early childhood, then there is a danger that the value of children's free mathematical, spontaneous play in early years' settings could be lessened. Ginsburg (2008) argues, in periods of free play teachers spend little

[9] Teachable moments are "where the teacher observes the children's play and other activities to identify and interact with spontaneously emerging situations that can be exploited to promote learning" (Helenius, 2008, p.187).

[10] Playful teaching is mostly adult-initiated activities that involve situations that might appeal to children (Walsh, McMillan & McGuiness, 2017)

time with children or are only concerned with managing their behaviour. Surely, the problem to address is, not spontaneous play and shifting pedagogy away from this kind of play to adult-directed playful teaching. Instead, the issue to concentrate on is, how teachers value and understand mathematical play and their possible interactions within play. My study may go some way towards addressing this concern of teachers' positioning in mathematical play.

From a different paradigm, within a socio-cultural lens, a recent study by Papandreou and Tsiouli (2020) observed *children's everyday mathematics* in free play, in early childhood classrooms. The study used a Funds of Knowledge Framework (Moll et al., 1992) "to investigate the content, the processes and the origin of children's mathematical knowledge in naturally occurring activities" (Papandreou & Tsiouli, 2020, p. 1). They highlighted the meaningful mathematical play that children engaged in within a school context when children use and recreate their funds of knowledge. An important part of the research was it centred on the cultural knowledge children bring to school and, similar to Worthington and van Oers' (2016) research, this was highly acknowledged as a useful factor in developing children's mathematical understanding in play.

Although there are many types of play (Moyles, 2014), Vygotsky identified pretend play as the leading activity of young children's learning. An important aspect of pretend play is that children use substitute objects and gestures instead of real objects. This is put forward as contributing to the development of abstract thinking (Vygotsky, 1978). Vygotsky saw representation as a critical element of young children's concept development (1978). van Oers (1996), a Vygotskian scholar, also puts significant weight on children's pretend play, and researching mathematical pretend play gave, what he termed, an example of good practice. The example was of an adult-initiated and led, shoe shop role play. In later work, van Oers (2013) reflected on the criticism of this type of play acknowledging that it may have seemed intrusive. He went on to explain it "as a format of cultural activity" (p.191) involving highly experienced players following some rules (explicitly or implicitly) and who have some freedom about the interpretation of the rules and the choices they make in their play. This kind of adult driven, themed role-play is very popular (Rose & Rogers, 2012) it is supported by the English Ofsted curriculum document on play in the early years (Ofsted, 2015) where mostly examples of adult set-up role play are presented. Heavily adult sculpted role-play does not necessarily give children the opportunity to pretend or use something to substitute for what is needed in play, at the time. This might be because the real things are all there, perfectly supplied by the adults.

It is this kind of play that could be said to be distant from the Vygotskian play that is significant for abstract thinking in young children.

Other mathematical researchers (Gifford, 2005; Williams, 2014) seeking mathematics in role-play (adult set up role-play) noted an absence of mathematics. Gifford (2005) explains that "children's role-play was concerned with the larger themes of life, like love and power, rather than mundane things like the price of potatoes" (p. 2). Williams' (2014) research on role-play eventually concluded that the Mantle of the Expert, a form of drama (Heathcote, 1976), provided more mathematical play experiences for children than set up role-play areas. Although this is a well-respected approach to drama (Taylor, 2016) it is not children's free pretend play and it seems to be more suitable for older children. Perhaps, one of the problems of seeing mathematics in child-led play is, it is hard to catch spontaneous pretend play episodes, especially if you are an outside researcher. To capture child-led play episodes, as they happen, it may be best if the researcher is situated in a nursery or school where the environment is conducive to play. Otherwise, if a researcher does not see children's self-initiated pretend play it could be argued to be almost impossible to comment on.

Broadhead and Burt (2012), in an attempt to counter the sterile adult-led and constructed role play areas they had encountered, and capture children's own imaginary play, recommended that the Reception teachers, in their study, set up a "the whatever you want it to be place" (p. 46). The area had ordinary equipment such as cardboard boxes, material, crates, barrels, cable reels, ropes and tubing for children to use in whatever way they needed for their play, instead of traditional play equipment. Their research showed how much children learn, when they take the lead in their play and the teachers build on the children's play ideas and themes, which includes aspects of mathematics. The main conclusion from this research is when children choose their own play themes and interests, in a mostly unrestricted environment, with an assortment of *loose parts*[11] then their play can become more challenging and fulfilling than anything that is adult-led or suggested. The researchers also question the idea of pretence because, through observing children closely, in pretend play, over a long period of time (they used set-up video cameras in the area) they concluded, what might be described as pretending to be somebody else is not what the children are doing. They are completely themselves and

[11] The term *loose parts* refers to materials which can be moved around, carried, stacked, lined up and manoeuvred in multiple ways to enhance creativity when playing. The materials do not have a definitive purpose (Bilton, 2018).

wholly committed to their play, it might be what Csikszentmihalyi (1991) describes as flow, the child is absorbed and not easily distracted by others. However, I add, the opportunity of the open-ended materials gave the children affordances to pretend that these materials were something else, even although their role within the play may not be seen as pretence.

Building on the research of Carruthers and Worthington (2005, 2006, 2011), later work by Worthington and van Oers (2017) looked at spontaneous pretend play in a Nursery School in England where the conditions were conducive to child-led play and the teachers were skilled in being with the children and following their current mathematical enquiries. The data came from the detailed teacher observations of children over a one-year period and from home visits and interviews with parents. Worthington and van Oers found that children were engaged in mathematics in pretend play and their play was situated in the social and cultural experiences of their homes and communities. Similar findings were reported by Papandreou and Tsiouli (2020), the children in their study also drew upon their home knowledge to develop pretend play situations, which included mathematical content, in their free play periods at school. Worthington and van Oers (2017) noted that the children engaged in many self-initiated literacy opportunities including the use of mathematical signs and symbols. Significantly, this study emphasised, there is a "compelling case for greater appreciation of pretence as a potentially valuable context for the enculturation of literacies" (p. 147) and this includes mathematics.

4.1.4 Children's mathematical communication

In summary, there has been a position built that children are mathematically competent, in a broad sense, moving away from focusing on the conceptual and perceptual limitations of children's mathematical abilities (Perry & Dockett, 2008). However, Perry and Dockett (2008) state that moving away from centring on children's mathematical deficits, the knowledge they have not already acquired, and looking at children as having a source of mathematical knowledge is a challenge. It means that mathematical educators must have the ability to listen to children and appreciate the social and cultural context they are in and are learning from (Rogoff, 2003). They also have to appreciate and accept the many ways that children communicate in mathematics. Kress's (1997) seminal work on multi-modality has been influential in seeing children as sense makers of their worlds. He presented intriguing ways that children use all the materials and communication avenues that are available to them, if given the opportunity. As referred to in 3.7 young children communicate their mathematics in many ways, for example through speaking, drawing, writing, and gesture (Flewitt et al., 2009; Anning & Ring, 2003). However, some ways of listening to children's mathematics are privileged. Certainly, as already put forward in 2.5, there has been and still is a prevalence of *practical mathematics* (Griffiths, Back & Gifford, 2017) in classrooms where children have resources to manipulate. The Williams Review of Primary and Early Years Mathematics (DCSF, 2008) reported that teachers found Numicon (Atkinson, Tacon & Wing, 2013), a set of plastic counting materials with holes, very effective in the teaching of mathematics. In certain periods the same popularity has been assigned to Dienes apparatus (Dienes, 1960) and Cuisenaire Rods (Cuisenaire, 1952). Speaking is also encouraged in schools as a major form of children's mathematical communication although that can sometimes be adult-orientated with direct answer-seeking (Littleton & Mercer, 2013).

There has been less research on children's own graphic representations of their mathematical thinking as communication; it is this form of children's thinking that this study is about and therefore, in the next section, I am discussing pertinent studies in this area.

4.2 Young Children's Representations of Their Mathematical Thinking

In this section I draw together the research on children's mathematical representations highlighting two of the main studies in this area, (Hughes, 1986; Carruthers & Worthington, 2006) which suggest that supporting children's own inscriptions may be a way to close the gap between the informal mathematics of young children and the abstract symbols and signs of standard mathematics. I also draw on the extensive work of van Oers and his use of semiotic theory within classroom contexts. Next, I point to other, smaller studies that have focused on children's mathematical representations, bringing different perspectives but also strengthening the case for schools to consider valuing and incorporating children's mathematical graphics in their work with young children. Finally, I discuss the research that highlights the possibilities of mathematical graphics in pretend play situations.

4.2.1 Introduction

Vygotsky (1978) saw representation as a way of knowing and highlighted the vital place of representation in young children's concept development, including their construction of mathematics. However, there has been little research on young children's own ways of representing mathematics compared to the wealth of literature on children's own early writing (Smith, 1998; Clay, 1975; Graves, 1983; Gerde, Bingham & Wasik, 2012). As discussed in chapter 3.7 listening to children has mainly been through spoken language and writing and there has been a paucity of research into children's communication of meaning in other modes, for example drawing, gesture and even less through mathematics (Matthews, 2003; Anning & Ring, 2004). However, there have been two significant studies that have influenced our understanding of young children's own mathematical representations and I explain these below.

4.2.2 Hughes' study

The first substantial study that foregrounds children's own mathematical representations is Hughes' study (1986) which came out of his previous research with Tizard (Tizard & Hughes, 1984) where he researched the nursery and home experiences of young *working- class* (a phrase that was used in the study) girls. He was surprised at the high level of mathematics some of these children were engaged in at home with their mothers, and he decided to investigate if, and how, young children represent quantities. He invented a game called the *tins game* which proved a successful way to give the children the opportunity to visualise a quantity they could not see (Hughes 1986). They, therefore, had to depict something in their mind and *put it on paper* (p. 64). The revelation was that these very young children (3 and 4 year-olds) did manage to draw, write and use their own invented symbols to represent quantities. Allardice (1977), using a different game, also reported that children did use their own invented symbols but gave few focused examples of children's own mathematical representations. Hughes, however, had examples from all the children in the study which he was able to categorise. This study was a watershed as it opened the door to the realisation that young children could use their own graphics to represent quantity. Later, smaller studies using the tins game, also had similar findings that young children had the ability to represent mathematical quantities (Vandersteen, 2002). Another significant finding of the Hughes' study was, when he used a similar game with older children, giving them addition and subtraction to represent, they did not choose to use the conventional arithmetic operation signs, even though they used them in school nearly every day in mathematics work books. Hughes (1986) concluded that children's understanding of mathematical symbols, "does not go beyond the context they are taught. There appears to be a serious and disturbing split between their use of symbols in the classroom and their ability to apply them to problems encountered elsewhere" (p.78).

The Hughes' study confirms, to some extent, that young children have some ideas of abstraction but the difficulties lie when older children have to use standard signs and symbols in mathematics even although they know the mathematical concept. For example, they can work out that 9 boats sailing in the ocean joined by 5 boats equals 14 but they do not seem to understand the written standard mathematical signs and symbols well enough to use them confidently i.e., 9+5=14. Hughes asserts, the key issue is the children do not seem to link the concrete (meaning what the children can see and touch) with the abstract and, as he further explains, children need "to develop links with this new language and their own concrete knowledge" (1986, p. 51). It appears that when children have to jump between practical mathematics to abstract signs, there seems to be a serious gap in understanding.

Hughes' work was celebrated and the important points noted but much of the research after and related to Hughes' study mostly concentrated on reproducing the tins game (Montague-Smith, 1997; Pound, 1999; Vandersteen, 2002). Although this research confirmed Hughes findings it failed to uncover the wider range of children's own graphics and the different ways that they might represent their graphics, for example, in classrooms and homes. Others have acknowledged, to some extent, the significance of children's own representations but mainly with older children (Cobb et al., 2000; Thompson, 2008; Terwel et al., 2009).

4.2.3 Children's mathematical graphics

Influenced by Hughes' work, the second significant study in the area of young children's mathematical representations, is Carruthers and Worthington's research on, what we term, *children's mathematical graphics* which are children's own inscriptions, signs and symbols of their mathematical thinking. Using qualitative research methodology this study uncovered children could represent other mathematical concepts beyond Hughes' discovery of young children representing quantities that are counted (see Carruthers & Worthington's 2005 Taxonomy, Appendix 2). The children chose to use, for example, tallies, arrows, spirals, hearts, and all kinds of scribbles to represent their own mathematical signs and symbols (Carruthers & Worthington, 2005, 2006). We drew from 700 samples of children's graphics, over twelve years, taken from classrooms and homes following children's mathematical enquiries in their free writing and drawing. Importantly, the children used their graphics in a variety of ways to solve mathematical problems which were adult or child-initiated. Carruthers and Worthington's (2006) data showed a rich range of children's own mathematical representations "no two pieces of children's thinking on paper are the same because no two children's minds are the same, we all have different experiences to form our thinking" although some of the children's strategies and ways of representing signs were similar (2006, p. 200). Notably, this study, for the first time, revealed a continuum from young children's earliest mathematical marks, to which they ascribed mathematical meaning, to written calculations (Carruthers & Worthington, 2005, 2006).

Similar to Hughes' study, Carruthers and Worthington (2006) also recognised the significance of the gap between children's informal home mathematics to using and understanding the standard symbols and signs of school mathematics. They pointed out that usually the child's developing mathematical understanding is moved on, in schools, from practical mathematics to children *copy recording*[12] the practical mathematics in standard ways. The teacher may expect the children to use standard mathematics and this might be done by copying standard sums from the board. In many schools this is a one-way process (practical to standard) and it rarely involves children's own representations of their graphics (Terwel et al., 2009). It also does not allow for children's cultural mathematical knowledge to be used and built on (Carruthers & Worthington, 2006; Worthington & van Oers, 2016). Carruthers and Worthington (2006) strongly emphasise that the gap between informal and formal mathematics could be bridged by supporting children's own mathematical graphics. Importantly, this is not a one-way process but it involves a transitional stage in which children can loop back and forth between home and school mathematics and between informal and formal notation, allowing concepts "to metamorphose: informal marks are gradually transformed into standard symbolism" (p. 81). As Hughes (1986) and Pimm (1987) have done Carruthers and Worthington (2006) also compare standard mathematical notation to learning another language,

> Children become what we term 'bi-numerate' and like bilinguals they come to use these two languages of mathematics fluently. Their understanding of the second language – the abstract mathematical language – will develop at a deep level since they will have constructed their own understanding of the roles and function of symbols. (p. 81)

[12] Copy recording is a term that has been introduced in this study. It means children copying a mathematical problem that has been previously worked out, either by the child using, for example, manipulatives or as a class problem they copy from the board. The distinction between children's own mathematics writing, where children use their graphics to work out a problem, and copy recording was first introduced by Carruthers and Worthington, (2005).

Drawing on Vygtosky's perspectives on bilingualism, and language and thought, John-Steiner adds, learners use 'their internal meaning system [...] they start the process of weaving the two meaning systems together" (John-Steiner, 1985, p. 364). This is a crucial part of the transition between children's informal mathematical signs and symbols and standard mathematical notation.

4.2.4 Schematising

van Oers has written extensively on young children's semiotic activity and the significance of children as symbol users and symbol inventors and the relevance of this to children's mathematical representations (van Oers, 2010; Poland & van Oers, 2007). However, his work uncovers only a few examples of children's mathematical inscriptions. His work with Poland concentrates on children's use of and understanding of symbols and signs, particularly dynamic representations, which include representations of transformations, change and movements (Poland & van Oers, 2007). They refer to this as schematising.[13] Their research design differs fundamentally from the Carruthers and Worthington research (2005) as they use clinical and quasi-experimental studies which elicit answers from children by direct questions. In the three schools Poland and van Oers (2007) researched in, a teacher trainer was employed as an interpreter "to translate the theoretical ideas of the researcher into practical ideas for teachers and children" within, what they termed, "meaningful play contexts" (p. 4). The teachers directed the children to signs they could use in their play designs of construction, although it was stressed the children did have a choice. Poland and van Oers concluded that children do gain from being taught schematising activities as they can explain and use some symbols, for example, arrows. They also proposed children do not spontaneously use symbols and signs in maps or play designs; it needs to be taught. Another central difference between the two studies is that in the Carruthers and Worthington study the practice and, following the child, comes first and the theory evolves from the practice and the children's actions. Whereas, in the Poland and van Oers study the theory comes first and is converted into practice; the signs were explicitly taught to the children, the researchers did not follow children's own intuitive understandings or ways of creating symbols. This might be the reason they could not uncover the spontaneous ideas or symbols of the children as the Carruthers and Worthington study did.

[13] Poland, van Oers and Terwel (2009) explain schematising as the construction and improvement of symbolic representations particlularly in diagrams or maps. (p. 307).

Nevertheless, the van Oers and Poland (2007) study importantly revisited the children in the following year and they discovered that the experimental children's mathematical thinking had improved significantly, compared with the control children. van Oers (2010) reflecting on this study (van Oers & Poland, 2007) and his previous studies argues that dynamic representations are crucial to the development of mathematical thinking and should be one of the main considerations of mathematics education. The importance of children understanding mathematical symbols and signs through their own mathematical graphics is key to my thesis.

4.2.5 Small studies highlighting children's mathematical representations

Carraher et al. (2006) assert young children often exceed adults' expectations of their ability to represent abstract symbolic thinking. They make a case for teaching algebra to children as young as seven. Their study supported and encouraged children's own algebraic notation. They found that children can combine both idiosyncratic and conventional notations that they understand and use to work out problems involving known and unknown quantities. Brizuella (2004) refers to this as a *meaningful transition* (p. 12) period to standard notation. It is this transition period, where children are afforded space and time to make meaning of mathematics that Carruthers and Worthington (2005, 2006) see as an important pathway into children's understanding of formal notation.

Several studies have identified that data handling is a useful area for young children to manipulate their own invented inscriptions and representations (Lehrer & Schauble, 2006; English, 2010; Levan & Hourigan, 2018; Papandreou, 2019). Using their own organisation and presentation of data contributes to children's meta-representational knowledge (diSessa, Hammer, Sherin & Kolpakowski, 1991). English (2010) adds,

> Children's developing inscriptional capacities provide a basis for their mathematical activity. Indeed, inscriptions are mediators of mathematical learning and reasoning; they not only communicate children's mathematical thinking but also, they shape it (p. 29).

However, English (2010) finds that in schools children are often taught a standard traditional lay out for data representations as single topics, at specific times in the curriculum, and the children do not really understand how to use data. They do not get a grasp of the importance of data handling and its relevance in society. Papandreou (2019) investigated signs children produce in their own data handling and the information they communicated through their graphical signs. She uncovered a range of ways children used their inscriptions in their data handling, for example, to indicate the class of data and to signify their inquiry results. Much of these findings support Carruthers and Worthington (2006) and Worthington, Dobber and van Oers' (2019) argument that children use their own invented signs and experiment with semiotic tools for a range of authentic activities, in this case, real data collecting situations.

In two separate studies, McDonald (2013) and Papandreou (2014) both consider drawing as a significant medium in children's developing mathematical semiotic activity. Drawings are a part of children's mathematical graphics, one of the mediums children use to create their mathematical thinking (Matthews, 2003; Carruthers & Worthington, 2006). McDonald (2013) draws our attention to the process of drawing, in illuminating what children already know, what they are challenged by and what they are interested in to explore further. She also explains that children sometimes make utterances to themselves as they draw, a running narrative, which can give a further window to our understanding of their mathematical drawings. She adds representations, "can capture the process of constructing a mathematical concept or relationship and can allow the creator to record and reflect on their thinking" (p. 70). Papandreou's study focused on young children (5 years-old) using mathematical graphics to solve a calculation problem, within the context of a story. Some of the children chose to invent their own symbols, to represent the animals in the story, instead of drawing the whole animal. This short-handing has also been described by Gifford (1990) and van Oers (2005), where children realise they do not need to draw the whole picture with details, a modified representation will equally convey the meaning. The children used their drawings to describe their thinking processes as they solved the mathematical problem. The drawing "became a tool for thinking at the various stages" of the drawing process (Papandreou, 2014, p. 97). Papandreou concluded that through drawings children co-construct mathematical meanings, solve semiotic problems and draw upon their mental activity to create new ideas and thinking.

Both studies above confirm that drawing can be a powerful part of children's mathematical graphics. The two studies concluded that schools might need to consider valuing and incorporating children's drawings as a mathematical thinking tool in their work with young children. Another area children have been observed using their mathematical graphics is in pretend play and I now discuss studies pertaining to this.

4.2.6 Children's mathematical representations in pretend play

Previously, I have addressed (4.1.3), to some extent, the debate around mathematical play also highlighted pretend play as a possible way that children use abstraction. I am now returning to address play more closely in regard to children's mathematical graphics in pretend play situations. Vygotsky saw pretend play as a "bridge" between "spontaneous and scientific concepts"(2.4.1). In pretend play the child substitutes and uses different cultural tools to signify meaning (Vygotsky, 1978, p. 238). Children's use of representation and mathematical understandings could be said to have its roots in pretend play where they explore meanings through objects, gestures, actions, words, artefacts, models, graphics and signs (Carruthers & Worthington, 2011; Worthington & van Oers, 2016). A recent longitudinal study tracing the development of children's graphics in pretend play situations, in one Nursery School, stated that "pretend play is a powerful context for the emergence of graphical signs that underpin the development of symbolic activities such as literacy, mathematics and numeracy" (Worthington, Dobber & van Oers, 2017, p. 150) Communication through talk and graphics, in this study, aided the development of children's mathematical thinking through their play narratives. Children drew on their cultural experiences at home as they played with others in their school. Analysing the signs and symbols, it was noted children used a range of symbols, for example, crosses to denote 'no more', zig-zags for a shopping list and ticks for collecting information. Children chose to communicate their literacies, with ease, in pretend play and out of 146 pretend play episodes 43.8% (65) included literacy events i.e., using graphics to communicate meaning. Another major finding was that the graphic communication increased significantly during the year. Papandreou and Tsiouli's (2020) study with older children (average age 5 years) concurs, to some extent, with the above study in that they found the source of children's school mathematical play episodes were often rooted in their home experiences. Within the play, some of which was pretend play, children used mathematical graphics as well as other modes of mathematical communication freely.

4.2.7 Summary

Research is increasingly uncovering that children can use their own mathematical graphics in a range of mathematical areas. Pretend play is a possible rich vehicle for children to use their own cultural knowledge to engage in mathematical play narratives, communicating through graphics. There is evidence that children have the capacity to develop their own ways of representing mathematics to problem solve, particularly in the area of number, including calculation. This is important for children to develop their own culturally based mathematical ideas, by using their own mathematical symbols and signs and gradually substituting these for more efficient standard signs. Children's own informal representations of their mathematical thinking could bridge the gap between children's informal mathematics and standard mathematical symbols and signs.

It seems children can and do use their own mathematical graphics to solve mathematical problems if given the opportunity. From the research evidence this seems to aid their understanding of mathematics. As put forward in 1.1.2, teachers need to value and support children's mathematical graphics otherwise children's growth in this area of developing an understanding of mathematical symbols and signs, may become stunted. Teachers' thinking and reflections on their mathematics teaching that support children's mathematical graphics may be vital to understanding pedagogies that encourage children's graphics. Therefore, in the next section, I analyse research that foregrounds teacher perspectives on mathematics teaching that allows children freedom to use their own mathematical representations.

4.3 Teacher Studies on Children's Graphic Mathematical Representations

In this section, I bring the literature research on pedagogy, listening to children, mathematics education, mathematical play and children's mathematical graphics to a tentative conclusion. I stress again that the research on children's mathematical graphics, although it has been a small but recognised field for more than thirty years, has not significantly impacted on classroom practice in the early years. I present five teacher studies that uncover some relevant practice and point to a possible direction for future research.

4.3.1 Introduction

Research studies both in the United Kingdom and internationally (Papandreou & Tsiouli, 2020; van Oers, 2010; Moffett & Eaton, 2018; Anantharajan, 2020) realise the strength of supporting children's own ways of representing mathematics to solve problems, and personally understand their mathematics, eventually accessing more abstract forms and higher-level mathematics. However, this has not had a notable impact on classrooms, it is not the norm in any school system (Moffett & Eaton, 2018). "Research takes a long time to get to classrooms" (Stipek, 2013, p. 434) and as Siraj (2017) states only a small percentage of university research makes an impact on classroom practice. My argument is that research studies wanting to make a difference to children’s mathematics involves listening to teachers’ understandings of children’s mathematical graphics. The emergent literacy movement (1989) with its focus on children’s own emerging writing had classroom practice at the heart, teachers were involved in experimenting and trying out and sharing their learning, they became what Pascal and Bertram (2018) term “praxeologists” straddling between practice and theory.

4.3.2 The issues with uncovering children’s mathematical graphics

It has been stated in this thesis, several times, the research into young children’s emerging mathematical graphics is meagre but studies concentrating on classroom teachers, engaged in developing children’s own ways of representing their mathematical thinking, are even rarer (Moffett & Eaton, 2018).

Just as teachers’ beliefs and attitudes seem to be influenced by traditional views of drawing (Matthews, 2003; Ring, 2006) it also appears that teachers’ beliefs about mathematical notation also keeps a traditional stance. Terwel et al. (2009) reflect there have been changes but there is not yet sufficient teaching practice for children to have opportunities to represent their mathematical thinking through their graphics. Stylianou (2010) found that children’s own mathematical representations did not play a major part in the teachers mathematical teaching, rather, representations were something to be taught directly, not as a way of approaching mathematical ideas. The issue remains and researchers (van Oers, 2010; Worthington, Dobber & van Oers, 2019) who want to analyse the children’s mathematical inscriptions also find it difficult to uncover enough classroom-based examples to examine trends and identify patterns, the exception being Hughes’ (1986) and Carruthers and Worthington’s (2006) studies. A later study by Worthington, building on her previous research with Carruthers (Carruthers & Worthington, 2005, 2006, 2011), on children’s mathematical semiotic activity was seriously hampered as there were insufficient examples, from schools and early years settings that the researchers approached in England and the Netherlands, to study development. Often, the prospective schools and early years’ settings produced only one or two graphic examples (Worthington, Dobber & van Oers, 2019). There seemed to be a dearth of knowledge and understanding; the culture to nourish and develop children’s mathematical graphics appeared lacking. Eventually, Worthington used only one Nursery School because it was “the only Nursery School identified that embodied such open approaches” (2019, p. 95).

The challenge, therefore, is moving the discourse into classrooms. This is not going to be straightforward as Donaldson (1968) states in the forward to Hughes' book,

> Hughes does not make extravagant claims for his proposals as to teaching methods. No one who takes Hughes' research seriously and thoughtfully will be left with the impression that this enterprise is ever likely to be an easy one. (p. 7)

More than thirty years later, pedagogy that gives opportunities for and acknowledges young children's own ways of representing mathematics are rare. Teaching mathematics, for the most part, is still about teaching only one way to do written mathematics and this therefore can be seen by children, as the *only* way. The danger is that children "ignore other possible ways to do it" (Tomasello, 2016, p. 4).

4.3.3 Teacher studies of children's mathematical graphics

There are teacher studies focusing on older children's mathematical representations (6 years and above) where the researchers discuss with the teachers what they often name children's written work in mathematics (Kazemi & Franke, 2004). There are also studies with teachers of older children where the researchers are specific about using the word representations for children's mathematical graphics (Terwel et al., 2009). In the Terwel et al. study, teachers, although part of the study, are not central to the study meaning their views are not a central source of the data collection. The teachers were given a manual of planned lessons and, "trained to assist students in generating representations in a process of guided co-construction" (Terwel et al., 2019, p. 26). Therefore, the teachers in this study were not generating their own ways of supporting children's mathematical representations but were following instructions by the researchers.

Teacher studies of young children's (2-5years) mathematical graphics are few and they usually focus on the children's graphics and not on teachers' perspectives and pedagogies about the graphics (Brizuella, 2004; Worthington & van Oers, 2016). I searched a range of research sources including Bristol University Online Library resources, Research Gate, Google Scholar, international early years' journals, and American and Australasian early years' mathematics networks. I found five studies that have included teachers' views, to some extent, about the pedagogy of children's mathematical graphics in early years' classrooms (Moffat & Eaton, 2018; Anantharajan, 2020; NCETM, 2009; Carruthers, 2012; Carruthers & Worthington, 2011). All these studies have recounted teachers' accounts of situations where they have uncovered young children's mathematical graphics and this has exemplified that it is possible in classrooms. I critically review these studies next.

The first study I am discussing is Carruthers and Worthington's (2011) study. We drew from our previous work (Carruthers & Worthington, 2006), taking most of the data from our own classrooms. Therefore, children's mathematical graphics theory was originally from a praxeology stance, rooted in classrooms. We gave pointers and strategies to teachers who may have wanted to work in a way that supported children's mathematics. This included an emphasis on teacher's modelling different ways to represent mathematical thinking through both standard notation and the informal signs and symbols that children presented to them. Even though the government guidance had recognised the importance of children's mathematical graphics and recommended practice that "encourages children to choose to use their own mathematical graphics to support their mathematical thinking and processes" (DCSF, 2008 p. 37), it was still rare to find practice that valued and used children's graphics. We reported in our study (Carruthers & Worthington, 2011), one of the main difficulties identified, seemed to be, teachers were uncertain of what children's mathematical graphics looked like. Many teachers showed examples of children copy recording, for example, where a child would work out an addition, using cubes and then would copy draw the cubes and the teacher would inform them of the symbols to use. In contrast children's mathematical graphics is about children using their own ways to represent and understand, as they work out the problem (Carruthers & Worthington, 2011). This can be further explained by Muira (2001) who expresses there are two types of representations that affect children's understandings and solutions to mathematical problems. She refers to these as *instructional* which are external to the student, for example direct teaching of representations, which could mean copy recording. The other is *cognitive* representations which she said are "constructed by the students themselves as they try to make sense of a mathematical concept" and the latter is similar to children's mathematical graphics (p. 53). Taking a pedagogical stance Carruthers and Worthington (2011) have consistently advocated, "it is what a teacher does deliberately, knowing why they are doing it, that makes a difference to a child's mathematical experience" and this is also true for incorporating a pedagogy that listens to young children's ways of expressing their mathematical thinking (p. 161).

The second teacher study of children's mathematical graphics I am highlighting, is a single case study of one Reception teacher's conversation, with the researcher, of her experience as she introduced the concept of children's mathematical graphics into her classroom (Carruthers, 2012). The teacher presented an abundance of graphics that children used to work out mathematical problems. She said children need problems that are challenging and that they cannot work out mentally. She stressed this is when the graphics are useful. This resonates with Vygotsky's view that useful and meaningful instruction is ahead of development (1978). This one long conversation, with a teacher who was working in a way that supported children's mathematical graphics, underlined how she had previously not noticed that some of the children were very able mathematicians. This teacher seemed to understand the power of valuing children's mathematical graphics, to the extent she started to influence her whole foundation team. My strong sense, at the time is, she was brimming over with excitement as she shared the children's mathematical strategies and talked about their graphics. I view this as similar to my feelings when, as a teacher, I discovered that children could use their own ways of writing, becoming independent writers, in my classroom, as I started to value their emergent writing (Carruthers & Worthington, 2006). I believe it is teachers, like the teacher I discussed above, we have much to learn from about the situated pedagogy of children's mathematical graphics in the classroom.

Another study that discussed teachers' engagement with children's mathematical graphics is the Promoting Early Number Talk (PENT) study in Northern Ireland (Moffett & Eaton, 2018). In this research, teachers felt that they were under pressure from school leaders to progress children from practical activities to formal standard recording. The researchers invited the teachers to use language and children's mathematical graphics, instead of moving instantly from practical mathematics to standard mathematical symbols and signs. The teachers trialled some of the strategies the researchers recommended, and the teachers related they would never have noticed the children's own ways of representing their mathematics before. They said they were overwhelmed by the variety of strategies the children used to represent quantities and addition. Being unaware of children's mathematical ability to represent mathematics in different ways was also a notable theme in the Carruthers (2012) study referenced above. A main conclusion of the PENT study, although predominately a mathematical language study, strongly asserted, "Greater prominence must be given to the development of children's own mathematical graphics" (Moffett & Eaton, 2018, p. 559).

The fourth study I am commenting on was not directly about teachers and children's mathematical graphics, it was looking at a range of successful mathematical professional development projects (NCETM, 2009). However, interestingly, it underlined one particular and comparatively different project. In comparison to nearly all the other projects in the study, which were mostly initiated and supported by outside funding and led by specialised trainers or recognised Continual Professional Development (CPD) leaders, this project had no funding. It was initiated by teachers from several schools after a few of them had been on a children's mathematical graphics course. They decided to form a collaborative support group to further develop their understanding of children's mathematical graphics. The researchers found, on visiting the teachers' classrooms, "the standard of mathematical understanding and thinking and reasoning that the displays (of the mathematical graphics) revealed was far higher than the specified curriculum objectives for children" (NCETM, 2009 p. 64). One teacher in the study relayed how she was hesitant to change her practice because, "she was uncertain about how much mathematics the children would engage with, if they were not filling in worksheets" (p. 63). However, after being in the teacher support group for some months and trialling new approaches in her classroom, she said the children were learning so much more than what she had previously observed. She asserted, "I could never go back to that now" (p. 63) meaning returning to how she formerly taught mathematics using worksheets.

It was also found that the teachers were involved in reading and reflecting on educational research findings, they were becoming researchers in their own classrooms and this facet seemed to contribute to their change of practice (NCETM, 2009, p. 62). This aspect was also recognised in the PENT study where teachers were "engaging with relevant literature" (Moffett and Eaton, 2018, p. 551). This report confirms Carruthers and Worthington's (2011) beliefs about the potential of empowering, trusting and giving teachers ownership of their professional development. Coupled with this, is the importance of an intellectual environment, not only for children (Siraj-Blatchford, 2002, p. 48) but there is an argument, as evidenced above, that teachers also need such an environment. This seems to support their understanding and subsequent pedagogy of young children's mathematics using graphics. From the research evidence, so far, this is more than tips for teachers.

The final teacher study reported here is of teachers supporting children's mathematical graphics in Kindergarten (4-6 year-olds). The study focused on teachers' awareness of the children's mathematical representations in counting (Anantharajan, 2020). The children counted a collection of objects and then recorded this (recording as in data handling), in whatever way they wanted. The teachers said, as they analysed the children's graphics, it heightened their understanding of the children's mathematical thinking. The children's representations supplemented the teachers' in-the-moment noticing with more reflective noticing as they had time, after school, to analyse the children's representations. The teachers were familiar with discussing the children's counting with the children, but they were unfamiliar with the task of looking and analysing the children's counting graphic representations.

The teachers reported that they saw many more mathematical ideas in the representations than when they just observed the children's counting and this was seen in all of the children's counting representations. For example, one teacher noticed "addition, composition and decomposition of numbers, skip counting and place value" (Anantharajan, 2020, p. 287). This gave the teachers additional insight into the children's mathematical thinking. Crucially, in this study, the teachers emphasised that these insights could not be observed only by watching and talking to the children when they were counting.

Four, out of the five, teacher studies discussed above were involved in professional development sessions about children's mathematical graphics, before obtaining data from classrooms. This strongly points to the premise teachers do need a professional learning input in this area, as supporting children's mathematical graphics is unlikely to instantly happen in classrooms by merely drawing attention to children's mathematical representations. What the most appropriate form of CPD for children's mathematical graphics is, has not been researched. Another key aspect, that is already known, about the pedagogy of children's mathematical graphics from previous studies, is that creating a democratic culture within classrooms is critical. This can allow children the freedom to think for themselves, engaging in metacognition and working out their own mathematical understandings (Carruthers & Worthington, 2011; Worthington & van Oers, 2016). A notable point from some of the studies presented is, teachers found that children benefit from using their own mathematical graphics when they have a problem to solve, a challenge that they cannot work out easily (Carruthers, 2012). As Hoyles (2019) explains, mathematics is still about problem solving, and van Oers (2001) adds, with symbolic tools.

The realities of the stress that teachers are under in UK schools from the performative agenda, for example, Ofsted expectations, assessments and many other organisational constraints cannot be underestimated (Ball, 2016). Classroom life is hectic and chaotic with many curricula demands. Mathematics, although privileged as a curriculum area, is only one strand of seven curriculum strands for the Early Years Foundations Stage, in England. There has been official government guidance and recognition of children's mathematical graphics, for example, in the Independent Review of Mathematics Teaching in Primary and Early Years Settings (DCSF, 2008) where it was identified that it is common for teachers to recognise and support children's early writing but, "It is comparatively rare, however, to find adults supporting children in making mathematical marks as part of their abilities to extend and organise their mathematical thinking" (DCSF, 2008, p. 34). The review felt that teachers and early years' practitioners were missing an opportunity to support children's understanding of mathematics, therefore, they commissioned publications for teachers to develop teachers' understanding of the value of children's mathematical graphics (DCSF, 2008; DCSF, 2009). Although, to some extent, this was going in a useful direction, the booklets were not accompanied by professional development guidance and teachers, who may have had questions or were confused by certain aspects, had no support. The first booklet was of a generic mark-making theme, promoting all the marks children make, not necessarily mathematical ones. This was missing the point of the review where it stressed that the materials should be focused on "mathematical mark-making and children's mathematical development which can be used to support early years' practitioners CPD." (DCSF, 2008, p. 32). The second booklet was focused on children's mathematical graphics but did not give sufficient examples of practice. Officially, in England, children's mathematical graphics were accepted and schools could have felt confident incorporating mathematical graphics into their teaching. However, as Bousted (2019) states governments can make recommendations but until they align with Ofsted inspections and inspectors are knowledgeable and understand, then teachers are reluctant to implement any changes for fear of failing the Ofsted teacher observations. Bousted goes on to say that teachers take this stand, even if they think it may not be what is best for children's learning. This tension was especially noted by one teacher, who was trying to develop children's mathematical graphics within her school. She expressed how this might go against the grain of Ofsted, "We are being battered to make sure our (mathematics) books are neat" (Carruthers, 2012). She further states,

> Representing (mathematical) thinking is not neat, it is not in boxes […] The focus should be on the mathematics, not the recording, not, is it on the line? but, what is the maths? Is there mathematical thinking going on? (p. 205)

This teacher resolutely wants the direction of how children's mathematics is observed to be firmly looking at children's mathematical thinking, not set ways to write down mathematics.

4.3.4 Conclusion

It seems teachers have certain obstacles to overcome before they can begin to implement changes that they feel would benefit children's mathematics. Importantly, teachers and those who lead schools and curriculum need to see the substantial benefits of teaching that supports children's own ways of representing mathematics because it may mean altering their teaching approach and understanding of how children learn mathematics (Carruthers, 2012). This could mean too much change for schools to consider, it may be too much of a risk as referred to above. However, as Guskey (2002) states, when teachers see a new approach working in their classrooms, change is more likely to follow. In all five teacher studies presented here teachers claimed, when they supported children's mathematical graphics, they saw a variety of children's mathematical strategies and they achieved higher-level mathematical thinking, than was previously noted. To what extent and how this is sustainable, within classrooms, is less clear.

Stenhouse (1975) reflects, it is teachers who will make an impact on education. It is the teachers who are firmly centred in the real classrooms of experience and are constantly faced with the everyday dilemmas of the teaching world. Listening to teachers' perspectives on trying to incorporate practice that supports children's own ways of thinking through their graphics, makes a connection to real-life pedagogy and the realities and possibilities of the classroom. Taking a note of what teachers say may give an understanding of what might work in classrooms and provides a window into classroom culture. Teachers need time to think through what works for them and their children (Moffett & Eaton, 2018).

Pedagogies that are democratic enough for teachers to listen to children's mathematical narratives, giving opportunities for children to use their own graphics, as already stated, may be a difficult option. I argue, it is only teachers in their classroom that will make a difference, in the end, as with emergent writing in the 80's and early 1990's when teachers began to acknowledge children's evolving writing and not only standard, correctly written scripts (National Curriculum Council, 1989). Therefore, the research I am proposing is about the pedagogy of children's mathematical graphics in number, from teachers' perspectives. What can we learn from teachers? How can listening to children's informal graphical ways of communicating be incorporated into early years' classrooms and be a part of everyday mathematical practice, rather than a scarce event? These questions will help me formulate the focus of my thesis.

4.3.5 My research questions

The teacher studies above, although they included teacher's view on children's mathematical graphics in their classrooms, it was not the main focus of the research. For example, Moffett and Eaton's study (2018) was mainly to do with spoken language and the NCETM (2009) research focused on teachers' mathematics professional development. The Anantharajan study (2020) was specifically about children's representations of counting guided by a specific programme (Cognitively Guided Instruction, Carpenter et al., 2015), it was not about how teachers might incorporate and support children's mathematics inscriptions, in general, in their classroom. Also, in most of these studies, they gave no specific examples of children's mathematical graphics which I find is crucial and gives clarity around the meanings and interpretations of the children's graphics. The Carruthers and Worthington (2011) study and the Carruthers (2012) study is based on descriptions of the teachers' examples of the children's graphics, analysing the graphics and understanding the children's meanings. Although some teachers within the study gave useful pointers to practice, it was not mainly focused on pedagogy. Empirical studies have not addressed how early years' teachers might incorporate and support children's mathematical graphics in their classrooms as regular practice. The study, I am proposing, may go some way to filling a gap in the literature, adding to our existing knowledge, through the small but growing area of research interested in young children's mathematical representations in classroom practice.

My broad and over-arching question in this study is:

- What do teachers consider cultivates the existence and understanding of young children's mathematical graphics in early years' classrooms?

My sub-questions are:

- What are the difficulties teachers face in teaching in a way that supports children's mathematical graphics and how can these difficulties be resolved?

- What are the conceptual shifts in practice that a teacher may have to consider as she/he embarks on supporting children's mathematical graphics?

- How do teachers feel the children benefit, if at all, from using their graphics to support their mathematical understanding?

In the literature review, it has been noted that studies have uncovered children use their mathematical graphics in pretend play (Worthington & van Oers, 2016) but there are no studies, that I have noted, that focus on how teachers enable and support children's pretend mathematical play. This is a crucial concern, as I underlined within the research, the adult's role within play, in general, is a contentious and puzzling issue. To address this, my question is:

- If teachers support pretend mathematical play, where children use their mathematical graphics, how do they interact with children to support and enhance their play and mathematical thinking?

Considering these research questions, the next chapter outlines the research design. Firstly, I state my ontological and epistemological positions.

Chapter 5: Methodology

In chapter 4, my literature review concluded with the premise that a research focus on pedagogies that support young children's mathematical graphics would be a useful addition to research on children's mathematics' representations. I now propose a methodology that will aid my research inquiry. This chapter is in two parts. In chapter 5.1 I discuss my ontological and epistemological position outlining social-cultural research. In chapter 5.2 I present the methodology and methods used in this thesis, stating a clear rationale.

5.1 Ontological and Epistemological Positions

This thesis is about mathematics education focusing on teachers' views about how they are reflecting on their classroom teaching, to include young children's understandings of mathematics, through their graphics. Firstly, therefore, I want to consider what mathematics is.

5.1.1 What is mathematics?

Mathematics is an important curriculum area in education and increasingly there is acknowledgement that it is also an important feature of early education (Perry & Dockett, 2008). Freudenthal (1973) viewed mathematics "not primarily as a body of knowledge, but as a human activity" (p.5), and he stressed that mathematics education should be similar. Hersh (1996) agrees, from a humanist point of view that mathematics is a human activity and a feature of human culture. He states that many eminent mathematicians believe in the absolute truth of mathematics. He struggles with their absolutism viewpoint and why they see something he considers "so unscientific, so far-fetched as an independent material world of mathematical truth" (p. 6). He continues, "If we give up the obligation of mathematics to be a source of indubitable truths, we can accept it as human activity" (p. 22). The stance of mathematics as a human activity is vital to this study and the socio-cultural theoretical underpinning. It gives the freedom to view mathematics from different human standpoints, including children's perspectives. Similar to Hersh, others question the absolutist philosophical stance. This has given the absolutist claim serious challenge. For example, Lakatos (1962) argues, from a fallibility perspective, any mathematical system depends on a set of assumptions and attempting to prove their certainty leads to a never-ending regression. There is no way of getting rid of the assumption. Without proof, the assumptions remain fallible beliefs and not absolute truth. That is also a position that Ernest (1991) puts forward, "mathematical truth is fallible and corrigible and can never be regarded as beyond revision and correction" (p. 46).

As mathematics is a human-made science, it is open to be challenged. It is important to question the absolutist definition of mathematics because otherwise we will find it hard to embrace the subjective element of mathematics. If we want to open up our observations of the development of mathematical knowledge and mathematical pedagogy then the subject must be seen as offering possibilities at every level; from young children making their own gross approximations, working through their own interpretations to develop their understanding of the mathematical world and their mathematical world, to adults rediscovering mathematical theorems, broadening mathematics and inventing new theorems. As Ernest (1991, p. 46) asserts if, "mathematics is a fallible social construct, then it is a process of inquiry and a coming to know, a continually expanding field of human creation and invention, not a finished product" (p. 46). This concurs with Benn and Burton's (1996) fallibilist position as they state, teachers who believe in this philosophical stance are more likely to "ensure mathematics is seen like other disciplines as a negotiated journey, a quest and a voyage of discovery" (p. 1). This is an exciting position for mathematics education.

This view of mathematics as a human activity is similar to a Vygotskian standpoint where mathematics is embedded in social practices, it is an active and cultural process. van Oers (2001), from a Vygotskian perspective, argues that "mathematics as a subject matter is really about problem solving activity with [the help of] symbolic tools" (p. 63). These symbolic tools appear in a range of ways, for example, in conversations, dialogue and representations. It is the children's mathematical representations that are steeped in their cultural understandings and this thesis is about acknowledging and supporting these understandings. My ontological stance is therefore that mathematics is a fallible, human creation, embedded in socio-cultural practices. To establish my epistemological position I now consider the question of how children (or adults) come to acquire mathematical knowledge.

5.1.2 How do children (or adults) acquire mathematical knowledge?

In the last century, traditional views of how children acquire mathematical knowledge were based on the behaviourist movement, for example Thorndyke's experiments on animals of stimulus response were translated into mathematics teaching (Thorndyke, 1905). For many years, therefore, mathematics, in particular, was not an education subject that was acknowledged as part of children's cultural knowledge. Mathematics was seen as an isolated piece of knowledge. This view has gradually declined and the need to make mathematics authentic and relevant has become part of many mathematics classrooms (Askew et al.,1997; Tucker, 2014). This has not necessarily resulted in acknowledging children's contributions or background home and community knowledge. The cultural aspect of young children's mathematical knowledge is still not a main thread of many mathematics curricula (Worthington & van Oers, 2016).

Vygotsky, towards the end of his life, highlights that the heart of doing and learning mathematics occurs as connections between people. Roth (2014) explains Vygotsky's theory further: "the essence of mathematical thinking and reasoning is social, not because people have socially constructed it, but because it is a phenomenon that only arises with society" (p. 12). From the previous literature, referred to in chapter 2, there is a strong case that children come to know mathematics by interacting and making meaning with the mathematical cultural tools and influences that surround them (Vygotsky, 1978). James and Pollard (2008) and Sfard (1998) term this, learning as participation. From a socio-cultural perspective, knowledge is created together with others as children participate as active agents in their cultural and social worlds; they play critical roles in shaping the interplay of social life and in determining individual activities (Sfard & Prusak, 2005). These ways of knowing capture beliefs, patterns of behaviour, rules, structures, interactions within their family and community lives. For mathematics this means that children do not just learn isolated skills, set apart from the context of their worlds, for example rote learning numbers 1-100 or reciting the multiplication tables. They become mathematicians and communicators through the multi-faceted experiences and "sustained engagement in repertoires of practice which can be adult or child-initiated" (Wood, 2010, p. 14). Skemp (1978) puts forward that learning accumulates and slowly becomes refined and organised into increasingly coherent concepts and schemata. Learning is therefore gradual, a coming to know and the learner is immersed in the experience and actively involved in their learning (Athey, 1990).

Learning, and learning mathematics within a socio-historical cultural perspective can be defined as enculturation (van Oers, 2019). Enculturation develops through interactions with other people in the context of cultural practices (Lave & Wenger 1991). As van Oers expresses, it is multifaceted and constantly evolving and includes the role of the adult and how they engage with children on the "basis of their pre-suppositions and ideologies" (van Oers, 2019, p. 433). However, Radford (2016) contests the concept of enculturation stating that it over emphasises the part of social practice, without critical debate; it also weakens the role of the individual. Radford also criticises Rogoff's (1990) theory of guided participation and analogy of apprenticeship, even although Rogoff provides a balance between the individual and the social influence on the child. Radford's argument is, "there is little room to investigate the individuals as agentic entities, such as the manners in which the individuals come to position themselves and be positioned in those practices" (2006, p. 201). Radford, therefore, stresses the important place and activity of the individual within the social learning sphere.

Radford (2016) in his Theory of Objectification, within a socio-cultural perspective on learning and knowledge, includes affective learning, not just as an add on, but following Vygotsky's observation, concludes emotions are an ever-present part of thinking, including mathematical thinking. This, Radford (2016) explains, is not usually an interwoven part of most educational theories but it is vital to his view on socio-cultural knowledge: "Emotions are ontological constituents of us, humans, as part of nature. Affect, that is, the capacity of being affected by things in our surroundings, on the other hand, is also part of our human makeup" (p. 191). Therefore, he stresses that learning mathematics is not just solely a mental undertaking but involves emotions "and affects in manners that affect us profoundly as human beings. This is why classrooms do not produce knowledge only; they produce subjectivities as well." (p. 191). From an early childhood perspective, I believe that the affective part of learning is a priority. This has been highlighted also in government policy documents (DfE, 2012) but it is less recognised as directly connected with young children coming to know mathematics. However, from a socio-cultural standpoint and because mathematics has been described as more than a set of skills to learn, then Radford's ideas open up the possibility of recognising an affective influence of children's mathematical behaviours, in developing knowledge in mathematics.

In summary, therefore, I adopt a socio-cultural perspective on teaching and also the pedagogy of mathematics, as the epistemological part of my theoretical frame, for seeing and understanding young children as social and cultural beings, who use all their experiences to learn and to come to know. A socio-cultural perspective does not take away the important part of the individual to synthesise that knowledge. Realising the implications of this perspective in the classroom, however, relies on teachers trusting children and knowing children well. My main argument is that teachers cannot uncover or know the children's mathematical graphics if they do not draw upon the children's social and cultural worlds. Knowing children's social and cultural worlds has implications for how children learn mathematics and how mathematics is taught. Therefore, the underpinning epistemology of this thesis is based on socio-cultural theory.

Socio-cultural research looks at the situated context holistically (Schoen, 2011). Socio-culturalism also acknowledges that human beings are "social and reflexive and that complexity in the social world alters human thought and behaviour" (Schoen, 2011, p. 12) It is also viewed as a naturalistic approach to research (Lincoln & Guba, 1985). It is rooted in the knowledge that human action and thought processes do not happen separately from each other but develop as humans interact (Vygotsky, 1978). This study is about researching pedagogy and therefore it is important to have discussed what pedagogy is, within a socio-cultural lens (see chapter 2) in order to engage in pedagogy research. Nind, Curtin and Hall (2016) state "pedagogy concerns how people are enabled, supported and constrained in how they participate in practices and activities, and how their histories mediate and are brought to bear by the teacher and by the setting" (p. 36). Therefore, from a socio-cultural perspective pedagogy is never neutral but bound by the experience and histories of those who engage in teaching.

Schoen (2011) outlines the value in socio-cultural research as he asserts it can uncover how thoughts are connected and make sense of the vast "barrage of research-based information today's practitioners face" (p. 30). However, Schoen (2011) cautions against over simplifying data and sees this as a major barrier to socio-cultural theorists. Simple interpretations of intricate data can occur because of the broadness of the required area in considering numerous sources of influence. In my study, to counter this over simplification of the data, in the next section, chapter 5.2, I define the methodology of my thesis explaining carefully how I undertook the analysis of the data, clearly explaining the issues involved and how I might overcome them.

5.2 The Methodological Design of the Thesis

Within this section, I outline the methodological design of the thesis. Firstly, I critically analyse the importance of qualitative research. I then describe my rationale for using a case study methodology. I examine the issue of the balance of power within the research. Next, I introduce the participants of the study and the position of the Masters' modules as a springboard for the teachers to consider changing their practice. To support the design of the study I explain the pilot study. The phases of the data collection are explained and justified, within this the data collection methods are explained and critiqued. The methods of analysis are discussed alongside the data collection. Finally, the validity and reliability of the study are questioned highlighting reflexivity as a way to capture subjectivity. The ethics of the study are discussed, not only as a section within this chapter but ethical considerations are commented on throughout my thesis.

5.2.1 Qualitative research

An overarching aim of this study is to capture the perspectives of teachers on their developing mathematical pedagogy, therefore, qualitative research, because it adheres to less restricted methods of collecting data (Atkins & Wallace, 2012) could give the teachers opportunities "to say what they want to say, and they will not, for example, be limited to responding to researchers' pre-established questions" (Yin, 2016, p. 14). Questions might open up the discussions, giving teachers avenues to pursue their own lines of thinking about their mathematical pedagogy. Therefore, this is a qualitative longitudinal study situated within an interpretative paradigm (Crotty,1998).

Qualitative research involves the study of a collection of a variety of empirical materials for example, personal experiences, interviews and documents that "describe routine and problematic moments and meaning in individual lives" (Denzin & Lincoln, 2000, p. 6).
This study draws upon qualitative case study methods which links with my established socio-cultural theoretical position explained in chapter 5.1.

Although Denzin and Lincoln (2006) and Hammersley (2008) are strong proponents of qualitative research they underline possible criticisms of this methodology. For example, a main criticism of qualitative research is the question of insufficient thoroughness as it is too subjective and the outcomes are not generalisable. These critiques base their notions on traditional views of positivist scientific inquiry (Crotty, 1998). However, there is increasing challenge over precise divisions between qualitative and quantitative research such traditional stances on research methodology are also increasingly questioned (Gage, 1989; Tashakkori & Teddie, 1998; Bryman, 2006). Instead, the potential of an interpretative paradigm is that it acknowledges the complexities and multiple meanings situated within the research field by both the participants and the researcher. It highlights the researcher's own interpretations and subjectivity and therefore challenges perspectives (Counsell, 2009). The interpretative approach is a tool for understanding the reality experienced by people. It gives you different points of view, you can have access to different aspects of reality (Denzin & Lincoln, 2008) and within this the individual is an active agent (Wotherspoon, 2004). However, as Counsell (2009) points out, critiques on an interpretative approach often state that one could perceive it as being too loose as anything can be claimed. This is because beliefs and perceptions are considered and, as Nietzsche (2003) adds, "no facts only interpretations" (p. 3). However, what matters is how it is reported on, how the data are drawn and although no firm claims can be made, insights into, for example, classroom life can be rich and informative through a qualitative approach.

5.2.2 Case study research

In this thesis I am using case study as the research methodology. Yin declares, "the distinctive need for case study arrives out of the desire to understand complex, social phenomena" (2018, p. 5). I am engaging with teachers to jointly understand how children's own mathematical inscriptions can be understood and supported, within early childhood classrooms. Case study research gives the scope to make in-depth observations "to retain a holistic and real-world perspective" (Yin, 2018, p. 5). Therefore, case study research gives the opportunity to study pedagogy in detail within the everyday experiences of classroom teachers, from their perspectives.

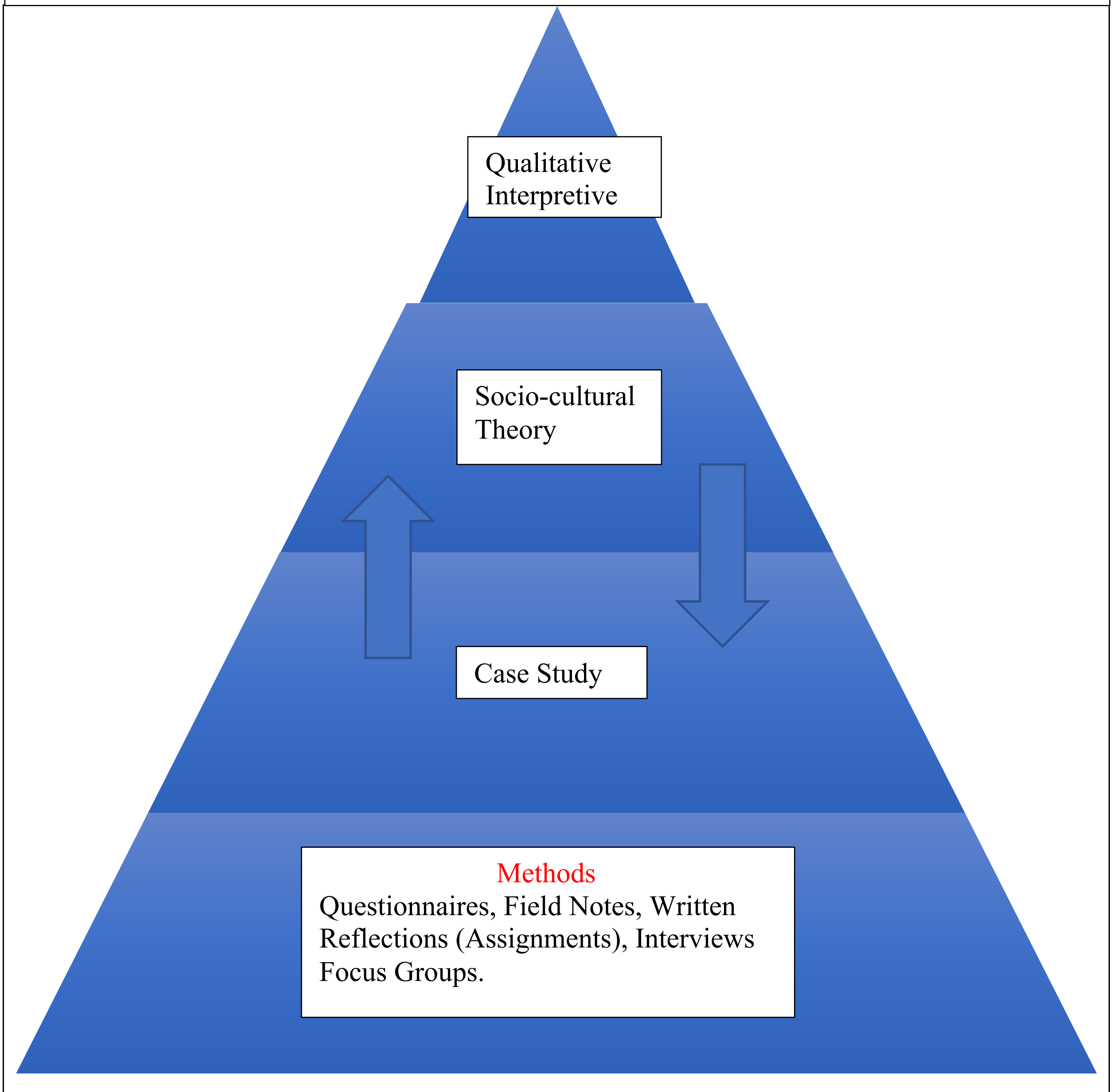
In considering the methodological framing of this study I initially considered ethnography as Willis (2007) affirms there are many similarities between case study and ethnography. For example, in both, you can gather detailed data in a natural setting; it can be about human behaviour in social contexts, and it can be done without pre-set hypothesis (p. 239). This dilemma about case study versus ethnography is also documented by White, Drew and Hay (2009). The difference, in my study, which could be said to be partly ethnographic, is that although from an ethnographic stance, the research is exploratory in nature, it departs from ethnography, because I am not going directly into the field, i.e., the teacher participants classrooms, to collect data. I am getting information about their classroom practice through the teachers' discussions, writings and interviews. It is the teachers' perspectives I want to know about to answer my research questions, not their classrooms per se. Ethnography is a natural process, as stated before, with no predetermined aims. This is different from this study, although the participants are following their own ideas, the modules presented to the teachers an aspect of mathematics education they had not considered. I aimed to be sensitively guiding them, to think about children's mathematical graphics; I am intervening in their practices in a manner that does not fit a typical ethnographic study. Therefore, although there are some elements of ethnography within the methodology, case study is the overarching methodology.

5.2.2.1 Perspectives on case study methodology

Yin is a seminal author on case study methodology and Yazan (2015) suggests that Yin shows positivist attributes in his views on case study. He explains it aligns with Crotty's (1998) perspective on positivism as Yin (2002) stresses that the researcher needs to, throughout the cases study research, keep an awareness of four essential factors: construct validity, internal validity, external validity, and reliability. Yin (2018) does not declare a philosophical stance and argues against those that make a distinct division between qualitative and quantitative case study methods emphasising the similarities between the two research traditions. However, Stake (2006), another seminal case study author, conversely urges case study researchers to maintain a qualitative stance. His philosophical view is constructivist and converges with Merriam (1998). Merriam asserts "the key philosophical assumption upon which all types of qualitative research are based is the view that reality is constructed by individuals interacting with social worlds" (Merriam 1998, p. 22). My underpinning epistemological view within the study is from a socio-cultural perspective (see 5.1) and is more aligned with Merriam (1998) and Stake's (2006) position on constructivism than Yin's positivism.

From a Vygostkyian (1978) standpoint, people are culturally based and therefore this study is taking a view that people's actions and thoughts in the world are embedded in culturally situated experiences. Culture is not a separate variable or an add-on but it is seen as "the essential medium of human understanding" (Saljo, 1991, p. 191). Yin's (2018) extensive work on case study design is useful, within this study, but his leanings towards positivism (Crotty, 1998) could run counter to a social-cultural paradigm. Wegerif (2004) argues that in an attempt to follow a highly systematic structure, within the research, there is a danger of operating double standards "where the truth claims of the subjects studied are bracketed out or ironised" (p. 153). Therefore, similar to Merriam (1998) and Stake (2006) this study takes a flexible approach to case study methodology and is also mindful of the participants' cultural base and understandings and their interpretations of their pedagogy. It is the pursuit of shared interests, in teachers reflecting together on their pedagogy, that supports children's own mathematics, which is vital in this study. Without collaboration, teachers' knowledge is not always recognised and often remains silent and stays within the teacher (Hatch, 2002). From a socio-cultural position, the methodology encourages teachers to voice their thoughts on young children's mathematics in a collaborative meaning-seeking situation. Each teacher's cultural perspective is different and they bring that diversity of knowledge to the group. Below, I illustrate the methodology for my study (figure 5.1).

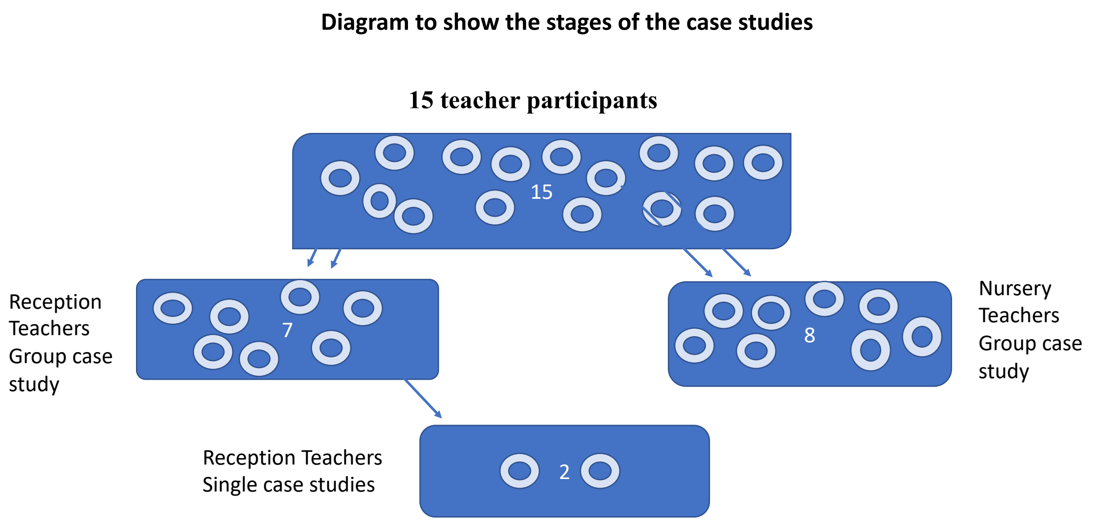
Figure 5.1: Diagram to show the methodology and flow of the study.



5.2.2.2 The case studies

This case study methodology is centring on both single and multiple case studies (Stake, 2006). I am considering the eight Nursery School teachers and the seven Reception teachers as two collective group case studies. I then extend this by drawing out and continuing two single case studies from the Reception teachers for further analysis (see diagram 5.2). Over the three years and five months of collecting and analysing data I managed to pace the study so that it became manageable. Originally, I had planned the Reception and Nursery School teachers as one whole group but as the data unfolded in the initial open coding it uncovered that these were two notably different groups and analysing them separately, at first, would reveal their distinct features. The necessity to do further single case studies arose as a result of the analysis of the two group case studies and the need to focus on certain aspects (this is further explained in the phases of the data collection in 5.2.7.3 and 8.1.) Although, the findings cannot be generalised they may offer a starting point for other researchers or provide a useful comparison for similar research (Atkins & Wallace, 2012). Therefore, case studies can be the foundations for insightful generalisations. Stake (2006) emphasises that one major feature of case study methodology is that different data collecting methods are combined to illuminate a case from different angles, to triangulate by combining methods and this is expanded on in 5.2.9.

Figure 5.2: Diagram to show the stages of the case studies.



5.2.3 Ethical considerations

In this section I discuss ethical issues focusing on relationships and power relations, gaining consent and the importance of confidentiality.

5.2.3.1 Relationships and power relations

Socio-cultural research is to do with human relationships and collaboration and inevitably there may be issues, for example of power relations (Atkins & Wallace, 2012). Foucault (1977) asserts “Power is often masked, consequently we take for granted the power structures in social institutions, social structures and social expectations” (p. 8). Teachers may feel in an inferior position compared to educational researchers thus creating an imbalance in power relations (Hatch, 2002). In my study, the issue of my position as headteacher of the school of two of the teacher participants could bring a reticence to the research. As Greishaber (2007) reflects, the power equality can continually go back and forth from participant to researcher in qualitative research, therefore, power relations are uncertain. However, reflecting on and discussing situations collaboratively and co-operatively can neutralise the power balance (Greishaber, 2007). For example, the two single case study Reception teachers will have the opportunity to take part in the knowledge production (Bergold, 2007) as they identify their own theories and add to the growing knowledge of mathematical teaching practice.

Frequently scientific research, as Greishaber (2007) declares, is *put on* research participants who become the *objects* of the research. In contrast, equitable research “involves notions of justice and fairness moving away from deficit models and the idea of homogeneity and the detection of bias” (Carruthers, 2015, p. 7). Greishaber underlines, “Research design that is informed by principles of equity is explicit in its political purpose of seeking socially just outcomes for the short and the long term” (p. 177). Important questions to consider are: in what ways will the research participants and others involved benefit from the research; who owns the knowledge? These questions are derived from ethical principles. From the perspective of equitable relationships between researcher and participants,

> my research design considers, from the beginning, that I am doing research *with* the teachers rather than *on* them. I, therefore, will involve them in discussions and collaborative understandings through recurrent conversations, checking their meanings and changing my interpretations of their data when needed. I intend to keep them up to date with my research writings and inform them of publications

and presentations asking for further contributions or changes. (Carruthers, 2015, p. 8)

Collaborating with research participants also requires ongoing reflexivity and sensitivity to emergent ethical issues (Atkins & Wallace, 2012). To address this, throughout the study, I will make written reflections upon current ethical tensions and how to deal with them as they arise.

5.2.3.2 Gaining consent

In considering the ethics of this study I was informed by the British Educational Research Association (2018) and the School of Education, Bristol University Guidelines (Appendix 3) Ethical approval for my research was granted by Bristol University. Consent forms and full explanations of this study were given to all the participants (Appendix 4). As Flewitt (2005a) states, it is the responsibility of the researcher "not only to establish a robust and negotiated ethical framework […] but also to ensure that these ethical principles are applied throughout all stages of the research process" (p. 10). This necessitated asking my prospective participants for consent and making them aware of their right to withdraw. I was aware that the two participants, who were teachers in my school, may be basing their decisions regarding consent on the foundations of trust developed through their relationship with me as head teacher. I made it clear that there was no obligation to take part and they knew they could withdraw at any time. These two participant teachers were also aware of ethical issues in research as they were also studying for a Masters' degree, which included a day's session on ethics as part of their thesis work.

Although no direct observations of children were made by me, the participant teachers discussed children within their assignments and in the interviews and focus groups. All permissions, for children's participation through the assignments, were granted by the participants' university's strict code on data protection. The interview transcripts did not name children directly and in the focus group the children were under the permission of the schools' data protection policy and the parents' informed consent.

5.2.3.3 Confidentiality

The importance of confidentially to protect participants is vital and therefore pseudonyms were used for the participants and any children mentioned within the data. All personal data has been stored safely and securely using password-protected files and in compliance with the Data Protection Act (1998). Data has only been shared with my research supervisors, at educational conferences and for educational purposes as relayed in the participants' consent forms (Appendix 4).

Throughout my proposal, in my research design and in each data chapter I consider the ethics of this research, "The study itself is about democratic practices that gives children freedom to explore their mathematics" (Carruthers, 2015, p. 9). For example, a chapter in the literature review is about the importance of listening to children and children's rights to be heard in whatever way they want to communicate (UNICEF, 2009). It was important to ensure that I did not encroach on the participant teachers' time unless it was beneficial to them, as Freire (2000) states, "Academics as anthropological tourists enter into communion with the people for a brief time while, for example, collecting data for their research projects, but leave the struggling community at its own mercy soon after" (p. 30). Similarly, Brownhill (2014) discusses the term non-maleficence which recommends you reflect on the possible harm to your research participants, the school and the community conducting your research can cause. As the research developed I was aware of other factors that could cause the participants harm. This was in the tensions that might occur between them and their school colleagues as they started to share their changing practice within their school teams and their leaders.

The considerable length of time the two case study teachers (chapters 8 and 9) participated in this longitudinal study made me consider that their involvement in my research project would need to have a substantial impact on their careers. Taking part in this study could be useful for their own professional learning and understanding of children's mathematics which would hopefully, in turn, benefit the children in their classes. Brownhill (2014) refers to this as beneficence which assists you in reflecting on the "value of your research, not just for yourself but for those participating in your project" (p. 45). Two of the participant teachers expressed that they were grateful for the focus group discussions as they made them focus their thinking, especially when they had to present to an audience at a conference or lead professional development for local teachers. All the participants gained a double Masters' module and four are now Specialist Leaders of Early Mathematics for our Teaching School (which included the two case study teachers). Therefore, engaging in this research has contributed positively to their career development. I have also encouraged and supported their projects, presentations and alerted and aided them to access national and local mathematics conferences and mathematics communities. As the data collecting phases drew to a close I supported three of the teachers to write an article, in a well-known national early years' teacher journal, about their mathematical pedagogy in which they explicitly related the positive changes to their teaching. The Bristol School of Education ethics form, which I completed in conjunction with a conversation with a PhD student, heightened my awareness of the ethical issues that may occur (Appendix 3). Regard for ethics has permeated the research at every stage, and key ethical issues are outlined within the data chapters (6,7, 8 and 9), including measures taken.

5.2.4 The pilot study

The pilot to this study has guided some of the research design and methods. To trial ways of researching pedagogies that support children's mathematical graphics I collaborated with a teacher from a three-day mathematics course on children's mathematical graphics (Carruthers, 2012). The teacher gave a detailed account of her changing practice and how she had influenced her colleagues; she said the three-day course was instrumental in her changing practice because she had never considered children could use their own ways to represent their mathematical thinking. In her class children wrote down their thinking freely in literacy and she accepted their emergent writing but she thought mathematics was different, school mathematics had to be correct and in standard mathematical form. The changes in the pilot study teacher's mathematics practice prompted me to use the early years mathematics Masters' modules (Appendix 5) as a space for teachers, to think through ways of approaching their mathematics teaching and perhaps consider means in which they could support children's mathematical graphics.

One viewpoint of the pilot study teacher's comments that struck me as crucial, was when she said, "To change your way of thinking that much, you really need to understand why" (Carruthers, 2012, p. 206). Taking this comment on board, I considered, for teachers to change their practice to supporting children's mathematical graphics they also needed time to consider what they are committing themselves to. I thought that the Masters' modules would enable teachers to have considerably more time, than the previous three-day mathematics course the pilot study teacher attended. This may give them the opportunity to deeply reflect on their mathematics teaching and perhaps utilise aspects of teaching in a way that encouraged children to use their mathematical graphics.

5.2.5 The participants

The participants in the study are fifteen teachers in the English School system: seven Reception (teaching 4-5 year-olds) and eight Nursery School teachers (teaching 3-4 year-olds). The research involved fourteen schools. The teachers completed a double Masters' early mathematics module, which included sessions on children's mathematical graphics. This was accredited by a local university and I was the lead tutor and planned the modules. The teachers were concurrently teaching in inner-city schools, in the same city, in the South-West of England with the exception of one who taught in a town nearby. My position about teachers concurs with Sharkey, Olarte and Ramfrez (2006) who state, "Teachers are legitimate contributors to the knowledge base, are experts of their particular contexts, capable of theorizing classroom practices, and contributing to reform/policy debates" (p.307). Increasingly, over the last twenty-five years, the English educational climate is one where teachers are monitored by senior leaders within their schools and Ofsted (Office for Standards in Education). Teachers' practice is under so much scrutiny that I did not want to add to this pressure, therefore, "I took an ethical stance not to make observations in their [the participant teachers] classrooms; their own autonomy was an important part of the process of possibly revealing rich data that was relatively authentic and not pressurised" (Carruthers, 2015, p. 10). The power balance may have been significantly unequal if I had taken observational notes of the teacher's classroom practice. The interpretations of their teaching would be from my view point and not theirs. "Their interpretations may reveal more insights into their thinking" (Carruthers, 2015, p. 10) and, as they did not teach with me observing their teaching they may feel less inhibited to discuss their practice. The disadvantages are, there are the crucial issues of subjectivity and also the accuracy of their stories. What the teachers believed they saw in their teaching can never be a complete story, stories are always fragmented, and may be especially so, given they are busy classroom teachers and could easily have missed some vital issue or insight into their teaching. How they viewed the world has a significant influence on their perspective (Brookfield, 1995).

The teachers' social and cultural contexts are important and their stories and discussions may reveal findings that could impact their practice. Individually, teachers have their own history of knowing and being, of situation and educational influences (Fives & Gill, 2015). As I am using qualitative data collecting methods (Manzo & Brightbill, 2007), for example, open questioning within the methods employed, the influence of the teacher's thoughts and ideas direct the research. This might uncover complex and perhaps chaotic research data (Yin, 2018) but the benefits might be that the open space and time they use, within an open framework, could give opportunities for deep consideration of their pedagogy.

5.2.5.1 The Masters' modules

In the Masters' modules key concepts were presented to the participants including, current theories in early childhood mathematics, early number (Gelman & Gallistell, 1978; Maclellan, 2008), children's mathematical graphics (Hughes, 1986; Carruthers & Worthington, 2005), and children's imaginary play (Worthington & van Oers, 2016) (Appendix, 5). I asked them to keep a portfolio to store any children's mathematical graphics' examples and anything connected to their teaching or information they disseminated to other teachers and early years' practitioners. Two tutors from the university presented ideas on critical reflection (Schon, 1987) and the requirements of academic writing. They also marked the assignments (Appendix, 10) as it was important that I remained neutral and did not influence the marking, in any way, which would have added to further possibilities of bias and subjectivity within my study as I analysed the teachers' assignments. The university permitted me to use the assignments if the students agreed and gave their permission, which they did, as has been previously mentioned. This was the first time that a Masters' course had been conducted within a Nursery School setting in the city, it was supported and funded by the local authority. My Nursery School was the lead Early Years Teaching School[14] and the plan was to build an understanding of early mathematics within early years' settings, including schools, in the city and these early years' mathematics modules would be the basis to develop this vision. The hope was that some of the teachers would apply for the Early Years' Mathematics Specialist Leaders of Education[15] within our Teaching School.

5.2.6 Data collection and analysis

I used five types of data collecting sources in this study: questionnaire, assignments, interviews, focus group discussions and field notes. These multiple sources were used to collect the data best suited to a particular stage of the research. To strengthen case study research Yin (2018) puts forward the need to use multiple sources of evidence as this provides an "in-depth study of phenomenon in its real world context" (p. 127). Yin extends this by relaying that "any case

[14] Teaching Schools were established by the English Government's Department of Education in 2011. Their remit is to support schools in their local area for example in research, professional development and leadership
[15] Specialist Leaders of Education were established as part of the work of the Teaching School. The SLE's support other teachers and schools providing professional development and individual school support.

study, finding or conclusion, is likely to be more convincing and accurate if it is based on several different sources of information, following a similar convergence" (2018, p. 128). As I outline each data collection phase of the study I will explain and critically analyse the data collection method used within that phase.

I present the data in two parts and in four chapters, 6, 7, 8 and 9. The first part is in chapters 6 and 7 where I analyse the data of the two case study groups, the Reception and the Nursery School teachers. In chapters 7 and 8 I present the second part of the study, the analysis of the two single case study Reception teachers. I am using grounded theory as a basis for analysis, "that through a series of carefully planned steps, develops theoretical ideas." (Crotty, 1998, p. 78). It strives to ensure that the theory surfaces only from the emerging data and no other source. Corbin and Strauss (2008) express the data collection and data analysis are not separate, detached parts of the research; I collected and analysed the data simultaneously. The four stages of data collection, as described,

> were analysed at the different stages based on the emerging directions in that analysis. I let the data guide the analysis, therefore, I also collected the data as I uncovered new connections or needed to probe further, as in the focus groups. (Carruthers, 2015, p. 13)

Theoretical sampling (Corbin & Strauss, 2008) was used, "where new categories are produced and relationships between categories are identified" (Carruthers, 2015, p. 13). I used Corbin and Strauss's (2008) coding system as I analysed the data. Firstly, I used *open* coding, scanning the data and searching for "all potentials and possibilities contained within them" (p. 160) (Appendix 6). Secondly, I used *conceptual coding*, categorising prominent themes (Corbin & Strauss, 2008) (Appendix 7). In this stage of the analysis you intensify your examination and scrutinise the data more closely, looking for the main emerging patterns; it requires an ability to disregard pre-conceptions (Wicker, 1985). The analysis of the data of the teacher's written reflections also benefited from the scrutiny of an expert early childhood mathematics researcher to generate coding reliability using inter-rater reliability (Creswell, 2013; Shoukri, 2010). I gave this researcher the coding explaining how I categorised the data and then she used the coding to code a copy of the raw data. We achieved 93% consensus. After all the stages of the data collection there was a final analysis to ensure no other prominent themes had been missed or data had been mis-interpreted.

The data was collected from:

1.Initial Questionnaire - September, 2013 (15 questionnaires collected)

2. Assignments - September, 2014 (15 assignments)

3. Interviews - November, 2014 (3 interviews)

4. Focus groups - March, 2016 to February, 2017 (4 participants, including me)

5. Field notes - October 2013 to February, 2017

Below is a chart to show the time line of the data collection all names are pseudonyms (figure 5.3)

Figure 5.3: Research data time line, September 2013 – February 2017.

Date	Data	Further Information
First Phase -September 2013	Questionnaire	3 questions about the participants existing knowledge of mathematical graphics. All participants, 8 Nursery School teachers and 7 Reception teachers.
October 2013 – June 2014	Mathematics Modules begin Field Notes	
Second Phase -September 2014	First assignment	All Participants written reflections on their mathematics teaching and new aspects of their teaching practice.
Third phase-November 2014	Interviews	3 Reception teachers, Zoe, Janine and Millie
June, 2014- June 2015	Field notes Modules ended	
July 2015	SLE Strategic meeting, Field Notes	Millie and Janine
November, 2015 November 2015- March, 2016 Break …	Mathematics conference, Field Notes Field notes	Millie Millie and Janine
Fourth Phase - March, 2016	Focus Group	Millie and Janine (Esme)
Fourth Phase - May, 2016	Focus Group	Millie and Janine (Esme)
Fourth Phase - October, 2016	Focus Group	Millie and Janine (Esme)
Fourth - Phase, February, 2017	Focus Group	Millie and Janine (Esme)

5.2.7 The phases of the data collection (see also figure 5.3)

In section 5.2.7 I offer more detail about each of the four phases of the data collection

5.2.7.1 Phase 1. questionnaires (2013)

I gave the participants an initial brief paper questionnaire that aimed to establish the teachers' understanding of children's mathematical graphics. This was given to all 15 teachers. The questions were:

1. Have you heard of the term children's mathematical graphics?
2. If you have, can you explain what it means to you?
3. Do children in your class solve problems, in their own ways, using their own representations?

My aim in this first phase was to collect some initial information from the participants. I wanted to know something about the participants' understanding of children's mathematical graphics. I did not know them well (except the two teachers who taught at my school) so I wanted to ask them questions that were not encroaching too much on their personal space and time as, for example, interviews could do. I chose a questionnaire as an impersonal method of collecting data (Patten, 2017) and a step towards my research problem and data collection. One of the drawbacks of questionnaires is the participants might not understand the questions and therefore they might not answer them in a way the researcher had intended. To counter this issue of participants not understanding or being unclear of the questions, I explained the questionnaire to the participants before the first session of the Masters began. I had a face-to-face discussion with all the participants individually about the general aspects of the modules and I also, at the same time, gave the participants a chance to ask anything they were not clear on which also included any questions about the questionnaire. Patten (2017) describes a drawback of questionnaires is that they only give a snapshot of the phenomena under research. However, this suited my aim for this beginning phase of my research, I was not seeking detailed responses. It was also a good time to meet the teachers and start to get to know them. I felt, at the time, this may help with making the modules more accessible as they may feel more comfortable having met me and some of the other participants.

5.2.7.2 Phase 2. Written reflections (2014-15)

In the second phase of data collecting I used the first main module assignments which were the written reflections of the fifteen teachers. The assignment asked the teachers to describe their present practice in a reflective commentary, some also described their already changing practice influenced by the course and suggested readings. There are many issues with using assignments as they wield a huge amount of data which makes it difficult to process. However, this is a way of collecting a lot of relevant data that is not time consuming (beyond the work needed to write the assignment of course). The assignments may have influenced the teacher's freedom to put down their own views as they may have been swayed in writing what they thought the tutors wanted although the marking criteria (Appendix 10) was clear that the assignments were about their reflections on their practice.

There are ethical issues that arise when using assignments which I discussed with another PhD student in an ethical discussion (Appendix 3). My particular concern was even although I had asked their permission, I felt using personal assignments was invading the teachers' privacy. To counter this, I decided, if I used personal extracts from any of the assignments I would ask their permission again. Transcripts would also be checked for accuracy, at the time, and changes made if needed. Constant checking on the participants meanings, revisiting conceptual understanding, was another way to make sure I was respectful to the participant's views and avoid over-interpretations.

Following Corbin and Strauss's (2008) open coding analysis, as described previously (5.2.6), of these Reception and Nursery School assignments which revealed themes and patterns that informed the next step of the analysis. Piantanida and Garman (1999) note that "as researchers become more acquainted with [...] the stable records, some information begins to emerge as more interesting or significant" (p. 170). For example, through this familiarity and systematic classification certain aspects of classroom practice could be highlighted.

At the next phase of the data collection because of the issues surfacing, which I describe in the data analysis (chapters 6, 7, 8 and 9) I decided to interview three of the Reception teachers in the hope I might uncover, in more depth, what the issues might be and ways forward. It is important to note, at this stage, that I was departing from collecting further data from the Nursery School group to focus on the case studies of the three Reception teachers. As already stated in the introduction of the thesis the mathematics pedagogy of the Reception teachers was a vital part of the study.

5.2.7.3 Phase 3. Interviews (2015)

From the original seven Reception teachers three volunteered to be interviewed and take part in the next phases of the study. The need to interview came out of the findings of the general analysis of phase two data, as described above (one teacher did not continue into phase four for personal reasons). I was also their mentor for the module so it was convenient as I was meeting them on a regular basis. Seidman states that the basic objective of qualitative interviewing is to pursue "an interest in understanding the lived experience of other people and the meaning they make of that experience" (2006, p. 9). Therefore, I was trying to understand the experiences of these teachers by listening to their views and stories of their practice in teaching mathematics.

Interviews can be classified as focused, unstructured, semi-structured or ethnographic interviews (Yin, 2016). I used semi-structured interviews. This method of interviewing, "does not use fixed questions, but aims to engage the interviewee in conversation to elicit their understandings and interpretations" (Liamputtong & Ezzy, 2005, p. 332). These interviews are characterised by active involvement in engaging the participant to converse about a particular topic or discussion relevant to the research questions or topic being explored. I wanted to delve a little further into the Reception teachers' understanding of the material presented in the modules. As my research unfolded and I had semi-analysed the second stage of data collecting, I identified some of the problems that were arising. I was, therefore, able to be more focused on specific aspects of the tensions the Reception classes faced and interviewing the three participants was a useful research tool to use. By this time, the participants seemed more comfortable with me as a researcher. Fielding and Thomas (2004) refer to the importance of the manner of the interviewer being 'relaxed and unselfconscious' (p. 127). This is important in order to retain honesty and frankness and for those interviewees not to feel they need to give you the answer that they think you want. I used a Dictaphone, this was a way to ensure that I had every word and this could add to a more accurate interpretation. Qualitative interviewing is not easy (Rubins & Rubins, 2012) as the interview can be more of a conversation mode and sometimes researcher and interviewee can be using the same vernacular but their meanings can be different. At times, I probed for the teachers to explain further, something I thought interesting, or to clarify a point they made to make sure that I was in line with their understandings. Another difficult part is really listening and reflecting, at the same time as developing the participants thinking in a pertinent area, by making utterances or asking useful questions (Rubins & Rubins, 2012). The role of the interviewer takes skill, I relied on my experience of being a part of many informal groups and situations to support and enhance educational thinking including educational research projects (Carruthers, 2008; Carruthers, 2012). I prepared three open questions which the teachers took in different directions. The interview session was informal, I did not need to ask all the interviewees all the questions as the starter question sometimes took over and then led to open discussions between myself, as the interviewer, and the participant.

Interviews, as a source of evidence, are verbal reports only (Yin, 2016) and as such are subject to bias and inaccurate information. However, this is not the only source of information within this study and the assignments, for example, corroborated the teacher's stories like many of the examples that they drew upon, from their classroom practice, they also wrote about in their assignments.

Ethically, I was aware of making the time short as I did not want to encroach on the teachers valued personal time and the interviews, therefore, took only about ten minutes. I was also mindful of how I conducted the interview in terms of being respectful and not probing so much that it became aggressive (Atkins & Wallace, 2012).

As the data, at this point, from the three Reception teachers produced only one example of children's mathematical graphics I had to reconsider my research plan and I will discuss these issues within the data chapters, 8 and 9. From here, there was an interval where I only collected a few field notes as I had to reassess, as a researcher, my position and where the research was going next. This also gave time to the participants to have a break from the research and perhaps reflect on the modules and the interviews. Etherington (2006) describes this as where research is "put on the backburner for a while, creating a space for new understanding to unfold" (p. 211). I describe this within my data analysis as a period of fermentation.

5.2.7.4 Phase 4. Focus groups (2015-17).

In phase 4, I was building on the analysis of the previous data, where it was identified that the case study Reception teachers were slowly developing their understanding of practice that might support children in using their mathematical graphics. The participants also gave useful insights into their classroom practice in supporting children's mathematical graphics although their classrooms were not yet producing children's graphics. Therefore, I decided to use focus groups (Cronin, 2004) for the next phase of my research, in the hope that this method would be fruitful for yielding illuminating data. focus groups can be collaborative, bringing in differing viewpoints more than, for example, interviews. I used notes and a Dictaphone to record the discussions. The Dictaphone helped in writing reliable transcripts of the discussions. The notes were a back-up in case the technology did not work.

These focus groups were what Cronin (2004) defines as a group discussion about reflecting on people's concepts and beliefs and clarifies their meanings. He emphasises that a focus group is dissimilar to a single interview or a focus group interview. An advantage of a focus group is social communication and collaboration which can activate new ideas and insights which is more difficult in an individual one-to-one interview. The drawback is that you gain less from one individual and this may result in less depth (Merton 1999). However, there were only four teachers in this focus group including myself as facilitator and, at times, I did contribute to the discussion. If facilitated well, each teacher would be given time to develop their point or elaborate on a classroom happening. Cronin (2004) describes the role of the facilitator as giving the participants the opportunity to lead but also questioning when needed. As Stake (2005) states if the focus group is too constrained then the data may be biased by a distinct perspective as new avenues of thought could be restricted. The focus groups started a year after the interviews had been conducted, and by that time the three teachers were used to working together and appeared comfortable and relaxed with each other and with me. This may have made the teachers less reticent to talk freely. The focus groups were, for me, a more comfortable way of collecting data. I was used to facilitating discussions through previous research projects (Carruthers, 2012) and within my role as headteacher, as I had based developing a research school around a discussion-based model of learning (Carruthers, 2008). However, my way of conducting the focus groups was different than my previous research where I had used one main single case study and conversational analysis (Woolfit, 2005) and the aim of the study was largely to benefit one institution.

We met for four focus group sessions that lasted forty minutes to an hour each time. At this time, the two case study Reception teachers and one Nursery School teacher, from the study, were all working on developing early mathematics across the local authority. I was their mentor and supported them in designing a strategy for early mathematics, across the city. This included professional development by supporting an early years' mathematics lead in every early years' setting. I documented parts of one meeting and a conference where the participants led a workshop. I used field notes. I explain more fully, in the next section (5.2.8.1) how this was recorded and analysed. The focus group as a data collecting method seemed to be timely as the participants had accumulated experience, knowledge and thinking time since the interview phase of the study.

5.2.8 Reflexivity

Throughout the three years and five months of collecting data for the study, I used a journal which included an array of field notes and provided a crucial means of written reflections on my assumptions and biases. Reflexivity was a key aspect of the research, for recognizing my own perspectives and subjectivities. Etherington (2004) declares a research journal "can provide a means of which we can make the most of our complexities" (p. 128) in the research arena in a methodological and consistent way.

In qualitative research "the researcher unavoidably serves as a research instrument" (Yin, 2018, p. 40). Yin (2018) considers reflexivity as a technique but Etherington (2006) puts forward that it is "an essential human capacity" (p. 33) and in discussion with her colleagues questioned "how it was impossible NOT to be reflexive" (p. 33 emphasis in the original). Reflexivity refers to the researcher's self-reflection as he/she reports social phenomenon. Within research accounts, reflexivity is a portrayal of the researchers' ideas and experiences, which can be used by readers to judge the possible impact of these influences on a study. The research journal/field notes became a way to capture the subjectivity along with a critical explanation of the process of analysis which helps illuminate the approach taken.

My bias inevitably influenced how I selected materials, focusing on some issues but not others. Acknowledging my own research lens and accounting for the potential influences was crucial, throughout the study, to offset subjective analysis and findings. From my theoretical perspective the collection and selection of evidence can only be partial and incomplete (Drummond, 2012). "The false ideal of a detached impersonal point of view must be replaced by the ideal of an essential human point of view which is within the limits of a human perspective, constantly trying to enlarge itself" (Manheim, 1936, pp. 266-7). In keeping with my socio-cultural perspective and interpretivist stance I recognise and use the advantages of "human intuition, insight, sensitivity to the nuances, complexity, and subtleties of social behaviour and communications" (Simpson & Tuson, 2003, p. 82). Therefore, a reflexive stance is vital, and analytical claims must be rigorously grounded in the data.

5.2.9 Validity and reliability

Validity can be a problem in qualitative research. Edwards (2010) puts forward that issues surface in the contending positions of how each research paradigm is precisely understood. Carruthers (2015) states, "in more qualitative approaches to research the meaning of validity is closely associated with philosophy and therefore the truth value in a statement" (p. 14). Edwards concludes, "Validity in qualitative research is a matter of being able to offer as sound a representation of the field of study as the research method allows" (2010 p. 124).

The terms validity and reliability are repositioned by Taylor (2007) who uses the term trustworthiness,

> She says, within her research, as she was challenging knowledge and truth itself then she would not be able to verify her findings. Validity for her became trustworthiness and working with the participants on their meanings and their interpretations and respecting their changing positions. (Carruthers, 2015, p. 14)

Similarly, Lather (1991), in her research, acknowledged different perspectives and opposing trends were underlined and not concealed. Therefore, I am considering validity and reliability from the position of trustworthiness. An issue with the data in my study is, are the teachers accounts of their classroom teaching adequately reliable as I did not observe their teaching. I feel that an important part of the ethical considerations is to trust these teachers and give them the opportunity to tell their stories and reflections without doubt. Part of my study was focusing on what the teachers were relaying in our conversations in the focus groups then for that purpose these teachers' stories are reliable enough. Through the focus group discussions, I asked them to clarify their meanings and gave them space to talk and within the analysis of the data I made sure I was not discounting aspects that did not fit into the major emerging theories. Certainly, a limitation was that it may have been a more detailed account if I had used a video to capture the teachers' teaching, but I feel the relationship between myself and the participants would have been strained. The importance of building a respectful partnership, with the participants, within my research, was paramount to keep the participants interested and, to an extent, equal partners in the research (Manzo & Brightbill, 2007). Later in the study, one participant did bring a video voluntarily to explain how some children in her school were using their graphics to support their mathematics, this, I felt, was also a mark of confidence and a developing trust in the researcher participant relationship.

In addressing the issue of reliability, quantitative researchers employ techniques to show that, if the work were repeated, in the same context, with the same methods and with the same participants, similar results would be obtained. This replicability is questioned in qualitative research because of the nature of the phenomena under study (Pandey & Patnaik, 2014). It is particularly difficult to repeat a qualitative case study exactly, as responses from individuals are different over time and conditions will inevitably change, especially in a longitudinal study (Yin, 2018). Ensuring trustworthiness in qualitative research Lincoln and Guba (1985) put forward the term dependability and suggest that an important part of ensuring reliability is reflexivity which I have already discussed (5.2.8.1) within this chapter.

To overcome some of the issues of dependability I have made the procedures I used as explicit as possible, detailing the phases of the study and explaining and critically analysing the methods used.

The trustworthiness of the research can be strengthened through triangulation (Denzin & Lincoln, 2000). This is an attempt to get a robust understanding of the field by looking at it from varying angles. The multiple modes of data collection within my study can be classed as data triangulation (Denzin, 1978; Yin, 2018). Triangulation can also reduce some aspects of bias, by supporting analytical insights and checking interpretations. There was also the use of more than one data source in the study. I have collected data from both Nursery School and Reception teachers and this gives different interpretations of the questions. Green (2005) affirms using multi-data collection methods "can create spaces for a full engagement with the challenges of understanding teaching and learning as complex processes" (p. 25). Using multiple sources illuminates the different challenges that teachers face. Denzin (1978) states that the conclusion can be more dependable if two or more types of data collected converge.

In conclusion, in this chapter, I have outlined the methodology of the research including all the research procedures. There will inevitably be challenges and limitations of this qualitative study and this I will discuss in the final chapter of the study.

In the next four chapters, 6,7,8 and 9 I present and analyse the data. Firstly, in the following chapter (6), I report on the data of the group case study of the Reception teachers.

Chapter 6: The Reception Teachers

The data is presented and analysed in four chapters, chapters 6,7,8, and 9 and is separated into four case studies. This first data chapter is a group case study of seven Reception teachers, the second data chapter is a group case study of eight Nursery School teachers and the third data chapter presents two single case studies of Reception teachers. The single case studies of the two Reception teachers are initially drawn from the group case study of the Reception teachers. The data was collected over three years and five months; the data from the two group studies were collected in the first two years of the study; the data from the case studies of the two Reception teachers were collected in the second two years of the data collection period.

Before I discuss the analysis of the data of the Reception teachers, I report briefly on the outcomes of a questionnaire given to all the teachers at the start of the project as part of the introduction to the case studies. I used a questionnaire to gather initial information to inform the research. I found that only four teachers had heard of the term Children's Mathematical Graphics and out of the four, only two seemed to have incorporated this into their teaching (the two were the teachers at my school). Although, later, in reading one of the reflections, it seemed that another of the four teachers previously mentioned was aware of children's own mathematical graphics before the modules and she explained her previous practice in her assignments. Most of the Reception teachers encouraged children to write down something after they had solved the mathematical problem, and this is what the teachers termed *recording*. It is not children using their graphics to help them solve the mathematical problem. It is crucial to the concept of children's mathematical graphics that the children do not already know the answer, because their graphics help them solve the problem (Carruthers & Worthington, 2006, 2011). The understanding that most of the teachers only used recording in mathematics was able to inform my planning of the modules and I emphasised the difference between recording, and children's own thinking, using their graphics. Later, the difference between children recording mathematics and children's mathematical graphics became puzzling, particularly to some of the Reception teachers as they continued to confuse children's recording for children's mathematical graphics. To clarify the distinction between the two, I introduced the term *copy recording* within the study to stress and make clear the difference between the meaning of the two. *Children's mathematical graphics* are when children use their own signs and symbols to work out mathematical problems they do not yet know the answer to or sometimes they generate these inscriptions in their play situations. *Copy recording* is what children are asked to do by the teacher after they have worked out the answer to a mathematical problem.

6.1 The Written Reflections of the Reception Teachers

This is the presentation of the data of the Reception teachers' reflective written accounts of their teaching and my personal field notes from the module class discussions (Appendix 8). Extracts from interviews of three of the seven Reception teachers are also included. The participants: Melinda, Kathy, Beatrice, Hilary, Milly, Janine, and Zoe (pseudonyms) were all reception class teachers, in Primary Schools maintained by a Local Authority. All of the teachers taught in the same city, in the South West of England except Hilary who taught in a nearby town. Three of the teachers also had senior leadership roles. Milly and Janine were also the case study teachers in the third and fourth data chapters, 8 and 9.

I am presenting the data, in this chapter, in four sections including a discussion and a conclusion. The data is organised by the emerging themes (Corbin & Strauss, 2008) I developed from the data as described in the methodology chapter (5.2.6). The first sections (6.1.1 and 6.1.2) outline the pedagogical issues the Reception class teachers' discussed, including the restrictions and barriers to more democratic practices and the tensions and dilemmas the teachers expressed about adult-directed and child-initiated learning. The teachers' assignment writing aim was to reflect on their present practice, however, as these assignments were written well into the modules the writing also showed how the modules were already impacting the Reception teachers' practice, therefore, the third sections (6.1.3, 6.1.4, 6.1.5 and 6.1.6) are about aspects of the teachers' changing practice. The final sections (6.1.7 and 6.1.8) focus on children's mathematical graphics, with an emphasis on the difference between children's 'copy' recording and children's own mathematical representations.

W.R.= written reflections of the Reception teachers from their first assignment: September, 2014

F.N. = my personal notes and observations written at the time of the modules: September, 2013 to March 2015.

Interview = The transcripts of the interviews of three Reception class teachers on the 22nd November, 2014

I have written the quotations from the teachers in italics to highlight their reflections. There are eight themes and I label each of them using the words of the teachers (Appendix 8).

6.1.1 "*The demands placed on Reception class teachers*" (Millie, F.N., 17/2/2014)

Six of the seven Reception teachers described their mathematics teaching in terms of following a set curriculum and objectives. From the conversations noted in the module sessions Kathy expressed "*I do not know if this is the way to do it; is this the way we are supposed to do it?*" (F.N., 17/2/214) and Melinda, *"We follow pressures from government imposed curriculum expectations"* (F.N.17/2/2014). For example, Millie talked about following the EYFS learning goals. "*I work first on numbers 0-5 and then 5-10, although now we can go up to twenty"* (Millie, W.R., 15/9/2014). This seems to be a carefully graded step-by-step approach to mathematics, going no further than what is stipulated by the EYFS learning goals. Melinda also wrote that she planned from the goals and she stated, "*The assessment points limited my expectations* […] *I would only expect my children to know numbers to ten and perhaps some of my higher ability to know numbers to twenty"* (Melinda, W.R., 15/9/2014).

Even although two of the Reception teachers stressed how open and free their play environments were, they admitted that this was not the case for their teaching of mathematics. For example, Hilary wrote about the focus on free play and the environment her Foundation Stage Unit had created, then she began to reconsider, "*When reflecting on practice within the unit* (Foundation Stage Unit which in this case is two Reception classes) *the emphasis has been predominantly focused on adult-led or adult-initiated activity"* (Hilary, W.R., 15/9/2014). This has impacted on the staff not knowing the children's home and community culture as she goes on to say, "*Significantly pieces of information about children's deeper level cognition and cultural influences have been unintentionally overlooked"*.(Hilary, W.R., 15/9/2014).

Melinda revealed that her teaching was restricting children to one way of working out addition and subtraction problems, "*As a practitioner, I wanted to see all the children in my class use the same strategy to get the 'right' answer. For example, I modelled one way to do a sum then all the children copied that way"* (Melinda, W.R., 15/9/2014).

Planning for children's learning with clear learning objectives taken from the EYFS was a main focus of all the Reception teachers. However, the EYFS (DfE, 2012) states teachers also need to plan from children's interests and this is not evident from the analysis of the data. The data revealed that there seemed to be an emphasis on planning from the stated objectives and adult-specific agendas in mathematics. Each of the Reception teachers wrote about doing focused planning linked to pre-defined objectives; planning issues also dominated many of the discussions at the beginning of the modules. Influenced by the module discussions and literature considering child-centred practices, Janine reflected on the necessity to also plan from children's ideas and interests, "*My planning also needs to be focussed on greater observations of children to help give more starting points and to provide more open-ended activities otherwise they will not be meaningful or relevant to the child"* (Janine W.R., 15/9/2014).

All the Reception teachers described, in some way, the pressures of the curriculum and the everyday demands of teaching which made it difficult for them to find a space, in their planning schedule, to reflect on their current practice and trial more open mathematics. Beatrice highlighted her difficulties as she wrote, "*The busy day to day life of being a teacher with wider school commitments can make it difficult to find the time to pause and reflect on classroom practice as the demands on a teacher's time may seem overwhelming".* (Beatrice, W.R., 15/9/2014) Beatrice considered what she has read from the recommended reading in the modules, and she wrote,

> *Adams (2007) describes various ways in which beginning teachers may develop a reflective approach to their teaching and identifies that such reflective practice requires considerable time. This is something which I identify with: as a member of the senior leadership team I feel the demands on my time pull me away from the classroom and I am left little time for deep critical reflection.* (W.R., 15/9/2014)

Beatrice reflected further and said she has become what Rose and Rogers (2012) describe as a *technical reflector* and she explains this is someone who is focused,

> *on the achievement of goals and is concerned with practical issues and as such the specified goal becomes the focus* […] *I feel that under pressurised time constraints this is the type of reflective practitioner I have become over recent years: focusing*

on whether an activity or provision has allowed children to reach the required standard. (W.R., 15/9/2014)

This also has resonance with Melinda's writing when she states, "*I only plan from the assessment goals"* (Melinda, W.R., 15/9/2014). The strains of the expected curriculum goals and top-down government initiatives and expectations has prioritised set programmes. In one of the taught module sessions Millie, in discussion with the other teachers about these external pressures, lamented to the teachers her feelings about the general situation, "*The demands placed on Reception class teachers perhaps cannot yield a more open mathematics curriculum"* (F.N., 17/2/2014). She further confirms these difficulties in her writing, "*I currently feel that I am not prioritising and valuing learning through play due to the constant struggle to ensure I am reading with all children, completing focus activities and observations"* (Millie W.R., 15/9/2014).

In summary, the data has revealed that there were tensions because of what is expected from the schools and government set goals for mathematics and this seemed to impinge on the Reception class teachers time to be more open and child-orientated in their teaching of mathematics. In the next section the Reception teachers also highlighted their dilemmas between adult led-learning and child-initiated learning.

6.1.2 "*A constant battle between child-initiated and adult-led learning*" (Melinda, W.R., 15/9/2014)

There were tensions and dilemmas facing the teachers in terms of who should be leading the learning, the child or the adult, and when, how long and the implications of these decisions. Terms like adult-led (where the adult has a teaching objective in mind and leads the activity) and child-initiated (where the child has their own agenda and leads their own learning) are used in the EYFS framework (DfE, 2012) which is statutory guidance for all Foundation Stage settings in England. Melinda wrote about her dilemma in understanding how to plan for this in the context of classroom practice. "*I have a constant battle between child-initiated and adult-led learning"* (W.R., 15/9/2014). Influenced by the literature, she states, "*It became apparent to us* [Reception colleagues in her school] *that we were pulling children away from valuable play situations* [child-initiated learning] *to complete the work we had set them* (Melinda, W.R.

15/9/2014). Melinda goes on to quote Rogers, this, she said, resulted in, *"fragmented narratives and closed opportunities"* (Rogers, 2010, p. 16 cited in Melinda, W.R., 15/9/2014).

Janine wrote at length about the meaning and the difference between adult-led and child-initiated learning. She believed that she does allow for free-flow play as described by Bruce (1992) which is child-initiated. She said, "*By giving children opportunities for child-initiated play I feel it is more likely that children's thinking will be given wider scope and children are more likely to use (mathematical) marks to express their ideas"* (Janine, W.R., 15/9/2014). However, she also admitted she has not seen children freely writing or drawing within the play. She considered this and resolved, "*I need to be more open to listening and learning from the children"* (Janine, W.R., 15/9/2014). Janine continued throughout her writing to tussle in her thinking between adult-initiated and child-initiated learning and play, she reflected,

> *I do feel that practitioners need to have aims in mind when planning activities to enable children to move on in their learning and thinking and for them to be challenged and that Fisher's (1996) contention that if there is a predetermined task or goal in mind no play can be said to take place is too narrow a definition. I would suggest as long as a task remains open and can be developed by the children it is worthwhile.* (Janine, W.R., 15/9/2014)

Kathy also challenged what she was reading in the literature,

> *I initially found her criticisms of* [Reception] *teaching and teaching practice unjust and insulting. Rogers (2010) comments that the Reception class teachers that she studied did not seem to value child-initiated play enough, and unnecessarily interrupted children's play particularly when they were needed to participate in group work or 1:1 reading* […] *however I think this is a necessary task for children's literacy development.* (W.R., 15/9/2014)

Kathy continued to debate in her writings about the value placed on play and then concluded by writing, "*the chapter (Rogers, 2010) did make me think more about when to hear children read, asking, instead of requesting and choosing children who appeared less engaged in the play"* (Kathy, W.R., 15/9/2014).

Janine acknowledged that within the statutory framework for the Early Years Foundation Stage it stated, and she quoted, "*Play is essential for children's development* […] *children learn by leading their own play*" (DfE 2012), and she wrote "*The role of the adult in the setting, I feel, is another hugely significant factor in creating the ethos to allow children to explore through their play*" (W.R., 15/9/2014). Similar to Melinda, she saw the role of the adult within the early years' classroom as complex and full of dilemmas as she stressed, "*The role of the adult is fraught with difficulties*" (Janine, W.R., 15/9/2014).

Millie commented on the lack of child-initiated mathematical opportunities in her class,

> *Although it is difficult to accept, there was very little child-initiated mathematics occurring within my classroom and the teaching of mathematics within my class, prior to starting this course, has been mostly adult-led, consisting of daily carpet sessions (whole class) and adult-led groups.* (W.R., 15/9/2014)

This was also the case for Kathy who said that she did not notice the mathematics in children's play and she thought that, "*quality mathematical observations needed to be teacher led or directed at the very least"* (W.R., 15/9/2014). However, she said, "*Since embarking on the modules and reflecting and listening to other teachers I have increasingly begun to notice how children's play is rich in mathematics"* (Kathy, W.R., 15/9/2014).

In summary, the Reception teachers considered their complex and difficult role within a play orientated classroom. Their reading of the literature on play and child-initiated practices seems to have put them in a state of disequilibrium. However, all of the Reception teachers seemed to be moving to change their practice and this change seemed not about merely adding resources or introducing a new game but rethinking their practice conceptually. The next theme deals with the Reception teachers' changing practice.

6.1.3 "*True mathematics comes out of freedom*" (Hilary, W.R., 15/9/2014)

It appeared that the Reception teachers did not, at first, see the children's mathematics because the pedagogy they described above, for the most part, was restricted to set objectives, especially for mathematics. Working with set objectives seemed not to allow the children to develop their own ways of mathematical thinking and, in turn, *their* mathematical graphics to thrive. The Reception teachers' writing indicated that they realised, in order to see children's mathematical graphics they had to move to more open ways of teaching mathematics. Each teacher has tackled the move to a more democratic approach in different ways, depending on their focus, what they did before and what seemed to interest them.

As the seven Reception class teachers changed their practice, Hilary focused on changing the environment and promoting everyday mathematical opportunities and observations of children with her colleagues. Although Beatrice and Janine had no case studies or specific examples of their changing practice they discussed how they were planning to alter their teaching. Kathy and Melinda wrote about small changes they had made and provided specific examples and Zoe and Millie gave substantial examples of their changing practice.

Hilary reflected on previous influences on her mathematical teaching, something she had pondered on for a while. The example she gave shows that teacher change is not always immediate and influences to practice sometimes can be sparked off by a statement or a conversation that stays with a teacher and puzzles them, sometimes for years; until there is a connection and the statement or conversation eventually makes sense. In Hilary's case changes in her teaching occurred partly because the module discussions reminded her that she had been inspired by a talk she had heard some years ago. She said the lecturer had made a strong statement that *"true mathematics comes out of freedom"*.

> *I have always been intrigued by this statement and throughout most of my teaching career I would say that I still had some confusion as what she had implied about early years mathematics and how this should impact on my work alongside young children.* (Hilary, W.R., 15/9/2014)

Hilary wrote that, "*Up until very recently, I always thought that my teaching and classroom provision enabled mathematical access, investigation and enquiry for all"* (W.R., 15/9/2014). However, through attending the modules she felt that, perhaps, her teaching of mathematics was not as open and free as she thought. She explained, "*During the past six months, since embarking on these early years' Mathematics Masters' Modules I have begun to explore what true early mathematics looks like".* Hilary wrote about how her reflections with the colleagues she worked with had continued to change their mathematical teaching practice "*As our current practice develops, the resources that we are providing the children are more carefully thought through"*. She gave an example, "*As well as more common mathematical resources we now have included real life resources, such as old clocks, sand, water, magazines and fabrics"*. (W.R., 15/9/2014) The discussions she had with practitioners and teachers in the unit about this change in the environment had "*raised awareness and understanding of what the children are showing us in their child-initiated play and we discuss collaboratively how we can support and scaffold learning experiences appropriately"* (Hilary, W.R., 15/9/2014). Hilary and her colleagues had shifted their practice to looking at children's mathematics and what mathematics is surfacing in their play.

6.1.4 "*The first part of the process will be to create a culture in the classroom in which children become used to using graphics to aid their mathematical thinking"* (Beatrice, W.R., 15/9/2014)

In their writing Janine and Beatrice had no specific examples of children's mathematics or open ways of working but both talked about how they were planning to change their practice. Janine wrote she had started to plan ways of making her classroom culture more open to possibilities for children's interests to be explored. She said she needs to listen to children more. "*My planning also needs to be focused on greater observations of children to help give more starting points and to provide more open-ended activities otherwise they will not be meaningful or relevant to the children*" (Janine, W.R., 15/9/2014). Janine's planning of play featured a lot in her writing, for example, she commented; "*I have begun to note down, on the planning sheets, where and how children are playing in the class, what their interests are and what next steps can be taken."* (Janine, W.R., 15/9/2014). She reflected that she needs to move away from set objectives and she wrote, "*By using this more* [observations of children's play] *rather than having fixed objectives, play can be promoted more in the setting and learning can become more focussed and specific to a child's needs"* (Janine, W.R., 15/9/2014)**.** Janine also considered she needed to be more self-assured. As well as experimenting with ideas from the case studies presented in the modules, she wrote, "*I hope to gain confidence to devise my own problems using the children's interests as starting points*" (W.R., 15/9/2014).

Beatrice's writing was theoretical and it was clear she had read beyond the set texts but there is only one area that she said she added to and that was to do with parent engagement.

She explains, "*To further support parents, in enhancing their children's learning, we now run a session to help them see the mathematical potential in their children's play at home"* (Beatrice, W.R., 15/9/2014). Beatrice stressed the reason she wanted to take the modules was to learn more about children's mathematical graphics and she wrote, "*I have identified an area of practice I do not feel satisfied with* [mathematical graphics]*"* (W.R., 15/9/2014). She wrote about understanding the potential of this and she asserts, "*The first part of the process will be to create a culture in the classroom in which children become used to using graphics to aid their mathematical thinking"* (W.R., 15/9/2014).

In summary, both Janine and Beatrice, before attempting to trial any different approaches to their teaching, saw the need for a change of culture and a reassessing of their practice. They seemed to be thinking these changes through and, also, as in Janine's case proactively finding out about the children's play in her classroom

6.1.5 *"Beyond the expectations of what the early years curriculum expects"* (Kathy, W.R., 15/9/2014)

Two of the teachers, Kathy and Melinda, moved on from thinking about planning to change their practice to actively making small changes, which incorporated examples of what the teachers termed *open mathematics*. It seemed problematic for Kathy, who wrote that she had signed on to the modules because she felt something was lacking in her teaching of mathematics. She describes how she felt when she was presented with possible ways into a more enquiry-based, or what the Reception teachers refer to as an *open* approach to mathematics,

> *I found it difficult to plan for open-ended and enquiry based mathematics* […] *on reflection and discussion with peers during the first few sessions of the course, it was clear I was in need of unlocking skills I already had and used in other areas of the curriculum.* (Kathy, W.R., 15/9/2014)

Kathy decided to take on board aspects of a more open approach to mathematics, "*During the course, inspired by seminar sessions and literature reading, I have attempted to make small changes"* (W.R., 15/9/2014). Kathy described two short sessions where she experimented with open questions and observed what the children said. For example, she decided to use a hundred square, this was because she wanted to go beyond what she thinks is expected in the Reception class (i.e. numbers to twenty). She then asked an open question, "*What do you notice?".* She followed this by asking, "*Do you notice any patterns?*" (Kathy W.R., 15/9/2014). She explained that the session went well even although she said she had no idea where this was going to go and had not expected the responses she got. She was particularly impressed by a child who is usually disengaged,

> *The final comment was made by a child who often displays disengaged behaviour, though her observation is one of the most interesting. The ability of the child who discusses numbers going up in tens, appears to show a greater understanding in number that is beyond the expectations of what the early years curriculum expects, thus showing that children are more highly skilled in number comprehension (than I would expect).* (Kathy, W.R., 15/9/2014)

Kathy's writing about the children's higher mathematics level and understanding continued in the next example she gave. She set up a toy shop with the children in her class and she observed,

> *The children had been given some brightly coloured labels to put prices in the toys, as facilitator I began to model the prices thinking that I needed to teach the children about coins, so I wrote 1p, 2p, 5p etc. on the labels. When the children started copying they mark-made and when asked what they had written, they started saying larger and more complex numbers such as £25.00, £35.50, etc., they also asked me to write these larger numbers if they were unwilling to do it themselves.* (Kathy, W.R., 15/9/2014)

Kathy commented that the observation showed the children's experience in the language of money and it also showed that they understood authentic prices rather than simple unrealistic prices. Inspired by this Kathy went on to do a mathematics topic on the book 'How big is a million' (Milbourne, 2007).

Melinda, similar to Kathy, discussed the limitations of her teaching regarding the graphics as she said, "*I still need to provide ways in which all the children can represent their mathematical thinking, which is something I was not doing before"* (Melinda, W.R., 15/9/2014). She also trialled ways of using more open questions, where children could participate more in their own mathematical thinking, in her classroom. She saw at least one child's own way of quickly mentally working out addition and she realised that she did not know that child's mathematical capability.

6.1.6 "*Transferring the ownership of the maths to the child*" (Millie, F.N., 12/6/2014)

Both Millie and Zoe wrote that they were hesitant about what the more open approaches to mathematics would look like. For Millie, allowing the children freedom to explore mathematics, in their own way was unknown territory to her, "*At the beginning of the year I had many questions about how this would look like in practice* (this open approach to mathematics)". Within Millie's writing she talked about trialling opening up the mathematics and she wrote, "*I decided to begin to create a mathematical culture within my class and to transfer the ownership of the maths from the adult to the child, allowing the children to lead their own mathematical enquiry"* (W.R., 15/9/2014). She provided two examples of this, one where she gave the children whiteboards and asked them to show her two numbers that made ten and she wrote "*they (the children) used different ways to work out numbers that made ten and graphically represented their working out"* This, she said, was "*a significant moment in my teaching*" as the children "*showed a much greater understanding of maths than I had realised*". Millie explained that she gave *"each child the opportunity to explain their graphics to an adult in the classroom"* (W.R., 15/9/2014).

Millie's second example described what she called a case study, of a whole class enquiry, which started with a question from a child who asked, "*How many children are in the whole school*?" (W.R., 15/9/2014). Millie asked, "*How will we find out and one child said*, *we could put all the children in the church and count them"* (W.R., 15/9/2014). Millie said, "*This was a dilemma, do you just ignore these questions because they were too big or do you just go with it"* (Millie, Interview, 22/11/2014). Millie added, "*I let the children lead this*" (Millie, Interview, 22/11/2014) and she asked the child, who started the enquiry, to go to the headteacher and ask her permission, if all the children could go to the church to be counted. The headteacher offered an alternative saying they could ask all the teachers how many children were in their class. This led into the whole class being involved in organising the count and Millie said this was "*exciting"* and she felt it was *"a key thing, the children organising the maths"* (Millie, Interview, 22/11/2014). Millie expressed, "*you have to understand the children's meaning rather than the child understanding the teacher's meaning. For me this is very powerful"* (Millie, Interview, 22/11/2014). Then she added, *"especially for those who are* *struggling mathematicians, that is the turning point for to get the teacher to understand what they* [the children's meanings] *are about"* (Millie, Interview, 22/11/2014).

Millie has recognised her changing practice,

> *Allowing the children ownership of the project ensured that their levels of engagement remained high for over a week, with them coming in each day excited to find out how many children were in the next year group. They saw themselves as mathematicians. As a Reception teacher it was also important that I was flexible and found the time to follow children's interests and allow them to develop beyond the set early learning goals, which cap learning to numbers to twenty.* (Millie, W.R., 15/9/2014)

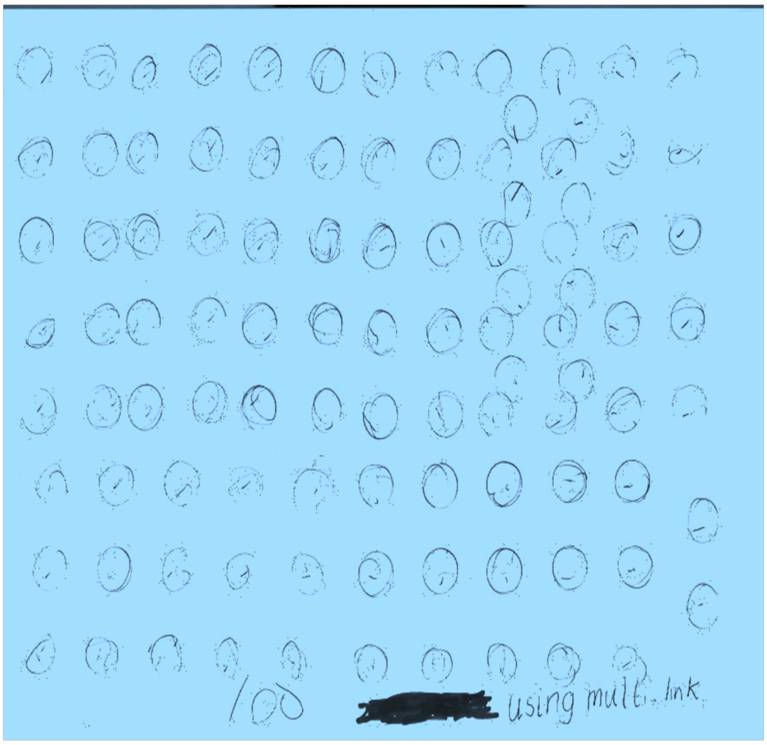
For Millie there was a significant conceptual change in practice,

> *At the beginning of the year the maths that was occurring in my classroom was of limited value, my practice has changed significantly and will continue to change. I have been particularly inspired by the idea that teachers should understand a child's mathematical graphics rather than a child understanding a teacher's mathematical graphics.* (Millie, W.R., 15/9/2014)

Interestingly, Millie did not encourage mathematical graphics in the mathematical enquiry she described above, instead, the children were given Unifix Cubes (Didatic Apparatus, 1972) to support their thinking.

Zoe, by the end of the first module assignment was thinking more about the graphics, having been intrigued that what she thought was mathematical graphics was children recording their mathematics after they had practically worked out a sum with cubes (copy recording). She said, "*On looking through our learning diaries there is much to do with number and observing if children can order and count numbers. We feel we need to give children opportunities to explore and investigate mathematics more*" (Zoe, W.R., 15/9/2014). Zoe went on to explain a recent mathematics session with children. She talked about the different ways she went about it, starting with an open question following on from children's interests in Numicon, *Children had chosen to play with Numicon and I asked the question 'What number can you make? Both boys were excited by this and shouted out many numbers before settling on investigating one hundred. One boy commented "I need lots" showing awareness that one hundred is a large number. I observed the children talking to each other using larger numbers and fitting the pieces of Numicon together on their books. One boy said "woa I can't count this, it's too much" so he decided to draw all the circles on his book. One boy finds it hard to remember which circles he has counted so he begins again and counts from the outside in and also draws a small line inside each one so he knows he has counted* it (figure, 6.1). *One boy returns to the activity later in the day and explores one hundred on the table, rather than his book because "I couldn't find it in my book". The next day these children and others played with the Numicon again. Another child uses all the ten plates to fill up the Numicon board. He counts them in tens and realises he has made one hundred. He shouts in excitement and shows me what he has achieved.* (Zoe, W.R., 15/9/2014)

Figure 6.1: 100 circles.



Zoe reflects on her changing practice,

> *When reflecting on this mathematics I realised that I had opened up the mathematics and by doing that had given the children a real opportunity for investigation and problem solving. It allowed the children to use larger numbers in their talk and play, rather than being restricted to smaller numbers they feel confident with. Children were solving their own problems of fitting shapes together and working out the most efficient way to count their marks.* (Zoe, W.R., 15/9/2014)

Zoe went on further to reflect on the literature that resounded with this mathematics scenario,

> *Gifford (2005) argues that "challenge and choice of method are therefore key characteristics of problems. If children already know the answer they are not problem solving (p. 150)". Children were obviously self-motivated by this activity as they returned to it later the same day and the next day, inspiring other children to join in.* (Zoe*,* W.R., 15/9/2014)

Zoe explained another three examples of children's mathematical investigations and she thought the benefits were, *"it allows the children to explore their own ideas and explore bigger numbers. It really challenged them* [the children] *and they were excited and it* [the investigation] *continued the next day, the next week."* (Zoe, Interview, 22/11/2014). At the beginning of the modules Zoe expressed her concern that her children would never choose to engage in mathematics, but she said "*we* [meaning the children] *are at the stage now investigators and explorers and finding out"* (Zoe, Interview, 22/11/2014).

In summary, all the teachers changed their practice to some degree and, it appeared, for three of the Reception teachers there have been significant changes. Six commented on the struggle of the change and it appears an ongoing process and two have identified the need for a culture change. However, those teachers who gave specific examples of working with children in this 'open' way expressed how beneficial it was to the children and their mathematics. They were surprised at how much mathematics the children knew and that they had not realised this before. They also commented on, and were surprised by, the children using their own strategies to work out problems. Two of the teachers expressed that the children were excited about their mathematics, there were high levels of engagement and children were inspiring other children. There was an emphasis on manipulatives, especially Numicon. One teacher encouraged the children to use Unifix Cubes to work out their problems and one teacher described children using a combination of Numicon and their mathematical graphics.

6.1.7 *The trouble is I want it* [children's mathematical graphics] *to happen now* (Janine, F.N., 13/6/2014)

Before the modules started the data from the questionnaire revealed that out of the seven Reception class teachers only two had heard of children's mathematical graphics, and of these two, both seemed to have limited understanding, confusing it with children's recording of mathematics. Millie wrote, "*There were isolated cases of children representing their mathematics* [in my classroom] *but it was not the norm"* (Millie, W.R., 15/9/2014).

All of the Reception teachers were interested in developing an understanding of children's mathematical graphics. For example, Beatrice said it was a core factor in taking the modules. *One of my main drivers in taking this module was to further my understanding of children's mathematical graphics* (Beatrice, W.R., 15/9/2014).

In the modules the teachers were presented with examples of children's mathematical graphics from adult-led sessions and in child-initiated learning. Zoe expressed her first reactions to the mathematical graphics, *"When I saw examples from you* [E.C.] *I thought to myself my children do not do this, they would never choose to do maths"* (Zoe, Interview, 22/11/2014). When one of the teachers (Janine) trialled including writing implements and blank paper about her classroom, intending the children to use it, she announced her despair to the other teachers. "*They* [the children] *don't do anything with it, I have tried* (Janine, F.N., 18/2/2014)".

At the end of the first module, although five out of the seven Reception teachers experimented with more open questions in mathematics and gave more opportunities for children to use paper and pencil and other graphic materials, only two gave examples of mathematical graphics. Two other Reception teachers gave examples of practice where the children did use their graphics however the graphics were not presented. The Reception teachers seemed to be going into unknown territory as Millie explains, "*At the beginning of the year allowing the children* *freedom to explore mathematics, in their own way, was daunting and I had many questions of how this would look in practice"* (Millie, W.R., 15/9/2014). Later on, in the module, Janine realised that children using their own mathematical graphics in her classroom may take a long time, as it was not an instant approach to pedagogy. This seemed frustrating for her as she said, "*The trouble is I want it* [children's mathematical graphics] *to happen now"* (Janine, F.N., 13/6/2014).

From the data, it appears practical mathematics, offering what is termed manipulatives (Griffiths et al., 2017) to the children to use in classroom mathematics sessions, was the way that all of these Reception teachers previously worked. For example, Beatrice states, "*In the past I have tended to focus more on children recording what they have done in practical mathematics activities rather than them using graphics to support the process of children's mathematical thinking"* (Beatrice, W.R., 15/9/2014). The manipulatives, for example, Numicon, became the resource given to the children to represent their mathematics rather than paper and pencil or other graphic materials. For example, both Zoe and Melinda used Numicon in their teaching, the children worked out their mathematics with Numicon and then recorded it in their books. In Millie's example above (6.1.6) about the children's own problem regarding the number of children in their school, the children were given Unifix Cubes to work out the problem and no other resource seem to be offered. Millie said this was an exciting example of following children enquiries but when she was asked why she had suggested to the children they used Unifix Cubes to help their thinking, she said they were used to using cubes in mathematical activities and, "*I never thought about suggesting anything else, perhaps because of time"* (Millie, Interview, 22/11/2014).

Melinda wrote about her previous experience as a classroom teacher. In play time one day a child had come up to her and said she had written one hundred and the child explained that she wrote one and two zeros. Melinda complimented the child on her knowledge and the child went on to write the pattern 200, 300. Melinda said she did not expect this and it did not fit into the curriculum and she stated that now she knew it was a child's mathematical graphics. Melinda went on to reflect on education literature she had read that argued writing materials need to be hidden and only be brought out, once the children have explored the apparatus so that they can record what they have done. Melinda goes on to say this is because the writer believes this encourages mathematical thinking as the children are, "*not focussed on writing things down*" (Melinda, W.R., 15/9/2014). Melinda challenged this as she was being confronted by the discussions in the module and the literature she was reading and her developing practice in this area, "*I was surprised to find that children who would normally need a lot of adult support found astonishing ways to represent their mathematical thoughts, if I had taken away paper and pens I would not have gained this kind of information"* (Melinda, W.R., 15/9/2014).

6.1.8 *"Recording of a product rather than aiding the thinking process" (*Beatrice, W.R., 15/9/2014)

Three of the module sessions (one of one day and two of twilight sessions) were dedicated to looking at written mathematics, highlighting the difference between recording and children's mathematical graphics. The difference between the two was reinforced in a discussion activity where the teachers had to sort the examples into two categories according to a set criterion, which was more of a guide and enhanced debate as they discussed each example. In the following sessions, the teachers had to bring examples of any graphics they had and discuss whether it was a good example of children's own mathematics. I took time, in the previous session, to talk extensively about the differences between children's mathematical graphics and recording and the teachers seemed to understand. However, all the Reception teachers, without exception, brought in children's mathematics recording where it was a copy or an imitation of the teacher's ideas. Zoe talked about the graphics she had brought into the mathematics session,

> *That was a good task on the course I said to myself, oh no* (she laughs) *because I thought I had a really good example! The example of maths graphics I brought in was not mathematical graphics because they had already done the maths practically with Numicon and I had asked them to put it in their books. I realised this was not mathematical thinking or extending their mathematics. This was just recording and they had already done the thinking in the previous discussion* (Zoe, Interview, 22/11/2014).

I asked Zoe if the difference between the children's mathematical graphics and copy recording was a challenge to her. She said, "*Definitely, yes, and still is, I really have to think what is the difference between maths graphics and recording"* (Zoe, Interview, 22/11/2014). The data revealed that throughout the modules the difference between copy recording and children's own mathematical graphics seemed difficult for the teachers to notice or instantiate in their own practice. Beatrice said she had mistaken recording for children's mathematical graphics as she was more concerned with the product rather than the process of the children using a writing tool to think through their own mathematics, "*In pausing to critically reflect, on my use of this approach, I feel I had started to become more concerned with the children's product, rather than the process, as this is the expectations for mathematics in year one and beyond*" (Beatrice, W.R., 15/9/2014). Beatrice saw the process of children thinking through their mathematics as a positive aspect of the children's graphics. This was also highlighted by Hilary, who, by the end of the module wrote that she had begun to display the children's graphics and talk them through with her colleagues and parents.

6.2 Summary Discussion

In summary, it appeared difficult, at first, for the Reception teachers to not only open up the mathematics to children's ideas and ways of seeing but, in turn, to know what was an example of children's mathematical graphics. Some of the set tasks in the module were successful in alerting the teachers to the difference between copy recording and children's mathematical graphics but it did not result in the Reception teachers sharing examples of children's mathematical graphics. However, through time, the teachers seemed to be changing their thinking and reflecting on identifying children's mathematical graphics. For example, Beatrice reconsidered a book that was recommended on her PGCE ten years previously. She wrote, "*rereading these chapters now I challenge whether the authors are in fact describing true mathematical graphics, as the explanations that accompany the children's graphics seem to fit more with recording of a product rather than aiding the thinking process*" (Beatrice, W.R., 15/9/2014).

In the first section of the data presented above, the Reception teachers relayed the barriers to their teaching of mathematics in terms of both the mathematics curriculum offered and the assessment points they needed to cover for the children to achieve the Early Learning Goals at the end of the Reception year (DfE, 2012). The EYFS (DfE, 2012) also states teachers need to plan from children's interests, from the data this is not evident in the teachers' descriptions of their practices in mathematics. This tension between adult-directed and child-initiated learning was a major theme captured in the data. Many of the discussions, at the beginning of the modules, were dominated by considering the blocks to child-centred practices. When the Reception teachers considered the literature on play and child-initiated practice it seemed some of them had uneasy pedagogies (Goouch, 2010), meaning they were partaking in practice that they felt uncomfortable with.

The teachers worked in varied ways to change their practice to more open and democratic approaches, opening up from a short, ‘how many ways can you make ten?’ adult question, to a more detailed and open story of following one child’s question and how that question encouraged other children’s questions. Following children’s own questions sustained the thread of children’s mathematical enquiry in which they are learning many skills not planned for. All the Reception teachers appeared, at first, not to give children the freedom to think about their own mathematics. However, they all reconsidered their practice, in some way, and there are many references, within the data, to what they refer to as *open* practice. Two teachers worked on play-orientated environments. These seven Reception teachers ventured into unknown territory. Goouch (2010) states, “it is a risky undertaking for a teacher just to see what happens and work in ways they are not sure of the outcome” (p. 45). It appeared they were going against the grain of what they knew, they were grappling with ideas and new conceptual practice. For example, when one of the children in Millie’s class suggested they put all the children in the school in the church to count them, Millie said this was a dilemma. Millie, having opened up her practice and let the children lead, did not know where to go next. This is a fragile place to be where the learning closes or continues. The teachers were challenging their previous thinking and seemed challenged by the literature they were reading. Both Melinda and Beatrice critically reconsidered previous literature they had read as students that did not now fit with their developing practice.

At the beginning the data revealed that the teachers were not familiar with children’s own written representations of their mathematical thinking and they struggled with identifying children’s mathematical graphics. The mathematical graphics appeared outside the teachers’ culturally influenced, professional zone. It seemed that it was difficult for them to crossover into alternative ways to teach mathematics when they were still being deeply influenced by the confines of their present school culture. There was a heavy emphasis on practical mathematical resources instead of graphic materials for the children to use in their mathematics. However, they put forward that they needed to change their practice to more child-orientated teaching before they could uncover children’s mathematical graphics. As identified in the data, when they did alter their practice, five of the teachers presented examples of teaching where children seemed to use their graphics to work out mathematical problems. However, only two of them included these examples in their assignments or portfolios.

I think that a barrier to most of the Reception class teachers showing mathematical graphics was that throughout the two years of the modules they were still unsure if the graphics the children were producing were children's own mathematics graphics. This may have made them reticent to show the graphics in the assignments. The knowledge they perceived I had may also have inhibited them. However, what seems more evident in the data is that an overriding factor in the lack of children's own mathematical graphics may have been because the teachers were concentrating on given children more freedom in mathematics. For example, they were changing closed questions to open questions, listening to children's own mathematics problems at the same time they were also reflecting on these changes to their practice. This may have distracted the teachers from giving the children opportunities to use graphic materials to think through their mathematics. The example of Millie is a case in point, where she opened up her mathematical teaching to explore a child's question about the number of children in the school, and at the same time constrained the ways in which the children could model and record their work on the problem to only offering the children Unifix.

The five Reception teachers who trialled opening up their teaching commented on the benefits of children's mathematics, as they started to see the children through a new lens. Some did not realise that the children knew so much and were surprised at how they rose to the challenge of the open practice. They observed that the children were going beyond what they expected and what was required from the Early Learning Goals (DfE, 2012). Moving restrictions in their teaching such as defined objectives and aims, five of the seven Reception teachers all said, the children were exceeding expectations and going beyond the curriculum goals.

All through the teachers' writings they described the pedagogy they wanted to aspire to as *open* and *opening up* the mathematics and this became a regular expression within the writing and class discussions and they seemed to have their own collective and individual understanding about the meaning of this. This is what they understood and this word open had resonance with them. This language of using the word open was, I believe, vital for the Reception teachers to access an unfamiliar pedagogy as this word open became a tool for thinking and communicating that thinking.

6.3 Conclusion

The changes that the teachers did make were the first steps towards the possibility for children to use their own graphics, to support them in their mathematical thinking. The changes also could be a step into these teachers knowing more about children's mathematics.

Despite the everyday barriers of their classroom practices, the Reception teachers took on a challenge and willingly thought about pedagogical change and gave examples of their changing practice and some had examples of their children's mathematical graphics. This appeared a demanding task and all of them referred, in their reflections, to the challenge of this change. They all, to different degrees and in different ways, did *open up* their teaching or planned to develop spaces for listening to the children's mathematics. Within this, what I heard the Reception teachers saying (although I understand the limits of my interpretation of their writing) that seemed to help them is;

- Opening up their mathematical teaching to the unexpected and giving children opportunities to experiment.
- Being flexible, making time for enquiry and not sticking to the set curriculum.
- Knowing the difference between mathematical graphics and children's copy recording.
- Observing and listening to the children's mathematical graphics and not expecting children to follow teachers' methods of written mathematics.
- Letting the children organise their mathematical enquiries.
- Thinking about children's play and the teacher's positioning within this.

In the next chapter, the data from the Nursery School teachers are analysed. The emerging themes of the writing and discussions in this chapter are different from the themes of the next chapter as the Nursery School teachers' perspectives and what they chose to write about are different.

Chapter 7: The Nursery School Teachers

In this chapter I analyse the reflections of the eight Nursery School teachers, Esme, Sue, Jess, Harry, Ella, Lynn, Nadira and Marcus (pseudonyms). They were teachers in Nursery Schools that are maintained by a Local Authority, in a city, in the South West of England. As well as having a full-time teaching commitment they were the mathematics leaders in their Nursery Schools and four of them had senior leadership roles.

7.1 The Nursery School Teachers' Reflections on Their Mathematics Teaching

The data is organised by the emerging themes (Corbin & Strauss, 2008) I developed from the data as described in the methodology chapter (5.2.6). I am presenting the data in main themes including a discussion and a conclusion. Firstly, I describe the Nursery School teachers' issues around their colleague's mathematical subject knowledge and confidence in mathematics. Secondly, I discuss the importance the Nursery School teachers place on their relationship with the children in their Nursery School. I go on to outline how the Nursery School teachers describe their play environment as central to the children's learning, explaining how the teachers view play. I analyse two extended examples of pretend play practice which includes children's mathematical graphics. Next, I present the Nursery School teachers' reflections on their teaching of mathematics supporting children's mathematical graphics. I then compare the Nursery School teachers and Reception teachers' data, analysing the pedagogy that could support children's mathematical graphics. Finally, I draw together all the sections, looking at a way forward in developing a pedagogy which supports children's mathematical graphics.

W.R. = written reflections of the Nursery School teachers from their first assignment: the 15th of September, 2015. F.N. = my personal notes and observations written at the time of the modules: September, 2013 to March 2015. The children's names are pseudonyms. I have written the quotations from the Nursery School teachers in italics to highlight their reflections. There are five main themes and I label each of them using the words of the teachers (Appendix 9).

7.1.1 *"A lot of maths that happens in play and child-initiated time gets missed"* (Lynn, W.R., 15/9/2014).

Five of the Nursery School teachers noted that mathematics was not a strong part of their Nursery school's curriculum although they individually felt they understood and were confident in mathematics. The early years' practitioners in these Nursery Schools, (these are not teachers but early years staff with qualifications in child care and education) were hesitant about mathematics and some even felt scared of mathematics. Marcus did an audit of what practitioners thought about mathematics, he stated, "*They ranged from a simple description such as 'number' or 'counting' to 'horrific', 'boring and 'difficult'* " (Marcus, W.R., 15/9/2014). Some were not aware of the mathematical potential of a resource or an activity and the opportunities to extend children's mathematical learning. Sue and Lynn both said their colleagues were hesitant about mathematics. For example, Lynn comments that her colleagues (early years practitioners) do not have the same enthusiasm for mathematics as they have for any of the other EYFS curriculum areas. Lynn's said, *"a lot of maths that happens in play and child-initiated time gets missed"* (Lynn, W.R., 15/9/2014). She gave an example of a child in water play and the practitioner missing what the child was doing and the possible mathematics that could come out of this. When Lynn discussed this water play activity, the practitioner was surprised that any mathematics could happen in the water tray. Lynn talked to her about volume, capacity, containers and filling and emptying. Lynn concluded, *"In the main our practitioners are not confident in firstly identifying and then extending the children's mathematical thinking, beyond straightforward counting and naming shapes. It is far more comfortable to stick with what they know"* (Lynn, W.R., 15/9/2014). Sue also commented on the lack of mathematical awareness from colleagues in her Nursery School and said they did not see the children's graphics as mathematical representations.

7.1.2 *"If children do not feel valued and secure, they will not engage with their surroundings"* (Tucker, 2014, p. 23, cited in Ella, W.R., 15/9/2014)

An important part of early childhood practice in England is the Key Person Approach. The principle of the Key Person, in a Foundation Stage setting, is that an adult is partnered with a child. The parents of the child know that the Key Person is the adult they can talk to about the child. In England it is a required part of the syllabus to learn about the Key Person Approach (Elfer, Goldschmied & Selleck, 2003) for early childhood degree programmes and for Initial Teacher Training Early Years' courses. The Key Person gets to know the child and the family much better than a traditional teacher; they visit the child's home and regularly listen to conversations about the child's home life with the parents. Elfer, Goldschmeid and Selleck (2003) state that the Key Person Approach is also a vehicle to support the child's well-being. The data revealed that all eight Nursery School teachers, in this study, were Key People (Elfer, Goldschmeid & Selleck, 2003) and were familiar with the concept of attachment (Bowlby, 1958)[16]. The Nursery School teachers seemed to place a great emphasis on the child's emotional well-being. Ella emphasised, within her writing, that the emotional environment is paramount and she stressed, *"if children do not feel valued and secure, they will not engage with their surroundings"* (Tucker, 2014, p. 23, cited in Ella, W.R., 15/9/2014). At times the mathematics was an integral support for the child. For example, Sue had been engaged with a child in his need to know about time so that he knew when his mother was coming to collect him, *"He has used his mathematical skills and learning to meet his need to know when his parents come to collect him, so regulating any anxiety he feels about waiting and therefore he is becoming more independent in his emotional well-being"* (Sue, W.R., 15/9/2014).

Harry makes the distinction, *"about gaining knowledge about the children at a deeper level as opposed to just having information about the children"* for example, just having information about the child might be knowing their date of birth and how many siblings in their family (Harry, W.R., 15/9/2014). He says he aspires to, *"attaining this level of relationship with all my key children, from learning as much as possible about the child during the home visit to developing my understanding of certain key aspects of their experience or knowledge"* (Harry, W.R., 15/9/2014). Harry gives an example of this co-engagement to gain a deeper knowledge of children through a home visit. During an initial home visit Harry spent time talking to his key child and uncovered Cameron's interest in the video game, 'Sonic the Hedgehog'. He gives an extract of this encounter from Cameron's learning diary where he talked to Cameron about the home visit,

> *You were really concentrating in what you were doing, using your thumbs to operate Sonic so the Eggman didn't get him. Sometimes you talked about what was happening. You said the Eggman was "dancing like a girl" as you made him jump and spin in the air. You said more than once, "what did Sonic just do!" You were*

[16] Bowlby 's (1958) research found that children appeared to be born with programming that helped them to form an attachment to others. This relationship Bowlby said was vital to the emotional wellbeing of the child as it provided a secure base which children needed as they developed into adulthood.

> *pleased when you made "2 and 3" Sonics go together and your mum said that you especially like the number 3, spotting it in other numbers you see about. You were proud of how you were getting better: "I've got new levels on Sonic running. I beating Eggman, daddy beating Eggman, I beating Eggman, I beating Eggman! Cameron learns about number sequences, adding groups of objects together, height, length and time through engaging in the game technology."* (Harry, W.R., 15/9/2014)

From this home encounter, armed with the knowledge of Cameron's enthusiasm for Sonic the Hedgehog, Harry downloaded the Sonic app. to his smart phone and wrote, "*He could now share this interest properly with me, as I was now a fellow player, not just an interested observer"* (Harry, W.R., 15/9/2014).

This was a genuine response to a child's engagement with his home mathematics and this mathematics became his curriculum at nursery. Tuning in and acknowledging children's meanings and ways of knowing is introduced as *attached teaching* within this study (see 7.4.4). It is in opposition to the *detached* teaching of adult-led pedagogies where the adult has the learning agenda, the context and the outcome in mind and they have little connection with the children's home mathematics worlds. Children's home mathematics was also acknowledged in Jess's observation of play where she described that Natneal drew upon his knowledge of having a party at home, where many people went into his house and he said to Jess, "*we had lots of people in our house*". Jess went on to describe the connection to Natneal's play, "*He directed his friends* [into the home corner[17]] *by saying you go in the house* […] *now you. He then began counting how many children were in the home corner"* (Jess, W.R., 15/9/2014).

Both Harry and Jess were aware of the mathematics in the children's play and how valuable the children's home knowledge was to nourish their mathematical play episodes. From Harry's play observation, I reflected that Harry was working pedagogically not only at an emotional level with Cameron, as he acknowledges Cameron's passion for the computer game but also at an intellectual level as he picked up on Cameron's knowledge and conversed with Cameron. Cameron showed Harry how to play the game thus the child led his learning and had to think

[17] The home corner is part of a play area in an English Nursery School where the teacher has set up small furniture and other objects found in a real home.

through the process of teaching his teacher. As in the play episodes of Sarah and Esme below (7.1.4) we also see, in Harry's teaching, the traditional position of teacher leading has changed to the child leading. This relationship highlighted that adults know how to share power so that children can take part fully in their own learning accessing a wide variety of opportunities and real choices.

7.1.3 *"The Nursery School values the* [play] *environment"* (Ella, W.R., 15/9/2014).

In the Nursery schools the data indicated that a large amount of time is allocated for children's free play, not structured play or planned play but what Bruce (1991) terms free-flow play. This is where the children have relative access and open rights to play outside or inside with anything or anyone they choose. The resources are planned for but the plans do not prescribe what the children do with them; there is relatively free choice. The play environment is a negotiated space where children also learn about social aspects of life i.e. turn-taking, sharing and accommodating other children in their play (Broadhead & Burt, 2012). The planned resources have mathematical possibilities, such as playing with shapes, water, sand and containers (volume and capacity). The Nursery School teachers in this study documented that they observed the children in play and responded to the children's interests and possible intentions, as they saw them. Therefore, there seemed many possibilities for children to play and explore. The Nursery School teacher's play environment could be described as rhizomatous where the learning can go anywhere, it is unpredictable and non-hierarchical (Franzen, 2015). All eight of the Nursery School teachers espoused that they valued play and wrote that the environment was central to this and they gave time, every day, for free-flow play. The weight they gave to the play environment is reflected in their writing. Ella wrote,

> *The children at the nursery have free flow access to the indoor and outdoor environment where they are able to become absorbed in play or provocations involving practitioners. The Nursery School values the environment and views it as the third teacher behind the family and key person. Children are seen as independent, confident co-creators in their learning and therefore they should have the freedom to access resources and use them for their own personal learning journeys.* (W.R., 15/9/2014)

Nadira's understanding of the environment concurs with Ella's as she wrote,

> *The environment is planned for with children's interests highlighted, attention is given to the layout and appeal of resources set out, and this planning is done weekly. However, care is taken to include spontaneous interests that children come in with and therefore the environment planner is not a fixed document but a flexible inventory of children's enquiries.* (W.R., 15/9/2014)

The Nursery School teachers observed and commented on their perspectives of the children's actions in the environment which would inform their planning, including their planning for mathematics. Marcus observed,

> *In the outside area children were playing with the petrol pump. Seb was looking at the dials on the pump windows. Seb says "That's not right. It doesn't just do 1,2,3,4,5. It goes up and up like this" (moves his arms around rapidly). He was referring to the numbers on a real petrol pump compared to the numbers on our role play petrol pump, which only went from 1-10. "It goes like this" he said and drew a large one with four small zeros. "It's a hundred and a hundred and a hundred" Seb clearly showed no fear of numbers larger than ten or twenty and could see the flaw in the design of the play petrol pump and point out the shortcomings!* (Marcus, W.R., 15/9/2014)

Ella reflected, "*This approach* [meaning the importance of planning the resources for the Nursery School environment] *is supportive of mathematical learning and development and encourages children to explore and use resources in different areas and in different ways* (Ella, W. R., 15/9/2014).

Ella went on to explain the mathematics area and her Nursery School's resources for play and child-initiated learning and how children used them,

> *Furthermore, the classrooms have a maths area where mathematical resources are available such as number fans, calculators, rulers, felt, beads, threading, and compare bears, however practitioners have observed more children transporting these resources to other areas to support imaginary play rather than sitting in the area to thread or count bears. This raises the question how valuable is the maths area in our nursery and how are the children using it?* (Ella, W.R., 15/9/2014)

Ella's observation highlighted that in a free play environment you cannot be certain by placing, what might be categorised as mathematical resources, that the children are always going to use them mathematically, especially the way the teacher intended. This aspect of children using mathematical resources in their own way and not what the teacher intended was also highlighted in the review of the literature (2.2). Ella and her colleagues, reflecting on their observations, decided to rethink what mathematical resources are and how they would place them in the environment,

> *The practitioners decided that more 'intelligent' maths resources that are 'culturally appropriate' would better support children's interests and mathematical development. We have started to collect resources such as clocks, sand timers, scales, shoe measures, shoes, real fruit and vegetables so through role play children can engage in meaningful everyday mathematics.* (Ella, W.R., 15/9/2014)

Ella's writing exemplified how in the Nursery School the environment is planned and the children are observed, within this environment. The children's behaviour dictated how the environment might be changed to accommodate their interests. The observations of the environment, as above, seemed crucial to support the teachers in planning an environment that supports children's mathematics. Jess added to this thinking about the environment and wrote, *"We are particularly skilled at following the children's interests, supporting children's learning through play and providing an environment that encourages children's own thinking".* (Jess W.R., 15/9/2014)

Harry also placed great value on the environment and especially the behaviour of the adult he wrote, *"opportunities given to children for engaging in a climate where understanding is co-constructed and questioning encouraged"* and *"Adults need to show, through their behaviour, that they really listen to what children have to say and value everything that they do"* (Harry, W.R., 15/9/2014). For the Nursery School teachers the environment seemed central to their educational ethos of child-initiated learning. It appeared that in the nursery environment children had time to follow their own enquiries and had the time, space and resources to do so in an emotionally supported environment. The Nursery School teachers changed the environment informed by the children's responses. They observed and interacted with the children at the free play times and how they might do this is exemplified by the observations of the pretend play scenarios of Sue and Esme below.

7.1.4 *"Yes half-past and it's run out of electricity and magic"* (Esme, W.R., 15/9/2014)

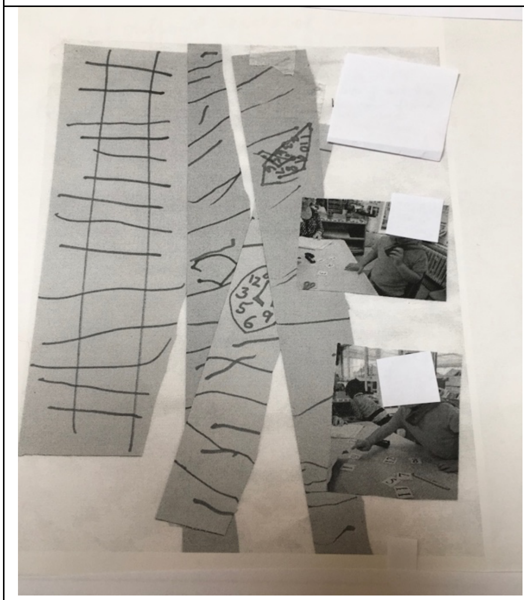
The scenarios below are of pretend play episodes and this particular type of play underlined the mathematical learning that spontaneously happened in the two Nursery Schools where Sue and Esme worked. Both play episodes are focused on aspects of time (coincidentally). Sue was involved with a nursery child in supporting him in making a watch for his Super Hero play. Esme was part of a spontaneous puppet show episode. In both episodes, children were thinking about time and they used their own graphics as a tool/resource within their play. I now give the full transcripts of the pretend play as described by Sue and Esme and the analysis follows. Sue wrote,

> *Ethan, wearing a superhero cape, is playing superheroes and stops to look for me. Ethan comes to find me to ask for help making a watch* (figure 7.1). *He says, "we need paper and colours". I hold the paper while he cuts a strip off using a two-handed grip. "We need the lines on it now, blue lines and green lines. That way, the lines got to go that way." Ethan then draws a circle in the centre of the strip. "The numbers go in the circle. Sue, do the numbers". I say 'You tell me what numbers to write". Ethan gets numbers from the velcro number strip and points to the ones he wants me to write and I write them in the circle as he says them. Ethan says, "five, seven, three, six, eight, that one (eleven), ten,* […] *is that enough?" Ethan looks at his watch and it appears he is not satisfied. He says "It's not right, it's too long." Ethan starts again and calls out numbers for me to write "Ten, nine eight, seven, six, five, four, three, two, one, blast off!". As Ethan is engaged in making the watch, an argument develops between Ethan and Sammy, a nursery child and friend of Ethan. Sammy, who is working at the table beside Ethan, spreads some postcards over the table. Pointing to the writing on the postcards, he says "this has got numbers". Ethan replies, "they are not numbers, they are letters". Sammy says, "no, they are numbers, a, b, n, s." Ethan says, these are letters. Ethan takes one of the watches and he asks me to puts it on his wrist (I do this quickly with Sellotape). He dashes off outside.* (Sue, W.R., 15/9/2014)

Sue went on to write about how she continued to support Ethan in his interest in clocks and this aided his emotional well-being,

> *I have extended the opportunity, in this play, by introducing Ethan to the classroom clock and also a visual time line of the session. He now anticipates or estimates how much time he has to play before lunch and also at the end of the session before he goes to tea-club "the thing that happens before my mummy comes". He has used his mathematical skills and learning to meet his need to know when his parents come to collect him, so regulating any anxiety he feels about waiting and therefore he is becoming more independent in his emotional well-being.* (Sue, W.R., 15/9/2014).

Figure 7.1: Ethan's Super Hero Watch.



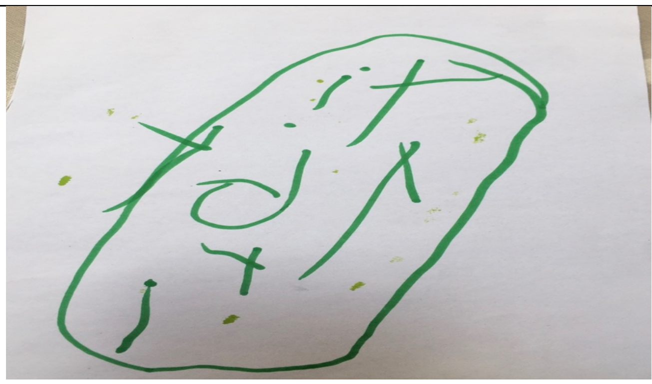
I now present Esme's scenario of mathematical pretend play, Esme wrote,

> *I was invited to play by a small group of children who were pretending to have puppet shows, they became focused on the times their shows started and told me I was a member of the audience. Ben represented a clock on paper to use as a symbolic tool to support the evidence within the play of the time* (figure 7.2). *This prompted Tiana to represent her very different clock* [using the letters in her name

to represent the numbers on the clock] *which she also referenced when talking about the time of her show* (figure 7.3). *A problem culminated when Tiana said her show did not start until four and Ben said his clock was broken and "all the numbers have gone wrong, it's gone round really fast now it's stopped, it stuck down here" he said to me. "It's stuck at half past" I replied. Ben said, "yes half-past and it's run out of electricity and magic". Ben collected more 'electricity' and pretended to open his clock by turning his paper over, he pretended to insert more electricity into the 'wires' of the clock. He said he was now able to reset his clock* (Esme, W.R., 15/9/2014)

Esme recalled how the children resumed their interests in clocks and time in future play episodes,

> *In subsequent imaginary play, Tiana and Ben have revisited clocks and time, a pivotal issue in reality, which children are acutely aware of. Recent problems have led to one hand of the clock moving anti-clockwise and the other clockwise, thus enabling the possibility to go back and forward in time, a response to the real problem of not being able to do this.* (Esme, W.R., 15/9/2014)

Figure 7.2: Ben's clock.	Figure 7.3: Tiana's clock.
	

The initial start to the adult being with the child in Sue and Esme's examples of pretend play began when the child asked the adult to be involved. The children led and gave the teacher a role within their play (see also an example of a child leading in child-initiated learning in 7.1.2). Esme was asked by the children to be involved in their play. Sue was asked to help with the resources for play and Ethan told her what resources were needed. The power immediately shifted from teacher to child within these Nursery School play contexts. Esme and Sue accepted and encouraged the child to lead. For example, when Ethan asked Sue to write down the numbers on the circle he made for his watch, she asked him to tell her what numbers to write, again she put the thinking back to him. This also confirmed to the child that he had useful mathematical knowledge within the learning situation at Nursery School. The children had also chosen the space to play, the materials and the process.

Esme was deeply involved in the free flow of the play whereas Sue was supporting child-initiated learning with the purpose that Isaac would eventually use the watch he had made in his superhero play. Esme provided a running commentary as she repeated what Ben said for clarity. She was an interested player, engaged and sought to understand the meanings in the children's play as she described in her reflections, *"When observing or joining children's play, from their inventions, I notice children flexibly move from reality to fantasy, in addition to naturally drawing upon multiple mathematical positions"* (Esme, W.R., 15/9/2014). Esme and Sue highlighted the mathematics within the play which showed that children had a connection with time and knew parts of the concept and were able to use their knowledge. For example, they knew that watches were needed to inform you of the time. Symbols (numbers, one child used letters) were used on time equipment to represent the time of the day. During the day, times in the clock and watches were linked to certain happenings in the day e.g. home time. In Esme's play scenario she highlighted the importance children placed on the time of their performance and this meant they understood the context of using time beyond set nursery times. Children played with the idea of time going backwards as well as forwards. They knew some of the language of time e.g. o'clock and half-past and they used these in context. Esme added that the children later extended this interest in clocks and time, *"Recent problems have led to one hand of the clock moving anti-clockwise and the other clockwise, thus enabling the possibility to go back and forward in time, a response to the real problem of not being able to do this"* (Esme, W.R., 15/9/2014).

In both play scenarios children engaged in understanding time in different ways and each with different problems. In Esme's vignette mathematical learning seemed so much broader than the mathematics encounters expected in the standard curriculum, especially when she described that the children were thinking about going forward and backward in time. The nursery children seemed to be encouraged to use their imagination and to go as far as they wanted to. The children's developing mathematical enquiry about clocks could be argued to be beginning to embrace a complex concept like time. Esme seemed to appreciate the intellectual learning of the situation as she sought to learn more about the children's mathematics and encouraged the children's continual interest in broader areas of time, rather than just telling the time. As Lynn put forward, "*the maths in imaginary play will not always be the counting of knives and forks or recognising the square window of the house but far more subtle*" (Lynn W.R., 15/9/2014). Lynn recognised that her colleagues might not be noticing these subtle or more complex mathematical enquiries of young children.

This pretend play afforded the children opportunities to use their own mathematical graphics which became a source of their mathematical thinking. Sue explained the problem for Ethan was the making of the watch and the recall of numbers and how you write them. It seemed important to him for his play that he had the correct written numerals and the right kind of watch. Esme explained that the problem for Ben and Tiana was the timing of the puppet show which Ben took in his own direction and drew a clock and then Tiana drew her "*different clock*" (Esme, W.R., 15/9/2014). Both teachers listened to the children's own thinking through their graphics and accepted, but perhaps not always understood all the idiosyncrasies that might bring. They did not seem pressurised by teaching objectives although the children were involved in school curriculum areas, for example number and measurement. They exceeded standard learning expectations for their age group; Ethan recognised and went beyond numbers to ten (DFE, 2017) and Ben was also engaged in concepts broader than the set curriculum as he talked about time going backwards and resetting clocks.

Within Esme's reflections, it seemed she understood the significance of the children's graphics as symbolic tools, not just fine motor skills or marks, but meaningful inscriptions as she wrote, *"Ben represented a clock on paper to use as a symbolic tool*" (Esme, W.R., 15/9/2014).

Sue's teaching underlined the important pedagogical role of modelling mathematical symbols and signs and she did this by writing the numbers for Ethan, and she had a useful class number line reference which he used. For Esme, in her nursery, in each group area there was an analogue clock placed at the children's height and this may have provided an indirect model for Ben and Tiana's clocks although she did not suggest this.

In both pretend play episodes, the teachers had noted that children used letters for numerals but they did not point that out to the child. Both Esme and Sue seem to understand children's emerging knowledge. Esme accepted Tiana's different clock perhaps knowing that children use all the knowledge they have, at the time, to represent their meaning. For example, Tiana used the letters of her name for numerals because perhaps these are letters she is very familiar with and can write quickly. Sue observed the children arguing about letters and numerals but she let the argument continue without interrupting their debate, for example by informing them of the difference between letters and numbers.

These two pretend play episodes, which I have discussed above, have presented children's engagement with mathematical graphics in the Nursery Schools. I now, in the next section, present further data of the reflections of the eight Nursery School teachers as they considered their children's mathematical graphics. [18]

7.1.5 *"Offering the children opportunities to show us their mathematical thinking"* (Lynn, W.R., 15/9/2014)

The Nursery School teachers' knowledge of mathematical graphics ranged from considering opening up their mathematical practice; to an emerging understanding; to being much more confident and already incorporating it within their practice. This first assignment, which I am drawing the data from, was well into the module and therefore their changing practice and thinking about the graphics was evident in their writing.

The data showed that Lynn and Nadira had no scenarios of children engaged in mathematics using their graphics. Lynn stated, in her writing, that her nursery colleagues did not have the confidence to follow children's enquiries and open up the mathematics, so they did not think about noticing or supporting children's mathematical graphics.

Nadira discussed the way the adults demonstrated how mathematics can be recorded; she did not offer any examples of mathematical graphics or any scenarios where children used their graphics. In Nadira's writing she said that it was difficult sometimes because she was dealing with large groups of twenty-six children and she then may have taken on more of a "*teacher at*

[18] Extracts from pages 125-131 Carruthers (2020) were used with kind permission from the editor of the *Review of Science, Mathematics and ICT Education Journal.*

the front role" with an early year's practitioner supporting her (Nadira, W.R., 15/9/2014). She stressed, *"it is largely impossible for the teacher to know what each child has understood in a whole class session, especially when the group size is sometimes as big as 26 children"* (Nadira, W.R., 15/9/2014). From these comments, I had the impression that Nadira had observed children's mathematical graphics can only be facilitated in small groups or individually and that was difficult in her teaching situation.

Both Nadira and Lynn wrote about changing their practice. Nadira stated she wanted to focus on *"the role of the adult in more detail in enabling children's learning of maths"* (Nadira, W.R., 15/9/2014). Lynn reflected, *"I think the way to unlock children's understanding will be to look carefully at the provocations we plan and really consider whether it is offering the children opportunities to show us their mathematical thinking"* (Lynn, W.R., 15/9/2014).

In Marcus, Ella and Sue's Nursery Schools they noted that their colleagues were beginning to notice the children's mathematical graphics within the play environment. They all had at least one example of mathematical graphics. Sue talked about her colleagues and their apprehension about mathematics. However, because of her discussions with them and the renewed focus on mathematics, inspired by the module, she saw more mathematical graphics in the setting. She exclaimed in her writing, "*now they are seeing*" (Sue, W.R., 15/9/2014). Her colleagues now encouraged and supported children in writing and they made available more opportunities to draw, write and use mathematical equipment; Sue said one practitioner commented, *"I didn't know my children* [her key children] *would know what to do with a tape measure"* (Sue, W.R., 15/9/2014). Sue said her colleagues discussed how children noticed numbers in the environment and made connections, for example one practitioner said, *"the children realised the numbers on the tape measures were the same as on the number line"* (Sue, W.R., 15/9/2014).

Ella led the mathematics in her setting and worked with her colleagues to observe what mathematics was happening. She said her colleagues noticed incidents where the children were using mathematical graphics which she said they shared and discussed at review time. Ella showed me a sample of a child's mathematical graphics and explained the child said it was, *"Lots of ladybirds"* (figure 7.4). Ella added, "*In the morning* […] *I put different shapes of paper out on the table with pens, pencils and different coloured markers. Astur drew a large* *ladybird* [figure 7.4] *and then carefully drew different sized ladybirds in the spaces*" (Ella, F.N., 12/11/2014).

Figure 7.4: Astur's Ladybirds.

Ella went on to say she thought Astur's drawing was in the category of *"quantities that are not counted"* (Carruthers & Worthington, 2005). Ella was not only noticing children's mathematical graphics but beginning to analyse the children's inscriptions (Ella, F.N., 12/11/2014).

Jess wrote that when she was teaching in a Reception class she saw the graphics that children made as only literacy. She described a scenario where a child, Hasana, was in the book corner and scribed on a piece of paper and Jess presumed it was a story. The next day, Jess observed Hasana in the book corner again and she drew, on paper, something that looked like circles and lines (figure 7.5). Jess explained her observation,

> *She (Hasana) pointed to the teddy bears that were on the window sill in the book corner. By looking closely at the marks on the paper, and by Hasana pointing to the bears; I soon discovered that she had drawn 6 bears (represented as 6 circles) and had been keeping a record of when the bears left the book corner. Some circles had a line through them and some didn't. Hasana was keeping a 'tally' of how many bears were in the book corner and when another peer had taken a bear out*

she would cross another bear off her list. On reflection I may have shut down Hasana's line of enquiry by referring to her marks as literacy based (Jess, W.R., 15/9/2014)

Jess wrote about the importance of open-mindedness. This, she said is, "*essential for rigorous reflection to take place*" (Dewey, 1933, p. 14, cited in Jess, W.R., 15/9/2014) and *"I was able to listen to alternative possibilities, I was able to value Hasana's line of mathematical enquiry"* (Jess, W.R., 15/9/2014). Without this further probing Jess reflected that she could have, "*simply described Hasana's marks*" as "*circular shapes with lines"* (W.R., 15/9/2014). Therefore, she said, *"Hasana's learning would have been a missed opportunity for her mathematical thinking to be evidenced and further developed"* (Jess, W.R., 15/9/2014).

Figure 7.5: Hasana's bears.

Jess, like Sue, highlighted the difficulty her colleagues had of seeing children's graphical marks as mathematics. Sue observed her colleagues viewed all the children's graphics as a generic mark-making (unaware of the children's possible intentions and meanings). Jess seemed to have been thinking about mathematical graphics for a longer time than Sue, even before the modules began and her writing conveyed her interest in this area of children's mathematics. Most of Jess's writing was about mathematical graphics and questioning what it is. She found it interesting and seemed to have a need to know. She commented about the complexity, just like the teacher in the pilot study (5.2.4). She also said she involved the early years' practitioners, within her setting, in what she described as training, "*The majority of* *practitioners in our setting have had training surrounding the important role mathematical graphics plays within a setting*" (Jess, W.R. 15/9/2014).

Esme and Harry (the teachers in my Nursery School) were well acquainted with the term and the meaning of mathematical graphics and had incorporated that knowledge within their practice. They were confident in their approach as they had experimented with supporting children's own representations in mathematics for five years. Harry, in discussion with the other teachers stated,

> *Nearly every day I write a message, comment or draw on the whiteboard. Often children will write on the board and use the space to discuss what they are thinking. We also have whiteboards outside for children to use. In our morning talk-time we set up the area sometimes with mathematical intent and see how the children engage. As well as mathematical resources, there are always paper and pencils and markers. Sometimes I have two standing whiteboards at children's heights* […] *We model mathematical notation, drawings or tallies for the children to think about and this gives a focal point for discussion. Every group area of the nursery has a whiteboard at child height.* (Harry, F.N., 15/2/2014)

Harry's use of modelling mathematics as a strategy that engaged children in thinking about mathematics is arguably a key means that children saw different ways to represent mathematics and could be a huge influence on their mathematical thinking as they began to solve their mathematics. The small child-sized whiteboards seemed to have become a vital communication tool, that provided a vehicle for the children and adults to communicate mathematically.

Harry and Esme's growing understanding of children's mathematical graphics seemed to develop through giving opportunities for the children to engage in their own graphics and seeing where the children took their thinking and how they used their graphics (see also Esme's example 7.1.4 and figures 7.2 and 7.3). For example, Harry responded to children's spontaneous enquiries and described such a situation in which the children were, during snack time, discussing counting and higher numbers,

> *It started with a child stating that that he could count to " zero, zero, zero, one" and developed to higher alternatives such as "58, 50, 10, 20!" Was there an even*

bigger number, I wondered. Leo responded with the observation that "[…] infinity is much bigger than one, it's a very big number, it just goes on and on it never stops!" Later, having provided a long roll of paper and some tape measures and calculators, three children including Leo decided to write infinity. Could infinity be made up of zeros alone, they questioned? They decided to use a combination of zeros and ones, each one drawing their own symbols onto a length of paper to make infinity (figure 7.6). When they had run out of floor space, the idea of sticking it high up along the length of the wall drew in the engagement of others, particularly when I brought in a step ladder for them to use. There were lots of questions concerned with practicalities of getting it up, including how many steps constituted a safe height to balance. (Harry, W.R. 15/9/2014)

Figure 7.6: "I never knew it could be that long. You can't beat infinity".

Harry's writing highlighted the importance of allowing children autonomy and stressed the collaboration between children and between children and teacher in their mathematical enquiries. Harry helped the scaffolding of the children's own problems in authentic situations and as this evolved the children naturally used graphics to support their thinking. Harry mentioned Katz (2000, p. 14) and what she describes as 'intellectual goals' such as, "*dispositions to make sense, ponder and synthesize which arose out of collaborative investigation"* (Harry W.R., 15/9/2014). The mathematics they investigated was not in the Foundation Stage mathematics goals and areas of development but it was at a higher intellectual search. The children grappled with huge ideas on how to define infinity. This was also seen in Esme's observation of play (7.1.4) where one child was thinking about time going backwards.

Another pedagogical point Harry stressed is, he acknowledged the importance of children putting up the display themselves and this made fertile ground for other mathematical opportunities. Harry wrote,

> *I resisted my instinct to put up the display myself at the end of the day and engaged the children in what turned out to be further mathematical opportunities. The impact on the children of being allowed to control their learning contexts was in some way empowering for them, and lead to further attempts to define infinity and reach the ceiling using other methods.* (Harry, W.R., 15/9/2014)

The impact of this mathematical episode was sustained, the display (children's mathematical graphics) the children had put up was also a reminder of the mathematical enquiry and prompted further thinking. Harry wrote, "*A couple of months later, after this event, Leo, looking up at the display, reviewed the earlier attempt to represent infinity by declaring that it was actually a zillion since infinity does indeed never stop and therefore cannot be written down"* (Harry, W.R., 15/9/2014). Harry's description of children's mathematics, as well as exemplifying the way children used their mathematical graphics, was also a notable example of broadening mathematical opportunities wider than the expected curriculum goals.

7.2 Summary Discussion

The Nursery School teachers identified the importance of being aware of mathematics in children's play and child-initiated learning. Three of the Nursery School teachers observed this was not an easy task for their colleagues as they said they did not notice the children's mathematics. One Nursery School teacher stated practitioner's lack of understanding children's mathematics provided an impoverished mathematical curriculum for their Nursery School. Pre-school practitioners' awareness of mathematics within play contexts has often been reported as a concern (Munn & Schaffer, 1993; Pound, 2006).

The setting up and planning of the environment was paramount to all the Nursery School teachers' pedagogy and they highlighted the importance of the selection and the placement of mathematical materials. The Nursery School teachers explained that their environment was set up for play and seven out of the eight Nursery School teachers gave examples of children's play as they explained their current practice in mathematics. The examples of play varied including play station games, super-hero play, water play and home corner play. Within this there were six examples where children were using pretence and two of the Nursery School teachers gave more detailed explanation of how they engaged with pretend play. These teachers provided and environment with easily accessible resources for play and useful mathematical references, for example, number lines and clocks. In both of the more detailed play episodes (7.1.4) the teachers engaged in pretence and children led and participated in "shaping the pedagogical" practices (Rogers, 2010, p.16). An important point is the teachers encouraged the children to lead and this seemed effective in the children needing to use their knowledge of mathematics to problems solve and perhaps feel confident to do so. This is vital, encouraging the children to take ownership and steer their learning enhances the children's meta-cognition (Whitebread & Neilson, 1999); the children are active within the learning process and are not passive receivers of knowledge. The children also used their graphics within their play. For example, the children's inscriptions on paper of watches and clocks became objects for their play. The clock, in the puppet show, went beyond a drawing on paper as the children transformed (Phal, 1999) their paper drawings to a real object within the pretend play scenario. These graphics seemed central to their mathematical problem solving and added to the meaning of their play. The teachers took the children's graphics seriously and enabled the play to continue by supporting the children's thinking around the area of their mathematical interest.

In both extended pretend play scenarios (7.1.4) the teachers not only noticed the mathematics but engaged generously with the children's ideas and tuned into the possibilities of mathematics. In one of the examples of pretend play, the teacher accommodated the children's thinking about their new mathematical inventions. The children generated their own problems and went beyond what they could do in reality. The pretend mathematical play seemed to give the teacher a window into children's thinking, giving an insight into their mental life. Vygotsky states that teachers knowing and understanding the high-level function that is possible in pretend play (Vygotsky, 1978) is essential to embrace children's mathematical thinking.

The Nursery School teachers accepted children's ways of representing their mathematics, for example, a large line of circles for infinity (7.1.5 figure 7.6) circles for bears (7.1.5 figure 7.5) and letters for numbers (7.1.4 figure 7.2). Children, at this age, often use letters for numerals especially in pretend play (Hall & Roberts, 2003) but it is not necessarily because they do not know the difference, instead as Tolchinsky (2003) found, children cross boundaries in informal situations. In some way, this may be part of the pedagogy, not to interfere but perhaps note this for later teaching and reflection. It seemed the Nursery School teachers were going with the direction of children's play seeing it as important not to upset the children's flow (Csikszantmihalyi, 1975).

The Nursery School teachers gave evidence that they were listening to the children at an emotional and intellectual level and this could be argued to enhance the mathematical development of the children. There were examples of this intellectual connection (meaning they discussed children's mathematical meanings with them) in Harry, Esme and Jess's engagement with the children and their graphics.

The teachers, in the child-initiated and play episodes listened to the children and unlike traditional transmission teaching where the child tries to understand the teacher, the role was reversed and the teacher tuned into the children's meanings. The Nursery School teachers' knowledge of the children mostly came, not from standardised tests, but knowing the child on their terms, by listening and observing them in play, child-initiated scenarios and knowing their contexts (Papandreou & Tsiouli, 2020). This was crucial to inform the planning of mathematics within their classroom. For example, one of the teachers (Sue) noted one of the children's emotional needs and how his mathematical knowledge made him feel more secure. What appears vital to the teacher-child relationship is that the Nursery Schools have embedded the Key Person approach and this is very much about emotional development but also impacts cognition (Elfer, et al., 2003). Again, the data revealed that having a close relationship with the child and the family, knowing their culture and ways of knowing (Gonzalez, 2005) may also be relevant to mathematical development. This was evidenced in the data when Harry, through a home visit, observed a child's access to mathematics at home. Later, Harry was able to use that knowledge of the child's home mathematics in the Nursery School to engage the child in a mutual sharing of mathematical thinking. In socially inclusive relationships the emotional aspects of children being loved and cared for are vital for trust and feelings of self and they are able to make the most of what is offered to grow learn and explore (Gillespie & Edwards, 2002). Lancaster (2003) concurs with this view and emphasises the socially inclusive relationship between the adult and child, in her study, impacted positively on the young child's impetus to freely use her own writing to communicate. This inclusiveness that seemed evident in the Nursery School data of aspects of play and child-initiated episodes appeared also to be a pivotal part in children communicating through their mathematical graphics.

7.3 Conclusion

The data revealed that most of the Nursery School teachers had a level of freedom within their classroom environment for uncovering children's mathematical graphics. All the teachers were at different understandings of developing children's mathematical graphics.
The children used their graphics in play and this helped their play themes and understanding of the world. Knowing the child both emotionally and intellectually and connecting pedagogically with them, in shared meanings and co-constructing understandings, seems to underpin the pedagogy that will support children's own mathematical thinking and, in turn, children will use their graphics as a tool to support their thinking.

The open play, child-initiated orientation of the Nursery Schools appeared fertile ground to support children's mathematics. The scenarios of play and child-initiated learning highlighted equal relationships between the teacher and the child and the teachers seemed "conceptually and contextually connected with the children" Hedegaard and Fleer (2013, p. 56). The children were given opportunities to choose what they do, within the play environment, and they led. In the play situations documented here, it appeared that the teachers had faith in the children's ability. Children chose to use their graphics, as a tool, within this unrestricted space. Play was highly valued and this is not rhetorical but lived through everyday practice. The teachers had a role within children's free play and child-initiated learning as they valued and responded to the children. Some key aspects of the Nursery School teachers' practice have been uncovered which could be the foundations of democratic pedagogies that enhance mathematical play and child-initiated learning where children can freely use their mathematical graphics they include:

- children leading their play, which means children choose the focus of their play and organise the players, including the adults;
- time and resources being easily available for children to choose and to make artefacts or graphics for their play;
- providing useful mathematical references within the environment that children can use within their play;
- accepting and tuning into the emergent learning of children's mathematics and the mathematics of their home and community;
- being emotionally and intellectually connected to children's mathematical thinking;
- realising children have their own mathematical perspectives within play and child-initiated learning. These might be unorthodox and different from the standard mathematics curriculum;
- being available for the unexpected in imaginary worlds where anything can happen;
- being ready to build on children's mathematical thinking and this may be some days or weeks after the original play or child-initiated enquiries, as they might return to similar mathematical themes;
- understanding that mathematical pretend play is complex and therefore there is a need for teachers to seek high-level professional opportunities that expand their existing knowledge of pretend play, mathematics and children's mathematical representations.;
- accepting the children's mathematical graphics as children's emergent thinking, even although the graphics might not be standard ways of representing mathematics;
- allowing children to continue their mathematical thinking by not intervening or resolving issues immediately.

Although the teaching points above are important the main and overriding pedagogical feature, was the psychological culture which the Nursery Schools provided in making the environment respectful and child-led where "play is king" (Paley, 2004, p. 4) and children are confident co-creators of their own learning. The pretend play and child-initiated pedagogy reported here depicted children highly engaged where they led rich mathematical learning, it was complex and was in juxtaposition to adult-led play pedagogies (Rogers, 2010) and traditional transmission teaching.

In this democratic practice that permeated, especially in Harry, Sue, Esme and Jess's examples of mathematics pedagogy, there was a reciprocity; the adult learned from the child and about the child, as well as the child learning from the adult. It is where there was equal partnership, in the sense that the adult and child were thinking together and sharing meanings. In some cases, the learning delved into spheres of higher thinking about possibilities beyond the everyday such as infinity and going back in time. However, for teachers to participate in a pedagogy that has uncertain outcomes, they must be brave enough to engage with children in mathematical thinking that they do not know the outcome of and may not be mathematically comfortable with. They must, "take a leap of faith and go with the children, have a receptivity to the unpredictable" (Fochi, 2019, p. 342). This democratic space does not place a limit on what the children will learn as they are not constrained by adult perceptions of what is important. The children's own graphics are a thinking tool that they can choose to use within their mathematical enquiries.

In the next section I discuss both the Reception and Nursery School teachers' practice and what can be learned about the pedagogy of children's mathematical graphics from their joint perspectives.

7.4 The Discussion of the Reception and Nursery School Teachers' Mathematics Pedagogy

The Reception and Nursey School teachers' data revealed the differing complexities of their classroom pedagogy. The Reception teachers struggled with accommodating mathematical play in their practice and providing open opportunities for children to use their mathematical graphics. This resulted in diminished opportunities for these Reception teachers to engage in children's play and the possibilities of mathematics within children's play worlds. Four of the Nursery School teachers were concerned with the practitioners in their nurseries not being aware of children's mathematical graphics and children's mathematical play. It was evident, within the data, that for the majority of the Nursery School teachers and all of the Reception teachers the changes that they made were a new way of thinking about mathematics teaching. It was not an easy change like using a new resource or using an idea for a mathematics activity, instead it was about conceptual change; for example, in the Reception teachers' reflections they stated they needed to think about a cultural change. I now consider both pedagogies in the light of what pedagogical approaches might support children's mathematical graphics. From the analysis of the Reception teachers' data (chapter 6) and the Nursery School teachers' data (chapter 7) I discuss the most prominent pedagogical themes which appeared useful in giving children opportunities to engage in their mathematics and their mathematical graphics. I now present these five pedagogical themes; children's play and open enquiries; pretence; the environment; detached/attached teaching and knowing the child.

7.4.1 Children's play and open enquiries

The Nursery School teachers reflected that they viewed play as a central vehicle of learning, all seven of the Nursery Schools considered play had the potential for developing children's mathematics and their graphics. This view of play as being central is in juxtaposition to the place of play in the curriculum in five out of the seven Reception classes in the study. In these Reception class teachers' stories of their mathematics classroom practice, they came to realise they did not give the time or value children's play because objectives were tightly planned for, and set outcomes were expected and as one Reception teacher commented she did not have time. It was noted (6.1.2) that in the restricted play times, in most of the Reception classes, play was not observed and the teachers did not support or interact with the children's play because the Reception teachers were engaged in directed teaching for set aims. This lack of teachers being involved in play concurs with the findings in the Rogers (2010) and the Moyles and Worthington (2013) studies of Reception class practice in England. It was uncomfortable for the Reception teachers to acknowledge that play was on the periphery of their pedagogy and they continued to wrestle with their pedagogical positioning in play throughout the eighteen months of the modules.

It could be thought that the Reception teachers did not have a good working knowledge of play and play behaviours as they did not take part in the children's play or plan the environment informed by the children's play as the Nursery School teachers did. They were not growing in their understanding of play by building up their professional and theoretical knowledge of play and this may have become a block to valuing play and also child-initiated learning. And, yet, play is one of the three Characteristics of Effective Learning highlighted in the Early Years Foundation Stage Framework (DfE, 2012) of which both the Nursery School and Reception teachers were working from. Through the evidence of their written reflections the Nursery School teachers were expanding their knowledge of play, in contrast, the Reception teachers seem to be drawn away from play orientated child-led practice which can be a place where children use their mathematical graphics.

The Reception teachers saw the potential for children's own mathematics to surface in play. Two of the Reception teachers on understanding their play practice needed reviewing were planning to observe the children's play and follow their lead, instead of planning play opportunities from the objectives alone. Five of the Reception teachers focused on their everyday group and whole-class teaching in which they were beginning to change their mathematical practice to use open questions. One Reception teacher decided on developing a culture where children were confident to ask their own mathematical questions. This change to open and democratic teaching, the Reception teachers believed, had the potential for children to use their mathematical graphics. The Reception teachers all reflected on changing their classroom culture to listening and responding to the children's mathematics. Most importantly they saw the benefits to the children from what they described as *open mathematics* or *opening up the mathematics*.

7.4.2 Pretence

A significant feature of the Nursery School teachers' play practice is that they observed and engaged in pretence. This seemed important because, as was evidenced in the data, the children not only used their mathematics to problem solve in the pretend play scenarios but they engaged in thinking about mathematics beyond what they could do in reality. It also provided an opportunity for children to use their mathematical graphics and in both examples of pretend play documented (7.1.4) children's graphics were central to their play. The psychological atmosphere of the Nursery School teachers' classrooms was conducive for this level of freedom and agency in play. Perhaps, for some critics of free play worlds the mathematics in the pretend play scenarios discussed could be seen as vague and without direct focus and next steps for learning. For example, in reviewing the literature (4.1.3) I reported that Ginsberg (2008) stated there was not sufficient mathematical learning going on in play as the children were not mathematising. However, from the writings of the Nursery School teachers in this study the children were making mathematical connections but in wider mathematical areas (clocks going backwards and needing resetting) and not just standard curriculum mathematics areas. Pretend play was broadening the children's mathematical thinking and I argue is an important vehicle for children's developing mathematical ideas and in turn their need to use their mathematical graphics.

7.4.3 The environment

A key point that was uncovered through the data from both the Reception class and Nursery School teachers was that the environment and the culture the teachers created was vitally important to support children's own mathematics and their mathematical graphics. The Reception teachers and some of the Nursery School teachers' colleagues at first did not notice children's mathematical graphics; perhaps because there were few opportunities within the Reception classes for children to experiment with graphics; and some of the Nursery School practitioners were not aware of the mathematics that children engaged in. Every Nursery School had a graphics area where children could freely go and choose to do their own graphics. Some of the Nursery School environments had abundant opportunities for graphics beyond the graphics area. For example, the Nursery Schools which seemed to have the most experienced practice in children's mathematical graphics always had papers and graphic equipment freely available, in a variety of forms both inside and outside the nursery, and this was encouraged as part of the play sessions. The Nursery School teachers' data reflected a democratic pedagogy and the Nursery School children, it appeared, had more ownership of their environment than the children in the Reception classes.

7.4.4 Detached/Attached teaching and knowing the child

The Nursery School teachers' knowledge of the children mostly came, not from standardised tests, but knowing the child on their terms, by listening and observing them in free range activity and in some of the teaching described, the children's home mathematics was valued and built upon. This was crucial to tune into the children's thinking. Within the Reception teachers' data there was no particular reference to children's cultural knowledge and home experiences. The one Reception teacher who did reflect on this, commented that the children's home and community experiences were not considered in their teaching team's mathematical pedagogy. Lack of knowing the children's mathematical knowledge outside school may create a gulf between the teacher/child mathematical relationship. Teachers who only seem to know the child's school mathematics (that is the standard curriculum set by the school) could be seen to have a one-way mathematical relationship with the children they teach, meaning predominately teacher-led and controlled, standard curriculum mathematics taught to children. At least at first, the Reception teachers were engaged in what I am seeing as detached teaching where the set objectives and timetables largely influence the pedagogy instead of the child inspiring the teaching. One of the Reception teachers stated that her teaching was similar to a technician, there was little personal connection, as Nias describes, "teaching as a technical act rather than personal activity" in mathematics (1989p. 43).

The concept of the Key Person approach is for the most part related to the emotional connection with children, in general. However, the data has revealed that it is relevant also to mathematical development. The data also points to the Key person, as well as having an emotional connection, with the children, also needing an intellectual connection. In Hughes and Tizard's study with four-year-olds, at home and nursery, they introduced a phrase they termed, "passages of intellectual search" (Tizard & Hughes, 1984, p. 114). This was when very young children pondered on a problem and asked questions about a concept (interestingly all examples of this were from the children's homes and not from their nurseries). For example, the children in Harry's class, from the Nursery School teachers group study, were also observed engaged in passages of intellectual search but within the area of mathematics (7.1.5). My thesis refers to these behaviours as 'higher intellectual search' moving beyond what is expected in the basic mathematics curriculum. This intellectual connection between children and teacher was uncovered in the data of four of the Nursery School teachers and it helped the teachers understand the children's mathematical perspectives. Day and Gu (2007, p. 1) state that "teaching effectiveness is underpinned by teachers who are able to be at their best emotionally and intellectually". However, the Nursery School teachers' data revealed that it is also vital to be *connected* emotionally *and* intellectually *with the children*. I am proposing the two, intellectual and emotional attachments, could bring about a strong psychological sphere for mathematical learning. Really knowing the children is a large part of this connection; as already stated, the Nursery School teachers appeared to know the children and their families and their knowledge beyond what was experienced by them in the Nursery School.

Young children have been at home and exposed to community and home mathematics for many years longer than they have been at school which means they come to school armed with their cultural and sometimes informal mathematical knowledge. They also have a deep bonding with their parents and carers and mathematics may have an emotional connection for many young children as can be seen in Sue's (7.1.4) story about a child needing to know the time his mummy was picking him up. When they enter school, the Nursery School teachers' writings confirmed that young children need to be listened to and nurtured, bonding with their teacher and/or their Key Person. I am arguing here that it may be beneficial for the children's mathematical understanding and development if teachers of young children know the children's cultural mathematics of home, school and community together with the *affective domain* of mathematics (McLeod, 1992 p. 576), which refers to a broad spectrum of beliefs, feelings and moods. This may give a revolving connection (home to school and school to home and back again) that may be needed to enhance and develop children's mathematics.

Throughout this study, I discussed how the voice of the child is significant but the voices of teachers are equally as important in understanding developing pedagogies. As the Reception teachers changed their mathematics teaching practice, they had useful insights into that change. The literature in some cases became a mirror to their practice and because they struggled to change their mathematics teaching, this tussle seemed to make them dig deeper in their thinking; they made key points that illuminated important pedagogical considerations. For example, they stressed the importance of opening up mathematical teaching to listen to children, to encourage children's mathematical questions and consider the children's mathematical graphics not always the standard mathematical notation teachers present in classrooms. The data uncovered the Reception and Nursery School teachers' differing understandings of children's mathematical graphics. This has given an insight into possible pedagogies that support children's mathematics. However, it has also highlighted the problems and dilemmas that both the Nursery School and Reception teachers faced when they reflected on their teaching. It is easy to forget past, less developed understandings and struggles; I believe it is knowing about these dilemmas and issues, that perhaps can help wider understanding of developing pedagogies of children's mathematics and their graphics, to reach a wider body of teachers.

7.5 Conclusion

The different pedagogies between the Reception and Nursery School teachers have been highlighted, the disparity between pre-school and school pedagogies is not new and has been recognised within the literature especially in the abundance of literature on transitions. Dockett et al. (2017) foregrounds these different pedagogies as a significant issue within the early years field. What has been less well researched are children's mathematical experiences from nursery to school. Transitions from nursery to school rarely focus on curriculum issues but on the social and emotional issues which may affect children's learning (O'Connor, 2018). However, recently there has been an international growing body of work focused on transitions from pre-school to school regarding children's experiences in mathematics (Perry, Macdonald,

& Gervasoni, 2015). As stated in 2.1 international perspectives and comparisons concerning nursery education may need to be compared cautiously, as the difference in children's ages when starting school in each country is variable.

In looking at the pedagogy that might support children's own ways of thinking about their mathematics, using their graphics, both the Reception and Nursery School teachers gave considered reflections on their pedagogy. The Nursery School teachers centred on play and child-initiated learning which enabled children to use their mathematical graphics. The Reception teachers focused on experimenting and trialling with what they described as *opening up the mathematics* and this gave the children, in their class, opportunities to engage in solving their own problems and, in some cases, use their mathematical graphics. This was not an easy pedagogy for the Reception teachers to undertake as they had many barriers that were beyond their control.

To conclude, I draw upon aspects of the pedagogy of Nursery School and Reception teachers practice, to inform us of how early years' teachers and early years' practitioners can uncover and develop children's mathematical graphics. One key aspect that was highlighted throughout is that the Nursery School pedagogy is child-orientated and the voice of the child is dominant. The Reception teachers' reflections were pivotal in understanding the complexities of their classroom practice and how children's graphics might be explored within this. The first list below informs the pedagogy from the child's perspective. This is my interpretation of the children's mathematical behaviours through the observations of the Nursery School teachers. I cannot express the children's views and intentions entirely and of course it is open to my misinterpretation, for example, Church and Bateman (2019) reflect, we rarely see young children's views. Instead, we see children's thoughts projected and sifted through the lens of adults. This is what I heard from the nursery child's perspective through the writings of the Nursery School teachers:

- Know me as an individual child as opposed to knowing the class as a generic group;
- I need free access to materials that will help me communicate through my graphics; for example, chalk, pencils and pens;
- I also need free movement around the class and outside to provide a space that I have the optimum opportunity to imagine in;

- Value the mathematics I communicate, in whatever form and way;
- Knowing me and my family and my community and the mathematics of my world, outside school, helps me to share my mathematical understandings and be understood;
- I operate best if I have the opportunity, every day, to have a long time to think for myself and to think with others including you, my teacher;
- Play with me and engage in my pretence when I request an adult, but do not take over my ideas or skew them in a different direction, to meet the curriculum objectives, as this may stunt the growth of my broader understanding of mathematical concepts.

From the data, I also heard from the Nursery School teachers these points that may further support the pedagogy of children's mathematical graphics:

- Learn about children's free play and especially their imaginary play because this will give an insight into the children's own mathematics that can be nurtured;
- Grow your own knowledge of children's learning and this seems to be best done within a professional learning community;
- Engage with children on an intellectual *and* emotional level in mathematics.

Added to the points already made from the Reception teacher's data (6.3) this is what I heard from the Reception teachers. I would categorise these as perhaps being helpful for those teachers who want to change their practice from adult-led mathematics teaching:

- Firstly, you must be clear of the difference between children recording their mathematics after they have already completed their thinking (copy recording) and children's mathematical graphics;
- Be flexible in space and time for children to follow their own enquiries;
- Open up the mathematics, do not be afraid to go beyond the standard curriculum goals, have courage!

The point below appeared to be a strong influence affecting conceptual change in the both the Nursery School and Reception teachers:

- Doing your own classroom enquiry on children's mathematical graphics and engaging with the literature helps develop your understanding of children's mathematics.

7.6 A Way Forward for the Next Part of the Study

Within the data presented there was only one Reception teacher who presented examples of children's mathematical graphics. By the end of the modules, three other Reception teachers were writing about situations where children may have used their graphics, but they did not present the children's graphics. It was evident in the analysis of the data that the Reception teachers struggled to uncover children's mathematical graphics in their classrooms. They seemed to be still working out how to open up their teaching and listen to the children's mathematics. Although the Reception teachers gave useful insights into how to develop democratic pedagogies only one of the teachers seemed to delve further by observing, describing and responding to children's mathematical graphics. One of the aims of this study was to gain perspectives from Reception teachers engaging with children and their mathematical graphics. Therefore, at this point of the study, I thought that I may have to change the direction of my research significantly and this was similar to the research issues that Davis et.al (2019) had in their study of mathematics teachers' changing pedagogies. In comparative research conditions and after two years of a mathematics Masters' level course, which presented the teachers in their study with a more democratic teaching style, Davis et al. found that the teachers had not, on the whole, changed their practice. Most were retaining more traditional practices and like the Reception teachers, in this study, were "occupying a more conflicted space" (p.6). I felt, therefore, I was not sufficiently answering the questions I had set out to answer within my study and reflected on why these Reception teachers thinking about children's mathematical graphics in the modules, did not develop in their classroom practice. Perhaps one of the reasons was that the Reception teachers were not as knowledgeable as the teacher in the pilot study about young children's early symbolic representations. The teacher in the pilot study was an early literacy specialist with a working understanding of children's own symbolic representations through their early writing. This may have made it easier for her to grasp the main aspects of supporting children's mathematical graphics in her classroom. The Reception teachers seemed to be coming to an understanding about children's own representations but this was not evident yet in their classroom practice. They were still working in pedagogical ways they were familiar with but with an "emergent reformist understanding" (Davis et al. 2019, p. 15). They had an insight and awareness of children's own ways of representing mathematics and they seemed to be slowly coming to know how they might incorporate children's mathematical graphics in their classroom. This, of course, is the vital pedagogical aspect that is the focus of my thesis and I decided I needed to probe further and extend my period of data collecting to focus specifically on Reception class practice. Therefore, in the next two chapters I critically analyse phase four of the data collection. For nineteen months I traced two Reception teachers' (from the Reception teacher group study, chapter 6) developing pedagogy that supported children's mathematics and, in turn, assisted children to use their mathematical graphics.

Chapter 8: The Case Studies of Two Reception Teachers, Millie and Janine

In this chapter and chapter 9, I present the analysis of the data of the two case study teachers' explanations and reflections about their mathematical teaching pertaining to childalren's mathematical graphics over nineteen months from 2015-2017 although I sometimes refer to earlier data. The data came from focus group sessions and field notes of a SLE meeting and conference where the two teachers led a mathematics workshop.

I introduce this chapter by recapping briefly on chapters 5 and 6, the analysis of the data from the fifteen Nursery School and Reception teachers. I then present the two Reception class case study teachers, Millie and Janine, and the rationale for including them in this part of the study. I go on to discuss the background information highlighting my changing role within the research, briefly discussing the pertinent influences that seemed to support the teachers' thinking in the previous data collecting phase of the study. After the introduction, I analyse the data referring to Millie's developing mathematical teaching concerning children's mathematical graphics. In 8.2 I focus on Janine's mathematical teaching again centring on children's mathematical graphics. Janine relates useful insights into four of her mathematics teaching sessions which uncover children's mathematical graphics over five months.

F.N. = field notes, which were my personal written reflections during the focus groups 2015-17

F.G. = focus group

The children's names are pseudonyms.

I have written quotations from the two Reception teachers and one Nursery School teacher in italics to highlight their spoken reflections. There are five themes in the first section, 8.1.2 to 8.1.6 and I label each of them using Millie's words. In the next part of the chapter, I foreground four of Janine's teaching stories and one longer child observation, 8.2.2 to 8.2.6. I label each section using the teachers' words.

8.1 Introduction

The previous two chapters presented data collected from 2013 to 2015. The analysis uncovered the contrasting pedagogy between the Nursery School and Reception teachers. Taking on a pedagogy that supports children's own mathematics, including their mathematical graphics, according to the data, seemed difficult for the Reception teachers. There was a scarcity of children's mathematical graphics presented from the Reception teachers, in the first eighteen months of the research. However, all the Reception teachers focused on *opening up* their mathematics pedagogy. *Open up mathematics* was a term that the Reception teachers (chapter 6) found useful to explain democratic teaching. For example, some Reception teachers gave examples of children engaging in open mathematics questions, others gave examples of children leading mathematical enquiries with their own questions and one teacher focused on how to include children's mathematical graphics in role play. Within the busy Reception class life, these teachers managed to make progress towards promoting children's mathematical thinking moving away from solely direct adult-led mathematics lessons. This could be seen as a starting point to support children's mathematical graphics. Building on these findings, I now go on to analyse more closely through field notes and focus group meetings the reflections of two Reception teachers, Millie and Janine, (who were part of the Reception teachers' group discussed above) as they developed a pedagogy that went further to supporting children's mathematical graphics from 2015 to 2017.

I asked the Reception case study group of teachers if anybody wanted to continue with my study as I was still collecting data. Millie, Janine and Zoe said they were interested but Zoe because of personal reasons had to drop out. Millie wanted to be involved in my research because she was interested in the children's mathematical graphics and particularly in understanding the children's mathematical meaning, "*you have to understand the children's meaning rather than the child understanding the teachers meaning. For me this is very powerful"* (Millie, Interview, 22/11/2014). Janine wanted to be involved in my research because she said she was actively trying different ways to get children to use their own mathematical graphics, working on planning for play and having graphic materials available for the children to use. She wrote, "*I needed to take a step back and reflect on what maths was hidden within my classroom*" (Janine, W.R., 15/9/2014). They were both enthusiastic and self-motivated and I considered these two teachers' insights to be potentially invaluable to understand a pedagogy of children's mathematical graphics especially within a Reception class. Listening to their teaching practice and reflections may pick up important points of their developing practice and theories which can often be forgotten about when teaching practice becomes more intuitive and every day (Sipman, et al. 2019).

At this fourth phase of data collection my relationship with Millie and Janine had changed on a professional level; they both eventually became Early Mathematics, Specialist Leaders of Education (SLE's) within our Early Years' Teaching School. This gave them times for frequent discussion together and with the other Specialist Leader of Education (Esme) who was one of the Nursery School teachers within the second phase of the study. My role had changed from tutor within a module that Millie and Janine attended, to their mentor within the Teaching School. We, therefore, had informal conversations about mathematics education and of children's mathematical graphics. This became a professional learning group, the three teachers together in which I was also an active participant. We met every six weeks as an SLE group and some of these times were set as a focus group for this study. Within this focus group my role as a researcher has changed to being an insider as well as an outsider in the research process (Sfard, 2008, p. 292). I am listening to the teachers, taking notes and audio recordings but I am also sometimes taking part in the conversations.

8.1.1 Considering factors that might support Millie and Janine's thinking on mathematical graphics

Drawing on the last chapter and the pointers coming from the Reception teachers on how children's mathematical graphics can be recognised and developed, it appeared that both Millie and Janine had already considered these points.

- Firstly, you must be clear of the difference between children recording their mathematics after they have already completed their thinking (copy recording) and children's mathematical graphics.
- Be flexible in space and time for children to follow their own enquiries.
- Open up the mathematics, do not be afraid to go beyond the standard curriculum goals, have courage! (7.5)

Chapter 6 highlighted that the Reception teachers' engagement with their own enquiries into their classroom mathematics' practice alongside reflecting on relevant literature aided their growing knowledge of children's mathematics. Therefore, within the next phase it was important for me to encourage them to continue to experiment within their own classrooms and to share relevant literature to do with early mathematics and areas related. This, in the modules, provided provocation, disequilibrium and challenge to the teachers' present practice. Another useful point from the previous set of data is that partaking in discussion with other teachers helped them grow their understanding of children's mathematics. The interchange of listening to others and reflecting on their own practice enhanced their understanding of possible ways forward for their own mathematics teaching. Therefore, I chose to use focus groups for this next part of the study because this may encourage further discussion and differing perspectives on mathematical practice. The focus groups which included Esme (a Nursery School teacher from the Nursery School teachers' group) could strengthen dialogues and possibly be essential to encourage Millie and Janine's evolving practice and perspectives on children's mathematical graphics

8.1.2 Ethical considerations

As my role had changed and I would see Millie and Janine regularly I was careful not to use the time we had just for my PhD research, relevant as it was, because their remit was also to develop professional development in early mathematics for teachers and practitioners across the Local Education Authority. I informed them after the focus group in February 2017 that this was my last PhD focus group meeting and I thanked them.

8.2 Millie

8.2.1 "*I was unsure where to take this* […] *what to do with it*" (Millie, F.N., 26/1/2014).

In this section I present the data that highlights Millie's examples of children's mathematical graphics and the aspects of her evolving mathematical classroom practice. Millie, similar to all the Reception teachers, by the end of the module had not yet produced many examples of children's mathematical graphics. She had said the children had produced some but she did not show me them. She may have been unsure whether they were children's graphics or not. However, Millie had considered times, that might arise, where she could give children an opportunity to write down their own mathematics. Her first trial of this, which she said was a significant moment in her teaching, was when she gave the children white boards in pairs to discuss and show her two numbers that make ten. Millie commented "*They showed me a greater understanding than I had realised and it also highlighted misconceptions that I could support*" (Millie, W.R., 15/9/2014). Next Millie trialled encouraging children to ask their own mathematical questions. She gave a lengthy account of a situation where the children had taken ownership and it was a child's question that led the mathematical enquiry (Millie, W.R., 15/9/2014). This was an exciting example of giving children more freedom to think through their mathematics. Millie did not give the children an opportunity to use graphical tools, to support their mathematical thinking, instead, the children used Unifix Cubes to support their enquiries. Millie said the children in her class were used to using practical equipment and manipulatives in mathematics and she never thought of providing graphic equipment. At first, Millie reflected that when she went with the children's questions, "*I was unsure where to take this* […] *what to do with it*" (Millie, F. N., 26/1/2014). It seemed Millie was concentrating on open questions and children's questions and how to manage this within a classroom situation. This may have been challenging enough for her without having to also think of how to support the children's mathematical graphics. Millie, now, in the next two years, continued to foster children's own mathematical questions in her mathematics teaching but she also started to encourage children to use their graphics to help them solve their mathematical questions. Millie's discourse had moved from the experimentation of open mathematical enquiries in her classroom to talking about children's mathematical graphics. She, similar to all the Reception teachers in this study, had identified that the culture needed to be open enough to give children opportunities to use their own mathematical graphics.

8.2.2 *"Valuing children's graphics and understanding the thinking behind it*" (Millie, F.N., 24/6/2015).

At a meeting of the SLE's Millie contributed to the discussion about the content of the slides for the mathematics professional development day for other teachers in the community. She said, "*Valuing children's graphics and understanding the thinking behind it"* (Millie, F.N., 24/6/2015). In the same meeting Millie took up the theme of problem solving and illustrated it through an example in her Reception class in which she introduced her way of supporting children's problem solving. She titled the slide "*How I engage with children and their problems*" (Millie, F.N., 24/6/2015). She followed this by putting in bullet points,

- Listen to the children

- Value what they say
- Respond quickly to their questions
- Plan from the children
- Allow children to share their ideas

  (Millie, F.N., 24/6/2015)

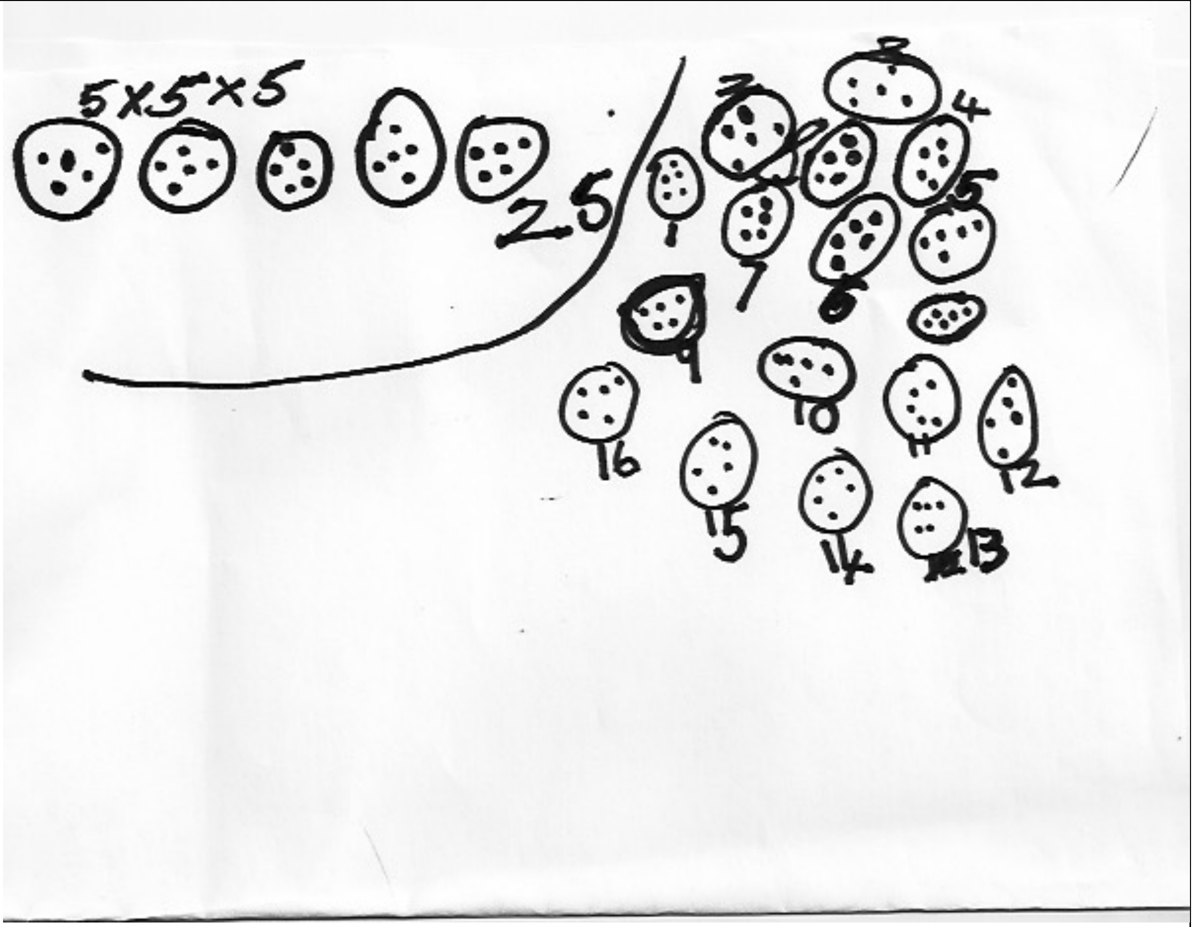
In this slide, Millie had used a discourse that centred on the children's ideas. Also, she used the word engage instead of saying, this is how I teach children. She talked about children and *their* problems not generic problem solving. Millie had already started thinking about children's questions in the modules; and she now continued this a year later, in this meeting, highlighting how she approached this and was confident to share it with others.

In the next slide she talked through a child's problem and the child had used their own graphics to think through their problem, "*How big would a classroom have to be if there were 108 children take away 8*" (Millie, F.N., 24/6/2015). She carefully talked through the strategies the child had used and pointed to the graphic and talked through the symbols he used. She said she was impressed by how the child kept going to find his answer. Millie said "He never quite finished but had thought through a lot of mathematical strategies" (Millie, F.N., 24/6/2015).

She also shared another child's mathematical question relating a story about when a child asked her "*How many children have you taught*? (Millie, F.N., 24/6/2015). She said it made her think about how she would work this out and started to see the significance of her modelling mathematical thinking (Millie, F.N., 24/6/2015). She followed this thinking on children's questions by saying that children's mathematical questions were different from adult's, "*They are more interesting and for me more challenging"* (Millie, F.N., 24/6/2015).

In a focus group meeting, in March 2016, Millie showed a short two-minute video of a year 2 (age 6–7 years) class in her school where she supported children's mathematical enquiry about measurement and the children used their graphics to work out the problem. This was not her class but she was demonstrating to another teacher how to support children in working out their self-chosen problem. This was to demonstrate to the Key Stage One class teachers, in her school, how children can use their own strategies to work out their mathematics and that the children's own graphics were important to their mathematical thinking. She had provided blank paper and pens. She said the children would normally be going through a mathematics work book that had set problems. Two children were working together, the measurement they wanted was 5x5x5 (figure 8.1). This was a more difficult calculation than usual as it was not a simple case of knowing the five times table because they also had to work out 25x5.

Figure 8.1: 5x5x5.



Millie, in showing us this video, seemed to be exhibiting confidence in her teaching of mathematics which allowed children's enquiries and their use of graphics. She also appeared to be accumulating more personal knowledge about children's mathematical graphics. After we watched the video Millie started by saying,

> *I believe the graphics have allowed the child to challenge herself and take on more. For example, she has used an iconic method to work out multiples of five; she put*

> *five dots in a circle and repeated this to solve the multiplication problem. She needed to do this, it made sense to her and it was clear. She needed to do this as she is not quite ready to take on a further abstraction like 5x5x5. Her graphics allowed her to take on a challenge and use a method she was comfortable with. The question is, is the child okay to keep on experimenting or should I be moving her on?* (Millie, F.G., 8/3/2016)

Millie went on to say, "*she* [the child in the video] *was internalising ideas and exploring it herself, in her own way"* (Millie, F.G., 8/3/2016). Millie is analysing the graphics and describing the child's method and symbols and forms of representation e.g. iconic. She is also asking questions of herself and her teaching, facing dilemmas and going into unknown territory. There seems to be a parallel between the child searching for meaning in their mathematics and the teacher searching for ways to understand how to support the children's graphics and also understand the children's graphics. She was looking at the children and their mathematics through a wider lens than at the beginning of the study when she focused on set mathematics skills to teach.

Although Millie did not share many examples of the children's mathematical graphics or explain the graphics, in the detail that Janine did (8.2), her reflections in the focus groups were insightful, particularly her reference to the children needing conceptual understanding (Millie, F. G., 15/5/2016) and how she was going to influence practice throughout her school. She also recognised, as did the other Reception teachers (chapter 6), that the culture had to change. One factor she said was really changing in her classroom culture was, the children were choosing to make drawings and all sorts of graphics inside and outside, "*My class are using paper and pencil all the time now, all kinds of paper and writing implements outside and in."* (Millie, F. G., 15/5/2016). She seemed to imply that this was an important factor in supporting children's mathematical graphics but she did not elaborate on this.

8.2.3 *"But I think that is the conceptual bit we are missing out, aren't we?"* (Millie, F.G., 15/5/2016).

As I stated, in the last paragraph, Millie often talked about conceptual understanding and how children needed time to understand mathematical concepts. At a focus group she said, *"But I think that is the conceptual bit we are missing out, aren't we?"* (Millie, F.G., 15/5/2016) and she gave us an example of a child who she said gradually, through the school year, began to understand mathematical symbols and signs. She said that she was, "*thinking about* […] *this understanding of symbolic representation, allowing children just to do this*" (Millie, F.G., 15/5/2016). She had observed a child, Milo, in her class who was not understanding, "*lots and lots of numbers from the board*" and "*the symbols were not making sense to him*" (Millie, F.G., 15/5/2016). She said she could have told him how to do the calculations on the board but she said he would not have understood it. Millie felt that this is what other schools do, "*I know that would happen in many other places* [schools] *but I purposely didn't*" [tell him how to do it]. *I wanted him to work it out and explore*" (Millie, F.G., 15/5/2016). Millie continued to talk about Milo and how she, "*just let him get on with it, I didn't disturb that too much"* and, *"I gave him time"* (Millie, F.G., 15/5/2016). She then went on to explain further,

> *Over time I just remember watching him through Christmas to the end of the year and over time, he slowly put the symbols in the correct spaces. I could have gone in, in January or December when he started doing it and gone oh no it is 10+2=12 you put 10 here and 2 here and so on. He was fascinating, I just watched him slowly change throughout the year where the symbols made sense to him.* (Millie, F.G., 15/5/2016)

Millie went on to emphasise, again, that this exploration and giving children time is, "the *conceptual bit, I think we miss out on in education"* (Millie, F.G., 15/5/2016). Millie added that this is something to consider beyond the Reception year, *"We can allow them* (the children) *more of this right up until year 2 and above, Key stage 2, I am still trying to unpick that"* (Millie, F.G., 15/5/2016). This conceptual understanding that Millie had emphasised as being important was about children slowly understanding the symbols and what they mean and this resonates with Gattegno's thinking, explained in 3.5. Gattegno put forward that young children need to have the time and freedom to explore mathematics in their own way to allow them "to use the basis of surety that exists in their perception [...] to believe in their sense of truth" (Gattegno, 2010, p. 19). As Millie explained, if you give children space to explore and not disturb their thinking too much then they may work out their mathematics, and importantly, be confident in their understanding.

Millie's changing pedagogy to facilitate the children's mathematical graphics seemed to be a slow meandering process that involved personal deliberation, rethinking and time. As she said it was not easy you have to persevere, "*It's not a quick fix* […] *you have to be strong"* (Millie, F.G., 8/3/2016). She also identified that the children needed resilience, she considered this carefully and there was a long pause before she spoke, *"Children need that resilience because it gets quite difficult as they challenge themselves"* (Millie, F.G., 8/3/2016).

8.2.4 *"If children could do graphics a bit more in Key Stage One then they would thrive and understand the maths more in Key Stage Two"* (Millie, F.G., 8/3/2016).

A notable ongoing reflection point for Millie, as a mathematics leader within her school, was influencing her school colleagues in supporting children's mathematical graphics. She had, from the beginning of the modules, kept her school informed of what she saw as the important aspects of children's mathematical graphics. Millie started to support other teachers in her school in what she called, "*opening up their mathematics*" (Millie, W.R., 15/9/2014). This was in terms of children's mathematical questions, listening to the children and for children to use their graphics to solve their mathematics problems. Millie said, "*I have collected some of their* [her colleagues] *learning completed and it doesn't show mathematical graphics, although it does show open questions thought of by the children*" (Millie, W.R., 15/9/2014). This seemed to match where Millie was in her teaching two years before.

Millie also expressed concern for the older children's understanding of mathematics. She put forward that children's mathematical graphics could help understanding, "*Deep conceptual understanding is needed earlier because children are finding maths difficult in the higher grades e.g. year six. Mathematical graphics helps conceptual understanding. Children are explaining **their** way"* (Millie, F.G., 8/3/2016). She expressed that for children to use their own mathematical graphics in Key Stage One classes in her school, then she would need the support of the SLT and they need to understand this new approach but she said they were finding it hard to understand, *"People find new approaches really hard. Senior leadership teams need the understanding of 'open maths'. And backing is needed from the SLT"* (Millie, F.G., 8/3/2016).

In a SLT team meeting at Millie's school one of the teachers expressed doubts about Millie's mathematics teaching and also questioned the mathematical graphics that the children had produced when Millie had been in the SLT team members' classroom. Millie did a demonstration lesson in the SLT members' year 2 class to show the teachers, who were observing her, that children can use their graphics to solve their own mathematical problems. Millie explained to the SLT that the children had revealed more mathematical knowledge than was usually expected of them. This raised tensions in the meeting. Millie said the teacher was not accepting the graphics, it seemed difficult for her to understand. Millie stated,

> *Actually, it is a little bit on what we reflected on at the conference* [Millie and Janine had been leading a mathematics workshop and they had noticed that some teachers had doubts about accepting children's mathematical graphics] *it is a very different way of thinking, maths graphics. The teacher involved she almost became more worried, I did not mean it that way. She became more worried about what she was doing and how she was doing it*. (Millie, F.G., 8/3/2016)

I asked if the SLT were a bit confused and Millie said,

> *No, not all, some are really supportive* […] *very supportive. I think I said to her* [the teacher who had doubts] *I don't think you understand it* […] *I know it makes sense to the children. The children understand it. She did not question me further or reply to me so I thought that is the case* [she does not understand it]. (Millie, F.G., 8/3/2016)

Millie paused, then she said,

> *I felt that if anybody came to the school I would have to defend this* [children's mathematical graphics] *and stick up for it and argue the case for it and say why other ways don't necessarily show a child's mathematical understanding. You would almost have to be there to stick up for it* (Millie, F.G., 8/3/2016).

This was a realisation that Millie had come to that children seemed to do more mathematical thinking, when they used their own personal graphics. She had discovered this in the modules that children uncovered what they knew through their own mathematical problems. I reflected, she kept going perhaps because she felt this was beneficial for children and had the potential for improving mathematics learning in her school. However, she was finding out that it was not easy for some of her colleagues to understand that children's mathematics and their graphics might be helpful for children's mathematical thinking. When she said, "*if anybody came into the school*" (see full quotation above) she may have meant Ofsted Inspectors and that would be a major concern for her school. As Millie said, *"it is a very different way of* *thinking, maths graphics"* (Millie, F.G., 8/3/2016) and it may be difficult for Ofsted Inspectors to see the value of children's mathematical graphics, especially if it was not working strongly through the whole school. For example, most of the teachers might be at the stage of beginning to experiment with trialling approaches that support children's mathematical graphics. The experimental stage of new ideas can look a bit haphazard, not perhaps what Ofsted Inspectors are looking for.

8.2.5 *"About the mathematical graphics, is that acceptable, is it not acceptable?"* (Millie, F.G., 8/3/2016).

On at least three occasions (Millie, F.G., 8/3/2016; Millie, F.G., 15/5/2016; Millie, F.N., 16/11/2015) Millie brought up the question of mathematical graphics not being acceptable to teachers, her school, Ofsted or the present government tests for Key Stage One and Two. She had experienced teachers at a workshop she had led being doubtful of children's own graphics, perhaps, to these teachers, mathematical graphics did not look right; they were not neat and not standard notation and therefore may not be acceptable. Millie explained, children's mathematical graphics could be seen as errors by other teachers because mathematical graphics are not always standard symbols and may not be recognised as children's emergent mathematical understanding (Millie, F.G., 8/3/2016). She went on to say, it is expected by her school that, "*You should pick up the misconceptions ready for the next day"* (Millie, F.G., 8/3/2016). Therefore, I think Millie was implying that children were not given the space, as in the case of Milo above, to have time to reflect and rethink over a long period. Millie has identified a barrier to accepting children's mathematical graphics and, therefore, not allowing for emergent mathematical thinking.

Millie continued to place an emphasis on child-led teaching and in the SLE strategic session she agreed with Jess (Nursery School teacher and SLE) that the phrase "*opening up the mathematics"* (Millie, F.N., 24/6/2015) was important to use to present to other teachers as they were trying to explain a more democratic mathematics approach to teaching. However, Esme (Nursery School teacher) did not agree, saying, "*it was more Janine and Millie's phrase*" (Esme, F.N., 24/6/2015). I considered this and I reflected that Esme probably did not need this term, as she worked in a different education environment to Millie and Janine, she did not constantly have to defend child-led teaching because that was already embedded within her Nursery School. Contrastingly, Janine and especially Millie seemed, from the data, to need sound explanations and theories to justify this change in their mathematical teaching;

something they understood enough to convince their colleagues and school leaders. What they collectively (Janine, Millie and Jess) now termed *open maths,* was a term they could use to explain to other teachers about democratic teaching. This phrase, *open maths*, could aid their explanation of possibly supporting other teachers to steer the way for children to use their own mathematical graphics, to solve their and teacher posed mathematics problems. I, similar to Esme, was not convinced that this phrase was helpful, but on listening to Millie and Janine I was beginning to see the term, *open mathematics*, as a step on the way in developing a pedagogy that perhaps could put children's mathematics at the centre of Reception class mathematical practice.

Millie's interest in children's mathematical questions and graphics was influencing the teachers in her school. It appeared she had a clear vision of how the graphics were not only an important tool for mathematical thinking in her class; but she could see the benefits of this for Key Stage One and Two children. However, to develop this further in her school, there were barriers she knew she had to overcome.

8.3 Janine

In this section I focus on Janine's descriptions of the children in her classroom's mathematical graphics. This was in one particular focus group session (15/5/2016). There were four people in the focus group (Janine, Millie, Esme and E.C.) listening to Janine. Janine related four mathematics teaching sessions and one long child observation.
Firstly, I discuss Janine's previous thinking and experiences about children's mathematical graphics.

8.3.1 *"My ideas seemed clearer and I could see where I wished to go*" (Janine, W.R., 15/9/2014)

During the first module when Janine trialled having paper and graphic materials around her class, for children to freely write or draw on, she exclaimed in one of our module discussions, *"They* [meaning the children] *don't do anything with it* [refers to paper, pens and other graphic materials], *I have tried"* (Janine, F.N., 18/2/2014)". She may have been, at this time, unaware of any factors that might be causing the children not to use the graphics equipment she had provided around the classroom. She said she had tried hard to engage the children in using their own graphics and they did not respond. Janine gave me the impression that children's mathematical graphics was not possible in her classroom. She also was unsure of what children's mathematical graphics were and looked like as she wrote, "*It also shows, by sharing my examples with others on the course, that some of what I perceived as mathematical graphics was recording*" (Janine, W.R., 15/9/2014). During the modules she was planning to develop an open approach to teaching including being more aware of children's play and mathematics. Janine considered this change in her teaching, to enable children's mathematical graphics, may take some time, "*The trouble is I want it* (children's mathematical graphics) *to happen now"* and "*By going through the process though, I understand that I may have to confront some of the methods I have been using*" (Janine, W.R., 15/9/2014).

By the end of the modules Janine continued reflecting on possible changes in her mathematics teaching, *"My ideas seemed clearer and I could see where I wished to go*" (Janine, W.R., 15/9/2014). She also reflected, "*we were going to encourage greater opportunities to allow more open-ended problem solving within our classroom"* (Janine, W.R., 15/9/2014)". Observing and listening to children were what she now was going to focus on, "*Children's interests are now at the forefront of our planning"* (Janine, W.R., 15/9/2014). Janine, similar to Millie, realised that she had to change her practice and class culture if she was to facilitate children using their own ways to represent their mathematics.

In this focus group session (F.G., 15/5/2016) I asked Janine and Millie to bring in examples of children's mathematical graphics. Janine brought a variety of examples of children's mathematical graphics from her classroom. Up until this group session Janine had not been forthcoming in showing what was happening with mathematical graphics in her classroom. However, she presented a range of children's graphics in this focus group session, which she explained with confidence. On reflection, this was a significant moment in the study.

In the next sections below (8.2.2 to 8.2.6) Janine explained her interactions with the children and the children's strategies and mathematical graphics; she also uncovered her own personal pertinent teaching strategies. Therefore, from the data presented, I highlighted the pedagogical shifts in her practice and the skilful and sometimes subtle ways she interacted with the children. I am presenting Janine's descriptions and analysis of the children's graphics in the order she presented them to us. There was a range of mathematical graphics the children in her class had produced over five months, from mid-November 2015 to the end of March 2016.

8.3.2 *"So literally it was just free activity on the floor with pens and paper" (*Janine, F.G., 15/5/2016).

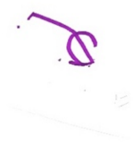
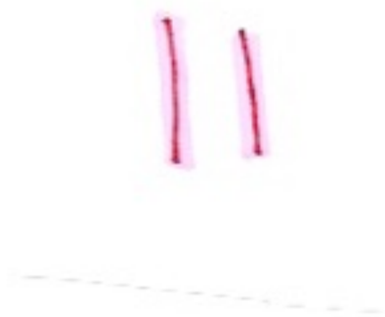
Janine explained the context of the mathematics teaching session she had with the children in her Reception class in mid-November 2015, the children would have been in school for about three months. Janine began,

> *One of the things that I did was go over counting and then left them with a large piece of paper and I put some beads out; so, literally it was just free activity on the floor with pens and paper. I just left the paper out with beads* […] *Some were just counting it, as you would expect, but some wanted to 'record it'* [pause] *'put it on paper'.* (Janine, F.G., 15/5/2016)"

Janine seemed to be thinking carefully about the language she used to describe what the children are doing and that is why she might have paused and said *put it on paper* (Janine, F.G., 15/5/2016). It could be she was hesitant about using the word *recording* as the children were not recording what they had previously done. Janine was giving the children the opportunity to think about their mathematics, in the moment, and use their graphics. As Janine said some wanted to put that thinking on paper.

Her pedagogy seemed less restricted as she was now giving the children freedom '*it was just free activity*'(Janine, F.G., 15/5/2016). Janine did not have a planned outcome which was also a shift in pedagogy. It seemed she was saying, just let's see what happens. There was also a choice to *put it on paper*. Having pens and paper immediately available gave the children the opportunity to put their mathematical thinking on paper. The choice is important, further exemplifying democratic teaching.

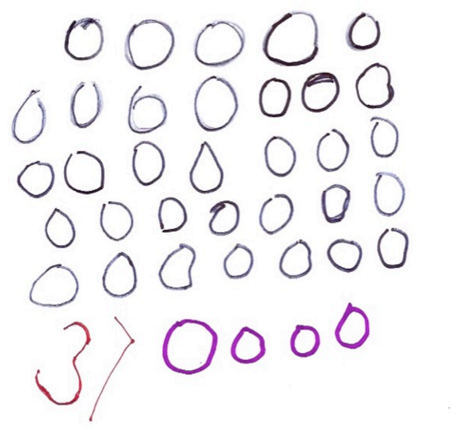
Janine discussed individual children's mathematical graphics, she started, "*This was interesting, this little boy* [child 1] *usually was not interested in 'recording' at all. He decided to see how many* [beads] *he got and so that is his 9* [figure 8.2: Child 1]. Janine seemed to be identifying and supporting children's attempts at having a go and acknowledged and accepted this child's way of writing a 9.

Figure 8.2: Child 1 *"that is his 9"*.	Figure 8.3: Child 2 *"She wanted to count into the teens"*.	Figure 8.4: Child 3 *"She wanted to write the number"*.
		

Janine discussed child 2, "*She had not shown an interest before, but she wanted to count into the teens and she needed support to carry on and we did that together"* [figure 8.3: Child 2] (Janine, F.G., 15/5/2016). Janine explained her collaborative and supportive style of teaching, where the child was leading and Janine encouraged the child's interest in counting beyond what she already knew and therefore extended her learning.

Janine continued, "*Another little girl* [child 3] *she counted and then she wanted to write the number and I was surprised she knew the number and could write it, 11, she counted up to this"* [figure 8.4: Child 3] (Janine, F.G., 15/5/2016). Again, this child was leading and that seemed to be important as Janine was uncovering what children knew about number that she was unaware of.

Janine gave another example, "*This was another girl* [child 4] *and again she surprised me, she was counting above 20 and she got up to 29 and needed a prompt to go on to the 30. I gave the number to her in the end as she started counting backwards"* (Janine, F.G., 15/5/2016). Janine went on to say*, "Then she counted on and put it down on the paper* [figure 8.5: Child 4] *she put 37 but when we recounted this it was only 33 and then she added those on. So, lots of things going on there"* (Janine, F.G., 15/5/2016). Again, Janine demonstrated collaborative teaching, waiting for the child to solve the problem, but did not disturb the flow of the child, she used teacher judgement.

Figure 8.5: Child 4 *"she added those on"*.	Figure 8.6: Child 5 "*how many fruit altogether"*.
	

Janine now explained, "*Then another girl* [child 5] *got out the fruit to count. She represented the fruit and crossed it off as she counted it. She then did an adding and decided which fruit she was going to buy. She then represented, how many fruit all together"* [figure 8.6 : Child 5] (Janine, F.G., 15/5/2016). Children developed their own strategies to check the count. Janine gave this child the freedom to use the fruit and to turn it into a play experience and therefore allowed the child a different way to respond.

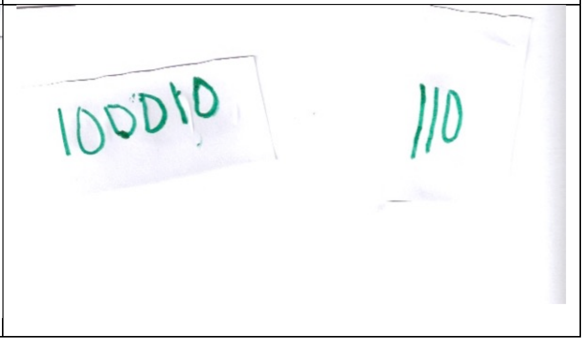
Janine gave another three examples,

> *That was that and then I think I had a few numbers out as well and one of them said* [child 6] *they wanted to talk about time. That was obviously what he was interested in and he was using o'clock. He did 11 o'clock and 10 o'clock so he was writing a number and then saying the o'clock. There are numbers there and…* [figure 8.7: Child 6]

and,

> *Another girl* [child 7] *put some numbers down and she got me to say them and then she randomly put 7 and 8 together and said 78. This was an EAL child and this was surprising that she knew that. I was quite surprised she knew that. She really enjoys her numbers.* [Janine, F.G., 15/5/2016]

Then this boy [child 8] *wanted to represent larger numbers and so he knew that was a thousand and he wanted to write it and knew he needed 4 digits to represent a thousand. He said that was a thousand and ten and we talked about what that would look like. Then he wrote one hundred and ten. He really was exploring numbers and we talked to the whole class about that*. [figure 8.8: Child 8]

Figure 8.7: Child 6 *"11 o'clock"*.	Figure 8.8: Child 8 "*He knew he needed 4 digits to represent a 1000"*.
	

All three examples showed children engaging in what they were interested in and leading. One child used Janine to help her read the numbers. Janine was an interested listener and supported each child in different ways.

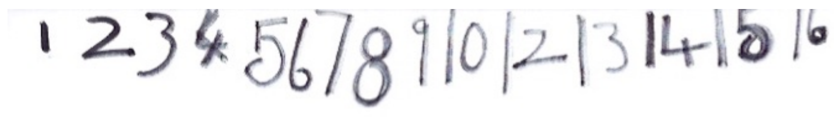
The pedagogy that Janine displayed throughout all the examples was varied and tailored to each child's individual need. Janine seemed very confident not only in her explanations of the children's graphics but in her pedagogy.

8.3.3 *'But then they started to move on*" (Janine, F.G., 15/5/2016).

Janine now showed us individual paper examples of children exploring numbers from the beginning of December to the end of January,

> *But then they started to move on, then they started writing numbers in order and Amber* [figure 8.9: Child 9] *is not confident and okay she did miss out the eleven there. Layla, she wanted to go further and again, at this point, she was not sure about going over the tens and we talked on how to represent that, on paper, what that would look like.* (Janine, F.G., 15/5/2016)

Figure 8.9: Child 9 *"She wanted to go further"*.



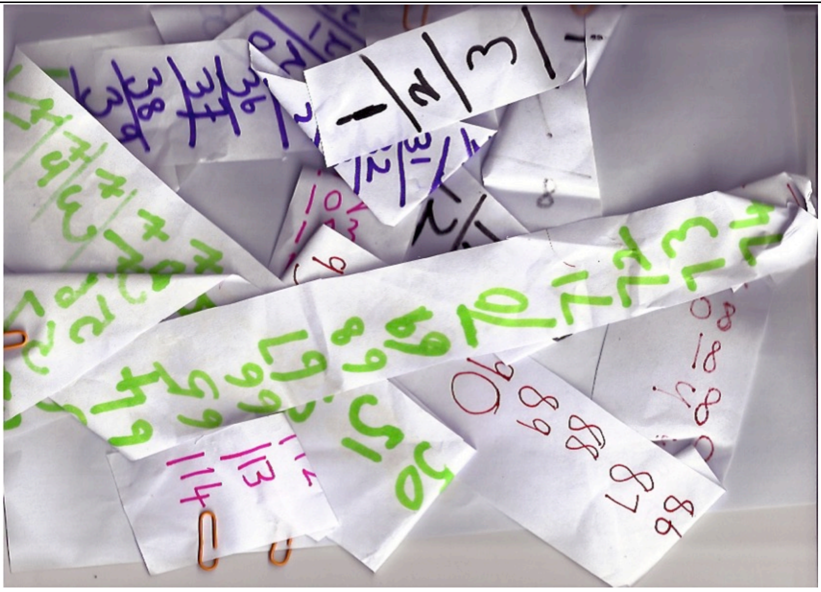
Janine displayed she was accepting the children's graphics and accepting that the child had missed out a number. At the time, I felt that Janine was almost defending the child who missed out number eleven, as if she could predict that some colleagues or senior leaders would not accept this. Perhaps, Janine thought it was more important for that child to feel confident to choose to write numbers, as far as she wanted to go, and missing one number out was not significant.

Janine was continuing to show us a range of individualised mathematical graphics, children going their own way and children leading their learning and engrossed.

8.3.4 *"It's the sort of thing that makes you excited to go to school"* (Esme F.G., 15/5/2016).
Janine unravelled a very, long, roll of paper, Millie, Esme and I immediately understood this was a good example of a child's mathematical thinking as we all gasped. Esme said, "*Amazing*" (Esme F.G., 15/5/2016).
Janine explained,

> *She has started it, as you can see on paper, like this and then she got up to 39* [figure 8.10: Child 11] *and then she could not go on and then she went back to it the next day and ran out of paper. So, I said use some more paper and she went and stuck it on the end and that went on for a bit and she added more paper and over the next few days she got really excited and carried on with her number line and eventually she got up to 114.* (Janine, F.G., 15/5/2016)

Figure 8.10: Child 11 “*and eventually she got up to 114”*.



I said, “*She must have loved this*” (E.C. F.G. 15/5/2016) Esme replied in agreeance, “*Yes, it was great that even although she stopped one day and abandoned it but she went back to it. It’s the sort of thing that makes you excited to go to school”* (Esme F.G., 15/5/2016). Janine agreed,

> *Exactly, then there were lots of discussion about that and how to write the numbers and they all* [the children] *just wanted to explore, partly because they wanted to do big numbers. Some of them just wanted to write 1 and 0 because they wanted to write a big number and knew 0’s were in large numbers. So, they were looking at numbers, sequencing and learning how to write them from other children. What does it mean etc*. (Janine, F.G., 15/5/2016)

Janine then said,

> *Oh yes*, *another boy Joshua* [child 12] *is good with numbers, mentally, and he is fascinated by buses and numbers on buses. He wanted to write something but he finds writing difficult, he wanted to write his own one* [number line]. *Using a pen is difficult for him”*. (Janine, F.G., 15/5/2016)

Janine had identified that children were socially constructing their understanding. The children in her class were being influenced by other children and this girl's passion for developing a long number line was taken up by other children even children who find writing numbers difficult. Janine is allowing children to lead and carry on thinking about their mathematics over days instead of in one teacher-directed lesson. She seemed to have created a culture where children felt confident to explore their mathematics and learn from other children. It appeared that she also understood the benefits of the mathematical learning that was happening. This episode of children's learning reminded me of Tovey's phrase, "*Learning should be joyous, meaningful and relevant. It should inspire further learning"* (2016, p. 126).

8.3.5 *"This is the building blocks isn't it? Children having a go"* (Millie, F.G., 15/5/2016).

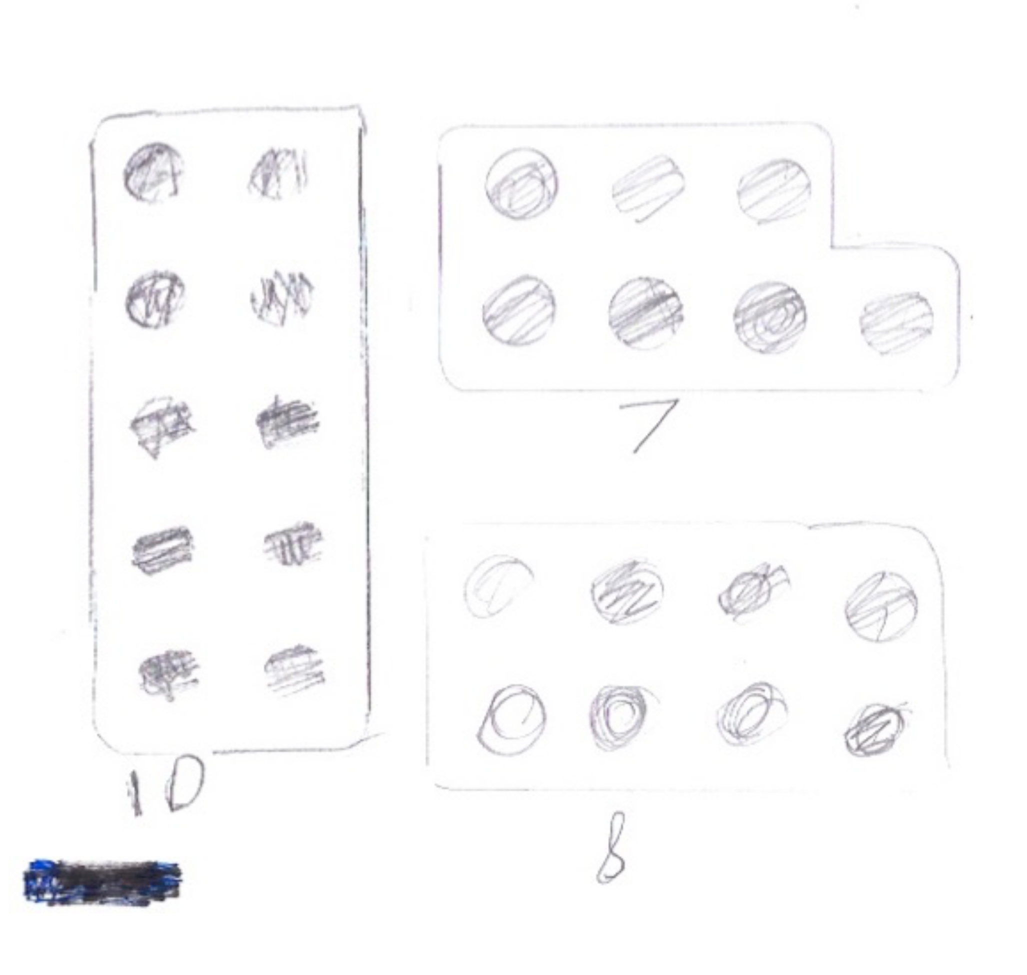
Janine explained another mathematics session with the children where she used Numicon (This mathematics session was in February, 2016),

> *We played a game of counting, adding on and we had been using Numicon for one more one less, I just put some Numicon out on the floor and I said, look I am adding these together, you can write it this way if you want this is what I am doing.*

Janine was modelling how she was doing the calculation but she was not expecting children to copy her.

Janine accommodated children's thinking on different levels for example she said, "*Some of them just wanted to draw round the Numicon and write down the numbers*" [figure 8.11: Child 12] (Janine, F.G., 15/5/2016) and she explained, "*But this is a boy* [child 13] *who is a very able mathematician he went from, he added,* [...] *then he said, I can do it a different way 10 add 10 and 10 make 30 and I do know that 10x10 =100, so he was building this up. Then this child* [child 14] [Janine pointed to the graphic] *is having a go, this was a less able child, but she was watching other children* [figure, child 14].

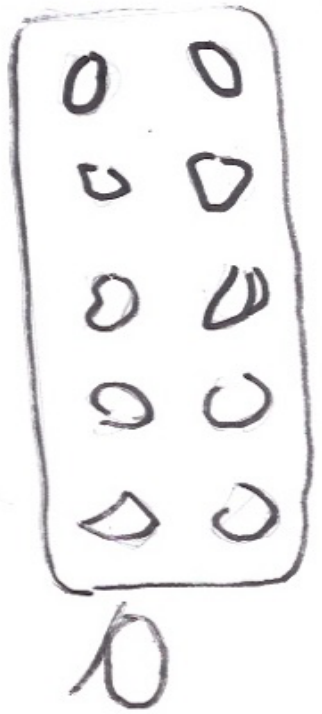
Figure 8.11: Child 12 *Drawing around the Numicon.*



The emphasis seemed to be on having a go and seeing where your thinking takes you and you can learn from others and everybody is successful.

Millie said, at this point, "*this is the building blocks isn't it? Children having a* go" (Millie, F.G., 15/5/2016) and Janine agreed, "*Yes, and she was influenced by the other children"* (Janine, F.G., 15/5/2016).

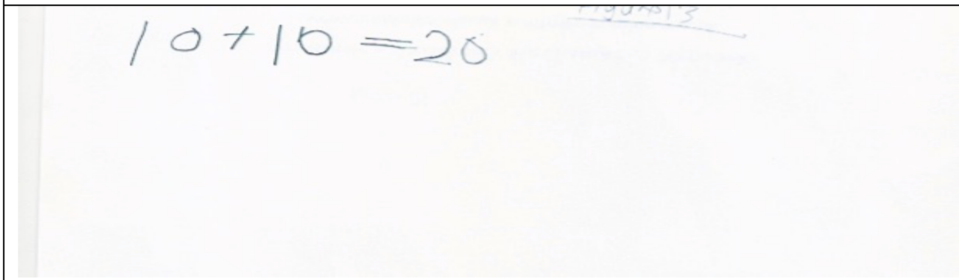
Figure 8.12: Child 14 " having a go".



There was also a lot of free drawing on the paper. Janine said, *"and some of the children got carried away actually drawing"* (Janine, F.G., 15/5/2016). Janine looked at another graphic and explained,

> *He knew the add sign and knew what that meant, and again that was there. Loretta* [child 15] *said 10 add 10 is 20, she did not bother with the Numicon she probably thought this was easier, quicker* [figure 8.13]. *She knew the sum and she counted it out in her head and you could see her and she just put the answer. It was really interesting to see that she did that calculation in her head and she did not need to use the representation and she didn't but she wanted to record the sum. I have not done any recording at all with adding, just games and that, yes, it is interesting that some have picked this up and want to use it.* (Janine, F.G., 15/5/2016)

Figure 8.13: Child 15 *"she just put the answer"*.



Janine was very positive about what the children knew mathematically as she described how they used strategies that make sense to them. Janine explained that this child (child 15) was using standard mathematical representations but she had not yet taught the class this. She also explained that other children were using their own ways of representing and not standard calculation. She accepted a variety of responses and strategies.

8.3.6 *"There was too much to think about doing it, in one day"* (Janine, F.G., 15/5/2016).
Janine now explained how she used the book, *One, is a snail, ten is a crab* (Sayre & Sayre, 2004), as a stimulus for the children to choose a number and then show on paper the number in animal legs. This was at the end of March, 2016. Janine said, "We had been working on counting in tens, and so I went over counting in tens. The children were really interested." (Janine, F.G., 15/5/2016**)** Janine continued,

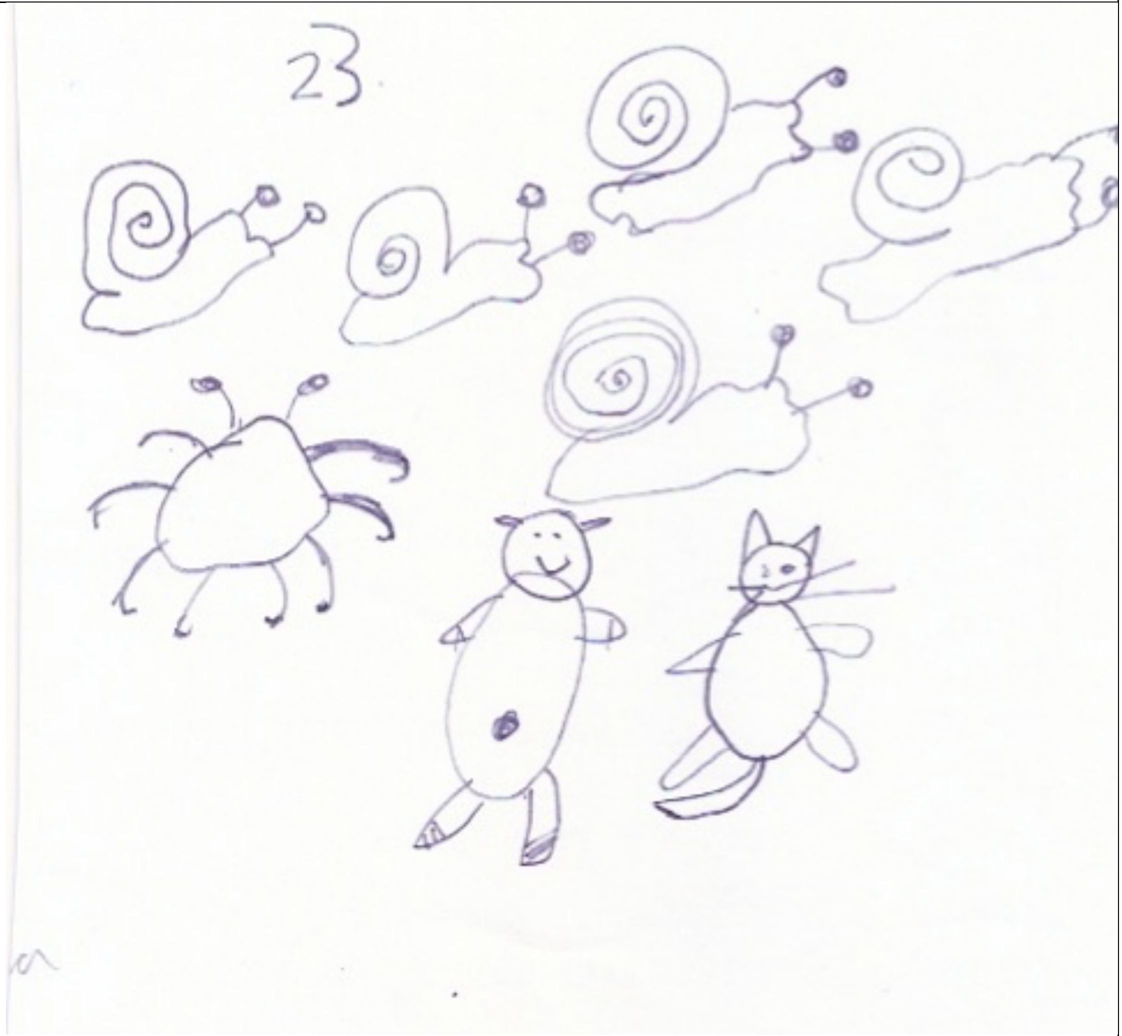
> *And then from there they were really keen to count in tens and they could choose which number they wanted and then they could decide how they wanted to represent it; and this was Leonard* [figure 8.14, Child 16] *and he decided he wanted to do a 100. He was really clear, he said I think I want to do 100 and I think I need 10 lots of 10 to make a hundred, so then he checked that he did and then he counted on and then he was really good at realising how many more crabs he needed. When he got to 5 then he was able to work out in his head and I am going to need 5 more now and then he got 8 and then he said 2 more and then he did it. He checked at the end. I think that Elsbeth* [child 17] *wanted to do 40 and she need a little more support to see where she was going but again she checked it at the end. She got to 30 and then needed help to see where she was going next.* (Janine F.G., 15/5/2016)

Figure 8.14: Child 16 *"I think I need 10 lots of 10".*

This further exemplified how Janine was encouraging children to extend their learning and lead their learning and challenge themselves. Janine continued,

> *And then there was Georgina* [figure 13: Child 18] *although she is quite a good mathematician and she wanted to do 23, she started with the snail and then she counted 5 snails and I said you have a long way to go, can you make it quicker? She then said, yes, I will use a crab, 10 legs, and at one point she worked out she needed four more and she used a cat, 4 legs. She was using a lot of skills, such as counting on.* (Janine F.G., 15/5/2016)

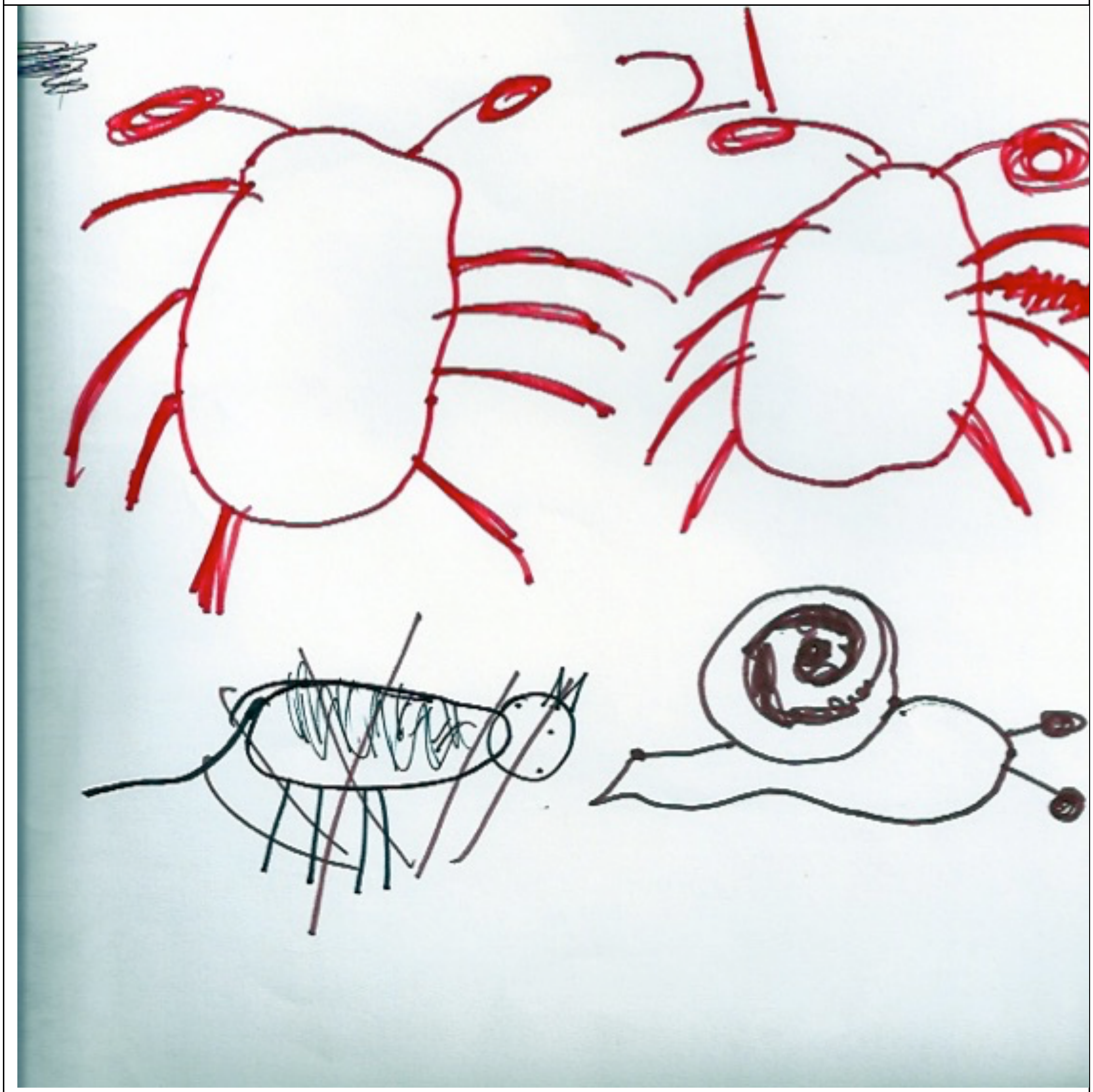
Figure 8.15: Child 18 "*and at one point she worked out she needed four"*.



Janine pointed to another child's graphic and said,

> *Danny* [figure 13: Child 19] *started with a crab* [pause] *and then he found it really difficult to work out 20 to 21. I asked him to go to the number line and find the number and I said how many jumps do you need? Once he saw he could do a jump then and it was one, he realised he could use a snail for that. But you can see he is not ready, he has not quite understood the tens and the counting on.* (Janine F.G., 15/5/2016)

Figure 8.16: child 19 "*he realised he could use a snail for that*".



Janine was assessing children's knowledge and acknowledging that Georgina used a lot of skills and Danny needed more support. Janine described another child's mathematical graphics.

> *Claudia* [child 20] *again, she just knew, she did 33 and she said I need 3 crabs. I said what do you need next and she said a crab, then she said no and she knew she needed 3 snails. Then Luca,* [child 21] *he knew that he needed 3 tens.* (Janine F.G., 15/5/2016)

In Janine's explanations of all of the children's graphics above she demonstrated that she was carefully listening, supporting, challenging and helping the children organise their thinking when they were trying to solve their problems. Janine concluded that these mathematical sessions were "*really interesting"* (Janine F.G., 15/5/2016) and *"It was a clear and quick way to show what children had the idea of the tens and how to make it a quicker way*" (Janine F.G., 15/5/2016). She again stated that she was surprised by the children's mathematical knowledge, "*Not that I expected the children to know so much"* (Janine F.G., 15/5/2016). Janine also reflected on the children who needed more help "*the children who needed more thinking about that* [...] *others still needed".* (Janine F.G., 15/5/2016)

Janine demonstrated that she was understanding children individually, knowing the children's mathematics. She was able to comment on how they approached mathematics and what they knew, "*Oh yes, and I know* […] *and I need to challenge him more and I now know that he is really systematic* (Janine F.G., 15/5/2016). She sometimes seemed surprised at what the children in her class did know. Janine used the word interesting a lot and I think this revealed that she was discovering the children's ways of representing mathematics and their individual mathematical thinking. Janine was also understanding that all the children had something to share.

Janine said that she found the book, *'One is a Snail, ten is a Crab'* (Sayre & Sayre, 2004) useful to give the children opportunities for mathematical problem solving and reflected that all the children could engage in problem solving and said, "*I have not done this with all the children yet I need to do it with the children who are not as able,* [pause] *not as confident in their maths"* (Janine F.G., 15/5/2016). Again, Janine is changing her use of language to be more positive about the children and changed her wording from, *not as able* to *not as confident*. In all of the transcript data above, Janine had not made a negative comment about the children and their mathematics. She was looking at children's learning positively and this was a major pedagogical shift.

Janine had used this book before with children (her previous class) and I asked her what had changed from the first time, "*what personally has changed from the first time, what has changed for you*" (E.C. F.G., 15/5/2016). Janine responded by saying,

> *This time I didn't do too much for them but read the story carefully and we did predictions and in the second session we did how could you make up a number. I gave them more input before I went over tens. This time I did it over two days as they had enough the first day thinking about prediction. My input was more*

successful for children this time although I do not think they approached it that differently. There was too much to think about doing it in one day (Janine F.G., 15/5/2016).

Janine again was giving children more time and being flexible. The first time she used this book for problem solving in mathematics Janine gave the children a model but this time, *"I said you can do any animals it does not need to be the ones in the book. Obviously, I suggested that a crab was good for ten because I could not think of anything else* [laughs] *but for four there are lots of animals they could choose* (Janine F.G., 15/5/2016).

Janine did not give the children a way to solve the problem which gave them more freedom to think of their own models. Her teaching was gradually changing to being confident enough to trust the children to explore mathematics in their own way.

8.4 "*One thing that is really coming out strongly is that this is supporting you in understanding children individually"* (Esme, F.G., 15/5/2016).

Janine had finished her explanation of the children's mathematical graphics in her classroom. Everybody acknowledged that Janine presented thought-provoking examples of children's mathematical graphics and it was fascinating listening carefully to Janine's explanation of the children's graphics. It seemed, at the time, that we were all excited and knew this group meeting was significant. Janine had suddenly revealed her knowledge, sharing reflections on her children's mathematical thinking and Esme noted two distinct aspects of Janine's practice that she thought had changed. Firstly, Janine, through focusing on children's mathematical graphics was now understanding children's individual mathematics, **"***One thing that is really coming out strongly is that this is supporting you in understanding children individually"* (Esme, F.G., 15/5/2016). Secondly, Esme said to both Janine and Millie, "*You are really talking more about what the children are doing and really listening"* (Esme, F.G., 15/5/2016).

In Janine's explanations of the children's mathematical graphics, she appears to be developing a sensitivity to children's own mathematical representations (Anders & Rossbach, 2015). Her pedagogical actions in these mathematics teaching sessions seemed fluent and responding to the children, in the moment, and trying to uncover their mathematical meanings and what mathematics they wanted to engage in.

As Esme had commented Janine was getting to know the children more by *really* (Esme, F.G., 15/5/2016) listening to *their* mathematics. Throughout Janine's discussion of each child's graphic she seemed to be uncovering children's mathematical funds of knowledge (Papandreou & Tsiouli, 2020).

She often, through the transcript, expressed her surprise at what the children could do. Janine knew the children individually, including their interests but she was also uncovering more about their mathematical knowledge and how they were working mathematically. She was individualising her teaching for every child and skilfully thinking through the input she was giving, sensitively supporting children, as she saw the need, as Esme previously said, "*beside the child, not over them*" (Esme, F.N., 16/2/2015). On more than one occasion Janine assisted children in writing their own number writing challenges and worked with that individual child, "*she needed support to carry on and we did that together"* (Janine F.G., 15/5/2016). Janine also discussed numbers with the children instead of immediately showing them the correct written form, *"He said that was a thousand and ten and we talked about what that would look like"* (Janine F.G., 15/5/2016). This perhaps gave the child a chance to think more about the number he wanted to represent. Janine understood when children were solving their own problems they may get frustrated and there comes a point you have to just tell them the number that they are seeking, so as not to disturb the flow of their thinking, *"I gave the number to her in the end as she started counting backwards"* (Janine F.G., 15/5/2016). That professional judgement of the right moment to step in is a skill to develop and may be needed in supporting children's mathematical graphics.

Janine seemed to be making the decisions and trialling and not relying on the set curriculum. She is developing her own understandings about the children's mathematics. Through researching her practice Janine created her own ways of knowing what works and how to support children's mathematical graphics. This seemed to have made her confident enough to show her children's mathematical graphics to us. At the beginning of this study, before the modules began, it appeared Janine had not thought about children's mathematical graphics, it may not have been something that was even a possible concept to her. Janine, in thinking together with the focus group, reading, trialling ways of teaching and not always following the set planned curriculum but following the children has led her towards personal ways of uncovering and encouraging children's mathematical graphics.

Not only did Janine's pedagogy change but as a result of this, the children in her class behaved differently now. Janine's comment about the children in her class three years before, when she gave them free access to pens and paper was, *"they* [the children] *do not do anything with it"* (Janine, F.N., 18/2/2014). Yet, now, children in her class were leading their learning, and experimenting; from the child who she said struggled with mathematics and the child who found writing difficult to the children who wanted to know more. The children were self-challenging; one child took three days on her own number line project. Importantly the children were using their own ways of representing and these were individual. It seemed they were not afraid to make a mistake because their mathematical graphics were not only accepted by their teacher but encouraged.

Although Janine was opening up her teaching and giving the children freedom to explore, for example, when she said, *"So, literally it was just free activity on the floor with pens and paper"* (Janine F.G., 15/5/2016), it was not laissez-faire teaching. Janine was sensitively listening and interacting with the children. She was honing into children's mathematical meanings, actively listening and skilfully responding. Janine throughout the explanations of all the graphics carefully wove in how she was interacting with the children as she said, "*this was my input*" (Janine F.G., 15/5/2016). At times, this input was deliberate and planned and at other times it appeared intuitive teaching, responding to the situation at the time. For example, Janine explained her initial input, the mathematical skill she was directly drawing the children's attention to, for example, counting, "*One of the things that I did was go over counting"* (Janine F.G., 15/5/2016). When she gave the children, in her class, the freedom to problem solve using a counting book she said, "*We had been working on counting in tens and so I went over counting in tens"* (Janine F.G., 15/5/2016). This input from Janine was relevant to the children's problem solving, at the time, and the children used their knowledge of counting in tens. When one child could not count on from twenty to twenty-one Janine supported him by showing him how he could work it out. Janine said, *"I asked him to go to the number line, find the number and I said how many jumps do you need?"* (Janine F.G., 15/5/2016). Showing the child a strategy, at the time he needed it, was intuitive teaching and not planned.

Janine used the children's ways of exploring as a teaching point for the class, *"He really was exploring numbers and we talked to the whole class about that"* (Janine F.G., 15/5/2016). Janine recognised the positive influence the children were having on each other and she encouraged this. "*Yes, and she was influenced by the other children"* (Janine F.G., 15/5/2016). and "*but she was watching other children"* (Janine F.G., 15/5/2016). Children were learning from other children and this appeared to inspire them to have a go at what the other children were doing in their mathematics.

Janine also modelled mathematics, *"I just put some Numicon out on the floor and I said, look I am adding these together, you can write it this way if you want, this is what I am doing*" (Janine F.G., 15/5/2016). Janine was moving away from intense direct modelling where children directly copy the teacher, this was highlighted when she was talking about children using addition and subtraction in the story book problem, *"I said you can do any animals it does not need to be the ones in the book"* (Janine F.G., 15/5/2016).

This focus group seemed a watershed moment when Janine revealed many examples of children's mathematical graphics and expertly talked through each of the examples.

8.5 Pedagogical Shifts

There were overarching pedagogical shifts that both Janine and Millie demonstrated, for example, slowly changing to a democratic approach in their mathematics teaching and within this giving more opportunities for child-initiated learning, children were leading their learning. Through the focus group session above, Janine, in particular, had demonstrated, in detail, teaching strategies on how she was teaching to support children's mathematics and their graphics.

Millie also revealed pedagogical shifts, some were similar to Janine's, for example, both were searching for children's mathematical meanings and they had adjusted their classroom environment, ways of working and overall culture to allow for children's mathematical thinking. Below is a chart demonstrating the more detailed pedagogical shifts that Janine and Millie appeared to be making in their teaching of mathematics (figure 8.17). I have compiled this list from the analysis of Janine and Millie's descriptions of their teaching in this chapter.

Figure 8.17: A chart to show Millie and Janine's pedagogical shifts.		
From	To	Examples of shifts in practice
1.Only group and whole class work direct teaching of mathematics, with expected outcomes	Democratic teaching, degrees of freedom, Trusting children.	*"It was just free activity"* and *"then I left them with a large piece of paper and pens I put some beads out"* (Janine F.G., 15/5/2016).
2.Copy recording	Supporting children's mathematical graphics. Using the language that give children the freedom to have a go at their own graphics and ways of writing mathematics.	*"Put it on paper"* (Janine F.G., 15/5/2016).
3.Calculation; getting frustrated that children '*don't get it'* when trying to directly teach to the objective. Step by step.	Understanding that children have knowledge of calculation that the teacher has not directly taught. Children's emerging knowledge.	Uncovering children's calculation, "*Then he said I can do it a different way 10 add 10 and 10 make 30. Actually, I do know that 10 x 10 =100. So, he was building this up"*. (Janine F.G., 15/5/2016).
4.Counting and number recognition, working with numbers Step by step approach teaching 0-5 and then 5-10. Numbers rarely get discussed beyond 10, sticking to the EYFS goals for Reception	Letting children explore with whatever numbers and areas of mathematics they choose. Supporting and extending. Not putting a cap on what they are exploring. Going beyond what is expected in the framework.	"*she was counting above 20 and she got up to 29 and needed a prompt to go on to the 30"* (Janine F.G., 15/5/2016). *"He said that was a thousand and ten and we talked about*

		what that would look like" (Janine F.G., 15/5/2016).
5. Knowing what children can do in set tasks.	Knowing the children's interests and how they work, individually.	"*another boy Joshua is good with numbers mentally and he is fascinated by buses and numbers on buses"* (Janine F.G., 15/5/2016).
6. The teacher plans and leads the mathematics lessons. The children follow her rules with few opportunities to follow their own agendas	Children leading and the teacher responding to this positively. The child knows the teacher is a resource that she can use.	*"Another girl put some numbers down and she got me to say them"* (Janine F.G., 15/5/2016).
7. Children being regulated to do the mathematics expected from the curriculum.	Children feeling confident and free to do things that are not on the teacher's agenda. The teacher allowing this. (a larger degree of freedom)	*"And some of the children actually got carried away with drawing"* (Janine F.G., 15/5/2016).
8. Calculation symbols and signs taught in set lessons and children copy.	Identified that some children take a very long time to acquire the understanding of conventional mathematical symbols and signs and allowing for this.	*"I could have sat Milo down and told him the right way to do it (standard calculation). Throughout the year he just got there himself. But I could have just sat him down and showed him the right way that then he would have copied and used it. But I think that is the conceptual bit we are missing out aren't we?"* *"the children who needed more thinking about that"*

		(Millie, F.G., 15/5/2016)
9. Set times for mathematics and few opportunities for individual children to explore their mathematical interests over an extended period.	The teacher allowing and encouraging the children to carry on with their mathematical thinking over days.	*"then she went back to it the next day and ran out of paper. So, I said use some more paper and she went and stuck it on the end and that went on for a bit and she added more paper and over the next few days she got really excited and carried on with her number line and eventually she got up to 114."* (Janine F.G., 15/5/2016).
10. Children work by themselves mostly after an input from the teacher.	Peer modelling; children learning from other children	*So, they were looking at numbers, sequencing and learning how to write them from other children.* (Janine F.G., 15/5/2016).
11. Detached teaching; The expected objectives take over and are impersonal.	Attached teaching; takes into consideration the ideas and interests of the child individually including their home and community life.	*That was obviously what he was interested in* [time, clocks] *and he was using o'clock. He did 11 o'clock and 10 o'clock so he was writing a number and then saying the o'clock.* (Janine F.G., 15/5/2016).

12. Only accepting the right symbols and traditional ways of representing mathematics (literal mathematics)	Accepting children's ways of representing including: scribbles; crossing out; missing out numbers; drawing etc.	*Amber is not confident and okay she did miss out the eleven there.* *And that is his 9* [not the conventional way of writing 9] (Janine F.G., 15/5/2016).
13. Direct modelling of mathematics e.g. the teacher writes 2+3=5 (perhaps drawing 2 apples and 3 apples) then the children copy this.	Indirect modelling The teacher models different ways to represent mathematics, informally throughout the day. The teacher does not expect the children to copy her representation or a set model (Carruthers and Worthington, 2006; 2011).	*You can do any animals it does not need to be the ones in the book.* *I just put some Numicon on the floor and I said, look I am adding these together. You can write it this way if you want, this is what I am doing.* (Janine F.G., 15/5/2016).
14. The teacher gives children set problems to solve.	The teacher encourages children to think about their own mathematical problems. She discusses and supports their own ways to solve problems.	One child asked, "*How big would a classroom have to be if there were 108 children take away 8*" (Millie, F.N., 24/6/ 2015). Another child asked, "*How many children have you taught*? (Millie, F.N.24/6/2015) Children's mathematical questions are seen as different from adult's, "*They are more interesting and for me more challenging"* (Millie, F.N., 24/6/2015)

15. Emphasis on mathematics as a subject discipline of right and wrong answers.	Looking at children's mathematical learning positively, valuing the emergence of children's mathematical thinking.	*"He was fascinating, I just watched him slowly change throughout the year where the symbols made sense to him"* (Millie, F.G., May 2016).
16. Only practical resources are provided for children to represent their mathematical thinking.	There are always opportunities for children to use graphical equipment to support their mathematical thinking.	"*My class are using paper and pencil all the time now, all kinds of paper and writing implements outside and in."* (Millie, F. G., 15/5/2016).
17. Children's mathematical graphics are not noticed.	Children's mathematical graphics are encouraged and the teacher notices and analyses the children's mathematical graphics.	'*She has used an iconic method to work out multiples of five, she put five dots in a circle and repeated this to solve the multiplication problem*" (Millie, F.G., 8/3/2016).
18. The language used about children's mathematics is a deficit model for children who are seen not to be doing as well mathematically as other children.	The language used about children and their mathematics comes from an additive perspective. All children are seen as competent.	"*I have not done this with all the children yet I need to do it with the children who are not as able,* [pause] *not as confident in their maths"* (Janine F.G., 15/5/2016).

8.6 Conclusion

This chapter uncovered Millie and Janine's pedagogical approaches to children's mathematical graphics; how they listened and what particularly they focused on. Some of Millie and Janine's pedagogical changes were similar to the play pedagogy of the Nursery School teachers in chapter 7. For example, Millie and Janine were engaged, to some extent, in child-initiated learning and similar to Esme and Sue's pretend play examples (7.1.4) the children were leading. It was also underlined that they both actively listened to the children's mathematics in their classes and in doing so were knowledgeable of the children's individual mathematical thinking. In uncovering the children's mathematical graphics in their classrooms, they were gradually gaining detailed knowledge of the graphics.

Although Millie centred on and reflected on her mathematics teaching she, as a leader of mathematics within her school, was also concerned with school politics and the wider issues that affected children's mathematics including government agendas and Ofsted requirements. The changes in her teaching were supported, in her school, but not embraced by everybody. It is these wider issues and dilemmas that are discussed by Millie and Janine in chapter 9.

Chapter 9: Millie and Janine Discuss Pedagogical Issues Around Children's Mathematical Graphics.

9.1 Introduction

In this chapter I selected the data where Janine and Millie focused on teaching dilemmas and wider considerations around the pedagogy of children's mathematical graphics this was also to reflect on one of my research questions. I posed the question in chapter 4 asking what difficulties teachers might face in teaching in a way that supports children's mathematical graphics. This data is mostly from the final focus group (26/2/2017) where my intention was also to give Millie and Janine space to finally reflect on what they thought was important in supporting children's mathematics which linked into my main research question which was; what cultivates the existence of mathematical graphics in early years' classrooms? Teaching issues arose, for example, space to listen and the importance they placed on small group teaching, trust, the new mathematics curriculum and the imbalance of the curriculum towards mathematics and literacy assessments.

W.R. = written reflections of the Reception and Nursey School teachers from their first assignments.
F.N. = field notes which are my personal notes and observations from November 2015 to February 2017.
F.G. = focus groups
I have written Millie and Janine's quotations in italics to highlight their reflections. There are six main themes and I label each of them using the words of the teachers.

9.1.2 *"They have brought a lot of stuff"* (Millie, F.N., 16/11/2015)

Millie, Janine and Esme were invited to do an early years' workshop at a local community mathematics conference. Millie and Janine were leading this with Esme supporting. Millie and Janine talked about the children's mathematics from their classrooms. This workshop brought up the issue of resources and practical equipment in mathematics teaching.

As Millie and Janine were getting ready for their workshop they looked in the Secondary School and Primary School mathematics workshop rooms and they were full of resources, mostly brightly coloured and plastic; it might have been what Jess described as, "*Catalogue Mathematics*" (Jess F.N., 24/6/2015) previously in an SLE meeting. She said this was where schools buy lots of materials from mathematics education catalogues to use in their mathematics teaching throughout the whole school including Reception. Millie and Janine were concerned that they had not any standard resources and that their workshop looked lacking, in some respect, and therefore they were a little anxious and nervous of the expectations of the teachers who would attend their workshop. Millie worriedly said, "*The teachers are probably looking for activities and games, look at the other rooms, they have brought a lot of stuff"* (Millie, F.N., 16/11/2015). Esme and I reassured Millie and Janine that their focus on children and their mathematics was going to be very useful and they might be giving, perhaps, an alternative perspective on early years' mathematics.

Millie began her presentation by talking to the audience about the differences in the Secondary School and Primary School workshops regarding resources compared to their workshop and said, "*We do not have lots of resources deliberately. I am going to talk about children's mathematics and their questions*" (Millie, F.N., 16/11/2015). During the talk a discussion arose when one of the teachers in the audience asked about practical resources. Janine suggested that practical resources came before you encouraged children to use their mathematical graphics, "*practical and then graphics"* (Janine, F.N., 16/11/2015). Esme put forward that she thought that children benefited if both practical and graphic resources were available, and, *"it is fluid and not step by step*" (Esme, F.N., 16/11/2015). However, Janine also said to the participants, "*access to pens, pencils and paper is crucial"* (Janine, F.N., 16/11/2015). Later she said she felt pressure by the workshop participants, reiterating what Millie had said previously, *"I think some of the teachers expected resources*" (Janine, F.N., 16/11/2015).

Janine and Millie appeared hesitant about what they were going to say at the workshop. From the field notes above, it seemed they were concerned that they were presenting something different to what teachers wanted. My feeling at the time was that teachers might be there to get ideas that can easily be incorporated into their teaching. Although, at this workshop, Janine stated you present children with practical opportunities first, in her own mathematics teaching (Janine F.G., 15/5/2016) she seemed to be using a combination of giving children opportunities to use their graphics and practical resources. This workshop highlighted the dilemma pertaining to where practical resources might fit into mathematics teaching. Teachers might expect that practical resources were a step before you support children's own graphics. This is something that was also discussed by one of the Reception teachers, Melinda, in the second data collecting phase of the study. Melinda disagreed with one writer's comment about giving children practical materials to work with first in mathematics before providing writing materials. Melinda explained, "*I was surprised to find that children who would normally need a lot of adult support found astonishing ways to represent their mathematical thoughts, if I had taken away paper and pens I would not have gained this kind of information"* (Melinda, W.R., 15/9/2014). Janine and Millie both seemed to be moving away from solely using practical resources as they now encouraged children to use graphic tools to represent their mathematical thinking. To embrace a pedagogy that supports children's mathematical graphics it seems to be important to question your thinking on the place of practical apparatus in teaching early mathematics and this was a topic of debate in the review of the literature (2.5).

9.1.3 "*It is a long way off graphics though*" (Millie, F.G., 26/2/2017)

This final focus group (F.G., 26/2/2017) was in short snippets, often disjointed as Janine and Millie were, I think, pondering different aspects of their practice and discussing mathematical graphics. Esme and I listened to Janine and Millie's thoughts. Both Janine and Millie considered the new curriculum. This is something that was recently introduced by the English government. Millie explained,

> *Part of the new curriculum states that maths needs to make sense to the children. I think that children's mathematical graphics helps the sense making. However, in the SATs* [Standard Attainments Tasks] *paper, don't expect that, they have to work the problem out in a set way not their* [the children's] *own way.* (Millie, F.G. 26/2/2017)

From Millie's explanation there seemed to be a contradiction, the official new curriculum was talking about children making sense of mathematics, contrastingly, it appeared, from Millie's explanation, that the children are not encouraged to make their own sense in the SATS. The children seemed to have to use set ways of working out mathematical problems (although it seems that any method with a correct answer gets full marks but if the answer is incorrect, only the set methods will gain a mark).

Janine has observed the change in her school due to the new curriculum as she said, "*I have noticed more of a move, within my school, to models* [meaning showing the children a set way to do, for example, addition and subtraction] *and manipulatives because of the new curriculum even in year one and two"* (Janine, F.G., 26/2/2017). I asked if the children were able to create their own models. Janine replied, *"I am not sure that they are allowed to do this. It looks like they are all doing the same thing*" (Janine, F.G., 26/2/2017). Millie added that the new curriculum was more enlightened. However, she continued, "*It is a long way off graphics though*". She went on to say, "*In my school it is using a set model for maths*" (Millie, F.G.., 26/2/2017). Millie again returned the conversation to how children's mathematical graphics might be important for the whole school and children's own sense making of mathematics which impacted also on Key Stage Two. After a short silence Millie ended the conversation with, "*They are using squared paper in Key Stage One to structure the maths more"* (Millie, F.G., 26/2/2017). This seemed a long way from the discussions in the previous year where Millie and the other SLE's agreed, "*blank paper is seen as an opportunity"* (Esme, F.G., 24/6/2015). From this meeting, Millie and Janine seemed to be seeing the new mathematics curriculum as hindering opportunities for children's mathematical graphics.

Although Millie claimed the new curriculum is, "*more enlightened*" (Millie, F.N., 26/2/2017) as it purported that children have to make sense of mathematics, it seemed to have restricted the children to set models and squared paper where the children have to put their numbers in boxes and this may have restricted the scope for children's drawings and invented symbols. The assessment and curriculum planners appeared not to be giving the children the freedom to choose their own sense making and use their own graphics to aid understanding. This could make it difficult for mathematics leaders, like Millie, to influence the whole school to take on board children's mathematics and their mathematical graphics.

Millie and Janine were continually building up their individual knowledge of children's mathematics and their graphics but there were still barriers to overcome within the new curriculum and convincing the SLT. Janine and Millie's own classroom research was the vehicle that drove their search for children's mathematics and children's mathematical graphics. It was not easy, as the government educational focus was turning to what Millie and Janine thought restricted children's own mathematics to thrive. They were surrounded by a teaching community that was caught up in focusing on top-down assessment procedures, school issues or the latest government initiative. In this pressurised political culture how sustainable is a mathematics pedagogy that support children's mathematics and their mathematical graphics?

9.1.4 *"I really do not want to go back to whole class teaching"* (Millie, F.G., 26/2/2017)

I asked Millie and Janine to think about where they were in their practice and their thinking about children's mathematical graphics. This provoked Janine and Millie to discuss practical issues in their teaching such as the importance of small groups, space to listen to children and the trust and autonomy they had within their school. At this focus group, reflecting on her practice, Janine started the conversation by saying,

> *I think I am working on* [...] *I think Millie is further on than I am on this, I think having more small group times so you can have more conversations with them* [the children]. *So that you can split up more and do something more specific and not so much large group times.* (Janine, F.G., 26/2/2017)

I think Janine was trying to structure her classroom for listening to the children's conversations so that she could have a deeper insight into their mathematics than she presently had. Janine went on to explain further, "*The student* [Post Graduate Certificate in Education student] *has helped, at times we have been able to split into three groups and that has been really helpful"* (Janine, F.G., 26/2/2017). I think Janine was meaning that there were three adults, including the student and the classroom assistant and each could have a group of children to listen to and she added, "*And, I think that the first activity time when they all come in again is a useful time to see what the children are doing and to have conversations with them*" (Janine, F.G., 26/2/2017).

Janine explained the importance of all children being involved and this, Janine believed, happens when you have smaller groups of children, "*Small groups are so much better. They can all have a go rather than having a few children up at the front. They can all have their hands on the resources. It is like what you do in the nursery with the key groups"* (Janine, F.G., 26/2/2017). Millie agreed with Janine, *"that's where it came from looking at nursery practice.* […] *I am really struggling because I do not want to go back to whole class teaching in year one* [Millie was going to be teaching in a year one class in the next term] *and that is what it is going to be*." This led Millie and Janine into a discussion about the need to have more adults in the class so the teacher can individualise and have conversations with children and they said that did not happen often in the Key Stage One classes. They both agreed that the ratio of children to adults made a difference, *"The issue is further up in the school* [...] *it's the numbers. I can split the class up, I have an LSA and a student now and I really see the difference* (Janine, F.G., 26/2/2017).

Following on and linked to their discussion of how beneficial small group teaching is, Janine discussed another aspect of her changing practice, the space you work in with the children,

> *That is another issue the environment and actually where you are going to work. We were doing a phonics session. One person went out in the cloakroom there is a little bay there. Two of us* [adults] *were on the carpet with a group of children and I was trying to play a game, an interactive game on the whiteboard* [Janine laughs] *and it was just quite difficult for the children to see and hear because there was another group really involved and the student was doing a great group game over there. It is just like you say having space. It is those things you have to consider as well, having space to concentrate.* (Janine, F.G., 26/2/2017)

Janine acknowledged that Millie had a good listening space, "*but you are lucky you have two separate rooms*" (Janine, F.G., 26/2/2017) and Millie agreed. Janine explained her concern, in trying to find a quiet space in her classroom to listen to the children's mathematics. Both teachers viewed having a quiet space to listen as a key aspect of their developing mathematics pedagogy. It is difficult to listen to children if you cannot hear them and for Janine, ways to structure her classroom for listening seemed vital to her mathematics teaching.

9.1.5 *"For some people it is letting go of that control"* (Millie, F.G., 26/2/2017)

Millie again pondered on her situation on teaching a year one class next year and stated that she would not be totally confident with the Learning Support Assistant taking a group and Janine explained, teachers are hesitant, *"to trust the other worker"* (Janine, F.G., 26/2/2017). Janine had confidence in her LSA and described her LSA's skills which I did not quite catch on the tape. "*You have to do it gently, and not put too much on the other staff as we are paid much more. You have to keep a balance* (Janine, F.G., 26/2/2017). Millie agreed with Janine and added that she would find it difficult to trust a LSA with leading a small group for mathematics, "*I am not sure I would be confident fifty/fifty say?"* (Millie, F.G., 26/2/2017).

Later, Millie went back to confirm, "*I like small groups*" and questioned why they did not focus on small group teaching before in her Reception class (Millie, F.G., 26/2/2017).

Millie put forward what she had observed, *"For some people it is letting go that control. What we experience is people like to be the one in front of the class"* (Millie F.G., 26/2/2017). Millie was talking about other teachers in her school and Janine nodded in agreement. Millie turned to Janine and confirmed with Janine that letting go of control, "*That's what makes the difference"* (Millie, F.G., 26/2/2017).

From these pieces of conversation there was an issue with assistants and pay and how much they do. There was also the issue of the teacher sharing the responsibility of hearing individual children and therefore giving the assistants more responsibility. Listening to the children's mathematics may mean having faith to spread that listening ear and trusting other adults in the class and that Millie and Janine thought that brought in complexities. Millie's point that some teachers found it difficult to let go of control and they preferred to be the *teacher at the front* was a phrase that Nadira (Nursery School Teacher in the study, chapter 6) also used in her writing as she stressed the difficulties of not hearing individual children because of having a large group of twenty-six children, "*it is largely impossible for the teacher to know what each child has understood in a whole class session, especially when the group size is sometimes as big as 26 children*" (Nadir, W.R., 15/9/2014).

It seemed that Millie and Janine changed their position in the classroom from teacher at the front to having small groups, which they said gave the children opportunities to have conversations with them and hear their mathematics.

9.1.6 *"I am trusted to do what I want to do"* (Millie, F.G., 26/2/2017)

Millie pinpointed that trust from the school is an aspect that is crucial to giving her autonomy to continue experimenting with listening to children's mathematics and encouraging their graphics. She said,

I think we have more autonomy in our Reception classes. I am trusted to do what I want to do, you probably have that too, [Janine nods in agreement] *from the fact that they, the management, do not know early years so it is handed over and if nothing goes wrong* *then that trust will stay. Whereas in other schools I do not know whether they get the same choices.* **(**Millie, F.G., 26/2/2017)

Millie and Janine considered the schools they knew and (they were both working in other schools on research projects) they identified one school they said had this trust they were talking about. They explained that this school had been able to engage more with accessing and working in ways that gave children the freedom to use their own graphics to work out mathematics problems. They thought the headteacher was extremely supportive of the Reception teachers and encouraged teacher initiatives. Janine remarked, "*Brownsgate* [pseudonym] *school has the trust, the head is so open to listening to the teachers*" (Janine, F.G., 26/2/2017).

9.1.7 *"Am I listening to them or am I taking it where I want to go?"* (Millie, F.G., 26/2/2017).
I wanted to delve a little further about listening as Millie and Janine both seemed to think that was important. I asked, "*So you are listening to the children?"* and *"How are you listening?"* (E.C., F.G., 26/2/2017). Janine replied by saying, "*You are trying to illicit something from them"* (Janine, F.G., 26/2/2017) and Millie responded, *"It depends whose agenda it is as well"*. She added, *"There is a fine line between scaffolding and where you are taking it. Am I listening to them or am I taking it to where I want to go?"* (Millie, F.G., 26/2/2017). Janine and Millie stated that literacy and mathematics were predominant in the Reception curriculum and they considered that had a negative effect on them listening to the children, possibly because there were set expectations for assessment, *"The literacy and the maths are prominent* " (Millie, F.G., 26/2/2017) and Janine emphasised, *"Especially now we are collecting data"* (Janine, F.G., 26/2/2017). Janine and Millie agreed that it was cumbersome to have another agenda imposed on them and Janine concluded, *"It is really hard, particularly when children have slower starting points, there is a big pressure to move them on*" (Janine, F.G., 26/2/2017).

9.2 Summary

In this chapter some of the complexities of Millie's and Janine's classroom life, that impacted on their mathematics teaching, surfaced. Listening was a theme that was dominant within Millie and Janine's discussions which included a quiet space to hear children. Millie and Janine highlighted that teaching small groups helped adults to listen to individual children's mathematics. They brought up the issue of teachers letting go of control and not always being the teacher at the front of the class and stressed the importance of having more adults in the class to tune in and listen to children mathematics. Assessments and data collection was another barrier in responding to children's mathematics and listening to their agendas. Janine suggested that designing your classroom for listening to the children's mathematics was an important consideration to benefit the children's access to their mathematics. Trust from the school leaders was also a factor that Millie and Janine considered vital to allow them to experiment with pedagogical practice that supported children's mathematical graphics.

9.3 Conclusion of the Two Reception Teacher Case Study Chapters

Millie and Janine's reflections and conversation in the focus groups were grounded in their everyday practice and knowledge of their schools and to a certain extent some schools in their community. Their pedagogy of children's mathematics evolved slowly over time as they had also time to reflect and consider mathematics teaching within a learning community, firstly through the modules and then through our focus group meetings. There were opportunities afforded to them to attend national mathematics conferences, research groups and support other teachers through their role as SLEs. This may have helped shape and enrich their own practice, over the three years and five months of the study, although that connection between their practice and other outside influences is less clear. In chapter 6, the Reception teachers group case study, Janine and Millie's written reflections reflected a curriculum-led approach to teaching; a model which privileged set objectives instead of listening to children and their ways of understanding mathematics. Over the data collecting period of three years and five months, Millie and Janine's mathematical teaching practice became increasingly less influenced by what they believed to be the expected school mathematics. Instead, the two Reception case study teachers became teacher-researchers as they researched their own practice and looked to develop their teaching. In the beginning they were following the set curriculum and ways of teaching that their school in general accepted for mathematics, this was particularly poignant when Millie voiced her thoughts about copying what others do and not questioning that. As teacher-researchers, within this study, they followed their individual pathways to realising children's mathematical graphics and adjusted their teaching, even though that may have been quite radical within their school situation. Millie and Janine were shaping their own mathematical pedagogies, in collaboration with others and individually. Their evolving practice and constant reflections brought up useful ways to work where children are enabled to use their graphics to solve their mathematical problems.

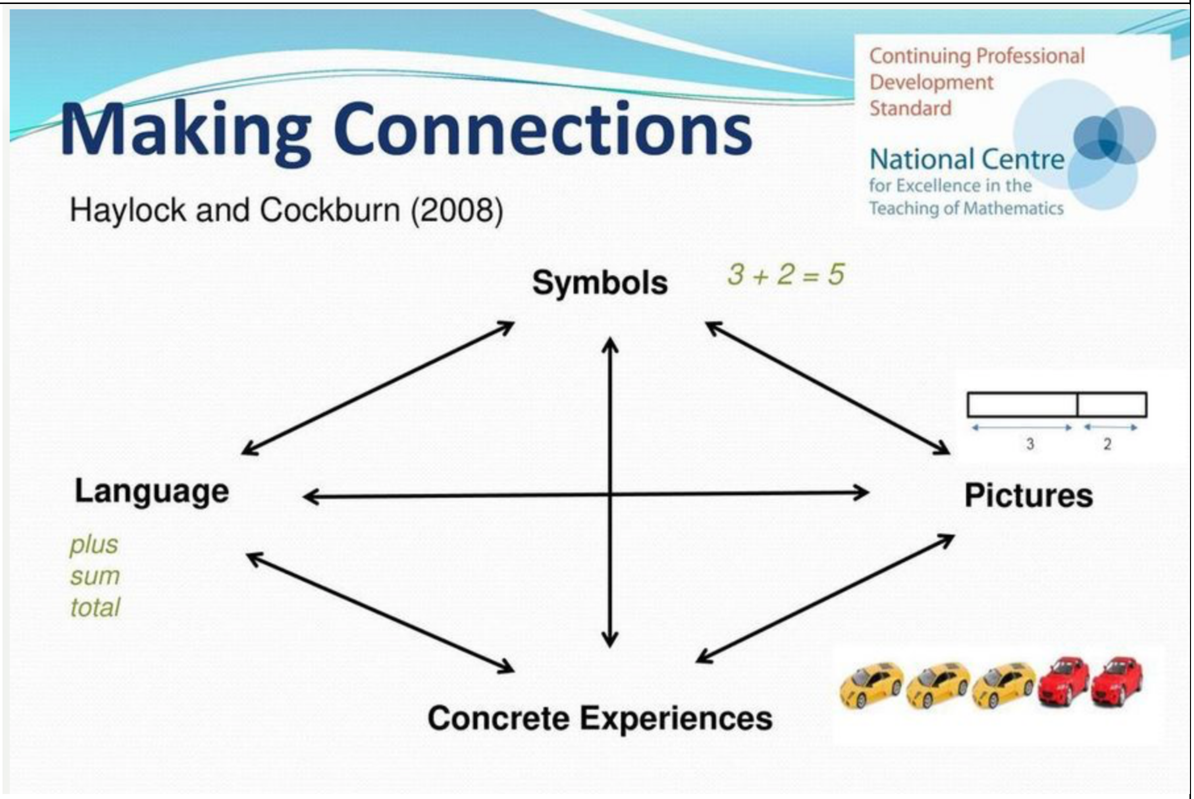
There were dilemmas, set-backs, and frustrations and it seemed that Millie and Janine were constantly swimming against the tide of government agendas, SATs, and imposed assessment procedures. These external and political barriers were also noted in chapter 6, from the seven Reception class teachers' data. Millie, especially, emphasised that she found her colleagues did not always understand children's mathematical graphics even although she pointed out how the children benefited and understood *their* mathematics. It seemed she was constantly trying to convince others of the advantages of supporting children's mathematical graphics. The struggle appeared to be that she was proposing to her colleagues fundamentally different mathematical teaching than they were engaged in, especially in Key Stage One and Two.

Both teachers, as their practice developed in supporting and understanding children's mathematical graphics, reported that the children, in their class, were more engaged in mathematics than they had experienced in their previous classes. Also, the two teachers often stated that the children's *understanding* of mathematics grew as they used their graphics as a tool for thinking.

There were many times over the nineteen months of talking and listening to the two Reception teachers that they expressed that children's mathematical graphics were radically different from what was happening in the mainstream of the current English education system in mathematics. Even when they said the new mathematics curriculum was more enlightened, the two teachers felt it was far removed from the more democratic teaching approaches needed for listening to children's mathematics.

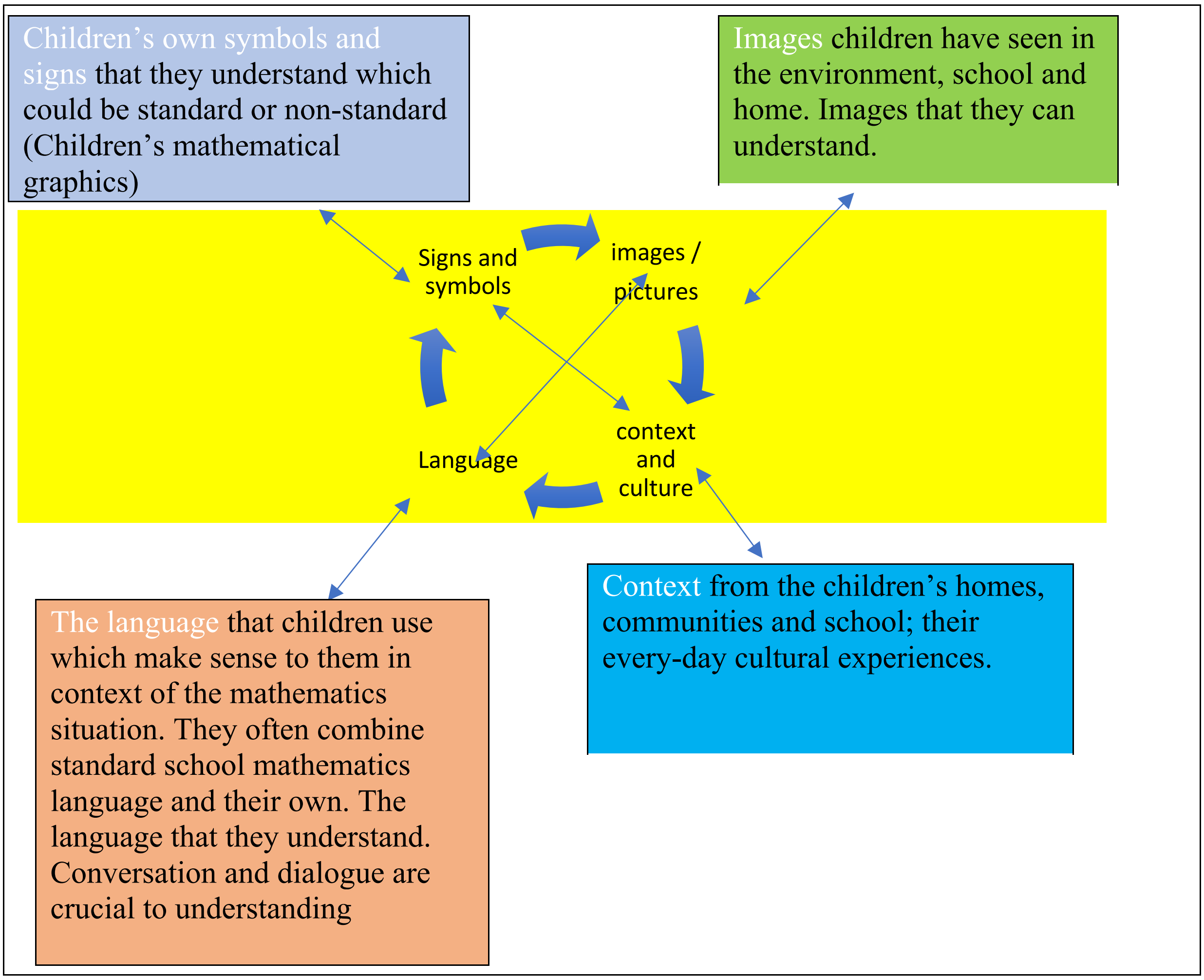
The major overarching shift that Janine and Millie appeared to make was, moving from an adult-centric approach to pedagogy to a child-centred socio-cultural model. This gave the children opportunities to develop their own mathematics through their own questions, signs and symbols. Many of the current mathematics teaching models that are advised in England both in Initial Teacher Training courses, some of which use Haylock's model (2010, p. 26) and the National Centre for Excellence in the Teaching of Mathematics (NCETM) which adapted Haylock's (2008) model (see figure 9.1) are adult-driven emphasising standard notation and copy recording where children are not using their own thinking and graphics.

Figure 9.1: Haylock and Cockburn Model (2008) adapted by the NCETM.



A later model of mathematics pedagogy posted on the NCETM website, reflecting the Mastery Teaching approach (Drury, 2018) although more detailed does not seem to encompass children's perspectives and this was similar to the adapted Haylock model as discussed previously. This later model is also a pedagogical adult-centric model which only appears to support standard mathematical symbols and signs and was the type of mathematical pedagogy that all the Reception class teachers seemed to be bringing to the conversation at the start of the mathematics modules. However, when the Reception teachers tuned in and listened to children and their mathematics and observed the children's perspectives they highlighted the importance of children's cultural knowledge and how they also learned from other children. I am proposing that Millie and Janine, for example, moved from a constructivist/connections pedagogical model to a child-centred socio-cultural pedagogical model of teaching mathematics. I illustrate this socio-cultural model below (figure 9.2) This is developed and expanded from figure 9.1: Haylock and Cockburn Model adapted by the NCETM, 2008.

Figure 9.2: My proposed child-centred socio-cultural mathematics teaching model.



The shifts Janine and Millie made from the adult centric Haylock/NCETM model to the child-centred socio-cultural model of mathematics teaching were as follows:

- Signs and symbols: From standard 1+2= 3 to include children's own representations of their mathematical thinking and the many ways children choose to represent
- From images and pictures: to include cultural contexts that are unique to the child, in their imagination which includes play.
- From the language of standard mathematics: to include children's language and conversation.
- From concrete images of objects in front of the child: to include cultural images and imaginary images that are not present. Practical resources are used in combination with opportunities for children to use their own graphical representations.

Millie and Janine, over three years and five months developed their individual ways to support children's mathematics and, in turn, children's mathematical graphics that were integral to their classroom practice. Their teaching was steeped in the authenticity of everyday classroom life and showed how it is possible, within Reception class teaching, to move towards a pedagogy that values, develops and listens to children's mathematics and their graphics.

Chapter 10: Conclusions, Main Findings, Implications and Recommendations

In this chapter, I address the findings and conclusions of my four research questions (10.1) and consider implications of the study. In section 10.1 I re-introduce my research questions and in 10.2 to 10.7 I present the findings. In 10.8 I summarise the findings regarding the discontinuity in mathematical pedagogy between Reception and Nursery Schools and offer recommendations. 10.8 to 10.11 puts forward the contribution to knowledge, future direction, challenges, ethics, limitations and conclusion of this study.

10.1 Introduction

The overall aim of this study, as stated in chapter 1, was to find out how teachers can support children's own mathematical graphics in the Early Years Foundation Stage (DFE, 2012) focusing on Reception and Nursery School teachers.

I was initially seeking *a* pedagogy of children's mathematical graphics but I realised through the study that all the teachers were approaching their teaching in *their* way. It became clear to me that the pedagogy of children's mathematical graphics, within this study, was individualised pedagogies that belonged to the teachers and best fitted their classrooms at the time. All fifteen teachers' perspectives added to the knowledge of pedagogies that support children's mathematics and children's mathematical graphics. It was evident that the two Reception case study teachers, in chapters 8 and 9, during the study had moved away from only transmission teaching where teachers impart a set mathematics curriculum. In taking a democratic approach to teaching they had uncovered children's mathematical knowledge and also gave insights on how to support children's mathematical graphics. I now address my research questions. The following questions were posed and addressed in this study:

Firstly, my broad and over-arching question was:

- What do teachers consider cultivates the existence and understanding of young children's mathematical graphics in early years' classrooms?

My sub-questions were:

- What are the difficulties teachers face in teaching in a way that supports children's mathematical graphics and how can these difficulties be resolved?

- What are the conceptual shifts in practice that a teacher may have to consider as she/he embarks on supporting children's mathematical graphics?

- How do teachers feel the children benefit, if at all, from using their graphics to support their mathematical understanding?

- If teachers support pretend mathematical play, where children use their mathematical graphics, how do they interact with children to support and enhance their play and mathematical thinking?

10.2 What Do Teachers Consider Cultivates the Existence and Understanding of Young Children's Mathematical Graphics in Early Years' Classrooms?

From the findings of the analysis of the data, I have summarised the key themes into six aspects of practice that the teachers' writings and conversations underlined developed the existence of children's mathematical graphics in their classrooms. These were: opening up mathematical teaching; listening; relationships; play; trust and professional learning. I discuss each of these aspects, in turn, below.

10.2.1 Open mathematics

Opening up mathematical teaching to the unexpected and giving children opportunities to experiment was prominent in the voices of the Reception class teachers. The language of *opening up* and *open mathematics* was regularly used when the Reception teachers were describing their changing practice. How they portrayed this change was, for example, turning closed questions to open questions, encouraging children to devise their own questions and moving beyond the set mathematics curriculum. The opening up of mathematics in their classrooms also meant that they needed to be flexible, making time for children's mathematical enquiries and allowing the children to organize their own enquiries. The open mathematics they reflected on became a significant cultural change and it appeared clear, within the study, that the teachers felt this open environment both physically and psychologically was needed first in order to cultivate the existence of children's mathematical graphics.

10.2.2 Listening

Listening to the children was a key theme arising from the data. An underlined aim of the two case study teachers was acquiring knowledge of children's mathematics by listening and tuning into their mathematical thinking. Knowing children's cultural mathematics was also embedded in five of the Nursery School teacher's mathematical practice. As Harry (7.1.2) said, it is more than just having information about the children but learning about their life outside school and what interests them. Knowing the children's cultural experiences aided the teachers understanding of the mathematics that children bring to school. Listening was also a major theme in the two case study Reception teachers as they discussed spaces to hear the children so that they could listen closely to the children's mathematics. As Millie (6.1.6) stated, it was crucial to listen to the children's mathematical agenda and not just what the set curriculum expects.

The teachers expressed that listening to each child individually was important. Janine and Millie thought that small groups were best because they said you can hear children individually and concentrate on what they are saying. Small groups and individualised teaching are adult intensive and this meant perhaps, that the teachers needed to share the listening with their teaching assistants but the teachers questioned if the assistants were skilled enough to do this. Small group and individualised teaching might be difficult for schools that rely on whole-class teaching and being what Nadira said, "*the teacher at the front*" (Nadira, W.R., 15//2014). Millie and Janine added that some teachers preferred whole class teaching because they did not want to lose control; they may feel less confident in giving more autonomy and trust to the children. Designing your classroom for listening to the children's mathematics meant that the teachers were in a better position to access the children's mathematics. This study confirmed that listening to children's contexts and prior to school and home learning are not seen as second to school learning but a crucial springboard for understanding their mathematics and possible trajectories.

10.2.3 Strong teacher-child relationships

Strong teacher-child relationships underscored the environment that supported children's mathematical graphics. How the teachers interacted and their connection with the children was vital, "*not over them but with them"* (Esme, F.N., 16/2/2015) is a phrase one of the Nursery School teachers used to emphasise equal relationships between teacher and child.

Similar to the examples of pretend play in the Nursery School teachers' group study, Janine, in the case study of the two Reception teachers, referred to children directing her within the classroom. The children found the teacher a useful resource and the children knew the teacher would help with their agenda even although the teacher may not know where the child was going with their thinking. Millie said children need resilience because mathematical problem solving can become quite difficult and challenging as the teacher probes and the children also self-challenge; sometimes staying with their line of mathematical thinking for days and returning after some time to think further. Therefore, the relationship has to be strong between the teacher and child as the child trusts the teacher to respond to and encourage their mathematics to keep their line of mathematical enquiry continuing.

The Key Person Approach, which was dominant in the Nursery Schools, also highlighted important relationships. For example, because of knowing the children and their family life, there seemed to be a closer bond between the teacher and the child than was portrayed in the Reception classes in this study. The Nursery School teachers tuned into the emotional side of children's mathematics forming close relationships and these relationships became part of their pedagogy. It appears that pedagogical perspectives need to work towards a broader analysis beyond school contexts and this includes emotional aspects of children's mathematical lives (Mortari, 2011). However, intellectual relationships were also prominent within the study; the Nursery School teachers gave evidence that there was also an intellectual connection where teachers discussed children's mathematical meanings with them.

10.2.4 Play

In the Nursery Schools (chapter 7) play seemed similar to what Goouche (2010) described, in her study of two Nursery School teachers, where play was, "assumed rather than planned or predicted" (p. 129). That is, all eight Nursery School settings were incontestably play arenas and the teachers were valued as play partners. The pretend play episodes were complicated and contrastingly different from adult-led pedagogies where children are given the mathematics to be learned and the teacher's agenda is at the forefront of the learning (Rogers, 2010). In the Nursery School pretend play episodes, children were involved in leading rich mathematical learning and this uncovered the children's mathematics and their mathematical graphics. This is in opposition to much of the research on play and mathematics (as reviewed in chapter 4) where, although, play is recognised to contain mathematical elements, it is not seen as a substantial learning vehicle for mathematics (Williams, 2014; Gifford, 2005; Ginsberg, 2006). For some, playful learning is posed as an answer to mathematical play (Ginsberg, 2006; Gifford, 2005) others put forward an in-role drama form of play (Fleer, 2011; Williams, 2014). All of these play approaches have some instances of children leading but for the most part it is adult sculpted and led. The play that emerged from the data in this research was initiated by the children and was spontaneous pretend play where mathematical learning, in a broad sense, was occurring. This kind of play may be hard to catch, as a researcher you perhaps need to be a daily part of the early years' setting to see this kind of play or, as in this study, listen to the teachers' stories of play. Therefore, a main finding of this study is that pretend play is proposed as a vehicle for cultivating the existence of children's mathematical graphics in early years' classrooms. One overriding factor is the knowledge the teacher has about children's play and how to engage in children's play. I would consider the teachers in the play sessions in this study to be tuned in to young children and highly-skilled, and the skill of the teacher might be a variable to consider in research on mathematics and play.

10.2.5 Trust

Trust from the school leaders was a prominent factor that the two case study Reception teachers considered as vital to allow them to experiment with pedagogical practice that supported children's mathematical graphics. These two Reception class teachers had the courage to seek children's mathematical points of view and they had, for the most part, the trust and freedom to research children's mathematical thinking within their schools. This trust afforded the teachers the opportunity to develop their teaching to support children's mathematical graphics. It appears therefore that teachers need to be trusted to have ownership of their practice just as children need ownership of their mathematical thinking. Schools that advocate pedagogic relationships that enable both the agency of the teacher and the child are vital for teachers to experiment and trial their ideas in supporting children's mathematical thinking to support their mathematical graphics.

10.2.6 Professional learning

A strong influence effecting conceptual change in both the Nursery School and Reception teachers' practice was teachers partaking in their own classroom enquiry and engaging with the literature which helped develop their understanding of children's mathematics.

Teaching in a way that supports children's mathematical graphics was not straight forward and linear; and the teachers' pathway to change their practice was complicated and tangled. It was clear the teachers needed time to reflect and they used pertinent literature to reflect, reconsider and reshape their practice. The literature gave them a voice, a way of articulating what they wanted to say. As stated in 7.4.4, the literature became a mirror to their practice. Reading other people's research and perspectives gave them an insight, another view and importantly it was not personal and therefore it made it easier for them to reconsider their practice in light of what they were reading.

Within a socio-cultural paradigm, it is understood that every human being views the world from his or her own personal cultural perspective and adds and widens that perspective through outside influences (Rogoff, 2003). In the case studies, teachers were discussing and thinking together about their pedagogies, attending conferences and seminars and reading literature to challenge and expand their views. This is a rich kind of professional development.

10.3 What are the Difficulties Teachers Face in Teaching in a Way that Supports Children's Mathematical Graphics and How Can These Difficulties be Resolved?

The teachers faced many issues as they developed their pedagogy of supporting children's mathematical graphics and I have organized these difficulties into four main themes; children's mathematical graphics; play; practical resources and political agendas.

10.3.1 Children's mathematical graphics

It was apparent in the analysis of the Reception group teachers' data (chapter 6) that the first difficulty the teachers had was recognising and understanding children's mathematical graphics. This difficulty was partly resolved by discussing the graphics and looking at examples of children's mathematical graphics although some of the Reception teachers continued to be confused between children's copy recording and children's mathematical graphics. One teacher particularly emphasized that the children's mathematical graphics were not always acceptable to other teachers. With the benefit of hindsight, I would have concentrated more on analysing the children's graphics with the teachers, so the teachers became more confident in explaining the children's graphics to other colleagues. It may have addressed this problem of not accepting the graphics, which could be linked to not understanding the children's mathematical graphics. Another significant difficulty was the Reception teachers all realised their classrooms were not conducive to children using their mathematical graphics and they emphasised that their classroom culture would have to change to take on children's mathematics. Cultural change is more problematic than, for example, introducing a new mathematical game or developing mathematical resources for a play area.

10.3.2 Play

As discussed in chapter 7 play was not a dominant part of the Reception teachers' pedagogy. To play with the children would have drawn the teachers away from the more objective- driven curriculum that their school had an obligation to promote. In the eighteen months of focus group discussions of the two case study Reception teachers, play was not a part of the conversations and there were no insights or discussion about play, although the children in both classes engaged in play and initially Janine said she wanted to focus on play and mathematics in her classroom. Play was the *elephant in the room*, meaning in the focus groups it was generally accepted that play was important but the subject was avoided. My feeling in the focus groups was the two Reception teachers perhaps thought it was too complex to attempt considering mathematical play pedagogy in their classrooms, at this time, as they were concentrating on child-initiated learning, small group and individualised teaching. Discussing play had been uncomfortable for the Reception teachers (group study, chapter 6) because as they read the literature on play they realised they were not valuing it as much as they thought they were.

Contrastingly, in the analysis of the Nursery School teachers' data as discussed in 7.1.4 it was evidenced that children engaged in mathematics in play and pretend play and they used their mathematical graphics in their play. Play was therefore a vehicle for children's mathematical graphics to flourish. I recommend that mathematical pretend play could be a future focus of research in Reception classes. However, I stress the type of pretend mathematical play I am putting forward is the kind that children lead and is distinctly different from the type of adult-led pretend mathematical play in which Williams advocates that, "children's attentions needs to be drawn towards the mathematics, prior to and after the play" (2014, p. 411). The pretend play episodes, reported in the Nursery School teachers' writings of play in this study, were child-led and the children sought the teacher to be involved in and support their play. The children's mathematics and their graphics arose because the children needed to use their graphics and this was part of their play. Play was on the peripheral of the Reception class teachers' pedagogy and this has implications for professional development to promote the importance of mathematical pretend play.

10.3.3 Practical resources

This study has highlighted the confusion that teachers might have about the place of practical materials and children's mathematical graphics in early mathematics teaching. It was evident, in the later descriptions of the two Reception case study teachers' teaching that they gave children opportunities to use both practical materials and graphic materials, at the same time. However, the place of practical mathematical teaching materials became perplexing and led to disagreement between two of the teachers; one who said you should give children practical materials before you provide graphic materials; and the other who stressed it was not strictly defined but fluid. At the beginning of the study all of the Reception teachers and some of the Nursery School teachers used only practical resources for children to represent their mathematics. The Reception teachers also said they asked the children to copy record after the child had worked out their mathematics using practical resources. Through the study the teachers realised that copy recording was not children's mathematical graphics. The place of practical resources in mathematics within a pedagogy that supports children's mathematical graphics needs further study.

10.3.4. Political agendas

For the Reception teachers, a major difficulty throughout the study was overcoming school-led and government set mathematics' curriculum initiatives and agendas of assessment. This finding concurs with much of the research and writings on the formality of Reception classes in England (Trevor, Ince & Ang, 2020). In the concluding focus group session, the two Reception teachers, although they found the latest government mathematics curriculum more amenable than the previous curriculum, they said it was far removed from what they had been developing to support children's mathematical graphics. On the one hand, the new curriculum was espousing that children should understand mathematics, and this is what the Reception teachers were striving for, but on the other hand, the new curriculum seemed to be recommending set models of mathematics for the children to learn. Similarly, Priestley, Biesta and Robinson (2015) have identified this tension in government policy, where one set of policies seems to be open to child and teacher agency and another set of policies erodes the first set. The two case study Reception teachers were constantly swimming against the tide of government set agendas that their school seemed to be obliged to follow. There was constant conflict for the Reception teachers because, what they believed was benefiting children's mathematics was not close to what the government seemed to be advocating. I conclude with the following question, in the English pressurised political culture, how sustainable is a mathematics pedagogy that support children's mathematics and their mathematical graphics?

10.4 What are the Conceptual Shifts in Practice That a Teacher May Have to Consider as She/He Embarks on Supporting Children's Mathematical Graphics?

In this section I put forward both the theoretical shift in teaching and the resulting practice that surfaced in this research. The points in 10. 2 also cross over to this section.

10.4.1 Theoretical shift in practice

First, from my analysis of the data the overriding and theoretical shift in practice that the two case study Reception teachers seemed to make, was moving from an adult led constructivist /connectivist model (chapter 9, figure 9.1) of pedagogy to a child-centred socio-cultural model (chapter 9, figure 9.2). The socio-cultural model adds a substantial layer to the constructivist model which invites teachers to not only have a subject pedagogy but also know about the children and their mathematics. There are ongoing and polarised debates between mathematical subject knowledge and child-centred practice (Hedges & Cullen, 2005; Rose & Rogers, 2012). The evidence from this study puts forward that these divergent opinions between the camps of child-centred teaching and mathematical subject knowledge need not be in total opposition if reconsidered from a sociocultural perspective. Socio-cultural theory as referenced in chapter 5 (5.1.2) includes the importance of all the children's experiences of knowledge and learning including adult-led teaching. The issue from this study that seemed clearer than the opposition between subject knowledge and child-centred teaching was that the children's home and outside school mathematics experiences are less valued and barely recognised in Reception classes. The two case study Reception teachers moved from detached teaching, where the adult structured and directly led mathematics lessons, to attached teaching where they started to know the children's contexts and supported them in their ways of knowing mathematics. This is vital because a major finding of this thesis is that pedagogies which support children to use their own ways of representing *their* mathematics are child orientated. It is this major conceptual shift to socio-cultural mathematics pedagogical practice that the two Reception case study teachers were noted to make in this study. This move is not easy if you are situated in a culture and belief system that is almost solely adult-focused. It is even more difficult when major institutions keep perpetuating an adult-orientated pedagogy of mathematics teaching as in the NCETM diagram (figure, 9.1).

Although the analysis of the data appeared to propose a socio-cultural model of mathematics teaching that encourages children's freedoms, to use their own ways of mathematical thinking and mathematical graphics as stressed in 8.3, it is not laissez-faire teaching. I stress the evidence of the study emphasises that part of pedagogies that support and develop children's mathematics were also adult-directed teaching. For example, Janine (8.3) deliberately stressed her direct input and Sue, Nursery School teacher in the study stated that in her nursery they still focused directly on counting skills to establish a basis for number but "*it is only the beginning of our children's mathematical journey*" (W.R., 15/9/2014). Therefore, adult-led teaching had not been abandoned by the teachers in the study but is only seen as part of a pedagogical approach that could support children's mathematics and their mathematical graphics.

10.4.2 Every day practical teaching shifts

The everyday teaching shifts in practice that the teachers in this study made are also situated in some of the points that surfaced from 10.2. which were: opening up mathematical teaching; listening; relationships and children's play. I now draw on the close analysis of the two case study Reception teacher's practice which I have put in a table (figure, 10.1) which is also in chapter 8 (figure 8.17). In chapter 8 Millie described episodes of her teaching, concentrating on listening to children's own problems and how they used their graphics to work out their mathematical problems. Janine described teaching sessions over five months that she explained in detail. The eighteen pedagogical shifts I identified, as the two Reception teachers described their teaching, are outlined below in figure 10.1.

Figure 10.1 A chart to show Millie and Janine's pedagogy.		
From	To	Examples of shifts in practice
1. Only group and whole class work direct teaching of mathematics, with expected outcomes	Democratic teaching, degrees of freedom, Trusting children.	*"It was just free activity"*and *"then I left them with a large piece of paper and pens I put some beads out"* (Janine F.G., 15/5/2016).
2. Copy recording	Supporting children's mathematical graphics. Using the language that give children the freedom to have a go at their own graphics and ways of writing mathematics.	"Put it on paper" (Janine F.G., 15/5/2016).
3. Calculation; getting frustrated that children '*don't get it'* when trying to directly teach to the objective. Step by step.	Understanding that children have knowledge of calculation that the teacher has not directly taught. Children's emerging knowledge.	Uncovering children's calculation, "*Then he said I can do it a different way 10 add 10 and 10 make 30. Actually, I do know that 10 x 10 =100. So, he was building this up"*. (Janine F.G., 15/5/2016).
4. Counting and number recognition, working with numbers Step by step approach teaching 0-5 and then 5-10. Numbers rarely get discussed beyond 10, sticking to the EYFS goals for Reception	Letting children explore with whatever numbers and areas of mathematics they choose. Supporting and extending. Not putting a cap on what they are exploring. Going beyond what is expected in the framework.	"*she was counting above 20 and she got up to 29 and needed a prompt to go on to the 30"* (Janine F.G., 15/5/2016). *"He said that was a thousand and ten and we talked about what that would look like"* (Janine F.G., 15/5/2016).
5. Knowing what children can do in set tasks.	Knowing the children's interests and how they work, individually.	"*another boy Joshua is good with numbers mentally and he is fascinated by buses and numbers on buses"* (Janine F.G., 15/5/2016).
6. The teacher plans and leads the mathematics lessons. The children follow her rules with few opportunities to follow their own agendas	Children leading and the teacher responding to this positively. The child knows the teacher is a resource that she can use.	*"Another girl put some numbers down and she got me to say them"* (Janine F.G., 15/5/2016).

7. Children being regulated to do the mathematics expected from the curriculum.	Children feeling confident and free to do things that are not on the teacher's agenda. The teacher allowing this. (a larger degree of freedom)	*"And some of the children actually got carried away with drawing"* (Janine F.G., 15/5/2016).
8. Calculation symbols and signs taught in set lessons and children copy.	Identified that some children take a very long time to acquire the understanding of conventional mathematical symbols and signs and allowing for this.	*"I could have sat Milo down and told him the right way to do it (standard calculation). Throughout the year he just got there himself. But I could have just sat him down and showed him the right way that then he would have copied and used it. But I think that is the conceptual bit we are missing out aren't we?"* *"the children who needed more thinking about that"* (Millie, F.G., 15/5/2016)
9. Set times for mathematics and few opportunities for individual children to explore their mathematical interests over an extended period.	The teacher allowing and encouraging the children to carry on with their mathematical thinking over days.	*"then she went back to it the next day and ran out of paper. So, I said use some more paper and she went and stuck it on the end and that went on for a bit and she added more paper and over the next few days she got really excited and carried on with her number line and eventually she got up to 114."* (Janine F.G., 15/5/2016).
10. Children work by themselves mostly after an input from the teacher.	Peer modelling; children learning from other children	*So, they were looking at numbers, sequencing and learning how to write them from other children.* (Janine F.G., 15/5/2016).
11. Detached teaching; The expected objectives take over and are impersonal.	Attached teaching; takes into consideration the ideas and interests of the child individually including their home and community life.	*That was obviously what he was interested in* [time, clocks] *and he was using o'clock. He did 11 o'clock and 10 o'clock so he was writing a number and then saying the o'clock.* (Janine F.G., 15/5/2016).

12. Only accepting the right symbols and traditional ways of representing mathematics (literal mathematics)	Accepting children's ways of representing including: scribbles; crossing out; missing out numbers; drawing etc.	*Amber is not confident and okay she did miss out the eleven there.* *And that is his 9* [not the conventional way of writing 9] (Janine F.G., 15/5/2016).
13. Direct modelling of mathematics e.g. the teacher writes 2+3=5 (perhaps drawing 2 apples and 3 apples) then the children copy this.	Indirect modelling The teacher models different ways to represent mathematics, informally throughout the day. The teacher does not expect the children to copy her representation or a set model (Carruthers & Worthington, 2006; 2011).	*You can do any animals it does not need to be the ones in the book.* *I just put some Numicon in the floor and I said, look I am adding these together. You can write it this way if you want, this is what I am doing.* (Janine F.G., 15/5/2016).
14. The teacher gives children set problems to solve.	The teacher encourages children to think about their own mathematical problems. She discusses and supports their own ways to solve problems.	One child asked, "*How big would a classroom have to be if there were 108 children take away 8*" (Millie, F.N., 24/6/ 2015). Another child asked, "*How many children have you taught*? (Millie, F.N.24/6/2015) Children's mathematical questions are seen as different from adult's, "*They are more interesting and for me more challenging"* (Millie, F.N., 24/6/2015)
15. Emphasis on mathematics as a subject discipline of right and wrong answers.	Looking at children's mathematical learning positively, valuing the emergence of children's mathematical thinking.	*"He was fascinating, I just watched him slowly change throughout the year where the symbols made sense to him"* (Millie, focus group, May 2016).
16. Only practical resources are provided for children to represent their mathematical thinking.	There are always opportunities for children to use graphical equipment to support their mathematical thinking.	"*My class are using paper and pencil all the time now, all kinds of paper and writing implements outside and in."* (Millie, F. G., 15/5/2016).

17. Children's mathematical graphics are not noticed.	Children's mathematical graphics are encouraged and the teacher notices and analyses the children's mathematical graphics.	'*She has used an iconic method to work out multiples of five, she put five dots in a circle and repeated this to solve the multiplication problem*" (Millie, F.G., 8/3/2016).
18. The language used about children's mathematics is a deficit model for children who are seen not to be doing as well mathematically as other children.	The language used about children and their mathematics comes from an additive perspective. All children are seen as competent.	"*I have not done this with all the children yet I need to do it with the children who are not as able,* [pause] *not as confident in their maths"* (Janine F.G., 15/5/2016).

Figure 10.1 exemplifies the multiple shifts in teaching that the two case study teachers made over the three years and five months of this research. The chart also shows how the teachers engaged with the children and how the children responded. The examples of the teachers' pedagogical shifts included: the language they used; how they engaged with the children's mathematics; their descriptions of children's mathematical learning; how they modelled mathematics; how they encouraged children's own problems and analysed the children's mathematical graphics. One of the major and overriding shifts identified was that the teachers moved to a positive stance when they observed and commented on the children's mathematics. When Janine and Millie were describing their observations of children's mathematics and their graphics there were no negative comments. They were looking at children's ways of expressing their mathematics both graphically and verbally from an additive rather than a deficit viewpoint. This constructive way of viewing children's mathematics seems to be a vital pedagogical lens, opening the way to understanding children's mathematics and their graphics.

10.5 How Do Teachers Feel the Children Benefit, If at all, from Using their Graphics to Support their Mathematical Understanding?

Both Reception case study teachers became strong in their convictions about the benefits of the children's mathematical graphics. These teachers often stated that the children's *understanding* of mathematics grew as they used their graphics as a tool for thinking. They also uncovered knowledge about the children's mathematics that they did not know about and this seemed especially prevalent for children who they viewed as developmentally not as confident in mathematics, compared to the other children in their class. The two case study

Reception teachers both said the children, in their classes, were more engaged in mathematics than they had experienced in their previous classes. At times it was observed the children were also excited about their mathematical learning and in some examples, the children were benefiting from listening to other children's mathematics. It was noted, children were engaged in higher-level thinking, beyond the set curriculum objectives, mainly because as children were allowed to explore their interests in mathematics, the mathematics curriculum became broader as children took their learning in different and wider directions. However, one teacher reported these wider mathematical encounters can become quite difficult for children as they self-challenge and she said, children need to be resilient. A pedagogy that supports children's mathematics may not be an easy style of pedagogy for some early years teachers to take on, however, it can lead to young children's broader mathematical thinking and may raise their confidence as young mathematicians.

10.6 If Teachers Support Pretend Mathematical Play, Where Children Use their Mathematical Graphics, How Do they Interact with Children to Support and Enhance their Play and Mathematical Thinking?

In the literature review the research pointed to the role of the adult in play as being a contentious and puzzling issue (4.3.5). In this section I present the main findings from the Nursery Teacher's pretend play episodes that may add to our understanding of how to engage with children's pretend play.

In the analysis of the Nursery School pretend play scenarios (7.1.4) main points were underlined as supporting mathematical pretend play, children's mathematics and their graphics. These key points did not highlight specific ways teachers might interact with children in play. In pretend play, within this study, the teacher's role seemed to be following the child and unlike adult-initiated or adult-directed teaching it is hard to be specific about questions the teacher would ask; it depends on the play and the children and if questions are appropriate. For much of the time, in the pretend play scenarios in this study, it was supporting children's ideas and encouraging them to take the lead. The pretend play, therefore, was all about the children leading *their* play and gave insights into how children's own mathematics can be nurtured. The first list of bullet points (figure 10.2), based on the teachers' writings, is how teachers might support the children's mathematical thinking within play. The second list of bullet points (figure 10.3) is the child's view and this might add to understanding the child's perspective. Both of these lists have already been presented in 7.4 and 7.6.

Figure 10.2: How teachers might support children's pretend play.
• children leading their play, which means children choose the focus of their play and organise the players, including the adults;
• time and resources being easily available for children to choose and to make artefacts or graphics for their play;
• providing useful mathematical references within the environment that children can use within their play;
• accepting and tuning into the emergent learning of children's mathematics and the mathematics of their home and community;
• being emotionally and intellectually connected to children's mathematical thinking;
• realising children have their own mathematical perspectives within play and child-initiated learning. These might be unorthodox and different from the standard mathematics curriculum;
• being available for the unexpected in imaginary worlds where anything can happen;
• being ready to build on children's mathematical thinking and this may be some days or weeks after the original play or child-initiated enquiries, as they might return to similar mathematical themes;
• accepting the children's mathematical graphics as children's emergent thinking, even although the graphics might not be standard ways of representing mathematics;
• allowing children to continue their mathematical thinking by not intervening or resolving issues immediately;
• understanding that mathematical pretend play is complex and therefore there is a need for teachers to seek high-level professional opportunities that expand their existing knowledge of pretend play, mathematics and children's mathematical representations.

The second list of bullet points below (Figure 10.3) are what I heard from the nursery child's perspective through the writings of the Nursery School teachers. These statements may further help teachers understanding how to interact sensitively in children's play and support children's mathematical graphics.

Figure 10.3: The Nursery School child's perspective
• Know me as an individual child as opposed to knowing the class as a generic group;
• I need free access to materials that will help me communicate through my graphics, for example, chalk, pencils and pens.
• I also need free movement around the class and outside to provide a space that I have the optimum opportunity to imagine in;
• Value the mathematics I communicate, in whatever form and way;
• Knowing me and my family and my community and the mathematics of my world, outside school, helps me to share my mathematical understandings and be understood;
• I operate best if I have the opportunity, every day, to have a long time to think for myself and to think with others including you, my teacher;
• Play with me and engage in my pretence when I request an adult, but do not take over my ideas or skew them in a different direction, to meet the curriculum objectives, as this may stunt the growth of my broader understanding of mathematical concepts.

The two tables above combine to show how pretend play can be understood and supported by the teacher. Pretend play, from the findings of this study, is unquestionably the domain of the child. I put forward a major finding of the study is that pedagogies that promote pretend play need to understand, respect and actively encourage children's perspectives. Also, it seems, taking into consideration the points in figure 10.3, children will use their mathematical graphics to enhance their mathematical pretend play.

10.7 Transitions

This research was not intended as a comparison study between the Nursery School teachers' group and the Reception teachers' group however, in analysing the data, there were transparent differences between the two groups' mathematical pedagogies. The Nursery School Teachers wrote and discussed their practice in terms of play (chapter 7) and child-initiated learning opportunities. The Reception teachers stressed the difficulties of the set curriculum they had to follow and concentrated on developing enquiry based mathematical learning, which they termed *open mathematics*. The two pedagogical perspectives gave a broad view of Foundation Stage mathematics and also brought up questions that have implications for transition from Nursery to Reception. It has already been reported in this study that this sharp difference in pedagogy between pre-school and school is not a new discovery (Dockett et al., 2017). Brooker (2002) wrote about the child's transition to school as "developmentally dramatic" (p. 24). From the findings of my study the child's experience in mathematics is also strikingly different and therefore creates a discontinuity in mathematical experiences from pre-school to school. This educational cultural change must be very confusing for children as mathematics could take on a very different and narrower agenda. Therefore, this study recommends that there needs to be a professional space where Nursery School and Reception teachers can discuss their practice and ethos and perhaps use this to influence the continuity of mathematical experiences as children enter Reception classes.

10.8 Contribution to Knowledge

This research will add to the ever-growing body of knowledge of democratic child-centred pedagogy, moving away from mainly detached teaching and constructivist pedagogical theory to attached teaching and socio-cultural theoretical models of pedagogy. This study provides an exemplification of what is possible at the classroom level, in supporting children's own mathematics and their mathematical graphics and enhances understanding of respecting children's emergent mathematical thinking. The research reported here also highlighted pretend play as a source of developing children's mathematics and their graphics and gives some indication in addressing the concerns of teachers' positioning in relation to mathematical pretend play. This research is original in that few studies concentrate solely on pedagogy that support children's mathematical graphics. I know of no other research that has emphasised the perspectives of early years teachers in England uncovering their developing pedagogical shifts in practice in the area of children's mathematics and their graphics.

10.9 Future Direction

The research area of children's mathematical graphics is a small but growing field and further research would be helpful and particularly important in the area of early calculation with Reception and Year One teachers. Focusing on these two groups would help the development and understanding of children's mathematical graphics beyond the pre-school years where children's early graphics are more acceptable. Research studies that focused on pretend play and children's mathematics in Reception classes would further our knowledge of how pretend play can be a valued part of Reception class mathematics. From the findings of this study there seems to be a need for teachers to understand children's mathematics and their graphics. Although quite a grand proposal, a future direction of research could be to have a national early mathematics graphics project, similar to the 1980's National Writing Project (National Curriculum Council, 1989). In the National Writing Project teachers were at the centre, involved in discussing and analysing children's early emergent writing and their developing pedagogy which led to a considerable number of teachers researching children's own written meanings.

10.10 Challenges, Ethical Considerations and Limitations of the Study

As I had been researching children's mathematical graphics for twenty years, a challenge I reflected on during the study and wrote in my field notes (17/11/2015) was that I might be too influential and unconsciously pushing Janine and Milly towards my way of thinking and therefore skewing the data. However, as the study progressed, I realised just as the answers they sought lay with the children, the answers I sought lay with them and I was mindful to listen afresh to their perspectives.

10.10.1 Ethical considerations

Throughout this study ethical issues were considered and addressed. One of the main issues was the length of time that I collected data because this meant that there was more scope for ethical dilemmas. For example, taking on the researcher role blurred the boundaries of my changing identities within the study for the participants. At first, I was the participants' tutor on a Masters' course and a local head teacher then I moved to being their mentor within our Early Years Teaching School whilst still being a researcher. Changing roles brought up predicaments, for example, the line between work and collecting data. To address these problems, firstly, I revisited the researcher's role in discussions with the participants and I also made explicit the timings of the focus group meetings and other times I would be collecting data with the participants. Secondly, I verbally renegotiated consent as I planned the next meeting with the participants. I was mindful of the fact that the participants were in job roles and pressurised by work commitments, therefore, I mostly collected data for short periods and breaks were essential. As the study progressed it was clear to me and the participants that the focus group discussions were supporting our collective work goal to improve early mathematical pedagogy. The research discussions became our professional development as part of the Teaching School and our work focus was also embedding a research mathematical culture as a vehicle for improving mathematics in the Foundation Stage in the city.

10.10.2 Limitations

This study although longitudinal and having multiple case studies was nevertheless small in scale and with limitations. An ongoing criticism of interpretive, qualitative research as highlighted in 5.2.1 is that it lacks generalisability beyond the circumstances in question and therefore as Noble and Smith (2015) point out, qualitative research can be judged as having limited application or general significance. However, this focus on statistical generalisablity in research has increasingly been confronted (Connelly & Clandinin, 1990; Feagin et al., 1991; Yin, 2016). Through a small longitudinal multiple case study, I attempted to extract the detail and complexity of the teachers' developing perspectives to uncover mathematical pedagogy that supported young children's mathematical graphics. I endeavoured to use these analyses to depict broader phenomena in an attempt to root out the typical from the unique (Burawoy, 2009). It is very unlikely that the exact writings and conversations of the teachers reported here would be observed happening anywhere else. However, the process of how I conducted the research as explained in chapter 5 may be replicable. Although results will differ, my analyses and interpretations of these observations will stand as useful for comparison and discussion in relation to observations in other situations.

10.11 Finally

In finding out how teachers can access a pedagogy that understands, nurtures and supports children's mathematical graphics it appears from this study, it is not a one size fits all pedagogy. The participant teachers had different teaching approaches and their school culture and personal understandings influenced the different pedagogies they developed. However, as previously discussed, there were common strands that perhaps teachers can take up to begin to access more democratic practices that give opportunities for children to experiment with their own symbolic mathematical understandings. From the evidence of this study, I propose a shift is needed in moving away from adult perspectives in early mathematics teaching, to authentically listening and noticing children's own mathematics and children's mathematical cultural knowledge that they bring with them to school. The teachers in this study made that shift to listening to children and becoming advocates for promoting children's mathematical views. I have questioned if moving to pedagogies that support children's mathematical perspectives is possible in England, where the voices of teachers are rarely heard within a tightly controlled government education system (Moss, 2019). I leave the final important point with the voice of a teacher in the study, as Millie stated and reflected, more than once, "*you have to understand the children's meaning rather than the child understanding the teacher's meaning. For me this is very powerful*" (Millie, Interview 22/11/2014).

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Appendices

Appendix 1: Children's Mathematical Graphics (Carruthers & Worthington, 2010)

Carruthers, E. and Worthington, M. (2010). *Children's Mathematical Graphics: Understanding the Key Concept*. [online] NRICH. Available at: https://nrich.maths.org/6894 (Accessed 12 June 2021).

The term *Children's* Mathematical Graphics was first introduced by myself and my co-researcher (Worthington & Carruthers, 2003). This term emphasises the word children because it is children's own representations of their mathematical thinking that is the important feature, rather than children producing standard adult taught algorithms.

Hughes Studies, 1984 and 1986

Children's mathematical graphics has its roots in Hughes (1986) ground -breaking work where, through clinical studies, his research revealed that young children, under five, could represent in their own ways quantities that are counted, through a range of self -invented marks. What is also a striking discovery in Hughes (1986) research is that when the researchers presented older children (six and seven year olds) with addition and subtraction problems, not one out of ninety six chose to use the standard symbols of '+ 'or '- 'yet these children were using these symbols regularly for addition and subtraction sums in school. Hughes therefore highlighted the difficulty of children understanding abstract symbols and proposed that the children did not see the value of using conventional symbols in mathematical problem solving. Ginsberg (1977) in similar clinical studies also identified that children have difficulty with standard written mathematics and similar to Hughes declared that there is a gap between children's informal knowledge and school standard written mathematics.

Hughes (1984;1986) describes the substantial knowledge about number that children develop naturally in everyday situations before they start school. He very clearly states that children need to build on their informal home mathematical knowledge making links to school mathematics. Hughes also touched on the fact that teachers do not identify or recognise or value children's own intuitive methods. Although he says teachers recognise that children have difficulties with standard written mathematics they conclude children are not yet ready for any written maths and as Carruthers and Worthington state they resort to 'practical mathematics' only (Carruthers & Worthington, 2006). A conclusion of the Hughes' research is, 'As with children's informal methods of calculation, their own invented symbolism must be given much greater prominence in the classroom' (Hughes 1986:177).

Hughes (1986) went on to describe possible teaching methods that might help the problem of children developing a better understanding of standard mathematical symbolism however he never focused on children's invented symbols that children make as central to the pedagogy of bridging the gap between formal and informal arithmetic, instead he focused on number games and Logo, a computer game. He made a few useful pointers regarding children's own mathematical representations for example that teachers should value children's mathematical invented symbols and build on children's own strategies. Children's own invented symbolism is what we put forward and name Children's Mathematical Graphics and we strongly make a case that this might be a stronger part of the solution (Worthington & Carruthers 2003, Carruthers & Worthington 2005, 2006, 2008 and 2011)

Children's Mathematical Graphics

There has been little research on young children's own ways of representing mathematics compared to the wealth of literature on children's own writing (Smith, 1998; Clay, 1975; Graves, 1983; Barrat-Pugh and Role, 2000) or drawing (Matthews 1999, 2003; Anning 2000; Anning and Ring 2004). The original research on Children's Mathematical Graphics (Worthington & Carruthers 2003 and Carruthers & Worthington 2005) consisted of 700 samples of children's graphics over a 12-year period. Most studies to date have concentrated on the analysis of children's number representations in clinically set-up tasks (Hughes, 1986; Sinclair, 1988; Munn, 1994). These studies have added to our knowledge and understanding of children's mathematical graphics.

Our research differs in that we based our study in children's homes, nursery and classroom contexts. Rather than being clinical researchers our role was that of participant observer, based on ethnographic research and grounded theory. This uncovered firstly as Hughes(1986) Ginsburg(1977) and Allardice (1997) had also discovered that children did have their own mathematical ways to represent their mathematical thinking and secondly our research, for the first time, traced the development of children's mathematical representations and showed the development from early marks to standard sums [see taxonomy, Appendix 2]. The more open methodology we used uncovered a wealth of data as we also collected children's responses and meanings they were making about their graphics. We had on our side the fact that these children knew the adult well, who was interested in their graphics, therefore the children we believe were more likely to respond openly rather than feeling awkward because a stranger was asking them questions such as in a clinical study. It is also what children chose to do rather than being asked to do and this has significance on the kind of data we uncovered.

Children's Mathematical Graphics are seen as important because they encourage children to develop their own ways of seeing and knowing mathematics and this is vital to bridge the gap between the informal maths of home and community and the more formal standard mathematics of school (Carruthers and Worthington, 2005; The Williams Review, 2008).

When we first put forward the premise that children need to explore their own ways of written mathematics to then develop standard and useful shorter methods of calculation teachers constantly came back to: but what do the graphics look like? It was clear that most teachers we encountered had not thought about this before and had not noticed the possibilities of children devising their own signs, drawings or symbols for mathematics. (Carruthers, 2012).

In the next part of this chapter I describe some examples of Children's Mathematical Graphics with illustrations. This is to show the range of graphics and some of the most important aspects of the children's mathematical representations. (A more in-depth account can be found in Carruthers and Worthington, 2006).

Tracing the development: Early marks (See the Taxonomy of Children's Mathematical Graphics for a fuller description in Carruthers and Worthington, 2006)

Early written numerals
Young children's early graphics or scribbles are often discarded by adults as being chaotic and random Athey (1990). When studying young children's marks Matthews concluded that many of these scribbles are not just haphazard actions but 'products of a systematic investigation' Matthews(1999:19). Malchioldi (1998) described children's very first marks on paper as a 'developmental landmark' as they now are making a connection with their marks on paper to the world around them.

From two and three years of age children start to make scribble-like marks but some also attribute numbers to these marks (see figure 1) and this is what is described as the early development of mathematical graphics (Worthington & Carruthers, 2003). For example, in figure 1 Molly aged 3years and 11 months whilst playing in the graphics area in the nursery writes, what Clay (1975) refers to as, letter-like marks. We would describe these marks as number-like marks as she ascribes numbers to her graphics.

Figure 1: Molly's numbers

Molly - "seven, six and number eight"

Representation of Quantities and Counting

One of the early strands of the development (see taxonomy Appendix xx) is where children are representing 'quantities that are *not* counted'. This is usually an energetic and dynamic representation of their thinking and they give a sense of the quantity. In figure xx the context was in a nursery class where Charlotte and her friend Jessica (both just turned 4 years of age) were playing with large felt tip markers on the graphics table and had two markers in each of their hands. They called over to their teacher as they made all sorts of coloured marks on their paper by drumming and shouting, in delight, "I've got hundreds and pounds, hundreds and pounds". Gardener (1993) celebrates what he refers to as children's unschooled minds,

'the five year old is in many ways an energetic, imaginative and integrating kind of learner; education should exploit the cognitive and affective powers of the five year old mind and attempt to keep it alive in all of us.'
(Gardener 1993: 32)

The marks Charlotte was making were unique, unrestricted, exciting and within a social context.

Figure 2: Charlotte

Representing quantities that are not counted - Charlotte: 'I've got hundreds and pounds!' - 4 yrs 2 months

Representing quantities that are counted

Children are experimenting with all sorts of number and counting procedures even before they are 4 years old. Jenna (see figure 3) in a nursery school context decided to represent raindrops in coloured pens. She wrote her name and from right to left and counted each stroke as she drew her raindrops. She also did this from right to left.

Figure 3: Jenna

Representing quantities that are counted
Jenna 's 'raindrops'- 3 yrs, 9 months

Written Calculations

Research into young children's calculation has found that children use counting strategies to work out small problems (Carpenter & Moss, 1984; Thompson, 1995) and they persist with counting strategies even if the teaching is focused on combining and separating sets (Orton,1992). In our study of young children's early calculation on paper we found that they do use an ever increasing range of strategies to represent number operations including counting continuously and separating sets, exploring symbols, standard symbolic calculations with smaller numbers and calculating with larger numbers supported by jottings. (Carruthers & Worthington, 2006). Instead of relating all the types I am going to give an example of one calculation strategy that children use when they work out a problem, counting continuously.

Counting Continuously

We use the term ‘counting continuously’ to describe this strand of children’s early representations of calculation (addition and subtraction) strategies. Children use both counting all items to be added or start at the end of the first group and count on. They use pictures icons or numerals to represent the two sets. They often set the graphics horizontally. In figure 4 Alison was counting two groups of things, children and toys, that were going to her class’s breakfast cafe. Alice used numerals to count and did not separate the toys from the children instead she counted one child and then the child’s toy and so forth until she produced a string of numerals. When she self-checked (a useful strategy) she found that she had counted too many and she solved the problem by using brackets. She drew a hand afterwards, but she did not say why. We found that many children used hands to signify take away or addition (Carruthers and Worthington 2005).

An important point to note is that although there are common elements to all the graphics, no two are the same. Children tackle the problems in different ways and use the graphics in ways that make sense to them. It is very individual, and it is easy to detect when a class are not using their own ways to represent their mathematical thinking because all the representations will be very similar using the same strategies.

Figure 4. Alice and the Breakfast Cafe

Counting continuously
Alison: ‘Seven toys and seven children’ - 5 yrs 1 month

Children’s meanings

It is now well documented that children are active in their learning bringing their own enquiries and understandings and that from birth children are constantly trying to make meaning of their world (Wells 1986; Rogoff 2003); for example, when a child might ask ‘what is the last number?’ as they think about ideas of infinity. This view of children as powerful learners is central to the concept of children’s mathematical graphics (Carruthers & Worthington, 2008). Grounded in a social culture perspective, children use all experiences they have within their own lives to make personal meaning. This perspective on cognition is wider than the idea that development consists of acquiring skills that are directly taught. The belief is that all higher order functions such, as learning, grow out of social interaction (Rogoff, 2003). When they are engaged in their own mathematical graphics children are communicating their own kinds of mathematical meaning, in personal ways.

Symbol use in Mathematics

In figure 6 the example of Jack and the grapes Jack used a combination of symbols to represent quantities. He used circles to represent the grapes which could be pictographic or iconic. He then uses standard symbols i.e. digits to represent the quantities again. Finally he chose to use a line to represent equals to show the final quantity. The use of symbols are important as John experiments and uses what he is comfortable with and understands. Combining and transforming symbols allows children to create and communicate complex meanings (Kress 1997; Pahl 1999). By exploring and encoding meanings in a range of contexts children can come to learn how graphicacy can be used flexibly to carry different meanings. It is the free and flexible use of signs and symbols that are important if children are to develop as proficient symbol users not only in mathematics but in all other literacies. As mentioned at the beginning of this chapter studies have emphasised the importance of children using mathematical notation with confidence and understanding (Hughes 1986, Ginsberg, 1977). When we discussed the research samples of children’s mathematical graphics then,

> one overriding point became clear: that the essential reason for teachers to encourage children to represent their mathematical understanding on paper, in their own terms, is that children will come to understand the abstract symbolism of mathematics at a deep level. (Carruthers & Worthington, 2005:21)

Recording or Representing thinking

Recording mathematics

As we talked to teachers about children’s mathematical graphics often they showed us examples that they thought were children’s mathematical graphics but many of them were ‘children’s recording’ Historically the emphasis has been on children 'doing' mathematics, working something out with practical resources and then 'recording' it afterwards (Tucker, 2016). However, our research has shown that this has limited value. *Recording* places the emphasis on marks, symbols and drawings as a *product*, and is a lower level of cognitive demand. The children copy draw what they have already done and very little mathematical thinking is involved. The National Numeracy Strategy advised that children do not need to record their mathematics if they can work something out mentally, neither do they need to record something they have already worked out in a practical context with resources as they have already worked out the maths. (QCA, 1999).

Example of mathematical recording

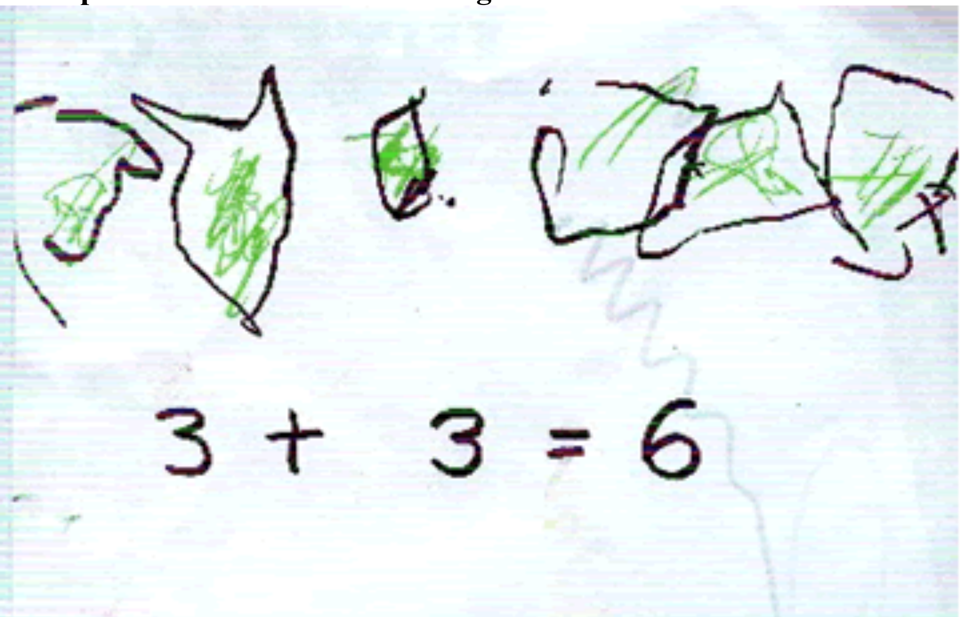


Figure 5: Recording - Leo (aged 4 years 6 months)

Figure 5 shows an example of a child's 'recording'. Leo's teacher showed the children addition using multilink. Giving each child three cubes to count and then three additional cubes, she asked the child how many cubes they had altogether. Leo counted from the first group "1,2,3" and then continued counting the cubes in the second group, "4,5,6". His teacher wrote the calculation on the white board commenting "3 add 3 equals 6". She asked the children to draw how many cubes they had altogether. Some of the children copied what the teacher had written: Leo hesitated so his teacher wrote "3+3=6" beneath the cubes Leo had drawn.

The children's drawings were nearly all the same: the only differences were the ways the children had drawn and coloured the cubes. In effect the teacher had done all of Leo's thinking for him; preventing him exploring possible ways that might make sense to him. An unintended consequence of working in this way is that it may lead to children's confusion about written calculations and a superficial understanding of the abstract written language of mathematics (Carruthers & Worthington, 2009).

Representing mathematical thinking

To differentiate between the early written mathematics that most young children experience in schools and *children's mathematical graphics*, we use the term 'representing': children represent their internal, mental representations, in effect - exploring their mental methods on paper. Children's own graphics support deepened thinking about the mathematics in which they are engaged, and significantly, about their use of symbols and other visual representations to signify meanings. They enable children to build on what they already know and understand.

Example of mathematical thinking (Children's Mathematical Graphics)

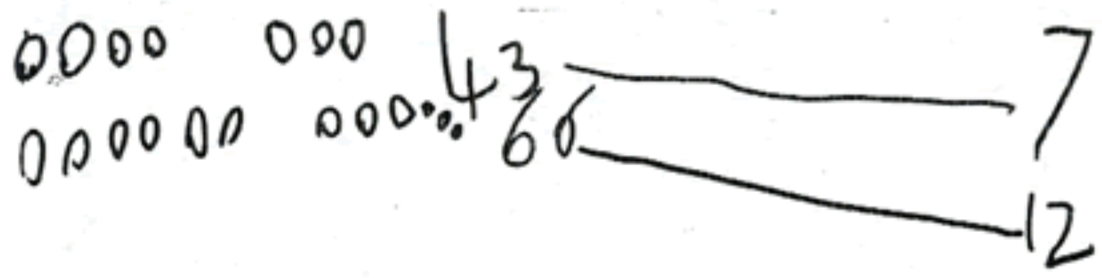


Figure 6: Children's mathematical graphics (representing personal mathematical thinking) - Jack (4 years 8 months)

Figure 6 shows an example of addition in a reception class where children were encouraged to *represent* their own mathematical thinking using their *mathematical graphics*.

Jack's teacher put a plate of grapes on the table where his group were sitting, adding some blank paper for any of the children who might want to explore their ideas on paper. She invited the children to choose a small quantity of grapes to put in each of two small dishes and work out how many grapes they had altogether. Jack chose to use paper and first drew two separate sets of grapes, leaving a gap that allowed his first calculation to be read as ' 4 and 3' (we term this an 'implicit symbol'). He confirmed the quantities by writing the numerals and drew a line between these and the final '7', using the line to signify an equals sign.

Since Jack represented his mathematical thinking in ways he chose, his calculations were personally meaningful. Two of the children worked out their calculations mentally, whilst others chose to explore their ideas on paper, connecting mental and written methods. They used a range of personal graphics including words, numerals, drawings and invented, mathematical symbols. Significantly *children's mathematical graphics* support children's processes ('using and applying mathematics'; DfES, 2006). The important point is that the children were making their own connections, building on their early understanding of marks and symbols and using them to make mathematical meanings.

The difference between the two examples one of recording and the other of mathematical graphics is quality and depth of thinking.

Understanding the difference between recording and representing seems to be a difficult concept for teachers to grasp yet it is a key concept in understanding children's own ways of representing their mathematical thinking (Carruthers & Worthington, 2011). However, as diSessa notes it is rare ' to find instruction that trusts students to create their own representations' (diSessa, 1991:156). More recently Terwel *et al* further reflects and states that 'although there have been positive changes in the past decades, we believe that today diSiessa's statements hold true for many classroom practices' (2009:28-9).

In England official recognition that young children could use their own ways in writing their mathematics was seen in the first Early Years Foundation Stage Document (DFEE, 2000) where they had acknowledged Hughes (1986) research and advised early years professionals to ask children to 'put something on paper' when they were engaged in mathematical learning. (DFEE,2000:71). They also gave an example of the Hughes (1986) tins game which he used in his original research with children under five (4.2.2). There was also some reference to supporting children's written methods in the National Numeracy Strategy (QCA, 1999).

In the Williams Review (2008), of mathematics teaching in early years settings, a strong case was made for early years professionals not to miss opportunities for children to experiment with early mathematical marks. Our taxonomy (Carruthers and Worthington 2006) was published and highlighted within this review and one of the major recommendations was to publish a set of materials on mathematical mark making and children's mathematical development. Two documents for teachers and early years practitioners followed. One firstly on Mark making (DCSF,2008) and secondly on developing children's mathematical thinking which included mathematical graphics (DCSF,2009). An issue with the Williams Review is that it only emphasised children in the foundation stage using their own graphics and it did not mention the development into key stage one although that was redressed, to some extent, in the materials following the report.

As discussed there is clear support for children to use their own mathematical graphics to aid their thinking in government documentation and I have seen a lot of enthusiasm from teachers but it is not a generally strongly understood or well-practiced concept. Mostly there is only superficial understanding that sometimes does not go beyond giving children more opportunities to write or draw.

Some of the barriers to practice are:

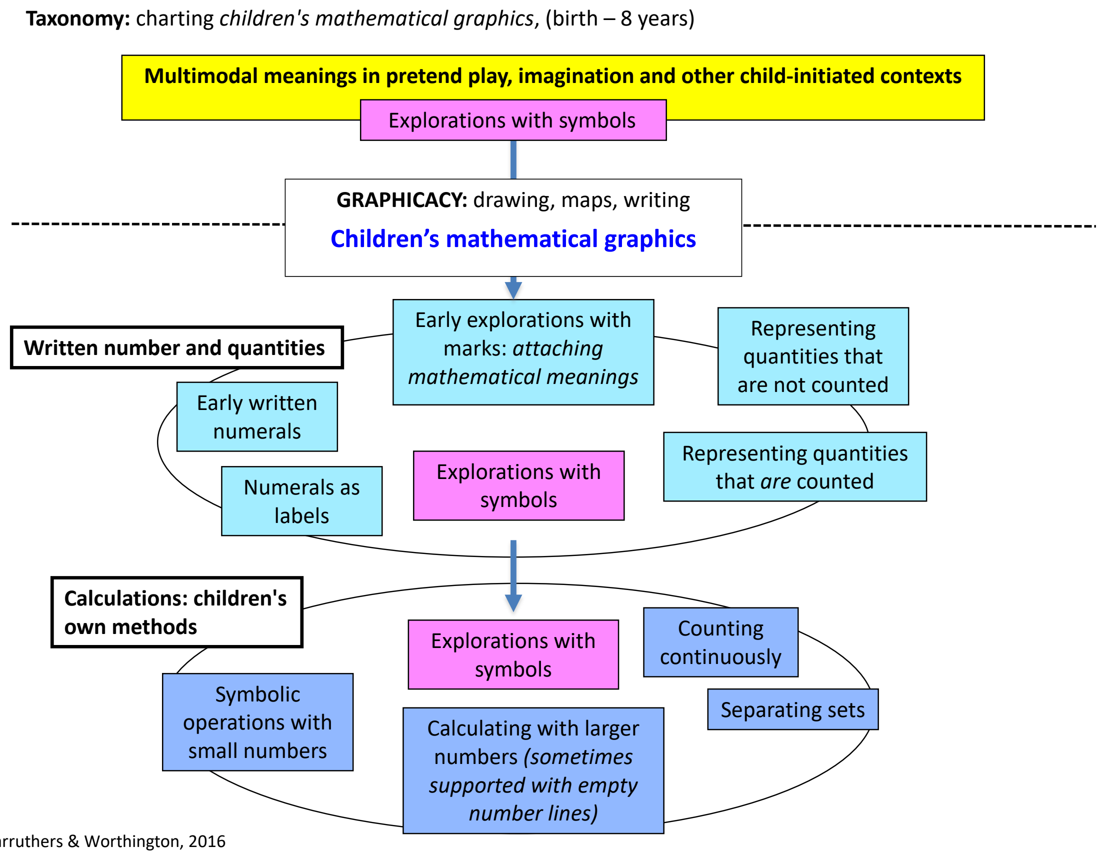
- Ofsted repeatedly raises concerns regarding children's over-reliance on formal written methods, emphasising the importance of children's informal methods of notation. Annual inspection reports highlight the importance of building links between children's mental and written methods and for greater opportunities for children to use and apply mathematics. However, inspectors still insist on neat and tidy sums in jotters and this has perpetuated a culture that even in year one children have to write maths problems in small, squared paper boxes which restrict the scope of their graphics (Carruthers, 2012).

- Lack of opportunities for teachers to reflect on the children's own mathematical thinking on paper and ways forward.

- Understanding of play and open democratic ways of working with children.

Within this study therefore an aim is to look more closely at the pedagogy of children's mathematical graphics through teachers' eyes and seeing their barriers and ways forward to support Children's Mathematical Graphics

Appendix 2: Carruthers & Worthington 2016 Taxonomy



Appendix 3: Ethical approval from Bristol University

The documentation below comprises the procedure for gaining ethical consent at the time it was submitted and approved, which was in 2016.

Extract to give background of the study

My research area is the pedagogy of Children's Mathematical Graphics. It is about teachers' perspectives on their mathematics teaching and how they can teach in a way that uncovers children's own representations of mathematics. This builds on my earlier work and co-research with Maulfry Worthington on Children's Mathematical Graphics. Children's Mathematical Graphics is a term first used by Worthington and Carruthers (2003) and Carruthers and Worthington (2005). The term is used to describe children's own ways of representing their mathematical thinking and this can be through any kind of signs, drawings scribble-like, marks, mathematical notation and writing. The research uncovered that children chose to use a range of graphics including their own invented mathematical symbols and standard symbols when they were working out a mathematical problem It has its roots in Hughes' (1986) work on children and number where he observed that four-year-olds could represent quantities in a variety of ways. The noteworthy aspect of this research was that children were using their own thinking; they were not told what to write or given a model to copy.

Our original study (Carruthers & Worthington, 2005) was mainly from the classes we had taught, ranging from Nursery through Key Stage One. When we first presented our findings at a research conference (EECERA, 2003) one of the professors in the audience Bert van Oers from the Free University, Amsterdam asked us how we got such rich examples of children's own mathematics. We could not really answer him as we were so focused on the meanings of the children's graphics. However, his question was an important and obvious question. Much of the research on children's mathematical graphics is to do with the analysis of the graphics and the children's thinking and although there is a connecting thread between the children's mathematical graphics and the pedagogy (Carruthers and Worthington, 2006 and 2011) there has not been sufficiently focused research on the pedagogy that uncovers and develops the children's own mathematical graphics.

There is a need for teachers to understand the pedagogical strategies that might be employed to uncover Children's Mathematical Graphics and develop children's mathematical thinking. Knowing that children can use their own methods, symbols and notation to help them solve mathematical problems is not enough. If we only know but do not support or value this, then it will become hidden and the knowledge we have about young's children's own mathematics will remain static. Therefore, the focus of this study is finding out how teachers can support children's own mathematical representations in the Foundation Stage (nursery and reception).

This is a longitudinal study originally with fifteen teachers and then with four out of the fifteen. Each part informs and builds on the findings from the last.

Ethical Discussion with my Research Colleague, December 2016

1. **Researcher access/ exit**

   I accessed the participants through a mathematics double module I had set up with Bath Spa University. Through the module I accessed data for my research. Most of the participants exited at the end of the 18 months of the module. Four agreed to continue as a focus group and case studies. I will close the project when I read the final transcripts back. I will be clear when I am finishing, informing my participants.

2. **Information given to participants**

   The Information form is included with a brief description of the research and participants' rights, for example, the right to withdraw at any time. I also had a verbal conversation with the participants outlining the main points and their right to withdraw and I discussed the power relationships and the issues. I said that at different times I felt that the power had swayed from researcher to participant. In the modules I felt uncomfortable especially because two of the teachers, in my school, were in the study. My research colleague and I had a long conversation about this and we felt it might be less of a power issue for heads of nurseries as the pressure was not so much to do with performance management and observations of classroom practice. My two teacher, colleague participants were not failing but strong in practice and pedagogy. These teachers plus the other focus group and case study teachers understood about ethics as they all had taken an ethics course for their masters' work. They could and do challenge me and continue to do so. This is all part of the richness of a learning community.

3. **Participants Right to Withdraw:** It is clear within the consent form that the participants have the right to withdraw at any time.

4. **Informed Consent:** The participants were given the consent forms plus an explanation of the proposed research. Throughout the research process I will be keeping them informed of the research and the outcomes. We will constantly discuss the implications for them. The nature of Participatory Research allows the participants to be informed as the research unfolds.

5. **Complaints procedure**
   I have given the university supervisor's name if the participants want to get in touch. I also stressed the BERA ethical guidelines and how they could access them. The case study participants are familiar with the guidelines as they have used them.

6. **Safety and well-being of participants/ researchers**
   I think that through the research I will be constantly reflecting with the participants and checking in if they want to ask questions. I think there is a dilemma in the research I propose, in that the researcher may be over anxious to obtain good data and yet trying to keep the welfare and rights of the participants in mind. It is a careful balance. I also think it is important to always confirm what the participants are saying so as not to misinterpret their meanings. I will be sharing drafts of the conversations and interviews with the participants to check for understanding. I am aware that the

teachers might be exposed, perhaps in a negative way, in their teaching which might not be what they feel comfortable with. I need to be aware of these issues and address them as I go along. We have created a research, learning community within our school and this is what I want to create in my research. This might address some issues with the power structure. My research colleague and I talked about the issue of using assignments as they are personal to the student and you can feel your invading the participants privacy. I explained that if I use any extracts from particular assignments then I will ask their permission again.

7. **Anonymity/ confidentiality**- This is semi-participatory as I made a decision that I did not want to do research *on* or *to* people. I intend to use initials of participants and also ask them if they would prefer me to use initials or a name of their choosing. The writing up of the study will eventually be narrowed mainly to about four case studies and I will inform the participants of any proposed changes. There is also general writing in the assignments and in my diary where it is less easy to identify the participant. As this study is semi-participatory, therefore, this will bring up other issues, for example, using too much of the participant's time. I will need to address these issues.

8. **Data collection:** I am using my diaries, interviews, focus group and participants' writing and examples of children's mathematical graphics. I have decided to use a dictaphone in focus group discussions. We are analysing the mathematical graphics together and then I am checking in with the participants as the study continues.

9. **Data analysis.** I am using thematic analysis and developing main points. I will have a research colleague check for interpretation and agreement of points. I think there is a dilemma when obtaining research data in being over anxious to obtain good data and still keeping the welfare and rights of the participants in mind. I also think it is important to always confirm what the participants are saying so as not to misinterpret their meanings. I am aware that the teachers might be exposed, in a negative way in their teaching, which might make them uncomfortable. I need to be aware of these issues. We have created a learning community within our school and this is what I want to create in my research. The power structure therefore could be more equal. If I use any extracts from particular assignments, then I will ask their permission.

10. Data storage I will store all data on my computer under a specific file, with initials only. I will anonymise the transcripts and keep a separate file with names and the codes. All my files will be password protected. The data will be kept for 6 months after I finish the thesis and after the viva.

11. Data Protection. All data will be handled and stored according to the Data Protection Act 2018.

12. Feedback. Throughout the project I will be checking and discussing the research project with them. The participants will have a chance to read and comment on all transcripts and I intend that we will publish together.

13. Responsibilities to colleagues/ academic community.

All interpretations of my qualitative analysis will be explored and discussed within the research. All collaborations with colleagues will be acknowledged especially my previous work with Maulfry Worthington. I will uphold the Bera Guidelines. Within my research the participants are part of the research community therefore I will name them, if they wish.

14. Reporting of research- I have discussed and written in the consent form how this will be disseminated through publications e.g. journal articles and chapters in books and also through conferences and seminars.

Signed by Elizabeth Carruthers and Anne Edwards, December, 2016.

Appendix 4: Consent Form

University of Bristol: Research led by Elizabeth Carruthers
Partly Funded by the Martin Hughes Memorial Scholarship

Dear

This year I am beginning to collect data for my PhD. I need a group of teachers to take part in my research, therefore I would really appreciate it if you would be willing to be part of this research. The data will be collected by using focus group discussions and individual meetings with the research teachers. It will include a collection of children's mathematical graphics. It will also include access to your assignments. It may also include classroom observations if you are willing.

This is ethical research and adheres to the BERA Ethical Research Guidelines and the Graduate School of Education, University of Bristol Ethical Guidelines. Data will be used for educational purposes only. If you agree to take part in this research you will be informed of procedures at all stages. Part of my thesis is about empowering teachers and through the research period I will continually listen to your views and your perspectives. You can withdraw from the research at any time.

The title of this research

The Pedagogy of Children's Mathematical Graphics in Number; Teacher Perspectives.

What you do for your assignment could also fit into this research. If you choose to focus an assignment on calculation/mathematical graphics then this would be perfect.

Please get in touch with me if you need more information. Telephone [redacted] *My supervisor is Rosamund Sutherland and she can be contacted on* [redacted]

With sincere thanks,

Elizabeth Carruthers

I agree to be a part of this research

Signature: ………………………………………………………

Date: ……………………………………………………………

Appendix 5: Masters Modules Brief

Logo of Nursery School deleted.

…………………Nursery School and National
Teaching School
Early Years Mathematics Masters / The Early Years Mathematics Specialist
Tutor: Elizabeth Carruthers
Venue: …………...Nursery School.

Who is this course aimed at?

This new, double mathematics education module is aimed at Early Years professionals who wish to understand more and develop children's mathematics in their settings and in their local area.

Children's Mathematics

With many pressures on the teaching of mathematics it can be difficult to appreciate just how much understanding of mathematics young children already have, and how this early knowledge can be supported in ways that are meaningful for children throughout the Foundation Stage.

The key areas of focus include:

* Promoting a classroom culture of enquiry, both physical and psychological.
* Pretend play and mathematics – links with children's home cultural knowledge.
* Children's mathematical graphics: introduction to key aspects of meaning making through children's own invented symbols, signs, drawings, writings and scribble like marks.
* From counting to calculation: analysing current pedagogical models
* Developing children's mathematical graphics – focus on calculations
* Assessment – how do you assess children's mathematical understandings in the Foundation Stage?
* Leading change and influencing wider school practice: coaching and leading theoretical input and activities. There will be an expectation that teachers will become leaders of mathematics pedagogy within their locality.

How will it help my practice?

This double Master's module aims to develop teachers' knowledge and experience in early years mathematics education, enabling them to work in a mentor /advisory role within the authority, to support Early Years mathematics within and beyond their schools, and to raise achievement in mathematics. As an integral feature of the course you will engage in a small research project in your own setting.

The course will focus on children's symbolic languages, encouraging critical discussion and reflection. Reference will be made to up-to-date theories and core texts on young children's mathematical learning. There will be an emphasis on the interactions of the adult and the children in negotiating mathematical understanding, and learning about children's learning.

The course will be a balance of discussion and direct tutor input to promote professional dialogue and further thinking. The classrooms and practice of the teachers involved will be a constant focus for discussion, analysis and points for change.

Appendix 6: Example of Initial Coding of Data.

The first analysis of the data from the assignments using initial open coding and writing down all the possible themes within the data. (all pseudonyms).

Zoe- Reception Teacher
Confronting areas of classroom practice that might benefit from development. Culture of play-long periods of uninterrupted sessions of play-free flow-strong ethos of child initiated learning. Strong emphasis on outdoor play-all adults working hard on what play looks like-staff training on how to ask open questions-the play environment, resources-Importance of the child's voice-Mathematics area to develop, needs more enquiries-examples of Teaching story changing practice, using numicon, children involved and returning to their problem later in the day- children solving their own problems after teacher initiated a question. Less restricting materials-children self-motivated-investigative mathematics-children using their graphics-talked about opening up the mathematics, open ended questions-changing practice, outdoor measuring. Children using data collection-children having a go, initiating their mathematics. Staff mostly on the edge of children's mathematical thinking-unaware of mathematics beyond number and counting and number thinking-asked for staff to have training on children's enquiries. Now seeing some exciting learning from children.

Melinda-Reception class Teacher
Thoughts on the current mathematics curriculum-Mathematical enquiry- verbal examples of graphics-expectations that children could only use numbers up to 10. "a child counting in 100's was far beyond my expectations'' excited by the mathematics. Direct modelling of the strategies. Changing practice -changing processes -moving to higher expectations for children for children- battle between child initiated play and adult directed learning interrupted play-changing play attitudes-"we were moving children from valuable platy situations"- change of focus, open ended mathematics-restrictions of the curriculum.

Kathy-Reception Teacher
Peer observations- 'a chance to see practice from children's eyes" questions the use of learning intentions-teacher led mathematics- changing practice-thinking about allowing children to develop their play-curriculum pressures-questions children's mathematical graphics-making small changes-open mathematics opportunities- discovering children know more about mathematics than she originally thought-using a book as a starter for a maths lesson-how big is a million-children having ownership of their mathematical learning-interesting questions about mathematics.

Beatrice-Reception class teacher
Revisiting her passion- re-examining her underlying pedagogy-underlines the phrase meaningful mathematical discovery-child centred mathematical discovery-child centred mathematical provision feels that is what the government advocates. Supports both a Piagetian and Vygotsky approach to play in her class (she believes)- the importance of parents. She asserts that one of her aims is to further her understanding of children's mathematical graphics-she says at present she focuses on recording not mathematical graphics-pressurised by year one expectations-insights on the literature regarding children's mathematical graphics-feels like she is a technician having to follow a set of rules. Set up a workshop on mathematics for parents.

Hilary-Reception class teacher
She aims to set mathematics free in her classroom. Changing practice-present practice uses themes for mathematics and sets up mathematical focused role play -now moving more to listening to children's interests-Acknowledges the limitations of adult set up role-play areas-engaging children in little child-initiated mathematics happening-moving to a balance of child-initiated and adult-initiated play. 'observations of children's 'true' mathematical knowledge in their play and mark making is now planned for'-moving to real life mathematics in everyday contexts. Moving away from mainly adult-directed or adult initiated activity.

Harry-Nursery School Teacher
Children's agency-imaginary pretend play-examples of pretend play- and the mathematics that arose from theses play episodes-how to interact in children's play-co-engagement-relationships with children-key person-interaction with child on home visit.-detailed observations of block play-children's real dilemmas and enquiries-culture of enquiry-pedagogical framing rather than pedagogical interactions-adults have to really listen to children. Sensitive to children's spontaneous enquiries-example of children's interest in large numbers-intellectual search-his philosophy in placing re. child at the centre , opening the mind to children's constructs.

Jess-Nursery School teacher
Mathematical graphics, understanding marks as mathematical thinking-responding to a child's mathematical graphics-discusses in length difference between children's recording and mathematical graphics with children's examples-the nursery is skilled at following the children's interests .. aim to follow mathematical graphics further. Definite about "mathematical graphics belonging within the play environment". Mathematical graphics has many layers. Three scenarios of pretend play.

Nadira-Nursery School teacher
Fundamental ethos, children learn through play. "I see play as child-initiated and child-led and the environment is paramount" -resources set out with children in mind. It is difficult to individualise teaching when you have 26 children in your nursery class - adults demonstrate to children how mathematical knowledge can be recorded in different ways- importance of extending children's thinking in mathematics-Flexible environment, attention given to the layout and appeal of resources.

Marcus-Nursery School teacher
Maths seen as correct or incorrect-in his setting maths often seen as boring or difficult by the practitioners-mathematics in his nursery. Separate and distinct area of the curriculum-examples of play, children engaging in mathematics-pedagogical relationships-key person-tuning in and really listening-co-construction-problem solving-moving between scaffolding and co-constructing-children's discussion about a number line.

Sue-Nursey School teacher
Practitioners views on mathematics are mostly negative-influence of teacher-EAL learners-mathematical graphics example -exploring counting, nursery rhymes-creating conditions for mathematics-current education pressures-child-initiated play-outside play-problem solving-was inspired by teacher talks—teaching measures and surprised at what the children knew about mathematical measures-changing practice-Ofsted testing children and closing down

play-influencing other staff members-examples of mathematical play and graphics-pretend play examples-examples of graphics.

Lynn-Nursery School teacher
Observed very little mathematics happening in the nursery. Early Years Practitioners reported mathematics as complicated and difficult, not for them. They reported being humiliated in maths lessons-questions what mathematics is-a lot of mathematics that happens during the child-initiated time gets lost and becomes a missed opportunity-children's thinking not identified as mathematics-gives examples of missed opportunities for mathematical learning in play-lack of subject knowledge-giving children genuine rather than contrived provocations-listening to families-need to identify the maths in play- or mathematics relies on closed activities-need to further embrace the Key Person relationship-creative play environment but practitioners are not aware of the mathematics -example of water play and teacher of

Ella-Nursey School teacher
The Nursery School values the play environment -communication friendly spaces creating a home environment -good quote about children being valued-children are seen as independent confident co-creators-freedom to access resources examples of child's mathematical play, transporting resources-quantities of resources in the environment so that can use them flexibly-changing mathematical resources to being authentic-children have long extended periods of play -play underpins all learning and development. Practitioners unsure of the right time to interact with children's play-discuss children's play theoretically-discuss observations of children's play and how reach it is with mathematical possibilities-feels that practitioners have no support to extend their subject knowledge-changing practitioners practice to a research and enquiry form of professional development based on everyday practice. This collaborative enquiry informed the practitioners of the mathematical knowledge the children already had from their home and community cultures- mathematical language-planning time to listen to children-change, developing understanding of children's mathematical graphics-the importance of reflecting during the Nursery School day not just after it.

Millie -Reception class Teacher
Current beliefs-play not valued-lack of child initiated mathematics-isolated cases of child-initiated mathematics-not prioritising play-significant moments of mathematics teaching-case study highlights the true importance of allowing children control of their learning and valuing their ideas-the children were now working at a higher level-children excited to find out. Support by head teacher gave me confidence-needing to learn more about play-following children's interest-questioning educational policy and how it sits with teacher beliefs-practical shifts in practice- Maths occurring in my class at the beginning of the year was of limited value.

Esme-Nursery School Teacher
Culture is of paramount importance, her key person group-culture children's questions, importance of adults listening, the community environment, environments that provoke mathematics, teacher challenged to question their own 'funds of knowledge'.
Children discovering their own theories, sharing in children's fascinations, teachers' role as 'guided interactions'. Children influencing teachers thinking-individual children's thinking, adult and child as equal partners-being with the children-communication as the driving force behind play-questioning curriculum aims, teaching story-data handling child-initiated

children's agency-children's play-questions that play is used for curriculum agenda, questions and issues in play-teachers story-who leads in play? Example of pretend play.

Janine – Reception class teacher
Changing practice to be more play orientated-one example of child's play-developing the graphics area, children becoming more interested in mark-making. Now displays show 'work in progress'-developing 'open' questions-discussions on adult -led, child-initiated activity, needs a balance-feels that an adult can initiate, and children can take it in their own direction-my planning needs to be focus on observations of children. Starting it right down in children's play-disagrees with Fisher's contention that, if there is a pre-determined task or goal in mind no play can take place-'the role of the adult is fraught with difficulties' there is a conundrum between what the EYFS(2012) says about play and the school readiness agenda. She hopes the changes in her classroom will support children's mathematical graphics. She says when she shared her examples of children's mathematical graphics on the course, some of what she perceived as mathematical graphics was recording

Appendix 7: Chart to show Recurring Themes of Reception teachers and Nursery School teachers' Reflections of their Mathematics Teaching Practice.

Recurring Themes

Nursery School Teachers	Reception Teachers
Colleagues, practitioners not confident in mathematics. A general dislike of mathematics. Not aware of mathematics.	Pressure to plan from set objectives-restricting children to set models for mathematics
Key person, attachment, emotional wellbeing.	Dilemmas between adult-led and child-initiated learning, pulling children away from play and child initiated opportunities. Not noticing the mathematics in children's play.
Importance of the play environment, valuing play.	Changing practice-observing child initiated play-open questions-open mathematics-thinking more about widening mathematical resources-planning to create a culture where children's mathematical graphics can flourish-Making small changes, examples of practice, open questions, limitations of their teaching.
Scenarios of pretend play children leading, children's mathematical graphics, teachers listening.	Examples of whole school enquiries encouraging children to lead
Children's mathematical graphics, developing practice, planning to change, beginning to notice children's mathematical graphics with two examples. Discussing graphics with the team.	Considerable conceptual changes in practice.
Examples of different types of play.	Difficult to see the difference between children recording their own mathematics and children's mathematical graphics.
Listening to children, making time to listen	Children excited to find out -working at a higher level.
Importance of reflecting	Surprised at how much children knew about mathematics.

Appendix 8: Chart to Show the Recurring Themes of the Reception Teachers Reflections on their Mathematics Teaching Practice Linked to the Headings of Data Chapter 6.

Recurring themes of the Reception Class teachers	**The headings of data chapter 6, The Reception Teachers.**
Pressure to plan from set objectives-restricting children to set models for mathematics	6.1.1 "*The demands placed on Reception Class teachers"* (Millie F.N., 17/2/2014)
Dilemmas between adult-led and child-initiated learning, pulling children away from play and child initiated opportunities. Not noticing the mathematics in children's play.	6.1.2 *"A constant battle between child-initiated and adult-led learning"* (Melinda, W.R., 15/9/2014)
Changing practice-observing child initiated play-open questions-open mathematics-thinking more about widening mathematical resources-planning to create a culture where children's mathematical graphics can flourish-Making small changes, examples of practice, open questions, limitations of their teaching.	6.1.3 "*True mathematics comes out of freedom"* (Hilary, W.R., 15/9/2014) 6.1.4 *"The first part of the process will be to create a culture in the classroom in which children become used to using graphics to aid their mathematical thinking"* (Beatrice, W.R., 15/9/2014) *6.1.5 "Beyond the expectations of what the early years curriculum expects"* (Kathy, W.R., 15/9/2014)
Examples of whole school enquiries encouraging children to lead.	6.1.6 *"Transferring the ownership of the maths to the child"* (Mille, F.N., 12/6/2014)
Considerable conceptual changes in practice.	6.1.6 *"Transferring the ownership of the maths to the child"* (Mille, F.N., 12/6/2014)
Difficult to see the difference between children recording their own mathematics and children's mathematical graphics.	6.1.7 *"The trouble is I want is I want it* [children's mathematical graphics] *to happen now* (Janine, F.N., 13/6/2014) 6.1.8 *"Recording of a product rather than aiding the thinking process"* (Beatrice, W.R., 15/9/2014
Children excited to find out -working at a higher level.	6.1.6 *"Transferring the ownership of the maths to the child"* (Mille, F.N., 12/6/2014) *6.1.5 "Beyond the expectations of what the early years curriculum expects"* (Kathy, W.R., 15/9/2014)
Surprised at how much children knew about mathematics.	*6.1.5 "Beyond the expectations of what the early years curriculum expects"* (Kathy, W.R., 15/9/2014) *6.1.6 "Transferring the ownership of the maths to the child"* (Mille, F.N., 12/6/2014)

Appendix 9: Chart to Show the Recurring Themes of the Nursery Teachers Reflections on their Mathematics Teaching Practice Linked to the Headings of Data Chapter 7.

Recurring themes of the Nursery School Teachers	**The headings of data chapter 7, The Nursery Teachers.**
Colleagues, practitioners not confident in mathematics. A general dislike of mathematics. Not aware of mathematics.	*"A lot of maths that happens in child-initiated play gets missed"* (Lynn, W.R., 15/9/2014).
Key person, attachment, emotional wellbeing.	7.1.2 "If children do not feel valued and secure, they will not engage with their surroundings" (Tucker, 2014, p.23, cited in Ella, W.R., 15/9/2014)
Importance of the play environment, valuing play.	7.1.3 *"The nursery school values the* [play] *environment*" (Ella, W.R., 15/9/2014)
Scenarios of pretend play children leading, children's mathematical graphics, teachers listening.	7.1.4 *"Yes half-past and it's run out of electricity and magic"* (Esme, W.R., 5/9/2014
Children's mathematical graphics, developing practice, planning to change, beginning to notice children's mathematical graphics with two examples. Discussing graphics with the team.	7.1.5 *"Offering the children opportunities to show us their mathematical thinking"* (Lynn, W.R., 15/9/2014).
Examples of different types of play.	7.1.3 *"The nursery school values the* [play] *environment*" (Ella, W.R., 15/9/2014)
Listening to children, making time to listen	7.1.3 *"The nursery school values the* [play] *environment*" (Ella, W.R., 15/9/2014) 7.1.4 *"Yes half-past and it's run out of electricity and magic"* (Esme, W.R., 5/9/2014 7.1.5 *"Offering the children opportunities to show us their mathematical thinking"* (Lynn, W.R., 15/9/2014).
Importance of reflecting	7.1.3 *"The nursery school values the* [play] *environment*" (Ella, W.R., 15/9/2014) 7.1.4 *"Yes half-past and it's run out of electricity and magic"* (Esme, W.R., 5/9/2014

Appendix 10: Criteria used by the University for marking the Reflective Commentary Assignment.

Criteria for Marking the Reflective Commentary of Current Teaching Practice Assignment		September 2014	
Areas for Assessment	Further Description		
Critical Analysis and reflection	Understanding of relevant theoretical frameworks, using the literature critically, developing different perspectives and arguments.		
Critical evaluation of practice and research	Locating own study in the context of wider issues and society.		
Structure and cohesion	Appropriate organization, building of argument and development of thinking leading to conclusion.		
Writing/presenting style	Suitably academic, properly referenced; use of academic voice and formal language.		

Appendix 11: Abbreviations and Acronyms

CPD - Continual Professional Development

DCSF - Department for Children, Schools and Families

DES – Department of Educations and Science

DfE - Department for Education

DfEE - Department for Education and Employment

DFES - Department for Education and Skills

EYFS - Early Years Foundations Stage

LSA - Learning Support Assistant

NCTEM - National Centre for Excellence in the Teaching of Mathematics

OFSTED - Office for Standards in Education

PGCE - Post Graduate Certificate in Education

SATs - Standard Attainment Tasks

SCAA - Schools Curriculum and Assessment Authority

SLE - Specialist Leader of Education

SLT - Senior Leadership team

TACTYC - Together and Committed to Today's Young Children

UNICEF - United Nations International Children's Emergency Fund