7-17-2017

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PME 41

Proceedings

Singapore | July 17-22, 2017

Of the 41st Conference of the International Group for the Psychology of Mathematics Education

Volume 3

Editors | Berinderjeet Kaur, Weng Kin Ho, Tin Lam Toh, Ban Heng Choy

41st PME Annual Conference
17-22 July 2017
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Cite as:
Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). (2017). Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education (Vol. 3). Singapore: PME.

Website: http://math.nie.edu.sg/pme41
Proceedings are also available via http://www.igpme.org

ISBN: 978-981-11-3742-6

Cover design: Mohamed Fadzli Ibrahim, National Institute of Education, Singapore
Printed by Welon Printing Co. Pte Ltd

MATHEMATICAL KNOWLEDGE AS MEMORIES OF MATHEMATICS

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I propose that an understanding of a mathematical concept is comprised of both a conceptual understanding of, and recollections of working with that concept. That is, a mathematical concept may not be immediately distilled in its abstract form from lived experience, didactical or otherwise, and this milleu is brought along in subsequent recollections of the concept. In an effort to balance pedagogical recommendations for increased conceptual teaching/understanding, I propose that memories of encountering a mathematical concept improve its utility in novel problem situations. I support this claim by drawing on the literature on episodic future thinking and on our developing understanding of how users of mathematics perform in authentic mathematical situations.

INTRODUCTION

Students receive a constant stream of experiences when learning mathematics — both external and mental, and taking the forms of sensory, cognitive, social, and otherwise — this stream of experience is interpreted and coloured by the student's current knowledge of and dispositions towards mathematics. From this they distil off mathematical knowledge. How exactly this process of mathematical knowledge formation takes place has long been a central subject of the cognitive psychology of mathematics learning. This article argues that mathematics education literature written from a cognitive psychology perspective has maintained too narrow a focus on the mathematical content of mathematics learning and ought to consider a student's broader remembered experience.

A discussion of the role of memory in mathematics learning is noticeably lacking in contemporary research despite advancements in memory research in psychology over the last two decades. This is not a new observation, having been identified in the mathematics education literature three decades ago (Byers & Erlwanger, 1985). This may be due in part to a conflation of memory and *memorizing*. I take the stand that, from a cognitive perspective, knowledge is a part of memory. Therefore, any serious discussion of the cognitive psychology of mathematical knowledge acquisition and development must consider the role of memory. My focus here is on the mathematical knowledge constucted in the mind of an individual. What led to this constructed knowledge, whether social interaction, bodily movements, individual reflection, or other situations, is not explicitly considered. I emphasise, however, that these modes of knowledge construction are not entirely disregarded; as will become clear, they form the substance of episodic memories.

2017. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 3, pp. 209-216. Singapore: PME.

The main impetus of this work is to gain insight into how a mathematics learner forms their idiosyncratic, personal understanding of mathematical concepts. That is, I am concerned with understanding the development of a student's *concept image* of a mathematical concept. Tall and Vinner define concept image as “the total cognitive structure associated with a mathematical concept” (Tall & Vinner, 1981). This is a powerful defintion, but has not realised its potential in the literature. I argue, and substantiate with examples in the Results section, that personal memories – so called *episodic memories* – of learning or working with the concept comprise a portion of this cognitive structure, especially during the initial stages of concept image formation. As such, these memories ought to receive greater attention in the mathematics education research literature. In the conclusion of this paper I conjecture that episodic memories associated with a student’s concept image are what may facilitate the utilization of the concept in novel situations. This conjecture meshes well with the current understanding of the role of episodic memories in planning for general, to-be-experienced events found in the psychology literature.

Episodic and Semantic Memory

The human memory system has been described by a number of qualitative models, the one used here – attibuted to Tulving (1983), having built on earlier work – has come to be widely accepted in the psychology community. In this model there is first a distinction between short- and long-term memory. Short-term memory — also often called *working memory* — lasts only less than a minute and is a key resource when interpreting a current experience. Long-term memory is partitioned into *implicit*, or *procedural*, and *explicit*, or *declarative*, memory systems. Implicit memory is the memory of rote tasks, those that are completed without conscious thought, such as walking, riding a bicycle, or teaching calculus. Declarative memories are those that can be explicitly recalled and stated. A further refinement of declarative memory into two qualitatively, and perhaps neurologically, different memory systems was proposed by Tulving (1983). *Semantic* memories are those that are not fixed to a particular individual's experience and can be known by anyone. That is, semantic memories are memories of shared, socially-available knowledge. *Episodic* memories are held by an individual and pertain to an event experienced by that individual. They are highly idiosyncratic, contain perceptual and temporal information, and can only be known by the individual. A memory of learning to ride a bicycle, for example, is episodic. Even though riding a bicycle is a fairly universal activity, each individual forms their own unique episodic memories of learning to do so. The analogy with constructs in the mathematics education literature is clear: episodic memories of a mathematical concept are a part of a student’s concept image of the underlying concept.

The purpose of this paper is to present evidence that students experience personal memories of mathematics when recalling mathematical concepts. These memories were often voiced freely, without prompting, by the student participants in this study, suggesting that the memories form a strong component of the students’ concept images. These personal memories, I argue, are valuable; in the wider field of the psychology of memory, episdoic memories are known to improve problem-solving ability (Taylor, et al., 1998; Schacter, 2012). Further, episodic memories may prove valuable to education theoreticians, insofar as they often reveal discrete moments in time in which a student's knowledge evolves.

METHODS

Student volunteers were recruited from two first-year mathematics courses, covering linear algebra and calculus, at a major, research-intensive New Zealand university. In total, 11 students volunteered; 9 from the general stream of the course, intended for science and business students, and 2 from the advanced stream for mathematics and science honours degree students.

Students were interviewed individually in two sessions. The results from the second set of interviews are reported in companion articles (Maciejewski & Barton, 2016; Maciejewski, Roberts, & Addis, 2016). In the first set of interviews, which forms the set of data used in the current study, each student was presented with a list of topics from their mathematics course and asked to rank them according to their own, personal familiarity with each; 1 for least familiar and 10 for most. The intention with the personal ranking of the topics was to have an increased diversity of episodic recollections. The researcher proceeded to ask the following set of questions for the topics ranked 1, 5, and 10.

For each topic, general questions about thoughts experienced by the participant when thinking about the given topic were asked first: "Describe the contents of your thoughts when thinking about [topic X] in as much detail as possible. Importantly, we are not looking for mathematical accuracy at this stage, we'd just like you to describe everything that comes to mind when you think about that topic. This may or may not include 1) mental imagery, 2) conceptual knowledge, and 3) personal memories."

More specific questions followed: "Can you describe how you came to understand (topic x) as you do now? Do you recall when you first encountered this topic? When was that? Can you describe that in detail? Do you understand this topic differently now than when you first encountered it? What led to this change? (If specific events are mentioned: Can you describe this event in detail?) Did these experiences come to mind in the first part of this study (even if you didn't talk about it)?"

Each of the interviews were recorded and subsequently transcribed. The transcriptions were analyzed from a *phenomenographical* perspective: a qualitative, interpretive methodology that seeks to understand individuals' idiosyncratic experiences of a common reality (Marton, 1986). The intention is to describe and categorize the range participants' experiences with, critically, equal weight given to each experience; no effort is made to identify which are the most prominent. In this way phenomenography is a powerful method for empirically uncovering possible lived experience.

The particular phenomenographical analysis is as follows. Student utterances were first categorized roughly as *episodic*, *semantic*, and *other*. Semantic utterances were those that contained "factual" information from an experience and no specific reference to the personal nature of the lived experience; semantic memories are abstract in the sense of not being tied to a particular experienced event. Semantic memories, and their associated utterances, are not the focus of this study and will fenceforth not be considered.

Episodic utterances were those that contained particular details specific to a lived experience. Explicit mention is made of some context of the experience. For example:

> Interviewer: When you think about these methods, do you have any images that come to mind?
>
> Participant [104]: Well, the image that comes to mind from that is the page in the course book and just the way the lecturer ... just explains it ... I can see the page and the way she laid it out and the method, how she goes step-by-step to solve it.

The participant recalls a specific instance of learning a mathematical method while sitting in class. They recall the lecturer talking and refering to the page in the textbook which, though not mentioned here, is displayed on a projector for the class.

Utterances classed as “other” were those that did not fit into either of the other two categories and often included clarifying statements, or comments unrelated to the questions.

The next stage of the analysis involved creating a categorization of the episodic utterances following phenomenographical methods (Marton, 1986). First, the collection of episodic utterances were read through and broad categories were formed. These are *temporal* – referencing the, at least relative, time an event occurred; *physical* – concerning the interaction of the participant and their environment; and *emotional* – the participant's recalled emotions during the experience. These three cateogories, though originating organically from the data, agree well with Tulving's description of episodic memories (Tulving, 1983).

Having formed these categories, each utterance was grouped into the category it best fit. Often an utterance contained elements of more than one category and was therefore duplicated in the corresponding categories. The categorisation part of the analysis was halted at this stage; further analysis appeared to lead to too fine categories which resisted succinct descriptors.

The subcategories were summarized by the researcher and combed for representative utterances. These are presented in the next section.

RESULTS

I present my analysis of the interview data according to the three identified categories – *temporal, physical,* and *emotional* – and present the corresponding subcategories. In each of the following subsections, I weigh the category topic against the students' self-reported familiarity of the mathematical topics. This allows for a richer analysis and informs conjectures about the role of memory in conceptual development we make in the subsequent section.

First, I comment on the prominence of these memories. All participants recalled episodic memories associated with at least one of the concepts. These recollections, notably the emotional recollections, were often offered by the participants without prompting. I take this as an indicator of how pervasive episodic memories are in students' mathematical conceptions.

Temporal Aspects

There were two types of responses in this category: 1) first encounter with the concept, 2) subsequent use of the concept. Participants generally recalled their first encounters with each concept. However, the vividness of these recollections varied with the participants' stated level of understanding. Less understood concepts tended to be associated with more vivid memories of a first encounter.

> I: Do you recall when you first encountered [least well understood concept]?
>
> S106: Yep. Because [the instructor's] accent just made it sound so cool, so, ya I do remember doing it and using matrices to solve something for it. But I don't remember what to do with it or anything.

More well-understood topics were associated with less vivid first-encounter memories.

> I: Do you remember when you first encountered [most well understood concept]?
>
> S106: I don't think so, actually ... I have a vague recollection of when I was supposed to have learned it.

Perhaps not surprisingly, more memories of subsequent work with the concept arose when the participants discussed more well-understood concepts. Of course, this may be because more well-understood concepts have been used more and so the participants have had greater opportunity to form memories of these concepts. This does not seem to be the case, however. Participants often mentioned using less understood concepts while solving problems, but these recollections were of "going through the motions" with the concept.

> S105: ...I came across a question in the assignment concerning Taylor Series, I had to answer it, so I kinda looked in the course book, I looked on the internet, asked my friend how to do it. He said it's kinda complicated so I looked on the internet and compared the answer, tried using one method to see if I got the answer...tried using another...just repeat that until I got the correct answer...I don't really like it, so after I answered that question, I sort of avoided Taylor Series.

Physical Aspects

Participants mentioned only a small number of physical aspects. Therefore, I present them here without grouping them further into larger categories. These are:

1) sitting in class:

> I: But what comes to mind when you think about [the concept]?
>
> S110: Our lecturer talking about it and me kinda not listening … it was a Friday morning, which isn't so conducive to learning. Um, and she was kinda talking about it and I wasn't listening 'cause they kept relating it to real life and I can't be bothered about real life … It's the last part of linear algebra as well and the test was coming up and I ignored it.

2) reading a book:

S104: I kinda associate these concepts more with the page in the textbook...I remember pictures. And like, certain bits of the course book that I thought were more important. So, I can visually remember how the things look on the page.

3) attending a tutorial:

S102: I remember … I was in the tutor room and one of the maths tutors actually thought I was stupid for asking such silly questions. And he came over and he actually explained step by step what he was doing for a couple of things, and he showed me a couple of little tricks … it clicked and then I could do those at least, and then it was just a case of applying that to everything else.

4) working with friends:

I: Can you think of what led to that change of understanding for you?

S100: When I was doing the Taylor Series with my friend … she was helping me … she had written a different number for the denominator, and I didn't understand why because from the example it seemed like what I was doing was correct, but then she explained to me that she was using factorial and I was 'timesing' the number, so that obviously made a difference.

and, 5) revising/reviewing/studying for an exam:

I: Do you recall specific events that led to this change in understanding?

S105: I think the assignment that we had and the test. 'cause I remember cramming for the test, 'cause I didn't understand, like, Taylor Series, really. I wasn't quite sure of them, so, studying for the test with friends and going over practice questions and searching examples online again.

Emotional Aspects

Though participants were not specificially asked about emotions they experienced when learning mathematics, all mentioned emotions in connection with at least one mathematical concept. These covered a wide range. Less understood concepts were associated with less favourable emotions, such as anxiety, trepidation, confusion.

S101: It was just, like, pretty overwhelming...I really was not looking forward to learning something new.

Perhaps not surprisingly, more well-understood concepts had more favourable associated emotions: familiarity, enjoyment, happiness, and confidence, for example.

S101: [I learned this concept on my own] because I was just bored...and I just was reading it and I kind of got it...and it felt...because, like, once you know how to do it, it becomes really easy, and it comes with practice, so yeah...I have an image of just sitting in class feeling pretty smug because I had already know how to do it...so yeah, it was a lot more pleasant than [the least understood concept]

DISCUSSION

The development of a student's mathematical knowledge may proceed episodically or semantically, or both. I propose that an exclusive accumulation of either one is necessarily undesirable. This is certainly the case for episodic memories; indeed, much of the research in mathematics education in the last half-century cautions against the accumulation of context-bound knowledge. Given that the same literature encourages the growth of context-transcending knowledge, it may seem an odd suggestion that students should not focus on acquiring semantic memories exclusively. I highlight a contemporary result from the psychology literature that may substantiate this claim, while being mindful of the need for further investigation in mathematics learning.

Contemporary reseach on the psychology of planning for to-be-experienced events indicates the humans often mentally simulate how an event might unfold and, in so doing, create episodic memories of the event before it takes place. This *episodic future thinking* (Atance & O'Neill, 2001) can facilitate planning and improve outcomes in general problem solving domains (Taylor, et al., 1998; Schacter, 2012). The key observation is that episodic future thinking relies on the same neurological regions and processes as are utilised in recalling episodic memories. Therefore, effective planning for to-be-experienced events is closely related, and influenced by, recollecting past events. Given the emphasis placed by some authors (Pólya, 1945; Schoenfeld, 1985) on *planning* in mathematical situations, it appears a worthwhile endeavour to investigate episodic future thinking in mathematical situations, its effect on planning, and how such future thinking is affected by episodic memories of mathematics.

Not much is known about episodic future thinking in specialized, context-specific domains, such as mathematics. There is emerging evidence that both mathematicians (Maciejewski & Barton, 2016) and mathematics students (Maciejewski, Roberts, & Addis, 2016) engage in episodic future thinking when solving mathematics problems. This is, of course, not the exclusive way of solving mathematics problems; some problems may invoke automaticity or an existing problem schema, or nothing at all (Maciejewski & Barton, 2016). What is needed is a better understanding of how a problem might relate to its solver and of which types of these relationships are likely to promote episodic future thinking. I conjecture that it is for those problem situations that are not too familiar to invoke a schema yet are familiar enough that progress can be made. This conjecture fits well with the literature on general problem solving behaviour, and further research in the context of mathematics is highly desirable.

One further point to be made is that episodic memories could act as signposts for the educational researcher. They signal discrete moments, locating the genesis of an idea or the punctuated evolution of understanding. Treating these memories as such may aid in deepening theoretical models of knowledge development.

CONCLUSION

This paper presents observations that personal experiences of mathematics pervade students' thoughts when recalling mathematical concepts. These episodic memories are a part of a student's understanding of the concept and present challenges and opportunities to educators. On the one hand, an exclusive reliance on episodic memories of using a concept could result in too-rigid knowledge without wide applicability. On the other, a diversity of rich episodic memories of mathematics may facilitate more effective planning in mathematical situations for a student. It is not clear to what extent educators ought to promote the formation of students' episodic memories of mathematics nor is it clear how best to do this. What is clear, however, is that students will continue to form episodic memories of mathematics whether or not they are attended to by educators. It is up to educational researchers to further investigate students' episodic memories of mathematics and ways in which they may be harnessed to aid students in reaching their potential.

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