POSET SHIFTED MATROIDS AND GRAPHS

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Abstract: We generalize the notion of shiftedness in simplicial complexes, matroids, and graphs. Using this generalization, called P-shiftedness, we give a condition for a matroid to be transversal in terms of a poset P. Our result both generalizes and recovers a similar result that characterizes all shifted matroids as transversal. Moreover, we characterize certain shifted and P-shifted simple graphic matroids. We also explore the properties of P-shifted graphs and classify certain P-shifted graph families.

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CHAPTER I

INTRODUCTION

A simplicial complex on n elements labeled with $[n] = \{1, 2, ..., n\}$ is shifted if we can replace higher labeled elements with lower labeled elements in a face. That is, for $i, j \in [n]$ and $i < j$, whenever $j \in F$ and $i \notin F$, the set $F - j \cup i$ is also a face of the simplicial complex. We can associate a shifted simplicial complex to every simplicial complex while also preserving some algebraic, combinatorial, and topological properties. Hence, shifted simplicial complexes have been studied in detail.

In this thesis, we consider a generalization of shiftedness that we call P-shiftedness. A simplicial complex on $[n]$ is P-shifted if only certain higher labeled elements can be exchanged with certain lower labeled elements in a face. The information about these exchanges can be stored in a partially ordered set P. That is, if F is a face; for $i, j \in [n]$ and $i <_P j$, whenever $j \in F$ and $i \notin F$, the set $F - j \cup i$ is also a face. Thus, shifted simplicial complexes can be thought of as C-shifted simplicial complexes, where C is the n-element chain. Furthermore, the independence complex of a matroid is a simplicial complex whose faces are exactly the independent sets of the matroid. Thus, a labeled matroid M on the ground set $[n]$ is P-shifted if its independence complex is P-shifted. We explore the characteristics of P-shifted matroids in terms of their bases, circuits, and flats. We also give a condition (that we call the Gale condition) that relates the order ideals in the poset with a presentation of the matroid. We show that if M is P-shifted, and M and P satisfy the Gale condition, then M is a transversal matroid. This result generalizes and recovers a similar result on shifted matroids.

One-dimensional simplicial complexes have the same structure as simple graphs. We study the properties of P-shifted graphs whose vertices are labeled with $[n]$. We provide a construction of the maximal poset for which a graph is P-shifted. We say a graph G belongs to a graph family $\mathcal{G}_P$ if P is the maximal poset for which G is P-shifted. It is known that the family of shifted graphs is exactly the family of threshold graphs. We show that $\mathcal{G}_P$ is a subfamily of split graphs for a particular poset $P = C - \{(i < i+1) \cup (j < j+1)\}$, where $i, i+1, j, j+1$ are all distinct. Since every threshold graph is also a split graph, our result extends the classical result on graph shiftedness.

CHAPTER II

PRELIMINARIES AND BACKGROUND

2.1 Matroids

Definition 2.1.1 *A matroid M is a pair $(E, \mathcal{I})$, where E is a finite set called the ground set and $\mathcal{I}$ is a collection of subsets of E called independent sets such that:*

1. *The empty set is independent: $\emptyset \in \mathcal{I}$,*

2. *A subset of an independent set is also an independent set: If $A \subset B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$,*

3. *Given two independent sets $A, B \in \mathcal{I}$ such that $|A| > |B|$, then there exists an $x \in A \backslash B$ such that $B \cup \{x\} \in \mathcal{I}$.*

A subset of E that is not independent is called a dependent set. A minimal dependent set is called a *circuit* of M, and a maximal independent set is called a *basis* of M.

The set $\mathcal{C}$ of circuits in a matroid satisfy the following axioms, which can be used as an alternate definition of a matroid:

1. $\emptyset \notin \mathcal{C}$,

2. If $C_1, C_2 \in \mathcal{C}$ such that $C_1 \subseteq C_2$, then $C_1 = C_2$.

3. If $C_1, C_2 \in \mathcal{C}$ are distinct circuits of a matroid M, and $e \in C_1 \cap C_2$, then there is a circuit C_3 such that $C_3 \subseteq (C_1 \cup C_2) \setminus e$.

The third axiom is called the circuit elimination axiom.

An element e in a matroid M is called a *loop* if $\{e\}$ is a circuit. $e \in M$ is called a *coloop* if $e \in B$ for every basis B in M. Two elements $e, f \in M$ are called parallel elements if $\{e, f\}$ is a circuit in M.

Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges in G.

Definition 2.1.2 *Let G be a graph. Then $M(G) = (E, \mathcal{I})$, where $\mathcal{I}$ is the set of forests in the graph, forms a matroid called the* graphic matroid.

The circuits of the graphic matroid are set of cycles in G. The graphic matroid is also referred to as the *cycle matroid of a graph.*

The following lemma is useful.

Lemma 2.1.1 *If B is a basis of a matroid M and $e \in E - B$, then $B \cup e$ contains a unique circuit C satisfying $e \in C$.*

Proof. By definition, $B \cup e$ contains a circuit C. Suppose C' is another circuit in $B \cup e$. Notice, $e \in C$ and $e \in C'$. Hence, by the circuit elimination axiom, there exists a circuit contained in $C \cup C' - e \subseteq B$ which is a contradiction. ■

This circuit is called the fundamental circuit completed by e with B.

The bases of a matroid are equicardinal, and we define the *rank* of a matroid as the size of its basis. Let $X \subset E$. Let $\mathcal{I}|X = \{I \in \mathcal{I} : I \subseteq X\}$. Then, $(X, \mathcal{I}|X)$ is a matroid called the restriction of M to X, denoted by $M|X$. We define the rank of X as the rank of the matroid $M|X$.

Let $\text{cl}(X)$ denote the closure of X, defined as $\text{cl}(X) = \{x \in E : \text{rank}(X \cup x) = \text{rank}(X)\}$. A subset $X \subseteq E$ is called a flat of the matroid M if $X = \text{cl}(X)$.

We define transversals in a set system and the associated transversal matroid. Let S be a set and $\mathcal{A} = \{A_1, ..., A_n\}$ be a family of subsets of S. A transversal, also called a system of distinct representatives, is a subset T of S such that each element in T is a distinct representative of the subsets $A_j, 1 \leq j \leq n$. More precisely, a transversal $T \subset S$ is a set such that there is a bijection $\phi : [n] \to T$ with $\phi(j) \in A_j$ for every $j \in [n]$.

Let $K \subset [n]$. A partial transversal is a subset T' of S such that there is a bijection $\phi : K \to T'$ with $\phi(k) \in A_k$ for every $k \in K$.

We can also define transversal matroids using bipartite graphs. Recall that a bipartite graph $G = (V, E)$ is one whose vertex set V can be partitioned into two independent sets of vertices V_1 and V_2 such that an edge in E connects a vertex in V_1 to a vertex in V_2.

Given a family $\mathcal{A} = \{A_1, ..., A_n\}$ of subsets of a S, and $J = \{1, ..., n\}$, we define a bipartite graph associated with $\mathcal{A}$ as follows: The vertex set is $S \cup J$ and the edge set is $\{xj : x \in A_j, j \in J\}$.

A matching in a bipartite graph is a subset of edges which do not share a vertex. A subset $T' \subseteq S$ is a partial transversal if and only if there is a matching where every edge has a vertex in T'.

Theorem 2.1.1 *[11] The partial transversals of a set system $\mathcal{A}$ form the independent sets of a matroid.*

Proof. Let $\mathcal{I}$ denote the set of partial transversals of $\mathcal{A}$. $\emptyset \in \mathcal{I}$ since the empty set is a transversal of the empty subfamily of $\mathcal{A}$. Thus Axiom 1 in the independent set definition of a matroid holds. Now, if I_1 is a partial transversal of $\mathcal{A}$, and $I_2 \subseteq I_1$, then I_2 is also a partial transversal of $\mathcal{A}$. Thus, Axiom 2 is true.

For Axiom 3, we will use the notion of transversals in a bipartite graph setting. Suppose I_1 and I_2 are partial transversals with $|I_1| < |I_2|$. Then, in $G(\mathcal{A})$, there are matchings W_1 and W_2 that match I_1 and I_2 into J respectively. We color the edges of $W_1 - W_2, W_2 - W_1$ and $W_1 \cap W_2$ with red, blue and purple respectively. Let W be a subgraph of $G(\mathcal{A})$ induced by edges that are red or blue. By assumption, there are more blue edges than red in W.

Since W_1 and W_2 are both matchings, the degree of any vertex in W is one or two. Thus, every connected component of W is either a cycle or a path. Since W is a bipartite graph, every cycle is of even length. As no like-colored edges meet at a vertex, there are equal number of red and blue edges in every cycle of W and in every even path. Since W has more blue edges than red, it must have some path of odd length whose first and last edges are colored blue. We call this path $\mathbf{p}$ with vertices $\{v_1, v_2, ..., v_{2k}\}$ in order. Since one of v_1 and v_{2k} is in S and other in J, we assume that $v_1 \in S$. As v_1 meets a blue edge and not a red edge, $v_1 \in I_2 - I_1$. Moreover, $\{v_3, v_5, .., v_{2k-1}\} \subseteq I_1 \cap I_2$ and $\{v_2, v_4, ..., v_{2k}\} \subseteq J$.

We now interchange the red and blue edges on $\mathbf{p}$ leaving the rest of the graph unchanged. In the new graph, there is an extra red edge than before. Also, every vertex of $I_1 \cup v_1$ is the end point of a red or purple edge, and this set of red and purple edges form a matching. Thus, $I_1 \cup v_1$ is a a partial transversal of $\mathcal{A}$. Thus, Axiom 3 holds.

■

Such a matroid $M(\mathcal{A})$ is called a transversal matroid. The bases of the matroid are the transversals of the set system $\mathcal{A}$. We call $\mathcal{A}$ as the presentation of M.

The following condition is well known to determine the existence of transversals in a set system.

Theorem 2.1.2 *[2] The set system $\mathcal{A} = \{A_1, ..., A_n\}$ has a transversal if and only if the following condition (called Hall's marriage condition) holds: for each subfamily $\mathcal{B} \subset \mathcal{A}$, $|\mathcal{B}| \leq |\bigcup_{A \in \mathcal{B}} A|$.*

Some examples of a transversal matroids are as follows.

Example 2.1.1 *Let $S = [7]$ and $\mathcal{A} = \{A, B, C, D\}$ where $A = \{1347\}, B = \{235\}, C = \{1267\}$ and $D = \{24567\}$. Notice that $\{3, 6, 7\}$ is a partial transversal since $A = \{1347\}, B = \{2\underline{3}5\}, C = \{12\underline{6}7\}$ and $D = \{2456\underline{7}\}$. The bijection with matchings in a bipartite graph can be observed in Figure 1.*

Example 2.1.2 *Let $S = \{1, 2, 3, 4, 5\}$ and $\mathcal{A} = \{\{1, 2\}, \{3, 4\}, \{1, 2, 3, 4, 5\}\}$. Here $T = \{1, 3, 5\}$ is a transversal in $\mathcal{A}$. We write $T = 135$ for simplicity.*

The bases of the transversal matroid $M(\mathcal{A})$ are all the transversals.

$$\mathcal{B} = \{123, 124, 134, 135, 145, 234, 235, 245\}.$$

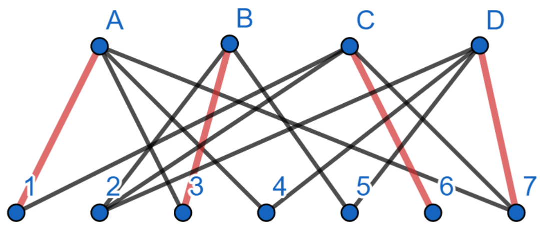

Figure 1: Transversals in a set system

Example 2.1.3 *The uniform matroid, donated by $U(k,n)$ is a matroid on the ground set with n elements where all subsets of the ground set containing at most k elements are independent. $U(k,n)$ is transversal for every k,n. The presentation for $U(k,n)$ is given by* $\underbrace{[n],[n],\cdots[n]}_{k\ times}$.

The presentation of a transversal matroid may not be unique. For instance, consider the following uniform matroid:

Example 2.1.4 *$U(2,4)$ is a uniform matroid. Two presentations for this matroid are* $\mathcal{A} = \{\{1,2,3,4\},\{1,2,3,4\}\}$ *and* $\mathcal{A}' = \{\{1,2,3\},\{1,2,3,4\}\}$.

We now define a simplicial complex. A k-simplex is defined as the convex hull of $k+1$ points in general position. For example, a 2-simplex is a solid triangle and a 3-simplex is a tetrahedron. The convex hull of any subset of the $k+1$ points in general position of the simplex is called a face of the simplex.

Definition 2.1.3 *A* simplicial complex Δ *is a finite collection of simplices satisfying*

- *If $F \in \Delta$ is a simplex, and $F_1 \subseteq F$ is a face of F, then $F_1 \in \Delta$.*
- *If F_1 and $F_2 \in \Delta$, then $F_1 \cap F_2$ is a face of both F_1 and F_2.*

Definition 2.1.4 *An* abstract simplicial complex *is a finite collection of subsets closed under set inclusion.*

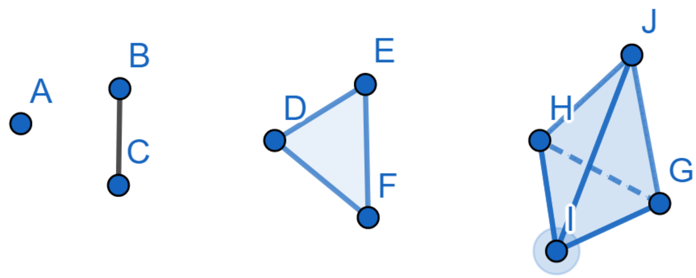

Figure 2: Simplices

The definition of an abstract simplicial complex is more useful in combinatorial settings. The k-simplices in a simplicial complex Δ are also called faces of Δ. A maximal face in Δ is called a facet. A simplicial complex is called pure if each of its facets have the same dimension. The combinatorial notion of an abstract simplicial complex is related to the geometric simplicial complex.

Theorem 2.1.3 *An abstract simplicial d-dimensional simplicial complex has a geometric realization in the Euclidean space* $\mathbf{R}^{2d+1}$.

Thus, we use the term simplicial complex to refer to the geometric as well as abstract simplicial complex.

There is an interesting connection between the independent sets of a matroid and the faces of a simplicial complex.

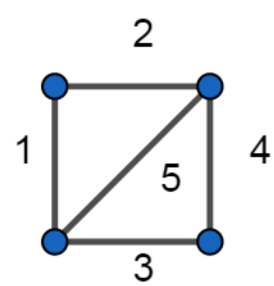

Figure 3: Matroid M

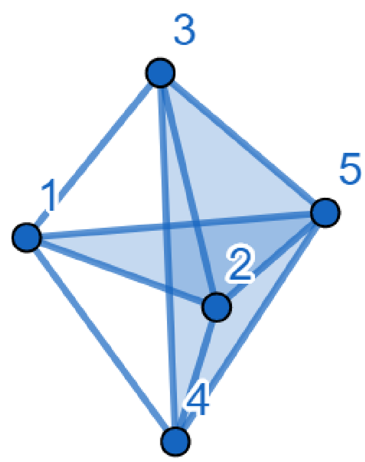

Figure 4: Δ - Independence complex of M.

Definition 2.1.5 *The independence complex* Δ *of a matroid is a pure simplicial complex whose faces are exactly the independent sets* $\mathcal{I}$ *of* M.

Example 2.1.5 *Consider the following matroid and simplicial complex in Figure 3 and Figure 4:*

We can check that Δ is the simplicial complex whose faces are exactly the independent sets of M. Δ is the independence complex of M.

Moreover, the facets of the independence complex are exactly the bases of the matroid and the minimal non-faces of the independence complex are the circuits in the matroid.

2.2 Shifted Complexes

Let M be a loopless matroid on the ground set $[n] = \{1, 2, ..., n\}$. We assume that the ordering on the ground set is fixed.

Definition 2.2.1 *A simplicial complex* Δ *on* $[n]$ *is shifted if the following condition holds for any* $i, j \in [n]$ *with* $i < j$: *if* F *is a face of* Δ *with* $j \in F$ *and* $i \notin F$, *then* $F \setminus \{j\} \cup \{i\}$ *is also a face of* Δ.

In a shifted simplicial complex, a vertex in a face can be replaced by a smaller labeled vertex.

Example 2.2.1 $\Delta = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 123\}$ *in Figure 5 is a shifted complex.*

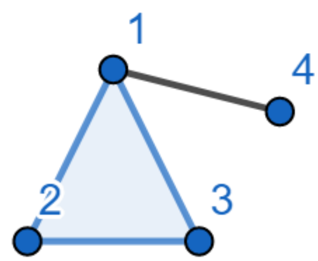

Figure 5: A shifted complex.

Shifted complexes are studied in detail (see [9]). Any simplicial complex K can be associated with a shifted complex $\Delta(K)$ using algebraic shifting. Algebraic shifting preserves many algebraic, topological and combinatorial properties.

The f-vector of a simplicial complex is the sequence

$$(f_{-1}, f_0, f_1, f_2, ...)$$

where f_i denotes the number of faces of dimension i. K and $\Delta(K)$ have the same f-vector.

Let $\tilde{H}_i(K)$ be the reduced homology group of the simplicial complex K in dimension i over a field k. The Betti numbers are defined as the dimension of these reduced homology groups.

$$\beta_i = \dim(\tilde{H}_i(K)).$$

Algebraic shifting preserves Betti numbers.

However, certain topological properties are not preserved. For instance, every shifted complex is homotopy equivalent to a wedge of spheres. To understand the combinatorial properties of shifted complexes see [10].

2.3 Borel Ideals

There is a correspondence between shifted complexes and square free Borel monomial ideals. We need some definitions before we can establish the connection between the combinatorics of complexes and algebra of ideals. We borrow the notations and terminology from [8].

Let $S = k[X] = k[x_1, ..., x_n]$ be a polynomial ring where k is a field.

A *monomial* of S is an element m which factors uniquely as a product of variables in X. A *square-free monomial* is one where the power of each factor in the product of variables is at most one. A *monomial ideal* I is defined as an ideal whose generating set consists of only monomials. A *square-free monomial ideal* is an ideal whose generating set consists of square-free monomials.

The Stanley-Reisner correspondence identifies a relationship between simplicial complexes and monomial ideals. Let $\{i_1, i_2, ..., i_t\}$ be a subset of $[n]$. We map this subset to a squarefree monomial ideal $x_{i_1}x_{i_2}\cdots x_{i_t}$.

Definition 2.3.1 *Let Δ be a simplicial complex on $[n]$. Then, the Stanley-Reisner ideal of Δ is the square free monomial ideal generated by the non-faces of Δ.*

$$I_\Delta = (m \subset X : m \notin \Delta)$$

Definition 2.3.2 *Let I be a square-free monomial ideal. Then, the Stanley-Reisner complex of I is the simplicial complex consisting of monomials not in I.*

$$\Delta_I = \{m \subset X : m \notin I\}$$

Note that the minimal generators of I_Δ are the minimal non-faces of Δ.

The dual complex $\Delta^\vee$, called the Alexander dual, is defined using the complements of non-faces of Δ.

Definition 2.3.3 *If Δ is a simplicial complex, the Alexander dual of Δ is defined as:*

$$\Delta^\vee = \{X \smallsetminus m : m \notin \Delta\}$$

Proposition 2.3.1 *Let $X = x_1 \cdots x_n$ be the product of variables. Let Δ be a simplicial complex. Then the facets of $\Delta^\vee$ are the monomials $\dfrac{X}{m}$, where m ranges over the generators of m.*

An important class of monomial ideals are strongly stable ideals, also called as 0-Borel fixed ideals, or in short Borel ideals.

Definition 2.3.4 *Let m be a monomial in the ring $S = k[x_1, ..., x_n]$. A Borel move is an operation on the monomial m that sends it another monomial $m \cdot \frac{x_i}{x_j}$ where $i < j$ and x_j divides m.*

Definition 2.3.5 *A squarefree monomial ideal B is a Borel ideal if B is closed under Borel moves.*

2.4 Shifted Matroids

A matroid M is shifted if its independence complex is shifted.

Definition 2.4.1 *A matroid M on the ground set $[n]$ is shifted if for any $i, j \in [n]$ with $i < j$: if A is an independent set in M, then the set $A \setminus \{j\} \cup \{i\}$ is also independent.*

Proposition 2.4.1 *A matroid M is shifted if its independence complex Δ is shifted.*

Proof. Since Δ on $[n]$ is shifted, for any $i, j \in [n]$ with $i < j$, if F is a face of Δ with $j \in F$ and $i \notin F$, then $F \setminus \{j\} \cup \{i\}$ is also a face of Δ. The faces of Δ are exactly the independent sets of M. Hence, if A is an independent set in M, then the set $A \setminus \{j\} \cup \{i\}$ is also independent. ■

Equivalently,

Definition 2.4.2 *A matroid M on the ground set $[n]$ is shifted if for any $i, j \in [n]$ with $i < j$: if B is a basis of M, then the set $B \setminus \{j\} \cup \{i\}$ is also a basis of M.*

Recall that an element e in a matroid M is called a loop if $\{e\}$ is a circuit. $e \in M$ is called a coloop if $e \in B$ for every basis B in M. Refer to Figure 6 for example. Suppose $e \in M$ is a loop. Then, e is not in any basis. Hence, any basis B in M is also a basis in $M - e$. Thus, deleting e does not impact shiftedness in M. Similarly, if e is a coloop, then $e \in B$ for every basis B in M. Thus, $B - e$ is a basis of the matroid $M - e$, and our discussion about shiftedness would be the same for M and $M - e$. Hence, we will assume M to contain no loops or coloops.

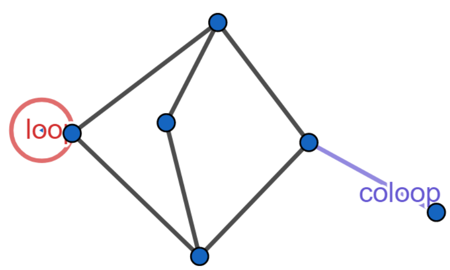

Figure 6: Loops and coloops

The following definitions helps us understand the characterization of shifted matroid complexes.

Theorem 2.4.1 *[3] For any matroid M on $[n]$, there is a basis G satisfying the following property: If B is any other basis of M and $g_1 < g_2 < ..., < g_r$ and $b_1 < b_2 < ... < b_r$ be the elements of G and B, respectively, written in increasing order; then $b_i \leq g_i$ for all i. The basis G is called the* Gale basis *of the matroid M.*

Example 2.4.1 *For the matroid in Figure 3, the bases are given by*

$$\mathcal{B} = \{123, 124, 134, 135, 145, 234, 235, 245\}.$$

Here, $G = 245$ is the Gale basis of M.

Notice that the Gale basis depends on the ordering of the ground set $[n]$. We consider an equivalent definition of shifted simplicial complexes and shifted matroids in terms of order ideals.

Definition 2.4.3 *We define a partial ordering on strings of integers as follows: we say* $(x_1 < x_2 < ... < x_r)$ *is less than* $(y_1 < y_2 < ... < y_r)$ *if* $x_i \leq y_i$ *for each* i*. We call this poset* Ω*.*

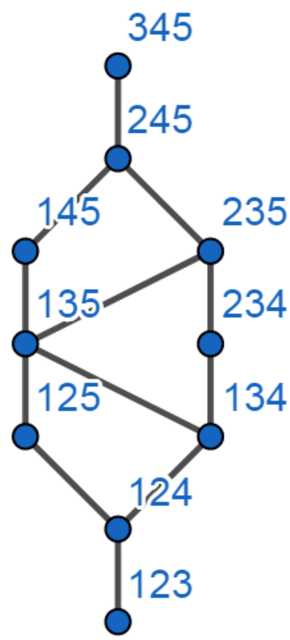

Figure 7: Poset Ω

The order ideals in this poset are the collections of facets of a shifted simplicial complex. We can say more about shifted matroids.

Proposition 2.4.2 *A shifted matroid* M *is given by a principal order ideal under the partial ordering* Ω*.*

Proof. Since M is shifted, M is an order ideal in Ω. Now the Gale basis G satisfies the property: If B is any other basis of M and $(g_1 < g_2 < ... < g_r)$ and $(b_1 < b_2 < ... < b_r)$ be the elements of G and B, respectively, written in increasing order; then $b_i \leq g_i$ for all i. This implies that $B <_\Omega G$ for any basis B in M.

Suppose $(x_1 < x_2 < ... < x_r) <_\Omega (g_1 < g_2 < ... < g_r)$. Then, $(x_1 < x_2 < ... < x_r)$ can be obtained through a series of shifts. Hence, M is given by the principal order ideal generated by the Gale basis $(g_1 < g_2 < ... < g_r)$. ■

Example 2.4.2 *The matroid M given by the bases $\{123, 124, 125, 134, 135\}$ is a shifted matroid. The Gale basis in M is 135. In Figure 7, the principal order ideal generated by $(1 < 3 < 5)$ is exactly the shifted matroid M.*

The Gale basis satisfies:

Theorem 2.4.2 *[3] If $\mathcal{B}$ is a set of size-r sets, then $\mathcal{B}$ is the set of bases of a matroid if and only if for each labeling of the ground set there exists a Gale basis G.*

We now state the characterization of shifted matroids.

Theorem 2.4.3 *[10] Let M be a shifted matroid with Gale basis $G = \{g_1, g_2, ..., g_r\}$. Then, M is a transversal matroid with presentation $[g_1], [g_2], ..., [g_r]$.*

Example 2.4.3 *If M is the shifted matroid with Gale basis $\{1, 3, 4\}$. Then, M is a transversal matroid with presentation*

$$\{1\}, \{1, 2, 3\}, \{1, 2, 3, 4\}.$$

We now discuss two interesting families of shifted matroids. We begin with lattice path matroids which were defined by Bonin, de Mier and Noy [1].

Definition 2.4.4 *A transversal matroid M with presentation $[a_1, c_1], [a_2, c_2], ..., [a_r, c_r]$ where each $[a_i, c_i]$ is an interval in the integers, $a_1 < a_2 < ... < a_r$ and $c_1 < c_2 < ... < c_r$, is called a lattice path matroid.*

We usually assume that $a_1 = 1$. There is a geometric interpretation to lattice path matroids.

A *lattice path* p from $(0, 0)$ to a point (n, r) is a particular sequence of steps in a lattice such that each step is directly north or directly east, and of unit length.

For a lattice path p, we define a set $B_p \subseteq [n{+}r]$ where $B_p = \{i :$ the i-th step of p is north$\}$. Moreover, given a basis B, we can associate to it a lattice path p_B such that $B_{p_B} = B$. Thus, p_B is a lattice path whose i-th step is north if and only if $i \in B$.

Proposition 2.4.3 *Suppose M is a lattice path matroid with presentation $[a_1, c_1], [a_2, c_2], ..., [a_r, c_r]$ where each $[a_i, c_i]$ is an interval in the integers, $a_1 < a_2 < ... < a_r$ and $c_1 < c_2 < ... < c_r$. Let B_a be the basis $\{a_1, a_2, ..., a_r\}$ and B_c be the basis $\{c_1, c_2, ..., c_r\}$. Then, B_p is a basis of M if and only if p is a lattice path within the region bounded by p_{B_a} and p_{B_c}.*

Proposition 2.4.4 *Every shifted matroid is a lattice path matroid*

Proof. Using Theorem 2.4.2, we know that a rank r shifted matroid M is a principal order ideal in the partial ordering Ω generated by the Gale basis $G = \{g_1, g_2, ..., g_r\}$. To show that this matroid is a lattice path matroid, we need to provide the bases B_a and B_c. $B_c = G$ and $B_a = \{1, 2, ..., r\}$ which is the smallest element in Ω.

■

We now compare the poset Ω with an ordering on monomials called the Borel order as defined in [6].

Notation 2.4.1 *Given a monomial m of degree d, we can write m uniquely as $m = \prod_{j=1}^{d} x_{i_j}$ with $i_1 \leq i_2 \leq ... \leq i_d$.*

Example 2.4.4 *The factorization of a^2bcd^3 is aabcddd.*

The Borel order $\Omega(\mathbf{B})$ is defined on monomials as follows:

Definition 2.4.5 *Let m_1 and m_2 be monomials. Factor $m_1 = \prod_{j=1}^{r} x_{i_j}$ and $m_2 = \prod_{j=1}^{s} x_{k_j}$. We say: $m_1 <_{\Omega(\mathbf{B})} m_2$ if $r \leq s$ and $i_j \leq k_j$ for all $j \leq s$.*

We can associate a square free Borel order ideal I to a shifted matroid M. Let $B = (b_1 < b_2 < b_3 < ... < b_r)$ be a basis of M. We map B to the monomial generator $(x_{b_1} \cdot x_{b_2-1} \cdot x_{b_3-2} \cdots x_{b_r-r+1})$. In particular, we map the Gale basis $(g_1 < g_2 < g_3 < ... < g_r)$ to the monomial $(x_{g_1} \cdot x_{g_2-1} \cdot x_{g_3-2} \cdots x_{g_r-r+1})$.

Example 2.4.5 *The borel order ideal associated with the shifted matroid in Example 2.4.2 is shown in figure 8.*

Figure 8: Borel order ideal

Proposition 2.4.5 *The poset* Ω *defined on the strings of integers is an interval in the Borel order* $\Omega(\mathbf{B})$.

Proof. Let $B = (b_1 < b_2 < b_3 < ... < b_r)$ and $D = (d_1 < d_2 < d_3 < ... < d_r)$ be two bases of M. Let f denote the function that maps B to the monomial $(x_{b_1} \cdot x_{b_2-1} \cdot x_{b_3-2} \cdots x_{b_r-r+1})$ in $\Omega(\mathbf{B})$. If $B <_\Omega D$ then $b_i \leq d_i$ for each i. We show that $f(B) <_{\Omega(\mathbf{B})} f(D)$. Notice that $(x_{b_1} \cdot x_{b_2-1} \cdot x_{b_3-2} \cdots x_{b_r-r+1})$ and $(x_{d_1} \cdot x_{d_2-1} \cdot x_{d_3-2} \cdots x_{d_r-r+1})$ are factored forms of the monomials. Since $b_i \leq d_i$, we have $b_i - i + 1 \leq d_i - i + 1$ and hence $x_{b_i-i+1} \leq x_{d_i-i+1}$. ■

2.5 Graphs and Graph Families

We define graphs and graph classes. We borrow the terminology from [4] here.

Definition 2.5.1 *Let* V *be a set of vertices. Let* $P_2(V)$ *denote the set of all 2-element subsets of* V. *A* graph $G = (V, E)$ *is a pair such that* $E \subseteq P_2(V)$ *where* E *is called the set of edges. If* x *and* y *are two vertices, then the edge joining them is denoted by* xy.

Definition 2.5.2 *A graph* $G' = (V', E')$ *is a* subgraph *if* $V' \subset V$ *and* $E' \subset E$.

Definition 2.5.3 *A subgraph G' is an* induced subgraph *of G if $E' = \{xy : xy \in E$ and $x, y \in V'\}$.*

Definition 2.5.4 *Let $G = (V, E)$ be a graph. A subset $V' \subseteq V$ is an* independent set *if for all $x, y \in V'$, $xy \notin E$.*

Definition 2.5.5 *Let $G = (V, E)$ be a graph. A subset $V' \subseteq V$ is a* clique *if for all $x, y \in V'$, $x \neq y$, then $xy \in E$.*

Graph families are extensively studied for their mathematical structure and uses in algorithmic problems. We discuss two such graph families.

A graph G is called as a threshold graph if there is a way to assign weights to each vertex such that the total weight of vertices in any independent set in the graph does not exceed a certain threshold.

Definition 2.5.6 *A graph $G = (V, E)$ is called a threshold graph if there exists non-negative real numbers t and w_v for $v \in V$ such that for any $U \subseteq V$,*

$$w(U) \leq t \text{ if and only if } U \text{ is an independent set}$$

where $w(U) = \sum_{v \in U} w_v$

Alternatively, threshold graphs can be defined in a constructive manner. We say a vertex v is an isolated vertex if it is not connected to any other vertex when added to a graph. On the other hand, we say that v is a dominating vertex if it is connected to every single vertex in the graph when added to a graph.

Definition 2.5.7 *A graph $G = (V, E)$ is called a threshold graph if it can be constructed from the one-vertex graph using a sequence of steps by adding either an isolated vertex or a dominating vertex in each step.*

Example 2.5.1 *In Figure 9, the threshold graph is constructed from the one-vertex graph by adding isolated and dominating vertices, labeled as I and D, respectively. Here, $D3$ denotes* *that the dominating vertex is the third vertex in the graph (or the second vertex added after the initial vertex).*

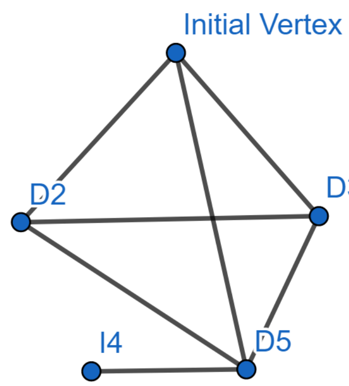

Figure 9: Threshold graph construction

The second family of graphs that we are interested in are called split graphs.

Definition 2.5.8 *A graph G is called a split graph if its vertices can be partitioned into a clique and an independent set.*

Example 2.5.2 *In Figure 10, the vertex set is partitioned into a clique on $\{A, B, C, D\}$ and an independent set on $\{E, F, G\}$.*

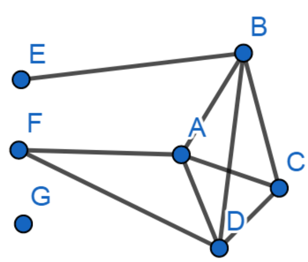

Figure 10: Split graph

The family of threshold graphs is contained in the family of split graphs.

Proposition 2.5.1 *If a graph $G = (V, E)$ is a threshold graph, then it is a split graph.*

Proof. Let G be a threshold graph. Then G can be constructed by adding isolated vertices and dominating vertices to the one-vertex graph. Let v be the initial vertex of the one-vertex graph. We construct a vertex partition in G, $V = V_1 \cup V_2$, where V_1 is the set of all the isolated vertices added to the one-vertex graph and V_2 is the set of all the dominating vertices added to the one-vertex graph and the vertex v. It is clear to see that V_1 is an independent set and V_2 is a clique in G. For any two vertices $x, y \in V_1$, $xy \notin E$ by the definition of the isolated vertex. On the other hand, for any two vertices $x, y \in V_2$, if x is added later than y then $xy \in E$ since x is a dominating vertex. Moreover, $xv \in E$ for all $x \in V_2$. Thus, G is a split graph. ■

Example 2.5.3 *In Figure 9, the vertex set of the graph can be partitioned into an independent set and a clique as $\{I4\} \cup \{Initial\ Vertex, D2, D3, D5\}$ respectively.*

CHAPTER III

P-SHIFTED MATROIDS

Our goal in this chapter is to generalize the notion of shiftedness to a larger class of simplicial complexes and matroids.

3.1 P-shiftedness

A *partial order* $(\leq)$ is a binary relation on a set P satisfying relexivity, anti-symmetry and transitivity. We can represent a partially ordered set (poset) using a Hasse diagram.

Example 3.1.1 *Consider the two posets in Figure 11. The first poset is the partial order on the divisors of 12 ordered by divisiblity. The second poset is a partial order on sets ordered by inclusion.*

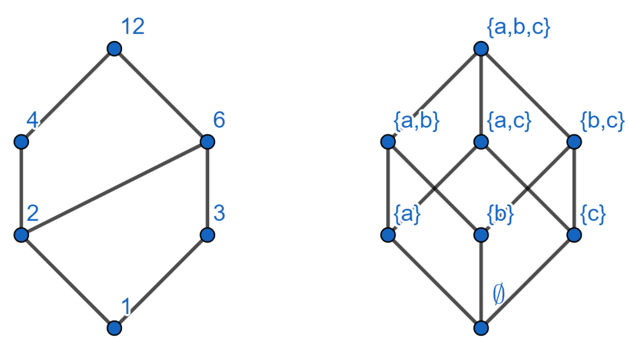

Figure 11: Hasse diagram of posets

We say a poset P labeled with $[n]$ is naturally labeled if it satisfies the order on natural numbers: $i <_P j$ then $i < j$. Equivalently, P on $[n]$ is naturally labeled if $1 < 2 < \cdots < n$ is an order completion of P.

We aim to generalize Theorem 2.4.3 to a larger class of matroids defined as follows:

Definition 3.1.1 *Let P be a naturally labeled poset on $[n]$. We say that a matroid M is P-*shifted *if the following holds for any $i, j \in [n]$ with $i <_P j$: For any independent set A of M with $j \in A$ and $i \notin A$, the set $A \smallsetminus \{j\} \cup \{i\}$ is independent.*

which is equivalent to

Definition 3.1.2 *Let P be a naturally labeled poset on $[n]$. We say that a matroid M is P-*shifted *if the following holds for any $i, j \in [n]$ with $i <_P j$: For any basis B of M with $j \in B$ and $i \notin B$, the set $B \smallsetminus \{j\} \cup \{i\}$ is a basis.*

Proposition 3.1.1 *The two definitions of P-shifted matroids in terms of independent sets and bases respectively are equivalent.*

Proof. Suppose for any $i, j \in [n]$ with $i <_P j$ and any independent set A of M with $j \in A$ and $i \notin A$, the set $A \smallsetminus \{j\} \cup \{i\}$ is independent. Let B be a basis in M. Since, B is also an independent set; given a basis B of M with $j \in B$ and $i \notin B$, the set $B \smallsetminus \{j\} \cup \{i\}$ is an independent set for any $i, j \in [n]$ with $i <_P j$. Since the sets are equicardinal, that is, $|B| = |B \smallsetminus \{j\} \cup \{i\}|$, the set $B \smallsetminus \{j\} \cup \{i\}$ is a basis in M.

For the other direction, let $i <_P j$ and suppose that A is an independent set of M with $j \in A$ and $i \notin A$. Then, A can be completed to a basis, that is, $A \subseteq B$ for some basis B in M. If $j \in B$ and $i \notin B$, using $i <_P j$, the set $B \smallsetminus \{j\} \cup \{i\}$ is a basis in M. In this case, $(A \smallsetminus \{j\} \cup \{i\}) \subseteq (B \smallsetminus \{j\} \cup \{i\})$. Thus, $A \smallsetminus \{j\} \cup \{i\}$ is an independent set. On the other hand, if $i \in B$, then $(A \smallsetminus \{j\} \cup \{i\}) \subseteq B$, and hence $(A \smallsetminus \{j\} \cup \{i\})$ is an independent set in M. ∎

Example 3.1.2 *Consider the matroid in Figure 12. The bases of the matroid are $\mathcal{B} = \{123, 124, 134, 135, 145, 234, 235, 245\}$. M is P-shifted for the poset in Figure 13. We can check for instance that $245 \in \mathcal{B}$ and since $3 <_P 4$, the set 235 is also a basis. However, $1 \not<_P 4$, and therefore the set 215 (or 125) is not necessarily a basis and in fact is not a basis of M.*

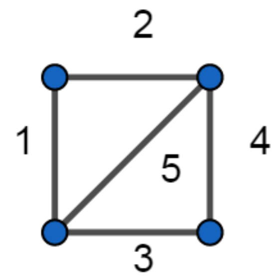

Figure 12: P-shifted matroid M

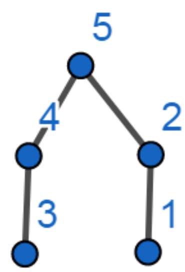

Figure 13: Poset P

Suppose $e \in M$ is a loop. Then, e is not in any basis. Hence, any basis B in M is also a basis in $M - e$. Thus, deleting e does not impact P-shiftedness in M. Similarly, if e is a coloop, then $e \in B$ for every basis B in M. Thus, $B - e$ is a basis of the matroid $M - e$, and our discussion about P-shiftedness would be the same for M and $M - e$. Hence, we will assume M to contain no loops or coloops. In such a setting, every element e in M is contained in some circuit C in M.

Shiftedness can be similarly generalized for simplicial complexes.

Definition 3.1.3 *Let P be a naturally labeled poset on $[n]$. A simplicial complex Δ on $[n]$ is P-shifted if the following condition holds for any $i, j \in [n]$ with $i <_P j$: if F is a face of Δ with $j \in F$ and $i \notin F$, then $F \setminus \{j\} \cup \{i\}$ is also a face of Δ.*

Proposition 3.1.2 *A matroid M is P-shifted if its independence complex Δ is P-shifted.*

Proof. Since Δ on $[n]$ is P-shifted, for any $i, j \in [n]$ with $i <_P j$, if F is a face of Δ with

$j \in F$ and $i \notin F$, then $F \setminus \{j\} \cup \{i\}$ is also a face of Δ. The faces of Δ are exactly the independent sets of M. Hence, if A is an independent set in M, then the set $A \setminus \{j\} \cup \{i\}$ is also independent. ■

The definition of P-shiftedness generalizes shiftedness. This can be observed by looking at shifted matroids as C-shifted matroids, where C is the n-element chain.

3.2 Q-Borel Ideals and P-shiftedness

The idea of P-shifted simplicial complexes has a naturally correspondence with the idea of square free monomial Q-Borel ideals defined by Francisco, Mermin and Schweig in [7]. We assume that all our complexes are pure and all ideals are squarefree monomial ideals.

Definition 3.2.1 *Let $I \subseteq k[x_1, ..., x_n]$ be a monomial ideal, and let Q be a naturally labeled poset on $\{x_1, ..., x_n\}$. An ideal I is Q-Borel if it satisfies the following condition: whenever $x_i <_Q x_j$ and $m \in I$ is a monomial divisible by x_j, then $m \cdot \frac{x_i}{x_j} \in I$.*

We define the notion of anti P-shifted complexes and anti Q-Borel ideals.

Definition 3.2.2 *Let P be a naturally labeled poset on $[n]$. A simplicial complex Δ on $[n]$ is anti P-shifted if the following condition holds for any $i, j \in [n]$ with $i <_P j$: if F is a face of Δ with $i \in F$ and $j \notin F$, then $F \setminus \{i\} \cup \{j\}$ is also a face of Δ.*

Definition 3.2.3 *Let $I \subseteq k[x_1, ..., x_n]$ be a monomial ideal, and let Q be a naturally labeled poset on $\{x_1, ..., x_n\}$. An ideal I is anti Q-Borel if it satisfies the following condition: whenever $x_i <_Q x_j$ and $m \in I$ is a monomial divisible by x_i, then $m \cdot \frac{x_j}{x_i} \in I$.*

Proposition 3.2.1 *If Δ is P-shifted then I_Δ is anti P-Borel.*

Proof. Let Δ be the independence complex of a matroid M. Let m be a monomial in I_Δ. Then, m is a non-face of Δ and thus a dependent set in M. Suppose $i <_P j$ and $i \in m$ and $j \notin m$. Using Proposition 3.5.2, if $C \subseteq m$ and $i \in C$ then $C - i \cup j$ is dependent and hence $m \cdot \frac{x_j}{x_i} \in I_\Delta$. If $i \notin C$, then $C \subset (m - i \cup j)$ and hence is dependent. That is, $m \cdot \frac{x_j}{x_i} \in I_\Delta$. ■

Proposition 3.2.2 *If I is a P-Borel ideal then Δ_I is anti P-shifted.*

Proof. Let Δ_I be the independence complex of a matroid M. Now, Δ_I is the simplicial complex consisting of monomials not in I. Suppose $i <_P j$ and $i \in F$ and $j \notin F$ for a face F of Δ_I. We claim that $F - i \cup j$ is a face of Δ_I. Suppose not. Then, $F - i \cup j$ is a monomial in I containing j and not containing i. Hence, $F - i \cup j - j \cup i = F \in I$ contradicting our assumption that F is not in I.

■

Example 3.2.1 *Consider the matroid and poset in Example 3.1.2. Let Δ be the independence complex of M and Δ is P-shifted. $I_\Delta = (125, 345, 1234)$ is generated by the minimal non-faces of Δ. Here, I_Δ is anti P-Borel. For instance, we can check that since $4 <_P 5$, the monomial generator $1234 \in I_\Delta$ forces $1235 \in I_\Delta$.*

Using Proposition 3.2.1 and Proposition 3.2.2, we can recover an association between shifted complexes and Borel ideals.

We say that a simplicial complex is antishifted if it is anti C-shifted. We say a squarefree monomial ideal is antiBorel if it is anti C-Borel, where C is the n-element chain.

Proposition 3.2.3 *If a simplicial complex Δ is shifted then the square-free monomial ideal I_Δ is antiBorel.*

Proposition 3.2.4 *If a squarefree monomial ideal I is Borel then Δ_I is an antishifted simplicial complex.*

3.3 The Maximal Poset

A simplicial complex Δ and a matroid M can be P-shifted for different choices of the poset P.

Example 3.3.1 *A shifted matroid is P-shifted for every naturally labeled poset P.*

Definition 3.3.1 *[5] For a fixed n, let Λ_n denote the poset of all the naturally labeled posets on $[n]$ ordered by inclusion of sets of relations. That is, if P and Q are two naturally labeled posets on $[n]$, then $P \leq_{\Lambda_n} Q$ if and only if $i <_P j$ implies $i <_Q j$ for all $i, j \in [n]$.*

Lemma 3.3.1 *Let P and Q be naturally labeled posets on $[n]$. If M is Q-shifted and $P \leq_{\Lambda_n} Q$, then M is also P-shifted.*

Proof. Suppose M is not P-shifted. Then, there exists some $i, j \in [n]$ for $i <_P j$ such that for an independent set A with $i \in A$, $j \notin A$, the set $A \setminus \{i\} \cup \{j\}$ is not independent. Since $P \leq_{\Lambda_n} Q$, $i <_P j$ implies $i <_Q j$. Thus, M is not Q-shifted. ■

The n-element chain is the unique maximal element and the n-element anti-chain is the unique minimal element in Λ_n.

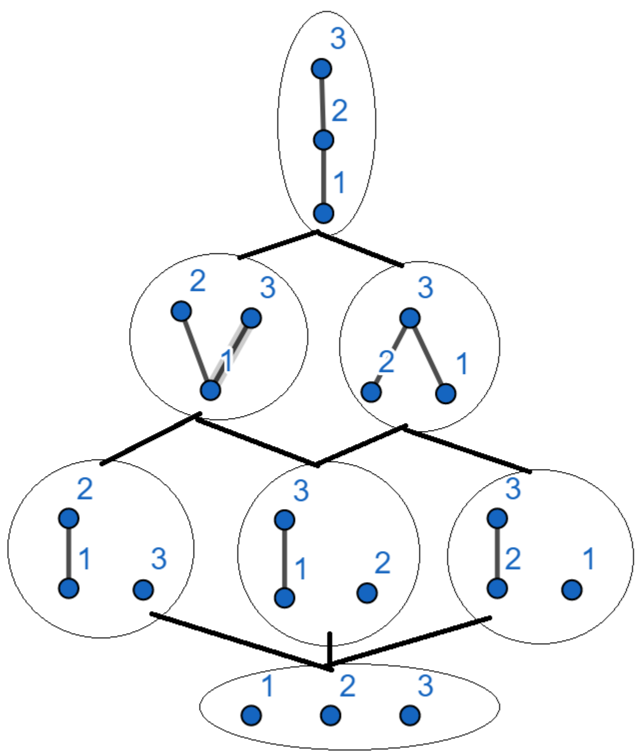

Figure 14: Λ_3, the lattice of naturally ordered posets on $n = 3$.

The meet of two naturally labeled posets $P \wedge Q$ in Λ_n is defined as the intersection of the sets of relations in P and Q.

Example 3.3.2 *In Figure 14, let P denote the naturally labeled poset with relations* $(1 < 2, 1 < 3)$*, and Q denote the naturally labeled poset with relations* $(1 < 3, 2 < 3)$*. Then, $P \wedge Q = 1 < 3$ which is the intersection of the set of relations in P and Q.*

We show that Λ_n is a lattice. In this direction, we recall that a poset Λ is a meet-semilattice if any two elements in Λ has a meet. We refer to the following proposition from [12].

Proposition 3.3.1 *Let Λ be a finite meet-semilattice with a unique maximal element $\hat{1}$. Then Λ is a lattice.*

Proposition 3.3.2 *For a fixed n, Λ_n is a lattice.*

Proof. For a fixed n, Λ_n is finite. Let P and Q be naturally labeled posets on $[n]$. Then, $P \wedge Q$ given by the intersection of sets of relations in P and Q is also a naturally labeled poset on $[n]$. Therefore, Λ_n is a meet-semilattice. Since the n-element chain is the unique maximal element, Λ_n is a finite meet-semilattice with $\hat{1}$, and is thus a lattice. ■

Let $P \vee Q$ denote the join of the naturally labeled posets P and Q in Λ_n defined as the transitive closure of the union of their relations.

Proposition 3.3.3 *If M is both P-shifted and Q-shifted, then M is also $P \vee Q$-shifted.*

Proof. Let $i, j \in [n]$ and $i <_{P \vee Q} j$. Since $P \vee Q$ is the transitive closure of the union of the set of relations in P and Q, there exists a sequence $i = k_0 < k_1 < k_2 < ... < k_{t-1} < k_t = j$ such that each inequality is either in P or Q. Let A be an independent set in M. Using induction, it follows that if $j \in A$ and $i \notin A$, then $A \cup \{i\} \setminus \{j\}$ is also independent. ■

To illustrate the above idea, consider $P_1 = 1 < 2$ and $P_2 = 2 < 3$, where $P_1, P_2 \in \Lambda_n$ for some fixed n. If we take the join of these two posets, $P_1 \vee P_2 = 1 < 2 < 3$. Let M be a matroid that is P_1-shifted and P_2-shifted. We show that M is $P_1 \vee P_2$-shifted as follows: Let A be an independent set such that $3 \in A$ but $1 \notin A$. There are two cases depending on whether $2 \in A$ or $2 \notin A$. Suppose $2 \notin A$, then by P_2-shiftedness, $A \cup \{2\} \setminus \{3\}$ is independent. Since $1 \notin A \cup \{2\} \setminus \{3\}$, $(A \cup \{2\} - \{3\}) \cup \{1\} - \{2\}$ is also indepedent by P_1-shiftedness. Thus, $A \cup \{1\} \setminus \{3\}$ is independent.

If $2 \in A$, then by P_1-shiftedness, $A \cup \{1\} \setminus \{2\}$ is independent. Since $2 \notin A \cup \{1\} \setminus \{2\}$, $(A \cup \{1\} - \{2\}) \cup \{2\} - \{3\}$ is also indepedent by P_2-shiftedness. Thus, $A \cup \{1\} \setminus \{3\}$ is independent.

Example 3.3.3 *Let $P_1 = 1 < 2$ and $P_2 = 2 < 3$. If we take the join of these two posets, $P_1 \vee P_2 = 1 < 2 < 3$ is the chain on 3 elements. Let M be a matroid on the ground set $\{1, 2, 3\}$ whose bases are given by $\mathcal{B} = \{12, 13\}$. M is P_1-shifted and P_2-shifted, and consequently $P_1 \vee P_2$-shifted. Moreover, since M is $P_1 \vee P_2$-shifted it is also P_3-shifted, where $P_3 = 1 < 2, 1 < 3$, since $P_3 \leq_{\Lambda_3} P_1 \vee P_2$ as observed in Figure 14.*

For naturally labeled posets $P_1, P_2, ..., P_t$, their join is defined as the transitive closure of the union of their sets of relations.

Proposition 3.3.4 *Let $K \subseteq \Lambda_n$ be the set of all posets P for which a matroid M is P-shifted. Then K has a unique maximal element with respect to Λ_n order.*

Proof. Let P_{max} denote the join of all the posets in K. Since M is P-shifted for all $P \in K$, M is also P_{max}-shifted, and $P_{max} \in K$. ■

Thus, for any matroid M, there is a unique maximal poset P_{max} such that if Q is another poset for which M is P-shifted, then $Q \leq_{\Lambda_n} P_{max}$.

Figure 15 is an example of a poset such that the matroid M is P_{max}-shifted.

3.4 Main Results

In this section, we prove the generalization of Theorem 2.4.3 for P-shifted matroids. We recall the standard definitions of an order filter and an order ideal in a poset P.

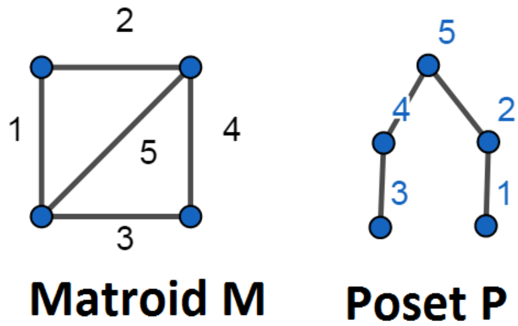

Figure 15: P_{max}

Definition 3.4.1 *A non-empty subset I of P is called an order ideal if it is closed downwards, that is, for every $x \in I$ and $y \in P$, if $y \leq_P x$ then $y \in I$.*

Definition 3.4.2 *An order ideal generated by a single element $x \in P$ is called a principal order ideal, and is denoted by $I(x)$.*

$I(x) = \{y : y \leq_P x\}$

Definition 3.4.3 *An order filter F is a non-empty subset of P that is closed upwards, that is, for every $x \in F$ and $y \in P$, if $x \leq_P y$ then $y \in F$.*

An order filter is the dual of an order ideal.

Definition 3.4.4 *Let $A \subset P$. The smallest order filter in P containing A is called the order filter generated by A, and is denoted by $F(A)$.*

Let G be the Gale basis of M, and let P be a poset for which M is P-shifted. For a set $A \subseteq [n]$, we define $G(A)$ as:

$$G(A) = G \cap F(A).$$

Thus, $G(A)$ denotes the elements of in the Gale basis closed upwards of A.

Definition 3.4.5 *We say that M and P satisfy the* Gale condition *if, for any $A \not\subseteq G$, the set $A \cup G(A)$ is dependent.*

In fact, to satisfy the Gale condition, we only need to check for single elements.

Proposition 3.4.1 *For any P-shifted matroid M with Gale basis G, M and P satisfy the Gale condition if and only if $\{a\} \cup G(a)$ is dependent for any $a \notin G$.*

Proof. Consider $A \not\subseteq G$, and let $a \in A \setminus G$. Then, $\{a\} \cup G(A)$ is dependent. Since $(\{a\} \cup G(a)) \subseteq (A \cup G(A))$, the set $A \cup G(A)$ is dependent as well. ■

Proposition 3.4.2 *If M and P satisfy the Gale condition, then all the maximal elements belong to G.*

Proof. Suppose not. Let x be a maximal element in P and $x \notin G$. Then, $G(x) = \emptyset$ and $\{x\} \cup G(x) = \{x\}$ which is dependent. Thus, x is a loop contradicting the assumption that M is loopless. ■

Proposition 3.4.3 *If M and G satisfy the Gale condition and $a \notin G$ is an element in M, then $a \cup G(a) = C$ is the unique circuit completed by a with G.*

Proof. Since $a \notin G$, a completes a unique circuit C with G. We claim that C contains all the elements in $G(a)$. If not, there exists some $x \in G(a)$ such that $x \notin C$. Now, $a <_P x$ and hence $C - a \cup x$ is dependent using Proposition 3.5.1. However,$C - a \cup x \subseteq G$ contradicting the fact that G is independent. Now, suppose $y \in C$ such that $y \in G - G(a)$. Then, $C - y$ is an independent set. However, $a \cup G(a) \subseteq C - y$ and $a \cup G(a)$ is dependent by assumption. Thus, such a y does not exist and $a \cup G(a) = C$. ■

We note that if M is P-shifted and M is a particular transversal matroid given by the presentation $\{I(x) : x \in G\}$ then M and P satisfy the Gale condition.

Proposition 3.4.4 *If M is P-shifted and M is a transversal matroid given by the presentation $\{I(x) : x \in G\}$, then M and P satisfy the Gale condition.*

Proof. Using Proposition 3.4.1, it is enough to prove that for any element $a \notin G$, the set $a \cup G(a)$ is dependent. Suppose on the contrary that there exists some $a \notin G$ such that $A = a \cup G(a)$ is an independent set. Let $A = (a, x_2, ..., x_k)$ for some k and $\{x_2, ..., x_k\} = G(a)$.

Here, the x_i's are all distinct. Since $\{I(x) : x \in G\}$ is a presentation of M, there exists $\{y_1, y_2, ..., y_k\} \in G$ such that $a <_P y_1$ and $x_i \leq_P y_i$ for all $2 \leq i \leq k$. Now, $y_i \in G(a)$ for all $1 \leq i \leq k$ since $x_i \in G(a)$ for all $2 \leq i \leq k$. Thus, $\{y_1, y_2, ..., y_k\} = G(a)$. However, this contradicts the cardinality of $G(a)$ and the result follows. ■

We now prove the main theorem of this section. Recall that a shifted matroid is a transversal matroid with presentation given by its Gale basis elements. We show that if a P-shifted matroid M satisfies the Gale condition, then M must be a transversal matroid. In fact, the presentation of the matroid can be given by the principal order ideals in the poset P generated by the Gale basis elements. Thus, conditions on the poset determines the presentation of the matroid.

Theorem 3.4.1 *If M and P satisfy the Gale condition, then M is a transversal matroid given by the presentation $\{I(x) : x \in G\}$, where G is the Gale basis and $I(x)$ is the principal order ideal generated by x.*

Proof. We show that M is transversal by proving that the set $\{I(x) : x \in G\}$ is a presentation for M.

Suppose $\{a_1, ..., a_k\}$ is an independent set in M. We need to show that for every i, there is a distinct element $x_i \in G$ such that $a_i \in I(x_i)$ for all $1 \leq i \leq k$.

We define a set A to be covered if $|A| \leq |G(A)|$. By Hall's marriage theorem, our proposition is true if every subset of any independent set is covered. Since a subset of an independent set is independent, we only need to prove that any independent set is covered.

We proceed by proof by contradiction. Suppose every independent set is not covered. Let k be the size of the largest independent set that is not covered and let A be the lexicographic largest such independent set of size k.

Since $|A| > |G(A)|$, it is possible that $G(A) \subset A$. Then $A = G(A) \cup A$ is independent contradicting the Gale condition. Thus $G(A) \not\subset A$. Let $x \in G(A) \setminus A$ be the largest labeled element in the set. We claim that x is labeled larger than any element in $A \setminus G(A)$. Suppose on the contrary that $a \in A \setminus G(A)$ is labeled greater than x. Notice that all elements in $G(a)$ are labeled greater than x since P is a naturally labeled poset. Since x is the largest labeled element in $G(A) \setminus A$, $G(a) \subset A$. This implies, $a \cup G(a) \subset A$. By the Gale condition, $a \cup G(a)$ is dependent contradicting the independence of A. Thus, x is labeled larger than any element in $A \setminus G(A)$.

Consider the set $A_1 = A \setminus \{a\} \cup \{x\}$ for some choice of $a \in A \setminus G(A)$. Notice that $|A_1| = |A|$. Since $x \in G(A), G(A_1) \subset G(A)$. Hence, $|G(A_1)| \leq |G(A)|$ and thus $|A_1| > |G(A_1)|$. Thus, A_1 is not covered. A_1 is larger than A in the lexicographic order and therefore A_1 is a dependent set. (since A is the lex largest size k independent set in M that is not covered.) Since A_1 is dependent, let $C_1 \subset A_1$ denote the unique circuit obtained by adding x to A. There is some $a' \in C_1$ for $a' \in A \setminus G(A)$. If not, $C_1 \subset G$. Let $A_2 = A_1 \setminus \{a'\} \cup \{x\}$. By the same argument, A_2 is dependent and contains a circuit C_2 containing x. For the circuits C_1 and C_2, there exists a circuit C such that

$$C \subset C_1 \cup C_2 \setminus \{x\} \subset A$$

which contradicts the hypothesis that A is an independent set.

■

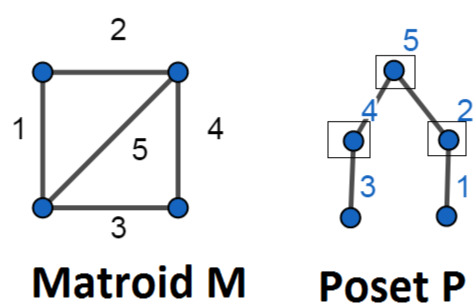

Figure 16: Gale condition

Let us look at the example in Figure 16. M is $P-$shifted with Gale basis $\{2, 4, 5\}$. We can check that $\{1\} \cup G(1) = \{1, 2, 5\}$ and $\{3\} \cup G(3) = \{3, 4, 5\}$ are dependent sets. Thus, P satisfies the Gale condition. Therefore, using Theorem 3.4.1, M is a transversal matroid.

The presentation of M is given by the order ideals in P generated by the elements of the Gale basis. Thus, $\mathcal{A} = \{\{1,2\},\{3,4\},\{1,2,3,4,5\}\}$ and $M = M(\mathcal{A})$.

If we change the labeling of the ground set in Figure 16 by swapping 3 and 5, we get a different matroid. The maximal poset for this matroid is shown in Figure 17. Here, $\{3\} \cup G(3) = \{3\}$ is independent. So, M and P do not satisfy the Gale condition.

We can recover Theorem 2.4.3 from Theorem 3.4.1 as a corollary.

Theorem 3.4.2 *Let M be a shifted matroid with Gale basis $G = \{g_1, g_2, ..., g_r\}$. Then, M is a transversal matroid with presentation $[g_1], [g_2], ..., [g_r]$.*

Proof. Since M is shifted, M is C-shifted, where C is the n-element chain. Let a be an element in M such that $a \notin G$. Let D be the unique circuit contained in $G \cup a$. We claim that a is labeled smaller than every element in D . If not, let $x \in D$ be an element labeled smaller than a. Then, the basis $G \setminus \{x\} \cup \{a\}$ contradicts our assumption that G is a Gale basis. Since C is the n-element chain, $D \subseteq F(a)$. Thus, $D \subseteq \{a\} \cup G(a)$. Since D is a circuit, $\{a\} \cup G(a)$ is dependent for every $a \notin G$. Thus, M and C satisfy the Gale condition and M is a transversal matroid with presentation $[g_1], [g_2], ..., [g_r]$. ■

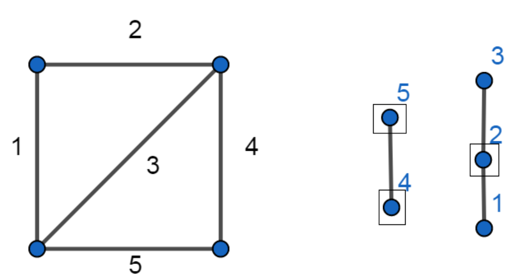

Figure 17: Non example

We define an order on the bases of M based on the P-shifts.

Definition 3.4.6 *Given bases $(x_1, ..., x_r)$ and $(y_1, ..., y_r)$ of a matroid M, we define an order $(x_1, ..., x_r) <_P (y_1, ..., y_r)$ if we can obtain $(x_1, ..., x_r)$ from $(y_1, ..., y_r)$ using P-shifts.*

This ordering defines a partially ordered set $\Omega(M)$ on the bases of a matroid.

Proposition 3.4.5 *Suppose M is P-shifted, and M and P satisfy the Gale condition. Then the Gale basis G is the unique maximum element in $\Omega(M)$.*

Proof. Let $G = (x_1, x_2, ..., x_r)$ be the Gale basis. Since M and P satisfy the Gale condition, M is a transversal matroid given by the presentation $\{I(x) : x \in G\}$. Let B be any other basis in M. We can write B as $(b_1, b_2, ..., b_r)$ where $b_i \leq_P x_i$ for all $1 \leq i \leq r$. Since M is P-shifted, $(b_1, x_2, ..., x_r)$ is also a basis in M since $b_1 \leq_P x_1$. Using the same idea, we can obtain $(b_1, b_2, ..., b_r)$ through a sequence of P-shifts as follows:

$$(x_1, x_2, ..., x_r) \rightarrow (b_1, x_2, ..., x_r) \rightarrow ... \rightarrow (b_1, b_2, ..., b_{r-1}, x_r) \rightarrow (b_1, b_2, ..., b_r)$$

Hence, $(b_1, b_2, ..., b_r) \leq_{\Omega(M)} (x_1, x_2, ..., x_r)$. ■

We can also say that M is the principal order ideal generated by the Gale basis in $\Omega(M)$ if M and P satisfy the Gale condition.

For the example matroid discussed in Figure 15, M is P-shifted and since M and P satisfy the Gale condition, M is a transversal matroid with the Gale basis $\{2, 4, 5\}$, written as 245 in short. In the partially ordered set $\Omega(M)$, of the bases of the matroid M ordered by P-shifts, 245 is the maximal element. We can observe this accurately in figure 18.

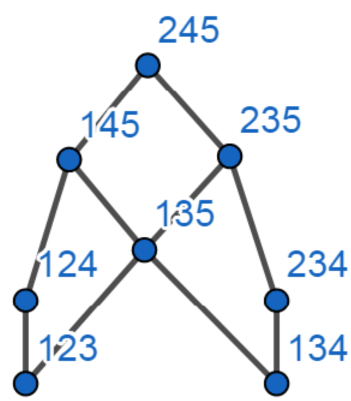

Figure 18: Gale basis is maximal

On the other hand, in our non-example as mentioned in Figure 17, M and P do not satisfy the Gale condition. Here, the Gale basis is not maximal in $\Omega(M)$.

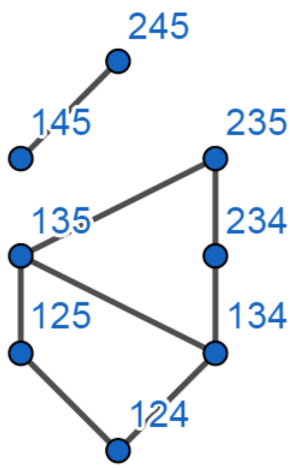

Figure 19: Gale basis is not maximal

Proposition 3.4.6 *If M is P-shifted, and M and P satisfy the Gale condition with Gale basis G. Then M has the (componentwise) smallest f-vector among all such matroids M' that are P-shifted with Gale basis G.*

Proof. Since M and P follow the Gale condition, and M is P-shifted, every basis B can be obtained from G using P-shifts. Moreover, any independent set can be obtained from a subset of B using P-shifts. Let M' be any other matroid that is P-shifted with the same Gale basis. Then, every independent set in M is also independent in M' since it can be obtained from G using P-shifts. Hence, M has the smallest f-vector. ■

Question 3.4.1 *If P is a fixed poset, is there a matroid M or simplicial complex Δ such that $P = P_{max}$ for M or Δ?*

The answer to Question 3.4.1 for a fixed dimension is no, and is discussed in detail in Chapter 3. However, for an arbitrary dimension, constructing such a matroid or simplicial complex is not known.

In [10], Klivans showed that the number of shifted matroids on $[n]$ of rank k is $\binom{n}{k}$. Can we obtain a general enumeration in case of P-shifted matroids?

Question 3.4.2 *If P is a fixed poset, can we enumerate the number of P-shifted simplicial* *complexes and P-shifted matroids of dimension d?*

Recall that the Hasse diagram of a poset is a graph with the vertex set V given by the elements of the poset, and the edge set E is given by the cover relations between two elements.

Question 3.4.3 *Can we characterize P-shifted matroids based on the connectedness of the Hasse diagram of P_{max}?*

The connectedness of the poset Q is helpful in the study of Q-Borel ideals and could play a role in P-shifted matroids.

3.5 Circuits and Flats

We present some results which are helpful in understanding the relationship between the matroid M and poset P. In this section, we assume M is a P-shifted matroid and that $P = P_{max}$, which means that P is the maximal poset for which M is P-shifted.

Definition 3.5.1 *Let $i, j \in [n]$ with $i < j$. Then $i <_P j$ if and only if, for every basis B of M with $j \in B$ and $i \notin B$, the exchange $B - j \cup i$ is a basis.*

Proposition 3.5.1 *Let $i, j \in [n]$ with $i < j$. Then $i <_P j$ if and only if, for any circuit C containing i,*

$$\text{rank}(C) = \text{rank}(C \cup j).$$

Proof. We first suppose that $\text{rank}(C) = \text{rank}(C \cup j)$ for every circuit C containing i. Consider a basis B containing j not containing i. The set $B \cup i$ contains a unique circuit C containing i. Suppose $j \notin C$. Then, $C \cup j - i$ is an independent set with $\text{rank}(C \cup j \setminus i) = |C|$ since $|(C \cup j \setminus i)| = |C|$. Thus, $\text{rank}(C \cup j) = |C|$. However, $\text{rank}(C) = |C| - 1$ contradicting our assumption. Hence, $j \in C$ and $B \cup i \setminus j$ is a basis.

On the other hand, if $\text{rank}(C) \neq \text{rank}(C \cup j)$ for some C containing i. $C \setminus i$ is independent and so is $C \cup j \setminus i$. Now, $(C \cup j \setminus i) \subset B$ for some basis B. Since the unique circuit formed by adding i to B is C and $j \notin C$, the set $B \cup i \setminus j$ is not a basis and $i \not<_P j$. ∎

Proposition 3.5.2 *For two elements i, j with $i < j$, we have $i <_P j$ if and only if, for each circuit C with $i \in C$ and $j \notin C$, $C - i \cup j$ is dependent.*

Proof. Suppose $i <_P j$, and let C be a circuit with $i \in C$ and $j \notin C$. If $C - i \cup j$ were independent, we could complete it to a basis B. But then $C \subseteq B - j \cup i$, meaning we cannot replace j with i in B.

For the other direction, suppose $i \not<_P j$. Then there would be a basis B with $j \in B$ and $i \notin B$, such that $B - j \cup i$ is not a basis. Let C be the unique circuit in $B - j \cup i$. Then $i \in C$, and $C - i \cup j$ is independent, as it is contained in B. ■

Proposition 3.5.3 *Suppose M is a graphic matroid with no parallel edges. If $i <_P j$, then for any circuit C with $i \in C$ and $j \notin C$, we can write $C \cup j = C_1 \cup C_2$, where C_1 and C_2 are circuits satisfying $C_1 \cap C_2 = \{j\}$*

Proof. Let C be a circuit containing i and not containing j. Using the above proposition, $C - i \cup j$ is dependent, and contains a unique circuit, say C_1. This circuit necessarily contains j.

Since M is a graphic matroid, we can express the circuit C as the sequence of vertices $v_0, v_1, ..., v_k$ where v_i are all distinct and $v_0 = v_k$. Let the edges $i = v_t v_{t+1}$ and $j = ww'$. where $v_0 \leq t < t + 1 \leq v_{k-1}$ Then, the vertices in $C - i \cup j$ are given by the set $\{v_0, v_1, ..., v_{k-1}\} \cup \{w, w'\} \setminus \{v_t, w_{t+1}\}$. Since C_1 is a cycle, $w = v_{i_1}$ and $w' = v_{i_2}$ for some $i_1, i_2 \in \{0, .., k\}$. Thus, $C_1 = v_{i_1}, v_{i_1+1}, ..., v_{i_2}, v_{i_1}$ and $C_2 = v_{i_1}, v_{i_1-1}, ..., v_{i_2}, v_{i_1}$ is also a circuit. That is, $C \cup j = C_1 \cup C_2$, where C_1 and C_2 are circuits satisfying $C_1 \cap C_2 = \{j\}$.

■

When the above proposition holds, we say that j splits the circuit C.

Let B be a fixed basis, and let G be the Gale basis. For an element x of $G \setminus B$, let C_x denote the unique circuit contained in $B \cup x$, and let $\bar{C_x} = C_x - x$.

Lemma 3.5.1 *Let $w \in G \setminus B$. If $x \in G \setminus B$ is in $F(\bar{C_w})$, then $\bar{C_x} \subseteq \bar{C_w}$.*

Proof. Since $x \in F(\bar{C_w})$, there is some $i \in \bar{C_w}$ with $i < x$. $C_w \cup x$ is the union of two circuits C_1 and C_2, with $C_1 \cap C_2 = \{x\}$. Since the only element in C_w not contained in B is w, one of these two circuits is contained in $B \cup x$, and the result follows. ■

The poset P is linked with the cyclic flats of M as follows.

Definition 3.5.2 (see, for instance, [2]) *A* cyclic flat *of M is a flat that is a union of circuits.*

Cyclic flats are studied in detail in [1]. Bonin and de Mier showed that every lattice is isomorphic to the lattice of cyclic flats of some matroid.

We prove the following two lemmas for completeness, but these do not refer to the relationship between M and P.

Lemma 3.5.2 *Suppose X is a union of circuits, and let i be an element of M with* $\text{rank}(X) = \text{rank}(X \cup i)$. *Then $X \cup i$ is a union of circuits.*

Proof. If $i \in X$, there is nothing to prove. Assume that $i \notin X$. Let Y be the smallest flat containing X, and consider the matroid M' obtained by restricting M to Y. Then $\text{rank}(Y) = \text{rank}(X)$, so X contains a basis B of M'. Then $B \cup i$ contains a circuit C. Since M' is the restriction of M to a flat, C is a circuit in M, and by construction $i \in C \subseteq X \cup i$.

■

Lemma 3.5.3 *Let C be a circuit. Then C must be contained in some cyclic flat X with* $\text{rank}(C) = \text{rank}(X)$.

Proof. If C is a flat, then C is a cyclic flat.

Suppose C is not a flat. Then, there is some $i \notin C$ such that $\text{rank}(C \cup i) = \text{rank}(C)$. Using Lemma 3.5.2, $C \cup i$ is a union of circuits. If $C \cup i$ is not a flat, then there exists some $i' \notin C \cup i$ such that $\text{rank}(C \cup i \cup i') = \text{rank}(C \cup i)$ and using 3.5.2, $C \cup i \cup i'$ is a union of circuits.

Since M is a finite matroid, we can repeatedly apply 3.5.2 to show that $C \subset X$ with $\text{rank}(C) = \text{rank}(X)$, where X is a cyclic flat. ■

Proposition 3.5.4 *Let X be a cyclic flat of M. Then X is an order filter of P.*

Proof. Given $i \in X$, we want to show that $j \in X$ for all $i \leq_P j$. If $\text{rank}(X) = \text{rank}(X \cup j)$, then $j \in X$ since X is a flat. Suppose not. Then, $\text{rank}(X \cup j) = \text{rank}(X) + 1$. Since X is a cyclic flat, there is a circuit $C \subset X$ with $i \in C$. $C - i$ is an independent set and can be completed to a basis B of X.

Since $\text{rank}(X \cup j) = \text{rank}(X) + 1$, $B \cup j$ is an independent set in M. Now, $C - i \cup j \subseteq B \cup j$ which contradicts Proposition 3.5.2 ($C - i \cup j$ is dependent.) ■

We note that non-cyclic flats may not be order filters. For instance, in Figure 15, the $\{1, 2\}$ is a non-cyclic flat which is not an order filter.

Proposition 3.5.5 *Let C be a circuit, and let $F(C)$ be the order filter in P generated by the elements of C. Then* $\text{rank}(F(C)) = \text{rank}(C)$.

Proof. By Lemma 3.5.3, C is contained in some cyclic flat X with $\text{rank}(C) = \text{rank}(X)$. Since X is an order filter, $C \subseteq F(C) \subseteq X$, and the result follows. ■

Recall that e is a loop if $\{e\}$ is a circuit and that two elements $e, f \in M$ are parallel if $\{e, f\}$ is a circuit. A matroid without loops or parallel elements is called a *simple* matroid.

We study simple graphic matroids in the following proposition. Recall that we already assumed that we are discussing matroids without loops or coloops.

Proposition 3.5.6 *Let M be a simple graphic matroid. If M and P satisfy the Gale condition and $x, y \notin G$, then x and y are incomparable in P.*

Proof. Assume to the contrary that $x <_P y$. Since M and P satisfy the Gale condition, $x \cup G(x)$ and $y \cup G(y)$ are dependent sets. Therefore, there is a circuit C_1 containing x such that $C_1 \subset x \cup G(x)$. and a circuit C_2 containing y such that $C_2 \subset y \cup G(y)$. Notice, $G(y) \subset G(x)$ since $x <_P y$.

Since $x <_P y$ and $y \notin C_1$, y splits C_1 using Proposition 3.5.3 . Thus, $y \cup C_1 = D_1 \cup D_2$ for the circuits D_1, D_2 satisfying $D_1 \cap D_2 = \{y\}$.

Assume $x \notin D_1$. Note that $y \in C_2$ and $y \in D_1$. Using the matroid circuit axioms, there exists a circuit C satisfying

$$C \subset C_2 \cup D_1 \setminus \{y\} \subset G$$

which contradicts the assumption that G is a basis. ■

Using Proposition 3.5.6, we can classify simple graphic matroids that are shifted.

Proposition 3.5.7 *Let M be a simple graphic matroid on $[n]$ that is shifted. Then, $M \cong M(C_n)$, where C_n is the n-cycle graph.*

Proof. Using Proposition 3.5.6, if $x, y \notin G$, then x and y are incomparable. Since, M is shifted, there are no such incomparable elements. Hence, $|G| = n$ or $|G| = n - 1$. Since M has no coloops, every element is contained in some circuit and $|G| = n - 1$. Hence, $M \cong M(C_n)$. ■

This result also characterizes the simple uniform matroids that are graphic since $U(k, n)$ is shifted for all k and n.

Corollary 3.5.1 *Let M be a simple uniform matroid $U(k, n)$. Then, M is graphic if and only if M is isomorphic to $U(n-1, n)$.*

Proof. Using Proposition 3.5.7, M is a rank $n - 1$ matroid. The result follows since $U(n-1, n) \cong M(C_n)$. ■

If we expand our attention to matroids with coloops then,

Observation 3.5.1 *Let M be a simple graphic matroid on $[n]$ that is shifted. Then, $M \cong M(C_k \cup T_{n-k})$ for $n \geq 0$, where T_k is the k-forest.*

Observation 3.5.2 *Let M be a simple uniform matroid $U(k, n)$. Then, M is graphic if and only if M is isomorphic to $U(n-1, n)$ or $U(n, n)$.*

The uniform matroid $U(n-1,n)$ is an n-cycle and $U(n,n)$ is a forest on n-edges.

We prove a special case for graphic matroids when P has a unique minimal element.

Corollary 3.5.2 *Let M be a simple graphic matroid on $[n]$ that is P-shifted. If P contains a unique minimal element, then M is isomorphic to $U(n-1,n)$.*

Proof. Suppose $1 \in P$ is the unique minimal element. If $1 \in G$, then $1 \in B$ for all bases B in M since G is maximal basis using Theorem 2.4.1. That means that $\{1\}$ is a coloop. Hence $1 \notin G$. Now, $1 <_P x$ for all $x \neq 1$. Using Proposition 3.5.6, $x \in G$ for all $x \neq 1$ or else there exist two comparable elements are not in the Gale basis. Since M is P-shifted, we can replace x with 1 to obtain a basis in M. Thus, every set of cardinality $(n-1)$ is a basis in M and M is isomorphic to $U(n-1,n)$. ■

The case for the shifted simple graphic matroids in Corollary 3.5.1 can be recovered from Corollary 3.5.2 since C, the n-element chain has a unique minimal element.

CHAPTER IV

P-SHIFTED GRAPHS

In this chapter, we consider the idea of shifted simplicial complexes and P-shifted simplicial complexes for small dimensions. In particular, one dimensional simplicial complexes are simple graphs. Thus, the study of one dimensional shifted and P-shifted simplicial complexes is the study of shifted and P-shifted simple graphs.

4.1 P-shiftedness

Definition 4.1.1 *A graph G on n vertices is* shifted *if there exists a labeling on the vertex set V with the set $[n]$ so that if jk is an edge of G, then ik is also an edge for all $i < j$.*

Note that in this definition of shiftedness, we don't require the vertex set to be pre-labeled. We can define shiftedness for labeled graphs.

Definition 4.1.2 *A labeled graph G on n vertices is* shifted *if the following condition holds: if jk is an edge of G, then ik is also an edge for all $i < j$.*

We assume all our posets P to be naturally labeled posets on $[n]$. Consider the following relaxation:

Definition 4.1.3 *A graph G on n vertices is* P-shifted *if there exists a labeling on the vertex set with the set $[n]$ so that if jk is an edge of G, then ik is also an edge for all $i <_P j$.*

Definition 4.1.4 *A labeled graph G on n vertices is* P-shifted *if the following condition holds: if jk is an edge of G, then ik is also an edge for all $i <_P j$.*

Here, the naturally labeled poset P captures the pairs of vertex exchange that is possible. A shifted graph is thus a C-shifted graph, where C is the chain of size n.

Example 4.1.1 *Consider the labeled graph G with $E = \{12, 14, 23, 24, 34\}$. Here, G is P-shifted for the given poset. For instance, 34 is an edge in G and following the poset relation $1 <_P 3$, we can check that 14 is also an edge in G. However, G is not C-shifted since $14 \in E$ but $13 \notin E$.*

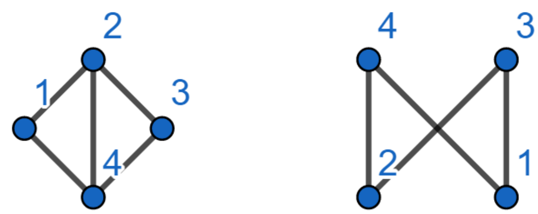

Figure 20: G is P-shifted

4.2 The Maximal Poset

Notice that a graph can be P-shifted for several choices of P. A shifted graph is P-shifted for all possible naturally labeled posets on $[n]$. Moreover, every graph is A-shifted, where A is the anti-chain. This begs the question:

Question 4.2.1 *If G is a labeled graph on $[n]$ vertices, is there a unique poset which captures the notion of P-shiftedness of G?*

Definition 4.2.1 *Fix n. For any two naturally labeled posets P and Q, we define $P <_{\Lambda_n} Q$ if $i <_P j \implies i <_Q j$ for any $i, j \in [n]$.*

Here, Λ_n is the poset of all naturally labeled posets, and as we have seen in Chapter 2, Λ_n satisfies the following properties.

- Λ_n is a lattice.
- The antichain A on $[n]$ is the unique minimal element in Λ_n.
- The chain C on $[n]$ is the unique maximal element in Λ_n

Notice that if a labeled graph G is P-shifted, then G is also Q-shifted for $P <_{\Lambda_n} Q$. The operation $P \vee Q$ denotes the transitive closure of the union of relations of P and Q . If G is P-shifted for the posets $P_1, ..., P_t$, it is also $(\bigvee_{i=1}^{k} P_i)$-shifted for the join $\bigvee_{i=1}^{k} P_i$ of the posets $P_1, ..., P_t$. Hence, for a labeled graph G, there is a unique maximal poset, denoted by P_{max}, which can be obtained by taking the join of all posets for which G is maximally P-shifted.

In example 4.1.1, the poset is P_{max} for the labeled graph. Let us relabel G by exchanging the labels 1 and 4. In this case, $E = \{12, 13, 14, 23, 24\}$ and G is C-shifted. Thus, P_{max} depends on the labeling of G.

We give the following optimal labeling of G that gives us a P_{max} with the highest rank (i.e. most relations) in Λ_n. We first need some standard definitions, for instance, from [4].

The *neighborhood* $N(i)$ of a vertex i in G is the set of all vertices adjacent to i. The *closed neighborhood* is defined as $N[i] = N(i) \cup i$. The binary relation $\preccurlyeq$ is defined on the vertex set V as follows: $i \preccurlyeq j \iff N(i) \subset N[j]$. This relation creates a preorder on V called as the vicinal preorder.

Definition 4.2.2 *The relation* $\preccurlyeq$ *satisfying:* $i \preccurlyeq j$ *if and only if* $N(i) \subset N[j]$ *is called the* vicinal preorder *on* V.

The vicinal preorder is not a partial order because there are vertices satisfying $N(i) \subset N[j]$ and $N(j) \subset N[i]$. For these vertices i, j, we write $i \sim j$.

Definition 4.2.3 *The optimal labeling on the vertices is as follows:*

- *If* $i \preccurlyeq j$ *then* j *is labeled with a smaller index than* i.
- *For* $i_1 \sim i_2 \sim \cdots \sim i_k$ *, label the elements in an increasing linear order:* $i_{l+1} = i_l + 1$ *for all* $1 \leq l < k$.

We note that the optimal labeling is not unique. In Figure 21, graph G can be labeled in two different ways, providing us with two optimal labelings and their corresponding maximal posets.

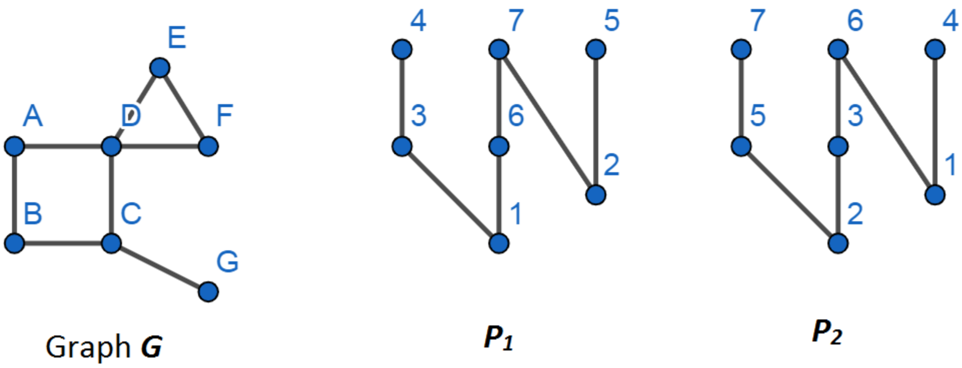

Figure 21: Optimal labelings are not unique

This labeling on the vertex set provides us a highest rank (i.e. with most relations) P_{max}, which is the maximal poset for which G is P-shifted. This highest ranked P_{max} can be defined as an reverse order extension of the vicinal preorder.

Definition 4.2.4 *The reverse order extension poset on* $[n]$ $(<_{P_{roe}})$ *is defined as:*

- *For* $i \preccurlyeq j$, *define* $j <_{P_{roe}} i$
- *For* $i_1 \sim i_2 \sim \cdots \sim i_k$, *define* $i_k \lessdot_{P_{roe}} i_{k-1} \lessdot_{P_{roe}} \ldots \lessdot_{P_{roe}} i_1$.

Proposition 4.2.1 *A graph* G *with the optimal labeling is* P_{roe}*-shifted. In fact,* $P_{roe} = P_{max}$

Proof. If $j \preccurlyeq i$ then j is labeled with a smaller index than i in G and $N(j) \subset N[i]$. That is, given $l \in V$ such that $jl \in E \implies il \in E$. Hence, G is shifted for a poset where i and j are not comparable and maximally shifted for a poset where $i < j$.

Suppose $j \sim i$. Then, we choose to label j with a smaller index than i in G. In this case, given $l \in V$, $jl \in E \Leftrightarrow il \in E$. Hence, G is shifted for a poset where $i < j$.

Suppose i and j are not comparable under $\preccurlyeq$. Then, $N(i) \not\subset N[j]$ and $N(j) \not\subset N[i]$. That is, there exists $l, l' \in V$ such that $il \in E$ but $jl \notin E$ and $jl' \in E$ but $il' \notin E$. Hence, G is shifted for a poset where i and j are not comparable.

These are precisely the relations in P_{roe} and thus G is $P_{roe}-$shifted and $P_{max} = P_{roe}$. ∎

4.3 Graph Classes

Our next goal is to classify graphs that are maximally P-shifted for a fixed P. We assume that every graph G is optimally labeled.

Definition 4.3.1 *Fix P. Let $\mathcal{G}_P$ be the family of graphs for which $P = P_{max}$.*

It is known that $\mathcal{G}_C$, namely the shifted graphs, is the family of threshold graphs [4], where C is the n-element chain.

Example 4.3.1 *In Figure 22, the threshold graph that was constructed from the one-vertex graph admits the following optimal labeling: $D5 = 1, D3 = 2, D2 = 3$, and initial vertex $= 4$, $I4 = 5$. This graph is C-shifted for the chain on 5 elements.*

We check that $E = \{12, 13, 14, 15, 23, 24, 34\}$ and G is C-shifted as in Figure 23.

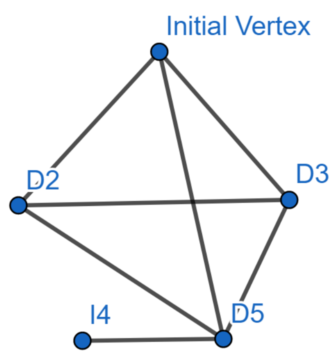

Figure 22: Constructing a threshold graph

In general, on a threshold graph with n vertices, an optimal labeling using $[n]$ can be obtained by first labeling the dominating vertices starting with the dominating vertex that is added last followed by the dominating vertex added second to last and so on. The isolated vertices are labeled next starting with the initial vertex followed by the isolated vertex added first. The isolated vertex that is added last gets labeled as n.

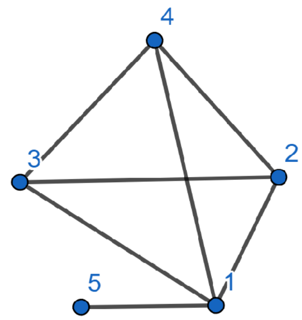

Figure 23: Threshold graph is C-shifted

Proposition 4.3.1 *[10] If $P = C$, the n-element chain, then $\mathcal{G}_C$ is the family of threshold graphs.*

We show that for $P = C - \{i < i+1\}$, a poset obtained by removing a relation in the chain, $\mathcal{G}_\mathcal{P} = \emptyset$, and for $P = C - \{(i < i+1) \cup (j < j+1)\}$, where $i, i+1, j, j+1$ are all distinct, $\mathcal{G}_\mathcal{P}$ is a subfamily of split graphs.

Proposition 4.3.2 *If $P = C - \{i < i+1\}$, where $i, i+1 \in [n]$, then $\mathcal{G}_\mathcal{P} = \emptyset$.*

Proof. Suppose not. Let $G \in \mathcal{G}_\mathcal{P}$. We know that $P_{max} = P_{roe}$. Since $i \not<_P i{+}1$, $N(i) \not\subset N[i{+}1]$ and $N(i+1) \not\subset N[i]$. Hence, there exists some $l \in V$ such that $il \in E$ but $(i+1)l \notin E$ and some $l' \in V$ such that $(i+1)l' \in E$ but $il' \notin E$. However, l and l' are related in P.

Consider the case when $l <_P l'$. The edge $(i+1)l' \in E$ forces $(i+1)l \in E$ which is a contradiction. By symmetry, assuming $l' <_P l$, $il \in E$ implies $il' \in E$ which is again a contradiction. Thus, our supposition that P is maximal is false and $\mathcal{G}_\mathcal{P} = \emptyset$. ■

We now classify the family of graphs shifted for $P = C - \{(i < i+1) \cup (j < j+1)\}$, where $i, i+1, j, j+1$ are all distinct. In that regards, consider the following characterization of the class of threshold graphs using induced subgraphs. Let P_4 denote the path on four vertices, C_4 denote the cycle on four vertices and $\bar{C}_4$ is the complement of the cycle on four vertices.

Theorem 4.3.1 *[4] A graph is threshold if and only if it has no P_4, C_4 or $\bar{C}_4$ as its induced subgraphs.*

Lemma 4.3.1 *Let $P = C - \{(i < i+1) \cup (j < j+1)\}$, where $i, i+1, j, j+1$ are all distinct. If $G \in \mathcal{G}_{\mathcal{P}}$, then there is an induced path on the four vertices $\{i, j, j+1, i+1\}$.*

Proof. We assume that $j+1 <_P i$ without loss of generality. We construct a path by noticing that certain edges cannot be part of the graph. Using the fact that $i \not<_P i+1$, $N(i) \not\subset N[i+1]$ and $N(i+1) \not\subset N[i]$. Hence, there exists some $l \in V$ such that $il \in E$ but $(i+1)l \notin E$ and some $l' \in V$ such that $(i+1)l' \in E$ but $il' \notin E$. We claim that $\{l, l'\}$ have to be exactly $\{j, j+1\}$. If not, l and l' are related in P. If $l <_P l'$: $(i+1)l' \in E$ implies $(i+1)l \in E$ whereas, if $l' <_P l$, then $il \in E$ implies $il' \in E$ which are both contradictions. Let $U = \{i, j, j+1, i+1\} \subset V$. We consider two cases and note that they are analogous.

Case 1: $l = j$ and $l' = j+1$. Thus, $ij \in E$ and $(i+1)(j+1) \in E$ but $i(j+1) \notin E$ and $(i+1)j \notin E$. Since $j+1 <_P i$, $ij \in E$ implies $(j+1)j \in E$. However, $i(i+1) \notin E$. If $i(i+1) \in E$, then $i(j+1) \in E$ through $j+1 <_P i+1$ which is a contradiction. Thus, on U, the induced subgraph is given by the edges $\{ij, j(j+1), (i+1)(j+1)\}$. This is a P_4.

We now construct a P_4 using similar arguments in case 2 as follows. Case 2: $l = j+1$ and $l' = j$. Thus, $i(j+1) \in E$ and $(i+1)j \in E$ but $ij \notin E$ and $(i+1)(j+1) \notin E$. Since $j <_P i$, $i(j+1) \in E \implies (j+1)j \in E$. However, $i(i+1) \notin E$. If $i(i+1) \in E$, then $ij \in E$ through $j <_P i+1$ which is a contradiction. Thus, on U, the induced subgraph is given by the edges $\{i(j+1), j(i+1), j(j+1)\}$ which is a P_4. ■

Proposition 4.3.3 *Let $P = C - \{(i < i+1) \cup (j < j+1)\}$, where $i, i+1, j, j+1$ are all distinct. If $G \in \mathcal{G}_{\mathcal{P}}$, then G is not a threshold graph.*

Proof. This follows directly from Theorem 4.3.1 and Lemma 4.3.1. ■

Definition 4.3.2 *A graph is a* split graph *if its vertices can be partitioned into a clique and an independent set.*

Proposition 4.3.4 *Let $P = C - \{(i < i+1) \cup (j < j+1)\}$ where $i, i+1, j, j+1$ are all distinct. If $G \in \mathcal{G}_{\mathcal{P}}$, then G is a split graph (that is not a threshold graph).*

Proof. Suppose $G \in \mathcal{G}_{\mathcal{P}}$. Without loss of generality, assume $j+1 < i$. Using Lemma 4.3.1, there is a path on $U = \{i, j, j+1, i+1\} \subset V$. We consider here the path given by the edges $\{ij, j(j+1), (i+1)(j+1)\}$. The case with path on the edges $\{i(j+1), j(i+1), j(j+1)\}$ will follow similarly.

We show G is a split graph by partitioning V into an independent set and a clique. We construct the independent set as follows.

Let $I_1 = \{k : k >_P (i+1)\}$. We claim I_1 is an independent set. To this end, if $kk' \in E$ for $k, k' \in I_1$, then $k(i+1) \in E$ through $k' >_P (i+1)$. Thus, $j(i+1) \in E$ using $k >_P j$ contradicting the induced path P_4 on U.

In general, $kt \notin E$ for any $t >_P j$ and $k \in I_1$. If not, $kj \in E$ and thus $(i+1)j \in E$. Moreover, since $i(i+1) \notin E$, $I_1 \cup \{i, i+1\}$ is an independent set.

Let $A = \{l : i >_P l >_P j+1\}$. For $l \in A$, $(i+1)(j+1) \in E$ implies $l(j+1) \in E$ and $ij \in E$ implies $lj \in E$. Now, $il \notin E$ and $(i+1)l \notin E$ for any $l \in A$. If not, $i(j+1) \in E$ and similarly, $(i+1)j \in E$ which is a contradiction to the induced P_4 in G.

Let H be the induced subgraph on A. We consider edges in H to be ordered pairs (f_a, s_a), where $f_a > s_a$. We call an edge (f_a, s_a) maximal if there is no other edge (α, β) in H such that $\alpha >_P f_a$ and $\beta >_P s_a$. Let E_1 be the set of maximal edges in H. Notice that for any two edges $(f_1, s_1) \in E_1$ and $(f_2, s_2) \in E_1$, if $f_1 <_P f_2$, then $s_1 >_P s_2$ by maximality of edges in E_1. Let $(f, s) \in E_1$ be the edge such that $f \leq f_a$ and $s \geq s_a$ for all $(f_a, s_a) \in E_1$. Consider the set $I_2 = \{s+1, ..., i-1\}$. We claim that I_2 is an independent set. If not, let qq' be an edge in H for $q, q' \in I_2$ such that $q >_P q'$. Either $qq' \in E_1$ or there is a maximal edge (f_a, s_a) such that $q <_P f_a$ and $q' <_P s_a$. If $qq' \in E_1$, then $q' \geq s$ which is contradicts the definition of (f, s). Else, we have the maximal edge (f_a, s_a) such that $q <_P f_a$ and $s < q' <_P s_a$ which contradicts the definition of (f, s).

Thus, $I = I_1 \cup \{i, i+1\} \cup I_2$ is an independent set.

Consider the set $V - I = \{1, ..., s\}$. We know that $(f, s) \in E_1$. Since $f >_P s-1$ by the definition of (f, s), $fs \in E \implies (s-1)s \in E$. This further forces all $mm' \in E$ for

$m, m' \in V - I$ since $s >_P m$ and $s - 1 >_P m'$. This shows that $V - I$ is a clique.

Thus, $V = I \cup (V - I)$, a partition of the vertex set into an independent set and a clique.

G is a split graph that is not a threshold graph. ■

Example 4.3.2 *Let G be an optimally labeled graph on the vertex set $\{1, 2, ..., 8\}$. The edge set of G is $E = \{83, 81, 72, 71, 63, 62, 61, 54, 53, 52, 51, 43, 42, 41, 32, 31, 21\}$. The maximal poset for which G is P_{max}-shifted is $C - \{(7 < 8) \cup (2 < 3)\}$ where C is the chain on 8 elements. Moreover, there is an induced path P_4 on $\{8, 7, 3, 2\}$ given by the edges $\{83, 32, 27\}$.*

Notice that G is a split graph with a vertex set partition $U_1 \cup U_2$ where $U_1 = \{8, 7, 6, 5\}$ is an independent set and $U_2 = \{4, 3, 2, 1\}$ is a clique. Thus, G satisfies Proposition 4.3.4.

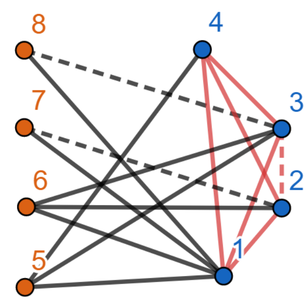

Figure 24: A split graph G

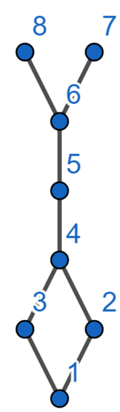

Figure 25: P_{max}: Split graph G is P_{max}-shifted

By Proposition 2.5.1, the threshold graphs family is a subclass of the split graphs family.

Our result gives an association between a fixed poset and a well studied graph family. Since there are other well known graph families such as chordal graphs and perfect graphs, we ask:

Question 4.3.1 *Is there a characterization of graph families or a subclass of graph families for a fixed poset P?*

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