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KF, PKF, AND REINHARDT'S PROGRAM

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Abstract. In "Some Remarks on Extending and Interpreting Theories with a Partial Truth Predicate", Reinhardt [21] famously proposed an instrumentalist interpretation of the truth theory Kripke–Feferman (KF) in analogy to Hilbert's program. Reinhardt suggested to view KF as a tool for generating "the significant part of KF", that is, as a tool for deriving sentences of the form $\mathrm{Tr}\ulcorner\varphi\urcorner$. The constitutive question of *Reinhardt's program* was whether it was possible "to justify the use of nonsignificant sentences entirely within the framework of significant sentences". This question was answered negatively by Halbach & Horsten [10] but we argue that under a more careful interpretation the question may receive a positive answer. To this end, we propose to shift attention from KF-provably true *sentences* to KF-provably true *inferences*, that is, we shall identify the significant part of KF with the set of pairs $\langle \Gamma, \Delta \rangle$, such that KF proves that if all members of Γ are true, at least one member of Δ is true. In way of addressing Reinhardt's question we show that the provably true inferences of suitable KF-like theories coincide with the provable sequents of matching versions of the theory Partial Kripke–Feferman (PKF).

§1. Introduction. Kripke's theory of truth [12] is a cornerstone of contemporary research on truth and the semantic paradoxes. The theory provides us with a strategy for constructing, that is defining, desirable interpretations of a self-applicable truth predicate as the fixed points of a certain monotone operator (so-called Kripke Jump). These fixed points can serve as interpretations of the truth predicate within non-classical models of the language, as in Kripke's original article, but can also be used in combination with classical models, so-called closed-off models.[1] Feferman [6] devised an elegant axiomatic theory of the Kripkean truth predicate of these closed-off fixed-point models. The theory is known as Kripke–Feferman (KF) and is still one of the most popular axiomatic truth theories in the literature formulated in classical logic.[2]

[1] This was also suggested by Kripke (cf. [12, p. 715]).

[2] In the literature the label KF is used for different theories. In particular, Reinhardt [20, 21] and Halbach & Horsten [10] use KF to include the axiom (Cons) while Halbach & Nicolai [11] and Nicolai [14] (see also [9]) use KF to refer to the theory Ref(PA) introduced by Feferman [6] that dispenses of (Cons). Moreover, Cantini [2] uses KF to denote a theory with a restricted form of induction (usually called internal induction). From Section 3 onward we commit to the latter use of KF, but at this point we allow KF to stand for either theory unless we refer to particular results in which case we defer to the conventions of the relevant author.

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Nonetheless, KF displays a number of unintended and slightly bizarre features, which it inherits from the behavior of the truth predicate in the closed-off fixed-point models. While in the non-classical fixed-point models the truth predicate is transparent, i.e., φ and $\mathrm{Tr}\ulcorner\varphi\urcorner$ will always receive the same semantic value (if they receive any), this no longer holds in the closed-off models. Rather, for each closed-off model there will be sentences φ, e.g., the Liar sentence, such that either φ and $\neg\mathrm{Tr}\ulcorner\varphi\urcorner$ will be true in the model, or $\neg\varphi$ and $\mathrm{Tr}\ulcorner\varphi\urcorner$ will be true in the model. As a consequence one can prove this counterintuitive disjunction in KF for the Liar sentence λ, i.e.,

$$\mathrm{KF} \vdash (\lambda \wedge \neg\mathrm{Tr}\ulcorner\lambda\urcorner) \vee (\neg\lambda \wedge \mathrm{Tr}\ulcorner\lambda\urcorner). \tag{$*$}$$

Since the transparency of truth seems to be one of the basic characteristics of the truth predicate, the aforementioned asymmetry puts the idea of understanding the closed-off models as suitable models of an intuitively acceptable truth predicate under some stress and alongside casts doubt on KF as an acceptable theory of truth. However, reasoning within the non-classical logic of the Kripkean fixed-points seems a non-trivial affair or, as Feferman [5, p. 95] would have it, "*nothing like sustained ordinary reasoning can be carried on*" in these non-classical logics. Giving up on KF thus hardly seems a desirable conclusion.

In reaction to the counterintuitive consequences of KF, Reinhardt [20, 21] proposed an instrumentalist interpretation of the theory in analogy to Hilbert's program. Famously, Hilbert proposed to justify number theory, analysis and even richer mathematical theories by finitary means. Without entering into Hilbert-exegesis, the main idea was of course to provide consistency proofs for these mathematical theories in a finitistically acceptable metatheory. From a finitist perspective this would turn the mathematical theories into useful tools for producing mathematical truths. But the fate of Hilbert's program, at least on its standard interpretation, is well known: Gödel's incompleteness theorems are commonly thought to be the program's coffin nail. Nonetheless Reinhardt [20, 21] was optimistic that his own program had greater chances of success.[3] Reinhardt proposed to view KF as a tool for deriving Kripkean truths in the same way Hilbert viewed, say, number theory as a tool for deriving mathematical truths. A Kripkean truth is a sentence that is true from the perspective of the Kripkean fixed points: if $\mathrm{KF} \vdash \mathrm{Tr}\ulcorner\varphi\urcorner$ ($\mathrm{Tr}\ulcorner\neg\varphi\urcorner$), then φ is true (false) in all non-classical fixed-point models, that is, we are guaranteed that φ receives a semantic value from the perspective of Kripke's theory of truth. This is not a general feature of the theorems of KF but peculiar to those sentences that KF proves true (false). The latter sentences Reinhardt called "*the significant part of* KF" [21, p. 242] and labelled the set of KF-significant sentences $\mathrm{KFS} := \{\varphi \mid \mathrm{KF} \vdash \mathrm{Tr}\ulcorner\varphi\urcorner\}$.[4] In light of this terminology the constitutive question of Reinhardt's program is whether it is possible "*to justify the use of nonsignificant sentences entirely within the framework of significant sentences*" [21, p. 225].

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[3] On page 225 Reinhardt [21] writes:

> "I would like to suggest that the chances of success in this context, where the interpreted or significant part of the language includes such powerful notations as truth, are somewhat better than in Hilbert's context, where the contentual part was very restricted."

[4] KFS is sometimes also called the *inner logic* of KF (cf. [10, p. 638]). We decided to stick with Reinhardt's original terminology.

But what would such a successful instrumentalist interpretation of KF and the use of non-significant sentences amount to? Reinhardt himself is scarce on the exact details and does not commit to a particular answer to this question. However, at the end of [20], without explicitly addressing the question of an instrumentalist interpretation of KF, Reinhardt asks the following question:

If $KF \vdash Tr\ulcorner\varphi\urcorner$ is there a KF-proof

$$\varphi_1, \ldots, \varphi_n, Tr\ulcorner\varphi\urcorner$$

such that for each $1 \leq i \leq n$, $KF \vdash Tr\ulcorner\varphi_i\urcorner$? (cf. [20, p. 239])[5]

If we were to answer this question positively, we could justify each Kripkean truth provable in KF by appealing solely to the significant fragment of KF: even though we have reasoned in KF, each step of our reasoning is part of KFS and hence remains "within the framework of significant sentences." It is then suggestive to take the above question to be constitutive of Reinhardt's program. Indeed, this interpretation of Reinhardt's program is adopted by Halbach & Horsten [10], who called the question REINHARDT'S PROBLEM. Unfortunately, as Halbach and Horsten convincingly argue, if understood in this way the instrumentalist interpretation of KF will fail. We refer to [10] for details but, in a nutshell, the reason for this failure is that the truth-theoretic axioms of KF will not be true in the non-classical fixed-point models and hence not be part of KFS, e.g., if $\varphi_i := \forall x(St_{Tr}(x) \rightarrow (Tr(x) \leftrightarrow Tr(\neg\neg x)))$,[6] then $\varphi_i \notin KFS$. Indeed, Halbach & Horsten [10, p. 684] take this to show "*that Reinhardt's analogue of Hilbert's program suffers the same fate as that of Hilbert's program*".

However, we think that this conclusion is premature and argue that, to the contrary, if suitably understood Reinhardt's program has good chances of succeeding. Our key point of contention is that Halbach & Horsten [10], arguably following Reinhardt, employ the perspective of classical logic when theorizing about the significant part of KF. But the logic of the significant part of KF is not classical logic but the logic of the non-classical fixed-point models, that is, a non-classical logic. This observation has two interrelated consequences for Reinhardt's program. First, contra Reinhardt, and contra Halbach and Horsten, we should not identify the significant part of KF exclusively with the set of significant *sentences*. Rather it also seems crucial to ask which *inferences* are admissible within the significant part of KF.[7] Of course, in classical logic this difference

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[5] To be precise, Reinhardt [20] asked whether there was a KF-proof $\varphi_1, \ldots, \varphi_n, \varphi$ rather than a KF-proof $\varphi_1, \ldots, \varphi_n, Tr\ulcorner\varphi\urcorner$. But this presupposes that all KF-provably true sentences are also theorems of KF. While this is true in the variants of KF Reinhardt considers, this is not the case in all versions of KF discussed in the literature. However, all remarks concerning our version of the question generalize to Reinhardt's original question.

[6] See Section 2 for details on notation.

[7] An anonymous referee suggests that since Reinhardt identifies the significant part of KF with the set of significant sentences and not with the set of admissible inferences, our proposal is not a different version of Reinhardt's program but a different project altogether. This really depends on what one takes Reinhardt's program to be, as Reinhardt himself remains rather vague and unspecific in this respect. On our view, Reinhardt's program is the project of giving an instrumentalist justification of KF. Whilst Reinhardt might not have thought of the particular instrumentalist justification we provide, there is nothing to suggest that our proposal is incompatible with the general outlines of the program sketched by Reinhardt and that it is a new project altogether. However, nothing hinges on this and readers who tend

collapses due to the deduction theorem but not necessarily so in non-classical logics. For example, the three-valued logic *Strong Kleene*, K3, has no logical truth, but many valid inferences—if, in this case, we were to focus only on the theorems of the logic, there would be no logic to discuss. Moreover, since the significant sentences can be retrieved from the significant inferences, that is the provably true inferences, we should focus on the latter rather than the former in addressing REINHARDT'S PROBLEM.[8] To this end, it is helpful to conceive of KF as formulated in a two-sided sequent calculus. Let Γ, Δ be finite sets of sentences and let $\text{Tr}\ulcorner\Gamma\urcorner$ be short for $\{\text{Tr}\ulcorner\gamma\urcorner \mid \gamma \in \Gamma\}$. The admissible inferences of the significant part of KF, which we label KFSI, can then be defined as follows:[9]

$$\text{KFSI} := \{\langle \Gamma, \Delta \rangle \mid \text{KF} \vdash \text{Tr}\ulcorner\Gamma\urcorner \Rightarrow \text{Tr}\ulcorner\Delta\urcorner\}.$$

Second, REINHARDT'S PROBLEM, according to the formulation of Halbach & Horsten [10], which admittedly was inspired by Reinhardt's [20] original question, conceives of KF-proofs as sequences of theorems of KF. But by focusing on sequences of theorems, we cannot fully exploit the significant part of KF, that is, KFSI for precisely the reasons Halbach & Horsten [10] used to rebut Reinhardt's program: while double negation introduction is clearly a member of KFSI, proving this fact by a sequence of theorems would take us outside of KFS since it would use the truth-theoretic axiom $\forall x(\text{St}_{\text{Tr}}(x) \to (\text{Tr}(x) \leftrightarrow \text{Tr}(\neg\neg x)))$, which is not a member of the significant part of KF. This suggests a reformulation of REINHARDT'S PROBLEM in terms of a notion of proof that focuses on inferences rather than theorems. To this end, it proves useful again to formulate KF in a two-sided sequent calculus and to conceive of proofs as derivation trees, where each node of the tree is labeled by a sequent. As a matter of fact, in this case we can distinguish between two versions of REINHARDT'S PROBLEM:

(1) For every KF-theorem of the form $\text{Tr}\ulcorner\varphi\urcorner$, is there a KF-derivation $\mathcal{D}$ of $\varnothing \Rightarrow \text{Tr}\ulcorner\varphi\urcorner$ such that every node d of $\mathcal{D}$ is a significant inference?

(2) For every KF-derivable sequent of the form $\text{Tr}\ulcorner\Gamma\urcorner \Rightarrow \text{Tr}\ulcorner\Delta\urcorner$, is there a KF-derivation $\mathcal{D}$ of $\text{Tr}\ulcorner\Gamma\urcorner \Rightarrow \text{Tr}\ulcorner\Delta\urcorner$ such that every node d of $\mathcal{D}$ is a significant inference?

The first question is a reformulation of Halbach & Horsten's [10] REINHARDT'S PROBLEM.[10] The second question, which we label GENERALIZED REINHARDT PROBLEM, asks whether all provably true inferences can be justified by appealing to the significant inferences only. Arguably, to deem Reinhardt's program successful one needs to give

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to agree with the referee should understand our version of "Reinhardt's program" as a new project inspired by Reinhardt.

[8] The provable true sentences can be viewed as inferences where the truth of the sentence follows from an empty hypothesis.

[9] Notice that moving to a two-sided sequent formulation of KF is not essential. Due to the deduction theorem we can also define KFSI by appeal to KF formulated in an axiomatic Hilbert-style calculus. In this case, the definition would amount to

$$\text{KFSI} := \{\langle \Gamma, \Delta \rangle \mid \text{KF} \vdash \bigwedge \text{Tr}\ulcorner\Gamma\urcorner \to \bigvee \text{Tr}\ulcorner\Delta\urcorner\}.$$

[10] Admittedly, our reformulation amounts to a proper weakening of the original formulation of REINHARDT'S PROBLEM, that is, the question asked in [20]. An affirmative answer to latter question implies an affirmative answer to its reformulation but not vice versa (for precisely the reasons given in the main text).

an affirmative answer to the GENERALIZED REINHARDT PROBLEM.[11] Otherwise, a proof of Tr$\ulcorner\varphi\urcorner$ could still rely on inferences that, whilst part of KFSI, cannot themselves be justified by appealing only to the significant inferences of KF. In this paper, we shall argue that on this more careful formulation, Reinhardt's program has good prospects of succeeding. We corroborate our assessment by giving an affirmative answer to the GENERALIZED REINHARDT PROBLEM for versions of KF with internal and restricted induction.

Arguably, giving an affirmative answer to the GENERALIZED REINHARDT PROBLEM is at best a partial completion of Reinhardt's program: what is still required is an independent axiomatization of the significant part of KF, for only this would prove KF dispensable. We will now take a fresh look at the question of an independent axiomatization, which Halbach and Horsten called REINHARDT'S CHALLENGE [10, p. 689]. This will prove instrumental in answering GENERALIZED REINHARDT PROBLEM for the aforementioned versions of KF. Reinhardt [20] asked for an independent axiomatization of the significant part of KF. More precisely, Reinhardt [20, p. 239] asked:

  (a) "Is there an axiomatization of $\{\sigma \mid \text{KF} \vdash \text{Tr}\ulcorner\sigma\urcorner\}$ which is natural and formulated entirely within the domain of significant sentences,...."
  (b) "Similarly for the relation $\Gamma \vdash_S \sigma$ defined by $\text{KF} + \{\text{Tr}\ulcorner\gamma\urcorner \mid \gamma \in \Gamma\} \vdash \text{Tr}\ulcorner\sigma\urcorner$."

Halbach & Horsten [10] proposed their theory Partial Kripke–Feferman (PKF) in way of answering REINHARDT'S CHALLENGE. PKF is formulated in a non-classical, two-sided sequent calculus and thus fits neatly with our observation that one should focus on the provably true inferences of KF rather than the provably true sentences. Moreover, PKF is arguably a natural axiomatization of Kripke's theory of truth. However, Halbach & Horsten [10] observed that there are sentences φ such that $\text{KF} \vdash \text{Tr}\ulcorner\varphi\urcorner$ but $\text{PKF} \nvdash \varphi$, which led them to conclude that REINHARDT'S CHALLENGE cannot be met. The reason for this asymmetry is due to the difference in proof-theoretic strength of KF and PKF: while KF proves transfinite induction for ordinals below ε_0, PKF only proves transfinite induction for ordinals smaller than ω^ω. As a consequence, there will be arithmetical sentences that KF proves true that PKF cannot prove. The story does not end there however. First, as Halbach & Nicolai [11] observe, the discrepancy between KF and PKF arises only if the rule of induction is extended beyond the arithmetical language, that is, if we restrict induction to the language of arithmetic—call the resulting theories KF$^-$ and PKF$^-$—then $\text{KF}^- \vdash \text{Tr}\ulcorner\varphi\urcorner$ if, and only if, $\text{PKF}^- \vdash \varphi$. This highlights that the asymmetry between KF and PKF is not due to the truth-specific principles but the amount of induction that is assumed in the respective theories. Second, corroborating the latter observation, Nicolai [14] showed that the asymmetry between KF and PKF is indeed solely due to the amount of induction available within the respective theories: Nicolai shows that if transfinite induction up to $< \varepsilon_0$ is added axiomatically to PKF— call this theory PKF$^+$—then $\text{KF} \vdash \text{Tr}\ulcorner\varphi\urcorner$ if, and only if, $\text{PKF}^+ \vdash \varphi$. Moreover, Nicolai [14] shows that independently of which version of induction is assumed in KF there will be a suitable PKF-style theory that has exactly the provably true sentences of

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[11] An affirmative answer to GENERALIZED REINHARDT PROBLEM also would also yield a positive answer to our reformulation of REINHARDT'S PROBLEM, as Γ is allowed to be empty.

the relevant KF-style theory as theorems. Nicolai took these results to "*partially* [*accomplish*] *a variant of a program sketched by Reinhardt*" [14, p. 103].

However, the work by Halbach and Nicolai provides at best a positive answer to Question (a). It does not—at least not immediately—yield an answer to Question (b). Indeed, it seems as if, despite working in a non-classical, two-sided sequent calculus, Halbach & Horsten [10] have largely neglected Question (b) of REINHARDT'S CHALLENGE and so have subsequent publications on this issue. Indeed the version of PKF originally proposed by Halbach & Horsten [10] failed to yield a positive answer to Question (b) for rather banal reasons—even for theories with restricted induction: the version of KF Halbach & Horsten [10] consider assumes the truth predicate to be consistent, which, as we shall explain in due course, means that the logic of KFSI is K3. But Halbach and Horsten formulate PKF in symmetric strong Kleene logic KS3 and as a consequence $\mathrm{KF} \vdash \mathrm{Tr}^{\ulcorner}\varphi^{\urcorner}, \mathrm{Tr}^{\ulcorner}\neg\varphi^{\urcorner} \Rightarrow \mathrm{Tr}^{\ulcorner}\psi^{\urcorner}$ while $\mathrm{PKF} \nvdash \varphi, \neg\varphi \Rightarrow \psi$. The main technical contribution of this paper is to clarify the situation and to show how a positive answer to Question (b) of REINHARDT'S CHALLENGE can be provided for KF-variants with restricted, or internal induction. To this end, we show how to pair the different variants of KF with a suitable PKF-style theory such that the provable sequents of the latter theory constitute exactly the significant inferences of the former theory. Moreover, it turns out that once we have an independent axiomatization of the set of significant inferences of the KF-variant under consideration, an affirmative answer to the GENERALIZED REINHARDT PROBLEM for the particular KF-variant will be nothing but a corollary. What remains then to be shown to give a complete answer to the GENERALIZED REINHARDT PROBLEM and bring Reinhardt's program to a successful completion is to show that PKF^+—or, possibly, some other non-classical truth theory—amounts to a positive answer to Question (b) in the case of KF with full induction. While to date we lack a proof to this effect, it seems very likely that our results can be extended to pair KF-variants with full induction with a suitable PKF^+-theory.[12] This would show that PKF^+ amounts indeed to a positive answer to Question (b) for KF with full induction and that Reinhardt's program, on our proposed understanding, can be brought to a successful completion.[13] Indeed, the results for KF-variants with restricted or internal induction proved in this paper show that there is an alternative (and arguably more adequate) understanding of Reinhardt's program. Taken together, these considerations shed new light on Reinhardt's instrumentalism and vindicate Reinhardt's optimism in regard to his program.

1.1. Plan of the paper. The paper starts by fixing some basic terminology and notation. More specifically, Section 2 introduces the language and the logical systems underlying PKF and its variants. That is, we introduce the logics FDE, KS3, K3 and LP. In the next section, Section 3, we introduce the relevant families of KF- and PKF-like theories and observe some basic properties of these PKF-systems. In Section 4 we prove the central technical results of this paper. We show that for each KF-like theory with restricted or internal induction we can find a PKF-counterpart such that the latter is an independent axiomatization of the significant inferences of the former.

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[12] The main problem in providing such a result is that the techniques used by Nicolai [14] to show that KF and PKF^+ prove the same sentences true cannot be straightforwardly applied to deal with sequents.

[13] One might take issue with this claim by questioning whether PKF^+ is indeed an independent axiomatization of KFSI. We will come back to this issue in the conclusion of the paper.

In other words, we show that the set of pairs $\langle \Gamma, \Delta \rangle$ such that $\mathrm{Tr}\ulcorner\Gamma\urcorner \Rightarrow \mathrm{Tr}\ulcorner\Delta\urcorner$ is derivable in a KF-like theory coincides with the set of pairs $\langle \Gamma, \Delta \rangle$ such that $\Gamma \Rightarrow \Delta$ is derivable in a corresponding PKF-like theory. As mentioned, a positive answer to GENERALIZED REINHARDT PROBLEM is but an immediate corollary of the existence of such an independent axiomatization. In the conclusion (Section 5) we reevaluate Reinhardt's program, that is, the prospect of an instrumentalist interpretation of KF in light of our technical results.

§2. Logics and formal notation.

2.1. Language and notation. The language $\mathcal{L}_{\mathrm{PA}}$ denotes the language of first-order *Peano arithmetic* (PA) in the signature $\{\bar{0}, ', +, \times\}$ expanded by finitely many function symbols for suitable primitive recursive (p.r.) functions. The language $\mathcal{L}_{\mathrm{Tr}}^{-}$ expands $\mathcal{L}_{\mathrm{PA}}$ by a unary truth-predicate Tr. Terms and formulae are generated by closing off under $\neg, \wedge, \forall$ ($\vee, \exists, \rightarrow, \leftrightarrow$ are defined according to the conventions of classical logic). By an $\mathcal{L}_{\mathrm{Tr}}^{-}$-expression we mean a term or a formula of $\mathcal{L}_{\mathrm{Tr}}^{-}$. We let $\bar{n}$ be the numeral corresponding to the number $n \in \omega$. We fix a canonical *Gödel numbering* of $\mathcal{L}_{\mathrm{Tr}}^{-}$-expressions. If e is an $\mathcal{L}_{\mathrm{Tr}}^{-}$-expression, the Gödel number ($=$ gn) of e is denoted by $\#e$ and $\ulcorner e \urcorner$ is the term representing $\#e$ in $\mathcal{L}_{\mathrm{PA}}$. The sets of terms, closed terms, variables, formulae, and sentences of $\mathcal{L}_{\mathrm{Tr}}^{-}$ are p.r., and our language contains function symbols representing them. In practice, we take the following $\mathcal{L}_{\mathrm{Tr}}^{-}$-predicates to abbreviate the equations for the (p.r.) characteristic functions for such sets. For example, $\mathrm{Ct}(x)$ abbreviates $f_{\mathrm{Ct}}(x) = 1$, where f_{Ct} is the characteristic function of the set of codes of closed terms:

$\mathrm{Tm}(x)$ $(\mathrm{Ct}(x)) := x$ is the gn of a (closed) term;

$\mathrm{Var}(x) := x$ is the gn of variable;

$\mathrm{Fml}_{\mathrm{Tr}}^{n}(x)$ $(\mathrm{St}_{\mathrm{Tr}}(x)) := x$ is the gn of a formula with at most n (0) free distinct variables;

$\mathrm{Eq}(x) := x$ is the gn of an equality between closed terms;

$\mathrm{Ver}(x) := x$ is the gn of true closed equality.

Additionally, $\mathcal{L}_{\mathrm{Tr}}^{-}$ contains function symbols for the following p.r. operations on Gödel numbers:

OPERATION	FUNCTION SYMBOL
$\#t, \#s \mapsto \#(t = s)$	$\dot{=}$
$\#\varphi \mapsto \#(\neg\varphi)$	$\dot{\neg}$
$\#\varphi, \#\psi \mapsto \#(\varphi \wedge \psi)$	$\dot{\wedge}$
$\#v_k, \#\varphi \mapsto \#(\forall v_k \varphi)$	$\dot{\forall}$
$n \mapsto \#\bar{n}$	num
$\#e, \#t, \#v_k \mapsto \#e[t/v_k]$	sb

The expression $e[t/v_k]$ is the result of replacing, in the expression e, each free occurrence of v_k by the term t. Since $\mathcal{L}_{\mathrm{Tr}}^{-}$ contains only finitely many function symbols, there exists a p.r. evaluation function VAL on codes of closed terms of $\mathcal{L}_{\mathrm{Tr}}^{-}$, such that $\mathrm{VAL}(\#t) \mapsto t^{\mathbb{N}}$,

where $t^{\mathbb{N}}$ is the value of the closed term t in the standard model. We let $\mathcal{L}_{\mathrm{Tr}}$ be the language $\mathcal{L}_{\mathrm{Tr}}^{-}$ augmented by a unary function symbol $\mathrm{val}(x)$, which represents VAL.[14]

We will make use of the following abbreviations:

$$x(u/y) := \mathrm{sb}(x, u, y) \qquad\qquad x(t) := x(t/y) \qquad\qquad \varphi(\vec{v}) := \varphi(v_1, \dots, v_k)$$
$$x \dot{=} y := \dot{=}(x, y) \qquad\qquad x \dot{\wedge} y := \dot{\wedge}(x, y) \qquad\qquad \dot{\forall} u.x := \dot{\forall}(u, x)$$
$$x(\dot{z}) := \mathrm{sb}(x, \mathrm{num}(z), y) \qquad \ulcorner\varphi(\dot{x})\urcorner := \mathrm{sb}(\ulcorner\varphi(v)\urcorner, \mathrm{num}(x), \ulcorner v\urcorner) \qquad x^{\circ} := \mathrm{val}(x).$$

When it is clear from the context which variable is being replaced, we write $\mathrm{sb}(x, t)$ instead of $\mathrm{sb}(x, t, v)$. The abbreviation $\ulcorner\varphi(\dot{x})\urcorner$ extends to the case of multivariables in the obvious way, and we write $\ulcorner\varphi(\dot{\vec{x}})\urcorner$ for $\ulcorner\varphi(\dot{x}_1, \dots, \dot{x}_n)\urcorner$. The Gödel numbering is canonical, so in particular we require that the following are provable in (a fragment of) PA expanded by defining axioms for additional function symbols:

$$\mathrm{PA} \vdash \mathrm{val}(\mathrm{num}(x)) = x \wedge \mathrm{Ct}(\mathrm{num}(x)),$$
$$\mathrm{PA} \vdash \mathrm{Fml}_{\mathrm{Tr}}^1(x) \to \forall z \mathrm{St}_{\mathrm{Tr}}(x(\dot{z})),$$
$$\mathrm{PA} \vdash \mathrm{Ct}(x) \wedge \mathrm{Ct}(y) \to (\mathrm{Ver}(x \dot{=} y) \leftrightarrow x^{\circ} = y^{\circ}).$$

2.1.1. Terminology and notation for Gentzen-systems. A sequent is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of $\mathcal{L}_{\mathrm{Tr}}$-formulae. Γ is called the *antecedent*; Δ is called the *succedent*. They are both referred to as *cedents*. Given a cedent $\Gamma := \{\gamma_1, \dots, \gamma_n\}$, we set $\neg\Gamma := \{\neg\gamma_i \mid \gamma_i \in \Gamma\}$ and $\mathrm{Tr}\ulcorner\Gamma\urcorner := \{\mathrm{Tr}\ulcorner\gamma\urcorner \mid \gamma \in \Gamma\}$. For t free for v in Γ (i.e., t is free for v for all members $\varphi_1, \dots, \varphi_n$ of Γ), we write $\Gamma[t/v]$ for $\{\varphi_1[t/v], \dots, \varphi_n[t/v]\}$.

A *derivation* of a sequent $\Gamma \Rightarrow \Delta$ is a tree with nodes labeled by sequents. The *height* of a derivation $\mathcal{D}$ is the maximum *length* of the branches in the tree, where the length of a branch is the number of its nodes minus 1.

In the rules of inference displayed below,

- formulae in Γ, Δ are called *side formulae*, or *context*,
- the formulae not in the context in the conclusion are called *principal formulae*, and
- the formulae in the premises from which the conclusion is derived (i.e., the formulae in the premises not in the context) are called *active formulae*.

A *literal* is an atomic formula or the negation of an atomic formula. The *cut rank of a formula* which is eliminated in a cut-rule is the positive complexity of the formula. The supremum of the cut ranks of a derivation $\mathcal{D}$ is called the *cut rank* of $\mathcal{D}$. The expression

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[14] Note that the $\mathcal{L}_{\mathrm{Tr}}^{-}$-predicates $\mathrm{St}_{\mathrm{Tr}}(x), \mathrm{Ct}(x)$, etc., represent the sets of codes of sentences, closed terms, etc., of the language $\mathcal{L}_{\mathrm{Tr}}^{-}$. Moreover, observe that the function VAL is p.r. for $\mathcal{L}_{\mathrm{Tr}}^{-}$ contains only finitely many function symbols (if $\mathcal{L}_{\mathrm{Tr}}^{-}$ had function symbols for all p.r. functions, then VAL would be recursive and PA-definable, but it would not be p.r.). For more details on the representation of syntax within PA, we refer the reader to [2] and [9, sec. 5]. Let us also mention that, in order to have a syntax theory, one can treat the additional function symbols as abbreviations for suitable $\mathcal{L}_{\mathrm{PA}}$-formulae representing the respective PA-definable predicates and operations (see [8] for more details and for a discussion of some subtle issues related to this). However, for our purposes it proves convenient to have function symbols as part of the language, as this facilitates both statement and proof of our Main Lemma 4.5. We thank an anonymous referee for urging us to be more explicit on this issue.

$S \left|\frac{n}{m}\right. \Gamma \Rightarrow \Delta$ denotes that $\Gamma \Rightarrow \Delta$ is derivable in S with a derivation of height $\leq n$ and cut rank $\leq m$.

2.2. Sequent calculi for FDE *and some of its extensions*. In this section we introduce the various logics underlying the systems of truth employed in the paper. We start with the two-sided sequent calculus of First Degree Entailment (FDE). For a general overview of the different non-classical logics employed in this section see [19].

DEFINITION 2.1 (FDE). *The logic of* FDE *consists of the following axioms and rules.*

$$\text{Ax} \quad \varphi, \Gamma \Rightarrow \Delta, \varphi \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'} \text{ Cut}$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\text{L} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge\text{R}$$

$$\frac{\varphi[t/v], \Gamma \Rightarrow \Delta}{\forall x \varphi, \Gamma \Rightarrow \Delta} \forall\text{L} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi[u/v]}{\Gamma \Rightarrow \Delta, \forall x \varphi} \forall\text{R}$$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\neg\neg\varphi, \Gamma \Rightarrow \Delta} \neg\neg\text{L} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi} \neg\neg\text{R}$$

$$\frac{\neg\varphi, \Gamma \Rightarrow \Delta \qquad \neg\psi, \Gamma \Rightarrow \Delta}{\neg(\varphi \wedge \psi), \Gamma \Rightarrow \Delta} \neg\wedge\text{L} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)} \neg\wedge\text{R}$$

$$\frac{\neg\varphi[u/v], \Gamma \Rightarrow \Delta}{\neg\forall x \varphi, \Gamma \Rightarrow \Delta} \neg\forall\text{L} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg\varphi[t/v]}{\Gamma \Rightarrow \Delta, \neg\forall x \varphi} \neg\forall\text{R} .$$

Conditions of application: φ literal in Ax; u eigenparameter.

REMARK 2.2. *By means of a standard induction, one can show that the sequents* $\varphi, \Gamma \Rightarrow \Delta, \varphi$ *are derivable for all* $\varphi \in \mathcal{L}_{\text{Tr}}$.

FDE is the base logic of PKF. Semantically, models of this logic admit both truth-value gluts (sentences which are both true and false) and truth-value gaps (sentences which are neither true nor false). Other PKF-variants are based on extensions of FDE, obtained by adding one additional rule which restricts the class of models. These additional rules are introduced below, and the extensions of FDE are the defined in Definition 2.4.

DEFINITION 2.3. *Let* φ *be an atomic* $\mathcal{L}_{\text{Tr}}$-*sentence. Then*

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \neg\text{L} \qquad\qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \neg\text{R} \qquad\qquad \frac{\psi, \Gamma \Rightarrow \Delta \qquad \neg\psi, \Gamma \Rightarrow \Delta}{\varphi, \neg\varphi, \Gamma \Rightarrow \Delta} \text{GG}.$$

The rule ¬L (¬R) restricts the class of models to those in which there is no glut (gap). The label GG stands for "gaps or gluts," as via this rule we can derive $\varphi, \neg\varphi, \Gamma \Rightarrow$

$\Delta, \psi, \neg\psi$, thereby excluding the simultaneous occurrence of gaps and gluts.[15] The FDE extensions are then defined as follows:

DEFINITION 2.4 (Extensions of FDE).

- *Classical Logic*, CL, *is the system given by* FDE *without* $\neg\circ M$, *for* $\circ \in \{\neg, \wedge, \forall\}$, $M \in \{L, R\}$, *and with the addition of unrestricted* $\neg L$ *and* $\neg R$.[16]
- *Strong Kleene*, K3, *is the system* FDE $+ \neg L$.
- *Logic of Paradox*, LP, *is the system* FDE $+ \neg R$.
- *Kleene's Symmetric Logic*, KS3, *is the system* FDE $+$ GG.[17]

We now extend the base logics with rules for identity.

DEFINITION 2.5 (Identity rules). *Let φ be a literal. Then*

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ Ref}$$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{s = t, \varphi(s), \Gamma \Rightarrow \Delta} \text{ RepL} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t) \qquad \Gamma \Rightarrow \Delta, s = t}{\Gamma \Rightarrow \Delta, \varphi(s)} \text{ RepR}$$

Each of the logic introduced in Definition 2.4 will now be extended with rules axiomatizing the identity predicate.

DEFINITION 2.6 (Logics with identity). *We set*

- CL$_=$ *is* CL $+$ Ref $+$ RepL.
- FDE$_=$ *is* FDE $+$ Ref $+$ RepL.
- K3$_=$ *is* K3 $+$ Ref $+$ RepL.
- LP$_=$ *is* LP $+$ Ref $+$ RepR.
- KS3$_=$ *is* KS3 $+$ Ref $+$ RepL.

REMARK 2.7.

- (RepL) *and* (RepR) *are equivalent over* FDE, *and they both yield the replacement schema*

$$s = t, \varphi(s), \Gamma \Rightarrow \Delta, \varphi(t),$$

  for all $\varphi \in \mathcal{L}_{\text{Tr}}$.
- *The reason for formulating* K3$_=$ *and* KS3$_=$ *with* RepL, *and* LP *with* RepR, *is to obtain a syntactic proof of full Cut elimination. Restrictions on* $\neg L$, $\neg R$, *and* GG

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[15] By induction on φ, one first shows that GG is admissible for all φ. For example, assuming by i.h. that from $\psi, \Gamma \Rightarrow \Delta$ and $\neg\psi, \Gamma \Rightarrow \Delta$ we can infer $\varphi(u), \neg\varphi(u), \Gamma \Rightarrow \Delta$ for some $u \notin \Gamma \cup \Delta$, we obtain $\forall x\varphi, \neg\forall x\varphi, \Gamma \Rightarrow \Delta$ by applying $\forall L$ and $\neg\forall L$ (in this order). One then obtains $\varphi, \neg\varphi, \Gamma \Rightarrow \Delta, \psi, \neg\psi$ by applying (the unrestricted version of) GG to the sequents $\psi, \Gamma \Rightarrow \Delta, \psi, \neg\psi$ and $\neg\psi, \Gamma \Rightarrow \Delta, \psi, \neg\psi$. We thank an anonymous referee for spotting an important typo in the definition of GG.

[16] By "unrestricted" $\neg L$ and $\neg R$, we mean rules formulated without the provisio that φ be an atomic sentence.

[17] For similar calculi defining the same logic see, for instance, [1, 22].

are justified in the same way.[18] *Unrestricted versions of these rules can be shown to be admissible in the respective system by induction on the positive complexity on φ.*[19]

- *It can easily be shown that* FDE (KS3) *and* BDM (SDM), *that is, the system(s) defined by Nicolai* [14], *are equivalent.*[20] *The system* KS3, *however, has the advantage of enjoying a cut elimination theorem. Note, though, that the same does not hold for systems extended with identity, that is,* FDE$_=$ (KS3$_=$) *and* BDM$_=$ (SDM$_=$) *are not equivalent. This is so because, via contraposition, identity behaves classically in* BDM$_=$ (SDM$_=$).

§3. KF-like and PKF-like theories. This section introduces the KF-like and PKF-like truth theories we are going to investigate. The theory KF, developed by Feferman in the 1980s and published in [6], has been studied extensively, e.g., by Reinhardt [20, 21], McGee [13], and Cantini [2, 3]. As mentioned in the Introduction, in this article we concentrate on KF-variants with induction restricted on the arithmetical vocabulary, and on KF-variants with internal induction. These were both introduced by Cantini [2]. The theory Partial Kripke–Feferman (PKF) may be seen as the non-classical counterpart to KF. It was developed by Halbach & Horsten [10], and variants of PKF have been introduced and studied by, e.g., Halbach & Nicolai [11], Nicolai [14], and Fischer & Gratzl [7].[21]

We begin by introducing different rules of induction employed in the formulation of the theories. To this end we fix a standard notation system of ordinals up to Γ_0.[22] We use $a, b, c, \ldots$ to denote the code of our notation system whose value is $\alpha, \beta, \gamma, \ldots \in On$

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[18] We remark that Cut elimination holds for the sequent calculi introduced above. The key observation is that the calculi defined in this article are designed so that if the cut formula is principal in both premises of Cut, then the complexity of φ is > 0, i.e., φ cannot be a literal. In fact, In FDE$_=$, K3$_=$, and KS3$_=$, there is no rule introducing a literal on the right—that is the reason why we formulated K3$_=$ and KS3$_=$ with RepL, as they both have one rule introducing literals on the left, i.e., ¬L and GG, respectively—in LP$_=$ there is no rule introducing a literal on the left—that is the reason why we formulated LP$_=$ via (RepR), as this calculus has one rule introducing literals on the right, i.e., ¬R. Two points are worth emphasizing. The first is that, as far as we know, the rule GG excluding the simultaneous occurrence of gaps and gluts is new and KS3 is the first sequent calculus for symmetric Strong Kleene logic admitting a syntactic proof of Cut elimination. The second is that we are also not aware of Gentzen style calculi for first-order FDE, K3, LP, or KS3 using so-called geometric rules for identity; hence the calculi above might be the first (at least in literature on truth and paradoxes) axiomatizing identity with rules instead of axioms. A notable exception is [16, 17], where a cut-free calculus for a logic of formal inconsistency extending LP is defined.

[19] The argument for the admissibility of GG has already been sketched (see footnote 15). Also for ¬L and ¬R one can reason by induction on φ. For the quantifier-free fragment, one can use inversion and Cut. For quantified sentences, one first operates a proof-search to obtain a quasi-inversion. For example, to show that ¬R is admissible for $\forall x\varphi$ in LP, assume that $\forall x\varphi, \Gamma \Rightarrow \Delta$ is derivable. By induction on the height of derivations, one first shows that if $\vdash^n \forall x\varphi, \Gamma \Rightarrow \Delta$, then $\vdash^n \varphi(t_1), \ldots, \varphi(t_n), \Gamma \Rightarrow \Delta$ for some terms $t_1, \ldots, t_n$ (which of course need not be unique). One then derives $\Gamma \Rightarrow \Delta, \neg\varphi(t_1), \ldots, \neg\varphi(t_n)$ by i.h. on φ, and finally $\Gamma \Rightarrow \Delta, \neg\forall x\varphi$ by n-applications of ¬∀R.

[20] The equivalence between KS3 and SDM follows from the fact that via GG one can derive the sequents $\varphi, \neg\varphi \Rightarrow \psi, \neg\psi$, as remarked in footnote 15.

[21] See [9] for a presentation and discussion of both theories.

[22] See, for instance, [4, 18].

(with the exception of ω and ε-numbers, for which we use the ordinals themselves), and we use $\prec$ to denote a standard primitive recursive ordering defined on codes of ordinals. The expression $\forall z \prec y (\varphi[z/v])$ is short for $\forall z(\neg(\mathrm{Ord}(z) \wedge \mathrm{Ord}(y) \wedge z \prec y) \vee \varphi[z/v])$, where Ord represents the set of codes of ordinals. For $\alpha < \Gamma_0$ and a formula $\varphi(v) \in \mathcal{L}_{\mathrm{Tr}}$ we let $\mathrm{TI}^{<\alpha}(\varphi)$ denote the schemata of *transfinite induction* up to any ordinal below α:

$$\frac{\forall z \prec y\, \varphi(z), \Gamma \Rightarrow \Delta, \varphi(y)}{\Gamma \Rightarrow \Delta, \forall x \prec \overline{b}\, \varphi(x)}$$

$$\frac{\neg\varphi(y), \Gamma \Rightarrow \Delta, \neg\forall z \prec y\, \varphi(z)}{\neg\forall x \prec \overline{b}\, \varphi(x), \Gamma \Rightarrow \Delta} \qquad (\mathrm{TI}^{<\alpha})$$

for $\beta < \alpha$ and y eigenvariable.

We are going to use three additional schemata of induction. First, we have *full induction*:

$$\frac{\varphi(u), \Gamma \Rightarrow \Delta, \varphi(u')}{\varphi(\overline{0}), \Gamma \Rightarrow \Delta, \varphi(s)} \qquad (\mathrm{IND})$$

for $\varphi \in \mathcal{L}_{\mathrm{Tr}}$ and u eigenvariable. Second, we have *internal induction*:

$$\frac{\mathrm{Tr}\,\mathrm{sb}(t, \mathrm{num}(u)), \Gamma \Rightarrow \Delta, \mathrm{Tr}\,\mathrm{sb}(t, \mathrm{num}(u)')}{\mathrm{Tr}\,\mathrm{sb}(t, \mathrm{num}(\overline{0})), \Gamma \Rightarrow \Delta, \mathrm{Tr}\,\mathrm{sb}(t, \mathrm{num}(z))} \qquad (\mathrm{IND}^{\mathrm{int}})$$

for u eigenvariable. Third, we have *restricted internal induction*:

$$\frac{\mathrm{Tr}\ulcorner\varphi(\dot{u})\urcorner, \Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner\varphi(\dot{u'})\urcorner}{\mathrm{Tr}\ulcorner\varphi(\overline{0})\urcorner, \Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner\varphi(s)\urcorner} \qquad (\mathrm{IND}^{\mathrm{int}}_{\mathcal{L}_{\mathrm{PA}}})$$

for $\varphi \in \mathcal{L}_{\mathrm{PA}}$ and u eigenvariable.

We now introduce the basic truth principles employed in the systems of truth we discuss in the paper.

DEFINITION 3.1 (Truth-axioms and truth rules). *The following are truth-theoretic initial sequents* (*or truth axioms*). *TrReg is often called regularity axiom* .[23]

$$
\begin{array}{ll}
\mathrm{Tr}\!=\! & \mathrm{Ct}(t), \mathrm{Ct}(s), t^\circ = s^\circ, \Gamma \Rightarrow \Delta, \mathrm{Tr}(t \,\dot{=}\, s) \\
& \mathrm{Ct}(t), \mathrm{Ct}(s), \mathrm{Tr}(t \,\dot{=}\, s), \Gamma \Rightarrow \Delta, t^\circ = s^\circ \\
\mathrm{Tr}\neg\!=\! & \mathrm{Ct}(t), \mathrm{Ct}(s), \neg(t^\circ = s^\circ), \Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\neg}(t \,\dot{=}\, s)) \\
& \mathrm{Ct}(t), \mathrm{Ct}(s), \mathrm{Tr}(\dot{\neg}(t \,\dot{=}\, s)), \Gamma \Rightarrow \Delta, \neg(t^\circ = s^\circ) \\
\neg\mathrm{Tr}\neg(i) & \mathrm{St}_{\mathrm{Tr}}(t), \mathrm{Tr}(\dot{\neg}t), \Gamma \Rightarrow \Delta, \neg\mathrm{Tr}(t) \\
(ii) & \mathrm{St}_{\mathrm{Tr}}(t), \neg\mathrm{Tr}(t), \Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\neg}t) \\
\mathrm{TrSt}_{\mathcal{L}_{\mathrm{Tr}}} & \mathrm{Tr}(t), \Gamma \Rightarrow \Delta, \mathrm{St}_{\mathrm{Tr}}(t) \\
& \neg\mathrm{St}_{\mathrm{Tr}}(t), \Gamma \Rightarrow \Delta, \neg\mathrm{Tr}(t) \\
\mathrm{TrReg} & \mathrm{Fml}^1_{\mathcal{L}_{\mathrm{Tr}}}(r), \mathrm{Ct}(t), \mathrm{Ct}(s), t^\circ = s^\circ, \mathrm{Tr}\,r(t) \Rightarrow \mathrm{Tr}\,r(s) \\
& \mathrm{Fml}^1_{\mathcal{L}_{\mathrm{Tr}}}(r), \mathrm{Ct}(t), \mathrm{Ct}(s), t^\circ = s^\circ, \neg\mathrm{Tr}\,r(t) \Rightarrow \neg\mathrm{Tr}\,r(s).
\end{array}
$$

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[23] TrReg was dropped in Feferman's formulation of KF since it is derivable by IND. It was introduced by Cantini [2] since it is not derivable in systems without full induction.

The following rules are called truth rules:

$$\frac{\mathrm{Tr}(t), \Gamma \Rightarrow \Delta}{\mathrm{Tr}(\mathrm{sb}(\ulcorner \mathrm{Tr}(v_0) \urcorner, \mathrm{num}(t))), \Gamma \Rightarrow \Delta} \ \mathrm{TrTrL} \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Tr}(t)}{\Gamma \Rightarrow \Delta, \mathrm{Tr}(\mathrm{sb}(\ulcorner \mathrm{Tr}(v_0) \urcorner, \mathrm{num}(t)))} \ \mathrm{TrTrR}$$

$$\frac{\mathrm{Tr}(\dot{\neg}t), \Gamma \Rightarrow \Delta \quad \neg\mathrm{St}_{\mathrm{Tr}}(t), \Gamma \Rightarrow \Delta}{\mathrm{Tr}(\mathrm{sb}(\dot{\neg}\ulcorner \mathrm{Tr}(v_0) \urcorner, \mathrm{num}(t))), \Gamma \Rightarrow \Delta} \ \mathrm{Tr}\neg\mathrm{TrL} \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\neg}t), \neg\mathrm{St}_{\mathrm{Tr}}(t)}{\Gamma \Rightarrow \Delta, \mathrm{Tr}(\mathrm{sb}(\dot{\neg}\ulcorner \mathrm{Tr}(v_0) \urcorner, \mathrm{num}(t)))} \ \mathrm{Tr}\neg\mathrm{TrR}$$

$$\frac{\mathrm{Tr}(t), \Gamma \Rightarrow \Delta}{\mathrm{St}_{\mathrm{Tr}}(t), \mathrm{Tr}(\dot{\neg}\dot{\neg}t), \Gamma \Rightarrow \Delta} \ \mathrm{Tr}\neg\neg\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Tr}(t)}{\mathrm{St}_{\mathrm{Tr}}(t), \Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\neg}\dot{\neg}t)} \ \mathrm{Tr}\neg\neg\mathrm{R}$$

$$\frac{\mathrm{Tr}(t), \mathrm{Tr}(s), \Gamma \Rightarrow \Delta}{\mathrm{St}_{\mathrm{Tr}}(t \dot{\wedge} s), \mathrm{Tr}(t \dot{\wedge} s), \Gamma \Rightarrow \Delta} \ \mathrm{Tr}\wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Tr}(t) \quad \Gamma \Rightarrow \Delta, \mathrm{Tr}(s)}{\mathrm{St}_{\mathrm{Tr}}(t \dot{\wedge} s), \Gamma \Rightarrow \Delta, \mathrm{Tr}(t \dot{\wedge} s)} \ \mathrm{Tr}\wedge\mathrm{R}$$

$$\frac{\mathrm{Tr}(\dot{\neg}t), \Gamma \Rightarrow \Delta \quad \mathrm{Tr}(\dot{\neg}s), \Gamma \Rightarrow \Delta}{\mathrm{St}_{\mathrm{Tr}}(t \dot{\wedge} s), \mathrm{Tr}(\dot{\neg}(t \dot{\wedge} s)), \Gamma \Rightarrow \Delta} \ \mathrm{Tr}\neg\wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\neg}t), \mathrm{Tr}(\dot{\neg}s)}{\mathrm{St}_{\mathrm{Tr}}(t \dot{\wedge} s), \Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\neg}(t \dot{\wedge} s))} \ \mathrm{Tr}\neg\wedge\mathrm{R}$$

$$\frac{\mathrm{Tr}(t(\dot{z})), \Gamma \Rightarrow \Delta}{\mathrm{St}_{\mathrm{Tr}}(\dot{\forall}v.t), \mathrm{Tr}(\dot{\forall}v.t), \Gamma \Rightarrow \Delta} \ \mathrm{Tr}\forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Tr}(t(\dot{u}))}{\mathrm{St}_{\mathrm{Tr}}(\dot{\forall}v.t), \Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\forall}v.t)} \ \mathrm{Tr}\forall\mathrm{R}$$

$$\frac{\mathrm{Tr}(\dot{\neg}t(\dot{u})), \Gamma \Rightarrow \Delta}{\mathrm{St}_{\mathrm{Tr}}(\dot{\forall}v.t), \mathrm{Tr}(\dot{\neg}\dot{\forall}v.t), \Gamma \Rightarrow \Delta} \ \mathrm{Tr}\neg\forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\neg}t(\dot{z}))}{\mathrm{St}_{\mathrm{Tr}}(\dot{\forall}v.t), \Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\neg}\dot{\forall}v.t)} \ \mathrm{Tr}\neg\forall\mathrm{R} \ .$$

Conditions of application: *u eigenvariable*.

We begin by introducing the relevant variants of KF, and we then define their non-classical counterparts, i.e., the PKF-like systems.

DEFINITION 3.2 (KF). *KF is obtained from* $\mathsf{CL}_=$ *by adding initial sequents of* PA (*see* [23, *Definition* 9.3]); *defining axioms for additional function symbols*; IND; *truth axioms and truth rules of Definition* 3.1, *except the initial sequents* ¬Tr¬.

DEFINITION 3.3 (KF-variants).

(i) $\mathrm{KF}_{\mathrm{cs}}$ *is obtained from* KF *by adding the initial sequent*

$$\mathrm{St}_{\mathrm{Tr}}(t), \mathrm{Tr}(\dot{\neg}t), \Gamma \Rightarrow \Delta, \neg\mathrm{Tr}(t). \tag{Cons}$$

(ii) $\mathrm{KF}_{\mathrm{cp}}$ *is obtained from* KF *by adding*

$$\mathrm{St}_{\mathrm{Tr}}(t), \neg\mathrm{Tr}(t), \Gamma \Rightarrow \Delta, \mathrm{Tr}(\dot{\neg}t).^{[24]} \tag{Comp}$$

(iii) KF_{S} *is obtained from* KF *by adding*

$$\mathrm{St}_{\mathrm{Tr}}(t), \mathrm{St}_{\mathrm{Tr}}(s), \mathrm{Tr}(t), \mathrm{Tr}(\dot{\neg}t), \Gamma \Rightarrow \Delta, \mathrm{Tr}(s), \mathrm{Tr}(\dot{\neg}s). \tag{GoG}$$

For $\mathrm{KF}_\star \in \{\mathrm{KF}, \mathrm{KF}_{\mathrm{cs}}, \mathrm{KF}_{\mathrm{cp}}, \mathrm{KF}_{\mathrm{S}}\}$,

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[24]   Note that Cons is (¬Tr¬)(*i*) and Comp is (¬Tr¬)(*ii*). Of course, over the non-classical logics studied in this chapter, (¬Tr¬) does not imply that the truth predicate is consistent and complete. (¬Tr¬) is just axiomatizing the well-known property of fixed-point models that the anti-extension A can be defined via the extension E as $A := \{\varphi \mid \neg\varphi \in E\}$.

(iv) $KF_\star^-$ *is obtained from* $KF_\star$ *by replacing* IND *with* $IND_{\mathcal{L}_{PA}}^{int}$.

(v) $KF_\star^{int}$ *is obtained from* $KF_\star$ *by replacing* IND *with* IND^{int}.

In what follows, theories introduced via items (iv) and (v) of Definition 3.3 will be referred to as KF-like theories (or systems) or as KF-variants.

REMARK 3.4. *Let* $KF^\circ \in \{KF^{int}, KF_{cs}^{int}, KF_{cp}^{int}, KF^-, KF_{cs}^-, KF_{cp}^-\}$. *Every axiom of* KF° *has the form* $\Theta, \Gamma \Rightarrow \Delta, \Lambda$. *Formulae in* Θ, Λ *are called active. Every active formula has positive complexity* 0.

Let us remark that our formulation of $KF_\star^-$ deviates from the standard formulation. Typically, the systems $KF_\star^-$ are defined by restricting the induction schema IND to the arithmetical vocabulary, as, e.g., in [2, 9]. The reason why we have introduced this different—yet equivalent, as shown in Observation 3.6—formulation is that our arguments below rely on partial Cut elimination, which fails in presence of induction, but which can be restored for KF-systems with internal induction, as well as for KF-systems with restricted internal induction.[25] In order to state this precisely, call a derivation $\mathcal{D}$ *quasi-normal* if $\mathcal{D}$ has cut rank 0; then, by application of standard techniques for Cut elimination we obtain:

PROPOSITION 3.5. *Let* $KF^\circ \in \{KF_\star^{int}, KF_\star^-\}$, *where* $KF_\star^{int}$ *and* $KF_\star^-$ *are as in Definition* 3.3. *Then every* KF°-*derivation* $\mathcal{D}$ *can be effectively transformed into a quasi-normal derivation* $\mathcal{D}'$ *of the same end sequent.*

OBSERVATION 3.6. *Let* $KF_\star^{int}$ *be as in Definition* 3.3, *and let* $KF_\star[IND{\upharpoonright}\,\mathcal{L}_{PA}]$ *be the theory obtained from* $KF_\star^{int}$ *by replacing* IND^{int} *with the schema* IND *restricted to* $\mathcal{L}_{PA}$-*sentences. Then* $KF_\star^-$ *and* $KF_\star[IND{\upharpoonright}\,\mathcal{L}_{PA}]$ *are equivalent.*

Proof Idea. Following [2], one first shows that both $KF_\star[IND{\upharpoonright}\,\mathcal{L}_{PA}]$ and $KF_\star^-$ prove the T-Schema on arithmetical formulae. That is to say one shows that both theories prove the sequents

$$\varphi(x), \Gamma \Rightarrow \Delta, Tr\ulcorner\varphi(\dot{x})\urcorner,$$
$$Tr\ulcorner\varphi(\dot{x})\urcorner, \Gamma \Rightarrow \Delta, \varphi(x),$$

if φ does not contain Tr. The proof then continues by induction on the height of derivations. Since $KF_\star[IND{\upharpoonright}\,\mathcal{L}_{PA}]$ and $KF_\star^-$ only differ for the employed induction schema, the just mentioned interderivability between $Tr\ulcorner\varphi(\dot{x})\urcorner$ and $\varphi(x)$ immediately yields, via Cut, the desired conclusion. $\square$

We can now move on to PKF-like theories.

DEFINITION 3.7 (PKF). PKF *is obtained from* $FDE_=$ *by adding*: *sequents* $\Gamma \Rightarrow \Delta$ *for* $\Gamma \Rightarrow \Delta$ *an initial sequent of* PA (*see* [23, *Definition* 9.3(1)]); *defining axioms for additional*

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[25] The same strategy has been used by Fischer & Gratzl [7] for defining PKF, that is to say, they replace PKF's full induction by the schema

$$\frac{Tr\ulcorner\varphi(\dot{u})\urcorner, \Gamma \Rightarrow \Delta, Tr\ulcorner\varphi(\dot{u'})\urcorner}{Tr\ulcorner\varphi(\bar{0})\urcorner, \Gamma \Rightarrow \Delta, Tr\ulcorner\varphi(t)\urcorner},$$

for $\varphi \in \mathcal{L}_{Tr}$, in order to obtain a system equivalent to PKF which however enjoys partial cut elimination.

function symbols; IND; *truth axioms and truth rules of Definition* 3.1; *and the following two rules requiring identity statements to behave classically*:

$$\frac{\Gamma \Rightarrow \Delta, s = t}{\neg(s = t), \Gamma \Rightarrow \Delta} \, =\!\neg\mathrm{L} \qquad\qquad \frac{s = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg(s = t)} \, =\!\neg\mathrm{R}\,.$$

DEFINITION 3.8 (PKF-variants). *We introduce variants of* PKF

   (i) $\mathrm{PKF_{cs}}$ *is obtained by adding* ¬L *to* PKF.
   (ii) $\mathrm{PKF_{cp}}$ *is obtained by adding* ¬R *to* PKF.
   (iii) $\mathrm{PKF_S}$ *is obtained by adding* GG *to* PKF.

For $\mathrm{PKF_\star} \in \{\mathrm{PKF}, \mathrm{PKF_{cs}}, \mathrm{PKF_{cp}}, \mathrm{PKF_S}\}$,

   (iv) $\mathrm{PKF_\star^-}$ *is obtained by replacing* IND *with* $\mathrm{IND}^{\mathrm{int}}_{\mathcal{L}_{\mathrm{PA}}}$.
   (v) $\mathrm{PKF_\star^+}$ *is obtained by extending* $\mathrm{PKF_\star}$ *with* $\mathrm{TI}^{<\varepsilon_0}$.[26]

Since our formulation of PKF deviates from the formulation in [10], we now show that PKF behaves classically on the Tr-free fragment of $\mathcal{L}_{\mathrm{Tr}}$ and that $\psi(\vec{x})$ and $\mathrm{Tr}\ulcorner \psi(\dot{\vec{x}})\urcorner$ are interderivable.

LEMMA 3.9. *Let* PKF° *be one of the* PKF-*like theories introduced in Definition* 3.8, $\varphi \in \mathcal{L}_{\mathrm{PA}}$, *and* $\psi(\vec{x}) \in \mathcal{L}_{\mathrm{Tr}}$. *Then*

   (i) $\mathrm{PKF}^\circ \vdash \Gamma \Rightarrow \Delta, \varphi, \neg\varphi$ *and* $\mathrm{PKF}^\circ \vdash \varphi, \neg\varphi, \Gamma \Rightarrow \Delta$;
   (ii) $\mathrm{PKF}^\circ \vdash \Gamma \Rightarrow \Delta, \psi(\vec{x})$ *iff* $\mathrm{PKF}^\circ \vdash \Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner \psi(\dot{\vec{x}})\urcorner$;
   (iii) *Unrestricted* ¬L *and* ¬R *are admissible in* PKF° *for* $\varphi \in \mathcal{L}_{\mathrm{PA}}$.

Proof. (i) and (ii) are shown by a straightforward induction on φ. For (iii), observe more generally that, if $\varphi, \neg\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \varphi, \neg\varphi$ are both derivable, then ¬L and ¬R are derived rules

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \neg\varphi, \Gamma \Rightarrow \Delta}{\neg\varphi, \Gamma \Rightarrow \Delta} \, \mathrm{Cut} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \, \mathrm{Cut}$$

$\square$

Finally, to complete the picture we note that contraposition is admissible in PKF and $\mathrm{PKF_S}$.

LEMMA 3.10 (Contraposition). *Contraposition*, *i.e.*, *the rule*

$$\frac{\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\Gamma}$$

is admissible in $\mathrm{PKF}(\mathrm{PKF}^-, \mathrm{PKF}^+)$ *and* $\mathrm{PKF_S}(\mathrm{PKF_S^-}, \mathrm{PKF_S^+})$.

Proof. The proof is by induction on the height of derivations. If $\Gamma \Rightarrow \Delta$ is an axiom of PA, the conclusion follows from Lemma 3.9(iii). For truth-theoretic initial sequents, the initial sequent ¬Tr¬ plays a central role. For derivations of height > 0, we show three crucial cases involving the rules GG, RepL, and IND. Suppose the derivation

---

[26] The idea of extending PKF with transfinite induction up to $< \varepsilon_0$ is due to [14].

ends with

$$
\frac{\psi, \Gamma \Rightarrow \Delta \qquad \neg\psi, \Gamma \Rightarrow \Delta}{\varphi, \neg\varphi, \Gamma \Rightarrow \Delta} \; \text{GG}
$$

where $\mathcal{D}_0$ and $\mathcal{D}_1$ are above $\psi, \Gamma \Rightarrow \Delta$ and $\neg\psi, \Gamma \Rightarrow \Delta$ respectively.

To begin with, using induction on $\mathcal{D}_1$ we first form

$$
\mathcal{D}'_1 \left\{ \frac{\neg\Delta \Rightarrow \neg\Gamma, \neg\neg\psi \qquad \dfrac{\psi, \neg\Delta \Rightarrow \neg\Gamma, \psi}{\neg\neg\psi, \neg\Delta \Rightarrow \neg\Gamma, \psi} \; \neg\neg\text{L}}{\neg\Delta \Rightarrow \neg\Gamma, \psi} \; \text{Cut} \right.
$$

with i.h. above $\neg\Delta \Rightarrow \neg\Gamma, \neg\neg\psi$.

We then continue as follows:

$$
\frac{\neg\Delta \Rightarrow \neg\Gamma, \neg\psi \qquad \dfrac{\neg\Delta \Rightarrow \neg\Gamma, \psi \qquad \dfrac{\varphi, \neg\Delta \Rightarrow \neg\Gamma, \varphi, \neg\varphi \qquad \neg\varphi, \neg\Delta \Rightarrow \neg\Gamma, \varphi, \neg\varphi}{\psi, \neg\psi, \neg\Delta \Rightarrow \neg\Gamma, \varphi, \neg\varphi} \; \text{GG}}{\neg\psi, \neg\Delta \Rightarrow \neg\Gamma, \varphi, \neg\varphi} \; \text{Cut}}{\dfrac{\neg\Delta \Rightarrow \neg\Gamma, \varphi, \neg\varphi}{\neg\Delta \Rightarrow \neg\Gamma, \neg\varphi, \neg\neg\varphi} \; \neg\neg\text{R}} \; \text{Cut}
$$

with i.h. on $\mathcal{D}_0$ above $\neg\Delta \Rightarrow \neg\Gamma, \neg\psi$, and $\mathcal{D}'_1$ above $\neg\Delta \Rightarrow \neg\Gamma, \psi$.

If the derivations ends with

$$
\frac{\varphi(t), \Gamma \Rightarrow \Delta}{s = t, \varphi(s), \Gamma \Rightarrow \Delta} \; \text{RepL}
$$

we reason as follows. We first derive

$$
\frac{\neg\Delta \Rightarrow \neg\Gamma, \neg\varphi(t) \qquad \dfrac{\neg\varphi(s), \neg\Delta \Rightarrow \neg\Gamma, \neg\varphi(s)}{s = t, \neg\varphi(t), \neg\Delta \Rightarrow \neg\Gamma, \neg\varphi(s)} \; \text{RepL}}{\dfrac{s = t, \neg\Delta \Rightarrow \neg\Gamma, \neg\varphi(s)}{\neg\Delta \Rightarrow \neg\Gamma, \neg\varphi(s), \neg(s = t)} \; =\neg\text{R}} \; \text{Cut}
$$

with i.h. above $\neg\Delta \Rightarrow \neg\Gamma, \neg\varphi(t)$.

If the derivation ends with

$$
\frac{\varphi(u), \Gamma \Rightarrow \Delta, \varphi(u + 1)}{\varphi(\overline{0}), \Gamma \Rightarrow \Delta, \varphi(t)}
$$

with u eigenvariable, we apply Parsons' [15] trick. By induction, we have

$$\neg\varphi(u+1), \neg\Gamma \Rightarrow \neg\Delta, \neg\varphi(u).$$

Replacing $t - (u+1)$ for u,

$$\neg\varphi((t-(u+1))+1), \neg\Gamma \Rightarrow \neg\Delta, \neg\varphi(t-(u+1)).$$

Since $(t-(u+1))+1 = t-u$, by replacement of identicals we obtain

$$\neg\varphi(t-u), \neg\Gamma \Rightarrow \neg\Delta, \neg\varphi(t-(u+1)),$$

and hence by IND

$$\neg\varphi(t-\overline{0}), \neg\Gamma \Rightarrow \neg\Delta, \neg\varphi(t-t), \text{ i.e.,}$$
$$\neg\varphi(t), \neg\Gamma \Rightarrow \neg\Delta, \neg\varphi(\overline{0}). \qquad \square$$

§4. Reinhardt's challenge. In this section we address REINHARDT'S CHALLENGE and show that, given any KF-like theory with a restricted form of induction, there is a corresponding PKF-like theory such that the set of inferences $\text{Tr}\ulcorner\Gamma\urcorner \Rightarrow \text{Tr}\ulcorner\Delta\urcorner$ provable in the PKF-like theory coincides with the set of significant inferences of the KF-like theory. This observation may be considered as a positive answer to Question (b) discussed in the Introduction with respect to the particular KF-like theories under consideration, that is, as providing an independent axiomatization of the significant inferences of the particular KF-like theories. Moreover, by axiomatizing the set of significant inferences of these KF-like theories, we obtain a positive answer to the GENERALIZED REINHARDT PROBLEM as an immediate corollary. More precisely, we show in Proposition 4.11 that every significant inference of a KF-like theory with a restricted form of induction has a significant derivation, i.e., whenever the theory proves $\text{Tr}\ulcorner\Gamma\urcorner \Rightarrow \text{Tr}\ulcorner\Delta\urcorner$, we can find a derivation $\mathcal{D}$ of $\text{Tr}\ulcorner\Gamma\urcorner \Rightarrow \text{Tr}\ulcorner\Delta\urcorner$ such that every node of $\mathcal{D}$ is itself a significant inference.

To begin with, we introduce the notion of significant inference.

DEFINITION 4.1 (Significant Inferences). *Let* Th *be an axiomatic truth theory formulated in* $\mathcal{L}_{\text{Tr}}$, *and* Γ, Δ *be finite sets of* $\mathcal{L}_{\text{Tr}}$-*sentences. The set of significant inferences of* Th *is defined as*

$$\text{ThSI} := \{\langle\Gamma, \Delta\rangle \mid \text{Th} \vdash \text{Tr}\ulcorner\Gamma\urcorner \Rightarrow \text{Tr}\ulcorner\Delta\urcorner\}.$$

Since the truth predicate of PKF-like theories is transparent (cf. Lemma 3.9(ii)) the significant inferences of any PKF-like theory will simply amount to the set of provable inferences of the theory. We also note in passing that the significant part of a truth theory (ThS) in the sense of [20, 21], that is the provably true sentences of the theory, can be retrieved from ThSI by setting

$$\text{ThS} := \{\varphi \in \text{St}_{\mathcal{L}_{\text{Tr}}} \mid \langle\varnothing, \{\varphi\}\rangle \in \text{ThSI}\}.$$

Let us now show that for every KF-like theory there is a PKF-like theory such that the provable sequents of the latter constitute exactly the significant inferences of the former.

DEFINITION 4.2 ($\mathrm{PKF^\circ, KF^\circ}$). *The pair* $(\mathrm{PKF^\circ, KF^\circ})$ *is a variable ranging over the following theory-pairs*:

$$(\mathrm{PKF}{\upharpoonright}, \mathrm{KF}{\upharpoonright}), (\mathrm{PKF_{cs}}{\upharpoonright}, \mathrm{KF_{cs}}{\upharpoonright}), (\mathrm{PKF_{cp}}{\upharpoonright}, \mathrm{KF_{cp}}{\upharpoonright}), (\mathrm{PKF_S}{\upharpoonright}, \mathrm{KF_S}{\upharpoonright})$$

$$(\mathrm{PKF}, \mathrm{KF^{int}}), (\mathrm{PKF_{cs}}, \mathrm{KF_{cs}^{int}}), (\mathrm{PKF_{cp}}, \mathrm{KF_{cp}^{int}}), (\mathrm{PKF_S}, \mathrm{KF_S^{int}}).$$

Moreover, let $\mathrm{Th} \in \{\mathrm{KF^{int}}, \mathrm{KF}{\upharpoonright}, \mathrm{PKF}{\upharpoonright}\}$. *Then* $\mathrm{Th_\star} \in \{\mathrm{Th}, \mathrm{Th_S}, \mathrm{Th_{cs}}, \mathrm{Th_{cp}}\}$.

Unless otherwise specified, we let $(\mathrm{PKF^\circ, KF^\circ})$ be as in Definition 4.2.

We can now start proving the principal result of this section, i.e., we can prove that $\mathrm{PKF^\circ = KF^\circ SI}$. We first show that $\mathrm{PKF^\circ \subseteq KF^\circ SI}$.

PROPOSITION 4.3. *If* $\mathrm{PKF^\circ} \vdash \Gamma(\vec{x}) \Rightarrow \Delta(\vec{y})$, *then* $\mathrm{KF^\circ} \vdash \mathrm{Tr}\ulcorner\Gamma(\dot{\vec{x}})\urcorner \Rightarrow \mathrm{Tr}\ulcorner\Delta(\dot{\vec{y}})\urcorner$.

Proposition 4.3 is essentially due to Halbach & Horsten [10], Halbach & Nicolai [11], and Nicolai [14], who proved the claim for theories without index cs or cp, that is, for pairs of theories that do not assume the truth predicate to be consistent or complete. It thus suffices to extend their result to these theories.

Proof of Proposition 4.3. The proof is straightforward.[27] For pairs extended with a consistency principle, it suffices to show that the KF-theory "internalizes" the soundness of ¬L. That is, it suffices to show that, e.g., if $\mathrm{KF_{cs}} \vdash \mathrm{Tr}\ulcorner\Gamma\urcorner \Rightarrow \mathrm{Tr}\ulcorner\Delta\urcorner, \mathrm{Tr}\ulcorner\varphi\urcorner$, then $\mathrm{KF_{cs}} \vdash \mathrm{Tr}\ulcorner\neg\varphi\urcorner, \mathrm{Tr}\ulcorner\Gamma\urcorner \Rightarrow \mathrm{Tr}\ulcorner\Delta\urcorner$. Symmetrically for theories extended with a completeness principle one needs to show that the KF-theory "internalizes" the soundness of ¬R.   □

In light of the definition of $\mathrm{KF^\circ SI}$, Proposition 4.3 immediately yields that the provable inferences of $\mathrm{PKF^\circ}$ are a subset of $\mathrm{KF^\circ SI}$:

COROLLARY 4.4. $\mathrm{PKF^\circ \subseteq KF^\circ SI}$.

4.1. From $\mathrm{KF^\circ}$*-significant inferences to* $\mathrm{PKF^\circ}$*-provable sequents.* The proof of the converse directions of Proposition 4.3 and Corollary 4.4 constitutes the main technical contribution of this article. The basic idea underlying the proof is to show that if $\mathrm{KF^\circ} \vdash \mathrm{Tr}\ulcorner\Gamma\urcorner \Rightarrow \mathrm{Tr}\ulcorner\Delta\urcorner$ then $\mathrm{PKF^\circ} \vdash \mathrm{Tr}\ulcorner\Gamma\urcorner \Rightarrow \mathrm{Tr}\ulcorner\Delta\urcorner$. This will be shown by proving a stronger claim, i.e., it will be shown that, whenever Γ, Δ contain only literals, then

$$\mathrm{KF^\circ} \vdash \Gamma \Rightarrow \Delta \quad \text{implies} \quad \mathrm{PKF^\circ} \vdash \Gamma^+, \Delta^- \Rightarrow \Gamma^-, \Delta^+,$$

where for a set of sentences Θ, and At the set of atomic sentences,

$$\Theta^+ := \{\varphi \in \mathrm{At} \mid \varphi \in \Theta\} \qquad \Theta^- := \{\varphi \in \mathrm{At} \mid \neg\varphi \in \Theta\}.[28]$$

This transformation is motivated by (i) the fact that identity behaves classically in $\mathrm{PKF^\circ}$ and (ii) the following semantic consideration: if a formula of the form $\neg\mathrm{Tr}(t)$ is classically false (true), then $\mathrm{Tr}(t)$ is either true (false) or both (neither) from the perspective of the non-classical theory of truth. As a consequence, moving $\mathrm{Tr}(t)$ in the succedent (antecedent) of the sequent will not interfere with the validity of the sequent from the perspective of the non-classical logics at stake. Moreover, if $\Gamma \Rightarrow \Delta$ is of the form $\mathrm{Tr}\ulcorner\Gamma'\urcorner \Rightarrow \mathrm{Tr}\ulcorner\Delta'\urcorner$, the transformation leaves the sequent unaltered; hence,

---

[27] Notice that the use of rules for identity instead of identity axioms does not impact the arguments due to [10, 11, 14].

[28] In other words, for ψ atomic, $\psi \in \Theta^-$ iff $\neg\psi \in \Theta$.

if we prove that for each KF°-provable sequent its transformation is PKF°-provable, we obtain our desired result as a corollary.

We first consider KF-systems with internal induction.

LEMMA 4.5 (Main Lemma). *For Γ, Δ sets of literals, for all $\star$*

$$\text{KF}^{\text{int}}_\star \vdash \Gamma \Rightarrow \Delta \ \text{ implies } \text{PKF}_\star \vdash \Gamma^+, \Delta^- \Rightarrow \Gamma^-, \Delta^+.$$

In the following proof, we implicitly use the fact that the replacement schema $s = t, \varphi(t), \Gamma \Rightarrow \Delta, \varphi(s)$ is derivable in each PKF-like system of Definition 3.8. Also, let us recall that active formulae in truth-axioms and truth rules are literals.

Proof of Lemma 4.5. The proof is by induction on the height n of $\text{KF}^{\text{int}}_\star$-derivations. Suppose first that $n = 0$, and that $\Gamma \Rightarrow \Delta$ is a $\text{KF}^{\text{int}}_\star$-initial sequent. For the pair KF^{int}-PKF, we first notice that every axiom of KF^{int} is an axiom of PKF. Hence for initial sequents not involving negated atomic formulae, the proof is immediate. The only truth-theoretic axiom of KF^{int} involving a negated atomic formula is $\text{Tr}\neg=$. We obtain the desired conclusion by Lemma 3.9(iii), e.g., from

$$\text{Ct}(t), \text{Ct}(s), \neg(t^\circ = s^\circ), \Gamma^+, \Delta^- \Rightarrow \Gamma^-, \Delta^+, \text{Tr}(\dot\neg(t \dot= s)),$$

which is an initial sequent of PKF, via $=\neg\text{R}$, $\neg\neg\text{L}$, and Cut, we can derive

$$\text{Ct}(t), \text{Ct}(s), \Gamma^+, \Delta^- \Rightarrow \Gamma^-, \Delta^+, \text{Tr}(\dot\neg(t \dot= s)), t^\circ = s^\circ.$$

As for theories with specific Tr-axioms, we want to show that PKF_{cp}, PKF_{cs}, and PKF_S derive, respectively, the transformation of Cons, Comp, and GoG. That is to say, we have to show (we omit context for readability)

$$\text{PKF}_{\text{cp}} \vdash \text{St}_{\text{Tr}}(t) \Rightarrow \text{Tr}(t), \text{Tr}(\dot\neg t), \tag{1}$$

$$\text{PKF}_{\text{cs}} \vdash \text{St}_{\text{Tr}}(t), \text{Tr}(t), \text{Tr}(\dot\neg t) \Rightarrow \varnothing, \tag{2}$$

$$\text{PKF}_S \vdash \text{St}_{\text{Tr}}(t), \text{St}_{\text{Tr}}(s), \text{Tr}(t), \text{Tr}(\dot\neg t) \Rightarrow \text{Tr}(s), \text{Tr}(\dot\neg s). \tag{3}$$

For (1)

$$\frac{\dfrac{\text{St}_{\text{Tr}}(t), \neg\text{Tr}(t) \Rightarrow \text{Tr}(\dot\neg t)}{\text{St}_{\text{Tr}}(t) \Rightarrow \text{Tr}(\dot\neg t), \neg\neg\text{Tr}(t)} \neg\text{R} \qquad \dfrac{\text{St}_{\text{Tr}}(t), \text{Tr}(t) \Rightarrow \text{Tr}(t)}{\text{St}_{\text{Tr}}(t), \neg\neg\text{Tr}(t) \Rightarrow \text{Tr}(t)} \neg\neg\text{L}}{\text{St}_{\text{Tr}}(t) \Rightarrow \text{Tr}(t), \text{Tr}(\dot\neg t)} \text{Cut}$$

For (2) we have

$$\frac{\text{St}_{\text{Tr}}(t), \text{Tr}(\dot\neg t) \Rightarrow \neg\text{Tr}(t) \qquad \dfrac{\text{St}_{\text{Tr}}(t), \text{Tr}(t) \Rightarrow \text{Tr}(t)}{\text{St}_{\text{Tr}}(t), \text{Tr}(t), \neg\text{Tr}(t) \Rightarrow \varnothing} \neg\text{L}}{\text{St}_{\text{Tr}}(t), \text{Tr}(t), \text{Tr}(\dot\neg t) \Rightarrow \varnothing} \text{Cut}$$

Finally, for (3)

$$\frac{\dfrac{\text{St}_{\text{Tr}}(t), \text{St}_{\text{Tr}}(s), \text{Tr}(s) \Rightarrow, \text{Tr}(s), \neg\text{Tr}(y) \qquad \text{St}_{\text{Tr}}(t), \text{St}_{\text{Tr}}(s), \neg\text{Tr}(s) \Rightarrow \text{Tr}(s), \neg\text{Tr}(y)}{\text{St}_{\text{Tr}}(t), \text{St}_{\text{Tr}}(s), \text{Tr}(t), \neg\text{Tr}(t) \Rightarrow \text{Tr}(s), \neg\text{Tr}(s)} \text{GG}}{\text{St}_{\text{Tr}}(t), \text{St}_{\text{Tr}}(s), \text{Tr}(t), \text{Tr}(\dot\neg t) \Rightarrow \text{Tr}(s), \text{Tr}(\dot\neg s)} \neg\text{Tr}\neg$$

In the last inference we have been sloppy. The double line indicates that from the upper sequent we can derive the lower sequent via some obvious additional steps involving the truth-theoretic initial sequents $\neg\mathrm{Tr}\neg\mathrm{Tr}$ and Cut.

Now suppose that $n = m + 1$ and that $\Gamma \Rightarrow \Delta$ has been derived. We distinguish two cases:

Case 1: $\Gamma \Rightarrow \Delta$ *contains no principal formula*. In this case, the last inference of the derivation of $\Gamma \Rightarrow \Delta$ must be either Ref or Cut. If the former, we obtain our desired conclusion by i.h. and Ref in $\mathrm{PKF}_\star$. Otherwise, the derivation ends with

$$
\frac{\begin{array}{ccc} \mathcal{D}_0 & & \mathcal{D}_1 \\ \vdots & & \vdots \\ \Gamma \Rightarrow \Delta, \varphi & \quad & \varphi, \Gamma \Rightarrow \Delta \end{array}}{\Gamma \Rightarrow \Delta} \; \text{Cut}
$$

For this to work, it is crucial that we are dealing with quasi-normal derivations. In this case, we can assume the cut formula to be a literal and thus apply i.h. and Cut on $\mathrm{PKF}_\star$. For example, if $\varphi \equiv \neg\psi$ for $\psi \in \mathrm{At}$, then we reason as follows:

$$
\frac{\begin{array}{ccc} \text{i.h. on } \mathcal{D}_1 & & \text{i.h. on } \mathcal{D}_0 \\ \vdots & & \vdots \\ \Gamma^+, \Delta^- \Rightarrow \Gamma^-, \Delta^+, \psi & \quad & \psi, \Gamma^+, \Delta^- \Rightarrow \Gamma^-, \Delta^+ \end{array}}{\Gamma^+, \Delta^- \Rightarrow \Gamma^-, \Delta^+} \; \text{Cut}
$$

If $\varphi \in \mathrm{At}$, then we use i.h. and cut on φ.

Case 2: $\Gamma \Rightarrow \Delta$ *contains a principal formula*. We reason by subcases, according to the last inference of the derivation. Logical rules for $\wedge$ and $\forall$ need not be taken into account as they cannot be the last rule of a quasi-normal derivation $\mathcal{D}$ that has only literals in the end-sequent. The rules having literals as principal formulae are the following: $\neg\mathrm{L}$, $\neg\mathrm{R}$, RepL, $\mathrm{IND}^{\mathrm{int}}$, and truth rules.

Case 2.1: *The last inference is* $\neg\mathrm{L}$ *or* $\neg\mathrm{R}$. Here the conclusion is immediate, e.g., suppose the $\mathrm{KF}^{\mathrm{int}}_\star$-derivation ends with

$$
\frac{\begin{array}{c} \vdots \\ \Gamma' \Rightarrow \Delta, \varphi \end{array}}{\neg\varphi, \Gamma' \Rightarrow \Delta} \; \neg\mathrm{L}
$$

If $\varphi \in \mathrm{At}$, then by induction we have $\mathrm{PKF}_\star \vdash \Gamma'^+, \Delta^- \Rightarrow \Gamma'^-, \Delta^+, \varphi$, which is our desired conclusion. The case where $\varphi \equiv \neg\psi$ for ψ atomic need not be taken into account, as we are dealing with derivations containing only literals in the end-sequent.

Case 2.2: *The last inference is* RepL. Suppose the $\mathrm{KF}_\star^{\mathrm{int}}$-derivation ends with

$$
\begin{array}{c}
\vdots \\[-2pt]
\dfrac{\varphi(t), \Gamma' \Rightarrow \Delta}{s = t, \varphi(s), \Gamma' \Rightarrow \Delta} \ \text{RepL}
\end{array}
$$

Assume first $\varphi \in$ At. For the pairs $\mathrm{KF}^{\mathrm{int}}$-PKF, $\mathrm{KF}^{\mathrm{int}}_{\mathrm{cs}}$-$\mathrm{PKF}_{\mathrm{cs}}$, and $\mathrm{KF}^{\mathrm{int}}_{\mathrm{S}}$-$\mathrm{PKF}_{\mathrm{S}}$, we just apply i.h. and RepL in each PKF-variant. For the pair $\mathrm{KF}^{\mathrm{int}}_{\mathrm{cp}}$-$\mathrm{PKF}_{\mathrm{cp}}$, since $\mathrm{PKF}_{\mathrm{cp}}$ does not have RepL, but RepR instead, we reason in $\mathrm{PKF}_{\mathrm{cp}}$ as follows, omitting context for readability:[29]

$$
\dfrac{
\dfrac{s = t, \varphi(s) \Rightarrow \varphi(s) \qquad \dfrac{\text{Symmetry of } =}{s = t, \varphi(s) \Rightarrow t = s}}{s = t, \varphi(s) \Rightarrow \varphi(t)} \ \text{RepR} \qquad
\begin{array}{c} \text{i.h.} \\ \vdots \\ \varphi(t) \Rightarrow \varnothing \end{array}
}{s = t, \varphi(s), \Rightarrow \varnothing} \ \text{Cut}
$$

For $\varphi \equiv \neg\psi$ with ψ atomic, we reason in an arbitrary $\mathrm{PKF}_\star$:

$$
\dfrac{
\begin{array}{c} \text{i.h.} \\ \vdots \\ \Gamma'^{+}, \Delta^{-} \Rightarrow \Gamma'^{-}, \Delta^{+}, \psi(t) \end{array}
\qquad
\dfrac{\text{Replacement Schema}}{s = t, \psi(t), \Gamma'^{+}, \Delta^{-} \Rightarrow \Gamma'^{-}, \Delta^{+}, \psi(s)}
}{s = t, \Gamma'^{+}, \Delta^{-} \Rightarrow \Gamma'^{-}, \Delta^{+}, \psi(s)} \ \text{Cut}
$$

Case 2.3: *The last inference is* $\mathrm{IND}^{\mathrm{int}}$. Suppose the $\mathrm{KF}_\star^{\mathrm{int}}$-derivation ends with

$$
\dfrac{\mathrm{Tr}(t(\dot{u})), \Gamma \Rightarrow \Delta, \mathrm{Tr}(t(\dot{u'}))}{\mathrm{Tr}(t(\mathrm{num}(\overline{0})/v)), \Gamma \Rightarrow \Delta, \mathrm{Tr}(t(\dot{z}))} \ \mathrm{IND}^{\mathrm{int}}
$$

with u eigenvariable. We use IND in $\mathrm{PKF}_\star$ as follows:

$$
\dfrac{
\begin{array}{c} \vdots \\ \mathrm{Tr}(t(\dot{u})), \Gamma^{+}, \Delta^{-} \Rightarrow \Gamma^{-}, \Delta^{+}, \mathrm{Tr}(t(\dot{u'})) \end{array}
}{\mathrm{Tr}(t(\mathrm{num}(\overline{0})/v)), \Gamma^{+}, \Delta^{-} \Rightarrow \Gamma^{-}, \Delta^{+}, \mathrm{Tr}(t(\dot{z}))} \ \mathrm{IND}
$$

Case 2.4: *The last inference is a truth-rule*. Finally, suppose the derivation ends with an application of a truth-rule. In this case, it essentially suffices to notice that each active formula of each truth-rule is an atomic formula. Hence, the desired conclusion follows by i.h. and the rule itself in $\mathrm{PKF}_\star$. As an example, we consider the truth-rule $\mathrm{Tr}\forall\mathrm{R}$. So

---

[29] Of course, since each PKF variant proves the replacement schema, then each PKF variant proves that identity is symmetric.

suppose the derivation ends with

$$\frac{\vdots \quad \vdots \quad \vdots}{\cfrac{\Gamma \Rightarrow \Delta', \mathrm{Tr}(t(\dot{u}))}{\mathrm{St}_{\mathrm{Tr}}(\forall v.t), \Gamma \Rightarrow \Delta', \mathrm{Tr}(\forall v.t)}} \; \mathrm{Tr}\forall\mathrm{R}$$

The conclusion follows immediately by i.h., i.e., we reason in $\mathrm{PKF}_\star$ as follows:

$$\frac{\vdots \quad \vdots \quad \vdots}{\cfrac{\Gamma^+, \Delta'^- \Rightarrow \Gamma^-, \Delta'^+, \mathrm{Tr}(t(\dot{u}))}{\mathrm{St}_{\mathrm{Tr}}(\forall v.t), \Gamma^+, \Delta'^- \Rightarrow \Gamma^-, \Delta'^+, \mathrm{Tr}(\forall v.t)}} \; \mathrm{Tr}\forall\mathrm{R}$$

$\square$

By inspecting the proof of the Main Lemma, it can be observed that we can immediately lift it to the pair $(\mathrm{KF}_\star^-, \mathrm{PKF}_\star^-)$, thus obtaining the following.

COROLLARY 4.6. *For Γ, Δ sets of literals, for all $\star$*

$$\mathrm{KF}_\star^- \vdash \Gamma \Rightarrow \Delta \quad implies \quad \mathrm{PKF}_\star^- \vdash \Gamma^+, \Delta^- \Rightarrow \Gamma^-, \Delta^+.$$

This immediately yields that the set of significant inferences of KF-like theories is contained in the set of provable sequents of appropriate PKF-like theories.

LEMMA 4.7. $\mathrm{KF}^\circ\mathrm{SI} \subseteq \mathrm{PKF}^\circ$.

Proof. If $\mathrm{KF}^\circ \vdash \mathrm{Tr}\ulcorner\Gamma\urcorner \Rightarrow \mathrm{Tr}\ulcorner\Delta\urcorner$, then $\mathrm{PKF}^\circ \vdash \mathrm{Tr}\ulcorner\Gamma\urcorner \Rightarrow \mathrm{Tr}\ulcorner\Delta\urcorner$ by Lemma 4.5 and Corollary 4.6. But the truth predicate of PKF° is transparent and thus $\mathrm{PKF}^\circ \vdash \Gamma \Rightarrow \Delta$. $\square$

Lemma 4.7 along with Corollary 4.4 shows that PKF° yields precisely the significant sentence of KF°. In other words we have answered Question (b) of REINHARDT'S CHALLENGE.

THEOREM 4.8 (REINHARDT'S CHALLENGE). $\mathrm{PKF}^\circ = \mathrm{KF}^\circ\mathrm{SI}$.

We remark that an answer to Reinhardt's Question (b) yields an answer to Question (a) as a corollary, that is, our result subsumes parts of the results provided by Halbach & Horsten [10], Halbach & Nicolai [11], and Nicolai 14].

COROLLARY 4.9. $\{\varphi \in \mathrm{Sent}_{\mathcal{L}_{\mathrm{Tr}}} \mid \mathrm{PKF}^\circ \vdash \varphi\} = \mathrm{KF}^\circ\mathrm{S}$.

Of course, this result also implies that, by means of a very simple observation connecting provably true KF-sequents to provable PKF-sequents, we have reduced questions regarding the proof-theoretic strength of PKF-like theories to questions regarding the proof-theoretic strength of KF-like theories.

COROLLARY 4.10. KF° *and* PKF° *are proof-theoretically equivalent, i.e.,*

$$\mathrm{KF}^\circ \equiv \mathrm{PKF}^\circ.$$

Concluding, we promised to give a positive answer to the GENERALIZED REINHARDT PROBLEM for KF-variants with restricted forms of induction, that is, the question

whether for any pair $\langle \Gamma, \Delta \rangle \in \mathrm{KF}^{\circ}\mathrm{SI}$ there is a KF°-derivation such that each node of the derivation is a member of $\mathrm{KF}^{\circ}\mathrm{SI}$. However, as anticipated in the Introduction (Section 1) to this paper, a positive answer to GENERALIZED REINHARDT PROBLEM follows rather immediately, once we have an independent axiomatization of $\mathrm{KF}^{\circ}\mathrm{SI}$:

COROLLARY 4.11 (GENERALIZED REINHARDT PROBLEM). *If* $\langle \Gamma, \Delta \rangle \in \mathrm{KF}^{\circ}\mathrm{SI}$, *then there is a* KF°-*derivation* $\mathcal{D}$ *of* $\Gamma \Rightarrow \Delta$ *such that for each node* d *of* $\mathcal{D}$, $d \in \mathrm{KF}^{\circ}\mathrm{SI}$.

Proof. If $\langle \Gamma, \Delta \rangle \in \mathrm{KF}^{\circ}\mathrm{SI}$, then by Proposition 4.8 $\mathrm{PKF}^{\circ} \vdash \Gamma \Rightarrow \Delta$ and hence $\mathrm{PKF}^{\circ} \vdash \mathrm{Tr}^{\ulcorner}\Gamma^{\urcorner} \Rightarrow \mathrm{Tr}^{\ulcorner}\Delta^{\urcorner}$. Now let $\mathcal{D}'$ be an arbitrary KF°-derivation of $\mathrm{Tr}^{\ulcorner}\Gamma^{\urcorner} \Rightarrow \mathrm{Tr}^{\ulcorner}\Delta^{\urcorner}$.

In order to obtain our desired derivation $\mathcal{D}$, it suffices to replace each node (including the leaves)

$$\Theta \Rightarrow \Lambda$$

of $\mathcal{D}'$ by

$$\mathrm{Tr}^{\ulcorner}\Gamma^{\urcorner}, \Theta \Rightarrow \Lambda, \mathrm{Tr}^{\ulcorner}\Delta^{\urcorner}.$$

Let $\mathcal{D}$ be the derivation resulting from this transformation. Since KF° is closed under weakening, $\mathcal{D}$ is a KF°-derivation of $\mathrm{Tr}^{\ulcorner}\Gamma^{\urcorner} \Rightarrow \mathrm{Tr}^{\ulcorner}\Delta^{\urcorner}$. But each node of $\mathcal{D}$ is also derivable in PKF°, too, since PKF° is closed under weakening. Hence every node of $\mathcal{D}$ is in $\mathrm{PKF}^{\circ} = \mathrm{KF}^{\circ}\mathrm{SI}$. □

Let us put our partial answer to the GENERALIZED REINHARDT PROBLEM in perspective: we have shown that for every significant inference there is a way to classically derive the sequent such that every node of the derivation is itself a significant inference and hence that every node of the proof is acceptable to the significant reasoner, i.e., the non-classical logician. This does not imply that the non-classical logician can follow the classical reasoning, i.e., that the KF°-proof is also a PKF°-proof. Our result only shows that KF-style theories with restricted forms of induction can be used instrumentally. It does not show that one can always reason non-classically within KF°. But if the latter were the case, it seems that KF° would deliver an independent axiomatization of its significant part in its own right.[30] Surely—while it is certainly an interesting question whether for every KF°-significant inference there is a KF°-derivation, which is also a PKF°-derivation—such a result is not required for an instrumental interpretation of KF° and left for future research.

§5. Conclusion. In this paper, we had a fresh look at Reinhardt's program and we proposed to focus on the provably true inferences of KF-like theories rather than on the provably true sentences only. We showed that if we conceive of the significant part of KF-like theories as the set of provably true inferences, then Reinhardt's program can be deemed successful for variants of KF with a restricted form of induction: we can remain within the significant part of the theories in proving their significant inferences. This answers the GENERALIZED REINHARDT PROBLEM for the aforementioned KF-variants and also shows that we need not step outside the significant part in

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[30] We note that even in the original formulation of REINHARDT'S PROBLEM due to [10] an affirmative answer would not have implied that the reasoning as such is non-classically acceptable.

proving theorems of the form $\text{Tr}\ulcorner\varphi\urcorner$, which was the content of the original formulation of REINHARDT'S PROBLEM. The use of the nonsignificant part of KF^{int} and KF^- is dispensable, and, in this sense, both theories can be given an instrumentalist interpretation as suggested by Reinhardt. Moreover, building on results by Halbach & Horsten [10], Halbach & Nicolai [11], and Nicolai [14], we have shown how to provide independent axiomatizations of the significant part of KF-like theories with restricted forms of induction in non-classical logic. This was precisely the content of REINHARDT'S CHALLENGE. Our results thus get us some way to completing Reinhardt's program and, at least as far as the KF-theories with restricted forms of induction are concerned, they fully "*justify the use of nonsignificant sentences entirely within the framework of significant sentences*" [21, p. 225].

However, to complete Reinhardt's program and prove it successful will require answering REINHARDT'S CHALLENGE for KF-like theories with full induction. To this effect one needs to pair KF-like theories with suitable non-classical truth theories. In light of Nicolai's [14] work, it seems very likely that these non-classical theories will be versions of PKF^+, although a proof of this conjecture is left for future research. Let us assume for now that PKF^+-like theories indeed answer REINHARDT'S CHALLENGE for KF theories with full induction. Should we conclude that we have justified the use of nonsignificant sentences entirely within the framework of significant sentences? Should we deem Reinhardt's program successful? In contrast to the case of KF-like theories with restricted forms of induction, an answer to this question seems to be less straightforward, as one may query whether PKF^+ amounts to an *independent* axiomatization of the significant part of KF. The crucial question is whether the rule $\text{TI}^{<\varepsilon_0}$ is available from within the significant framework. Ultimately, an answer to this question will depend on the role the theory of truth is supposed to play within one's theoretical framework. If, for instance, one takes the theory to play an important role in the foundations of mathematics and an important role in singling out the limits of predicativity Feferman [6], then one should arguably refrain from thinking that $\text{TI}^{<\varepsilon_0}$ can be assumed without further justification from within the significant framework. But, to the contrary, if the theory of truth is to play no role in the foundations of mathematics and classical mathematical theorizing is freely available from within the significant framework, then it is hard to see why $\text{TI}^{<\varepsilon_0}$ should not be considered as fully justified from within the significant perspective.[31] In this case it would seem that Reinhardt's program needs to be deemed successful once Theorem 4.8 has been extended to the theory pair $\langle\text{KF}, \text{PKF}^+\rangle$. Of course, such a generalization of Theorem 4.8 is yet to be provided, but the understanding of Reinhardt's program developed in this paper together with the results for KF-variants with restricted forms of induction suggests there is no principled obstacle in way of a successful completion of Reinhardt's program. Thus, after all, it seems that Reinhardt was right in claiming that "*the chances of success in this context* $(\cdots)$ *are somewhat better than in Hilbert's context*" [21, p. 225].

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[31] A proper philosophical assessment of the rule of transfinite induction up to $< \varepsilon_0$ from the perspective of Reinhardt's program is, however, beyond the scope of the paper.

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