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Flexibility in generating sets of finite groups

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Abstract. Let G be a finite group. It has recently been proved that every nontrivial element of G is contained in a generating set of minimal size if and only if all proper quotients of G require fewer generators than G. It is natural to ask which finite groups, in addition, have the property that any two elements of G that do not generate a cyclic group can be extended to a generating set of minimal size. This note answers the question. The only such finite groups are very specific affine groups: elementary abelian groups extended by a cyclic group acting as scalars.

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1. Introduction. Generating sets for finite groups have attracted the attention of many authors for decades. In recent years, major developments have been made drawing on our extensive knowledge of the generating sets of finite simple groups. One natural question on this topic is: for which finite groups G is every element contained in a generating set of minimal size $d(G)$? Completing a long line of research, this question was recently proved to admit a very simple answer: the finite groups G such that $d(G/N) < d(G)$ for all $1 \neq N \trianglelefteqslant G$. Acciarri and Lucchini [2] proved the case with $d(G) \geqslant 3$ and Burness, Guralnick, and Harper [4] completed the theorem by proving an influential conjecture of Breuer, Guralnick, and Kantor [3] regarding the case with $d(G) \leqslant 2$.

The most straightforward example with this generation property is the elementary abelian group p^r. Considered as a vector space, we see that not only is every nonzero vector contained in a generating set of minimal size (a basis), but moreover any collection of linearly independent vectors can be extended to such a generating set. This motivates the following definition.

Definition. Let G be a finite group. Let $d(G)$ be the minimal size of a generating set for G. For an integer $1 \leqslant k \leqslant d(G)$, we say that G is *k-flexible* if for any $x_1, \dots, x_k \in G$ such that $d(\langle x_1, \dots, x_k \rangle) = k$, there exists $x_{k+1}, \dots, x_{d(G)} \in G$ such that $\langle x_1, \dots, x_{d(G)} \rangle = G$.

We can now rephrase the result of Acciarri–Lucchini [2] and Burness–Guralnick–Harper [4].

Theorem 1. *A finite group G is 1-flexible if and only if $d(G/N) < d(G)$ for all $1 \neq N \trianglelefteqslant G$.*

Remark 1. Let us describe the groups in Theorem 1. (We use ATLAS notation, so p denotes both a prime and the cyclic group of that order.)

(i) If $d(G) = 1$, then $G = p$ with p prime.
(ii) If $d(G) = 2$, then, by Lemma 2.1 in [4] and the discussion that follows, G is one of:
  (a) p^2 for p prime,
  (b) $p^r{:}\langle g \rangle$ for p prime and $g \in \mathrm{GL}_r(p)$ irreducible,
  (c) $T^r.\langle (g, 1, \dots, 1)\sigma \rangle$ for a nonabelian simple group T, $g \in \mathrm{Aut}(T)$, and an r-cycle $\sigma \in S_r$.
(iii) If $d(G) \geqslant 3$, then, by [5, Theorem 1.4],
$$G = \{(g_1, \dots, g_r) \in L^r \mid Mg_1 = \dots = Mg_r\}$$
for a primitive monolithic group L with monolith M and $r = f(d(G)-1)$ where f is the function given in [5, Theorem 2.7].

This paper shows that startlingly few 1-flexible groups are also 2-flexible.

Theorem 2. *Let G be a finite group with $d(G) \geqslant 3$. Then the following are equivalent:*

(i) *G is 1-flexible and 2-flexible,*
(ii) *G is k-flexible for all $1 \leqslant k < d(G)$,*
(iii) *$G = p^r{:}\langle g \rangle$ for a prime p, a scalar $g \in \mathrm{GL}_r(p)$ and $r = d(G) - d(\langle g \rangle)$.*

While the class of 1-flexible groups is a large and rich class of soluble and insoluble groups, including all finite simple groups, adding the assumption of 2-flexibility restricts to a narrow class of supersoluble groups and guarantees k-flexibility for all $1 \leqslant k < d(G)$.

Remark 2. In Theorem 2, we assume that $d(G) \geqslant 3$. For $d(G) = 1$, 2-flexibility is not interesting. For $d(G) = 2$, we will see in Lemma 2.7 that G is 1-flexible and 2-flexible if and only if $G = p{:}\langle g \rangle$ where p is prime and $g \in \mathrm{GL}_r(p)$ is a scalar of *prime order*.

Remark 3. Note that 2-flexibility does not imply 1-flexibility. For example, if G is the quaternion group Q_8, then every proper subgroup of G is cyclic, so G is 2-flexible, but the nontrivial central element of G is not contained in any generating pair for G, so G is not 1-flexible. However, observe that Q_8 is a cyclic central extension of 2^2, which is 1-flexible. Indeed, by Corollary 2.6, every 2-flexible group is a cyclic central extension of a 1-flexible group.

2. Proofs. For this section, let G be a finite group such that $d(G) \geqslant 2$. We begin with some preliminaries.

Lemma 2.1. *Let $1 \leqslant k \leqslant d(G)$. Assume that G is k-flexible. Let $N \trianglelefteq G$ such that $d(G/N) = d(G)$. Then G/N is k-flexible.*

Proof. Write $d = d(G) = d(G/N)$. Let $Nx_1, \ldots, Nx_k \in G/N$ and assume $d(\langle Nx_1, \ldots, Nx_k\rangle) = k$. Then $d(\langle x_1, \ldots, x_k\rangle) = k$, so, as G is k-flexible, there exist $x_{k+1}, \ldots, x_d \in G$ such that $\langle x_1, \ldots, x_d\rangle = G$. Therefore, we deduce $\langle Nx_1, \ldots, Nx_d\rangle = G$, so G/N is k-flexible. □

To obtain a partial converse to Lemma 2.1, in Lemma 2.4, let us recall that the *cycliciser* of G is

$$\mathrm{Cyc}(G) = \{c \in G \mid \langle c, g\rangle \text{ is cyclic for all } g \in G\}.$$

The fact that the cycliciser $\mathrm{Cyc}(G)$ is a (cyclic) subgroup of G is a consequence of [1, Lemma 32], which states that if $x, y, z \in G$ are such that each of $\langle x, y\rangle$, $\langle x, z\rangle$, $\langle y, z\rangle$ are cyclic, then $\langle x, y, z\rangle$ is also cyclic. Lemma 2.2 characterises $\mathrm{Cyc}(G)$.

Lemma 2.2. *The cycliciser $\mathrm{Cyc}(G)$ is the smallest normal subgroup $N \trianglelefteq G$ such that $\mathrm{Cyc}(G/N)$ is trivial.*

Proof. Let $N = \mathrm{Cyc}(G)$. Let $g \in G$ such that $Ng \in \mathrm{Cyc}(G/N)$. Let $h \in G$ be arbitrary. Then $\langle Ng, Nh\rangle$ is cyclic, so there exists $k \in G$ with $\langle Ng, Nh\rangle = \langle Nk\rangle$. Therefore, $\langle N, g, h\rangle = \langle N, k\rangle$. Now [1, Lemma 32] implies that $\langle N, k\rangle$ is cyclic, so $\langle N, g, h\rangle$ is cyclic and thus $\langle g, h\rangle$ is cyclic. This means that $g \in N$, which proves that $\mathrm{Cyc}(G/N)$ is trivial.

Let $N \trianglelefteq G$ with $\mathrm{Cyc}(G/N) = 1$. Let $x \in \mathrm{Cyc}(G)$. For all $g \in G$, $\langle Nx, Ng\rangle$ is cyclic as $\langle x, g\rangle$ is cyclic. Hence, $Nx \in \mathrm{Cyc}(G/N)$, so $x \in N$ as $\mathrm{Cyc}(G/N) = 1$. Therefore, $\mathrm{Cyc}(G) \leqslant N$. □

Lemma 2.3. *Assume that $d(G) \geqslant 2$. Then $d(G/\mathrm{Cyc}(G)) = d(G)$.*

Proof. Write $\mathrm{Cyc}(G) = N$ and write $G/N = \langle Ng_1, \ldots, Ng_{d(G/N)}\rangle$. Then $G = \langle N, g_1, \ldots, g_{d(G/N)}\rangle$. As $d(G) \geqslant 2$, the quotient G/N is nontrivial, so $d(G/N) \geqslant 1$. Since $\langle N, g_1\rangle$ is cyclic, fix $h \in G$ such that $\langle N, g_1\rangle = \langle h\rangle$. Then $G = \langle h, g_2, \ldots, g_{d(G/N)}\rangle$, which gives us the bound $d(G) \leqslant d(G/N)$. Clearly, $d(G) \geqslant d(G/N)$, so $d(G) = d(G/N)$. □

Lemma 2.4. *Let $2 \leqslant k < d(G)$. Then G is k-flexible if and only if $G/\mathrm{Cyc}(G)$ is k-flexible.*

Proof. Write $\mathrm{Cyc}(G) = N$. First assume that G is k-flexible. Since $d(G) \geqslant 2$, Lemma 2.3 gives $d(G/N) = d(G)$, so Lemma 2.1 gives that G/N is k-flexible.

Now assume that G/N is k-flexible. Let $x_1, \ldots, x_k \in G$ such that $d(\langle x_1, \ldots, x_k\rangle) = k$ and write $H = \langle x_1, \ldots, x_k\rangle$. The image of H in G/N is isomorphic to $H/(H \cap N)$. Since $H \cap N \leqslant \mathrm{Cyc}(H)$ and $d(H) = k \geqslant 2$, Lemma 2.3 implies that $d(\langle Nx_1, \ldots, Nx_k\rangle) = d(H/(H \cap N)) = d(H) = k$. Since G/N is k-flexible, there exist elements $x_{k+1}, \ldots, x_{d(G)} \in G$ such that $\langle Nx_1, \ldots, Nx_{d(G)}\rangle = G/N$. Since $\langle N, x_{d(G)}\rangle$ is cyclic, fix $y \in G$ such that $\langle N, x_{d(G)}\rangle = \langle y\rangle$. Now $G = \langle x_1, \ldots, x_{d(G)-1}, y\rangle$, so G is k-flexible. □

The following lemma indicates that the case $k = 1$ does not follow the pattern described by Lemma 2.4.

Lemma 2.5. *Assume that G is 2-flexible. Then G is 1-flexible if and only if $\mathrm{Cyc}(G) = 1$.*

Proof. First assume that $\mathrm{Cyc}(G)$ is trivial. Let $1 \neq x_1 \in G$. As $\mathrm{Cyc}(G)$ is trivial, $x_1 \notin \mathrm{Cyc}(G)$, so there exists $x_2 \in G$ such that $\langle x_1, x_2\rangle$ is noncyclic. Since G is 2-flexible, there exist elements $x_3, \dots, x_{d(G)} \in G$ such that $\langle x_1, \dots, x_{d(G)}\rangle = G$. Therefore, G is 1-flexible.

Now assume that $\mathrm{Cyc}(G)$ is nontrivial. Suppose that G is 1-flexible. Let $1 \neq x_1 \in \mathrm{Cyc}(G)$. Then there exist elements $x_2, \dots, x_{d(G)} \in G$ such that $\langle x_1, \dots, x_{d(G)}\rangle = G$. Since $x_1 \in \mathrm{Cyc}(G)$, there exists $y \in G$ such that $\langle x_1, x_2\rangle = \langle y\rangle$, so $G = \langle x_1, \dots, x_{d(G)}\rangle = \langle y, x_3, \dots, x_{d(G)}\rangle$, which is impossible as G has no generating set of size $d(G) - 1$. Therefore, G is not 1-flexible. □

Corollary 2.6. *Assume that G is 2-flexible. Then $G/\mathrm{Cyc}(G)$ is 1-flexible.*

Proof. By Lemma 2.2, $\mathrm{Cyc}(G/\mathrm{Cyc}(G)) = 1$, so by Lemma 2.5, $G/\mathrm{Cyc}(G)$ is 1-flexible. □

We first handle the case where $d(G) = 2$.

Lemma 2.7. *Assume that $d(G) = 2$. Then G is 2-flexible if and only if $G = p^2$, $G = Q_8$, or G is presented as $\langle a, b \mid a^p = b^{q^m} = b^{-1}aba^{-r}\rangle$ for $p \neq q$ primes and $r > 1$ satisfying $r \mid q - 1$ and $p \mid r^q - 1$.*

Proof. Observe that G is 2-flexible if and only if every proper subgroup of G is cyclic. The result now follows from the main theorem of [7]. □

Lemma 2.8. *Assume that $d(G) = 2$. Then G is 1-flexible and 2-flexible if and only if $G = p^2$ or $G = p{:}\langle g\rangle$ where $g \in \mathrm{GL}_1(p)$ has prime order.*

Proof. By Remark 1 and Lemma 2.7, the groups in the statement are 1-flexible and 2-flexible. For the converse, consulting Lemma 2.7 and noting that p^2 is 1-flexible and Q_8 is not 1-flexible, it remains to show that if $G = \langle a, b \mid a^p = b^{q^m} = b^{-1}aba^{-r}\rangle$ with p, q distinct primes and $r > 1$ satisfying $r \mid q - 1$ and $p \mid r^q - 1$, then $m = 1$. To do this, note that if $m > 1$, then $1 \neq b^q \in Z(G)$ but G is nonabelian, so b^q is not contained in a generating pair of G. □

We now turn to the case where $d(G) \geqslant 3$. We begin with two examples.

Example 2.9. Let p be prime. Let G be the elementary abelian group p^r, so $d(G) = r$. Let $1 \leqslant k \leqslant r$. As p^r is a vector space, given $x_1, \dots, x_k \in G$, if $d(\langle x_1, \dots, x_k\rangle) = k$, then $x_1, \dots, x_k$ are linearly independent, so they can be extended to a basis $x_1, \dots, x_r$ for G. Thus, G is k-flexible.

Example 2.10. Let $r \geqslant 2$ and p be prime. Let $G = p^r{:}\langle g\rangle$ for a nontrivial scalar $g \in \mathrm{GL}_r(p)$. Note that $d(G) = r + 1$ (see [5, Theorem 2.7]). Let $1 \leqslant k \leqslant r$. We claim that G is k-flexible.

Let $x_1, \dots, x_k \in G$ such that $d(\langle x_1, \dots, x_k\rangle) = k$. Observe that there are exactly p^r distinct G-conjugates of $\langle g\rangle$ and that any two of these conjugates intersect trivially. Hence, fix a conjugate H of $\langle g\rangle$ such that $x_1,\dots,x_k \notin H$. Write $x_i = n_i h_i$ where $n_i \in p^r$ and $h_i \in H$; note that $n_i \neq 1$ as $x_i \notin H$. Fix $x_{k+1},\dots,x_r \in p^r$ such that $\langle n_1,\dots,n_k,x_{k+1},\dots,x_r\rangle = p^r$ and let x_{r+1} generate H. Let $X = \langle x_1,\dots,x_{r+1}\rangle$. Then $x_{r+1} \in X$, so $H \leqslant X$, and for each $1 \leqslant i \leqslant k$, we have $h_i \in \langle x_{r+1}\rangle \leqslant X$ and hence $n_i = x_i h_i^{-1} \in X$, so $p^r = \langle n_1,\dots,n_k,x_{k+1},\dots,x_r\rangle \leqslant X$, from which we can conclude that $G = p^r{:}H = X = \langle x_1,\dots,x_{r+1}\rangle$.

Corollary 2.11. *Assume $d(G) \geqslant 3$ and $G/\mathrm{Cyc}(G)$ is $p^r{:}\langle g\rangle$ where p is prime and $g \in \mathrm{GL}_r(p)$ is a scalar. Then G is k-flexible for all $2 \leqslant k < d(G)$.*

Proof. This is a consequence of Lemma 2.4 and Examples 2.9 and 2.10. □

We now show Examples 2.9 and 2.10 to be, in essence, the only examples.

Lemma 2.12. *Assume $d(G) \geqslant 3$ and G is 2-flexible. Then every minimal normal subgroup of G is cyclic.*

Proof. Let N be a minimal normal subgroup of G. For a contradiction, suppose that N is noncyclic. By [1, Lemma 32], there exist $x_1, x_2 \in N$ such that $\langle x_1, x_2\rangle$ is noncyclic. By assumption, there exists $x_3,\dots,x_{d(G)} \in G$ such that $\langle x_1,\dots,x_{d(G)}\rangle = G$, so $\langle Nx_1,\dots,Nx_{d(G)}\rangle = \langle Nx_3,\dots,Nx_{d(G)}\rangle = G/N$, which implies that $d(G/N) \leqslant d(G) - 2$. The main theorem of [6] implies that G is a nonabelian simple group, which is a contradiction since $d(G) \geqslant 3$. We conclude that N is cyclic. □

Lemma 2.13. *Assume $d(G) \geqslant 3$ and G is both 1-flexible and 2-flexible. Then $G = p^r{:}\langle g\rangle$ for a prime p, a scalar $g \in \mathrm{GL}_r(p)$ and $r = d(G) - d(\langle g\rangle)$.*

Proof. As G is 1-flexible, as discussed in Remark 1, we know $d(G/N) < d(G)$ for all $1 \leqslant N \trianglelefteq G$, so by [5, Theorem 1.4],

$$G = \{(g_1,\dots,g_r) \in L^r \mid Mg_1 = \cdots = Mg_r\}$$

with L a primitive monolithic group with monolith M and $r = f(d(G) - 1)$ where f is the function given in [5, Theorem 2.7]. Since G is 2-flexible, by Lemma 2.13, every minimal normal subgroup of G is cyclic, so $M = p$ and we deduce that $L = p{:}\langle\lambda\rangle$ where $\lambda \in \mathrm{GL}_1(p)$, so $G = p^r{:}\langle g\rangle$ where $g \in \mathrm{GL}_r(p)$ is a scalar. It remains to note that either $g = 1$ and $r = d(G)$ or otherwise $g \neq 1$ and [5, Theorem 2.7] implies that $r = d(G) - 1$. □

Theorem 2.14. *Let G be a finite group with $d(G) \geqslant 3$. Then the following are equivalent:*

(i) *G is 2-flexible,*
(ii) *G is k-flexible for all $2 \leqslant k < d(G)$,*
(iii) *$G/\mathrm{Cyc}(G) = p^r{:}\langle g\rangle$ for prime p, scalar $g \in \mathrm{GL}_r(p)$ and $r = d(G) - d(\langle g\rangle)$.*

Proof. (iii) $\Longrightarrow$ (ii): This is Corollary 2.11. (ii) $\Longrightarrow$ (i): This is immediately clear. (i) $\Longrightarrow$ (iii): Lemma 2.1 implies $G/\mathrm{Cyc}(G)$ is 2-flexible and Lemma 2.2 implies $\mathrm{Cyc}(G/\mathrm{Cyc}(G))$ is trivial, whence Lemma 2.5 implies $G/\mathrm{Cyc}(G)$ is 1-flexible, so Lemma 2.13 implies $G/\mathrm{Cyc}(G)$ has the claimed structure. □

Proof of Theorem 2. This is an immediate consequence of Lemma 2.5 and Theorem 2.14. □

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