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MATHEMATICAL FORESIGHT: THINKING IN THE FUTURE TO WORK IN THE PRESENT

WES MACIEJEWSKI, BILL BARTON

Think about the following:

> You're playing a mathematical game with a large number of people, all of whom are reading this article. Each of you needs to select a natural number between 0 and 100, inclusive. The winner of this game is the person, or persons, who selects the number closest to two-thirds the average of all numbers selected. Assuming you'd like to win, what number should you choose?

How did you solve this problem? "Assuming everyone selects a number at random," you think, "the average would be 50. Two-thirds of 50 is about 33. I should select that!" But your other mental voice interjects, "Ah! But everyone else is thinking the same! Therefore, they all select 33. Two-thirds of 33 is 22. That's the number…" "Wait!" Back and forth your mind lobs. "I can see this continuing…I'll select 0."

Phrased in mathematical jargon, the above problem is asking you to find the Nash Equilibrium of the given situation. If we had written the problem in such terms, some of you would not have made any progress while others would perhaps have selected 33 or one of the numbers along the sequence. Still others would have seen that the answer is 0 and engaged in the type of iterative thinking caricatured above. The present article expands on this last type of experience, of solving a mathematical problem by first seeing a resolution and a path leading to that resolution unfold into the near, problem-solving future.

The above game is seemingly a small, one-off problem. We claim, however, that this kind of thinking—seeing a resolution and way to that resolution—is a general phenomenon in certain mathematical work, that in which no solution schema is readily available. There is a sense in which mathematicians (broadly construed as users of mathematics) engaged in this type of work are imagining future activity and this "future thinking" drags their present actions forward, motivates them, and helps them to persist. Where do these abilities come from? Experience? What aspect of experience? Can these abilities be taught?

Perhaps the question that first needs asking is: how can we best characterize these future-thinking abilities? There are many ideas in the mathematics education literature that may aid in a description—intuition, strategic knowledge, an aesthetic sense, experience, heuristics, problem solving, meta-cognition, and many more. We argue here that none of these existing constructs quite captures a mathematician's future-thinking processes. Though each of these constructs can weigh in on features of future thinking, they each have come to encompass so much and in such diverse mathematical settings that we find we desire a precise, restricted description of future-thinking processes in mathematics. In this article, we therefore introduce and elaborate the construct of *mathematical foresight*, drawing on a series of interviews with mathematicians to do so.

Mathematical foresight

We describe mathematical foresight as a process that a mathematician may engage in when faced with a new problem, hypothesis, or situation, and when needing to work towards a solution, a proof, some clarity, or a resolution. From their starting point, we argue that a broad direction, or trajectory, can be described in which the mathematician might travel, although the exact path, with its detailed twists and turns, may initially be unknown. At the end of this avenue, we suggest, a hazy shape might be seen of what the solution, proof, or resolution will look like. We refer to this hazy shape as the *sphere of resolution.* We consider mathematical foresight as the mathematician's developing awareness of this broad *resolution trajectory* and sphere of resolution.

In our model, depicted in Figure 1 (overleaf), both the trajectory and destination are taken together. Usually, the trajectory a mathematician pursues does not exist without some idea of a destination, except in the case of mathematical exploration. And, as we discuss below, this idea is more than an "intuition" of how to proceed: if pressed, the mathematician can articulate their choice of trajectory and destination.

Mathematical foresight comes before the mathematician sets out on their specific path. During subsequent work, situations are encountered where decisions need to be made about what action should be performed next. These choices may deviate from the initially conceived trajectory, hence moving it, and possibly also shifting the sphere of resolution. In this way, mathematical foresight is a dynamic process: each step taken may revise the trajectory forward and will slowly bring into more focus the sphere of resolution.

The choices made along the way involve Schoenfeld's (1985) notion of control: "the way people use the information potentially at their disposal" (Schoenfeld, 1985, p. 27). His characterization of control as "decisions about what paths to take (which also, therefore, determine what paths are not

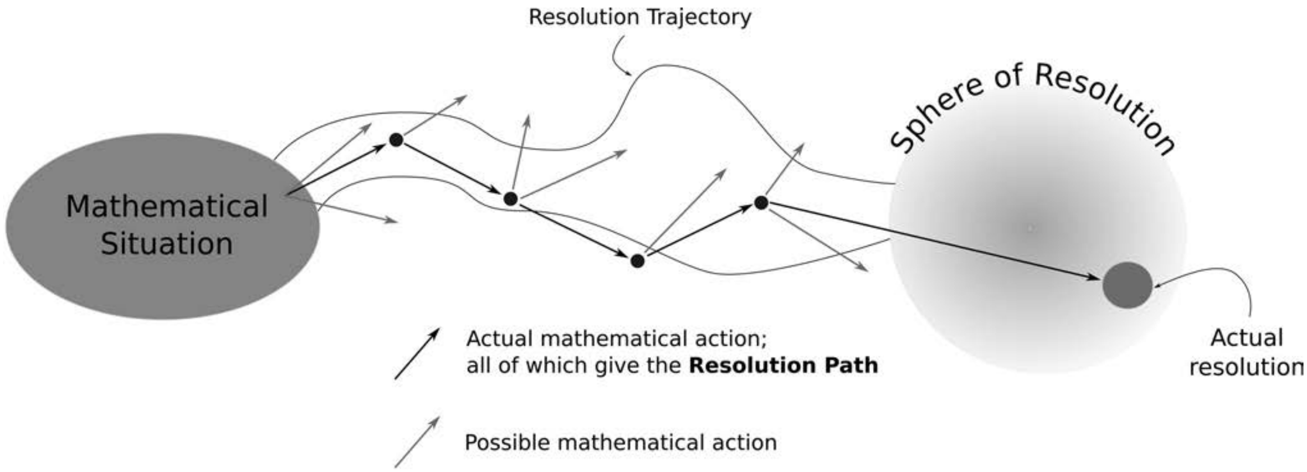

Figure 1. *Mathematical foresight is conceptualized as being able to describe in advance the general direction of the resolution trajectory and the shape of the sphere of resolution.*

taken)" (p. 27) meshes well with our notion here. Mathematical foresight entails control: being able to see a resolution and a path facilitates productive, disciplined mathematical behaviour, although the converse need not hold.

Our interviews with mathematicians confirm that they recognise and value this process. Five mathematician colleagues, one female and four male, each from a different field of pure or applied mathematics, each individually conversed with us about our initial foresight model. We asked if they recognized such a process in their own work and what it looked and felt like to them. All told us that they regularly engage in an activity accurately described by mathematical foresight, and have given us examples that have refined its characteristics. One mathematician described their experience with mathematical foresight as follows:

> For me it sort of resonated a little bit in terms of, you *see* things when you're trying to get to some result or so, you know that this is a dead end well before you reached it. And you know when things are going to work out well before you've really worked out the details. But, as I started to think about why and how do you actually know this. The more it's, it's experience! It's like you've seen it before, which is weird because you're doing something new, so you can't have seen it before. But somehow it feels familiar. (M1)

They told us that foresight typically happens quite quickly on first meeting the mathematical situation and contributes to the mathematician's anticipation of how to proceed. Boero (2001) writes, for example, in the context of algebraic problem solving:

> A common ingredient of all the processes of transformation […] is *anticipation*. In order to direct the transformation in an efficient way, the subject needs to foresee some aspects of the final shape of the object to be transformed related to the goal to be reached, and some possibilities of transformation. This "anticipation" allows planning and continuous feed-back. (p. 99)

Thus, the act of choosing a problem to work on is one of the functions of foresight:

> I can tell you exactly what is the right problem for me to work on now. But in fact there are 10 people working on it now who possibly have more time or whatever. I can identify the problem as important. I can identify that in principle I could contribute to it, but I will choose not to work on it until I have got really an angle that I think "Ah ha I don't think anyone working on it is doing that". […] Having a strategy is part of how you choose a problem. Not the other way around, that you would choose a problem then think of a good strategy. (M1)

> I can't imagine surviving as a pure mathematician without [foresight] because you need to come up with new projects and so on … and so you need to move in directions from which new things can bubble off easily. And I think that seeing a direction will be rich from your own point of view, but also then rich for other points of view, and will link in to things that other people are interested in and know about. (M2)

Mathematical foresight is just one process in which a mathematician may engage when encountering a mathematical situation. There are others and we imagine these processes as residing on a continuum, like that represented in Figure 2. On one end there is *automaticity*, in which the mathematician responds to the situation without conscious thought. Automaticity may occur with simple arithmetic, for example. Next to situations that evoke automaticity are those that evoke *problem schema:* a means to resolve a problem situation that has been reinforced with prior success. The first-year calculus instructor has seen, and assigned to their students, practically every possible undergraduate differential calculus problem and their process of solving any given one requires only the activation of the appropriate problem schema. On the other end of the continuum is *inertness*: the

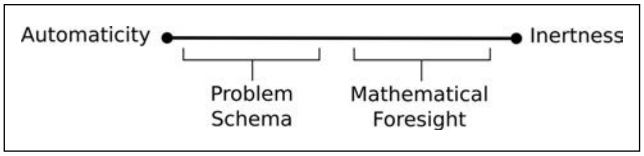

Figure 2. Ways in which a mathematician might act upon encountering a mathematical situation. The line segment represents a continuum of mathematical situations and is labeled according to possible actions taken by an individual mathematician.

total inability to navigate the mathematical situation. Inertness could arise, for example, when a mathematician encounters a situation completely removed from their area of expertise. A mathematical biologist is unlikely to gain any traction on a problem in algebraic number theory, let alone understand, or indeed recognize the terms involved.

Situations likely to invoke mathematical foresight reside toward the right of our imaginary continuum (see Figure 2). These situations are not so familiar to the mathematician that they invoke problem schema but are familiar enough to be understandable and potentially resolved.

Examples of mathematical foresight

Mathematical foresight is one process among many in which a mathematician may engage during a mathematical activity. As mentioned above, research-level mathematics often involves foresight, but what other types of activities are likely to elicit foresight?

A mathematician colleague gave an example of having foresight when lecturing. He had not fully prepared the lecture but was confident that he could prove the relevant theorem. It had four parts: the first was straightforward, the second he had recognized as suitable to hold back as a problem for the students, and the third was a direct argument. Then the fourth part he realized he did not know; however, he just thought "I'll use proof by contraposition", simply because he recognized that that was appropriate: "It just appeared that this is somehow the right way to go through it. It is a familiar thing to do, but it was not like 'I remember this is the right way to do it'." He linked this teaching experience to research: "it felt the same as when you are doing research. You'd be like 'Umm I think this is the way to go'. And that's not memory, when you're doing research it is not memory."

Another example is the act of forming a conjecture. Conjectures are anticipated mathematical relationships lacking formal justification. But conjectures are not arrived at only by happenstance. They are often presented alongside strong empirical evidence of their veracity and/or the poser's program for establishing the relationship rigorously. Of course, conjecturing does take on many forms. Fermat's last conjecture (Theorem) was likely arrived at empirically but did not have accompanying suggestions for its proof (aside from "buy a book with larger margins"), while many contemporary conjectures are buttressed by both evidence and a research program. An interesting example is William Thurston's Geometrization Conjecture, a corollary of which is the Poincaré Conjecture. Prosaically, the conjecture states that there are only so many geometries, like the familiar Euclidean and spherical geometry, in three dimensions. Thurston had an approach to the conjecture, and applied it to a specific class of geometric objects. Richard Hamilton subsequently contributed what became the key insight—considering the "Ricci flow" on a given object—and provided a program with which the conjecture could be proved. Two decades later, Grigori Perelman was able to carry out this program. We take this example as one of collective mathematical foresight: a community of mathematicians *sees* the resolution and a likely path leading to it. This *collective* mathematical foresight is a "scaled up" version of what a mathematician may experience in solitude.

Foresight and related concepts

How does foresight relate to other mathematical processes? In this section, we focus specifically on strategic thinking, heuristics, problem solving, and intuition and, in so doing, we necessarily leave out a number of other relevant ideas. At this stage, we encourage the reader to think of how other ideas might relate to mathematical foresight.

We take *strategic thinking* to be, following Weber's (2001) definition of strategic knowledge, choosing an action among many that is likely to lead to progress toward a desired end state. With this definition, strategic thinking is what occurs after the foresight process; foresight lays a wide path and strategic thinking traverses it. In our diagram (Figure 1), it is the process of deciding on the short arrows that make up the resolution path.

Weber (2001) identified four types of strategic knowledge needed for proving: knowledge of i) typical proofs in the domain, ii) relevant theorems and when to apply them, iii) when to use syntactic and semantic approaches, and iv) relevant facts. In this way, strategic thinking is making progress on a mathematical problem by allowing the solver to relate the current problem to previously solved problems and to recognize and activate relevant knowledge.

The difference between strategic thinking and foresight is primarily one of scale. Hence, they can coincide when the situation or problem is not complex for the solver. For example, proving that the shortest distance between two points is a straight line requires only appealing to the triangle inequality. Recognizing that this is a likely path to a proof is an instance of mathematical foresight, but also fits comfortably in Weber's definition of strategic knowledge. What is needed of a mathematical situation to distinguish foresight and strategic thinking is sufficient distance between the starting state and the desired end state; a characteristic of the problems described in the right-half region of Figure 2. Such a problem would have many components and stopping places at which the solver must choose a way forward. Mathematical foresight is a preliminary, global view of the resolution destination and trajectory and may aid in making strategic decisions along the way.

How is foresight different from *heuristics*, or general problem-solving tactics? We use Schoenfeld's (1985) definition: "The use of a general problem-solving strategy is heuristic if the problem solver is having difficulty, and there is no reason to suspect that taking this particular approach might help" (p. 60). Heuristics are therefore more closely related to, but distinct from, strategic thinking, amounting to a set of generalised ways of acting in a class of situations that may contribute to strategic decisions. Unlike foresight, heuristics are not dependent on the specific context, and will not contribute to understanding the nature of a resolution to a problem, although they might guide some aspects of the resolution trajectory.

We turn now to the most closely-related and nebulous construct: *intuition*. Mathematicians often speak of having an "intuitive understanding" of a concept, and of the "intuition" behind a proof. Intuition has featured prominently in writings about the nature of mathematical work (see, for example, Poincaré, 1905), and many mathematicians do recognize the importance of intuition in their practice (Burton, 2004). But despite this high degree of interest, intuition has come to be used colloquially as a catch-all phrase for non-rigorous mathematical activity and understanding.

In an effort to operationalize intuition, Fischbein (1987) identified five features:

1. Immediacy: a conclusion is reached instantly from limited perceptual data.
2. Self-evidence: the conclusion is formed without any further analytical thinking.
3. Extrapolativeness: the conclusion goes beyond what is present in the perceived situation.
4. Coerciveness: the intuitor is convinced of the veracity of the conclusion.
5. Globality: the conclusion transcends the given situation.

To see the role of each of these features, consider which arise when answering the question, "what is the shortest distance between two points?"

These characteristics allowed Fischbein to identify four types of intuitions. The first is *affirmatory*, or statements that are "obviously" true ("The shortest distance between two points is a straight line"). The second, *conjectural*, is the production of believed-to-be true statements that may or may not be able to be substantiated ("I know the Riemann Hypothesis is true," or, "you will make an excellent mathematics teacher"). *Conclusive* intuitions summarize a mathematical event and provide a global overview ("the intuition behind this proof is…").

The final type, *anticipatory*, is closest to mathematical foresight. What defines an anticipatory intuition? In one context, Fischbein writes, "anticipatory intuitions represent the preliminary, global view which precedes the analytical, fully developed solution to a problem" (Fischbein, 1987, p. 61), while, in another, he writes, "while striving to solve a problem one suddenly has the feeling that one has grasped the solution even before one can offer any explicit, complete justification for that solution" (Fischbein, 1982, p. 10). But some of the features of intuition do not apply to foresight. Foresight is not necessarily immediate and is a continuously evolving process, not a single state. It may involve some analytical thinking, and is not held with total certainty. For us, another difference is that foresight is potentially justified. Mathematicians speak as if they have grounds for their foresight and could give those grounds if necessary, by referring to previous experience, citing similar situations, recognizing the situation as being in a class of problems, or drawing on features of the given situation that suggest a way forward.

The psychological nature of foresight

One mathematician suggested that foresight requires an "information field" which may be made up of many different sorts of things: experiences, metaphoric references, prior knowledge, and so on, all of which makes "foresighting" a personal experience. This idea reinforced the way mathematics researchers seemed to personalise their experiences of foresight, phrasing it as "I would need that kind of math and I would do it this way" rather than, "that math is needed". For example:

> And so what I see as a path, and is crystal clear to me […] is only so to me and not necessarily to another person who's working on the problem […] you think about it and say, well if I would do it, then I would go about it this way. (M3)

> You get a sense of what things you'll be able to solve so […] I think with things you understand very well, even if you have a new idea in that area you sort of think, well I know this area so well I'll be able to sort it out one way or another […] I'll corner this thing and sort it out. (M2)

Statements like the above led us to consider that the mental processes involved in mathematical foresight are analogous to those involved in planning any to-be-experienced event. When planning a meal, for example, a chef forms a mental image of what dishes will be served, how they will taste, what ingredients are required, and the order in which each should be cooked, so that finishing times are coordinated. This planning is all done prior to grocery shopping or turning on the stove. Such thinking led us into discussions with psychologists concerned with memory, and some of the associated literature.

The example of a chef is an instance of someone engaging in *episodic future thinking* (Atance & O'Neill, 2001). While planning, the chef is not only making a grocery list, they are pre-experiencing an event by projecting themselves into the future and forming a memory of an event that has yet to occur. This ability to form future memories is intimately related to episodic memory: the memory of events already experienced (Schacter, Addis & Buckner, 2007). The more experience the chef has in the kitchen, the more able they are to pre-experience preparing a meal. The key, here, is that the to-be-experienced event does not necessarily have to be too similar to any actually previously experienced event; it can be entirely novel.

Episodic memory is one of two types of declarative memory, the other being semantic memory: the memory of facts (Tulving, 1983). These two types of memory are distinct: I can remember that water boils at 100°C but may not have experienced establishing this fact. Semantic memories can aid in the construction and recall of future memories, but are limited in their capacity to do so. To form effective future memories, with sufficient detail to be useful, both episodic and semantic memories are needed. In the context of mathematics, these two memory systems working in concert are the hallmark of the mathematician's work. A mathematician is good at mathematics because they both know and have experienced a great deal of mathematics.

Educational implications

Having identified and described an apparently valuable mathematical process, we are now concerned with how it develops in undergraduate students, how we might observe foresight, and whether we, as educators, can assist students with its development. The mathematicians we interviewed recognized the development of their own mathematical foresight and some were able to locate its genesis:

> In some senses it is linked to communication. I think I became aware of this in undergraduate. And I think it would come through tutoring. It comes through tutoring, the awareness. I am sure my memory was not that good, and I was tutoring when I was 20 years old. They were doing first year mathematics, they said they don't know how to solve a problem and I was like "well you try to get a feeling of this is the right sort of thing to do" and I would have some kind of foresight. Even it's a very simple thing, this sort of integration you know, do you use integration by parts, do you use substitution. Most people at some point get a feeling of how to solve certain thing. I think I could recognize that way back from my undergraduate. But I suspect that through interacting with others, if I was doing the problem on my own it's not something I would think about. But as soon as you start teaching, communicating, trying to help with someone who is stuck. It sort of raises up on your mind "how do I know it?" (M1)

We note that well-intentioned attempts to teach "soft" mathematical skills like intuition, creativity, heuristics, or problem solving have not consistently proven successful. We offer a response to Burton's (1999) question, "Why is *intuition* so important to mathematicians but missing from mathematics education?" Intuition is not taught because it cannot be taught. Neither can problem solving or creativity. These constructs are too broad and have come to encompass so much that they have become unworkable in education. Aspects of each can be taught, such as the five heuristic strategies in Schoenfeld's (1985) intensive, small-scale study, but teaching "problem solving" remains akin to teaching "mathematics".

Will attempts to teach foresight fail in the same way? Can we avoid the generalization trap that is described above, or can the context-specific and personal nature of foresight be utilized in a different way so that students' abilities in the area are enhanced? If intuition is un-grounded feelings that something is the case, can we help students use what knowledge and experience they have to create grounded feelings? We note that some aspects of Olympiad training do just this: students are alerted to "big ideas" that recur, such as the pigeon-hole principle, and the class of problems with which it is likely to be useful. Perhaps simply alerting students to the existence and nature of foresight will help them to use their mathematical information fields in productive ways?

In psychology, the development of general episodic, non-mathematical foresight has been recognized in young children. The emergence of domain-specific foresight, like in the chef example above or in mathematical foresight, has not been investigated, although the assumption that such foresight arises from greater experience in the given domain appears to be implicit. This leads us to ask, does comparable experience result in comparable foresight? If not, where does the variation come from?

When learning mathematics at any given stage, there is a central concept at hand to be learned. In the words of Vinner and Hershkowitz (1980) and Tall and Vinner (1981), for example, there is a central *concept definition.* Students will all experience the learning of, and come to know, this concept in idiosyncratic ways; they each form a personal *concept image.* When working in a mathematical problem situation, students often rely strictly on their concept images. At the outset, they may or may not see how to make progress on the problem. That is, they may or may not have foresight. In psychological studies of foresight, people were more able to simulate future events, and have these simulations later recalled, when they had increased familiarity with the "actors" in those events: the people, places, and things involved (McLelland, Devitt, Schacter & Addis, 2014). In fact, the process of imagining future events has been shown to activate the same neural processes as remembering past events (Schacter, Addis & Buckner, 2007). Taking this perspective, a student's (in)ability to engage in mathematical foresight may serve as an indicator of the richness of their concept images.

The converse relationship, mathematical foresight from understanding, is less clear. Our interviews with mathematicians suggest that mathematical foresight does not readily develop through engaging in more mathematical activity. A couple of our mathematician colleagues indicated that their mathematical foresight developed through conscious, deliberate effort; experience tills the soil, and knowledge is the seed, but much more is needed for a good crop. As Schoenfeld (1985) recognizes, "experience" is a part of becoming a better problem solver, but it is insufficient as a complete explanation. What it is about experience that makes it productive and shifts students to more mathematical thinking and behaviour is an open question.

We hope that our characterization of mathematical foresight is sufficient that it can be identified in student activity and its development scaffolded. We have begun work in this direction (Maciejewski & Barton, 2016; Maciejewski, Roberts & Addis, 2016). In a first attempt to observe mathematical foresighting by students, we have presented student volunteers with a number of mathematical situations and asked them to think about how they would resolve each. An analysis of students' inscriptions and utterances made while thinking about solving problems indicates that students do engage in a form of mathematical foresight, though many of our volunteers did struggle with forming resolutions to the situations. What remains is to chart how mathematical foresight develops in students. We are motivated by the observation that mathematical foresight appears to be strong in mathematicians and not as much in students. Where along the way in a mathematical education is this skill developed or strengthened?

What is the benefit of developing students' mathematical foresight? If mathematical foresight is an essential part of mathematicians' practice, what benefit does the development of mathematical foresight have for those students who take mathematics courses yet will never become mathematicians; that is, the majority of our students? At a practical level, we see mathematical foresight making a contribution in encouraging persistence in working on mathematical problems of all kinds in all contexts; that is, not just research mathematics but in any mathematical career. We believe that a student who is able to see the general shape of the resolution of a mathematical situation and the general trajectory of the path to that resolution is more likely to start on, and to persist in, working mathematically. Our grander hope is that the development of a student's mathematical foresight will improve their autonomy as a mathematics learner. Mathematicians use foresight to choose interesting problems and directions to head, to solve these problems, and to persist and make progress. Might the same happen for our students?

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We see the lives of others through lenses of our own grinding and [...] they look back on ours through ones of their own. That this led some to think the sky was falling, solipsism was upon us, and intellect, judgment, even the sheer possibility of communication had all fled is not surprising. The repositioning of horizons and the decentering of perspectives has had that effect before [...] as someone has remarked of the Polynesians, it takes a certain kind of mind to sail out of sight of land in an outrigger canoe.

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