An Initial Study of Multimodality in Wind Farm Layout Optimization Problems

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A study into the phenomena of multimodality in wind farm layout optimisation problems is presented. Little previous work has studied multimodality so this work aims to begin providing quantifiable results on multimodality in farm layout optimisation. The problems of optimising the location of a four and nine turbine farm within a bound circular domain are considered. Multi-start gradient-based optimisation is used to determine optima present in each case. Clear multimodality was found in all cases tested. With increasing farm size, the number of optima present increases, however the difference in performance of those optima is small. Therefore, while multimodality is present, for large farms, the chance of locating a local minima that has a detrimental performance compared to the global optimum, is small.

I. Introduction

Climate change, driven primarily by man-made increases in greenhouse gas concentrations,[1] is one of the most important factors in governmental policy making globally. This is set against the backdrop of global population increase (which is set to reach 11 billion by the end of the century[2]) and the continual increase in global living standards causing an exponential increase in the desire for energy. As such, a substantial drive in the development of low-emission energy production methods has been occurring. For example, in the United Kingdom, low-emission sources account for 37% of the UK energy production.[3] By far the most productive energy source is wind, which accounts for approximately a quarter of the total UK energy production.[3]

Power can be produced from wind by using a single turbine but in order to produce power on a large scale, multiple turbines are placed together in a wind farm. However, this introduces many associated problems which can reduce the overall efficiency of total power production. A problem that wind farms face is a reduction in efficiency due to wake interference. A wake is a zone of turbulent air which is slower than the surrounding flow and is created due to the turbine extracting kinetic energy from the air. If a turbine is placed within the wake of another turbine in the wind farm, then this turbine will produce less power than if it were in freestream flow;[4,5] for example, Barthelmie et al.[6] calculate an average energy loss of 12% in an offshore wind farm due to wake interaction effects. Furthermore, the increased turbulence and shear present within the wake may also increase the dynamic loads on this turbine.

Wind farm layout optimisation (WFLO) is the process whereby the locations of wind turbines within a farm are optimised to maximise some objective function, for example, total power output; see,[7,8] for example. WFLO requires the coupling of a simulation method to determine farm outputs as a function of turbine placements (and other factors such as flow conditions, turbine power curves, terrain, etc.), with a numerical optimisation algorithm that alters a set of design variables to maximise some objective. The turbine locations often take the form of design variables.

The choice of optimisation algorithm is drive by a number of factors. One primary factor is the degree of multimodality present in the problem. Multimodality is a reference to the number of local minima present in the problem i.e. high multimodality means that there are lots of local minima, whereas low multimodality

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means that there are few local minima. For a problem with high multimodality, it is common to use global search algorithms. For example, WFLO problems have been solved with a variety of global search methods, for example;[9–11] a comprehensive review of WFLO being solved with these types of algorithms is given by Khan and Rehman.[12] Global search methods attempt to locate the global (i.e. best) minimum from the large number of local minima. However, these methods often come at the expense of cost, and require a large number of objective function evaluations. Furthermore, these methods also suffer from the so-called '*curse of dimensionality*',[13] where the cost scales poorly with increased number of dimensions (design variables).

Given the limitations of global search methods, knowledge of multimodality is key to understanding whether these methods should be used, how they should be implemented, and expected performance. For WFLO, multimodality is generally assumed. In deed, Stanley and Ning[14] recently showed, through a simple study of moving one turbine in a 100 turbine optimisation problem, that the problem was highly multimodal. However, beyond that, no study has attempted to investigate multimodality in WFLO. There are many unanswered questions, such as is multimodality necessary a problem(?), to what degree are WFLO problems multimodal(?), which factors affect multimodality and how(?), is multimodality affected by different farm simulation approaches(?), can multimodality be avoided through clever use of parameterisation techniques(?), etc. etc. One fundamental question is whether there is a need to consider global search methods at all. Specifically, even if the WFLO problem is multimodal, if most optima have similar performance, then there is less reason for running a global optimisation method.

The work presented in this paper will begin to investigate multimodality in WFLO. This is performed through the study of a simple model problem. This work will not be able to answer all of the open questions posed above, but will begin the study of this important topic.

II. Problem Definition

As an initial study into multimodality, the model problem considered in this work is the optimisation of a small number of turbines, bounded within a circle. The set-up was inspired by Baker *et al.*,[15] who compared different optimisation results on a similar problem.

II.A. Optimisation Definition

The problem is parameterised in polar coordinates so the design variables are the radius and azimuth of each turbine. For a farm with N turbines, this leads to $2N$ design variables. For the i-th turbine ($i \in [1, N]$), the radius is the scaled distance from the origin and is scaled by the maximum radius of the farm, $\bar{r}_i = r_i/R$ so $\bar{r}_i \in 0, 1$, and the azimuth is the scaled angle (measured anti-clockwise from the positive x-axis) scaled by 2π, $\bar{\theta}_i = \theta_i/2\pi$. To avoid boundary issues on the angle, periodicity is exploited by leaving $\bar{\theta}_i$ unbounded i.e. if $\bar{\theta}_i = 1.1$, this is the same angle as if $\bar{\theta}_i = 0.1$. The design variable vector and its bounds can be written as:

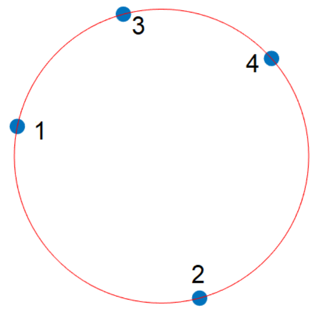
$$\begin{pmatrix} 0 \\ -\infty \\ 0 \\ -\infty \\ \vdots \\ 0 \\ -\infty \end{pmatrix} \leq \begin{pmatrix} \bar{r}_1 \\ \bar{\theta}_1 \\ \bar{r}_2 \\ \bar{\theta}_2 \\ \vdots \\ \bar{r}_N \\ \bar{\theta}_N \end{pmatrix} \leq \begin{pmatrix} 1 \\ +\infty \\ 1 \\ +\infty \\ \vdots \\ 1 \\ +\infty \end{pmatrix}$$

or more compactly in vector form:

$$\mathbf{L} \leq \boldsymbol{\alpha} \leq \mathbf{U}$$

An example layout for a four turbine farm with a given design vector is shown in figure 1.

The annual energy production (AEP) is the objective to be maximised. Due to most optimisers dealine with minimisation, this is transformed into -AEP for minimisation. There are no constraints beyond the



$$\begin{pmatrix} \bar{r}_1 \\ \bar{\theta}_1 \\ \bar{r}_2 \\ \bar{\theta}_2 \\ \bar{r}_3 \\ \bar{\theta}_3 \\ \bar{r}_4 \\ \bar{\theta}_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.47 \\ 1 \\ 0.79 \\ 1 \\ 0.29 \\ 1 \\ 1.11 \end{pmatrix}$$

Figure 1: Example of the graphical and numerical representations of a 4-turbine wind farm

design space bounds. Therefore, the optimisation problem is given as:

$$\begin{aligned} \underset{\boldsymbol{\alpha} \in \Re^{2N}}{\text{minimise}} \quad & -AEP \\ \text{subject to} \quad & \mathbf{L} \leq \boldsymbol{\alpha} \leq \mathbf{U} \end{aligned} \tag{1}$$

In the paper, a four turbine case is studies initially, followed by a nine turbine case.

II.B. Turbine Characteristics

For this work, a fixed turbine is considered. As noted in the introduction, further studies on the effect of turbine characteristics on multimodality as required, but for simplicity, this work is considering one turbine model.

The turbine considered in this report is the NREL 5MW turbine. This is a theoretical reference turbine that was developed by Jonkman et al.[16] in order to assess offshore wind technology. The characteristics of this turbine are shown in Table 1.

Table 1: Characteristics of the NREL 5MW turbine

Parameter	Value
Rotor Diameter (m)	126.0
Hub Height (m)	90.0
Tip speed ratio	8.0
Blade Count	3
Blade Pitch (°)	0.0
Yaw Angle (°)	0.0

II.C. Farm Conditions

As noted above, the farm is circular. The maximum radius is fixed, though the effect of this is studied later. The oncoming flow is split into 30° bins. It was chosen to avoid variation in the oncoming flow direction probability to reduce the number of factors that were being studied, so each bin has an equal probability. The speed from each direction is fixed at 8m/s.

III. Tools Used

A WFLO process couples together an optimisation algorithm with a farm flow solver through a parameterisation. The problem parameterisation is described above and the optimiser and solver are described here. The AEP for the farm is determined through the FLOw Redirection and Induction in Steady-state (FLORIS) tool, which is described below. To determine multimodality, multiple gradient-based optimisation runs are performed on a given problem. This optimisation strategy is described below.

III.A. Flow Solver - FLORIS

There exists a wide variety of methods allowing for the evaluation of the AEP of a given wind farm layout. These methods are commonly referred to as Farm Flow Solvers. Methods vary from high-fidelity Computational Fluid Dynamics (CFD) methods, where accuracy is given at the expense of high cost, to lower-fidelity Engineering Wake Models (EWM),[17–19] which try to approximate the flow using empirical or analytical fits leading to rapid evaluation of AEP.

A Farm Flow Solver of particular interest is the FLOw Redirection and Induction in Steady-state (FLORIS) model developed by Gebraad et al.[20] The FLORIS model functions by combining analytical relationships for wake decay, wake deflection and wake expansion as factors of the turbine and oncoming flow directions to determine turbine powers. These are then combined to output the AEP of the entire farm. To make the FLORIS model more suitable for gradient-based optimisation, Thomas *et al.*[21] make a series of modifications to remove discontinuities in the model. This was then shown to make FLORIS better suited for gradient-based optimisation.[22]

III.B. Optimiser - SLSQP

The problem given in equation 1 is defined as a bound-constrained problem, which is more often termed an unconstrained problem. The unconstrained refers to the lack of nonlinear constraints which can often be present in optimisation problems. The bound constraints are linear, and constrain the solution to lie within a D-dimensional orthotope. However, as the bound constraints could have a significant effect on the final solution (and this is something that has been found, see later), proper treatment of these is key. As such, an optimiser that can suitably handle both linear and non-linear constraints is chosen.

The optimiser used in this report is Sequential Least Squares Programming (SLSQP).[23] SLSQP is a sequential quadratic programming (SQP)[24] algorithm that allows nonlinearly constrained gradient-based optimisation. SQP algorithms optimise successive second-order approximations of the objective function. SLSQP replaces the quadratic programming subproblem with a linear least squares problem in order to improve efficiency. The Hessian inverse is approximated through successive BFGS updates (see,[25] for example). Gradients are evaluated through a second order central difference approximation.

The SLSQP implementation in SciPy v1.6.3 is used. It should be noted that the L-BFGS-B algorithm from SciPy was also tested. However, convergence issues with L-BFGS-B were found with many runs; it was typical that less than 50% of runs using L-BFGS-B would fully converge, whereas for SLSQP typically over 90% of runs converge.

IV. Determining Multimodality

To determine the level and characteristics of multimodality present in the WFLO problem given in equation 1, the approach taken is to run a large number of gradient-based optimisation runs from randomly generated initial design variable values. A gradient-based optimisation strategy was used as this is the most robust way to locate an optima; if a gradient-based run converges, then by definition this must be at a local optimum.

For the problems tested here, each one is tested with 100 independent optimisation runs. Any non-converged runs are discarded to leave only converged results. Convergence is determined as when optimality falls below 10^{-6}. The start point of the optimisations is determined by performing a Latin hypercube sampling (LHS) of the space.

Once the full suite of converged runs is obtained, two separate analyses are performed. The first is to determine similarity of the runs (this is especially useful where looking at larger numbers of design variables), and the second is simply to determine all the optima.

IV.A. Average Minimum Distance

The first method developed is able to calculate a measure of similarity between two layouts. This measure is called the Minimum Distance. The method to find the Minimum Distance between two layouts is as follows:

1. Overlay two layouts, A and B, and find the distances between a given turbine in layout A and every turbine in layout B.

2. Of these distances, select the smallest distance.
3. Iterate the previous two steps for every turbine in layout A.
4. Calculate an average of the selected distances to give the Minimum Distance between the two layouts.
5. Divide this distance by the wind farm radius in order to normalise it and allow comparison with wind farms of different radii.

Due to the use of polar coordinates, designs can exhibit rotational symmetry so care must be taken here. The method was further developed by rotating layout B through 360 degrees and calculating the distances at each step in order to find the angle of rotation at which the two layouts are most similar. This is a necessary step as layouts that are identical apart from the angle of rotation can be considered as the same solution given that the wind farm is circular.

Equation 2 shows how to calculate the Minimum Distance between two optimised layouts with the same parameters (circle size, number of turbines). In this equation, i and j represent the indices of the two selected layouts within the larger set. N is the number of turbines in the layout whilst T and t are the turbine indices for the first and second layout. $R(\mathbf{x},\theta)$ is a function that outputs the coordinates of a given turbine if the layout has been rotated by an angle of θ.

$$D_{i,j} = \min_{\theta \in [0, 2\pi)} \left\{ \frac{1}{N} \sum_{T=1}^{N} \min_{\substack{t \in [1,N] \\ T \neq t}} \|\mathbf{x}_T^i - R(\mathbf{x}_t^j, \theta)\|_2 \right\} \tag{2}$$

IV.B. Number of Optima

The second method developed is a series of transformations to determine all the optima that exist. Due to the use of polar coordinates, designs can exhibit rotational symmetry. Furthermore, the order of turbines can also be different though the overall design is the same. For example, in figure 1, if turbines 1 and 4 were transposed, even though the design variable vector would be different, the layouts are identical. As such, the following steps are followed to determine all of the identical local minima:

1. Rotate all the optimum layouts so that they are all similarly orientated. This similar orientation of the layouts is achieved by ensuring the Minimum Distances between the layouts are at a minimum.
2. Reorder the turbines in each layout such that θ goes from smallest to largest.
3. Compare the layouts to each other and sort these layouts into groups. This sorting process involves calculating the Minimum Distance between the layouts again. If the Minimum Distance is below a certain threshold, the layouts are grouped together[a]. Otherwise, a new group is created.
4. The number of optima present in the set is now known alongside the shape of these optima and the number of layouts that match each optimum.

V. Results

Results are presented on the multimodality present in the four-and nine-turbine wind farm optimisation cases. Maximum radius of the wind farm was also studied to determine how this affected the multimodality. 100 runs, initiated through a LHS of the space were performed for each farm size. If a run did not converge then it was randomly reinitialised and run again. This was repeated until 100 converged solutions were obtained.

[a]It is rare for layouts to be identical so it is necessary to introduce a threshold distance where if the distance is small enough, layouts are considered to belong to the same optima.

V.A. Four Turbine Case

Table 2 presents a summary of the final results obtained for varying wind farm size for the four turbine case. Evidently, these results show a strong presence of multi-modality within all the considered WFLO problems. If multi-modality were not present, the number of optima for each setup would be 1 and the Average Minimum Distance for each setup would be 0. Although this is an entirely expected outcome, it is still important to prove its existence definitively.

Table 2: Results for 4-turbine wind farms

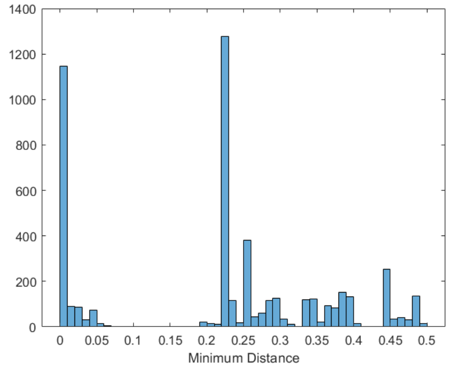
Wind Farm Radius (m)	% of Minimum Distances less than 0.01	Average AEP (MWh)	AEP Std. Dev (MWh)	No. of Optima
200	23.192%	1243.38	26.55	5
250	17.212%	1318.12	32.05	9
300	8.727%	1357.69	54.93	18
375	6.020%	1408.30	27.11	29
450	3.030%	1438.98	24.69	37
500	1.758%	1445.76	45.70	48
750	1.212%	1476.70	11.24	56
1000	0.788%	1482.66	6.93	59

Secondly, it can be seen that as the radius of the wind farm increases, so does the amount of multi-modality present. This is shown by the fact that as the wind farm radius increases, the Average Minimum Distance and the number of optima increases. The percentage of minimum distances less than 0.01 gives a measure of how many of the runs produced the same results (accounting for rotational symmetry). It is clear that the larger the farm, the fewer the number of runs that converged to the same minimum. The AEP results show that the optima of larger farms are better, though this is not surprising since a larger farm allows turbines to be places further apart to reduce wake interference. Furthermore, the AEP standard deviation (which measures how different each of the optima are) shows that for larger farms, although there are more optima, these optima have similar performance.

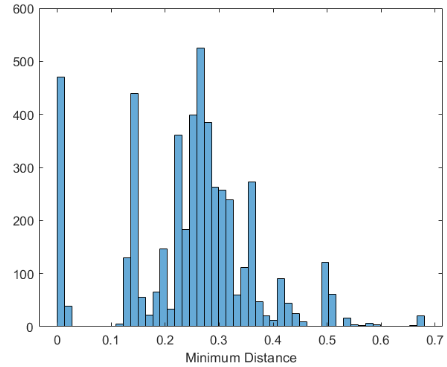
The trends from table 2 can be expanded on by considering Figure 2. These figures show histograms that group and count the entries of the Minimum Distance matrix calculated for each WFLO problem. It can be expected that as the amount of multi-modality increases, and therefore the number of optima present increases, the distribution of Minimum Distances in a set of layouts will become more normal. It is clear that as the wind farm radius increases, the distribution of Minimum Distances does indeed become more normal. It can also be seen that the number of Minimum Distances which are 0 reduces as the wind farm radius increases. A Minimum Distance of 0 occurs when two layouts are identical. The height of this bar in the histograms is analogous with the '% of Minimum Distances less than 0.01' column found in Table **??**.

An interesting conclusion that can be drawn from this study is that even though the multimodality of the problem increases with increasing farm size, the '*risk*' of converging to a local minimum that is vastly different in performance from the global optimum is low.

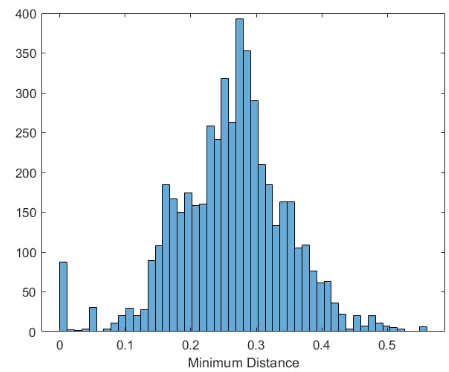
Figures 3 and 4 show the optima from the 200m and 250m cases. While it is difficult to make any quantitative conclusions from the layouts, there are a number of trends that are observed. There is a clear tendency to have optima where one turbine is in the centre of the farm, and the others are on the edge (this is similar to the optimum results found in Baker *et al.*[15]). Otherwise, the other type of optima is where none are on the edge and the others are distributed into a square-like pattern at about 50% of the maximum radius. The other optima are all variations of these.



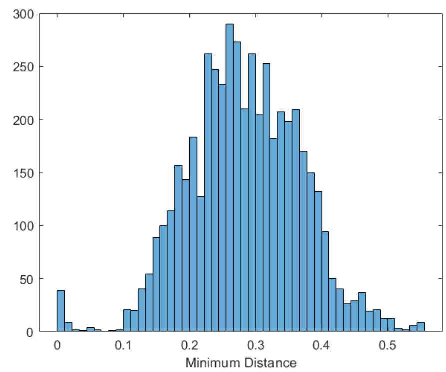
(a) 200m



(b) 300m



(c) 500m



(d) 1000m

Figure 2: Distribution of Minimum Distances for 4-turbine wind farms

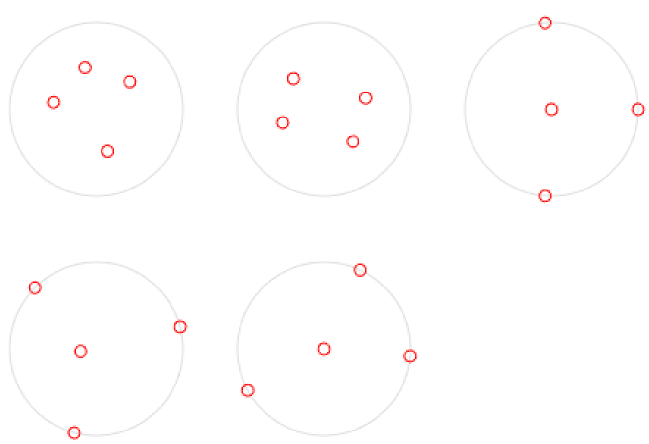


Figure 3: The 5 optima present in 100 runs of 4-turbine wind farms (200m radius)

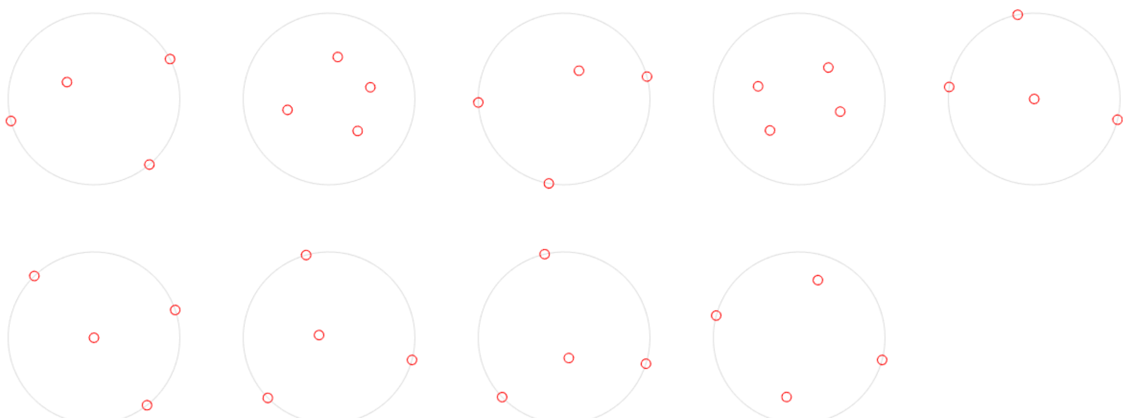


Figure 4: The 9 optima present in 100 runs of 4-turbine wind farms (250m radius)

V.B. Nine Turbine Case

It has been shown in the four turbine case that there exists clear multimodality. The multimodality worsens with increasing farm size, though each different optima have similar performance. Table 3 shows the equivalent results for the nine-turbine case. It should be noted that 50 optimisation runs were performed for this case. The same trends as found in the four turbine case are also observed here, though the multimodality is clearly more severe for nine turbines.

Table 3: Results for 9-turbine wind farms

Wind Farm Radius (m)	**% of Minimum Distances less than 0.01**	**Average AEP (MWh)**	**AEP Std. Dev (MWh)**	**No. of Optima**
375	3.173%	1133.10	24.35	27
750	0.327%	1372.40	28.92	48
1000	0.000%	1420.20	18.52	50
1500	0.000%	1465.93	9.01	50

VI. Conclusions and Future Work

This paper has presented an initial study into multimodality in wind farm layout optimisation problems. The locations of wind turbines were optimized to maximise the annual energy production from a simple circular farm. An engineering wake model was used to simulate the farm. A multi-start gradient-based optimisation strategy was employed where 100 independent runs of the SLSQP gradient-based optimiser were performed. A four turbine case was considered as well as a nine turbine case.

It was found that irrelevant of the size of the wind farm, the problem was multimodal. The number of optima in the problem increased as the maximum farm size increased due to having more locations that the optimiser could place turbines. However, while the number of optima increased, the difference between those optima decreased. Hence, while for large farms the problem is severely multimodal, the risk of finding a poor local minimum from a single optimisation run is low. It is likely that the design space is a large, flat region with small oscillations, likely coming about from the model.

As stated in the introduction, little work has attempted to quantify the effect of multimodality in wind farm layout optimisation. As such, there are many open questions which future work will look to consider.

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