STABILITY AND WELL-POSEDNESS PROBLEMS ON THE PARTIALLY DISSIPATED BOUSSINESQ EQUATIONS AND THE MICROPOLAR EQUATIONS

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Abstract:

Fluid Mechanics is a central theme of science concerned with the study of the behavior of fluids when they are in state of motion or rest. When the density of the fluid is constant or its change with the pressure is so small that can be neglected, the fluid is said to be incompressible. Examination of such fluid flow phenomena is carried out with the help of the incompressible Navier-Stokes equations. These fundamental equations provide a mathematical model of the motion of the fluid. In this direction, this thesis is concerned with the study of two closely associated systems, the micropolar equations and the Boussinesq equations. The work being conducted in this thesis includes four main chapters. In the first chapter, we give a small introduction to the concerned equations. The second chapter is devoted to show the existence and uniqueness of the weak solutions to the d-dimensional micropolar equation with general fractional dissipation. Additionally, in the third chapter, we focus first on the stability problem of the 2D Boussineq equations with vertical dissipation and horizontal thermal diffusion in $\mathbb{R}^2$, then we present some decay properties of the corresponding linearized system. Lastly, the fourth chapter investigates the stability and large-time behavior of the solutions to the 2D Boussinesq equations with horizontal dissipation and vertical thermal diffusion in two different spatial domains.

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CHAPTER I

INTRODUCTION

1.1 Navier-Stokes Equations

The Navier-Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes, model the motion of viscous fluids, which may be liquids or gases. These nonlinear partial differential equations govern, for example, the movements of air in the atmosphere, ocean currents, the flow of water in a pipe, blood flow, and many other fluid flow phenomena. When dealing with incompressible fluids, the Navier-Stokes equations can be written as

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + f, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1.1}$$

where u denotes the velocity field, p represents the pressure, ν is the kinematic viscosity and f denotes the external force. The first equation of (1.1.1) is known as the momentum equation, which is the statement of Newton's second law. The second equation $\nabla \cdot u = 0$, which expresses incompressibility, is derived from the conservation of mass. In the paricular case when $\nu = 0$, the system (1.1.1) is reduced to Euler equations. Here, and throughout this dissertation, we denote

$$\partial_i := \frac{\partial}{\partial x_i} \quad \text{and} \quad \partial_t := \frac{\partial}{\partial t}.$$

The Navier–Stokes equations describe the physics of many phenomena of scientific and engineering interest. They help with the analysis of pollution, the study of blood flow, the design of cars and aircraft, and much more things. Also, in a purely mathematical sense, the Navier-Stokes equations have long been a topic of extensive research. As any PDE, the central questions about existence and uniqueness of the solutions to the Navier-Stokes equations have been the subject of great research interest. Suitable answers to those questions were obtained in two dimensions by the celebrated works of Leray [42], Lions-Prodi [46] and Ladyzhenskaya [43]. However, in three dimensions, it has not yet been proven that solutions always exist. In this case, Leray [42] and Hopf [33] established the existence of global weak solutions, but no results on the uniqueness of the Leray-Hopf weak solutions have yet been achieved. So far, the existence and uniqueness of local solutions to the 3D Navier-Stokes equations was proved (see, e.g., [37]). In addition, global existence of strong solutions have been shown only, when initial and external forces data are sufficiently smooth (see, e.g., [60],[31]). Proving whether smooth solutions always exist in three dimensions remains to be answered. This basic open question is one of the seven Clay Foundation Millennium problems, and solving it carries a prize of one million-dollar.

1.2 Micropolar Equations

The micropolar equations were first derived in 1965 by C.A Eringen to modal micropolar fluids which are fluids with microstructure such as suspensions, liquid crystals and animal blood (see, e.g., [10],[29],[28]). Micropolar fluids belong to a class of fluids with nonsymmetric stress tensor called polar fluids. In particular, they include the viscous fluids modeled by the Navier–Stokes equations (see, e.g., [10],[29],[28],[41]). The standard 3D incompressible micropolar equations are given by

$$\begin{cases} \partial_t \mathbf{u} - (\nu+\kappa)\Delta \mathbf{u} - 2\kappa \nabla \times \mathbf{w} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla \Pi = 0, \quad x \in \mathbb{R}^3,\ t>0, \\ \partial_t \mathbf{w} - \gamma \Delta \mathbf{w} + 4\kappa \mathbf{w} - \mu \nabla\nabla\cdot\mathbf{w} - 2\kappa\nabla\times\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{w} = 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{1.2.1}$$

where $\mathbf{u} = \mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the fluid velocity, $\mathbf{w} = \mathbf{w}(x,t) = (w_1(x,t), w_2(x,t), w_3(x,t))$ the field of microrotation representing the angular velocity of the rotation of the fluid particles, $\Pi(x,t)$ the scalar pressure, κ denotes the micro-rotation viscosity, ν the Newtonian kinematic viscosity, and γ and μ the angular viscosities.

The first equation in (1.2.1) represents the conservation of linear momentum, the second one expresses the conservation of angular momentum and the third equation reflects the incompressibility condition for the fluid.

Note that when the microrotation effects are neglected, namely $w = 0$, (1.2.1) reduces to the incompressible Navier–Stokes equations (1.1.1). Further more, if we assume that the velocity component in the x_3-direction is zero and the axes of rotation of particles are parallel to the x_3-axis, that is

$$\mathbf{u} = (u_1(x_1,x_2,t), u_2(x_1,x_2,t), 0), \quad \Pi = \Pi(x_1,x_2,t), \quad \mathbf{w} = (0,0,w(x_1,x_2,t)),$$

the 3D micropolar equations reduce to the 2D micropolar equations given by

$$\begin{cases} \partial_t u - (\nu+\kappa)\Delta u - 2\kappa\nabla\times w + u\cdot\nabla u + \nabla\Pi = 0, \quad x\in\mathbb{R}^2,\ t>0, \\ \partial_t w - \gamma\Delta w + 4\kappa w - 2\kappa\nabla\times u + u\cdot\nabla w = 0, \\ \nabla\cdot u = 0, \end{cases} \tag{1.2.2}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t))$ is the 2D fluid velocity vector and $w = w(x,t)$ is a scalar function representing the angular velocity of the rotation of the fluid particles. Here, we remember that, in the 2D case,

$$\nabla\times u = \partial_1 u_2 - \partial_2 u_1$$

is a scalar function representing the vorticity, and $\nabla\times w = (\partial_2 w, -\partial_1 w)$.

Besides its rich physical background, the micropolar equations have been one of the most commonly studied models in mathematical fluid dynamics for many decades. In the recent years, extensive works have been done on the global regularity, well-posedness and large-time decay problems on the micropolar equations (see, e.g.,[4],[15],[32],[39],[40],[41],[51],[53],[66],[16],[12],[28],[41],[13]).

We mention in particular some results on the global regularity of the 3D micropolar equations (1.2.1). In [32], Galdi and Rionero were the first who studied the weak solutions of the initial boundary-value problem for the system (1.2.1). Then, Lukaszewicz established the global existence of weak solutions for sufficiently regular initial data in [39] and the local and global well-posedness results under asymmetric conditions in [40]. Further, Boldrini, Durán and Rojas-Medar showed the local existence and uniqueness of strong solutions to the initial and boundary-value problem for bounded or unbounded domains in their work [4]. In addition, Yamaguchi established the global existence of small classical solutions in bounded domains in [66].

There is also an array of important results on the global regularity of the 2D micropolar equations (1.2.2). (see,e.g., [22],[27],[54]). We mention in particular some results on the 2D micropolar equations (1.2.2) with partial dissipation. In [27], Dong and Zhang established the global regularity for the system (1.2.2) without the micro-rotation viscosity (when $\gamma = 0$). Then, Xue restudied the system (1.2.2) with $\nu = 0, \gamma > 0, \kappa > 0$ and $\kappa \neq \gamma$ in his paper [65] and obtained the global well-posedness in some Besov space settings. In [22], Dong, Li and Wu studied the global well-posedness problem and the large-time behavior of (1.2.2) with no velocity dissipation. See also [35], [48].

1.2.1 Fractional Dissipative Micropolar Equations

Despite all the efforts mentioned above, the global regularity of the weak solution of the system (1.2.1) with general initial data remains an open problem. When trying to solve this issue by applying the standard techniques, the main difficulty that encounters us is that the Laplacian dissipation is not sufficient to control the nonlinearity. Naturally, this leads us to ask the question of how much dissipation one requires in order to get the global regularity. To answer this question, we are led to replace the standard Laplacian dissipations $-\Delta u$ and $-\Delta w$ by general fractional Laplacian dissipations $(-\Delta)^\alpha u$ and $(-\Delta)^\beta w$. Doing so, we get the following 3D standard fractional micropolar equations

$$\begin{cases} \partial_t \mathbf{u} + (\nu+\kappa)(-\Delta)^\alpha \mathbf{u} - 2\kappa \nabla \times \mathbf{w} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\Pi = 0, \quad x\in\mathbb{R}^3,\ t>0, \\ \partial_t \mathbf{w} + \gamma(-\Delta)^\beta \mathbf{w} + 4\kappa\mathbf{w} - \mu\nabla\nabla\cdot\mathbf{w} - 2\kappa\nabla\times\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{w} = 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{1.2.3}$$

where the fractional powers α and β are nonnegative and the fractional Laplacian operator $(-\Delta)^\gamma$ is defined via the Fourier transform

$$\widehat{(-\Delta)^\gamma} f(\xi) = |\xi|^{2\gamma}\widehat{f}(\xi).$$

The generalization to include fractional Laplacian dissipations $(-\Delta)^\alpha u$ and $(-\Delta)^\beta w$ allows simultaneous study of a family of equations and is relevant in some physical circumstances.

Fundamental issues such as the well-posedness problem on the micropolar equations with fractional dissipation have recently attracted considerable interest and an array of important results have been established. Among those results, we mention the recent work [26] of Dong, Wu, Xu and Ye who studied the 2D fractional micropolar equations and established the global well-posedness for the fractional powers α and β in suitable ranges. We mention also the work of Wang, Wu and Ye [63], who proved that, if $\alpha \geq \frac{5}{4}$, $\beta \geq 0$ and $\alpha+\beta \geq \frac{7}{4}$, the fractional 3D micropolar equations always possess a unique global classical solution for any sufficiently smooth data.

In this dissertation, we are concerned with the following d-dimensional ($d = 2$ or

$d = 3$) incompressible micropolar equations with fractional dissipation

$$\begin{cases} \partial_t u + (\nu + k)(-\Delta)^\alpha u + u \cdot \nabla u + \nabla \Pi - 2k \nabla \times w = 0, \quad x \in \mathbb{R}^d, \ t > 0, \\ \partial_t w + 4kw + 2k \nabla \times u + u \cdot \nabla w + \gamma (-\Delta)^\beta w = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x)\,, \ \ w(x,0) = w_0(x), \end{cases} \tag{1.2.4}$$

where in the above system, $u = u(x,t) \in \mathbb{R}^d$ stands for the fluid velocity, $w = w(x,t) \in \mathbb{R}^d$ denotes the field of microrotation representing the angular velocity of the rotation of the fluid particles, $\Pi = \Pi(x,t)$ stands for the pressure in the fluids, and the parameter ν denotes the Newtonian kinematic viscosity, k the microrotation viscosity and γ the angular viscosity. Here the fractional Laplacian operator $(-\Delta)^\alpha$ (which is also referred as the Riesz potential operator) is defined via the Fourrier transform

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi),$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

In our joint work with Wu [8], that will be discussed in chapter II, we are interested in proving the local existence and uniqueness of weak solutions to the system (1.2.4) in the frame work of inhomogeneous Besov spaces. We mention here that definitions and some fundamental results about inhomogeneous Besov spaces that are mentioned in [8] and again prominently in Chapter II can be found in Subsection 2.1.1. In our research [8], we established that, when $\alpha \geq \frac{1}{2}$ and $\beta \geq \frac{1}{2}$, any initial data (u_0, w_0) in the critical Besov space $u_0 \in B^{1+\frac{d}{2}-2\alpha}_{2,1}(\mathbb{R}^d)$ and $w_0 \in B^{1+\frac{d}{2}-2\beta}_{2,1}(\mathbb{R}^d)$ yields a unique local weak solution. Due to this result, taking in particular $\alpha = \beta = 1$, one can deduce that the 2D micropolar equations with the standard Laplacian dissipation, have a unique local weak solution for $(u_0, w_0) \in B^0_{2,1}$. Additionally, we proved in [8] that for $\alpha \geq 1$ and $\beta = 0$, any initial data $u_0 \in B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)$ and $w_0 \in B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)$ also leads to a unique local weak solution as well. We should mention here that, the regularity indices obtained in the above stated Besov spaces appear to be optimal and can not be lowered in order to achieve the uniqueness results.

For the proof of the above results and further details, we refer the reader to Chapter II.

1.3 Boussinesq Equations

The Boussinesq equations were first introduced in 1872 by Joseph Valentin Boussinesq to reflect the basic physics laws obeyed by buoyancy-driven fluids. These fundamental equations are frequently used in modeling many geophysical flows, such as atmospheric fronts and oceanographics flows, and serve in the study of the Rayleigh-Bénard convection (see, e.g., [14],[20],[50],[52],[5],[30]).

In the 2-dimensional case, the Boussinesq equations can be written as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \mathbf{p} + \nu \, \Delta \mathbf{u} + \boldsymbol{\theta} \mathbf{e}_2, \quad x \in \mathbb{R}^2, \; t > 0, \\ \partial_t \boldsymbol{\theta} + \mathbf{u} \cdot \nabla \boldsymbol{\theta} = \eta \, \Delta \boldsymbol{\theta}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{1.3.1}$$

where $\mathbf{u} = \mathbf{u}(x,t) = (\mathbf{u}_1(x,t), \mathbf{u}_2(x,t))$ denotes the fluid velocity, $\mathbf{p} = \mathbf{p}(x,t)$ the scalar pressure, $\boldsymbol{\theta} = \boldsymbol{\theta}(x,t)$ the temperature, $\nu > 0$ the kinematic viscosity, and η the thermal diffusivity. Here $\mathbf{e}_2 = (0,1)$ is the unit vector in the vertical direction.

The first equation in (1.3.1) represents the Navier-Stokes equations with an extra buoyancy term $\boldsymbol{\theta} e_2$ that reflects the influence of the gravity and the stratification on the motion of the fluid. The second equation in (1.3.1) corresponds to the temperature equation. Lastly, the third equation $\nabla \cdot \mathbf{u} = 0$ is the mass continuity equation for incompressible fluid.

In addition to their geophysical applications, the 2D Boussinesq equations serve mathematically as a lower dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. Namely, the inviscid ($\nu = \eta = 0$) 2D Boussinesq equations can be identified as the Euler equations for the 3D axisymmetric swirling flows [5]. Due to these similarities, the study of the 2D Boussinesq equations, may in particular shed light on the global existence and the regularity open problems on the 3D Navier-Stokes and Euler equations. Despite the huge efforts done on the study of the 2D Boussinesq equations, in the non-dissipative case (where $\nu = \eta = 0$), the global regularity remains an open problem. When dealing with the desipative Boussinesq equations cases, studies show that the dissipation terms $\nu\Delta\mathbf{u}$ and $\eta\Delta\boldsymbol{\theta}$ in (1.3.1) play an important part in controlling the long time behavior of the system.

Over the past few years, fundamental issues on the 2D Boussinesq systems such as the global well-posedness problem have attracted considerable interests (see, e.g., [1], [2],[17], [9], [23], [34], [45]).

1.3.1 Partial Dissipative Boussinesq Equations

In order to model anisotropic flows (the thermal diffusivity and the viscosity are different in the vertical and the horizontal directions), (1.3.1) is generalized to the form

$$\begin{cases} \partial_t \mathbf{u}_1 + \mathbf{u}_1\,\partial_1\mathbf{u}_1 + \mathbf{u}_2\,\partial_2\mathbf{u}_1 = -\partial_1\mathbf{p} + \nu_1\,\partial_{11}\mathbf{u}_1 + \nu_2\,\partial_{22}\mathbf{u}_2, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t \mathbf{u}_2 + \mathbf{u}_1\,\partial_1\mathbf{u}_2 + \mathbf{u}_2\,\partial_2\mathbf{u}_2 = -\partial_2\mathbf{p} + \nu_3\,\partial_{11}\mathbf{u}_2 + \nu_4\,\partial_{22}\mathbf{u}_2 + \boldsymbol{\theta}, \\ \partial_t\boldsymbol{\theta} + \mathbf{u}\cdot\nabla\boldsymbol{\theta} = \eta_1\,\partial_{11}\boldsymbol{\theta} + \eta_2\,\partial_{22}\boldsymbol{\theta}, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{1.3.2}$$

Note that, when $\nu_1 = \nu_2 = \nu_3 = \nu_4$ and $\eta_1 = \eta_2$, the above system reduces to (1.3.1). Writing down the 2D Boussinesq equations in its general form (1.3.2) allow us to consider the horizontal and vertical dissipations separately.

What we care about here, is whether the dissipative models of (1.3.2) always have a global unique solution when the initial data $\mathbf{u}(x,0) = (\mathbf{u}_1(x,0), \mathbf{u}_2(x,0)) = \mathbf{u}_0(x)$ is sufficiently smooth, and if so, we are concerned also with its large-time behavior. These issues were resolved in several cases among which we summarize the following:

(i) When $\nu_1 = \nu_3 > 0, \nu_2 = \nu_4 > 0, \eta_1 > 0, \eta_2 > 0$ (Fully dissipative system). The global existence of unique solutions was obtained for any suffciently smooth data $(\mathbf{u}_0, \boldsymbol{\theta}_0) \in H^2(\mathbb{R}^2)$ (see, e.g.,[44]).

(ii) When $\nu_1 > 0, \nu_2 > 0, \nu_3 > 0, \nu_4 > 0, \eta_1 > 0, \eta_2 > 0$ (Both dissipation and thermal diffusion). The global existence of classical solutions was achieved in a similar way as for the 2D Navier–Stokes equations (see, e.g.,[20],[5]).

(iii) When $\nu_1 = \nu_2 = \nu_3 = \nu_4 > 0, \eta_1 = \eta_2 = 0$ (Velocity dissipation only). The global regularity was obtained. (see, e.g., [9], [34]). The uniform-in-time boundedness of kinetic energy for initial-boundary value problems on 2D bounded smooth domains for large data was shown in [47].

(iv) When $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 0, \eta_1 > 0, \eta_2 > 0$ (Thermal diffusion only). The global regularity was obtained (see, e.g., [9]) and the large-time behavior was studied in [69] in the particular case when $\eta_1 = \eta_2 > 0$.

(v) When $\nu_1 = \nu_3 > 0, \nu_2 = \nu_4 = \eta_1 = \eta_3 = 0$, (Horizontal velocity dissipation only). The global existence of suitably regular solutions was proved in [23]. Furthermore, the uniqueness was established in [45] under the assumption that $\boldsymbol{\theta}_0 \in L^\infty(\mathbb{R}^2)$.

(vi) When $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 0, \eta_1 > 0, \eta_2 = 0$, (Horizontal thermal diffusion only). The global existence of suitably regular solutions was achieved (see, e.g., [23],[45]).

(vii) When $\nu_1 = \nu_3 = \eta_1 = 0, \nu_2 = \nu_4 > 0, \eta_2 > 0$ (Vertical velocity and vertical thermal diffusion only). The global existence of classical solutions was established by Cao and Wu in [17].

In the last few years, the study of the stability problem on the 2D Boussinesq equations has gained momentum. Current investigations focused in particular on the stability near the hydrostatic equilibrium (the status of a fluid when it is at rest)

$$\mathbf{u}_{he} = 0, \quad \boldsymbol{\theta}_{he} = x_2, \quad \mathbf{p}_{he} = \frac{1}{2}x_2^2.$$

Hydrostatic equilibrium or hydrostatic balance occurs when the gravity is balanced out by the pressure-gradient force. The stability problem on perturbations near this particular steady state is one of the most prominent topics in fluid dynamics, astrophysics and atmospherics. Indeed, our atmosphere is mostly in the hydrostatic equilibrium.

To understand the desired stability, we write the equation of the perturbation denoted by (u, p, θ), where

$$u = \mathbf{u} - \mathbf{u}_{he}, \quad p = \mathbf{p} - \mathbf{p}_{he} \quad \text{and} \quad \theta = \boldsymbol{\theta} - \boldsymbol{\theta}_{he}.$$

It follows easily from (1.3.2) that the perturbation (u, p, θ) satisfies

$$\begin{cases} \partial_t u_1 + u_1\, \partial_1 u_1 + u_2\, \partial_2 u_1 = -\partial_1 p + \nu_1\, \partial_{11} u_1 + \nu_2\, \partial_{22} u_2, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t u_2 + u_1\, \partial_1 u_2 + u_2\, \partial_2 u_2 = -\partial_2 p + \nu_3\, \partial_{11} u_2 + \nu_4\, \partial_{22} u_2 + \theta, \\ \partial_t \theta + u \cdot \nabla\theta + u_2 = \eta_1\, \partial_{11}\theta + \eta_2\, \partial_{22}\theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.3.3}$$

The only difference between (1.3.2) and (1.3.3) is an extra term u_2 (the vertical component of u) in (1.3.3), which plays a very important role in balancing the energy.

The answer to the stability problem near the hydrostatic balance of the 2D Boussinesq equations (1.3.3) was first initiated by the work [24] of Doering, Wu, Zhao and Zheng for the particular case when $\nu_1 = \nu_2 = \nu_3 = \nu_4 > 0, \eta_1 = \eta_2 = 0$ (only velocity dissipation). Then, Tao, Wu, Zhao and Zheng established the large-time behavior and the temperature profile in their paper [59]. Further, the stability and large-time behavior of the 2D inviscid Boussinesq equations with velocity damping term was established by Castro, Córdoba and Lear in [11]. In this context, more recent works have been done (see, e.g., [6], [62], [64], [68], [49], [25]).

In our paper [6] that will be discussed in Chapter III, we are concerned with the study of the 2D Boussinesq equations with vertical dissipation and horizontal thermal diffusion (system (1.3.3) with $\nu_1 = \nu_3 = \eta_2 = 0, \nu_2 = \nu_4 := \nu > 0, \eta_1 := \eta > 0$) given by

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{22} u + \theta \mathbf{e}_2, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \eta\, \partial_{11} \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.3.4}$$

We obtained in [6] three main results. The first result established the H^2 nonlinear stability on (1.3.4). In the second result, we precised the anisotropic large-time behavior of the solution of the linearized system (see Theorem 3.3.4 in Chapter III). In the third main result, we proved that the Fourier frequency piece of the solutions (u, θ) of the linearized system away from the two axes of the frequency space decays exponentially in time to zero (see Theorem 3.3.6 in Chapter III).

Additionally, we studied in our research [7] the 2D Boussinesq equations with horizontal dissipation and vertical thermal diffusion (system (1.3.3) with $\nu_2 = \nu_4 =$

$\eta_1 = 0, \nu_1 = \nu_3 := \nu > 0, \eta_2 := \eta > 0)$ written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{11} u + \theta \mathbf{e}_2, \quad x \in \Omega,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \eta\, \partial_{22} \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.3.5}$$

When studying the 2D Boussinesq equations (1.3.5), we have observed that the type of the spatial domain Ω plays an essential role in the resolution of the stability problem. Indeed, when the spatial domain Ω is taken to be the whole plane $\mathbb{R}^2$, the stability problem in the Sobolev space H^2 remains an open problem. In this case, we are able to show that any small H^1 initial data leads to a global H^1 weak solution. However, it does not appear to be possible to show that the H^1-solutions are unique. While in the second situation, when the spatial domain is $\Omega = [0,1] \times \mathbb{R}$, we succeeded to solve the global H^2 stability of (1.3.5). In addition, we specified some decay properties in time of the solutions to the system (1.3.5). These results will be discussed in further detail in Chapter IV.

CHAPTER II

ON THE D-DIMENSIONAL MICROPOLAR EQUATION WITH FRACTIONAL DISSIPATION

2.1 Introduction

Recall that the d-dimensional ($d = 2$ or $d = 3$) incompressible micropolar equations with fractional dissipation is given as

$$\begin{cases} \partial_t u + (\nu + k)(-\Delta)^\alpha u + u \cdot \nabla u + \nabla \Pi - 2k\nabla \times w = 0, \quad x \in \mathbb{R}^d, \ t > 0, \\ \partial_t w + 4kw + 2k\nabla \times u + u \cdot \nabla w + \gamma(-\Delta)^\beta w = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x)\,, \quad w(x,0) = w_0(x). \end{cases} \tag{2.1.1}$$

As outlined in the introduction, this Chapter is concerned with the study of (2.1.1) in the frame work of inhomogeneous Besov spaces. A review of the Besov spaces and related facts is provided in Subsection 2.1.1. The main goal of our study is to prove the existence and uniqueness of solutions to (2.1.1) in a weakest possible functional setting for the largest possible ranges of α and β.

In this chapter, we present two main results from the author's joint work in [8]. Each result will be discussed and proved in a separate section. Further, each section is naturally split into two subsections with one devoted to the existence and the other to the uniqueness. Our process involves various analysis tools and techniques which will be provided in Subsection 2.1.1.

2.1.1 Preliminaries

This subsection is a collection of notations, definitions and Lemmas that will be used in the proofs of our main results. The definition of the Besov space and related simple facts presented here can be found in [3]. We refer the reader to [36, Lemma A.5] for a detailed proof of Lemma 2.1.6. In what follows, $\mathcal{S}(\mathbb{R}^d)$ stands for the Schwartz class and $\mathcal{S}'(\mathbb{R}^d)$ its dual, the space of tempered distributions.

Definition 2.1.1 (Inhomogenous Besov space $B^s_{p,q}$) *$f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $B^s_{p,q}$ with $s \in \mathbb{R}$ and $1 \le p \le q \le \infty$ if*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{sj}\|\Delta_j f\|_{L^p}\|_{l^q} = \begin{cases} \Big(\sum\limits_{j=-1}^{+\infty} (2^{sj}\|\Delta_j f\|_{L^p})^q\Big)^{\frac{1}{q}} & \text{if} \quad q < \infty, \\ \sup\limits_{j\ge -1} \; 2^{sj}\|\Delta_j f\|_{L^p} & \text{if} \quad q = \infty \end{cases}$$

is finite.

Lemma 2.1.2 *Let $B(0,r)$ and $C(0,r_1,r_2)$ denote the standard ball and the annulus, respectively,*

$$B(0,r) = \left\{\xi \in \mathbb{R}^d, \;\; |\xi| \le r\right\}, \quad C(0,r_1,r_2) = \left\{\xi \in \mathbb{R}^d, \;\; r_1 \le |\xi| \le r_2\right\}.$$

There are two compactly supported smooth radial functions ϕ and ψ satisfying

$$\operatorname{supp}\phi \subset B(0,\frac{4}{3}), \quad \operatorname{supp}\psi \subset C(0,\frac{3}{4},\frac{8}{3}),$$
$$\phi(\xi) + \sum_{j\ge 0}\psi(2^{-j}\xi) = 1 \quad \text{for all } \; \xi \in \mathbb{R}^d. \tag{2.1.2}$$

For a detailed proof of Lemma 2.1.2, we refer to [3, p.59].

Notations 2.1.2.1 *We use $\widetilde{h}$ and h to denote the inverse Fourier transforms of ϕ and ψ respectively*

$$\widetilde{h} = \mathcal{F}^{-1}\phi, \quad h = \mathcal{F}^{-1}\psi.$$

We write $\psi_j(\xi) = \psi(2^{-j}\xi)$. By a simple property of the Fourier transform,

$$h_j(x) := \mathcal{F}^{-1}\psi_j(x) = 2^{dj}h(2^jx).$$

Definition 2.1.3 *The inhomogeneous dyadic block operator* Δ_j *are defined as*

$$\begin{aligned}
&\Delta_j f = 0 && \text{for } j \leq -2, \\
&\Delta_{-1} f = \widetilde{h} * f = \int_{\mathbb{R}^d} f(x-y)\widetilde{h}(y)\,dy\,, && \\
&\Delta_j f = h_j * f = 2^{dj} \int_{\mathbb{R}^d} f(x-y)h(2^j y)\,dy && \text{for } j \geq 0.
\end{aligned}$$

The corresponding inhomogeneous low frequency cut-off operator S_j *is defined by*

$$S_j f = \sum_{k \leq j-1} \Delta_k f.$$

Remarks 2.1.3.1 *For any function* f *in the usual Schwarz class* $\mathcal{S}$, *(2.1.2) implies*

$$\widehat{f}(\xi) = \phi(\xi)\widehat{f}(\xi) + \sum_{j \geq 0} \psi(2^{-j}\xi)\widehat{f}(\xi),$$

or in terms of the inhomogenous dyadic block operators

$$f = \sum_{j \geq -1} \Delta_j f \quad \text{or} \quad Id = \sum_{j \geq -1} \Delta_j,$$

where Id *denotes the identity operator. For generality, for any* F *in the space of tempered distributions* $\mathcal{S}'$,

$$F = \sum_{j \geq -1} \Delta_j F \quad \text{or} \quad Id = \sum_{j \geq -1} \Delta_j \quad \text{in} \quad \mathcal{S}'. \tag{2.1.3}$$

(2.1.3) *is referred to as the Littlewood-Paley decomposition for tempered distributions.*

Definition 2.1.4 *In terms of inhomogeneous dyadic block operators, we can write the standard product in terms of the paraproducts, namely*

$$FG = \sum_{|j-k|<2} S_{k-1}F\Delta_k G + \sum_{|j-k|<2} \Delta_k F S_{k-1} G + \sum_{k \geq j-1} \Delta_k F \widetilde{\Delta}_k G,$$

where $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. *This is the so-called Bony decomposition.*

The following Lemma provides Bernstein type inequalities for fractional derivatives.

Lemma 2.1.5 *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

(1) If f satisfies

$$\operatorname{supp} \widehat{f} \subset \left\{ \xi \in \mathbb{R}^d, \quad |\xi| \leq K 2^j \right\},$$

for some integer j and a constant $K > 0$ then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq c_1 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) If f satisfies

$$\operatorname{supp} \widehat{f} \subset \left\{ \xi \in \mathbb{R}^d, \quad K_1 2^j \leq |\xi| \leq K_2 2^j \right\},$$

for some integer j and constants $0 < K_1 \leq K_2$ then

$$c_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq c_2 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where c_1, c_2 are constants depending only on α, p, q.

In the next Lemma, we state bounds for the triple products involving Fourier localized functions.

Lemma 2.1.6 *Let $j \geq 0$ be an integer. Let Δ_j be the inhomogeneous Littlewood-Paley-localization operator. For any vectors field F, G, H with $\nabla \cdot F = 0$ we have*

$$\begin{aligned} |\int_{\mathbb{R}^d} \Delta_j (F \cdot \nabla G) \cdot \Delta_j H \, dx| \leq c \|\Delta_j H\|_{L^2} \Big(2^j \sum_{m \leq j-1} 2^{\frac{d}{2} m} \|\Delta_m F\|_{L^2} \sum_{|j-k| \leq 2} \|\Delta_k G\|_{L^2} \\ + \sum_{|j-k| \leq 2} \|\Delta_k F\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m G\|_{L^2} + \sum_{k \geq j-1} 2^j 2^{\frac{d}{2} k} \|\Delta_k F\|_{L^2} \|\widetilde{\Delta}_k G\|_{L^2} \Big) \end{aligned}$$

and

$$\begin{aligned} |\int_{\mathbb{R}^d} \Delta_j (F \cdot \nabla G) \cdot \Delta_j G \, dx| \leq c \|\Delta_j G\|_{L^2} \Big(\sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m F\|_{L^2} \sum_{|j-k| \leq 2} \|\Delta_k G\|_{L^2} \\ + \sum_{|j-k| \leq 2} \|\Delta_k F\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m G\|_{L^2} + \sum_{k \geq j-1} 2^j 2^{\frac{d}{2} k} \|\Delta_k F\|_{L^2} \|\widetilde{\Delta}_k G\|_{L^2} \Big). \end{aligned}$$

For a detailed proof of Lemma 2.1.6, we refer the reader to [[21], Lemma 2.3].

2.2 System with Full Fractional Dissipation

Our first result established in [8] can be stated as follows.

Theorem 2.2.1 *Consider (2.1.1) with* $\alpha \geq \frac{1}{2}$ *and* $\beta \geq \frac{1}{2}$*. Assume the initial data* u_0 *and* w_0 *satisfy*

$$\nabla \cdot u_0 = 0, \quad u_0 \in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d), \quad w_0 \in B_{2,1}^{\frac{d}{2}+1-2\beta}(\mathbb{R}^d).$$

Then there exist $T > 0$ *and a unique weak solution* (u, w) *of (2.1.1) on* $[0, T]$ *satisfying*

$$u \in L^\infty(0, T, B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)) \cap L^1(0, T, B_{2,1}^{1+\frac{d}{2}}(\mathbb{R}^d)), \tag{2.2.1}$$

$$w \in L^\infty(0, T, B_{2,1}^{1+\frac{d}{2}-2\beta}(\mathbb{R}^d)) \cap L^1(0, T, B_{2,1}^{1+\frac{d}{2}}(\mathbb{R}^d)). \tag{2.2.2}$$

Here and throughout the present Chapter, $B_{p,q}^r$ denotes the inhomogeneous Besov space. As a direct application of Theorem 2.2.1, the 2D micropolar equations with standard Laplacian dissipation, namely $\alpha = \beta = 1$, have a unique local solution (u, w) in the critical Besov space $L^\infty(0, T; B_{2,1}^0(\mathbb{R}^2))$. Further more, the 3D micropolar equations with the standard Laplacian dissipation, possess a unique local solution in the critical Besov space $L^\infty(0, T; B_{2,1}^{\frac{1}{2}}(\mathbb{R}^3))$.

2.2.1 Local Existence of Weak Solutions

This subsection deals with the proof of the existence part of Theorem 2.2.1. This process starts with the construction of a successive approximation sequence $(u^{(n)}, w^{(n)})$ which iteratively solves systems close to (2.1.1). Then, the next step is to show that this successive approximation sequence is uniformly bounded in suitable Besov spaces via an iterative process. These uniform bounds allow us to extract a subsequence, which converges weakly to a limit. Finally, using the Aubin-Lions Lemma stated in the Appendix A.2, the weak limit is then shown to be the weak solution of the system (2.1.1). This process is detailed as bellow.

Proof. The existence of weak solutions to the system (2.1.1) is proven through a successive approximation process, which starts with the construction of a successive approximation sequence $\{(u^{(n)}, w^{(n)})\}$ satisfying

$$\begin{cases} u^{(1)} = S_2 u_0, \qquad w^{(1)} = S_2 w_0, \\ \partial_t u^{(n+1)} + (\nu + k)(-\Delta)^\alpha u^{(n+1)} = \mathbb{P}(-u^{(n)} \cdot \nabla u^{(n+1)}) + 2k\nabla \times w^{(n)}, \\ \partial_t w^{(n+1)} + \gamma(-\Delta)^\beta w^{(n+1)} = -4kw^{(n+1)} - 2k\nabla \times u^{(n)} - u^{(n)} \cdot \nabla w^{(n+1)}, \\ u^{(n+1)}(x,0) = S_{n+1} u_0, \qquad w^{(n+1)}(x,0) = S_{n+1} w_0, \end{cases} \tag{2.2.3}$$

where $\mathbb{P} := I - \nabla(-\Delta)^{-1}\text{div}$ is the standard Leray Projection. For

$$M = 2(\|u_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + \|w_0\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}}),$$

$T > 0$ sufficiently small and $0 < \delta < 1$ (to be specified later), we set

$$Y \equiv \Big\{ (u,w) \mid \quad \|u\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \le M \ , \ \ \|w\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})} \le M, \\ \|u\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \le \delta \ , \ \ \|w\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \le \delta \Big\}. \tag{2.2.4}$$

We show that $\{(u^{(n)}, w^{(n)})\}$ has a subsequence that converges to the weak solution of (2.1.1). This process is based on three main steps. The first step is to show that $\{(u^{(n)}, w^{(n)})\}$ is uniformly bounded in Y. The second step is to extract a strongly convergent subsequence via the Aubin-Lions Lemma (stated in the Appendix). While the final step is to show that the limit is indeed a weak solution of (2.1.1).

We start by proving the uniform bound for $\{(u^{(n)}, w^{(n)})\}$ in Y by induction. Clearly,

$$\|u^{(1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} = \|S_2 u_0\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \le M,$$
$$\|w^{(1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})} = \|S_2 w_0\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})} \le M.$$

If $T > 0$ is sufficiently small, then

$$\|u^{(1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \le T\|S_2 u_0\|_{B_{2,1}^{1+\frac{d}{2}}} \le T\, c\, \|u_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \le \delta,$$

$$\|w^{(1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \leq T\|S_2 w_0\|_{B_{2,1}^{1+\frac{d}{2}}} \leq T\, c\, \|w_0\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} \leq \delta.$$

Assuming that $(u^{(n)}, w^{(n)})$ obeys the bounds defined in Y, namely

$$\|u^{(n)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M\ ,\ \ \|w^{(n)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})} \leq M,$$
$$\|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \leq \delta\ ,\ \ \|w^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \leq \delta,$$

we prove that $\{(u^{(n+1)}, w^{(n+1)})\}$ obeys the same bound for suitably selected $T > 0$, $M > 0$ and $\delta > 0$. For the sake of clarity, we provide the proof of the four bounds in the following four steps.

Step 1: The estimate of $u^{(n+1)}$ in $B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)$.

Let $j \geq 0$ be an integer. Applying Δ_j to the second equation in (2.2.3) and then dotting with $\Delta_j u^{(n+1)}$, yield

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u^{(n+1)}\|_{L^2}^2 + (\nu + k)\|\Lambda^\alpha \Delta_j u^{(n+1)}\|_{L^2}^2 := A_1 + A_2, \tag{2.2.5}$$

with

$$A_1 := \int_{\mathbb{R}^d} 2k\Delta_j(\nabla \times w^{(n)}) \cdot \Delta_j u^{(n+1)}\, dx,$$
$$A_2 := -\int_{\mathbb{R}^d} \Delta_j(u^{(n)} \cdot \nabla u^{(n+1)})\Delta_j u^{(n+1)}\, dx,$$

where for the sake of conciseness, we denote here and throughout the proof, $\Lambda := (-\Delta)^{\frac{1}{2}}$. Note that the projection operator $\mathbb{P} := I - \nabla(-\Delta)^{-1}\mathrm{div}$ has been eliminated in (2.2.5) due to the divergence-free condition $\nabla \cdot u^{(n+1)} = 0$.

By Lemma 2.1.5, the dissipative part admits a lower bound

$$(\nu + k)\|\Lambda^\alpha \Delta_j u^{(n+1)}\|_{L^2}^2 \geq c_0\, 2^{2\alpha j}\|\Delta_j u^{(n+1)}\|_{L^2}^2,$$

where $c_0 > 0$ is a constant. According to Hölder's inequality and Lemma 2.1.5,

$$
\begin{aligned}
|A_1| &= \Big| \int_{\mathbb{R}^d} 2k\, \Delta_j(\nabla \times w^{(n)}) \cdot \Delta_j u^{(n+1)}\, dx \Big| \\
&\leq 2k\, \|\Delta_j(\nabla \times w^{(n)})\|_{L^2} \|\Delta_j u^{(n+1)}\|_{L^2} \\
&\leq c\, 2^j\, \|\Delta_j w^{(n)}\|_{L^2} \|\Delta_j u^{(n+1)}\|_{L^2}.
\end{aligned} \tag{2.2.6}
$$

Due to Lemma 2.1.6,

$$
\begin{aligned}
|A_2| &= \Big| - \int_{\mathbb{R}^d} \Delta_j(u^{(n)} \cdot \nabla u^{(n+1)}) \cdot \Delta_j u^{(n+1)}\, dx \Big| \\
&\leq c\, \|\Delta_j u^{(n+1)}\|_{L^2}^2 \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2} \\
&\quad + c\, \|\Delta_j u^{(n+1)}\|_{L^2} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n+1)}\|_{L^2} \\
&\quad + c \sum_{k \geq j-1} 2^j 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} \|\Delta_j u^{(n+1)}\|_{L^2}.
\end{aligned} \tag{2.2.7}
$$

Inserting (2.2.6) and (2.2.7) in (2.2.5) and eliminating $\|\Delta_j u^{(n+1)}\|_{L^2}$ from the both sides, we get

$$
\frac{d}{dt} \|\Delta_j u^{(n+1)}\|_{L^2} + c_0 2^{2\alpha j} \|\Delta_j u^{(n+1)}\|_{L^2} \leq J_1 + J_2 + J_3 + J_4, \tag{2.2.8}
$$

where

$$
\begin{aligned}
J_1 &:= c\, \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2}, \\
J_2 &:= c\, \|\Delta_j u^{(n)}\|_{L^2} \sum_{m \leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n+1)}\|_{L^2}, \\
J_3 &:= c\, 2^j \sum_{k \geq j-1} 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2}, \\
J_4 &:= c\, 2^j \|\Delta_j w^{(n)}\|_{L^2}.
\end{aligned}
$$

Integrating the estimate (2.2.8) in time yields

$$
\|\Delta_j u^{(n+1)}\|_{L^2} \leq e^{-c_0 2^{2\alpha j} t} \|\Delta_j u_0^{(n+1)}\|_{L^2} + \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} (J_1 + \cdots + J_4)\, d\tau. \tag{2.2.9}
$$

Multiplying (2.2.9) by $2^{(1+\frac{d}{2}-2\alpha)j}$ and summing over j, we obtain

$$\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \le \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + \sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} (J_1 + \cdots + J_4)\, d\tau. \tag{2.2.10}$$

Using the inductive assumption on $u^{(n)}$, the term with J_1 on the right hand side of (2.2.10) can be bounded for any $t \le T$ by,

$$\begin{aligned}
&\sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_1\, d\tau \\
&\qquad \le c \int_0^t \sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \\
&\qquad \le c \int_0^t \|u^{(n+1)}(\tau)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \|u^{(n)}(\tau)\|_{B^{1+\frac{d}{2}}_{2,1}}\, d\tau \\
&\qquad \le c\, \|u^{(n+1)}\|_{L^\infty(0,t,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \|u^{(n)}\|_{L^1(0,t,B^{1+\frac{d}{2}}_{2,1})} \\
&\qquad \le c\, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \le c\,\delta\, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})}.
\end{aligned} \tag{2.2.11}$$

By Young's inequality for series convolution, the term involving J_2 admits the following bound

$$\begin{aligned}
&\sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_2\, d\tau \\
&\qquad \le c \int_0^t \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j} 2^{2\alpha(m-j)} 2^{(1+\frac{d}{2}-2\alpha)m} \|\Delta_m u^{(n+1)}(\tau)\|_{L^2}\, d\tau \\
&\qquad \le c \int_0^t \|u^{(n)}(\tau)\|_{B^{1+\frac{d}{2}}_{2,1}} \|u^{(n+1)}(\tau)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}\, d\tau \\
&\qquad \le c \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \le c\,\delta\, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})}.
\end{aligned} \tag{2.2.12}$$

Similarly, the term associated with J_3 obeys the same bound,

$$\sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_3\, d\tau$$

$$\begin{aligned}
&= \int_0^t \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \sum_{k\geq j-1} c\, 2^j 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2}\, d\tau \\
&= c\int_0^t \sum_{j\geq -1}\sum_{k\geq j-1} 2^{(2+\frac{d}{2}-2\alpha)(j-k)} 2^{(1+\frac{d}{2})k} \|\Delta_k u^{(n)}\|_{L^2} 2^{(1+\frac{d}{2}-2\alpha)k} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\, d\tau \\
&\leq c\int_0^t \|u^{(n)}(\tau)\|_{B^{1+\frac{d}{2}}_{2,1}} \|u^{(n+1)}(\tau)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}\, d\tau \\
&\leq c\, \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \leq c\ \delta\, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})}.
\end{aligned} \tag{2.2.13}$$

It remains to bound the term with J_4,

$$\sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_4\, d\tau$$

$$\begin{aligned}
&= \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} c\, 2^j \|\Delta_j w^{(n)}\|_{L^2}\, d\tau \\
&\leq \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t c\, 2^j\, \|\Delta_j w^{(n)}\|_{L^2}\, d\tau = c\int_0^t \sum_{j\geq -1} 2^{(2+\frac{d}{2}-2\alpha)j} \|\Delta_j w^{(n)}\|_{L^2}\, d\tau \\
&\underset{\text{since } \alpha\geq\frac{1}{2}}{\leq} c\int_0^t \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j w^{(n)}\|_{L^2}\, d\tau = c\, \|w^{(n)}\|_{L^1(0,t,B^{1+\frac{d}{2}}_{2,1})} \leq c\,\delta.
\end{aligned} \tag{2.2.14}$$

Combining the bounds in (2.2.11), (2.2.12), (2.2.13), (2.2.14) and inserting them in (2.2.10), we have for any $t \leq T$

$$\|u^{(n+1)}(t)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} \leq \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + c\,\delta\, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} + c\,\delta.$$

It follows,

$$\|u^{(n+1)}(t)\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \leq \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + c\,\delta\, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} + c\,\delta.$$

Choosing $\delta > 0$ such that $c\,\delta \leq \min(\frac{1}{4}, \frac{M}{4})$ we obtain

$$\|u^{(n+1)}(t)\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \leq \frac{M}{2} + \frac{1}{4}\|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} + \frac{M}{4},$$

which gives

$$\|u^{(n+1)}(t)\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \le M.$$

Step 2: The estimate of $\|u^{(n+1)}\|_{L^1(0,T,B^{1+\frac{d}{2}})}$.

We multiply (2.2.9) by $2^{(1+\frac{d}{2})j}$, sum over j and integrate in time to get

$$\begin{aligned}\|u^{(n+1)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \le &\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} e^{-c_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\,dt \\ &+\int_0^T\int_0^s \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} e^{-c_0 2^{2\alpha j}(s-\tau)}(J_1+\cdots+J_4)\,d\tau\,ds.\end{aligned} \tag{2.2.15}$$

We estimate the terms on the right hand side of (2.2.15) and start with the first term.

$$\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} e^{-c_0 2^{2\alpha j}t}\|\Delta_j u_0^{n+1}\|_{L^2}\,dt = c\sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j}(1-e^{-c_0 2^{2\alpha j}T})\|\Delta_j u_0^{(n+1)}\|_{L^2}.$$

Since $u_0 \in B^{(1+\frac{d}{2}-2\alpha)}_{2,1}$, it follows from the Dominated convergence Theorem that

$$\lim_{T\to 0}\sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j}(1-e^{-c_0 2^{2\alpha j}T})\|\Delta_j u_0^{(n+1)}\|_{L^2} = 0.$$

Hence, we can choose T sufficiently small such that

$$\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} e^{-c_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\,dt \le \frac{\delta}{4}. \tag{2.2.16}$$

Due to Young's inequality for the time convolution, the term invoking J_1 can be bounded as follows

$$\begin{aligned}&\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j}\int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_1\,d\tau\,ds \\ &\quad = c\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j}\int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)}\|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(m)}\|_{L^2}\,d\tau\,ds \\ &\quad \le c\sum_{j\ge -1} 2^{(1+\frac{d}{2})j}\int_0^T \|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(m)}\|_{L^2}\,d\tau\int_0^T e^{-c_0 2^{2\alpha j}s}\,ds.\end{aligned}$$

Further, using the fact that there exists $c_2 > 0$ satisfying for $j \geq 0$,

$$\int_0^T e^{-c_0 2^{2\alpha j} s} ds \leq c\, 2^{-2\alpha j}(1 - e^{-c_2 T}), \tag{2.2.17}$$

we obtain

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_1 \, d\tau \, ds \\
&\qquad \leq c\,(1 - e^{-c_2 T}) \int_0^T \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(m)}\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1 - e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \\
&\qquad \leq c\,\delta\,(1 - e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})}.
\end{aligned} \tag{2.2.18}$$

The terms related to J_2 and J_3 obey the same bound,

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_2 \, d\tau \, ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n+1)}\|_{L^2} \, d\tau \, ds \\
&\quad \leq c \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^T \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n+1)}\|_{L^2} \, d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds.
\end{aligned}$$

Then, due to (2.2.17) and the above inequality, we get

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_2 \, d\tau \, ds \\
&\qquad \leq c(1 - e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n+1)}\|_{L^2} \, d\tau \\
&\qquad \leq c(1 - e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\qquad \leq c(1 - e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \\
&\qquad \leq c(1 - e^{-c_2 T})\, \delta\, \|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})}.
\end{aligned} \tag{2.2.19}$$

Similarly,

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau \, ds \\
&\qquad = c \int_0^T \sum_{j\geq -1} 2^{(2+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} \, d\tau \, ds \\
&\qquad \leq c \sum_{j\geq -1} 2^{(2+\frac{d}{2})j} \int_0^T \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds.
\end{aligned}$$

Then, owing to (2.2.17) and the above inequality, we obtain

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau \, ds \\
&\qquad \leq c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{(2+\frac{d}{2}-2\alpha)j} \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{(2+\frac{d}{2}-2\alpha)j} \|\widetilde{\Delta}_j u^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{(\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \|\widetilde{\Delta}_j u^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \\
&\qquad \leq c\,\delta\,(1-e^{-c_2 T})\; \|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})}. \qquad (2.2.20)
\end{aligned}$$

It remains to bound the term associated with J_4,

$$\begin{aligned}
\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_4 \, d\tau \, ds &= c \int_0^T \sum_{j\geq -1} 2^{(2+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \, ds \\
&\leq c \sum_{j\geq -1} 2^{(2+\frac{d}{2})j} \int_0^T \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds \\
&\leq c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{(2+\frac{d}{2}-2\alpha)j} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \\
&\underset{\text{since}\,\alpha\geq\frac{1}{2}}{\leq} c\,(1-e^{-c_2 T}) \underbrace{\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau}_{=\|w^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})}} \\
&\leq c\,\delta\,(1-e^{-c_2 T}). \qquad (2.2.21)
\end{aligned}$$

Inserting the estimates (2.2.16), (2.2.18), (2.2.19), (2.2.20), (2.2.21) in (2.2.15) yields,

$$\begin{aligned}\|u^{(n+1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} &\le \frac{\delta}{4} + c\,\delta(1-e^{-c_2T})\|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}})} + c\,\delta\,(1-e^{-c_2T}) \\ &\le \frac{\delta}{4} + c\,\delta(1-e^{-c_2T})M + c\,\delta\,(1-e^{-c_2T}).\end{aligned}$$

Hence, choosing T sufficiently small such that $c\,(1-e^{-c_2T}) \le \min(\frac{1}{4M}, \frac{1}{2})$, we obtain

$$\|u^{(n+1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \le \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{2} = \delta.$$

Step 3: The estimate of $w^{(n+1)}$ in $L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}(\mathbb{R}^d))$.

Applying Δ_j to the third equation in (2.2.3) and then dotting with $\Delta_j w^{(n+1)}$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta_j w^{(n+1)}\|_{L^2}^2 &+ (c_1 2^{2\beta j} + 4k)\|\Delta_j w^{(n+1)}\|_{L^2}^2 \\ &\le -2k\int \Delta_j(\nabla\times u^{(n)})\Delta_j w^{(n+1)}\,dx \\ &\quad - \int \Delta_j(u^{(n)}\cdot\nabla w^{(n+1)})\Delta_j w^{(n+1)}\,dx \\ &:= B_1 + B_2, \end{aligned} \tag{2.2.22}$$

where

$$\begin{aligned}B_1 &:= -2k\int \Delta_j(\nabla\times u^{(n)})\Delta_j w^{(n+1)}\,dx, \\ B_2 &:= -\int \Delta_j(u^{(n)}\cdot\nabla w^{(n+1)})\Delta_j w^{(n+1)}\,dx.\end{aligned}$$

By Hölder's inequality and Lemma 2.1.5, B_1 can be bounded by

$$\begin{aligned}|B_1| &:= |-2k\int \Delta_j(\nabla\times u^{(n)})\Delta_j w^{(n+1)}\,dx| \\ &\le 2k\,\|\Delta_j(\nabla\times u^{(n)})\|_{L^2}\|\Delta_j w^{(n+1)}\|_{L^2} \\ &\le c\,2^j\,\|\Delta_j u^{(n)}\|_{L^2}\|\Delta_j w^{(n+1)}\|_{L^2}. \end{aligned} \tag{2.2.23}$$

Thanks to Lemma 2.1.6, B_2 is bounded by

$$
\begin{aligned}
|B_2| &:= |-\int \Delta_j(u^{(n)}\cdot\nabla w^{(n+1)})\Delta_j w^{(n+1)}\,dx| \\
&\le c\,\|\Delta_j w^{(n+1)}\|_{L^2}^2 \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2} \\
&+ c\,\|\Delta_j w^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j} 2^{(1+\frac{d}{2})m}\|\Delta_m w^{(n+1)}\|_{L^2} \\
&+ c\,\|\Delta_j w^{(n+1)}\|_{L^2} 2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}. \qquad (2.2.24)
\end{aligned}
$$

Inserting (2.2.23), (2.2.24) in (2.2.22) and eliminating $\|\Delta_j w^{(n+1)}\|_{L^2}$ from both sides of the inequality, we get

$$
\frac{d}{dt}\|\Delta_j w^{(n+1)}\|_{L^2} + (c_1 2^{2\beta j} + 4k)\|\Delta_j w^{(n+1)}\|_{L^2} \le K_1 + K_2 + K_3 + K_4, \qquad (2.2.25)
$$

where

$$
\begin{aligned}
K_1 &= c\,2^j\,\|\Delta_j u^{(n)})\|_{L^2}, \\
K_2 &= c\,\|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}, \\
K_3 &= c\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j} 2^{(1+\frac{d}{2})m}\|\Delta_m w^{(n+1)}\|_{L^2}, \\
K_4 &= c \sum_{k\ge j-1} 2^j 2^{\frac{d}{2}k}\|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}.
\end{aligned}
$$

Integrating (2.2.25) in time yields, for any $t\le T$,

$$
\|\Delta_j w^{(n+1)}\|_{L^2} \le e^{-(c_1 2^{2\beta j})t}\|\Delta_j w_0^{(n+1)}\|_{L^2} + \int_0^t e^{-(c_1 2^{2\beta j})(t-\tau)}(K_1+\cdots+K_4)\,d\tau. \qquad (2.2.26)
$$

Multiplying (2.2.26) by $2^{(1+\frac{d}{2}-2\beta)j}$ and summing over j, we have

$$
\|w^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} \le \|w_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} + \sum_{j\ge -1}\int_0^t e^{-(c_1 2^{2\beta j})(t-\tau)} 2^{(1+\frac{d}{2}-2\beta)j}(K_1+\cdots+K_4)d\tau. \qquad (2.2.27)
$$

The terms invoking K_1 through K_4 on the right hand side of (2.2.27) can be estimated suitably as follows. Starting with the term containing K_1, we have

$$\begin{aligned}\sum_{j\geq -1}\int_0^t 2^{(1+\frac{d}{2}-2\beta)j}K_1\,d\tau &= \int_0^t \sum_{j\geq -1} c\,2^{(2+\frac{d}{2}-2\beta)j}\|\Delta_j u^{(n)}(\tau)\|_{L^2}\,d\tau\\ &\underset{\text{since }\beta\geq\frac{1}{2}}{\leq} c\underbrace{\int_0^t\sum_{j\geq -1}2^{(1+\frac{d}{2})j}\|\Delta_j u^{(n)}(\tau)\|_{L^2}\,d\tau}_{=\|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})}} \leq c\,\delta. \qquad (2.2.28)\end{aligned}$$

The term with K_2 admits the following bound

$$\begin{aligned}\sum_{j\geq -1}2^{(1+\frac{d}{2}-2\beta)j}\int_0^t K_2\,d\tau &= c\int_0^t\sum_{j\geq -1}2^{(1+\frac{d}{2}-2\beta)j}\|\Delta_j w^{(n+1)}\|_{L^2}\sum_{m\leq j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}\,d\tau\\ &\leq c\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\beta}_{2,1})}\|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})}\\ &\leq c\,\delta\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\beta}_{2,1})}. \qquad (2.2.29)\end{aligned}$$

Similarly, the terms associated with K_3 and K_4 obey the same bound. In fact, for the term with K_3, we write

$$\begin{aligned}\sum_{j\geq -1}2^{(1+\frac{d}{2}-2\beta)j}\int_0^t K_3\,d\tau &= c\int_0^t\sum_{j\geq -1}2^{(1+\frac{d}{2}-2\beta)j}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\leq j}2^{(1+\frac{d}{2})m}\|\Delta_m w^{(n+1)}\|_{L^2}\,d\tau\\ &\leq c\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\beta}_{2,1})}\|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})}\\ &\leq c\,\delta\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\beta}_{2,1})}. \qquad (2.2.30)\end{aligned}$$

Also, for the term with K_4, we have

$$\begin{aligned}\sum_{j\geq -1}2^{(1+\frac{d}{2}-2\beta)j}\int_0^t K_4\,d\tau &= c\int_0^t\sum_{j\geq -1}\sum_{k\geq j-1}2^{(2+\frac{d}{2}-2\beta)j}2^{\frac{d}{2}k}\|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}\,d\tau\\ &\leq c\int_0^t\sum_{j\geq -1}\sum_{j\geq -1}2^{(2+\frac{d}{2}-2\beta)j}2^{\frac{d}{2}j}\|\widetilde{\Delta}_j w^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2}\,d\tau\\ &= c\int_0^t\sum_{j\geq -1}\sum_{j\geq -1}2^{(1+\frac{d}{2}-2\beta)j}2^{(1+\frac{d}{2})j}\|\widetilde{\Delta}_j w^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2}\,d\tau\\ &\leq c\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\beta}_{2,1})}\|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})}\\ &\leq c\,\delta\|w^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\beta}_{2,1})}. \qquad (2.2.31)\end{aligned}$$

Collecting the estimates (2.2.28), (2.2.29), (2.2.30), (2.2.31) and inserting them in (2.2.27), we get for any $t \le T$,

$$\|w^{(n+1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} \le \|w_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} + c\,\delta + c\,\delta \|w^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})}.$$

Then, choosing $c\,\delta \le \min(\frac{1}{4}, \frac{M}{4})$, we obtain

$$\|w^{(n+1)}(t)\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})} \le \frac{M}{2} + \frac{M}{4} + \frac{1}{4}\|w^{(n+1)}(t)\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})},$$

which implies

$$\|w^{(n+1)}(t)\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})} \le M.$$

Step 4: The estimate of $\|w^{(n+1)}(t)\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})}$.

We multiply (2.2.26) by $2^{(1+\frac{d}{2})j}$, sum over j and integrate in time to get

$$\begin{aligned}\|w^{(n+1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \le &\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} e^{-c_1 2^{2\beta j} t} \|\Delta_j w_0^{(n+1)}\|_{L^2}\, dt \\ &+ \int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} (K_1 + \cdots + K_4)\, d\tau\, ds.\end{aligned} \tag{2.2.32}$$

Clearly,

$$\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} e^{-c_1 2^{2\beta j} t} \|\Delta_j w_0^{(n+1)}\|_{L^2}\, dt = c \sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\beta)j} (1 - e^{-c_1 2^{2\beta j} T}) \|\Delta_j w_0^{(n+1)}\|_{L^2}.$$

Since $w_0 \in B_{2,1}^{1+\frac{d}{2}-2\beta}$, we have by the Dominated Convergence Theorem,

$$\lim_{T\to 0} \sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\beta)j} (1 - e^{-c_1 2^{2\beta j} T}) \|\Delta_j w_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small so that

$$\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} e^{-c_1 2^{2\beta j} t} \|\Delta_j w_0^{(n+1)}\|_{L^2}\, dt \le \frac{\delta}{2}. \tag{2.2.33}$$

Applying Young's inequality for the time convolution, the term associated with K_1 can be bounded by

$$\begin{aligned}\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_1\, d\tau\, ds &\leq c \int_0^T \sum_{j\geq -1} 2^{(2+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} \|\Delta_j u^{(n)}\|_{L^2}\, d\tau\, ds \\ &\leq c \sum_{j\geq -1} 2^{(2+\frac{d}{2})j} \int_0^T \|\Delta_j u^{(n)}\|_{L^2}\, d\tau \cdot \int_0^T e^{-c_1 2^{2\beta j} s}\, ds.\end{aligned}$$

Using the fact that there exists $c_3 > 0$ satisfying for all $j \geq 0$,

$$\int_0^T e^{-c_1 2^{2\beta j} s} ds \leq c\, 2^{-2\beta j} (1 - e^{-c_3 T})\,, \tag{2.2.34}$$

we obtain

$$\begin{aligned}\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_1\, d\tau\, ds &\leq c\,(1 - e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{(2+\frac{d}{2}-2\beta)j} \|\Delta_j u^{(n)}(\tau)\|_{L^2}\, d\tau \\ &\leq c\,(1 - e^{-c_3 T}) \underbrace{\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}(\tau)\|_{L^2}\, d\tau}_{=\|u^{(n+1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})}} \\ &\leq c\,\delta\,(1 - e^{-c_3 T}).\end{aligned} \tag{2.2.35}$$

Due to Young's inequality for the time convolution, the term invoking K_2 is bounded by

$$\begin{aligned}&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_2\, d\tau\, ds \\ &\quad = c \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} c\,\|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau\, ds \\ &\quad \leq c \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^T \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \Big(\int_0^T e^{-c_1 2^{2\beta j} s} ds\Big) \\ &\quad \leq c\,(1 - e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\beta)j} \|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \\ &\quad \leq c\,(1 - e^{-c_3 T}) \underbrace{\|w^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})}}_{\leq M} \underbrace{\|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})}}_{\leq \delta} \\ &\quad \leq c\,\delta M\,(1 - e^{-c_3 T}).\end{aligned} \tag{2.2.36}$$

The terms related to K_3 and K_4 admit also the same bound,

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_3 \, d\tau \, ds \\
&\qquad = c\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} c \, \|\Delta_j u^{(n)}(\tau)\|_{L^2} \Big(\sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m w^{(n+1)}(\tau)\|_{L^2} \Big) \, d\tau ds \\
&\qquad \leq c \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^T \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m w^{(n+1)}(\tau)\|_{L^2} \, d\tau \Big(\int_0^T e^{-c_1 2^{2\beta j} s} ds \Big)
\end{aligned}$$

Hence, thanks to (2.2.34) and the above estimate, we have

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_3 \, d\tau \, ds \\
&\qquad \leq c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\beta)j} \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m w^{(n+1)}(\tau)\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\beta)j} \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \, d\tau \\
&\qquad = c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\beta)j} \|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1-e^{-c_3 T}) \underbrace{\|w^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})}}_{\leq M} \underbrace{\|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})}}_{\leq \delta} \\
&\qquad \leq c\,\delta M\,(1-e^{-c_3 T}).
\end{aligned} \tag{2.2.37}$$

It remains to bound the term containing K_4,

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_4 \, d\tau \, ds \\
&\qquad = c\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} 2^j 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} \, d\tau ds \\
&\qquad \leq c \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^T \sum_{k\geq j-1} 2^j 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} \, d\tau \Big(\int_0^T e^{-c_1 2^{2\beta j} s} ds \Big).
\end{aligned}$$

Then, owing to (2.2.34) and the above estimate, we get

$$
\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_1 2^{2\beta j}(s-\tau)} K_4 \, d\tau \, ds \\
&\qquad \leq c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\beta)j} 2^j \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} \, d\tau \\
&\qquad = c\,(1-e^{-c_3 T}) \int_0^T \sum_{j\geq -1} \sum_{k\geq j-1} 2^{(1+\frac{d}{2}-2\beta)(j-k)} 2^{(1+\frac{d}{2})k} \|\Delta_k u^{(n)}\|_{L^2} 2^{(1+\frac{d}{2}-2\beta)k} \|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2} \, d\tau \\
&\qquad = c\,(1-e^{-c_3 T}) \int_0^T \|u^{(n)}\|_{B_{2,1}^{1+\frac{d}{2}}} \|w^{(n+1)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} \, d\tau \\
&\qquad \leq c\,(1-e^{-c_3 T}) \underbrace{\|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})}}_{\leq \delta} \underbrace{\|w^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})}}_{\leq M} \\
&\qquad \leq c\,\delta M\,(1-e^{-c_3 T}). \qquad\qquad (2.2.38)
\end{aligned}
$$

Combining the bounds (2.2.33), (2.2.35), (2.2.36), (2.2.37), (2.2.38) and inserting them in (2.2.32), yield

$$
\|w^{(n+1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \leq \frac{\delta}{2} + c\,(1-e^{-c_3 T})\delta + c\,(1-e^{-c_3 T})\delta M.
$$

Therefore, choosing T sufficiently small such that $c\,(1-e^{-c_3 T}) \leq \min(\frac{1}{4M}, \frac{1}{4})$, we obtain,

$$
\|w^{(n+1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \leq \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta.
$$

These uniform bounds allow us to extract a weakly convergent subsequence. That is, there is $(u,w) \in Y$ such that a subsequence of (u^n, w^n) (still denoted by (u^n, w^n)) satisfies

$$
\begin{aligned}
u^n &\overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}), \\
w^n &\overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}).
\end{aligned}
$$

In order to show that (u,w) is a weak solution of (2.1.1) we need to further extract a subsequence which converges strongly to (u,w). We use the Aubin-Lions Lemma.

We can show by making use of the equation (2.2.3) that $(\partial_t u_n, \partial_t w_n)$ is uniformly bounded in

$$\partial_t u^n \in L^1(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}) \cap L^2(0,T,B_{2,1}^{\frac{d}{2}+1-3\alpha}),$$
$$\partial_t w^n \in L^1(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta}) \cap L^2(0,T,B_{2,1}^{\frac{d}{2}+1-3\beta}).$$

Since we are in this case in the whole space $\mathbb{R}^d$, we need to combine Cantor's diagonal process with the Aubin-Lions Lemma to show that a subsequence of a weakly convergent subsequence, still denoted by (u^n, w^n), has the following strongly convergent property

$$u^n \longrightarrow u \quad \text{in} \quad L^2(0,T,B_{2,1}^{1+\frac{d}{2}-\gamma_1}(Q)), \quad w^n \longrightarrow w \quad \text{in} \quad L^2(0,T,B_{2,1}^{1+\frac{d}{2}-\gamma_2}(Q)),$$

where $\alpha \le \gamma_1 \le 3\alpha$, $\beta \le \gamma_2 \le 3\beta$ and $Q \subset \mathbb{R}^d$ is a compact subset. This strong convergence property would allow us to show that (u,w) is indeed a weak solution of (2.1.1). This completes the proof for the existence part of Theorem 2.2.1. ■

2.2.2 Uniqueness of Weak Solutions

In this subsection we present the proof of the uniqueness part of Theorem 2.2.1.

Proof. Assume that $(u^{(1)}, w^{(1)})$ and $(u^{(2)}, w^{(2)})$ are two solutions of (2.1.1) in the regularity class in (2.3.2) and (2.3.3). Their difference $(\widetilde{u}, \widetilde{w})$ with

$$\widetilde{u} = u^{(2)} - u^{(1)} \quad \text{and} \quad \widetilde{w} = w^{(2)} - w^{(1)}$$

satisfies

$$\begin{cases} \partial_t \widetilde{u} + (\nu + k)(-\Delta)^\alpha \widetilde{u} = -\mathbb{P}(u^{(2)} \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u^{(1)}) + 2k \nabla \times \widetilde{w}, \\ \partial_t \widetilde{w} + \gamma(-\Delta)^\beta \widetilde{w} = -4k\widetilde{w} - 2k\nabla \times \widetilde{u} - u^{(2)} \cdot \nabla \widetilde{w} - \widetilde{u} \cdot \nabla w^{(1)}, \\ \nabla \cdot \widetilde{u} = 0, \\ \widetilde{u}(x,0) = 0, \quad \widetilde{w}(x,0) = 0. \end{cases} \tag{2.2.39}$$

We estimate the difference $(\widetilde{u}, \widetilde{w})$ in $L^2(\mathbb{R}^d)$. Dotting (2.2.39) by $(\widetilde{u}, \widetilde{w})$ and applying the divergence-free condition $\nabla \cdot \widetilde{u} = 0$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{w}\|_{L^2}^2\Big) + (\nu + k)\|\Lambda^\alpha \widetilde{u}\|_{L^2}^2 + \gamma\|\Lambda^\beta \widetilde{w}\|_{L^2}^2 + 4k\|\widetilde{w}\|_{L^2}^2 \\
&\quad = -\int u^{(2)} \cdot \nabla \widetilde{u} \cdot \widetilde{u}\, dx - \int \widetilde{u} \cdot \nabla u^{(1)} \cdot \widetilde{u}\, dx \\
&\qquad - \int u^{(2)} \cdot \nabla \widetilde{w} \cdot \widetilde{w}\, dx - \int \widetilde{u} \cdot \nabla w^{(1)} \cdot \widetilde{w}\, dx \\
&\quad := L_1 + L_2 + L_3 + L_4,
\end{aligned}
$$

with

$$
\begin{aligned}
L_1 &:= -\int u^{(2)} \cdot \nabla \widetilde{u} \cdot \widetilde{u}\, dx, \\
L_2 &:= -\int \widetilde{u} \cdot \nabla u^{(1)} \cdot \widetilde{u}\, dx, \\
L_3 &:= -\int u^{(2)} \cdot \nabla \widetilde{w} \cdot \widetilde{w}\, dx, \\
L_4 &:= -\int \widetilde{u} \cdot \nabla w^{(1)} \cdot \widetilde{w}\, dx,
\end{aligned}
$$

where we recall that $\Lambda := (-\Delta)^{\frac{1}{2}}$.

As $\nabla \cdot u^{(2)} = 0$, it is easy to show via integration by parts that $L_1 = 0$. In fact,

$$
\begin{aligned}
L_1 &:= -\int u^{(2)} \cdot \nabla \widetilde{u} \cdot \widetilde{u}\, dx \\
&= -\int u^{(2)} \cdot \nabla (\frac{1}{2}|\widetilde{u}|^2)\, dx \\
&= -\int \nabla \cdot (u^{(2)} \frac{1}{2}|\widetilde{u}|^2)\, dx \\
&= 0.
\end{aligned} \tag{2.2.40}
$$

By the same reason,

$$
L_3 := -\int u^{(2)} \cdot \nabla \widetilde{w} \cdot \widetilde{w}\, dx = 0. \tag{2.2.41}
$$

Applying Hölder's inequality and Lemma 2.1.5,

$$
\begin{aligned}
|L_2| &= |-\int \widetilde{u}\cdot\nabla u^{(1)}\cdot\widetilde{u}\,dx| \\
&\le \|\nabla u^{(1)}\|_{L^\infty}\|\widetilde{u}\|_{L^2}^2 \\
&\le \sum_{j\ge -1}\|\Delta_j\nabla u^{(1)}\|_{L^\infty}\|\widetilde{u}\|_{L^2}^2 \\
&\le c\underbrace{\sum_{j\ge -1}2^{dj(\frac{1}{2}-\frac{1}{\infty})}2^j\|\Delta_j u^{(1)}\|_{L^2}}_{=\|u^{(1)}\|_{B_{2,1}^{1+\frac{d}{2}}}}\|\widetilde{u}\|_{L^2}^2 \le c\,\|u^{(1)}\|_{B_{2,1}^{1+\frac{d}{2}}}\|\widetilde{u}\|_{L^2}^2. \qquad (2.2.42)
\end{aligned}
$$

To bound L_4, we set

$$
\frac{1}{p}=\frac{1}{2}-\frac{\beta}{d}\,,\quad \frac{1}{q}=\frac{\beta}{d}\ \ (or\ \frac{d}{q}=\beta).
$$

Due to Hölder's inequality and Lemma 2.1.5,

$$
\begin{aligned}
|L_4| &= |-\int \widetilde{u}\cdot\nabla w^{(1)}\cdot\widetilde{w}\,dx| \\
&\le \|\widetilde{u}\|_{L^2}\|\nabla w^{(1)}\|_{L^q}\|\widetilde{w}\|_{L^p} \\
&\le \sum_{j\ge -1}\|\Delta_j\nabla w^{(1)}\|_{L^q}\|\widetilde{u}\|_{L^2}\|\widetilde{w}\|_{L^p} \\
&\le c\sum_{j\ge -1}2^j2^{dj(\frac{1}{2}-\frac{1}{q})}\|\Delta_j w^{(1)}\|_{L^2}\|\widetilde{u}\|_{L^2}\|\widetilde{w}\|_{L^p} \\
&\le \sum_{j\ge -1}2^{j+\frac{dj}{2}-\beta j}\|\Delta_j w^{(1)}\|_{L^2}\|\widetilde{w}\|_{L^p}\|\widetilde{u}\|_{L^2} \\
&\le c\,\|w^{(1)}\|_{B_{2,1}^{1+\frac{d}{2}-\beta}}\|\widetilde{u}\|_{L^2}\|\Lambda^\beta\widetilde{w}\|_{L^2} \\
&\le \frac{\gamma}{2}\|\Lambda^\beta\widetilde{w}\|_{L^2}^2 + c\,\|w^{(1)}\|_{B_{2,1}^{1+\frac{d}{2}-\beta}}^2\|\widetilde{u}\|_{L^2}^2, \qquad (2.2.43)
\end{aligned}
$$

where in the last inequality we have made use of the Sobolev's inequality,

$$
\|\widetilde{w}\|_{L^p}\le c\,\|\Lambda^\beta\widetilde{w}\|_{L^2}.
$$

Collecting the estimates (2.2.40), (2.2.41), (2.2.42) and (2.2.43) yields,

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{w}\|_{L^2}^2\Big)+2(\nu+k)\|\Lambda^\alpha\widetilde{u}\|_{L^2}^2+\gamma\,\|\Lambda^\beta\widetilde{w}\|_{L^2}^2+8k\,\|\widetilde{w}\|_{L^2}^2 \\
&\qquad \le \Big(c\,\|u^{(1)}\|_{B_{2,1}^{1+\frac{d}{2}}}+c\,\|w^{(1)}\|_{B_{2,1}^{1+\frac{d}{2}-\beta}}^2\Big)\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{w}\|_{L^2}^2\Big). \qquad (2.2.44)
\end{aligned}
$$

Since $u^{(1)} \in L^1(0,T,B_{2,1}^{1+\frac{d}{2}})$ and $w^{(1)} \in L^1(0,T,B_{2,1}^{1+\frac{d}{2}}) \cap L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})$,

$$\int_0^T \|u^{(1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}}} dt < \infty,$$

$$\begin{aligned}\int_0^T \|w^{(1)}(t)\|^2_{B_{2,1}^{1+\frac{d}{2}-\beta}} dt &\le \int_0^T \|w^{(1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}}} \|w^{(1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}-2\beta}} dt \\ &\le \|w^{(1)}(t)\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\beta})} \int_0^T \|w^{(1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}}} dt < \infty.\end{aligned}$$

Applying Gronwall's inequality to (2.2.44), we get

$$\|\widetilde{u}\|_{L^2} = \|\widetilde{w}\|_{L^2} = 0,$$

which leads to the desired uniqueness. This completes the proof of the uniqueness part of Theorem 2.2.1. ■

2.3 System with no Diffusion for the Angular Velocity

Our second main result established in [8] is the following.

Theorem 2.3.1 *Consider (2.1.1) with $\alpha \ge 1$ and $\beta = 0$. Assume the initial data (u_0, w_0) satisfies $\nabla \cdot u_0 = 0$, and is in the following Besov spaces*

$$u_0 \in B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d), \quad w_0 \in B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d). \tag{2.3.1}$$

Then there exist $T > 0$ and a unique weak solution (u, w) of (2.1.1) on $[0, T]$ satisfying

$$u \in L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)) \cap L^1(0,T,B_{2,1}^{1+\frac{d}{2}}(\mathbb{R}^d)), \tag{2.3.2}$$

$$w \in L^\infty(0,T,B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)). \tag{2.3.3}$$

In comparison with Theorem 2.2.1, the Besov space of u remains the same, however due to the lack of diffusion in the equation of the angular velocity w, the setting for w needs to be in a more regular Besov space. In fact, the regularity index $\frac{d}{2}$ in the Besov space $B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)$ of w can not be lowered in order to obtain the uniqueness of solutions.

This section is divided into two subsections. Subsection 2.3.1 deals with the existence part of Theorem 2.3.1. Subsection 2.3.2 is devoted to the uniqueness part.

Before starting the proof of Theorem 2.3.1, we recall that we are concerned with the system of equations (2.1.1) with $\beta = 0$, namely

$$\begin{cases} \partial_t u + (\nu + k)(-\Delta)^\alpha u + u \cdot \nabla u + \nabla \Pi - 2k \nabla \times w = 0, \\ \partial_t w + (4k + \gamma) w + 2k \nabla \times u + u \cdot \nabla w = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \;\; w(x,0) = w_0(x). \end{cases} \tag{2.3.4}$$

2.3.1 Local Existence of Weak Solutions

This subsection is concerned with the existence part of Theorem 2.3.1. Using the same approach as in the proof of Theorem 2.2.1, we construct a successive approximation sequence and show that the limit of a subsequence actually solves (2.3.4) in the weak sense. To avoid repetitions, we will refer next to some inequalities already showed in the proof of Theorem 2.2.1.

Proof. We start by considering a successive approximation sequence $\{(u^{(n)}, w^{(n)})\}$ satisfying

$$\begin{cases} u^{(1)} = S_2 u_0 \,, \qquad w^{(1)} = S_2 w_0, \\ \partial_t u^{(n+1)} + (\nu + k)(-\Delta^\alpha) u^{(n+1)} = \mathbb{P}(-u^{(n)} \cdot \nabla u^{(n+1)}) + 2k \nabla \times w^{(n)}, \\ \partial_t w^{(n+1)} = -(4k + \gamma) w^{(n+1)} - 2k \nabla \times u^{(n)} - u^{(n)} \cdot \nabla w^{(n+1)}, \\ u^{(n+1)}(x,0) = S_{n+1} u_0, \qquad w^{(n+1)}(x,0) = S_{n+1} w_0, \end{cases} \tag{2.3.5}$$

where $\mathbb{P} := I - \nabla(-\Delta)^{-1}\mathrm{div}$ is the standard Leray Projection. For

$$M = 2(\|u_0\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + \|w_0\|_{B^{\frac{d}{2}}_{2,1}}),$$

$T > 0$ being sufficiently small and $0 < \delta < 1$ (to be specified later), we set

$$Y \equiv \Big\{ (u,w) \mid \;\; \|u\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \le M, \;\; \|w\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \le M, \\ \|u\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \le \delta, \;\; \|w\|_{L^1(0,T,B^{\frac{d}{2}}_{2,1})} \le \delta \Big\}. \tag{2.3.6}$$

We show that $\{(u^{(n)}, w^{(n)})\}$ has a subsequence that converges to the weak solution of (2.3.4). This process consists of three main steps. The first step is to show that $\{(u^{(n)}, w^{(n)})\}$ is uniformly bounded in Y. The second step is to extract a strongly convergent subsequence via the Aubin-Lions Lemma. While the last step is to show that the limit is indeed a weak solution of (2.3.4).

Our main effort is devoted to show inductively that $\{(u^{(n)}, w^{(n)})\}$ is bounded uniformly in Y. Recall that (u_0, w_0) is in the regularity class (2.3.1). Clearly,

$$\|u^{(1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} = \|S_2 u_0\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M,$$
$$\|w^{(1)}\|_{L^\infty(0,T,B_{2,1}^{\frac{d}{2}})} = \|S_2 w_0\|_{L^\infty(0,T,B_{2,1}^{\frac{d}{2}})} \leq M.$$

If $T > 0$ is sufficiently small, then

$$\|u^{(1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \leq T\|S_2 u_0\|_{B_{2,1}^{1+\frac{d}{2}}} \leq T\,c\|u_0\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \leq \delta,$$

$$\|w^{(1)}\|_{L^1(0,T,B_{2,1}^{\frac{d}{2}})} \leq T\|S_2 w_0\|_{B_{2,1}^{\frac{d}{2}}} \leq T\,c\|w_0\|_{B_{2,1}^{\frac{d}{2}}} \leq \delta.$$

Assuming that $(u^{(n)}, w^{(n)})$ obeys the bounds defined in Y, namely

$$\|u^{(n)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M, \quad \|w^{(n)}\|_{L^\infty(0,T,B_{2,1}^{\frac{d}{2}})} \leq M,$$
$$\|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \leq \delta, \quad \|w^{(n)}\|_{L^1(0,T,B_{2,1}^{\frac{d}{2}})} \leq \delta.$$

we prove that $\{(u^{(n+1)}, w^{(n+1)})\}$ admits the same bound for suitably selected $T > 0$, $M > 0$ and $\delta > 0$. For sake of clarity, the proof of the four bounds is achieved in the following four steps.

Step 1: The estimate of $u^{(n+1)}$ in $L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d))$.

Following the same process as in the proof of the first step of Theorem 2.2.1, we write the inequality

$$\|u^{(n+1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \le \|u_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + \sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)}(J_1+\cdots+J_4)\,d\tau, \tag{2.3.7}$$

where

$$J_1 := c\,\|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2},$$

$$J_2 := c\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n+1)}\|_{L^2},$$

$$J_3 := c\,2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2},$$

$$J_4 := c\,2^j\|\Delta_j w^{(n)}\|_{L^2}.$$

The terms on the right hand side of (2.3.7) can be estimated as follows. Recalling the definition of J_1 above and using the inductive assumption on $u^{(n)}$, we have for any $t\le T$,

$$\begin{aligned}
&\sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^t e^{c_0 2^{2\alpha j}(t-\tau)} J_1\,d\tau \\
&\quad= c\int_0^t \sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j}\|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}(\tau)\|_{L^2}\,d\tau \\
&\quad\le c\int_0^t \underbrace{\sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j}\|\Delta_j u^{(n+1)}\|_{L^2}}_{=\|u^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}}} \underbrace{\sum_{j\ge -1} 2^{(1+\frac{d}{2})j}\|\Delta_j u^{(n)}(\tau)\|_{L^2}\,d\tau}_{=\|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})}} \\
&\quad\le c\,\|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})}\|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \\
&\quad\le c\,\delta\,\|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})}.
\end{aligned} \tag{2.3.8}$$

The terms with J_2 and J_3 share the same upper bound. In fact, by Young's inequality for series convolution,

$$\begin{aligned}
&\sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_2\, d\tau \\
&\quad \leq c \int_0^t \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{2\alpha(m-j)} 2^{(1+\frac{d}{2}-2\alpha)m} \|\Delta_m u^{(n+1)}(\tau)\|_{L^2}\, d\tau \\
&\quad \leq c \int_0^t \|u^{(n)}(\tau)\|_{B^{1+\frac{d}{2}}_{2,1}} \|u^{(n+1)}(\tau)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}\, d\tau \\
&\quad \leq c\, \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \\
&\quad \leq c\,\delta\, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})}. \qquad (2.3.9)
\end{aligned}$$

Similarly the term related to J_3 is bounded by

$$\begin{aligned}
&\sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j}(t-\tau)} J_3\, d\tau \\
&\quad = \int_0^t \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \sum_{k\geq j-1} 2^j 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2}\, d\tau \\
&\quad = c \int_0^t \sum_{j\geq -1} \sum_{k\geq j-1} 2^{(2+\frac{d}{2}-2\alpha)(j-k)} 2^{(1+\frac{d}{2})k} \|\Delta_k u^{(n)}\|_{L^2} 2^{(1+\frac{d}{2}-2\alpha)k} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\, d\tau \\
&\quad \leq c \int_0^t \|u^{(n)}(\tau)\|_{B^{1+\frac{d}{2}}_{2,1}} \|u^{(n+1)}(\tau)\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}}\, d\tau \\
&\quad \leq c\, \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \\
&\quad \leq c\,\delta\, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})}. \qquad (2.3.10)
\end{aligned}$$

We now examine the term involving J_4,

$$\begin{aligned}
\sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j(t-\tau)}} J_4\, d\tau &= \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-c_0 2^{2\alpha j(t-\tau)}} c 2^j \|\Delta_j w^{(n)}\|_{L^2}\, d\tau \\
&\leq \sum_{j\geq -1} \int_0^t c\, 2^{(2+\frac{d}{2}-2\alpha)j} \|\Delta_j w^{(n)}\|_{L^2}\, d\tau \\
&\underbrace{\leq}_{\text{since } \alpha\geq 1} c\, \|w^{(n)}\|_{L^1(0,T,B^{\frac{d}{2}}_{2,1})} \\
&\leq c\,\delta. \qquad (2.3.11)
\end{aligned}$$

Collecting the bounds (2.3.8), (2.3.9), (2.3.10), (2.3.11) and inserting them in (2.3.7), we obtain for any $t \leq T$

$$\|u^{(n+1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \leq \|u_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + c\,\delta\,\|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} + c\,\delta.$$

Hence, it results

$$\|u^{(n+1)}(t)\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq \|u_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + c\,\delta\|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} + c\,\delta.$$

Consequently, choosing δ such that $c\,\delta \leq \min(\frac{1}{4}, \frac{M}{4})$, we get

$$\|u^{(n+1)}(t)\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq \frac{M}{2} + \frac{1}{4}\|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} + \frac{M}{4},$$

which implies

$$\|u^{(n+1)}(t)\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \leq M.$$

Step 2: The estimate of $\|u^{(n+1)}\|_{L^1(0,T,B^{1+\frac{d}{2}})}$.

Following the same method as in the proof of the second step of Theorem 2.2.1, we write the inequality

$$\begin{aligned}\|u^{(n+1)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \leq \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} e^{-c_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\,dt \\ + \int_0^T\int_0^s \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} e^{-c_0 2^{2\alpha j}(s-\tau)}(J_1+\cdots+J_4)\,d\tau\,ds.\end{aligned} \tag{2.3.12}$$

We estimate the terms on the right and start with the first term,

$$\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} e^{-c_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\,dt = c\sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j}(1-e^{-c_0 2^{2\alpha j}T})\|\Delta_j u_0^{(n+1)}\|_{L^2}.$$

Since $u_0 \in B_{2,1}^{1+\frac{d}{2}-2\alpha}$, then by the Dominated Convergence Theorem

$$\lim_{T\to 0}\sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j}(1-e^{-c_0 2^{2\alpha j}T})\|\Delta_j u_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small such that

$$\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} e^{-c_0 2^{2\alpha j} t} \|\Delta_j u_0^{(n+1)}\|_{L^2} \leq \frac{\delta}{4}. \tag{2.3.13}$$

Due to Young's inequality for the time convolution, we have

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_1 \, d\tau \, ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2} \, d\tau \, ds \\
&\quad \leq c \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^T \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2} \, d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds.
\end{aligned}$$

Further, using the fact that there exists $c_2 > 0$ satisfying for $j \geq 0$,

$$\int_0^T e^{-c_0 2^{2\alpha j} s} ds \leq c\, 2^{-2\alpha j} (1 - e^{-c_2 T}), \tag{2.3.14}$$

we obtain

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_1 \, d\tau \, ds \\
&\qquad \leq c\,(1 - e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(m)}\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1 - e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\qquad \leq c\,(1 - e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})} \|u^{(n)}\|_{L^1(0,T,B_{2,1}^{1+\frac{d}{2}})} \\
&\qquad \leq c\,\delta\,(1 - e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T,B_{2,1}^{1+\frac{d}{2}-2\alpha})}.
\end{aligned} \tag{2.3.15}$$

The terms associated with J_2 and J_3 can be similarly estimated and obey the same bound. In fact,

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_2 \, d\tau \, ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n+1)}\|_{L^2} d\tau \, ds \\
&\quad \leq c \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^T \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n+1)}\|_{L^2} d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds.
\end{aligned}$$

Then, owing to (2.3.14),

$$
\begin{aligned}
&\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_2 \, d\tau \, ds \\
&\qquad \le c(1-e^{-c_2 T}) \int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n+1)}\|_{L^2} d\tau \\
&\qquad \le c(1-e^{-c_2 T}) \int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} d\tau \\
&\qquad \le c(1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \\
&\qquad \le c(1-e^{-c_2 T})\, \delta \, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})}. \qquad (2.3.16)
\end{aligned}
$$

Similarly, the term with J_3 is bounded by

$$
\begin{aligned}
&\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau \, ds \\
&\qquad = c \int_0^T \sum_{j\ge -1} 2^{(2+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \sum_{k\ge j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} \, d\tau \, ds \\
&\qquad \le c \sum_{j\ge -1} 2^{(2+\frac{d}{2})j} \int_0^T \sum_{k\ge j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds.
\end{aligned}
$$

Using (2.3.14) and the above inequality leads to,

$$
\begin{aligned}
&\int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau \, ds \\
&\qquad \le c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\ge -1} 2^{(2+\frac{d}{2}-2\alpha)j} \sum_{k\ge j-1} 2^{\frac{d}{2}k} \|\Delta_k u^{(n)}\|_{L^2} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} \, d\tau \\
&\qquad \le c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\ge -1} 2^{(2+\frac{d}{2}-2\alpha)j} \|\widetilde{\Delta}_j u^{(n+1)}\|_{L^2} \sum_{j\ge -1} 2^{(\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\qquad \le c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\ge -1} 2^{(1+\frac{d}{2}-2\alpha)j} \|\widetilde{\Delta}_j u^{(n+1)}\|_{L^2} \sum_{j\ge -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\qquad \le c\,(1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \\
&\qquad \le c\,(1-e^{-c_2 T})\, \delta \, \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})}. \qquad (2.3.17)
\end{aligned}
$$

Now, it remains to control the term invoking J_4,

$$
\begin{aligned}
\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} J_4 \, d\tau \, ds \\
&= c \int_0^T \sum_{j\geq -1} 2^{(2+\frac{d}{2})j} \int_0^s e^{-c_0 2^{2\alpha j}(s-\tau)} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \, ds \\
&\leq c \sum_{j\geq -1} 2^{(2+\frac{d}{2})j} \int_0^T \|\Delta_j w^{(n)}\|_{L^2} d\tau \int_0^T e^{-c_0 2^{2\alpha j} s} \, ds \\
&\leq c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{(2+\frac{d}{2}-2\alpha)j} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \\
&\underset{\text{since}\,\alpha\geq 1}{\leq} c\,(1-e^{-c_2 T}) \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \|\Delta_j w^{(n)}\|_{L^2} \, d\tau \\
&= c(1-e^{-c_2 T}) \|w^{(n)}\|_{L^1(0,T,B^{\frac{d}{2}}_{2,1})}. \qquad (2.3.18)
\end{aligned}
$$

Taking into account the estimates (2.3.13), (2.3.15), (2.3.16), (2.3.17), (2.3.18) and (2.3.12), we obtain

$$
\begin{aligned}
\|u^{(n+1)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} &\leq \frac{\delta}{4} + c\,\delta(1-e^{-c_2 T}) \|u^{(n+1)}\|_{L^\infty(0,T,B^{1+\frac{d}{2}-2\alpha}_{2,1})} \\
&\quad + c\,(1-e^{-c_2 T}) \|w^{(n)}\|_{L^1(0,T,B^{\frac{d}{2}}_{2,1})} \\
&\leq \frac{\delta}{4} + c\,\delta(1-e^{-c_2 T}) M + c\,(1-e^{-c_2 T})\delta.
\end{aligned}
$$

As a result, choosing T sufficiently small such that $c(1-e^{-c_2 T}) \leq \min(\frac{1}{4M}, \frac{1}{2})$ yields,

$$
\|u^{(n+1)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \leq \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{2} = \delta.
$$

Step 3: The estimate of $w^{(n+1)}$ in $L^\infty(0,T,B^{\frac{d}{2}}_{2,1}(\mathbb{R}^d))$.

Applying Δ_j to the third equation in (2.3.5) and then dotting with $\Delta_j w^{(n+1)}$, we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} \|\Delta_j w^{(n+1)}\|_{L^2}^2 + (4k+\gamma) \|\Delta_j w^{(n+1}\|_{L^2}^2 &= -2k \int \Delta_j(\nabla \times u^{(n)}) \Delta_j w^{(n+1)} dx \\
&\quad - \int \Delta_j(u^{(n)} \cdot \nabla w^{(n+1)}) \Delta_j w^{(n+1)} dx \\
&:= B_1 + B_2, \qquad (2.3.19)
\end{aligned}
$$

where

$$B_1 := -2k \int \Delta_j(\nabla \times u^{(n)})\Delta_j w^{(n+1)} dx,$$
$$B_2 := -\int \Delta_j(u^{(n)} \cdot \nabla w^{(n+1)})\Delta_j w^{(n+1)} dx.$$

By Hölder's inequality and Lemma 2.1.5,

$$\begin{aligned} |B_1| &= |-2k \int \Delta_j(\nabla \times u^{(n)})\Delta_j w^{(n+1)} dx| \\ &\leq 2k \,\|\Delta_j(\nabla \times u^{(n)})\|_{L^2}\|\Delta_j w^{(n+1)} dx\|_{L^2} \\ &\leq c\, 2^j \,\|\Delta_j u^{(n)}\|_{L^2}\|\Delta_j w^{(n+1)}\|_{L^2}. \end{aligned} \tag{2.3.20}$$

According to Lemma 2.1.6,

$$\begin{aligned} |B_2| &= |-\int \Delta_j(u^{(n)} \cdot \nabla w^{(n+1)})\Delta_j w^{(n+1)} \, dx| \\ &\leq c\,\|\Delta_j w^{(n+1)}\|_{L^2}^2 \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2} \\ &+ c\,\|\Delta_j w^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m}\|\Delta_m w^{(n+1)}\|_{L^2} \\ &+ c\,\|\Delta_j w^{(n+1)}\|_{L^2} 2^j \sum_{k\leq j-1} 2^{\frac{d}{2}k}\|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}. \end{aligned} \tag{2.3.21}$$

Inserting the estimates (2.3.20), (2.3.21) in (2.3.19) and eliminating $\|\Delta_j w^{(n+1)}\|_{L^2}$ from both sides of the inequality, we get

$$\begin{aligned} \frac{d}{dt}\|\Delta_j w^{(n+1)}\|_{L^2} + (8k+2\gamma)\|\Delta_j w^{(n+1)}\|_{L^2} \leq & c\, 2^j \|\Delta_j u^{(n)}\|_{L^2} \\ &+ c\,\|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2} \\ &+ c\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m}\|\Delta_m w^{(n+1)}\|_{L^2} \\ &+ c\, 2^j \sum_{k\geq j-1} 2^{\frac{d}{2}k}\|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}. \end{aligned} \tag{2.3.22}$$

Integrating (2.3.22) in time yields, for any $t \le T$,

$$\|\Delta_j w^{(n+1)}\|_{L^2} \le e^{-(8k+2\gamma)t}\|\Delta_j w_0^{(n+1)}\|_{L^2} + \int_0^t e^{-(8k+2\gamma)(t-\tau)}(K_1 + \cdots + K_4)d\tau, \tag{2.3.23}$$

where

$$\begin{aligned}
K_1 &:= c\,2^j\|\Delta_j u^{(n)}\|,\\
K_2 &:= c\,\|\Delta_j w^{(n+1)}\|_{L^2}\sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2},\\
K_3 &:= c\,\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j} 2^{(1+\frac{d}{2})m}\|\Delta_m w^{(n+1)}\|_{L^2},\\
K_4 &:= c\,2^j\sum_{k\ge j-1} 2^{\frac{d}{2}k}\|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}.
\end{aligned}$$

Multiplying (2.3.23) by $2^{\frac{d}{2}j}$ and summing over j yield,

$$\|w^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} \le \|w_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + \sum_{j\ge -1}\int_0^t e^{-(8k+2\gamma)(t-\tau)}2^{\frac{d}{2}j}(K_1 + \cdots + K_4)\,d\tau. \tag{2.3.24}$$

Starting with the term involving K_1, we write

$$\begin{aligned}
\sum_{j\ge -1} 2^{\frac{d}{2}j}\int_0^t e^{-(8k+2\gamma)(t-\tau)}K_1\,d\tau &\le \int_0^t \sum_{j\ge -1} c\,2^{(1+\frac{d}{2})j}\|\Delta_j u^{(n)}(\tau)\|_{L^2}\,d\tau\\
&\le c\,\|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})}\\
&\le c\,\delta.
\end{aligned}\tag{2.3.25}$$

The terms containing K_2 through K_4 on the right of (2.3.23) can be bounded suitably and share the same bound. Indeed, for the term with K_2 we have,

$$\begin{aligned}
&\sum_{j\geq -1} 2^{(\frac{d}{2})j} \int_0^t e^{-(8k+2\gamma)(t-\tau)} c\,\|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\,d\tau \\
&\leq c\int_0^t \sum_{j\geq -1} 2^{(\frac{d}{2})j}\|\Delta_j w^{(n+1)}(\tau)\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}(\tau)\|_{L^2} d\tau \\
&\leq c\int_0^t \|w^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} \|u^{(n)}\|_{B^{1+\frac{d}{2}}_{2,1}}\,d\tau \\
&\leq c\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \\
&\leq c\,\delta\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})}. \qquad (2.3.26)
\end{aligned}$$

In the same way, the term involving K_3 is bounded by

$$\begin{aligned}
&\sum_{j\geq -1} 2^{(\frac{d}{2})j} \int_0^t e^{-(8k+2\gamma)(t-\tau)} c\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m}\|\Delta_m w^{(n+1)}\|_{L^2} d\tau \\
&\leq c\int_0^t \sum_{j\geq -1} 2^{(\frac{d}{2})j}\|\Delta_j u^{(n)}\|_{L^2} \sum_{j\geq -1} 2^{(1+\frac{d}{2})j}\|\Delta_j w^{(n+1)}\|_{L^2}\,d\tau \\
&\leq c\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \\
&\leq c\,\delta\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})}. \qquad (2.3.27)
\end{aligned}$$

Further, for the term associated with K_4 we have,

$$\begin{aligned}
&\sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^t c\,2^j \sum_{k\geq j-1} \|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} 2^{\frac{dk}{2}}\,d\tau \\
&\leq c\int_0^t \sum_{j\geq -1} 2^{(1+\frac{d}{2})j}\|\Delta_j u^{(n)}\|_{L^2} \sum_{j\geq -1} 2^{(\frac{d}{2})j}\|\widetilde{\Delta}_j w^{(n+1)}\|_{L^2} d\tau \\
&\leq c\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \\
&\leq c\,\delta\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})}. \qquad (2.3.28)
\end{aligned}$$

The estimates in (2.3.25), (2.3.26), (2.3.27) and (2.3.28) taken with (2.3.24), allow us to establish that for any $t \leq T$,

$$\|w^{(n+1)}(t)\|_{B^{\frac{d}{2}}_{2,1}} \leq \|w_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + c\,\delta + c\,\delta\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})}.$$

As a consequence,

$$\|w^{(n+1)}(t)\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \leq \|w_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + c\,\delta + c\,\delta\,\|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})}.$$

Choosing $c\,\delta \leq \min(\frac{1}{4}, \frac{M}{4})$ yields

$$\|w^{(n+1)}(t)\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \leq \frac{M}{2} + \frac{M}{4} + \frac{1}{4}\|w^{(n+1)}(t)\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})},$$

which implies

$$\|w^{(n+1)}(t)\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \leq M.$$

Step 4: The estimate of $\|w^{(n+1)}(t)\|_{L^1(0,T,B^{\frac{d}{2}}_{2,1})}$.

Multiplying (2.3.23) by $2^{\frac{d}{2}j}$, summing over j and integrating in time yield

$$\begin{aligned}\|w^{(n+1)}\|_{L^1(0,T,B^{\frac{d}{2}}_{2,1})} \leq &\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} e^{-(8k+2\gamma)t}\|\Delta_j w_0^{(n+1)}\|_{L^2}\,dt \\ &+ \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j}\int_0^s e^{-(8k+2\gamma)(s-\tau)}(K_1+\cdots+K_4)\,d\tau\,ds.\end{aligned} \tag{2.3.29}$$

Clearly,

$$\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} e^{-(8k+2\gamma)t}\|\Delta_j w_0^{(n+1)}\|_{L^2}dt = c\sum_{j\geq -1} 2^{\frac{d}{2}j}(1-e^{-(8k+2\gamma)T})\|\Delta_j w_0^{(n+1)}\|_{L^2}.$$

Since $w_0 \in B^{\frac{d}{2}}_{2,1}$, it follows from the Dominated Convergence Theorem that

$$\lim_{T\to 0}\sum_{j\geq -1} 2^{\frac{d}{2}j}(1-e^{-(8k+2\gamma)T})\|\Delta_j w_0^{(n+1)}\|_{L^2} = 0.$$

From this, we can choose T sufficiently small such that

$$\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} e^{-(8k+2\gamma)t}\|\Delta_j w_0^{(n+1)}\|_{L^2}\,dt \leq \frac{\delta}{2}. \tag{2.3.30}$$

Applying Young's inequality for the time convolution, the term related to K_1 can be bounded by

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} K_1 \, d\tau \, ds \\
&\qquad = c \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \, ds \\
&\qquad \leq \Big(c \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^T \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \Big) \Big(\int_0^T e^{-(8k+2\gamma)s} ds \Big) \\
&\qquad \leq c\, (1 - e^{-(8k+2\gamma)T}) \underbrace{\int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau}_{= \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})}} \\
&\qquad \leq c\, \delta \, (1 - e^{-(8k+2\gamma)T}). \qquad (2.3.31)
\end{aligned}$$

Applying again Young's inequality for the time convolution, the term associated with K_2 is bounded by

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} K_2 \, d\tau \, ds \\
&\quad = c \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2} d\tau ds \\
&\quad \leq \Big(c \sum_{j\leq -1} 2^{\frac{d}{2}j} \int_0^T \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2} \, d\tau \Big) \Big(\int_0^T e^{-(8k+2\gamma)s} ds \Big) \\
&\quad \leq c \Big(1 - e^{-(8k+2\gamma)T}\Big) \int_0^T \sum_{j\leq -1} 2^{\frac{d}{2}j} \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2} \, d\tau \\
&\quad \leq c\, (1 - e^{-(8k+2\gamma)T}) \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \|\Delta_j w^{(n+1)}\|_{L^2} \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \, d\tau \\
&\quad \leq c\, (1 - e^{-(8k+2\gamma)T}) \|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \\
&\quad \leq c\, \delta M \, (1 - e^{-(8k+2\gamma)T}). \qquad (2.3.32)
\end{aligned}$$

Similarly, due to Young's inequality for the time convolution, the term with K_3 admits the same bound

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} K_3\, d\tau ds \\
&= c\Big(\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} \|\Delta_j u^{(n)}\|_{L^2}\Big)\Big(\sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m w^{(n+1)}\|_{L^2}\Big)\, d\tau\, ds \\
&\leq \Big(c \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^T \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m w^{(n+1)}\|_{L^2}\, d\tau\Big)\Big(\int_0^T e^{-(8k+2\gamma)s} ds\Big) \\
&\leq c\,(1-e^{-(8k+2\gamma)T}) \int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j} 2^{(1+\frac{d}{2})m} \|\Delta_m w^{(n+1)}\|_{L^2}\, d\tau \\
&\leq c\,(1-e^{-(8k+2\gamma)T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{j\geq -1} 2^{\frac{d}{2}j} \|\Delta_j w^{(n+1)}\|_{L^2}\, d\tau \\
&\leq c(1-e^{-(8k+2\gamma)T}) \|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \\
&\leq c\,\delta M\,(1-e^{-(8k+2\gamma)T}).
\end{aligned} \tag{2.3.33}$$

Finally, it remains to bound the last term with K_4,

$$\begin{aligned}
&\int_0^T \sum_{j\geq -1} 2^{\frac{d}{2}j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} K_4\, d\tau ds \\
&\qquad = c \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^s e^{-(8k+2\gamma)(s-\tau)} \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} d\tau ds \\
&\qquad \leq \Big(c \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \int_0^T \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} d\tau\Big)\Big(\int_0^T e^{-(8k+2\gamma)s} ds\Big) \\
&\qquad \leq c(1-e^{-(8k+2\gamma)T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \sum_{k\geq j-1} 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k w^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2} d\tau \\
&\qquad \leq c(1-e^{-(8k+2\gamma)T}) \int_0^T \sum_{j\geq -1} 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{j\geq -1} 2^{\frac{d}{2}j} \|\widetilde{\Delta}_j w^{(n+1)}\|_{L^2} d\tau \\
&\qquad \leq c(1-e^{-(8k+2\gamma)T}) \|w^{(n+1)}\|_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})} \|u^{(n)}\|_{L^1(0,T,B^{1+\frac{d}{2}}_{2,1})} \\
&\qquad \leq c\,\delta M\,(1-e^{-(8k+2\gamma)T}).
\end{aligned} \tag{2.3.34}$$

In view of (2.3.29), collecting the estimates (2.3.30), (2.3.31), (2.3.32), (2.3.33) and (2.3.34) yield,

$$\|w^{(n+1)}\|_{L^1(0,T,B^{\frac{d}{2}}_{2,1})} \leq \frac{\delta}{2} + c\,\delta\,(1 - e^{-(8k+2\gamma)T}) + c\,\delta M\,(1 - e^{-(8k+2\gamma)T}).$$

Choosing T sufficiently small such that $c(1 - e^{-(8k+2\gamma)T}) \leq \min(\frac{1}{4M}, \frac{1}{4})$, we obtain

$$\|w^{(n+1)}\|_{L^1(0,T,B^{\frac{d}{2}}_{2,1})} \leq \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta.$$

These uniform bounds allow us to extract a weakly convergent subsequence. That is there is $(u, w) \in Y$ such that a subsequence of (u^n, w^n) (still denoted by (u^n, w^n)) satisfies

$$\begin{aligned} u^n &\overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0, T, B^{1+\frac{d}{2}-2\alpha}_{2,1}), \\ w^n &\overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty(0, T, B^{\frac{d}{2}}_{2,1}). \end{aligned}$$

In order to show that (u, w) is a weak solution of (2.3.4) we need to further extract a subsequence which converges strongly to (u, w). We use the Aubin-Lions Lemma. We can show by making use of the equation (2.3.5) that $(\partial_t u^n, \partial_t w^n)$ is uniformly bounded in

$$\begin{aligned} \partial_t u^n &\in L^1(0, T, B^{1+\frac{d}{2}-2\alpha}_{2,1}) \cap L^2(0, T, B^{1+\frac{d}{2}-3\alpha}_{2,1}), \\ \partial_t w^n &\in L^1(0, T, B^{\frac{d}{2}-2\alpha}_{2,1}) \cap L^2(0, T, B^{\frac{d}{2}-\alpha}_{2,1}). \end{aligned}$$

Since we are in this case in the whole space $\mathbb{R}^d$, we need to combine Cantor's diagonal process with the Aubin-Lions Lemma to show that a subsequence of a weakly convergent subsequence, still denoted by (u^n, w^n), has the following strongly convergent property

$$u^n \longrightarrow u \quad \text{in} \quad L^2(0, T, B^{1+\frac{d}{2}-\gamma_3}_{2,1}(Q)), \quad w^n \longrightarrow w \quad \text{in} \quad L^2(0, T, B^{\frac{d}{2}-\gamma_4}_{2,1}(Q)),$$

where $\alpha \leq \gamma_3 \leq 3\alpha$, $0 \leq \gamma_4 \leq 2\alpha$ and $Q \subset \mathbb{R}^d$ is a compact subset. This strong convergence property would allow us to show that (u, w) is indeed a weak solution of (2.3.4). This completes the proof for the existence part of Theorem 2.3.1. ■

2.3.2 Uniqueness of Weak Solutions

This subsection proves the uniqueness part of Theorem 2.3.1.

Proof. Assume that $(u^{(1)}, w^{(1)})$ and $(u^{(2)}, w^{(2)})$ are two solutions of (2.3.4) in the regularity class in (2.3.2) and (2.3.3). Their difference $(\widetilde{u}, \widetilde{w})$ with

$$\widetilde{u} = u^{(2)} - u^{(1)} \quad \text{and} \quad \widetilde{w} = w^{(2)} - w^{(1)}$$

satisfies

$$\begin{cases} \partial_t \widetilde{u} + (\nu + k)(-\Delta)^\alpha \widetilde{u} = -\mathbb{P}(u^{(2)} \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u^{(1)}) + 2k\nabla \times \widetilde{w}, \\ \partial_t \widetilde{w} = -(4k+\gamma)\widetilde{w} - 2k\nabla \times \widetilde{u} - u^{(2)} \cdot \nabla \widetilde{w} - \widetilde{u} \cdot \nabla w^{(1)}, \\ \nabla \cdot \widetilde{u} = 0, \\ \widetilde{u}(x,0) = 0, \quad \widetilde{w}(x,0) = 0. \end{cases} \tag{2.3.35}$$

We estimate the difference $(\widetilde{u}, \widetilde{w})$ in $L^2(\mathbb{R}^d)$. Dotting (2.3.35) by $(\widetilde{u}, \widetilde{w})$ and applying the divergence-free condition of $\widetilde{u}$, namely $\nabla \cdot \widetilde{u} = 0$, yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{w}\|_{L^2}^2\Big) + (\nu+k)\|\Lambda^\alpha \widetilde{u}\|_{L^2}^2 + (4k+\gamma)\|\widetilde{w}\|_{L^2}^2 \\ &\quad = -\int u^{(2)} \cdot \nabla \widetilde{u} \cdot \widetilde{u}\, dx - \int \widetilde{u} \cdot \nabla u^{(1)} \cdot \widetilde{u}\, dx \\ &\qquad - \int u^{(2)} \cdot \nabla \widetilde{w} \cdot \widetilde{w}\, dx - \int \widetilde{u} \cdot \nabla w^{(1)} \cdot \widetilde{w}\, dx \\ &\quad := L_1 + L_2 + L_3 + L_4, \end{aligned} \tag{2.3.36}$$

with

$$\begin{aligned} L_1 &:= -\int u^{(2)} \cdot \nabla \widetilde{u} \cdot \widetilde{u}\, dx, \\ L_2 &:= -\int \widetilde{u} \cdot \nabla u^{(1)} \cdot \widetilde{u}\, dx, \\ L_3 &:= -\int u^{(2)} \cdot \nabla \widetilde{w} \cdot \widetilde{w}\, dx, \\ L_4 &:= -\int \widetilde{u} \cdot \nabla w^{(1)} \cdot \widetilde{w}\, dx, \end{aligned}$$

where we denote $\Lambda := (-\Delta)^{\frac{1}{2}}$.

Due to $\nabla \cdot u^{(2)} = 0$, one can easily check by integration by parts that

$$L_1 = L_3 = 0. \tag{2.3.37}$$

As in (2.2.42),

$$|L_2| \le c\|u^{(1)}\|_{B^{1+\frac{d}{2}}_{2,1}} \|\widetilde{u}\|^2_{L^2}. \tag{2.3.38}$$

In order to estimate L_4, we set

$$\frac{1}{p} = \frac{1}{2} - \frac{\alpha}{d}, \quad \frac{1}{q} = \frac{\alpha}{d} \ \ (or\ \frac{d}{q} = \alpha).$$

Applying Hölder's inequality and Cauchy's inequality with epsilon,

$$\begin{aligned}
|L_4| &= | - \int \widetilde{u} \cdot \nabla w^{(1)} \cdot \widetilde{w}\, dx| \\
&\le \|\widetilde{w}\|_{L^2} \|\nabla w^{(1)}\|_{L^q} \|\widetilde{u}\|_{L^p} \\
&\le \sum_{j \ge -1} \|\Delta_j \nabla w^{(1)}\|_{L^q} \|\widetilde{w}\|_{L^2} \|\widetilde{u}\|_{L^p} \\
&\le c \sum_{j \ge -1} 2^j 2^{dj(\frac{1}{2} - \frac{1}{q})} \|\Delta_j w^{(1)}\|_{L^2} \|\widetilde{w}\|_{L^2} \|\widetilde{u}\|_{L^p} \\
&= \sum_{j \ge -1} 2^j 2^{\frac{dj}{2} - \frac{d}{q} j} \|\Delta_j w^{(1)}\|_{L^2} \|\widetilde{w}\|_{L^2} \|\widetilde{u}\|_{L^p} \\
&\underset{\text{since}\, \alpha \ge 1}{\le} c \sum_{j \ge -1} 2^{\frac{dj}{2}} \|\Delta_j w^{(1)}\|_{L^2} \|\widetilde{w}\|_{L^2} \|\widetilde{u}\|_{L^p} \\
&\le c\, \|w^{(1)}\|_{B^{\frac{d}{2}}_{2,1}} \|\widetilde{w}\|_{L^2} \|\Lambda^\alpha \widetilde{u}\|_{L^2} \\
&\le \frac{(\nu + k)}{2} \|\Lambda^\alpha \widetilde{u}\|^2_{L^2} + c\|w^{(1)}\|^2_{B^{\frac{d}{2}}_{2,1}} \|\widetilde{w}\|^2_{L^2},
\end{aligned} \tag{2.3.39}$$

where in the last inequality we make use of

$$\|\widetilde{u}\|_{L^p} \le c\, \|\Lambda^\alpha \widetilde{u}\|_{L^2}.$$

Combining the estimates (2.3.37), (2.3.38), (2.3.39) and inserting then in (2.3.36), we obtain,

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{w}\|_{L^2}^2\Big)+(\nu+k)\|\Lambda^\alpha\widetilde{u}\|_{L^2}^2+(8k+2\gamma)\|\widetilde{w}\|_{L^2}^2$$
$$\le\Big(c\|u^{(1)}\|_{B^{1+\frac{d}{2}}_{2,1}}+c\,\|w^{(1)}\|^2_{B^{\frac{d}{2}}_{2,1}}\Big)\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{w}\|_{L^2}^2\Big). \tag{2.3.40}$$

Since $u^{(1)}\in L^1(0,T,B^{1+\frac{d}{2}}_{2,1})$ and $w^{(1)}\in L^1(0,T,B^{\frac{d}{2}}_{2,1})\cap L^\infty(0,T,B^{\frac{d}{2}}_{2,1})$, we have

$$\int_0^T\|u^{(1)}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}dt<\infty \qquad \text{and} \qquad \int_0^T\|w^{(1)}(t)\|^2_{B^{\frac{d}{2}}_{2,1}}\,dt\le T\|w^{(1)}(t)\|^2_{L^\infty(0,T,B^{\frac{d}{2}}_{2,1})}.$$

Applying Gronwall's inequality to (2.3.40), we conclude that

$$\|\widetilde{u}\|_{L^2}=\|\widetilde{w}\|_{L^2}=0,$$

which leads to the desired uniqueness. This completes the proof of the uniqueness part of Theorem 2.3.1. ∎

CHAPTER III

STABILIZATION OF THE 2D BOUSSINESQ EQUATIONS WITH VERTICAL DISSIPATION AND HORIZONTAL THERMAL DIFFUSION

3.1 Introduction

As outlined in the introduction, we focus on the following special two-dimensional Boussinesq system with partial dissipation

$$\begin{cases} \partial_t U + U \cdot \nabla U = -\nabla P + \nu\, \partial_{22} U + \Theta \mathbf{e}_2, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t \Theta + U \cdot \nabla \Theta = \eta\, \partial_{11} \Theta, \\ \nabla \cdot U = 0. \end{cases} \tag{3.1.1}$$

Our study in this Chapter intends to reveal and rigorously prove the fact that the temperature can actually have a stabilizing effect on the buoyancy-driven fluids. More precisely, we aim to understand the stability and large-time behavior of perturbations near the hydrostatic equilibrium $(U_{he}, \Theta_{he}, P_{he})$ with

$$U_{he} = 0, \quad \Theta_{he} = x_2, \quad P_{he} = \frac{1}{2} x_2^2.$$

In order to understand the desired stability, we consider the equation of the perturbation denoted by (u, p, θ), where

$$u = U - U_{he}, \quad p = P - P_{he} \quad \text{and} \quad \theta = \Theta - \Theta_{he},$$

given as

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{22} u + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \eta\, \partial_{11} \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{3.1.2}$$

This Chapter is based on the author's joint work [6]. Its main three results are organized as follows. In Section 3.2, we establish the H^2 nonlinear stability on (3.1.2). In Section 3.3, we derive a precise anisotropic large-time behavior of the solutions to the linearized system of (3.1.2) and we prove that the Fourier frequency piece of the solutions (u, θ) to the linearized system away from the two axes of the frequency space decays exponentially in time to zero.

In order to establish the H^2 nonlinear stability, one needs to prove that the solution (u, θ) of (3.1.2) corresponding to any sufficiently small initial perturbation (u_0, θ_0) (measured in the Sobolev norm $H^2(\mathbb{R}^2)$) remains small for all time.

In our situation, we can obtain a uniform bound on the L^2-norm of the vorticity $\omega := \nabla \times u$ itself which satisfies

$$\partial_t \omega + u \cdot \nabla \omega = \nu\, \partial_{22} \omega + \partial_1 \theta, \quad x \in \mathbb{R}^2,\ t > 0, \tag{3.1.3}$$

but controlling the growth of the the L^2-norm of the gradient of the vorticity, $\nabla \omega$ does not appear to be an easy task due to the lack of horizontal dissipation.

In particular, taking θ identically zero, (3.1.3) turns to the 2D Navier-Stokes equation with degenerate dissipation ,

$$\partial_t \omega + u \cdot \nabla \omega = \nu\, \partial_{22} \omega, \quad x \in \mathbb{R}^2,\ t > 0. \tag{3.1.4}$$

(3.1.4) always has a unique global solution ω for any initial data $\omega_0 \in H^1(\mathbb{R}^2)$, but the issue of whether $\|\nabla \omega(t)\|_{L^2}$ for the solution ω of (3.1.4) grows as a function of t remains an open problem. In another particular case, when dealing with the 2D

Euler vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad x \in \mathbb{R}^2,\ t > 0.$$

it was shown in many works (see, e.g., [19],[38],[67]), that $\nabla \omega(t)$ can grow even double exponentially in time. In contrast, it has been shown that the solutions to the 2D Navier-Stokes equations with full dissipation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \, \Delta \omega, \quad x \in \mathbb{R}^2,\ t > 0,$$

always decay in time (see, e.g., [56],[57]). The lack of the horizontal dissipation in (3.1.4) prevents us from following the same approach used for the fully dissipative 2D Navier-Stokes equations. Indeed, when we estimate the L^2-norm of $\nabla \omega$, the issue is how to proceed from the energy equality

$$\frac{1}{2}\frac{d}{dt}\|\nabla \omega(t)\|_{L^2}^2 + \nu \|\partial_2 \nabla \omega(t)\|_{L^2}^2 = -\int \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dx.$$

Making use of the anisotropic dissipation, we can further decompose the nonlinear right hand side term in the above equation as follows,

$$\begin{aligned} \int \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dx &= \int \partial_1 u_1 \, (\partial_1 \omega)^2 \, dx + \int \partial_1 u_2 \, \partial_1 \omega \, \partial_2 \omega \, dx \\ &\quad + \int \partial_2 u_1 \, \partial_1 \omega \, \partial_2 \omega \, dx + \int \partial_2 u_2 \, (\partial_2 \omega)^2 \, dx. \end{aligned} \tag{3.1.5}$$

Due to the lack of control on the horizontal derivatives in the dissipation, the first two terms in (3.1.5) do not appear to admit suitable bounds. Fortunately, it is the smoothing and stabilization effects of the temperature through the coupling and interaction that allow us to establish the H^2 nonlinear stability of (3.1.2). To reveal these effects, we need first to eliminate the pressure term from the first equation of (3.1.2) and provide the explicit wave-type equations satisfied by the velocity u, the temperature θ and the vorticity $\omega := \nabla \times u$. Applying the Helmholtz-Leray projection $\mathbb{P} = I - \nabla \Delta^{-1} \nabla \cdot$ to the velocity equation of (3.1.2) we obtain

$$\partial_t u = \nu \partial_{22} u + \mathbb{P}(\theta \mathbf{e}_2) - \mathbb{P}(u \cdot \nabla u). \tag{3.1.6}$$

Then, using the definition of $\mathbb{P}$,

$$\mathbb{P}(\theta \mathbf{e}_2) = \theta \mathbf{e}_2 - \nabla \Delta^{-1} \nabla \cdot (\theta \mathbf{e}_2) = \begin{bmatrix} -\partial_1 \partial_2 \Delta^{-1} \theta \\ \theta - \partial_2^2 \Delta^{-1} \theta \end{bmatrix}. \tag{3.1.7}$$

Inserting (3.1.7) in (3.1.6) and writing (3.1.6) in terms of its component equations, we get

$$\begin{cases} \partial_t u_1 = \nu \, \partial_{22} u_1 - \partial_1 \partial_2 \Delta^{-1} \theta + N_1, \\ \partial_t u_2 = \nu \, \partial_{22} u_2 + \partial_1 \partial_1 \Delta^{-1} \theta + N_2, \end{cases} \tag{3.1.8}$$

where N_1 and N_2 are the following nonlinear terms,

$$N_1 = -(u \cdot \nabla u_1 - \partial_1 \Delta^{-1} \nabla \cdot (u \cdot \nabla u)), \quad N_2 = -(u \cdot \nabla u_2 - \partial_2 \Delta^{-1} \nabla \cdot (u \cdot \nabla u)).$$

Differentiating the first equation of (3.1.8) with respect to t yields

$$\partial_{tt} u_1 = \nu \partial_{22} \partial_t u_1 - \partial_1 \partial_2 \Delta^{-1} \partial_t \theta + \partial_t N_1.$$

Then using the equation of θ, namely $\partial_t \theta = \eta \, \partial_{11} \theta - u_2 - u \cdot \nabla \theta$ we get

$$\partial_{tt} u_1 = \nu \partial_{22} \partial_t u_1 + \partial_1 \partial_2 \Delta^{-1} u_2 - \eta \, \partial_{11} \partial_1 \partial_2 \Delta^{-1} \theta + \partial_1 \partial_2 \Delta^{-1} (u \cdot \nabla \theta) + \partial_t N_1.$$

Additionally, substituting $\partial_1 \partial_2 \Delta^{-1} \theta$ using the first equation of (3.1.8), namely

$$-\partial_1 \partial_2 \Delta^{-1} \theta = \partial_t u_1 - \nu \, \partial_{22} u_1 - N_1,$$

yields

$$\begin{aligned} \partial_{tt} u_1 \quad &= \quad \nu \partial_{22} \partial_t u_1 + \partial_1 \partial_2 \Delta^{-1} u_2 + \eta \, \partial_{11} (\partial_t u_1 - \nu \, \partial_{22} u_1 - N_1) \\ & \quad + \partial_1 \partial_2 \Delta^{-1} (u \cdot \nabla \theta) + \partial_t N_1. \end{aligned} \tag{3.1.9}$$

Due to the divergence-free condition $\partial_2 u_2 = -\partial_1 u_1$, it follows from (3.1.9),

$$\partial_{tt} u_1 - (\eta \partial_{11} + \nu \partial_{22}) \partial_t u_1 + \nu \eta \partial_{11} \partial_{22} u_1 + \partial_{11} \Delta^{-1} u_1 = N_3, \tag{3.1.10}$$

where the nonlinear term N_3 is given by,

$$N_3 = (\partial_t - \eta \partial_{11}) N_1 + \partial_1 \partial_2 \Delta^{-1} (u \cdot \nabla \theta).$$

Following a similar process, one can easily show that u_2 and θ satisfy

$$\begin{aligned} &\partial_{tt} u_2 - (\eta\partial_{11} + \nu\partial_{22})\partial_t u_2 + \nu\eta\partial_{11}\partial_{22} u_2 + \partial_{11}\Delta^{-1} u_2 = N_4, \\ &\partial_{tt}\theta - (\eta\partial_{11} + \nu\partial_{22})\partial_t\theta + \nu\eta\partial_{11}\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_5 \end{aligned} \tag{3.1.11}$$

where the nonlinear terms N_4 and N_5 are defined by

$$\begin{aligned} N_4 &= (\partial_t - \eta\partial_{11})N_2 - \partial_1\partial_1\Delta^{-1}(u\cdot\nabla\theta), \\ N_5 &= -(\partial_t - \nu\partial_{22})(u\cdot\nabla\theta) - N_2. \end{aligned}$$

Therefore, combining (3.1.10) and (3.1.11), we have converted (3.1.2) into the following new system

$$\begin{cases} \partial_{tt} u - (\eta\partial_{11} + \nu\partial_{22})\partial_t u + \nu\eta\partial_{11}\partial_{22} u + \partial_{11}\Delta^{-1} u = N_6, \\ \partial_{tt}\theta - (\eta\partial_{11} + \nu\partial_{22})\partial_t\theta + \nu\eta\partial_{11}\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_5, \end{cases} \tag{3.1.12}$$

where

$$N_6 = (N_3, N_4) = -(\partial_t - \eta\partial_{11})\mathbb{P}(u\cdot\nabla u) + \nabla^\perp\partial_1\Delta^{-1}(u\cdot\nabla\theta)$$

with $\nabla^\perp = (\partial_2, -\partial_1)$.

Further, applying the curl $\nabla\times$ to the first equation in (3.1.12), we can also convert (3.1.12) into a system of vorticity $\omega := \nabla\times u$ and the temperature θ,

$$\begin{cases} \partial_{tt}\omega - (\eta\partial_{11} + \nu\partial_{22})\partial_t\omega + \nu\eta\partial_{11}\partial_{22}\omega + \partial_{11}\Delta^{-1}\omega = N_7, \\ \partial_{tt}\theta - (\eta\partial_{11} + \nu\partial_{22})\partial_t\theta + \nu\eta\partial_{11}\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_5, \end{cases}$$

where

$$N_7 = -(\partial_t - \eta\partial_{11})(u\cdot\nabla\omega) - \partial_1(u\cdot\nabla\theta).$$

Amazingly, we have found that all physical quantities u, θ and ω satisfy the same damped degenerate wave equation only with various nonlinear terms N_5, N_6 and N_7.

The new system of wave type equations in (3.1.12) exhibits much more smoothing and stabilization properties hidden in the original system (3.1.2). In fact, the velocity in (3.1.2) involves only vertical dissipation, but the wave structure actually implies that the temperature can stabilize the fluids by creating the horizontal regularization via the coupling and interaction. These properties allow us to establish the desired global stability result and provide some decay properties of the solutions to the linearized system of (3.1.2).

3.2 The H^2 Nonlinear Stability

We now state our first main result.

Theorem 3.2.1 *Consider (3.1.2) with* $\nu > 0$ *and* $\eta > 0$*. Assume the initial data* (u_0, θ_0) *is in* $H^2(\mathbb{R}^2)$ *with* $\nabla \cdot u_0 = 0$*. Then there exists* $\varepsilon = \varepsilon(\nu, \eta) > 0$ *such that, if* (u_0, θ_0) *satisfies*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

then (3.1.2) has a unique global solution (u, θ) *satisfying, for any* $t > 0$,

$$\begin{aligned} &\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu \int_0^t \|\partial_2 u\|_{H^2}^2 \, d\tau \\ &\qquad +2\eta \int_0^t \|\partial_1 \theta\|_{H^2}^2 \, d\tau + C(\nu, \eta) \int_0^t \|\partial_1 u_2\|_{L^2}^2 \, d\tau \leq C\, \varepsilon^2, \end{aligned}$$

where $C(\nu, \eta) > 0$ *and* $C > 0$ *are constants.*

The idea of the proof of Theorem 3.2.1 is based on the construction of a suitable energy functional

$$\begin{aligned} E(t) &= \max_{0 \leq \tau \leq t} \left(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2 \right) + 2\nu \int_0^t \|\partial_2 u\|_{H^2}^2 d\tau \\ &\qquad + 2\eta \int_0^t \|\partial_1 \theta\|_{H^2}^2 d\tau + \delta \int_0^t \|\partial_1 u_2\|_{L^2}^2 \, d\tau \\ &:= E_1(t) + \delta\, E_2(t), \end{aligned} \tag{3.2.1}$$

where

$$E_1(t) := \max_{0\le\tau\le t}\left(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2\right) + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2\,d\tau + 2\eta\int_0^t \|\partial_1\theta\|_{H^2}^2\,d\tau,$$

$$E_2(t) := \int_0^t \|\partial_1 u_2\|_{L^2}^2\,d\tau,$$

and $\delta > 0$ is a suitably selected parameter. We should mention here that the control on the time integral of the horizontal derivative of the velocity field, namely

$$\int_0^t \|\partial_1 u_2(\tau)\|_{L^2}^2 d\tau. \tag{3.2.2}$$

plays an important role in the proof. Due to the special coupling in the system (3.1.2), which allows us to transfer the time integrability from one function in the system to another, we are able to prove that $E(t)$ satisfies

$$E(t) \le C_1 E(0) + C_2 E(t)^{\frac{3}{2}}. \tag{3.2.3}$$

Once (3.2.3) is established, an application of the bootstrapping argument, which general statement can be found in the Appendix A.2, implies that if $E(0)$ is sufficiently small or equivalently

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \le \varepsilon$$

for some sufficiently small $\varepsilon > 0$, then $E(t)$ remains uniformly small for all time, namely

$$E(t) \le C\,\varepsilon^2$$

for a constant $C > 0$ and for all $t \ge 0$.

Before starting the proof of Theorem 3.2.1, we need to state the following useful lemmas.

Lemma 3.2.2 *Assume that f, g, $\partial_2 g$, h and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then, for some constant $C > 0$,*

$$\int_{\mathbb{R}^2} |fgh|\,dx \le C\|f\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_1 h\|_{L^2}^{\frac{1}{2}}.$$

A detailed proof of Lemma 3.2.2 can be found in [18].

Lemma 3.2.3 *Assume that f is in $L^q(\mathbb{R}^2)$,*

$$\|f\|_{L^q} \leq C\,\|f\|_{L^2}^{\frac{2}{q}}\,\|\nabla f\|_{L^2}^{1-\frac{2}{q}},$$

for $2 < q < \infty$.

Lemma 3.2.4 *The following estimates hold when the right-hand sides are all bounded.*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\,\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\,\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\,\|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.$$

We now detail the proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. We define $E(t)$ as in (3.2.1). Our main efforts are devoted to establishing (3.2.3).

We recall that, for a divergence-free vector field u, namely $\nabla \cdot u = 0$, we have

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2}, \quad \|\Delta u\|_{L^2} = \|\nabla \omega\|_{L^2},$$

where $\omega := \nabla \times u$ is the vorticity.

Taking the inner product of (u, θ) with the first two equations in (3.1.2) yields

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_2 u(\tau)\|_{L^2}^2\, d\tau + 2\eta \int_0^t \|\partial_1 \theta(\tau)\|_{L^2}^2 d\tau \\ &= \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \end{aligned} \tag{3.2.4}$$

To estimate the L^2-norm of $(\omega, \nabla\theta)$, we make use of the vorticity and the temerature equations,

$$\begin{aligned} &\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{22} \omega + \partial_1 \theta, \\ &\partial_t \theta + u \cdot \nabla \theta + u_2 = \eta \partial_{11} \theta. \end{aligned} \tag{3.2.5}$$

Taking the inner product of $(\omega, \Delta\theta)$ with the system (3.2.5), we get

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \nu\|\partial_2 \omega\|_{L^2}^2 + \eta\|\partial_1 \nabla\theta\|_{L^2}^2 := I_1 + I_2, \tag{3.2.6}$$

where

$$I_1 := \int (\partial_1\theta\,\omega - \nabla u_2 \cdot \nabla\theta)\,dx, \quad I_2 := -\int \nabla\theta \cdot \nabla u \cdot \nabla\theta\,dx.$$

Owing to $\nabla \cdot u = 0$, one can write ω and u in terms of the stream function ψ, namely $\omega = \Delta\psi$ and $u = \nabla^\perp\psi := (-\partial_2\psi, \partial_1\psi)$, to obtain

$$\begin{aligned} I_1 &:= \int (\partial_1\theta\,\omega - \nabla u_2 \cdot \nabla\theta)\,dx = \int (\partial_1\theta\Delta\psi - \nabla\partial_1\psi \cdot \nabla\theta)\,dx \\ &= \int (-\theta\,\Delta\partial_1\psi + \Delta\partial_1\psi\,\theta)\,dx = 0. \end{aligned} \tag{3.2.7}$$

To bound I_2, we write it as,

$$\begin{aligned} I_2 &:= -\int (\partial_1 u_1(\partial_1\theta)^2 + \partial_1 u_2\partial_1\theta\partial_2\theta + \partial_2 u_1\partial_1\theta\partial_2\theta + \partial_2 u_2(\partial_2\theta)^2)\,dx \\ &:= I_{21} + I_{22} + I_{23} + I_{24}. \end{aligned} \tag{3.2.8}$$

We now estimate the terms on the right-hand side of (3.2.8). The effort is devoted to obtaining an upper bound that is time integrable for each term.

According to Lemma 3.2.2,

$$\begin{aligned} I_{21} &:= -\int \partial_1 u_1(\partial_1\theta)^2\,dx \\ &\le C\,\|\partial_1 u_1\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\theta\|_{L^2}^{\frac{1}{2}} \\ &\le C\,\|\partial_1 u_1\|_{L^2}\,\|\partial_1\theta\|_{L^2}\,\|\partial_1\nabla\theta\|_{L^2} \\ &\le C\,\|u\|_{H^1}\|\partial_1\theta\|_{H^1}^2. \end{aligned} \tag{3.2.9}$$

Due to Lemma 3.2.2 and Young's inequality,

$$\begin{aligned} I_{22} &:= \int \partial_1 u_2\partial_1\theta\partial_2\theta\,dx \\ &\le C\,\|\partial_1\theta\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\theta\|_{L^2}^{\frac{1}{2}} \\ &\le C\,\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\,\|\partial_2\theta\|_{L^2}^{\frac{1}{2}}\,\|\partial_1\theta\|_{L^2}\,\|\partial_2\nabla u\|_{L^2}^{\frac{1}{2}}\,\|\partial_1\nabla\theta\|_{L^2}^{\frac{1}{2}} \\ &\le C\,\|u\|_{H^1}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^1}^{\frac{1}{2}}\|\partial_1\theta\|_{H^1}^{\frac{3}{2}} \\ &\le C\,\|u\|_{H^1}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\Big(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2\Big). \end{aligned} \tag{3.2.10}$$

By Lemma 3.2.2 and Cauchy's inequality,

$$\begin{aligned}I_{23} &:= \int \partial_2 u_1 \partial_1\theta \partial_2\theta \, dx \\ &\leq C\, \|\partial_2\theta\|_{L^2} \|\partial_1\theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\theta\|_{L^2}^{\frac{1}{2}} \|\partial_2 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_2 u_1\|_{L^2}^{\frac{1}{2}} \\ &\leq C\, \|\nabla\theta\|_{L^2} \|\partial_2 u\|_{H^1} \|\partial_1\theta\|_{H^1} \\ &\leq C\, \|\nabla\theta\|_{L^2} \Big(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2 \Big). \qquad (3.2.11)\end{aligned}$$

Using respectively, integrating by parts, the divergence-free condition $\nabla \cdot u = 0$, Lemma 3.2.2 and Young's inequality yield,

$$\begin{aligned}I_{24} &:= \int \partial_1 u_1 (\partial_2\theta)^2 \, dx = -2 \int u_1 \, \partial_2\theta \, \partial_1\partial_2\theta \, dx \\ &\leq C\, \|\partial_1\partial_2\theta\|_{L^2} \|\partial_2\theta\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_2\theta\|_{L^2}^{\frac{1}{2}} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 u_1\|_{L^2}^{\frac{1}{2}} \\ &= C\, \|u_1\|_{L^2}^{\frac{1}{2}} \, \|\partial_2\theta\|_{L^2}^{\frac{1}{2}} \, \|\partial_2 u_1\|_{L^2}^{\frac{1}{2}} \, \|\partial_1\nabla\theta\|_{L^2}^{\frac{3}{2}} \\ &\leq C\, \|u\|_{H^1}^{\frac{1}{2}} \|\nabla\theta\|_{L^2}^{\frac{1}{2}} \Big(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2 \Big). \qquad (3.2.12)\end{aligned}$$

Clearly, each upper bound above is expressed in terms of favorable derivatives (∂_1 on θ and ∂_2 on u) and are time integrable.

Collecting the bounds (3.2.9), (3.2.10), (3.2.11), (3.2.12) and inserting them in (3.2.8), we obtain

$$I_2 \leq C \Big(\|u\|_{H^1} + \|\nabla\theta\|_{L^2} \Big) \Big(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2 \Big). \qquad (3.2.13)$$

In view of (3.2.7), (3.2.13) and (3.2.6), we get

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + 2\nu\|\partial_2\nabla u\|_{L^2}^2 + 2\eta\|\partial_1\nabla\theta\|_{L^2}^2 \\ &\qquad \leq C\,(\|u\|_{H^1} + \|\nabla\theta\|_{L^2})\, \left(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2\right). \qquad (3.2.14)\end{aligned}$$

Integrating (3.2.14) over $[0,t]$ and combining with (3.2.4), we obtain

$$\begin{aligned}&\|(u,\theta)\|_{H^1}^2 + 2\nu \int_0^t \|\partial_2 u(\tau)\|_{H^1}^2 d\tau + 2\eta \int_0^t \|\partial_1\theta(\tau)\|_{H^1}^2 d\tau \\ &\leq \|(u_0,\theta_0)\|_{H^1}^2 + C \int_0^t (\|u\|_{H^1} + \|\nabla\theta\|_{L^2})\, \left(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\theta\|_{H^1}^2\right) d\tau \qquad (3.2.15) \\ &\leq E(0) + C\, E(t)^{\frac{3}{2}}. \qquad (3.2.16)\end{aligned}$$

A simple consequence of (3.2.15) is that any initial small H^1 initial data leads to a global H^1 weak solution. However, it does not appear possible to show that H^1-solutions are unique. In fact, when we evaluate the difference $(\widetilde{u}, \widetilde{\theta})$ of two solutions $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$, the terms generated by the nonlinearity

$$\int \widetilde{u} \cdot \nabla u^{(1)} \cdot \widetilde{u} \, dx \quad \text{and} \quad \int \widetilde{u} \cdot \nabla \theta^{(1)} \cdot \widetilde{\theta} \, dx$$

are hard to deal with. When the solutions are only at the H^1-level, it does not appear possible to bound them suitably. This is one of the reasons that we are seeking global H^2-solutions.

In order to estimate the H^2-norm of (u, θ), it remains to bound the L^2-norm of $(\nabla\omega, \Delta\theta)$. Taking the inner product of $(\Delta\omega, \Delta^2\theta)$ with the system (3.2.5), yields

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2) + \nu\|\partial_2\nabla\omega\|_{L^2}^2 + \eta\|\partial_1\Delta\theta\|_{L^2}^2 := J_1 + J_2 + J_3, \tag{3.2.17}$$

where

$$\begin{aligned} J_1 &:= \int (\nabla\partial_1\theta \cdot \nabla\omega - \Delta u_2 \Delta\theta) \, dx, \\ J_2 &:= -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx, \\ J_3 &:= -\int \Delta\theta \cdot \Delta(u \cdot \nabla\theta) dx. \end{aligned}$$

Due to the divergence-free condition of u, one can write ω and u in terms of the stream function ψ, namely $\omega = \Delta\psi$ and $u_2 = \partial_1\psi$, to obtain

$$\begin{aligned} J_1 &:= \int (\nabla\partial_1\theta \cdot \nabla\omega - \Delta u_2 \Delta\theta) \, dx = \int (\nabla\partial_1\theta \cdot \nabla\omega - \Delta\partial_1\psi\Delta\theta) \, dx \\ &= \int (\nabla\partial_1\theta \cdot \nabla\omega - \partial_1\omega \, \Delta\theta) \, dx = \int (\nabla\partial_1\theta \cdot \nabla\omega + \partial_1\nabla\omega \cdot \nabla\theta) \, dx \\ &= \int \partial_1(\nabla\theta \cdot \nabla\omega) \, dx = 0. \end{aligned} \tag{3.2.18}$$

After integration by parts, we can decompose J_3 as,

$$\begin{aligned}J_3 :=& -\int \Delta\theta\, \Delta u_1\, \partial_1\theta\, dx - \int \Delta\theta\, \Delta u_2\, \partial_2\theta\, dx\\&- 2\int \Delta\theta\, \nabla u_1 \cdot \partial_1\nabla\theta\, dx - 2\int \Delta\theta\, \nabla u_2 \cdot \partial_2\nabla\theta\, dx\\:=& J_{31} + J_{32} + J_{33} + J_{34}.\end{aligned} \tag{3.2.19}$$

Thanks to Lemma 3.2.2 and Young's inequality,

$$\begin{aligned}J_{31} :=& -\int \Delta\theta\, \Delta u_1\, \partial_1\theta\, dx\\\le& C\, \|\partial_1\theta\|_{L^2}\, \|\Delta\theta\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\Delta\theta\|_{L^2}^{\frac{1}{2}}\, \|\Delta u_1\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\Delta u_1\|_{L^2}^{\frac{1}{2}}\\\le& C\, (\|\Delta\theta\|_{L^2} + \|\Delta u_1\|_{L^2})\, \|\partial_1\theta\|_{H^2}^{\frac{3}{2}}\, \|\partial_2\Delta u_1\|_{L^2}^{\frac{1}{2}}\\\le& C\, \Big(\|u\|_{H^2} + \|\theta\|_{H^2}\Big)\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big).\end{aligned} \tag{3.2.20}$$

Further, using the divergence-free condition of u, namely $\nabla \cdot u = 0$ and integration by parts, we split J_{32} into two terms,

$$\begin{aligned}J_{32} :=& -\int \Delta\theta\Delta u_2\partial_2\theta dx\\=& -\int \partial_1\partial_1\theta\, \Delta u_2\, \partial_2\theta dx - \int \partial_2\partial_2\theta\, \Delta u_2\, \partial_2\theta\, dx\\=& -\int \partial_1\partial_1\theta\, \Delta u_2\partial_2\, \theta\, dx + \frac{1}{2}\int \Delta\partial_2 u_2\, (\partial_2\theta)^2\, dx\\=& -\int \partial_1\partial_1\theta\, \Delta u_2\, \partial_2\theta\, dx - \frac{1}{2}\int \Delta\partial_1 u_1\, (\partial_2\theta)^2 dx\\=& -\int \partial_1\partial_1\theta\, \Delta u_2\, \partial_2\theta\, dx + \int \Delta u_1\, \partial_2\theta\, \partial_1\partial_2\theta dx.\end{aligned} \tag{3.2.21}$$

Therefore, according to Lemma 3.2.2 and Young's inequality,

$$\begin{aligned}J_{32} \le& C\, \|\partial_1\partial_1\theta\|_{L^2}\|\Delta u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\theta\|_{L^2}^{\frac{1}{2}}\\&+ C\|\partial_1\partial_2\theta\|_{L^2}\|\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\Delta u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta u_1\|_{L^2}^{\frac{1}{2}}\\\le& C\, (\|\partial_2\theta\|_{L^2} + \|\Delta u\|_{L^2})\, \|\partial_1\nabla\theta\|_{L^2}^{\frac{3}{2}}\, \|\partial_2\Delta u\|_{L^2}^{\frac{1}{2}}\\\le& C\, \Big(\|u\|_{H^2} + \|\theta\|_{H^2}\Big)\Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big).\end{aligned} \tag{3.2.22}$$

To bound J_{33}, we again apply Lemma 3.2.2 and Young's inequality,

$$\begin{aligned} J_{33} &:= -2\int \Delta\theta\,\nabla u_1\cdot\partial_1\nabla\theta\,dx \\ &\le C\,\|\partial_1\nabla\theta\|_{L^2}\,\|\Delta\theta\|_{L^2}^{\frac12}\,\|\partial_1\Delta\theta\|_{L^2}^{\frac12}\,\|\nabla u_1\|_{L^2}^{\frac12}\,\|\partial_2\nabla u_1\|_{L^2}^{\frac12} \\ &\le C\,(\|\Delta\theta\|_{L^2}+\|\nabla u_1\|_{L^2})\,\|\partial_1\theta\|_{H^2}^{\frac32}\,\|\partial_2\nabla u_1\|_{L^2}^{\frac12} \\ &\le C\Big(\|u\|_{H^2}+\|\theta\|_{H^2}\Big)\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \end{aligned} \tag{3.2.23}$$

After integration by parts,

$$\begin{aligned} J_{34} &:= -2\int \Delta\theta\,\nabla u_2\cdot\partial_2\nabla\theta\,dx \\ &= -2\int(\partial_1 u_2\partial_1\partial_2\theta\Delta\theta+\partial_2 u_2\partial_2\partial_2\theta\Delta\theta)\,dx \\ &= -2\int\partial_1 u_2\,\partial_1\partial_2\theta\,\Delta\theta\,dx+2\int\partial_1 u_1\,\partial_2\partial_2\theta\,\Delta\theta\,dx \\ &= -2\int\partial_1 u_2\,\partial_1\partial_2\theta\,\Delta\theta\,dx-2\int u_1\,\partial_1\partial_2\partial_2\theta\,\Delta\theta\,dx-2\int u_1\,\partial_2\partial_2\theta\,\partial_1\Delta\theta\,dx \\ &:= J_{341}+J_{342}+J_{343}. \end{aligned} \tag{3.2.24}$$

By Lemma 3.2.2 and Young's inequality, the terms on the right-hand side of (3.2.24) can be bounded as follows,

$$\begin{aligned} J_{341} &:= -2\int\partial_1 u_2\,\partial_1\partial_2\theta\,\Delta\theta\,dx \\ &\le C\,\|\partial_1\partial_2\theta\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac12}\|\partial_2\partial_1 u_2\|_{L^2}^{\frac12}\|\Delta\theta\|_{L^2}^{\frac12}\|\partial_1\Delta\theta\|_{L^2}^{\frac12} \\ &\le C\,(\|\Delta\theta\|_{L^2}+\|\partial_1 u_2\|_{L^2})\,\|\partial_1\theta\|_{H^2}^{\frac32}\,\|\partial_2\nabla u_2\|_{L^2}^{\frac12} \\ &\le C\Big(\|u\|_{H^2}+\|\theta\|_{H^2}\Big)\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big), \end{aligned} \tag{3.2.25}$$

$$\begin{aligned} J_{342} &:= -2\int u_1\,\partial_1\partial_2\partial_2\theta\,\Delta\theta\,dx \\ &\le C\|\partial_1\partial_2\partial_2\theta\|_{L^2}\|\Delta\theta\|_{L^2}^{\frac12}\|\partial_1\Delta\theta\|_{L^2}^{\frac12}\|u_1\|_{L^2}^{\frac12}\|\partial_2 u_1\|_{L^2}^{\frac12} \\ &\le C\,(\|\Delta\theta\|_{L^2}+\|u_1\|_{L^2})\,\|\partial_1\theta\|_{H^2}^{\frac32}\,\|\partial_2 u_1\|_{L^2}^{\frac12} \\ &\le C\Big(\|u\|_{H^2}+\|\theta\|_{H^2}\Big)\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big), \end{aligned} \tag{3.2.26}$$

$$
\begin{aligned}
J_{343} &:= -2\int u_1\,\partial_2\partial_2\theta\,\partial_1\Delta\theta\,dx \\
&\le C\,\|\partial_1\Delta\theta\|_{L^2}\|\partial_2\partial_2\theta\|_{L^2}^{\frac12}\|\partial_1\partial_2\partial_2\theta\|_{L^2}^{\frac12}\|u_1\|_{L^2}^{\frac12}\|\partial_2 u_1\|_{L^2}^{\frac12} \\
&\le C\,(\|\Delta\theta\|_{L^2}+\|u_1\|_{L^2})\,\|\partial_1\theta\|_{H^2}^{\frac32}\,\|\partial_2 u_1\|_{L^2}^{\frac12} \\
&\le C\Big(\|u\|_{H^2}+\|\theta\|_{H^2}\Big)\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.2.27)
\end{aligned}
$$

Combining the estimates (3.2.25), (3.2.26), (3.2.27) and inserting them in (3.2.24) yields,

$$
J_{34} \le C\Big(\|u\|_{H^2}+\|\theta\|_{H^2}\Big)\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.2.28)
$$

Putting (3.2.20), (3.2.22), (3.2.23) and (3.2.28) together, we get

$$
J_3 \le\ C\Big(\|u\|_{H^2}+\|\theta\|_{H^2}\Big)\Big(\|\partial_2 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2\Big). \qquad (3.2.29)
$$

We now turn to the estimate of J_2. As outlined before, we need the help of the extra regularization term

$$
E_2(t) := \int_0^t \|\partial_1 u_2\|_{L^2}^2\,d\tau. \qquad (3.2.30)
$$

To make full use of the anisotropic dissipation, we further write J_2 as

$$
\begin{aligned}
J_2 :=& -\int \partial_1 u_1\,(\partial_1\omega)^2\,dx - \int \partial_1 u_2\,\partial_1\omega\,\partial_2\omega\,dx \\
& -\int \partial_2 u_1\,\partial_1\omega\,\partial_2\omega dx - \int \partial_2 u_2\,(\partial_2\omega)^2 dx \\
=& \int \partial_2 u_2\,(\partial_1\omega)^2 dx - \int \partial_1 u_2\,\partial_1\omega\,\partial_2\omega\,dx \\
& -\int \partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,dx - \int \partial_2 u_2\,(\partial_2\omega)^2\,dx \\
:=& J_{21}+J_{22}+J_{23}+J_{24}. \qquad (3.2.31)
\end{aligned}
$$

To bound the first two terms, we need to make use of the term in (3.2.30). Using integration by parts, Lemma 3.2.2 and Young's inequality, yields

$$
\begin{aligned}
J_{21} &:= \int \partial_2 u_2 \, (\partial_1 \omega)^2 dx \\
&= -2 \int u_2 \, \partial_1 \omega \, \partial_2 \partial_1 \omega \, dx \\
&\leq C \|\partial_2 \partial_1 \omega\|_{L^2} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \\
&\leq C \, \|u\|_{H^2} \, \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{3}{2}} \, \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \\
&\leq C \, \|u\|_{H^2} \Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 \Big).
\end{aligned} \tag{3.2.32}
$$

According to Lemma 3.2.2 and Cauchy's inequality,

$$
\begin{aligned}
J_{22} &:= - \int \partial_1 u_2 \, \partial_1 \omega \, \partial_2 \omega \, dx \\
&\leq C \, \|\partial_1 u_2\|_{L^2} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \omega\|_{L^2}^{\frac{1}{2}} \\
&\leq C \, \|\nabla \omega\|_{L^2} \, \|\partial_2 \partial_1 \omega\|_{L^2} \, \|\partial_1 u_2\|_{L^2} \\
&\leq C \, \|u\|_{H^2} \Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 \Big).
\end{aligned} \tag{3.2.33}
$$

Thanks to Lemma 3.2.2,

$$
\begin{aligned}
J_{23} &:= - \int \partial_2 u_1 \, \partial_1 \omega \, \partial_2 \omega \, dx \\
&\leq C \, \|\partial_2 u_1\|_{L^2} \, \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \, \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{1}{2}} \, \|\partial_2 \omega\|_{L^2}^{\frac{1}{2}} \, \|\partial_1 \partial_2 \omega\|_{L^2}^{\frac{1}{2}} \\
&\leq C \, \|\nabla \omega\|_{L^2} \, \|\partial_2 \partial_1 \omega\|_{L^2} \, \|\partial_2 u_1\|_{L^2} \\
&\leq C \, \|u\|_{H^2} \|\partial_2 u\|_{H^2}^2.
\end{aligned} \tag{3.2.34}
$$

Similarly, due to Lemma 3.2.2,

$$
\begin{aligned}
J_{24} &:= - \int \partial_2 u_2 \, (\partial_2 \omega)^2 \, dx \\
&\leq C \, \|\partial_2 u_2\|_{L^2} \, \|\partial_2 \omega\|_{L^2}^{\frac{1}{2}} \, \|\partial_1 \partial_2 \omega\|_{L^2}^{\frac{1}{2}} \, \|\partial_2 \omega\|_{L^2}^{\frac{1}{2}} \, \|\partial_2 \partial_2 \omega\|_{L^2}^{\frac{1}{2}} \\
&\leq C \, \|\nabla \omega\|_{L^2} \, \|\partial_2 \nabla \omega\|_{L^2} \, \|\partial_2 u_2\|_{L^2} \\
&\leq C \, \|u\|_{H^2} \|\partial_2 u\|_{H^2}^2.
\end{aligned} \tag{3.2.35}
$$

Collecting the bounds (3.2.32), (3.2.33), (3.2.34), (3.2.35) and inserting them in (3.2.31) yield,

$$J_2 \leq C \|u\|_{H^2} \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 \right). \tag{3.2.36}$$

Inserting $J_1 = 0$, (3.2.29) and (3.2.36) in (3.2.17), we get

$$\begin{aligned} &\frac{d}{dt}(\|\Delta u\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2) + 2\nu\|\partial_2 \Delta u\|_{L^2}^2 + 2\eta\|\partial_1 \Delta\theta\|_{L^2}^2 \\ &\qquad \leq C \left(\|u\|_{H^2} + \|\theta\|_{H^2} \right) \left(\|\partial_1 \theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 \right). \end{aligned} \tag{3.2.37}$$

Integrating (3.2.37) over the time interval $[0, t]$ yields

$$\begin{aligned} &\|\Delta u(t)\|_{L^2}^2 + \|\Delta\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_2 \Delta u\|_{L^2}^2 d\tau + 2\eta \int_0^t \|\Delta \partial_1 \theta\|_{L^2}^2 d\tau \\ &\qquad \leq \|\Delta u_0\|_{L^2}^2 + \|\Delta\theta_0\|_{L^2}^2 \\ &\qquad\quad + C \int_0^t \left(\|u\|_{H^2} + \|\theta\|_{H^2} \right) \left(\|\partial_1 \theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 \right) d\tau \\ &\qquad \leq E(0) + C\, E(t)^{\frac{3}{2}}. \end{aligned} \tag{3.2.38}$$

Combining (3.2.16) and (3.2.38) leads to,

$$\begin{aligned} E_1(t) &:= \max_{0 \leq \tau \leq t} \left(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2 \right) + 2\nu \int_0^t \|\partial_2 u\|_{H^2}^2 \, d\tau + 2\eta \int_0^t \|\partial_1 \theta\|_{H^2}^2 \, d\tau \\ &\leq CE(0) + CE(t)^{\frac{3}{2}}. \end{aligned} \tag{3.2.39}$$

The next major step is to bound the last piece in $E(t)$ defined by (3.2.1), namely

$$E_2(t) := \int_0^t \|\partial_1 u_2\|_{L^2}^2 \, d\tau.$$

Applying ∂_1 to the second equation in (3.1.2) yields,

$$\partial_1 u_2 = -\partial_t \partial_1 \theta - \partial_1 (u \cdot \nabla\theta) + \eta \partial_{111} \theta. \tag{3.2.40}$$

Multiplying (3.2.40) with $\partial_1 u_2$ and then integrating over $\mathbb{R}^2$, we get

$$\begin{aligned} \|\partial_1 u_2\|_{L^2}^2 &= -\int \partial_t \partial_1 \theta \, \partial_1 u_2 \, dx - \int \partial_1 u_2 \, \partial_1 (u \cdot \nabla\theta) \, dx + \eta \int \partial_1 u_2 \, \partial_{111} \theta \, dx \\ &:= K_1 + K_2 + K_3. \end{aligned} \tag{3.2.41}$$

Even though the estimate of K_3 appears to be easy, the term with unfavorable derivative $\partial_1 u_2$ will be absorbed by the left-hand side,

$$K_3 := \eta \int \partial_1 u_2 \, \partial_{111}\theta \, dx \le \eta \|\partial_1 u_2\|_{L^2} \, \|\partial_{111}\theta\|_{L^2} \le \frac{1}{2}\|\partial_1 u_2\|_{L^2}^2 + C\, \|\partial_1\theta\|_{H^2}^2. \quad (3.2.42)$$

We shift the time derivative in K_1,

$$K_1 = -\frac{d}{dt}\int \partial_1\theta \, \partial_1 u_2 \, dx + \int \partial_1\theta \, \partial_1\partial_t u_2 \, dx := K_{11} + K_{12}. \quad (3.2.43)$$

Making use of the equation for the second component of the velocity, we have

$$\begin{aligned} K_{12} &:= -\int \partial_1\partial_1\theta \, \partial_t u_2 \, dx \\ &= -\int \partial_{11}\theta(-(u\cdot\nabla)u_2 - \partial_2 p + \nu\partial_{22}u_2 + \theta) \, dx \\ &= \int \partial_{11}\theta \, (u\cdot\nabla)u_2 \, dx \; + \int \partial_{11}\theta \; \partial_2 p \, dx \\ &\quad - \nu \int \partial_{11}\theta \; \partial_{22}u_2 \, dx \; - \int \partial_{11}\theta \; \theta \, dx. \end{aligned}$$

We further replace the pressure term. Applying the divergence operator to the velocity equation in (3.1.2) yields,

$$p = -\Delta^{-1}\nabla\cdot(u\cdot\nabla u) + \Delta^{-1}\partial_2\theta.$$

Therefore,

$$\begin{aligned} K_{12} :=& \int \partial_{11}\theta \, (u\cdot\nabla)u_2 \, dx \; + \int \partial_{11}\theta \, (-\partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)) \, dx \\ & - \nu \int \partial_{11}\theta \; \partial_{22}u_2 \, dx \; - \int \partial_{11}\theta \; \partial_{11}\Delta^{-1}\theta \, dx \\ :=& \, K_{121} + K_{122} + K_{123} + K_{124}. \end{aligned} \quad (3.2.44)$$

According to the boundedness of the double Riesz transform (see, e.g., [55]),

$$\|\partial_{11}\Delta^{-1} f\|_{L^q} \le C\, \|f\|_{L^q}, \qquad 1 < q < \infty,$$

we have

$$K_{124} := \int \partial_1\theta \, \partial_{11}\Delta^{-1}\partial_1\theta \, dx \le C\, \|\partial_1\theta\|_{L^2}^2 \le C\, \|\partial_1\theta\|_{H^2}^2. \quad (3.2.45)$$

Due to Hölder's inequality and Cauchy's inequality, K_{123} can be bounded by,

$$\begin{aligned}K_{123} &:= -\nu \int \partial_{11}\theta \, \partial_{22}u_2 \, dx \\ &\leq C \|\partial_{11}\theta\|_{L^2} \|\partial_{22}u_2\|_{L^2} \\ &\leq C \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2 \right). \end{aligned} \tag{3.2.46}$$

Using Hölder's inequality, the boundedness of the double Riesz transform, integration by parts, Lemma 3.2.3, Lemma 3.2.4 and Cauchy's inequality,

$$\begin{aligned}K_{122} &:= -\int \partial_1\theta \, \partial_{12}\Delta^{-1}\nabla \cdot (u \cdot \nabla u) \, dx \\ &\leq \|\partial_1\theta\|_{L^2} \, \|\Delta^{-1}\partial_{12}\nabla \cdot (u \cdot \nabla u)\|_{L^2} \\ &\leq C \|\partial_1\theta\|_{L^2} \|\partial_2(u \cdot \nabla u)\|_{L^2} \\ &\leq C \|\partial_1\theta\|_{L^2} \|\partial_2 u \cdot \nabla u + u \cdot \nabla \partial_2 u\|_{L^2} \\ &\leq C \|\partial_1\theta\|_{L^2} \left(\|\partial_2 u\|_{L^4} \|\nabla u\|_{L^4} + \|u\|_{L^\infty} \|\nabla\partial_2 u\|_{L^2} \right) \\ &\leq C \|\partial_1\theta\|_{L^2} \|\partial_2 u\|_{H^1} \|\nabla u\|_{H^1} + C \|\partial_1\theta\|_{L^2} \|u\|_{H^2} \|\nabla\partial_2 u\|_{L^2} \\ &\leq C \|u\|_{H^2} \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2 \right). \end{aligned} \tag{3.2.47}$$

To estimate K_{121}, we write it explicitly as,

$$\begin{aligned}K_{121} &:= \int \partial_{11}\theta (u_1 \partial_1 u_2 + u_2 \partial_2 u_2) dx \\ &= \int \partial_{11}\theta \, u_1 \, \partial_1 u_2 \, dx + \int \partial_{11}\theta \, u_2 \, \partial_2 u_2 \, dx. \end{aligned}$$

Thanks to Lemma 3.2.2, Lemma 3.2.4 and Cauchy's inequality,

$$\begin{aligned}K_{121} \leq\ & C \|\partial_{11}\theta\|_{L^2} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \\ & + C \|u_2\|_{L^\infty} \|\partial_{11}\theta\|_{L^2} \|\partial_2 u_2\|_{L^2} \\ \leq\ & C \|u\|_{H^1} \|\partial_2 u\|_{H^1} \|\partial_{11}\theta\|_{L^2} + C \|u\|_{H^2} \|\partial_2 u\|_{L^2} \|\partial_{11}\theta\|_{L^2} \\ \leq\ & C \|u\|_{H^2} \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2 \right). \end{aligned} \tag{3.2.48}$$

The bounds for K_{12} in (3.2.45), (3.2.46), (3.2.47) and (3.2.48) lead to,

$$K_{12} \leq C \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2 \right) + C \|u\|_{H^2} \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2 \right). \tag{3.2.49}$$

It remains to bound K_2. After integration by parts, we can write K_2 as,

$$K_2 := -\int \partial_1 u_2\, \partial_1 u_1\, \partial_1 \theta\, dx - \int \partial_1 u_2 u_1 \partial_1 \partial_1 \theta\, dx$$
$$- \int \partial_1 u_2 \partial_1 u_2 \partial_2 \theta\, dx - \int \partial_1 u_2 u_2 \partial_1 \partial_2 \theta\, dx.$$

By Lemma 3.2.2 and Young's inequality,

$$\begin{aligned}
K_2 \le\ & C\, \|\partial_1 u_2\|_{L^2} \|\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1\theta\|_{L^2}^{\frac{1}{2}} \\
& + C\, \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1\theta\|_{L^2} \\
& + C\, \|\partial_1 u_2\|_{L^2} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\theta\|_{L^2}^{\frac{1}{2}} \\
& + C\, \|\partial_1\partial_2\theta\|_{L^2} \|u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \\
\le\ & C \Big(\|u\|_{H^2} + \|\theta\|_{H^2}\Big) \Big(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big).
\end{aligned} \tag{3.2.50}$$

Collecting (3.2.42), (3.2.43), (3.2.49) and (3.2.50), we get

$$\begin{aligned}
\frac{1}{2}\, \|\partial_1 u_2\|_{L^2}^2 \le\ & C\, \|\partial_1\theta\|_{H^2}^2 - \frac{d}{dt}\int \partial_1\theta\, \partial_1 u_2\, dx + C\Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big) \\
& + C \Big(\|u\|_{H^2} + \|\theta\|_{H^2}\Big) \Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big).
\end{aligned} \tag{3.2.51}$$

Integrating (3.2.51) over $[0,t]$ and then applying Hölder's inequality and Cauchy's inequality, yield

$$\begin{aligned}
E_2(t) :=\ & \int_0^t \|\partial_1 u_2\|_{L^2}^2\, d\tau \\
\le\ & C \int_0^t \|\partial_1\theta\|_{H^2}^2\, d\tau - 2\int \partial_1\theta\, \partial_1 u_2\, dx + 2\int \partial_1\theta_0\, \partial_1 u_{02}\, dx \\
& + C \int_0^t \Big(\|\partial_1\theta\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\Big)\, d\tau \\
& + C \int_0^t \Big(\|u\|_{H^2} + \|\theta\|_{H^2}\Big) \Big(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{H^2}^2\Big)\, d\tau \\
\le\ & C \int_0^t \|\partial_1\theta\|_{H^2}^2\, d\tau + C \int_0^t \|\partial_2 u\|_{H^2}^2\, d\tau + C\Big(\|u\|_{H^2}^2 + \|\theta\|_{H^2}^2\Big) \\
& + C\Big(\|u_0\|_{H^2}^2 + \|\theta_0\|_{H^2}^2\Big) + C\, E(t)^{\frac{3}{2}}.
\end{aligned} \tag{3.2.52}$$

Adding (3.2.39) and δ (3.2.52) leads to

$$
\begin{aligned}
E(t) &:= E_1(t) + \delta\, E_2(t) \\
&= \max_{0\le\tau\le t}\left(\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2\right) + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2 d\tau + 2\eta\int_0^t \|\partial_1\theta\|_{H^2}^2 d\tau + \delta\int_0^t \|\partial_1 u_2\|_{L^2}^2 d\tau \\
&\le C\,E(0) + C\,E(t)^{\frac{3}{2}} + C\,\delta\,(\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2) + C\,\delta\,(\|u_0\|_{H^2}^2 + \|\theta_0\|_{H^2}^2) \\
&\quad + C\,\delta\int_0^t \|\partial_2 u\|_{H^2}^2 d\tau + C\,\delta\int_0^t \|\partial_1\theta\|_{H^2}^2 d\tau + C\,\delta\,E(t)^{\frac{3}{2}}. \qquad (3.2.53)
\end{aligned}
$$

Now, we need to eliminate the quadratic terms on the right-hand side of (3.2.53) by the corresponding terms on the left-hand side. To do so, it suffices to choose $\delta > 0$ sufficiently small, say

$$
C\,\delta \le \frac{1}{2}, \quad C\,\delta \le \nu, \quad C\,\delta \le \eta,
$$

so that, (3.2.53) is reduced to

$$
E(t) \le C_1\,E(0) + C_2\,E(t)^{\frac{3}{2}}, \qquad (3.2.54)
$$

where C_1 and C_2 are positive constants. An application of the bootstrapping argument to (3.2.54) then leads to the desired stability result. In fact, if the initial data (u_0, θ_0) is sufficiently small,

$$
\|(u_0, \theta_0)\|_{H^2} \le \varepsilon := \frac{1}{4\sqrt{C_1}C_2},
$$

then (3.2.54) allows us to show that

$$
\|(u(t), \theta(t))\|_{H^2} \le \sqrt{2C_1}\,\varepsilon.
$$

The bootstrapping argument starts with the ansatz that, for $t < T$

$$
E(t) \le \frac{1}{4C_2^2} \qquad (3.2.55)
$$

and show that

$$
E(t) \le \frac{1}{8C_2^2} \quad \text{for all } t \le T. \qquad (3.2.56)
$$

Then the bootstrapping argument would imply that $T = \infty$ and (3.2.56) actually holds for all t. (3.2.56) is an easy consequence of (3.2.54) and (3.2.55). Inserting (3.2.55) in (3.2.54) yields

$$\begin{aligned} E(t) &\leq C_1\, E(0) + C_2\, E(t)^{\frac{3}{2}} \\ &\leq C_1\, \varepsilon^2 + C_2\, \frac{1}{2C_2}\, E(t). \end{aligned}$$

That is,

$$\frac{1}{2} E(t) \leq C_1\, \varepsilon^2 \quad \text{or} \quad E(t) \leq 2\, C_1\, \frac{1}{16 C_1 C_2^2} = \frac{1}{8C_2^2} = 2C_1\, \epsilon^2,$$

which is (3.2.56). This completes the proof of the global stability.

Finally we briefly explain the proof of the uniqueness part of Theorem 3.2.1. It is very easy to check that any two solutions $(u^{(1)}, p^{(1)}, \theta^{(1)})$ and $(u^{(2)}, p^{(2)}, \theta^{(2)})$ to (3.1.2) with one of them in the H^2-regularity class, say $(u^{(1)}, \theta^{(1)}) \in L^\infty(0, T; H^2)$ must be unique. In fact, the difference between the two solutions $(\widetilde{u}, \widetilde{p}, \widetilde{\theta})$ with

$$\widetilde{u} = u^{(2)} - u^{(1)}, \quad \widetilde{p} = p^{(2)} - p^{(1)} \quad \text{and} \quad \widetilde{\theta} = \theta^{(2)} - \theta^{(1)}$$

satisfies

$$\begin{cases} \partial_t \widetilde{u} + u^{(2)} \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u^{(1)} + \nabla \widetilde{p} = \nu \partial_{22} \widetilde{u} + \widetilde{\theta} \mathbf{e}_2, \\ \partial_t \widetilde{\theta} + u^{(2)} \cdot \nabla \widetilde{\theta} + \widetilde{u} \cdot \nabla \theta^{(1)} + \widetilde{u}_2 \ = \eta \partial_{11} \widetilde{\theta}, \\ \nabla \cdot \widetilde{u} = 0, \\ \widetilde{u}(x, 0) = 0, \quad \widetilde{\theta}(x, 0) = 0. \end{cases} \tag{3.2.57}$$

We estimate the difference $(\widetilde{u}, \widetilde{p}, \widetilde{\theta})$ in $L^2(\mathbb{R}^2)$. Taking the L^2-inner product of (3.2.57) with $(\widetilde{u}, \widetilde{\theta})$ and applying the divergence-free condition $\nabla \cdot \widetilde{u} = 0$, we find

$$\frac{1}{2} \frac{d}{dt} \|(\widetilde{u}, \widetilde{\theta})\|_{L^2}^2 + \nu \|\partial_2 \widetilde{u}\|_{L^2}^2 + \eta \|\partial_1 \widetilde{\theta}\|_{L^2}^2 = - \int \widetilde{u} \cdot \nabla u^{(1)} \cdot \widetilde{u}\, dx - \int \widetilde{u} \cdot \nabla \theta^{(1)} \cdot \widetilde{\theta}\, dx.$$

Using respectively, Lemma 3.2.2, the uniformly global bound for $\|(u^{(1)}, \theta^{(1)})\|_{H^2}$ and

Cauchy's inequality with epsilon, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\widetilde{u},\widetilde{\theta})\|_{L^2}^2+\nu\|\partial_2\widetilde{u}\|_{L^2}^2+\eta\|\partial_1\widetilde{\theta}\|_{L^2}^2\\
&\le C\,\|\widetilde{u}\|_{L^2}\|\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla u^{(1)}\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u^{(1)}\|_{L^2}^{\frac{1}{2}}\\
&+C\,\|\widetilde{\theta}\|_{L^2}\|\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla \theta^{(1)}\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla \theta^{(1)}\|_{L^2}^{\frac{1}{2}}\\
&\le C\,\|\widetilde{u}\|_{L^2}^{\frac{3}{2}}\|\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}}+C\,\|\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\widetilde{\theta}\|_{L^2}\\
&\le\frac{\nu}{2}\|\partial_2\widetilde{u}\|_{L^2}^2+C\,\|(\widetilde{u},\widetilde{\theta})\|_{L^2}^2.
\end{aligned}$$

It then follows from Gronwall's inequality that

$$\|\widetilde{u}(t)\|_{L^2}=\|\widetilde{\theta}(t)\|_{L^2}=0.$$

That is, these two solutions coincide. This completes the proof of Theorem 3.2.1. ■

3.3 Decay Results for the Linearized System

In this section, we focus on the following linearized system

$$\begin{cases}
\partial_{tt}u-(\eta\partial_{11}+\nu\partial_{22})\partial_t u+\nu\eta\partial_{11}\partial_{22}u+\partial_{11}\Delta^{-1}u=0,\\
\partial_{tt}\theta-(\eta\partial_{11}+\nu\partial_{22})\partial_t \theta+\nu\eta\partial_{11}\partial_{22}\theta+\partial_{11}\Delta^{-1}\theta=0,\\
u(x,0)=u_0(x),\quad \theta(x,0)=\theta_0(x).
\end{cases}\tag{3.3.1}$$

Note that (3.3.1) is the corresponding linearized system of (3.1.12). Our efforts are devoted here to analyze the large-time behavior of the system (3.3.1). To do so, we first represent the solution of (3.3.1) explicitly in terms of kernel functions and the initial data in Subsection 3.3.1. Second, we find suitable upper bounds for the kernel functions in Subsection 3.3.2. We use these upper bounds to obtain precise decay rates of the solution in Subsection 3.3.3. Finally, in Subsection 3.3.4 we establish that the frequencies away from the two axes in the frequency space decay exponentially to zero in time.

3.3.1 Representation of the Solutions in Terms of the Kernel Functions

In the following Proposition, we solve the system (3.3.1) and express the solution in terms of kernel functions and the initial data.

Proposition 3.3.1 *The solution of (3.3.1) can be explicitly represented as*

$$u_1(t) = K_1(t)\, u_{10} + K_2(t)\, \theta_0, \tag{3.3.2}$$

$$u_2(t) = K_1(t)\, u_{20} + K_3(t)\, \theta_0, \tag{3.3.3}$$

$$\theta(t) = K_4(t)\, u_{20} + K_5(t)\, \theta_0, \tag{3.3.4}$$

where K_1 through K_5 are Fourier multiplier operators with their symbols given by

$$K_1(\xi,t) = G_2(\xi,t) - \nu\xi_2^2 G_1(\xi,t), \qquad K_2(\xi,t) = -\frac{\xi_1\xi_2}{|\xi|^2}\, G_1(\xi,t), \tag{3.3.5}$$

$$K_3(\xi,t) = \frac{\xi_1^2}{|\xi|^2} G_1(\xi,t), \quad K_4 = -G_1, \quad K_5(\xi,t) = G_2(\xi,t) - \eta\xi_1^2 G_1(\xi,t). \tag{3.3.6}$$

Here G_1 and G_2 are two explicit symbols involving the roots λ_1 and λ_2 of the characteristic equation

$$\lambda^2 + (\eta\xi_1^2 + \nu\xi_2^2)\lambda + \nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2} = 0$$

or

$$\lambda_1 = -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) - \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)},$$

$$\lambda_2 = -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) + \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}.$$

More precisely, when $\lambda_1 \neq \lambda_2$,

$$G_1(\xi,t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \qquad G_2(\xi,t) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}. \tag{3.3.7}$$

When $\lambda_1 = \lambda_2$,

$$G_1(\xi,t) = te^{\lambda_1 t}, \qquad G_2(\xi,t) = e^{\lambda_1 t} - \lambda_1 t\, e^{\lambda_1 t}. \tag{3.3.8}$$

The proof of Proposition 3.3.1 relies on the following lemma that solves the degenerate damped wave equation explicitly via the method of operator splitting.

Lemma 3.3.2 *Assume that f satisfies the damped degenerate wave type equation*

$$\begin{cases} \partial_{tt} f - (\nu\partial_{22} + \eta\partial_{11})\partial_t f + \eta\nu\partial_{11}\partial_{22} f + \partial_{11}\Delta^{-1} f = F, \\ f(x,0) = f_0(x), \quad (\partial_t f)(x,0) = f_1(x). \end{cases} \tag{3.3.9}$$

Then f can be explicitly represented as

$$f(t) = G_1(t)\, f_1 + G_2(t)\, f_0 + \int_0^t G_1(t-\tau)\, F(\tau)\, d\tau, \tag{3.3.10}$$

where G_1 and G_2 are two Fourier multiplier operators with their symbols given by

$$G_1(\xi,t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \qquad G_2(\xi,t) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} \tag{3.3.11}$$

with λ_1 and λ_2 being the roots of the characteristic equation

$$\lambda^2 + (\eta\xi_1^2 + \nu\xi_2^2)\lambda + \nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2} = 0 \tag{3.3.12}$$

or

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) - \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}, \\ \lambda_2 &= -\frac{1}{2}(\eta\xi_1^2 + \nu\xi_2^2) + \frac{1}{2}\sqrt{(\eta\xi_1^2 + \nu\xi_2^2)^2 - 4\left(\nu\eta\xi_1^2\xi_2^2 + \frac{\xi_1^2}{|\xi|^2}\right)}. \end{aligned} \tag{3.3.13}$$

When $\lambda_1 = \lambda_2$, (3.3.10) remains valid if we replace G_1 and G_2 in (3.3.11) by their corresponding limit form, namely,

$$G_1(\xi,t) = \lim_{\lambda_2\to\lambda_1} \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = te^{\lambda_1 t}$$

and

$$G_2(\xi,t) = \lim_{\lambda_2\to\lambda_1} \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} = e^{\lambda_1 t} - \lambda_1 t e^{\lambda_1 t}.$$

Proof of Lemma 3.3.2. We first consider the case when $F \equiv 0$. Since $\lambda_1(\xi)$ and $\lambda_2(\xi)$ are the roots of the characteristic equation in (3.3.12), we can decompose the second-order differential operator as follows,

$$(\partial_t - \lambda_1(D))(\partial_t - \lambda_2(D)) f = 0 \tag{3.3.14}$$

and

$$(\partial_t - \lambda_2(D))(\partial_t - \lambda_1(D))f = 0, \tag{3.3.15}$$

where $\lambda_1(D)$ and $\lambda_1(D)$ are the Fourier multiplier operators with their symbols given by $\lambda_1(\xi)$ and $\lambda_2(\xi)$, or

$$\lambda_1(D) = \frac{1}{2}(\nu\partial_{22} + \eta\partial_{11}) - \frac{1}{2}\sqrt{(\nu\partial_{22} + \eta\partial_{11})^2 - 4(\nu\eta\partial_{1122} + \partial_{11}\Delta^{-1})},$$
$$\lambda_2(D) = \frac{1}{2}(\nu\partial_{22} + \eta\partial_{11}) + \frac{1}{2}\sqrt{(\nu\partial_{22} + \eta\partial_{11})^2 - 4(\nu\eta\partial_{1122} + \partial_{11}\Delta^{-1})}.$$

Obviously, we can rewrite (3.3.14) and (3.3.15) into the following two systems,

$$\begin{cases} (\partial_t - \lambda_1(D))g = 0, \\ (\partial_t - \lambda_2(D))f = g \end{cases} \tag{3.3.16}$$

and

$$\begin{cases} (\partial_t - \lambda_2(D))h = 0, \\ (\partial_t - \lambda_1(D))f = h. \end{cases} \tag{3.3.17}$$

Taking the difference of the second equations of (3.3.16) and (3.3.17), yields

$$(\lambda_1(D) - \lambda_2(D))f = g - h$$

or equivalently

$$f = ((\lambda_1(D) - \lambda_2(D)))^{-1}(g - h). \tag{3.3.18}$$

Further, solving the first equations of (3.3.16) and (3.3.17), we get

$$g(t) = g(0)\, e^{\lambda_1(D)\, t} = ((\partial_t f)(0) - \lambda_2(D)f(0))\, e^{\lambda_1(D)t} \tag{3.3.19}$$

and

$$h(t) = h(0)\, e^{\lambda_2(D)\, t} = ((\partial_t f)(0) - \lambda_1(D)f(0))\, e^{\lambda_2(D)t}, \tag{3.3.20}$$

where we have used the second equations of (3.3.16) and (3.3.17) to obtain the initial data $g(0)$ and $h(0)$.

Plugging (3.3.19) and (3.3.20) into (3.3.18), we conclude that

$$\begin{aligned} f(t) &= (\lambda_1(D) - \lambda_2(D))^{-1} \Big((e^{\lambda_1(D)t} - e^{\lambda_2(D)t}) \, (\partial_t f)(0) \\ &\quad + (\lambda_1(D) e^{\lambda_2(D)t} - \lambda_2(D) e^{\lambda_1(D)t}) \, f(0) \Big) \\ &= G_1 \, f_1 + G_2 \, f_0, \end{aligned}$$

where

$$G_1 = \frac{e^{\lambda_1(D)t} - e^{\lambda_2(D)t}}{\lambda_1(D) - \lambda_2(D)}, \qquad G_2 = \frac{\lambda_1(D) e^{\lambda_2(D)t} - \lambda_2(D) e^{\lambda_2(D)t}}{\lambda_1(D) - \lambda_2(D)}.$$

In the case when F is not identically zero in (3.3.9), the formula in (3.3.10) is obtained via Duhamel's principle. This completes the proof of Lemma 3.3.2. ■

We are now ready to prove Proposition 3.3.1.

Proof of Proposition 3.3.1. Applying Lemma 3.3.2, we have

$$u(t) = G_2(t) \, u_0 + G_1(t) \, (\partial_t u)(x, 0), \quad \theta(t) = G_2(t) \, \theta_0 + G_1(t) \, (\partial_t \theta)(x, 0). \tag{3.3.21}$$

Since u and θ satisfy the original linearized system,

$$\begin{cases} \partial_t u_1 = \nu \, \partial_{22} u_1 - \partial_1 \partial_2 \Delta^{-1} \theta, \\ \partial_t u_2 = \nu \, \partial_{22} u_2 + \partial_{11} \Delta^{-1} \theta, \\ \partial_t \theta = \eta \, \partial_{11} \theta - u_2, \end{cases}$$

we obtain

$$\begin{cases} (\partial_t u_1)(x, 0) = \nu \, \partial_{22} u_{10} - \partial_1 \partial_2 \Delta^{-1} \theta_0, \\ (\partial_t u_2(x, 0) = \nu \, \partial_{22} u_{20} + \partial_{11} \Delta^{-1} \theta_0, \\ (\partial_t \theta)(x, 0) = \eta \, \partial_{11} \theta_0 - u_{20}. \end{cases} \tag{3.3.22}$$

Inserting (3.3.22) in (3.3.21), yields

$$\begin{aligned} u_1(t) &= (G_2(t) \, + \nu \partial_{22} G_1) \, u_{10} - \partial_1 \partial_2 \Delta^{-1} \, G_1 \, \theta_0, \\ u_2(t) &= (G_2(t) \, + \nu \partial_{22} G_1) \, u_{20} + \partial_{11} \Delta^{-1} \, G_1 \, \theta_0, \\ \theta(t) &= -G_1 \, u_{20} + (G_2 + \eta \partial_{11} G_1) \theta_0, \end{aligned}$$

which are the representations in (3.3.2), (3.3.3) and (3.3.4). This completes the proof of Proposition 3.3.1. ■

3.3.2 Upper Bounds for the Kernel Functions

In this subsection, we provide upper bounds for the kernels $K_1(\xi,t)$ through $K_5(\xi,t)$. Since these kernel functions rely on the Fourier frequencies ξ, we need to split the frequency space into three subdomains, S_{11}, S_{12}, S_2, and analyze the behavior of the kernel functions in each of these subdomains. Details of these upper bounds are given in the following Proposition.

Proposition 3.3.3 *Assume the kernel functions K_1 through K_5 are given by (3.3.5) and (4.3.158) with G_1 and G_2 defined in (3.3.7) and (3.3.8). Set*

$$S_1 = \left\{ \xi = (\xi_1,\xi_2) \in \mathbb{R}^2,\ \nu\eta\xi_1^2\,\xi_2^2 + \xi_1^2\,|\xi|^{-2} \geq \frac{3}{16}(\nu\xi_2^2+\eta\xi_1^2)^2 \right\},$$
$$S_2 = \mathbb{R}^2 \setminus S_1.$$

The kernel functions K_1 through K_5 can then be bounded as follows.

(a) Let $\xi \in S_1$. Then

$$Re\lambda_1 \leq -\frac{1}{2}(\nu\xi_2^2+\eta\xi_1^2), \qquad Re\lambda_2 \leq -\frac{1}{4}(\nu\xi_2^2+\eta\xi_1^2),$$

where Re denotes the real part, and, for constants $c_0>0$ and $C>0$,

$$|K_1(\xi,t)|, |K_5(\xi,t)| \leq C\,e^{-c_0|\xi|^2 t}, \tag{3.3.23}$$

$$|K_2(\xi,t)|, |K_3(\xi,t)|, |K_4(\xi,t)| \leq C\,t\,e^{-c_0|\xi|^2 t}. \tag{3.3.24}$$

(b) Let $\xi \in S_2$. Then

$$\lambda_1 \leq -\frac{3}{4}(\nu\xi_2^2+\eta\xi_1^2), \qquad \lambda_2 \leq -\frac{\nu\eta\xi_1^2\xi_2^2+\xi_1^2|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2},$$

$$|K_1|,\ |K_5| \leq C\,e^{-\frac{3}{4}(\nu\xi_2^2+\eta\xi_1^2)t} + C\,e^{-\frac{\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2}t} \tag{3.3.25}$$

and

$$|K_2| \le \frac{C|\xi_1||\xi_2|}{|\xi|^4}\, e^{-c_0\,|\xi|^2 t} + \frac{C|\xi_1||\xi_2|}{|\xi|^4}\, e^{-c_0 \frac{\xi_1^2\xi_2^2}{|\xi|^2} t}\, e^{-c_0\, \frac{\xi_1^2}{|\xi|^4} t}, \tag{3.3.26}$$

$$|K_3| \le \frac{C|\xi_1|^2}{|\xi|^4}\, e^{-c_0\,|\xi|^2 t} + \frac{C|\xi_1|^2}{|\xi|^4}\, e^{-c_0 \frac{\xi_1^2\xi_2^2}{|\xi|^2} t}\, e^{-c_0\, \frac{\xi_1^2}{|\xi|^4} t},$$

$$|K_4| \le \frac{C}{|\xi|^2}\, e^{-c_0\,|\xi|^2 t} + \frac{C}{|\xi|^2}\, e^{-c_0 \frac{\xi_1^2\xi_2^2}{|\xi|^2} t}\, e^{-c_0\, \frac{\xi_1^2}{|\xi|^4} t}.$$

In view of the above Proposition, one can see that the bounds for these kernel functions K_1 throught K_5 are anisotropic and are not uniform in different directions. We now prove Proposition 3.3.3.

Proof. To prove the bounds in (a), we need first to divide S_1 into two subsets,

$$S_{11} = \left\{\xi \in S_1, (\nu\xi_2^2 + \eta\xi_1^2)^2 \ge 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2})\right\},$$

$$S_{12} = S_1 \setminus S_{11}.$$

For any $\xi \in S_{11}$,

$$0 \le (\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2}) \le \frac{1}{4}(\nu\xi_2^2 + \eta\xi_1^2)^2.$$

According to the formula for λ_1 and λ_2 in (3.3.13), λ_1 and λ_2 are real and satisfy

$$\lambda_1 \le -\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2), \qquad \lambda_2 \le -\frac{1}{4}(\nu\xi_2^2 + \eta\xi_1^2).$$

It then follows by the mean-value theorem that

$$|G_1| = \left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right| \le t\, e^{-C\,|\xi|^2 t}, \tag{3.3.27}$$

for some constant $C > 0$.

Writing G_2 in (3.3.7) in terms of G_1,

$$G_2 = e^{\lambda_1 t} - \lambda_1 G_1$$

and using the simple fact that $x^m e^{-x} \le C(m)$ for any $x \ge 0$ and $m \ge 0$, we can bound K_1 and K_5 in S_{11} as follows,

$$\begin{aligned}
|K_1| &\le |G_2| + \nu|\xi_2^2|\,|G_1| \le e^{-c_0|\xi|^2 t} + C\,|\xi|^2\,t\,e^{-C\,|\xi|^2 t} + \nu|\xi_2^2|\,t\,e^{-C\,|\xi|^2 t} \\
&\le C\,e^{-c_0|\xi|^2 t}, \\
|K_5| &\le |G_2| + \eta|\xi_1^2|\,|G_1| \le C\,e^{-c_0|\xi|^2 t},
\end{aligned}$$

where $C > 0$ and $c_0 > 0$ are constants. The bounds K_2, K_3 and K_4 follow directly from (3.3.27). For any $\xi \in S_{12}$,

$$(\nu\xi_2^2 + \eta\xi_1^2)^2 < 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2})$$

and, as a consequence, λ_1 and λ_2 are complex numbers,

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2) - \frac{i}{2}\sqrt{4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2}) - (\nu\xi_2^2 + \eta\xi_1^2)^2}, \\
\lambda_2 &= -\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2) + \frac{i}{2}\sqrt{4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2}) - (\nu\xi_2^2 + \eta\xi_1^2)^2}.
\end{aligned}$$

Then

$$Re\,\lambda_1 = Re\,\lambda_2 = -\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2).$$

Further, we have

$$\begin{aligned}
|G_1| &= \left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right| \\
&= e^{-\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2)t}\left|\frac{\sin\left(t\sqrt{4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2}) - (\nu\xi_2^2 + \eta\xi_1^2)^2}\right)}{\sqrt{4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2}) - (\nu\xi_2^2 + \eta\xi_1^2)^2}}\right| \\
&\le t\,e^{-\frac{1}{2}(\nu\xi_2^2 + \eta\xi_1^2)t}.
\end{aligned}$$

The desired upper bounds for K_1 through K_5 then follow as before.

We now prove the bounds in part (b) of Proposition 3.3.3.

For any $\xi \in S_2$, we have

$$(\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2}) \ge \frac{1}{4}(\nu\xi_2^2 + \eta\xi_1^2)^2. \qquad (3.3.28)$$

Then λ_1 and λ_2 are both real. Obviously, λ_1 satisfies

$$\lambda_1 \le -\frac{3}{4}(\nu\xi_2^2 + \eta\xi_1^2). \tag{3.3.29}$$

In addition, λ_2 is bounded by

$$\begin{aligned} \lambda_2 &= -\frac{1}{2}\left((\nu\xi_2^2 + \eta\xi_1^2) - \sqrt{(\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2})}\right) \\ &= -2\frac{\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2 + \eta\xi_1^2 + \sqrt{(\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2})}} \\ &\le -\frac{\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2 + \eta\xi_1^2}. \end{aligned} \tag{3.3.30}$$

Hence, it results from (3.3.28), (3.3.29) and (3.3.30) that

$$\begin{aligned} |G_1| &\le \frac{1}{\sqrt{(\nu\xi_2^2 + \eta\xi_1^2)^2 - 4(\nu\eta\xi_1^2\xi_2^2 + |\xi_1|^2|\xi|^{-2})}} \\ &\quad \times \left(e^{-\frac{3}{4}(\nu\xi_2^2+\eta\xi_1^2)t} + e^{-\frac{\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2}t}\right) \\ &\le \frac{2}{\nu\xi_2^2 + \eta\xi_1^2}\left(e^{-\frac{3}{4}(\nu\xi_2^2+\eta\xi_1^2)t} + e^{-\frac{\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2}t}\right) \\ &\le \frac{C}{|\xi|^2}\, e^{-c_0\,|\xi|^2 t} + \frac{C}{|\xi|^2}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t}, \end{aligned}$$

where $C > 0$ and $c_0 > 0$ are constants. Therefore,

$$|K_2| \le \frac{C|\xi_1||\xi_2|}{|\xi|^4}\, e^{-c_0\,|\xi|^2 t} + \frac{C|\xi_1||\xi_2|}{|\xi|^4}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t},$$

$$|K_3| \le \frac{C|\xi_1|^2}{|\xi|^4}\, e^{-c_0\,|\xi|^2 t} + \frac{C|\xi_1|^2}{|\xi|^4}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t}$$

and

$$|K_4| \le \frac{C}{|\xi|^2}\, e^{-c_0\,|\xi|^2 t} + \frac{C}{|\xi|^2}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t}.$$

K_1 is bounded by

$$\begin{aligned} |K_1| &\le |G_2| + \nu|\xi_2^2|\,|G_1| \le e^{\lambda_1 t} \le e^{\lambda_1 t} + |\lambda_1||G_1| + \nu|\xi_2^2|\,|G_1| \\ &\le C\, e^{-\frac{3}{4}(\nu\xi_2^2+\eta\xi_1^2)t} + C\, e^{-\frac{\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2}t}. \end{aligned}$$

K_5 obeys the same bound. This completes the proof of Proposition 3.3.3. ■

3.3.3 Large-Time Behavior in the Standard Homogeneous Sobolev Space

By making use of the upper bounds for the kernel functions K_1 through K_5 in Proposition 3.3.3, we are able to derive the precise large-time behavior of the solutions to (3.3.1). To reflect the anisotropic behavior of the solutions, we need to use anisotorpic Sobolev type spaces defined as follows. For $s \geq 0$ and $\sigma \geq 0$, the anisotropic Sobolev space $\mathring{H}_1^{s,-\sigma}(\mathbb{R}^2)$ consists of functions f satisfying

$$\|f\|_{\mathring{H}_1^{s,-\sigma}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\xi|^{2s}\, |\xi_1|^{-2\sigma} |\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}} < \infty.$$

Similarly, $\mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)$ consists of functions f satisfying

$$\|f\|_{\mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\xi|^{2s}\, |\xi_2|^{-2\sigma} |\widehat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}} < \infty.$$

In addition, we write $\mathring{H}^{s,-\sigma}(\mathbb{R}^2) = \mathring{H}_1^{s,-\sigma}(\mathbb{R}^2) \cap \mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)$ with the norm given by

$$\|f\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)} = \|f\|_{\mathring{H}_1^{s,-\sigma}(\mathbb{R}^2)} + \|f\|_{\mathring{H}_2^{s,-\sigma}(\mathbb{R}^2)}.$$

Theorem 3.3.4 *Consider the linearized system in (3.3.1) with the initial data u_0 and θ_0 satisfying $\nabla \cdot u_0 = 0$ and*

$$u_0 \in \mathring{H}^{0,-\sigma} \cap \mathring{H}^{s,-\sigma} \cap \mathring{H}^{s-2,-\sigma}, \quad \theta_0 \in \mathring{H}^{0,-\sigma} \cap \mathring{H}^{s,-\sigma} \cap \mathring{H}^{s-1,-\sigma},$$

where $s \geq 0$ and $\sigma \geq 0$ satisfy $s + \sigma \geq 2$. Then the corresponding solution (u, θ) to (3.3.1) satisfies, for some constant $C > 0$,

$$\begin{aligned}
\|u_1(t)\|_{\mathring{H}^s} &\leq C\, t^{-\frac{1}{2}(s+\sigma)}\, \|u_{10}\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{\sigma}{2}}\, \|u_{10}\|_{\mathring{H}^{s,-\sigma}} \\
&\quad + C\, t^{-\frac{1}{2}(s+\sigma)+1}\, \|\theta_0\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{1}{2}-\frac{\sigma}{2}}\, \|\theta_0\|_{\mathring{H}^{s-1,-\sigma}}, \\
\|u_2(t)\|_{\mathring{H}^s} &\leq C\, t^{-\frac{1}{2}(s+\sigma)}\, \|u_{20}\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{\sigma}{2}}\, \|u_{20}\|_{\mathring{H}^{s,-\sigma}} \\
&\quad + C\, t^{-\frac{1}{2}(s+\sigma)+1}\, \|\theta_0\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-1-\frac{\sigma}{2}}\, \|\theta_0\|_{\mathring{H}^{s,-\sigma}}, \\
\|\theta(t)\|_{\mathring{H}^s} &\leq C\, t^{-\frac{1}{2}(s+\sigma)+1}\, \|u_{20}\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{\sigma}{2}}\, \|u_{20}\|_{\mathring{H}^{s-2,-\sigma}} \\
&\quad + C\, t^{-\frac{1}{2}(s+\sigma)}\, \|\theta_0\|_{\mathring{H}^{0,-\sigma}} + C\, t^{-\frac{\sigma}{2}}\, \|\theta_0\|_{\mathring{H}^{s,-\sigma}},
\end{aligned}$$

where $\mathring{H}^s$ *denotes the standard homogeneous Sobolev space with its norm defined by*

$$\|f\|_{\mathring{H}^s} = \||\xi|^s\,|\widehat{f}(\xi)|\|_{L^2(\mathbb{R}^2)}.$$

Before proving Theorem 3.3.4, we recall the next lemma that provides an explicit decay rate for the heat kernel associated with a fractional Laplacian Λ^α ($\alpha \in \mathbb{R}$) which is defined through the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi). \tag{3.3.31}$$

Lemma 3.3.5 *Let* $\alpha \geq 0$, $\beta > 0$ *and* $1 \leq q \leq p \leq \infty$. *Then there exists a constant* C *such that, for any* $t > 0$,

$$\|\Lambda^\alpha e^{-\Lambda^\beta t} f\|_{L^p(\mathbb{R}^d)} \leq C\, t^{-\frac{\alpha}{\beta} - \frac{d}{\beta}(\frac{1}{q} - \frac{1}{p})}\, \|f\|_{L^q(\mathbb{R}^d)}.$$

The proof of the Lemma can be found in [61].

In addition to the fractional operator written as in (3.3.31), we also make use of the fractional operators Λ_i^σ with $i = 1, 2$ defined by

$$\widehat{\Lambda_i^\sigma f}(\xi) = |\xi_i|^\sigma \widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2).$$

The following proof of Theorem 3.3.4 is based on Proposition 3.3.3 and Lemma 3.3.5.

Proof of Theorem 3.3.4. Taking the $\mathring{H}^s$-norm of u_1 in (3.3.2), applying Plancherel's theorem and dividing the spatial domain $\mathbb{R}^2$ as in Proposition 3.3.1, yield

$$\begin{aligned}
\|u_1(t)\|_{\mathring{H}^s(\mathbb{R}^2)} &\leq \|\Lambda^s K_1(t) u_0\|_{L^2(\mathbb{R}^2)} + \|\Lambda^s K_2(t)\theta_0\|_{L^2(\mathbb{R}^2)} \\
&\leq C\,\||\xi|^s\, K_1(\xi,t)\widehat{u_0}(\xi)\|_{L^2(S_1)} + C\,\||\xi|^s\, K_1(\xi,t)\widehat{u_0}(\xi)\|_{L^2(S_2)} \\
&+ C\,\||\xi|^s\, K_2(\xi,t)\widehat{\theta_0}(\xi)\|_{L^2(S_1)} + C\,\||\xi|^s\, K_2(\xi,t)\widehat{\theta_0}(\xi)\|_{L^2(S_2)}.
\end{aligned} \tag{3.3.32}$$

The terms on the right-hand side of (3.3.32) can be bounded as follows.

Using Proposition 3.3.3, Plancherel's theorem and Lemma 3.3.5, we get

$$\begin{aligned}
\||\xi|^s K_1(\xi,t)\widehat{u_0}(\xi)\|_{L^2(S_1)} &\le C\,\||\xi|^s e^{-c_0|\xi|^2 t}\,\widehat{u_0}(\xi)\|_{L^2(S_1)} \\
&= C\,\||\xi|^s\,|\xi_1|^\sigma e^{-c_0|\xi|^2 t}\;|\xi_1|^{-\sigma}\widehat{u_0}(\xi)\|_{L^2(S_1)} \\
&\le C\,\||\xi|^{s+\sigma} e^{-c_0|\xi|^2 t}\;|\xi_1|^{-\sigma}\widehat{u_0}(\xi)\|_{L^2(S_1)} \\
&= C\,\|\Lambda^{s+\sigma} e^{c_0\Delta t}\;\Lambda_1^{-\sigma} u_0\|_{L^2(\mathbb{R}^2)} \\
&\le C\,t^{-\frac{1}{2}(s+\sigma)}\,\|\Lambda_1^{-\sigma} u_0\|_{L^2(\mathbb{R}^2)}. \qquad (3.3.33)
\end{aligned}$$

Invoking (3.3.25) in Proposition 3.3.3,

$$\begin{aligned}
\||\xi|^s K_1(\xi,t)\widehat{u_0}(\xi)\|_{L^2(S_2)} &\le C\,\||\xi|^s e^{-c_0|\xi|^2 t}\,\widehat{u_0}(\xi)\|_{L^2(S_2)} \\
&\quad + C\,\||\xi|^s e^{-\frac{\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2}t}\widehat{u_0}(\xi)\|_{L^2(S_2)}. \qquad (3.3.34)
\end{aligned}$$

The first part on the right-hand side of (3.3.34) can be bounded similarly as (3.3.33). To deal with the second piece, we need first to split the domain S_2 into two subdomains S_{21} and S_{22} where

$$S_{21} = \{\xi \in S_2,\ |\xi_1| \ge |\xi_2|\}, \quad S_{22} = \{\xi \in S_2,\ |\xi_1| < |\xi_2|\}.$$

Clearly, for any $\xi \in S_{21}$,

$$-\frac{\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2} \le -C\,|\xi_2|^2 - C\,|\xi_1|^2|\xi|^{-4} \le -C\,|\xi_2|^2 \qquad (3.3.35)$$

and for any $\xi \in S_{22}$,

$$-\frac{\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2} \le -C\,|\xi_1|^2 - C\,|\xi_1|^2|\xi|^{-4} \le -C\,|\xi_1|^2. \qquad (3.3.36)$$

It follows that,

$$\begin{aligned}
&\||\xi|^s e^{-\frac{\nu\eta\xi_1^2\xi_2^2+|\xi_1|^2|\xi|^{-2}}{\nu\xi_2^2+\eta\xi_1^2}t}\widehat{u_0}(\xi)\|_{L^2(S_2)} \\
&\le C\,\||\xi|^s e^{-C\,|\xi_2|^2 t}\,\widehat{u_0}(\xi)\|_{L^2(S_{21})} + C\,\||\xi|^s e^{-C\,|\xi_1|^2 t}\,\widehat{u_0}(\xi)\|_{L^2(S_{22})} \\
&\le C\,\||\xi|^s\,|\xi_2|^\sigma e^{-C\,|\xi_2|^2 t}\,|\xi_2|^{-\sigma}\,\widehat{u_0}(\xi)\|_{L^2(S_{21})} \\
&+\ C\,\||\xi|^s\,|\xi_1|^\sigma e^{-C\,|\xi_1|^2 t}\,|\xi_1|^{-\sigma}\,\widehat{u_0}(\xi)\|_{L^2(S_{22})} \\
&\le C\,t^{-\frac{\sigma}{2}}\,\|u_0\|_{\mathring{H}^{s,-\sigma}}. \qquad (3.3.37)
\end{aligned}$$

We now estimate $\||\xi|^s\, K_2(\xi,t)\widehat{\theta}_0(\xi)\|_{L^2(S_1)}$. Owing to (3.3.24) in Proposition 3.3.3 and following the same process as in (3.3.33), we have

$$\begin{aligned}\||\xi|^s\, K_2(\xi,t)\widehat{\theta}_0(\xi)\|_{L^2(S_1)} &\le C\, t\, \||\xi|^s\, e^{-c_0|\xi|^2 t}\, \widehat{\theta}_0(\xi)\|_{L^2(S_1)} \\ &\le C\, t^{-\frac{1}{2}(s+\sigma)+1}\, \|\Lambda_1^{-\sigma}\theta_0\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.3.38}$$

We now bound $\||\xi|^s\, K_2(\xi,t)\widehat{\theta}_0(\xi)\|_{L^2(S_2)}$. According to (3.3.26) in Proposition 3.3.3,

$$\begin{aligned}\||\xi|^s\, K_2(\xi,t)\widehat{\theta}_0(\xi)\|_{L^2(S_2)} &\le C\, \||\xi|^s\, \frac{\xi_1\xi_2}{|\xi|^4} e^{-c_0|\xi|^2 t}\widehat{\theta}_0(\xi)\|_{L^2(S_2)} \\ &\quad + C\left\| |\xi|^s\, \frac{|\xi_1||\xi_2|}{|\xi|^4}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t}\, \widehat{\theta}_0(\xi)\right\|_{L^2(S_2)}. \end{aligned} \tag{3.3.39}$$

The first piece on the right-hand side of (3.3.39) can be bounded in the same way as in (3.3.33) and (3.3.38),

$$\begin{aligned}\||\xi|^s\, \frac{\xi_1\xi_2}{|\xi|^4} e^{-c_0|\xi|^2 t}\widehat{\theta}_0(\xi)\|_{L^2(S_2)} &\le \||\xi|^{s-2}\, e^{-c_0|\xi|^2 t}\widehat{\theta}_0(\xi)\|_{L^2(\mathbb{R}^2)} \\ &\le C\, t^{-\frac{1}{2}(s+\sigma)+1}\, \|\Lambda_1^{-\sigma}\theta_0\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.3.40}$$

To bound the second part on the right-hand side of (3.3.39), we use the simple fact that $x^m\, e^{-x} \le C(m)$ valid for any $m \ge 0$ and $x \ge 0$, and proceed as in (3.3.35) and (3.3.36) to get

$$\begin{aligned}|\xi|^s\, \frac{|\xi_1||\xi_2|}{|\xi|^4}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t} &= |\xi|^{s-1}\frac{|\xi_2|}{|\xi|}\, e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t}\, t^{-\frac{1}{2}}\, \frac{|\xi_1|t^{\frac{1}{2}}}{|\xi|^2}\, e^{-c_0\,\frac{\xi_1^2}{|\xi|^4}t} \\ &\le C\, t^{-\frac{1}{2}}\, |\xi|^{s-1} e^{-c_0\frac{\xi_1^2\xi_2^2}{|\xi|^2}t} \\ &\le \begin{cases} C\, t^{-\frac{1}{2}}\, |\xi|^{s-1}\, e^{-C\xi_2^2 t} & \text{for } \xi \in S_{21}, \\ C\, t^{-\frac{1}{2}}\, |\xi|^{s-1}\, e^{-C\xi_1^2 t} & \text{for } \xi \in S_{22}. \end{cases} \end{aligned}$$

It results that,

$$
\begin{aligned}
&\left\| |\xi|^s \, \frac{|\xi_1||\xi_2|}{|\xi|^4} \, e^{-c_0 \frac{\xi_1^2 \xi_2^2}{|\xi|^2} t} \, e^{-c_0 \, \frac{\xi_1^2}{|\xi|^4} t} \, \widehat{\theta}_0(\xi) \right\|_{L^2(S_2)} \\
&\le \left\| |\xi|^s \, \frac{|\xi_1||\xi_2|}{|\xi|^4} \, e^{-c_0 \frac{\xi_1^2 \xi_2^2}{|\xi|^2} t} \, e^{-c_0 \, \frac{\xi_1^2}{|\xi|^4} t} \, \widehat{\theta}_0(\xi) \right\|_{L^2(S_{21})} \\
&\quad + \left\| |\xi|^s \, \frac{|\xi_1||\xi_2|}{|\xi|^4} \, e^{-c_0 \frac{\xi_1^2 \xi_2^2}{|\xi|^2} t} \, e^{-c_0 \, \frac{\xi_1^2}{|\xi|^4} t} \, \widehat{\theta}_0(\xi) \right\|_{L^2(S_{22})} \\
&\le C \, t^{-\frac{1}{2}} \, \| |\xi|^{s-1} \, e^{-C \xi_1^2 t} \, \widehat{\theta}_0(\xi) \|_{L^2} + C \, t^{-\frac{1}{2}} \, \| |\xi|^{s-1} \, e^{-C \xi_2^2 t} \, \widehat{\theta}_0(\xi) \|_{L^2} \\
&\le C \, t^{-\frac{1}{2} - \frac{\sigma}{2}} \, \|\theta_0\|_{\mathring{H}^{s-1,-\sigma}}. \qquad (3.3.41)
\end{aligned}
$$

Inserting the upper bounds (3.3.40) and (3.3.41) in (3.3.39), we find

$$
\| |\xi|^s \, K_2(\xi, t) \widehat{\theta}_0(\xi) \|_{L^2(S_2)} \le C \, t^{-\frac{1}{2}(s+\sigma)+1} \, \|\Lambda_1^{-\sigma} \theta_0\|_{L^2(\mathbb{R}^2)} + t^{-\frac{1}{2} - \frac{\sigma}{2}} \, \|\theta_0\|_{\mathring{H}^{s-1,-\sigma}}. \quad (3.3.42)
$$

In view of (3.3.32), (3.3.33), (3.3.37), (3.3.38), (3.3.40) and (3.3.41), we obtain

$$
\begin{aligned}
\|u_1(t)\|_{\mathring{H}^s(\mathbb{R}^2)} \le\; & C \, t^{-\frac{1}{2}(s+\sigma)} \, \|\Lambda_1^{-\sigma} u_{10}\|_{L^2(\mathbb{R}^2)} + C \, t^{-\frac{\sigma}{2}} \, \|u_{10}\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)} \\
& + C \, t^{-\frac{1}{2}(s+\sigma)+1} \, \|\Lambda_1^{-\sigma} \theta_0\|_{L^2(\mathbb{R}^2)} + C \, t^{-\frac{1}{2} - \frac{\sigma}{2}} \, \|\theta_0\|_{\mathring{H}^{s-1,-\sigma}(\mathbb{R}^2)}.
\end{aligned}
$$

Using the same techniques, $\|u_2(t)\|_{\mathring{H}^s(\mathbb{R}^2)}$ and $\|\theta(t)\|_{\mathring{H}^s(\mathbb{R}^2)}$ can be bounded as follows

$$
\begin{aligned}
\|u_2(t)\|_{\mathring{H}^s(\mathbb{R}^2)} \le\; & C \, t^{-\frac{1}{2}(s+\sigma)} \, \|\Lambda_1^{-\sigma} u_{20}\|_{L^2(\mathbb{R}^2)} + C \, t^{-\frac{\sigma}{2}} \, \|u_{20}\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)} \\
& + C \, t^{-\frac{1}{2}(s+\sigma)+1} \, \|\Lambda_1^{-\sigma} \theta_0\|_{L^2(\mathbb{R}^2)} + C \, t^{-1 - \frac{\sigma}{2}} \, \|\theta_0\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)},
\end{aligned}
$$

and

$$
\begin{aligned}
\|\theta(t)\|_{\mathring{H}^s(\mathbb{R}^2)} \le\; & C \, t^{-\frac{1}{2}(s+\sigma)+1} \, \|\Lambda_1^{-\sigma} u_{20}\|_{L^2(\mathbb{R}^2)} + C \, t^{-\frac{\sigma}{2}} \, \|u_{20}\|_{\mathring{H}^{s-2,-\sigma}(\mathbb{R}^2)} \\
& + C \, t^{-\frac{1}{2}(s+\sigma)} \, \|\Lambda_1^{-\sigma} \theta_0\|_{L^2(\mathbb{R}^2)} + C \, t^{-\frac{\sigma}{2}} \, \|\theta_0\|_{\mathring{H}^{s,-\sigma}(\mathbb{R}^2)}.
\end{aligned}
$$

This completes the proof of Theorem 3.3.4. ■

3.3.4 Exponential Decay Away from the Two Axes of the Frequency Space

When the Fourier frequencies are away from the two axes, the solution of the linearized system (3.3.1) actually decays exponentially to zero in time. To state our result more precisely, we define a frequency cutoff function, for $a_1 > 0$ and $a_2 > 0$,

$$\widehat{\varphi}(\xi) = \widehat{\varphi}(\xi_1, \xi_2) = \begin{cases} 0, & \text{if} \quad |\xi_1| \le a_1 \text{ or } |\xi_2| \le a_2, \\ 1, & \text{otherwise.} \end{cases} \tag{3.3.43}$$

Theorem 3.3.6 *Let $\nu > 0$ and $\eta > 0$. Consider the linearized system in (3.3.1) or equivalently*

$$\begin{cases} \partial_t u_1 = \nu\, \partial_{22} u_1 - \Delta^{-1}\partial_1\partial_2\theta, \\ \partial_t u_2 = \nu\, \partial_{22} u_2 + \Delta^{-1}\partial_1\partial_1\theta, \\ \partial_t \theta = \eta\, \partial_{11}\theta - u_2, \\ (u_1, u_2, \theta)(x, 0) = (u_{01}, u_{02}, \theta_0). \end{cases}$$

Let (u, θ) be the corresponding solution. The Fourier frequency piece of (u, θ) away from the two axes of the frequency space decays exponentially in time to zero. More precisely, if $(u_0, \theta_0) \in H^2(\mathbb{R}^2)$ with $\nabla\cdot u_0 = 0$, then there is constant $C_0 = C_0(\nu, \eta, a_1, a_2)$ such that, for all $t \ge 0$,

$$\|\partial_t(\varphi * u)(t)\|_{L^2}^2 + \|(\varphi * u)(t)\|_{H^1}^2 \le C\,(\|\varphi * u_0\|_{H^2}^2 + \|\varphi * \theta_0\|_{L^2}^2)\, e^{-C_0 t}, \tag{3.3.44}$$

$$\|\partial_t(\varphi * \theta)(t)\|_{L^2}^2 + \|(\varphi * \theta)(t)\|_{H^1}^2 \le C\,(\|\varphi * \theta_0\|_{H^2}^2 + \|\varphi * u_0\|_{L^2}^2)\, e^{-C_0 t}, \tag{3.3.45}$$

where φ is as defined in (3.3.43) and $C = C(\nu, \eta, a_1, a_2) > 0$ is a constant.

Proof of Theorem 3.3.6. Let $\widehat{\varphi}$ be the Fourier cutoff function defined in (3.3.43). Taking the convolution of φ with the velocity equation in the system (3.3.1), yields

$$\partial_{tt}(\varphi * u) - (\eta\partial_{11} + \nu\partial_{22})\partial_t(\varphi * u) + \nu\eta\partial_{11}\partial_{22}(\varphi * u) + \partial_{11}\Delta^{-1}(\varphi * u) = 0. \tag{3.3.46}$$

Dotting (3.3.46) with $\partial_t(\varphi * u)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2\right)$$
$$+\nu\|\partial_2\partial_t(\varphi * u)\|_{L^2}^2 + \eta\|\partial_1\partial_t(\varphi * u)\|_{L^2}^2 = 0, \tag{3.3.47}$$

where $\mathcal{R}_1 := \partial_1(-\Delta)^{-\frac{1}{2}}$ denotes the Riesz transform.

Taking the L^2-inner product of (3.3.46) with $\varphi * u$ leads to

$$\frac{1}{2}\frac{d}{dt}(\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \eta\|\partial_1(\varphi * u)\|_{L^2}^2) + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \nu\eta\|\partial_{12}(\varphi * u)\|_{L^2}^2 + \int \partial_{tt}(\varphi * u)\cdot(\varphi * u)dx = 0.$$

Further, integrating by parts, we find

$$\frac{1}{2}\frac{d}{dt}\left(\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \eta\|\partial_1(\varphi * u)\|_{L^2}^2 + 2(\partial_t(\varphi * u), (\varphi * u))\right) + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \nu\eta\|\partial_{12}(\varphi * u)\|_{L^2}^2 - \|\partial_t(\varphi * u)\|_{L^2}^2 = 0, \tag{3.3.48}$$

where (f, g) stands for the L^2-inner product.

For $\lambda > 0$, adding (3.3.47) and λ (3.3.48), we obtain

$$\frac{d}{dt}A(t) + 2B(t) = 0, \tag{3.3.49}$$

where for notational convenience, we have set

$$A(t) := \|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2 + \lambda\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u)\|_{L^2}^2 + 2\lambda(\partial_t(\varphi * u), (\varphi * u)), \tag{3.3.50}$$

$$B(t) := \nu\|\partial_2\partial_t(\varphi * u)\|_{L^2}^2 + \eta\|\partial_1\partial_t(\varphi * u)\|_{L^2}^2 + \lambda\eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2 - \lambda\|\partial_t(\varphi * u)\|_{L^2}^2 + \lambda\|\mathcal{R}_1(\varphi * u)\|_{L^2}^2. \tag{3.3.51}$$

The next step is to show that, by selecting suitable $\lambda = \lambda(\nu, \eta, a_1, a_2) > 0$, there exists a constant $C_0 = C_0(\nu, \eta, a_1, a_2) > 0$ such that, for any $t \geq 0$,

$$B(t) \geq C_0\, A(t),$$

where, we recall that $a_1 > 0$ and $a_2 > 0$ are the parameters mentioned in the definition of the frequency cutoff function defined by (3.3.43).

Applying Plancherel's theorem and using the dfinition of $\widehat{\varphi}$ in (3.3.43),

$$\|\partial_2\partial_t(\varphi * u)\|_{L^2}^2 = \int_{|\xi_1|\geq a_1, |\xi_2|\geq a_2} |\xi_2\, \partial_t(\widehat{\varphi}\widehat{u}(\xi, t))|^2\, d\xi \geq a_2^2\, \|\partial_t(\varphi * u)\|_{L^2}^2. \tag{3.3.52}$$

Proceeding as in (3.3.52), we find

$$\|\partial_1\partial_t(\varphi * u)\|_{L^2}^2 \geq a_1^2\, \|\partial_t(\varphi * u)\|_{L^2}^2, \quad \|\partial_{12}(\varphi * u)\|_{L^2}^2 \geq a_1^2\, \|\partial_2(\varphi * u)\|_{L^2}^2, \tag{3.3.53}$$

$$\|\partial_{12}(\varphi * u)\|_{L^2}^2 \geq a_2^2\, \|\partial_1(\varphi * u)\|_{L^2}^2, \quad \|\partial_{12}(\varphi * u)\|_{L^2}^2 \geq a_1^2\, a_2^2\, \|\varphi * u\|_{L^2}^2. \tag{3.3.54}$$

Choosing $\lambda > 0$ such that

$$\lambda \leq \frac{1}{2}(\nu\, a_2^2 + \eta\, a_1^2),$$

we deduce from (3.3.52), (3.3.53), (3.3.54) and (3.3.51) that

$$\begin{aligned} B(t) \geq\; & (\nu\, a_2^2 + \eta\, a_1^2)\|\partial_t(\varphi * u)\|_{L^2}^2 - \lambda\|\partial_t(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2 \\ & + \frac{1}{4}\lambda\eta\nu\, a_1^2\, \|\partial_2(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu\, a_2^2\, \|\partial_1(\varphi * u)\|_{L^2}^2 \\ & + \frac{1}{4}\lambda\eta\nu\, a_1^2\, a_2^2\, \|\varphi * u\|_{L^2}^2 + \lambda\|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 \\ \geq\; & \frac{1}{2}\,(\nu\, a_2^2 + \eta\, a_1^2)\|\partial_t(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2 \\ & + \frac{1}{4}\lambda\eta\nu\, a_1^2\, \|\partial_2(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu\, a_2^2\, \|\partial_1(\varphi * u)\|_{L^2}^2 \\ & + \frac{1}{4}\lambda\eta\nu\, a_1^2\, a_2^2\, \|\varphi * u\|_{L^2}^2 + \lambda\|\mathcal{R}_1(\varphi * u)\|_{L^2}^2. \end{aligned}$$

In addition, invoking the Cauchy-Schwarz inequality and Cauchy's inequality, we have

$$\begin{aligned} & \frac{1}{4}\,(\nu\, a_2^2 + \eta\, a_1^2)\|\partial_t(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu\, a_1^2\, a_2^2\, \|\varphi * u\|_{L^2}^2 \\ & \geq \frac{1}{2}\sqrt{\nu\, a_2^2 + \eta\, a_1^2}\,\sqrt{\lambda\eta\nu\, a_1^2\, a_2^2}\,(\partial_t(\varphi * u), \varphi * u). \end{aligned}$$

Thus,

$$\begin{aligned} B(t) \geq\; & \frac{1}{4}\,(\nu\, a_2^2 + \eta\, a_1^2)\|\partial_t(\varphi * u)\|_{L^2}^2 + \lambda\|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 \\ & + \frac{1}{4}\lambda\eta\nu\|\partial_{12}(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu\, a_1^2\, \|\partial_2(\varphi * u)\|_{L^2}^2 + \frac{1}{4}\lambda\eta\nu\, a_2^2\, \|\partial_1(\varphi * u)\|_{L^2}^2 \\ & + \frac{1}{2}\sqrt{\nu\, a_2^2 + \eta\, a_1^2}\,\sqrt{\lambda\eta\nu\, a_1^2\, a_2^2}\,(\partial_t(\varphi * u), \varphi * u). \end{aligned}$$

Therefore choosing C_0 satisfying,

$$C_0 = \frac{1}{4}\min\left\{(\nu\, a_2^2 + \eta\, a_1^2),\ \lambda,\ \eta a_1^2,\ \nu a_2^2,\ \frac{1}{\sqrt{\lambda}}\sqrt{\nu\, a_2^2 + \eta\, a_1^2}\,\sqrt{\eta\nu\, a_1^2\, a_2^2}\right\},$$

we get

$$B(t) \geq C_0\, A(t). \tag{3.3.55}$$

Inserting (3.3.55) in (3.3.49) leads to

$$\frac{d}{dt}A(t) + C_0\, A(t) \leq 0, \qquad \text{or equivalently,} \qquad A(t) \leq A(0)e^{-C_0 t}. \tag{3.3.56}$$

In order to establish the desired inequality (3.3.44), we derive a lower bound for $A(t)$ as follows.

Owing to (3.3.54) and the Cauchy-Schwarz inequality,

$$\begin{aligned}
A(t) &\geq \|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \eta\nu\, a_1^2\, a_2^2\, \|\varphi * u\|_{L^2}^2 \\
&\quad + \lambda\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u)\|_{L^2}^2 - \frac{1}{2}\|\partial_t(\varphi * u)\|_{L^2}^2 - 2\lambda^2\|\varphi * u\|_{L^2}^2 \\
&= \frac{1}{2}\|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + (\eta\nu\, a_1^2\, a_2^2 - 2\lambda^2)\|\varphi * u\|_{L^2}^2 \\
&\quad + \lambda\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u)\|_{L^2}^2,
\end{aligned}$$

where, we recall that $\mathcal{R}_1 := \partial_1(-\Delta)^{-\frac{1}{2}}$ stands for the Riesz transform.

If we take $\lambda > 0$ such that

$$\eta\nu\, a_1^2\, a_2^2 - 2\lambda^2 \geq \frac{1}{2}\eta\nu\, a_1^2\, a_2^2 \quad \text{or} \quad \lambda \leq \frac{1}{2}\sqrt{\eta\,\nu}\, a_1\, a_2,$$

we get

$$\begin{aligned}
A(t) &\geq \frac{1}{2}\|\partial_t(\varphi * u)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u)\|_{L^2}^2 + \frac{1}{2}\eta\nu\, a_1^2\, a_2^2\, \|\varphi * u\|_{L^2}^2 \\
&\quad + \lambda\nu\|\partial_2(\varphi * u)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u)\|_{L^2}^2 \\
&\geq C\,(\|\partial_t(\varphi * u)\|_{L^2}^2 + \|\varphi * u\|_{L^2}^2 + \|\nabla(\varphi * u)\|_{L^2}^2),
\end{aligned} \tag{3.3.57}$$

where $C = C(\nu, \eta, a_1, a_2) > 0$ is a constant.

Finally, it remains to derive an upper bound for $A(0)$. Recalling that (u, θ) satisfies

$$\begin{cases}
\partial_t u_1 = \nu\partial_{22}u_1 - \Delta^{-1}\partial_1\partial_2\theta, \\
\partial_t u_2 = \nu\partial_{22}u_2 + \Delta^{-1}\partial_1\partial_1\theta, \\
\partial_t\theta = \eta\,\partial_{11}\theta - u_2,
\end{cases}$$

we have

$$\partial_t u_1(0) = \nu \partial_{22} u_{01} - \Delta^{-1}\partial_1\partial_2\theta_0, \qquad \partial_t u_2(0) = \nu\partial_{22}u_{02} + \Delta^{-1}\partial_1\partial_1\theta_0$$

and thus

$$\|(\partial_t(\varphi * u)(0)\|_{L^2}^2 \le 2\nu^2 \|\partial_{22}(\varphi * u_0)\|_{L^2}^2 + 2\|\varphi * \theta_0\|_{L^2}^2, \tag{3.3.58}$$

where we have used the fact that Riesz transforms are bounded in L^q with $1 < q < \infty$ (see, e.g., [55]),

$$\|\Delta^{-1}\partial_1\partial_2 f\|_{L^q} \le C\,\|f\|_{L^q}.$$

In addition, invoking the basic inequality,

$$2\lambda(\partial_t(\varphi * u), (\varphi * u)) \le \|\partial_t(\varphi * u)\|_{L^2}^2 + \lambda^2\|\varphi * u\|_{L^2}^2,$$

we obtain,

$$\begin{aligned} A(0) := {} & \|\partial_t(\varphi * u)(0)\|_{L^2}^2 + \|\mathcal{R}_1(\varphi * u_0)\|_{L^2}^2 + \eta\nu\|\partial_{12}(\varphi * u_0)\|_{L^2}^2 \\ & + \lambda\nu\|\partial_2(\varphi * u_0)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u_0)\|_{L^2}^2 + 2\lambda(\partial_t(\varphi * u)(0), (\varphi * u_0)), \\ \le {} & 4\nu^2\|\partial_{22}(\varphi * u_0)\|_{L^2}^2 + 4\|\varphi * \theta_0\|_{L^2}^2 + (1+\lambda^2)\|\varphi * u_0\|_{L^2}^2 \\ & + \eta\nu\|\partial_{12}(\varphi * u_0)\|_{L^2}^2 + \lambda\nu\|\partial_2(\varphi * u_0)\|_{L^2}^2 + \lambda\eta\|\partial_1(\varphi * u_0)\|_{L^2}^2 \\ \le {} & C\,(\|\varphi * u_0\|_{H^2}^2 + \|\varphi * \theta_0\|_{L^2}^2). \end{aligned} \tag{3.3.59}$$

In view of (3.3.56), (3.3.57) and (3.3.59), we conclude that

$$\begin{aligned} & \|\partial_t(\varphi * u)(t)\|_{L^2}^2 + \|(\varphi * u)(t)\|_{L^2}^2 + \|\nabla(\varphi * u)(t)\|_{L^2}^2 \\ & \quad \le C\,(\|\varphi * u_0\|_{H^2}^2 + \|\varphi * \theta_0\|_{L^2}^2)\,e^{-C_0 t}. \end{aligned}$$

This completes the proof of the inequality (3.3.44). The estimate for θ in (3.3.45) is very similar. In fact, since θ satisfies the same wave equation as u, most of the lines for u remain valid when we replace u by θ and replace the bound in (3.3.58) by

$$\|(\partial_t(\varphi * \theta)(0)\|_{L^2}^2 \le 2\eta^2\|\partial_{11}(\varphi * \theta_0)\|_{L^2}^2 + 2\|\varphi * u_{02}\|_{L^2}^2.$$

This finishes the proof of Theorem 3.3.6. ■

CHAPTER IV

STABILIZATION OF THE 2D BOUSSINESQ EQUATIONS WITH HORIZONTAL DISSIPATION AND VERTICAL THERMAL DIFFUSION

4.1 Introduction

In this Chapter we study the following special two-dimensional (2D) Boussinesq system with partial dissipation

$$\begin{cases} \partial_t U + U \cdot \nabla U = -\nabla P + \nu\, \partial_{11} U + \Theta \mathbf{e}_2, \quad x \in \Omega,\ t > 0, \\ \partial_t \Theta + U \cdot \nabla \Theta = \eta\, \partial_{22} \Theta, \\ \nabla \cdot U = 0, \end{cases} \tag{4.1.1}$$

where we will consider two different spatial domains $\Omega = \mathbb{R}^2$ and $\Omega = \mathbb{T} \times \mathbb{R}$ where $\mathbb{T} = [0,1]$ being a 1D periodic box and $\mathbb{R}$ being the whole line. The first case when $\Omega = \mathbb{R}^2$ will be discussed in Section 4.2. While the second case when $\Omega = \mathbb{T} \times \mathbb{R}$ will be studied in Section 4.3. For both situations we aim to understand the stability of the perturbations near the hydrostatic equilibrium $(U_{he}, \Theta_{he}, P_{he})$ with

$$U_{he} = 0, \quad \Theta_{he} = x_2, \quad P_{he} = \frac{1}{2} x_2^2.$$

Clearly, it follows from (4.1.1) that the perturbation (u, θ, p) given by

$$u = U - U_{he}, \quad p = P - P_{he} \quad \text{and} \quad \theta = \Theta - \Theta_{he}.$$

satisfies the following equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{11} u + \theta \mathbf{e}_2, \quad x \in \Omega,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \eta\, \partial_{22} \theta, \\ \nabla \cdot u = 0, \end{cases} \tag{4.1.2}$$

As outlined in the Introduction, the type of the spatial domain Ω plays an important role in solving the stability problem. In fact, when the spatial domain is the whole plane $\mathbb{R}^2$, the H^2 stability issue remains an open problem. In this situation, we are only able to show the existence of the H^1 global weak solutions (see Section 4.2). However, we do not know the uniqueness of H^1-level solutions. Indeed, when we evaluate the difference(u^*, θ^*) of two solutions $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$, the terms generated by the nonlinearity

$$\int_{\mathbb{R}^2} u^* \cdot \nabla u^{(1)} \cdot u^* \, dx \quad and \int_{\mathbb{R}^2} u^* \cdot \nabla \theta^{(1)} \cdot \theta^* \, dx$$

are hard to deal with. Further more, when trying to seek for global H^2 solutions we encounter a difficulty when estimating $\|(\nabla\omega, \Delta\theta)\|_{L^2}$ where $\omega := \nabla \times u$. The issue is how to proceed from the energy equality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\omega(t)\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\Big) + \nu\|\partial_1\nabla\omega(t)\|_{L^2}^2 + \eta\|\partial_2\Delta\theta(t)\|_{L^2}^2 \\ &\quad = \int \Big(\nabla\partial_1\theta \cdot \nabla\omega - \Delta u_2 \Delta\theta\Big)\, dx - \int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx - \int \Delta \cdot \Delta(u \cdot \nabla\theta)\, dx \\ &\quad := A + B + C. \end{aligned}$$

Using the divergence free condition of u namely $\nabla \cdot u = 0$, one can easily show that $A = 0$. The integral C can be bounded suitably. However, if we write B as follows

$$\begin{aligned} B &:= -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx \\ &= \int \partial_2 u_2\, (\partial_1\omega)^2 dx - \int \partial_1 u_2\, \partial_1\omega\, \partial_2\omega \, dx \\ &\quad - \int \partial_2 u_1\, \partial_1\omega\, \partial_2\omega \, dx - \int \partial_2 u_2\, (\partial_2\omega)^2 \, dx \end{aligned} \tag{4.1.3}$$

then the last two terms in (4.1.3) do not appear to admit suitable bounds due to the lack of control on the vertical derivatives in the dissipation.

In contrast, when the spatial domain is $\Omega = \mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 1]$ being the 1D periodic box and $\mathbb{R}$ being the whole line, we are able to establish the H^2 stability of the system (4.1.2). Working in the domain $\Omega = \mathbb{T} \times \mathbb{R}$ allows us to separate the horizontal average $(\overline{u}, \overline{\theta})$ from the corresponding oscillation part $(\widetilde{u}, \widetilde{\theta})$, where for any function $f = f(x_1, x_2)$ integrable in x_1 on $\mathbb{T}$, we define the horizontal average $\overline{f}$ by

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)dx_1, \tag{4.1.4}$$

and we write

$$f = \overline{f} + \widetilde{f}. \tag{4.1.5}$$

Clearly, $\overline{f}$ also represents the zeroth horizontal Fourier mode of f. This decomposition is very useful due to some of the associated fine properties. For example, $\overline{f}$ and $\widetilde{f}$ are orthogonal in L^2, namely the inner product $(\overline{f}, \widetilde{f}) = 0$ and as a consequence, for any $f \in L^2(\Omega)$,

$$\|f\|_{L^2(\Omega)}^2 = \|\overline{f}\|_{L^2(\Omega)}^2 + \|\widetilde{f}\|_{L^2(\Omega)}^2 .$$

In addition, a strong Poincaré type inequality holds,

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C\|\partial_1 \widetilde{f}\|_{L^2(\Omega)}.$$

By applying this decomposition to the velocity field, namely writing $u = \overline{u} + \widetilde{u}$ and taking advantage of the special properties of $\widetilde{u}$ such as the Poincaré type inequality, we are able to establish suitable upper bounds for the nonlinear terms in (4.1.3), which in turn leads to a global and uniform upper bound for $\|u\|_{H^2(\Omega)}$. This explicit upper bound also implies the stability of perturbations near the hydrostatic equilibrium. In addition, by writing the evolution equations of the oscillations $\widetilde{u}$ and $\widetilde{\theta}$, we are also able to prove that the norms $\|\widetilde{u}\|_{H^1(\Omega)}$ and $\|\widetilde{\theta}\|_{H^1(\Omega)}$ decay algebraically to zero in time. Details of these results will be discussed in Section 4.3 of the present Chapter.

4.2 System in the Spatial Domain $\mathbb{R}^2$

In this section we study the following 2D Boussinesq equations with partial dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{11} u + \theta \mathbf{e}_2, \quad x \in \Omega,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \eta\, \partial_{22} \theta, \\ \nabla \cdot u = 0, \end{cases} \tag{4.2.1}$$

where the spatial domain is $\Omega = \mathbb{R}^2$.

4.2.1 Existence of H^1 Global Weak Solutions

The following Theorem proves the existence of the H^1 global weak solutions to the system (4.2.1).

Theorem 4.2.1 *Consider (4.2.1) with $\nu > 0$ and $\eta > 0$. Assume $u_0, \theta_0 \in H^1(\mathbb{R}^2)$ and $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^1} + \|\theta_0\|_{H^1} \le \varepsilon,$$

then (4.2.1) has a global weak solution (u, θ) satisfying for all time,

$$\|u(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 + 2\nu \int_0^t \|\partial_1 u(\tau)\|_{H^1}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta(\tau)\|_{H^1}^2 d\tau \le C\varepsilon^2,$$

for some constant $C > 0$.

Proof. We define the natural energy functional,

$$E(t) := \max_{0 \le \tau \le t} (\|u(\tau)\|_{H^1}^2 + \|\theta(\tau)\|_{H^1}^2) + 2\nu \int_0^t \|\partial_1 u(\tau)\|_{H^1}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta(\tau)\|_{H^1}^2 d\tau. \tag{4.2.2}$$

Our main efforts are devoted to show that

$$E(t) \le C_1 E(0) + C_2 E(t)^{\frac{3}{2}}, \tag{4.2.3}$$

for some constants $C_1, C_2 > 0$ and for all $t \geq 0$. The bootstrapping argument then allows us to conclude the desired result.

The proof of (4.2.3) is achieved in the following two steps.

Step 1: Showing that

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_1 u(\tau)\|_{L^2}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \tag{4.2.4}$$

Applying the Leray Projection $\mathbb{P} = I - \nabla(-\Delta)^{-1}\text{div}$ to the first equation in (4.2.1) we obtain

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) = \nu \partial_{11} u + \mathbb{P}(\theta e_2). \tag{4.2.5}$$

Dotting (4.2.5) by u, integrating in space and using the divergence free condition $\nabla . u = 0$ we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \underbrace{\int_{\mathbb{R}^2} \mathbb{P}(u \cdot \nabla u) u dx}_{=0} &= -\nu \|\partial_1 u\|_{L^2}^2 - \int_{\mathbb{R}^2} \mathcal{R}_1 \mathcal{R}_2 \theta u_1 dx \\ &\quad + \int_{\mathbb{R}^2} \theta u_2 dx - \int_{\mathbb{R}^2} \mathcal{R}_2^2 \theta u_2 dx \\ &= -\nu \|\partial_1 u\|_{L^2}^2 + \int_{\mathbb{R}^2} \theta u_2 dx, \end{aligned} \tag{4.2.6}$$

where $\mathcal{R}_i = \partial_1 (-\Delta)^{-\frac{1}{2}}$ for $i = 1$ or 2.

Now dotting the second equation of (4.2.1) by θ, integrating in space and using the divergence free condition $\nabla . u = 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \underbrace{\int_{\mathbb{R}^2} u \cdot \nabla \theta \cdot \theta dx}_{=0} + \int_{\mathbb{R}^2} \theta u_2 dx + \eta \|\partial_2 \theta\|_{L^2}^2 = 0. \tag{4.2.7}$$

Adding the two equations (4.2.6) and (4.2.7) we get

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2) + \nu \|\partial_1 u(\tau)\|_{L^2}^2 + \eta \|\partial_2 \theta(\tau)\|_{L^2}^2 = 0. \tag{4.2.8}$$

Integrating the equation (4.2.8) in time yield

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_1 u(\tau)\|_{L^2}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2, \tag{4.2.9}$$

for all $t \geq 0$.

Step 2: Showing that

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_1 \nabla u(\tau)\|_{L^2}^2 d\tau \\ + 2\eta \int_0^t \|\partial_2 \nabla\theta(\tau)\|_{L^2}^2 d\tau \leq cE(t)^{\frac{3}{2}} + \|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2, \end{aligned} \tag{4.2.10}$$

holds for some constant $c > 0$ and for all $t \geq 0$. To prove (4.2.10) we resort to the equation of the vorticity $\omega := \nabla \times u$ and the second equation in (4.2.1)

$$\begin{cases} \partial_t \omega + u \cdot \nabla\omega = \nu\, \partial_{11}\omega + \partial_1\theta, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t\theta + u \cdot \nabla\theta + u_2 = \eta\, \partial_{22}\theta. \end{cases} \tag{4.2.11}$$

Taking the gradient of the second equation in (4.2.11), dotting with $\nabla\theta$, and multiplying the first equation by ω we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \eta\|\partial_2\nabla\theta\|_{L^2}^2 + \nu\|\partial_1\omega\|_{L^2}^2 \\ &\quad = -\int \nabla\theta \cdot \nabla u \cdot \nabla\theta dx + \int (\partial_1\theta \cdot \omega - \nabla u_2 \cdot \nabla\theta) dx \\ &\quad := A + B. \end{aligned} \tag{4.2.12}$$

Using $\nabla \cdot u = 0$, there exists a stream function ψ such that $u = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)$ and $\Delta\psi = \omega$. Then

$$\begin{aligned} B := \int (\partial_1\theta \cdot \omega - \nabla u_2 \cdot \nabla\theta) dx &= \int (\partial_1\theta\Delta\psi - \nabla\partial_1\psi \cdot \nabla\theta) dx \\ &= \int (-\theta\Delta\partial_1\psi + \Delta\partial_1\psi\theta) dx \\ &= 0. \end{aligned} \tag{4.2.13}$$

Writing A explicitly as follows

$$
\begin{aligned}
A &:= -\int \nabla\theta \cdot \nabla u \cdot \nabla\theta dx \\
&= -\int \Big(\partial_1 u_1 (\partial_1\theta)^2 + \partial_1 u_2 \partial_1\theta\partial_2\theta + \partial_2 u_1 \partial_1\theta\partial_2\theta + \partial_2 u_2 (\partial_2\theta)^2\Big) dx \\
&:= A_1 + A_2 + A_3 + A_4,
\end{aligned} \tag{4.2.14}
$$

one has to bound the four integrals A_1, A_2, A_3 and A_4.

Using respectively the divergence free condition of u, integration by parts, Lemma 3.2.2, and Young's inequality yield

$$
\begin{aligned}
A_1 &:= -\int \partial_1 u_1 (\partial_1\theta)^2 dx \\
&= \int \partial_2 u_2 (\partial_1\theta)^2 dx \\
&= -2\int u_2 \partial_1\theta\partial_2\partial_1\theta dx \\
&\le c\|\partial_2\partial_1\theta\|_{L^2} \|\partial_1\theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\theta\|_{L^2}^{\frac{1}{2}} \|u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \\
&\le c\|\partial_2\theta\|_{H^1} \|\theta\|_{H^1}^{\frac{1}{2}} \|\partial_2\theta\|_{H^1}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}} \|\partial_1 u\|_{H^1}^{\frac{1}{2}} \\
&\le c\|\partial_2\theta\|_{H^1}^{\frac{3}{2}} \|\partial_1 u\|_{H^1}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}} \|\theta\|_{H^1}^{\frac{1}{2}} \\
&\le c\|u\|_{H^1}^{\frac{1}{2}} \|\theta\|_{H^1}^{\frac{1}{2}} \Big(\|\partial_2\theta\|_{H^1}^2 + \|\partial_1 u\|_{H^1}^2\Big).
\end{aligned} \tag{4.2.15}
$$

To bound A_2, we apply Lemma 3.2.2 and Young's inequality

$$
\begin{aligned}
A_2 &:= -\int \partial_1 u_2 \partial_1\theta\partial_2\theta dx \\
&\le c\|\partial_2\theta\|_{L^2} \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1\theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\theta\|_{L^2}^{\frac{1}{2}} \\
&\le c\|\partial_2\theta\|_{H^1} \|u\|_{H^1}^{\frac{1}{2}} \|\partial_1 u\|_{H^1}^{\frac{1}{2}} \|\theta\|_{H^1}^{\frac{1}{2}} \|\partial_2\theta\|_{H^1}^{\frac{1}{2}} \\
&\le c\|\partial_2\theta\|_{H^1}^{\frac{3}{2}} \|\partial_1 u\|_{H^1}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}} \|\theta\|_{H^1}^{\frac{1}{2}} \\
&\le c\|u\|_{H^1}^{\frac{1}{2}} \|\theta\|_{H^1}^{\frac{1}{2}} \Big(\|\partial_2\theta\|_{H^1}^2 + \|\partial_1 u\|_{H^1}^2\Big).
\end{aligned} \tag{4.2.16}
$$

For A_3, we divide it into two integrals

$$\begin{aligned}
A_3 &:= \int \partial_2 u_1 \partial_1\theta\partial_2\theta dx \\
&= \int u_1 \partial_2(\partial_1\theta\partial_2\theta) dx \\
&= \int u_1 \partial_2\partial_1\theta\partial_2\theta dx + \int u_1\partial_1\theta\partial_2\partial_2\theta dx \\
&= A_{31} + A_{32}.
\end{aligned} \tag{4.2.17}$$

We start with A_{31}. By Lemma 3.2.2 and Young's inequality we get

$$\begin{aligned}
A_{31} &:= \int u_1 \partial_2\partial_1\theta\partial_2\theta dx \\
&\leq c\|\partial_2\partial_1\theta\|_{L^2}\|\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \\
&\leq c\|\partial_2\theta\|_{H^1}\|\theta\|_{H^1}^{\frac{1}{2}}\|\partial_2\theta\|_{H^1}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}}\|\partial_1 u\|_{H^1}^{\frac{1}{2}} \\
&\leq c\|\partial_2\theta\|_{H^1}^{\frac{3}{2}}\|\partial_1 u\|_{H^1}^{\frac{1}{2}}\|\theta\|_{H^1}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}} \\
&\leq c\|u\|_{H^1}^{\frac{1}{2}}\|\theta\|_{H^1}^{\frac{1}{2}}\Big(\|\partial_2\theta\|_{H^1}^2 + \|\partial_1 u\|_{H^1}^2\Big).
\end{aligned} \tag{4.2.18}$$

Similarly,

$$\begin{aligned}
A_{32} &:= \int u_1 \partial_1\theta\partial_2\partial_2\theta dx \\
&\leq c\|\partial_2\partial_2\theta\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \\
&\leq c\|\partial_2\theta\|_{H^1}\|\theta\|_{H^1}^{\frac{1}{2}}\|\partial_2\theta\|_{H^1}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}}\|\partial_1 u\|_{H^1}^{\frac{1}{2}} \\
&\leq c\|\partial_2\theta\|_{H^1}^{\frac{3}{2}}\|\partial_1 u\|_{H^1}^{\frac{1}{2}}\|\theta\|_{H^1}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}} \\
&\leq c\|u\|_{H^1}^{\frac{1}{2}}\|\theta\|_{H^1}^{\frac{1}{2}}\Big(\|\partial_2\theta\|_{H^1}^2 + \|\partial_1 u\|_{H^1}^2\Big).
\end{aligned} \tag{4.2.19}$$

Combining (4.2.18) and (4.2.19) yield

$$A_3 \leq c\|u\|_{H^1}^{\frac{1}{2}}\|\theta\|_{H^1}^{\frac{1}{2}}\Big(\|\partial_2\theta\|_{H^1}^2 + \|\partial_1 u\|_{H^1}^2\Big). \tag{4.2.20}$$

Due to Lemma 3.2.2,

$$
\begin{aligned}
A_4 &:= \int \partial_2 u_2 (\partial_2\theta)^2 dx \\
&\le c\|\partial_2 u\|_{L^2} \|\partial_2\theta\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_2\theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_2\theta\|_{L^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^1} \|\partial_2\theta\|_{H^1}^2. \qquad (4.2.21)
\end{aligned}
$$

Collecting (4.2.15), (4.2.16), (4.2.20) and (4.2.21) and inserting them in (4.2.14) we get

$$
A \le c\|(u,\theta)\|_{H^1} \Big(\|\partial_2\theta\|_{H^1}^2 + \|\partial_1 u\|_{H^1}^2 \Big). \qquad (4.2.22)
$$

Thus, in view of (4.2.22), (4.2.13) and (4.2.12) we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2 \Big) + \nu\|\partial_1\nabla u(\tau)\|_{L^2}^2 + \eta\|\partial_2\nabla\theta(\tau)\|_{L^2}^2 \\
\le c\|(u,\theta)\|_{H^1} \Big(\|\partial_2\theta\|_{H^1}^2 + \|\partial_1 u\|_{H^1}^2 \Big). \qquad (4.2.23)
\end{aligned}
$$

Integrating (4.2.23) over $[0,t]$ yields

$$
\begin{aligned}
\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_1\nabla u(\tau)\|_{L^2}^2 d\tau \\
+ 2\eta \int_0^t \|\partial_2\nabla\theta(\tau)\|_{L^2}^2 d\tau \le cE(t)^{\frac{3}{2}} + \|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2. \\
(4.2.24)
\end{aligned}
$$

Finally, combining (4.2.9) and (4.2.24) leads to

$$
E(t) \le C_1 E(0) + C_2 E(t)^{\frac{3}{2}}. \qquad (4.2.25)
$$

for some constants $C_1, C_2 > 0$ and for all $t \ge 0$. Then the bootstrapping argument implies that if $E(0)$ is sufficiently small namely

$$
\|(u_0,\theta_0)\|_{H^1} \le \varepsilon,
$$

for some sufficiently small $\varepsilon > 0$, then $E(t)$ remains uniformly small for all time, namely

$$
E(t) \le C\,\varepsilon^2,
$$

for a constant $C > 0$ and for all $t \ge 0$. This completes the proof of Theorem 4.2.1.

■

4.3 System in the Spatial Domain $\Omega = [0,1] \times \mathbb{R}$

In this section we study the following 2D Boussinesq equations with partial dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{11} u + \theta \mathbf{e}_2, \quad x \in \Omega,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \eta\, \partial_{22} \theta, \\ \nabla \cdot u = 0, \end{cases} \tag{4.3.1}$$

where the spatial domain is $\Omega = \mathbb{T} \times \mathbb{R}$, with $\mathbb{T} = [0,1]$ being a 1D periodic box and $\mathbb{R}$ being the whole line.

This section is based on the author's joint work [7]. It is divided into three main subsections. Subsection 4.3.1 makes necessary preparations for the subsequent subsections. Subsection 4.3.2 proves the H^2 nonlinear stability of the system (4.3.1). Lastly, Subsection 4.3.3 derives some decay rates for the solution to (4.3.1).

4.3.1 Preliminaries

We recall that for any function $f = f(x_1, x_2)$ that is integrable in x_1 over the 1D periodic box $\mathbb{T} = [0,1]$, its horizontal average $\overline{f}$ is given by

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1. \tag{4.3.2}$$

We decompose f into $\overline{f}$ and the corresponding oscillation portion $\widetilde{f}$,

$$f = \overline{f} + \widetilde{f}. \tag{4.3.3}$$

The following lemmas provide a few properties of $\overline{f}$ and $\widetilde{f}$ to be used in the proof of our main results.

Lemma 4.3.1 *Assume that the 2D function f defined on $\Omega = \mathbb{T} \times \mathbb{R}$ is sufficiently regular, say $f \in H^2(\Omega)$. Let $\overline{f}$ and $\widetilde{f}$ be defined as in (4.3.2) and (4.3.3).*
(a) $\overline{f}$ and $\widetilde{f}$ obey the following basic properties,

$$\overline{\partial_1 f} = \partial_1 \overline{f} = 0, \quad \overline{\partial_2 f} = \partial_2 \overline{f}, \quad \overline{\widetilde{f}} = 0, \quad \widetilde{\partial_2 f} = \partial_2 \widetilde{f}.$$

(b) If f is a divergence-free vector field, namely $\nabla \cdot f = 0$, then $\overline{f}$ and $\widetilde{f}$ are also divergence-free,

$$\nabla \cdot \overline{f} = 0 \quad \textit{and} \quad \nabla \cdot \widetilde{f} = 0.$$

(c) $\overline{f}$ and $\widetilde{f}$ are orthogonal in L^2, namely

$$(\overline{f}, \widetilde{f}) := \int_\Omega \overline{f}\widetilde{f} dx = 0, \quad \|f\|^2_{L^2(\Omega)} = \|\overline{f}\|^2_{L^2(\Omega)} + \|\widetilde{f}\|^2_{L^2(\Omega)}.$$

In particular,

$$\|\overline{f}\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \quad \textit{and} \quad \|\widetilde{f}\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}.$$

The orthogonality is actually more general and holds for any integrable functions,

$$\int_\Omega \overline{f} \cdot \widetilde{g}\, dx = 0.$$

The properties given in Lemma 4.3.1 can be easily verified via (4.3.2) and (4.3.3).

Lemma 4.3.2 *For any 1D function $f \in H^1(\mathbb{R})$,*

$$\|f\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\, \|f\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \|f'\|^{\frac{1}{2}}_{L^2(\mathbb{R})}. \tag{4.3.4}$$

For any bounded domain such as $\mathbb{T} = [0,1]$ and $f \in H^1(\mathbb{T})$,

$$\|f\|_{L^\infty(\mathbb{T})} \le \sqrt{2}\, \|f\|^{\frac{1}{2}}_{L^2(\mathbb{T})} \|f'\|^{\frac{1}{2}}_{L^2(\mathbb{T})} + \|f\|_{L^2(\mathbb{T})}, \tag{4.3.5}$$

in particular, if the function f has mean zero such as the oscillation part $\tilde{f}$,

$$\|f\|_{L^\infty(\mathbb{T})} \le C\, \|f\|^{\frac{1}{2}}_{L^2(\mathbb{T})} \|f'\|^{\frac{1}{2}}_{L^2(\mathbb{T})}. \tag{4.3.6}$$

Lemma 4.3.3 *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big|\int_\Omega fgh\, dx\Big| \le C\|f\|^{\frac{1}{2}}_{L^2}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}}\|g\|^{\frac{1}{2}}_{L^2}\|\partial_2 g\|^{\frac{1}{2}}_{L^2}\|h\|_{L^2}\,. \tag{4.3.7}$$

For any $f \in H^2(\Omega)$, we have

$$\begin{aligned}\|f\|_{L^\infty(\Omega)} \le& C\|f\|^{\frac{1}{4}}_{L^2}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{4}}\|\partial_2 f\|^{\frac{1}{4}}_{L^2} \\ &\times (\|\partial_2 f\|_{L^2} + \|\partial_1\partial_2 f\|_{L^2})^{\frac{1}{4}}.\end{aligned} \tag{4.3.8}$$

The upper bound for the triple product in (4.3.7) on the whole $\mathbb{R}^2$ was proved in [18] while the inequality (4.3.7) is a different version on the domain Ω and can be shown using (4.3.4) and (4.3.5).

Replacing f in Lemma 4.3.3 by the oscillation part $\tilde{f}$, yields to the following Lemma.

Lemma 4.3.4 *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\Big|\int_\Omega \widetilde{f} g h \, dx\Big| \leq C\|\widetilde{f}\|_{L^2}^{\frac{1}{2}}\|\partial_1 \widetilde{f}\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}. \tag{4.3.9}$$

For any $f \in H^2(\Omega)$, we have

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C\|\widetilde{f}\|_{L^2}^{\frac{1}{4}}\|\partial_1 \widetilde{f}\|_{L^2}^{\frac{1}{4}}\|\partial_2 \widetilde{f}\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2 \widetilde{f}\|_{L^2}^{\frac{1}{4}}. \tag{4.3.10}$$

Lemma 4.3.5 *Let $\overline{f}$ and $\widetilde{f}$ be defined as in (4.3.2) and (4.3.3). If $\|\partial_1 \widetilde{f}\|_{L^2(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C\|\partial_1 \widetilde{f}\|_{L^2(\Omega)},$$

where C is a pure constant. In addition, if $\|\partial_1 \widetilde{f}\|_{H^1(\Omega)} < \infty$, then

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C\|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

As a direct consequence of Lemma 4.3.5 and the inequality (4.3.9), one has

$$\Big|\int_\Omega \widetilde{f} g h \, dx\Big| \leq C\|\partial_1 \widetilde{f}\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}. \tag{4.3.11}$$

4.3.2 The H^2 Nonlinear Stabilty

We are able to prove the following result on the stability problem of (4.3.1).

Theorem 4.3.6 *Let $\mathbb{T} = [0, 1]$ be a 1D periodic box and let $\Omega = \mathbb{T} \times \mathbb{R}$. Assume $u_0, \theta_0 \in H^2(\Omega)$ and $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

then (4.3.1) has a unique global solution that remains uniformly bounded for all time,

$$\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + 2\nu \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau$$
$$+ 2\eta \int_0^t \|\partial_2 \theta(\tau)\|_{H^2}^2 d\tau + \int_0^t \|\partial_1 \theta(\tau)\|_{L^2}^2 d\tau \le C\varepsilon^2,$$

for some constant $C > 0$.

The proof Theorem 4.3.6, is based on the construction of a suitable energy functional

$$\begin{aligned} E(t) :=& \max_{0\le\tau\le t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu \int_0^t \|\partial_1 u\|_{H^2}^2 d\tau \\ &+ 2\eta \int_0^t \|\partial_2 \theta\|_{H^2}^2 d\tau + \int_0^t \|\partial_1 \theta\|_{L^2}^2 \, d\tau \\ :=& E_1(t) + E_2(t), \end{aligned} \tag{4.3.12}$$

with

$$E_1(t) := \max_{0\le\tau\le t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu \int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta\|_{H^2}^2 d\tau,$$

and

$$E_2(t) := \int_0^t \|\partial_1 \theta\|_{L^2}^2 \, d\tau.$$

The energy functional E_1 is natural and comes from the H^2-norm of (u, θ) and the corresponding time integral part due to the partial dissipation, while the part $E_2(t)$ is needed and comes from the regularization that $(\omega := \nabla \times u, \theta)$ satisfies the wave equation

$$\begin{cases} \partial_{tt}\omega - (\eta\partial_{22} + \nu\partial_{11})\partial_t\omega + \nu\eta\partial_{11}\partial_{22}\omega + \partial_{11}\Delta^{-1}\omega = N_1, \\ \partial_{tt}\theta - (\eta\partial_{22} + \nu\partial_{11})\partial_t\theta + \nu\eta\partial_{11}\partial_{22}\theta + \partial_{11}\Delta^{-1}\theta = N_2, \end{cases}$$

where

$$N_1 := -(\partial_t - \eta\partial_{22})(u \cdot \nabla\omega) - \partial_1(u \cdot \nabla\theta),$$
$$N_2 := (\nu\partial_{11} - \partial_t)(u \cdot \nabla\theta) + (u \cdot \nabla u_2 - \partial_2\Delta^{-1}\nabla \cdot (u \cdot \nabla u)).$$

Then the idea consists on using the bootstrapping argument after showing that $E(t)$ satisfies

$$E(t) \leq c_1 E(0) + c_2 E(t)^{\frac{3}{2}}, \tag{4.3.13}$$

for some positive constants c_1 and c_2 and for all $t > 0$. The proof of (4.3.13) consists of two main parts. The first part focuses on the estimate of E_1 and we obtain

$$E_1(t) \leq E_1(0) + c_3\, E_1(t)^{\frac{3}{2}} + c_4\, E_2(t)^{\frac{3}{2}}. \tag{4.3.14}$$

The second part proves

$$E_2(t) \leq c_5 E_1(0) + c_6\, E_1(t) + c_7\, E_1(t)^{\frac{3}{2}} + c_8\, E_2(t)^{\frac{3}{2}}, \tag{4.3.15}$$

where c_1 through c_8 are all constants. Adding (4.3.14) with $1/(2c_6)$ of (4.3.15) yields the desired inequality in (4.3.13). More details are given in the following proof.

Proof. We define the energy functional $E(t)$ as in (4.3.12). Our main effort is devoted to prove (4.3.13).

First of all, we have the global L^2-bound

$$\|(u(t), \theta(t))\|_{L^2}^2 + 2\nu \int_0^t \|\partial_1 u\|_{L^2}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta\|_{L^2}^2 d\tau = \|(u_0, \theta_0)\|_{L^2}^2. \tag{4.3.16}$$

Next, we compute the H^1-norm using the temperature equation and the vorticity equation associated with the velocity equation in (4.3.1),

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \partial_{11} \omega + \partial_1 \theta, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \eta \partial_{22} \theta, \end{cases} \tag{4.3.17}$$

where $\omega := \nabla \times u$.

Taking the inner product of $(\omega, \nabla\theta)$ with the equations of ω and $\nabla\theta$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \eta \|\partial_2 \nabla\theta\|_{L^2}^2 + \nu \|\partial_1 \omega\|_{L^2}^2 & \\ = -\int \nabla\theta \cdot \nabla u \cdot \nabla\theta \, dx + \int (\partial_1 \theta \cdot \omega - \nabla u_2 \cdot \nabla\theta)\, dx & \\ := I_1 + I_2. & \end{aligned} \tag{4.3.18}$$

Due to the divergence free condition of u, namely $\nabla \cdot u = 0$, there exists a stream function ψ so that $u = \nabla^{\perp}\psi = (-\partial_2\psi, \partial_1\psi)$ and $\Delta\psi = \omega$. Hence

$$\begin{aligned} I_2 := \int (\partial_1\theta \cdot \omega - \nabla u_2 \cdot \nabla\theta)dx &= \int (\partial_1\theta\Delta\psi - \nabla\partial_1\psi \cdot \nabla\theta)dx \\ &= \int (-\theta\Delta\partial_1\psi + \Delta\partial_1\psi\theta)dx \\ &= 0. \end{aligned} \tag{4.3.19}$$

We further split I_1 into four integrals as follows

$$\begin{aligned} I_1 :&= -\int \nabla\theta \cdot \nabla u \cdot \nabla\theta \, dx \\ &= -\int \partial_1 u_1(\partial_1\theta)^2 dx - \int \partial_1 u_2 \partial_1\theta\partial_2\theta dx - \int \partial_2 u_1 \partial_1\theta\partial_2\theta dx - \int \partial_2 u_2(\partial_2\theta)^2 dx \\ &:= I_{11} + I_{12} + I_{13} + I_{14}. \end{aligned} \tag{4.3.20}$$

The terms on the right-hand side of (4.3.20) can be bounded as follows. The key point here is to obtain upper bounds that are time integrable.

Using respectively, $\nabla \cdot u = 0$, integration by parts, the fact that $\overline{u_2} = 0$, Lemma 4.3.4 and Young's inequaliy

$$\begin{aligned} I_{11} :&= -\int \partial_1 u_1(\partial_1\theta)^2 dx \\ &= -2\int u_2\partial_1\theta\partial_1\partial_2\theta dx \\ &= -2\int \widetilde{u_2}\partial_1\theta\partial_1\partial_2\theta dx \\ &\le c\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\theta\|_{L^2} \\ &\le c\|u\|_{H^2}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}} \\ &\le c\|u\|_{H^2}\Big(\|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big). \end{aligned} \tag{4.3.21}$$

Due to the fact that $\overline{u_2}=0$, Lemma 4.3.4 and Cauchy's inequality,

$$\begin{aligned}
I_{12} :&= -\int \partial_1 u_2 \partial_1\theta\partial_2\theta dx \\
&= -\int \partial_1 \widetilde{u_2} \partial_1\theta\partial_2\theta dx \\
&\le c\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2} \\
&\le c\|\partial_1 u\|_{H^2}\|\partial_2\theta\|_{H^2}\|\theta\|_{H^2} \\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.22)
\end{aligned}$$

To bound I_{13}, we first invoke the decompositions $u=\tilde{u}+\bar{u}$ and $\theta=\tilde{\theta}+\bar{\theta}$ to write it into four integrals

$$\begin{aligned}
I_{13} :&= -\int \partial_2 u_1 \partial_1\theta\partial_2\theta dx \\
&= -\int \partial_2\overline{u_1}\partial_1\widetilde{\theta}\partial_2\overline{\theta}dx - \int \partial_2\widetilde{u_1}\partial_1\widetilde{\theta}\partial_2\overline{\theta}dx \\
&\quad - \int \partial_2\overline{u_1}\partial_1\widetilde{\theta}\partial_2\widetilde{\theta}dx - \int \partial_2\widetilde{u_1}\partial_1\widetilde{\theta}\partial_2\widetilde{\theta}dx \\
&:= I_{131}+I_{132}+I_{133}+I_{134}. \qquad (4.3.23)
\end{aligned}$$

Due to Lemma 4.3.1,

$$I_{131} := -\int_\Omega \partial_2\overline{u_1}\partial_1\widetilde{\theta}\partial_2\overline{\theta}dx = \int_{\mathbb{R}} \partial_2\overline{u_1}\partial_2\overline{\theta}\int_{\mathbb{T}}\partial_1\widetilde{\theta}\,dx_1dx_2 = \int_{\mathbb{R}}\partial_2\overline{u_1}\partial_2\overline{\theta}\partial_1\overline{\widetilde{\theta}}dx_2 = 0. \qquad (4.3.24)$$

According to Lemma 4.3.4 and Young's inequaliy

$$\begin{aligned}
I_{132} :&= -\int_\Omega \partial_2\widetilde{u_1}\partial_1\widetilde{\theta}\partial_2\overline{\theta}dx \\
&\le c\|\partial_2\overline{\theta}\|_{L^2}\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c\|\partial_2\theta\|_{H^2}\|u\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}} \\
&\le c\|\partial_2\theta\|_{H^2}^{\frac{3}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_2\theta\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\Big). \qquad (4.3.25)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
I_{133} &= -\int_\Omega \partial_2\overline{u_1}\partial_1\widetilde{\theta}\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_2\overline{u_1}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.26)
\end{aligned}
$$

I_{134} can be similarly bounded as I_{133}. In fact

$$
\begin{aligned}
I_{134} &:= -\int_\Omega \partial_2\widetilde{u_1}\partial_1\widetilde{\theta}\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_1}\|_{L^2} \\
&\le c\|u\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.27)
\end{aligned}
$$

Inserting (4.3.24), (4.3.25), (4.3.26) and (4.3.27) in (4.3.23), we get

$$
I_{13} \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big). \qquad (4.3.28)
$$

Using respectively, the divergence free condition of u, Lemma 4.3.1, Lemma 4.3.4 and Cauchy's inequality yield

$$
\begin{aligned}
I_{14} &: = -\int \partial_2 u_2(\partial_2\theta)^2 dx \\
&= \int \partial_1\widetilde{u_1}(\partial_2\theta)^2 dx \\
&\le c\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2} \\
&\le c\|\theta\|_{H^2}\|\partial_1 u\|_{H^2}\|\partial_2\theta\|_{H^2} \\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.29)
\end{aligned}
$$

Collecting the bounds obtained in (4.3.21), (4.3.22), (4.3.28), (4.3.29) and inserting them in (4.3.20) we get

$$
I_1 \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big). \qquad (4.3.30)
$$

It follows from (4.3.30), (4.3.19) and (4.3.18),

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \eta\|\partial_2\nabla\theta\|_{L^2}^2 + \nu\|\partial_1\omega\|_{L^2}^2$$
$$\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big). \tag{4.3.31}$$

Integrating (4.3.31) over $[0,t]$ yields,

$$\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + 2\eta\int_0^t \|\partial_2\nabla\theta\|_{L^2}^2 d\tau + 2\nu\int_0^t \|\partial_1 w\|_{L^2}^2 d\tau$$
$$\le \|\nabla u_0\|_{L^2}^2 + \|\nabla\theta_0\|_{L^2}^2 + c\,E_1(t)^{\frac{3}{2}} + c\,E_2(t)^{\frac{3}{2}}. \tag{4.3.32}$$

Applying ∇ to the first equation of (4.3.17) and dotting with $\nabla\omega$ and applying Δ to the second equation of (4.3.17) and dotting with $\Delta\theta$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2) + \eta\|\partial_2\nabla\theta\|_{L^2}^2 + \nu\|\partial_1\nabla\omega\|_{L^2}^2$$
$$= -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\, dx - \int \Delta\theta\cdot\Delta(u\cdot\nabla\theta)dx + \int(\nabla\partial_1\theta\cdot\nabla\omega - \Delta u_2\cdot\Delta\theta)dx$$
$$:= J_1 + J_2 + J_3. \tag{4.3.33}$$

Since $\nabla\cdot u = 0$, there exists a stream function ψ such that we can write $u = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)$ and $\Delta\psi = \omega$. Hence,

$$J_3 := \int(\nabla\partial_1\theta\cdot\nabla\omega - \Delta u_2\Delta\theta)\,dx = \int(\nabla\partial_1\theta\cdot\nabla\omega - \Delta\partial_1\psi\Delta\theta)\,dx$$
$$= \int(\nabla\partial_1\theta\cdot\nabla\omega - \partial_1\omega\,\Delta\theta)\,dx = \int(\nabla\partial_1\theta\cdot\nabla\omega + \partial_1\nabla\omega\,\cdot\nabla\theta)\,dx$$
$$= \int \partial_1(\nabla\theta\cdot\nabla\omega)\,dx = 0. \tag{4.3.34}$$

Integrating by parts, one can write J_2 as follows

$$J_2 := -\int \Delta\theta\cdot\Delta(u\cdot\nabla\theta)dx$$
$$= -\int \Delta\theta\Delta u_1\partial_1\theta dx - \int \Delta\theta\Delta u_2\partial_2\theta dx$$
$$- 2\int \Delta\theta\nabla u_1\cdot\partial_1\nabla\theta dx - 2\int \Delta\theta\nabla u_2\cdot\partial_2\nabla\theta dx$$
$$:= J_{21} + J_{22} + J_{23} + J_{24}. \tag{4.3.35}$$

To deal with J_{21}, we invoke the decompositions $u = \overline{u} + \widetilde{u}$ and $\theta = \overline{\theta} + \widetilde{\theta}$ to write it into four terms,

$$\begin{aligned}
J_{21} : &= -\int \Delta\theta \Delta u_1 \partial_1 \theta dx = -\int \Delta\theta \Delta u_1 \partial_1 \widetilde{\theta} dx \\
&= -\int \Delta\overline{u_1} \partial_1 \widetilde{\theta} \Delta\overline{\theta} dx - \int \Delta\overline{u_1} \partial_1 \widetilde{\theta} \Delta\widetilde{\theta} dx - \int \Delta\widetilde{u_1} \partial_1 \widetilde{\theta} \Delta\overline{\theta} dx - \int \Delta\widetilde{u_1} \partial_1 \widetilde{\theta} \Delta\widetilde{\theta} dx \\
&:= J_{211} + J_{212} + J_{213} + J_{214}.
\end{aligned} \tag{4.3.36}$$

According to Lemma 4.3.1,

$$J_{211} := -\int \Delta\overline{u_1} \partial_1 \widetilde{\theta} \Delta\overline{\theta} dx = \int_{\mathbb{R}} \Delta\overline{u_1} \Delta\overline{\theta} \int_{\mathbb{T}} \partial_1 \widetilde{\theta} \, dx_1 dx_2 = \int_{\mathbb{R}} \Delta\overline{u_1} \Delta\overline{\theta} \partial_1 \overline{\widetilde{\theta}} dx_2 = 0. \tag{4.3.37}$$

Further, we write J_{212} explicitly as follows,

$$\begin{aligned}
J_{212} &:= -\int \Delta\overline{u_1} \partial_1 \widetilde{\theta} \Delta\widetilde{\theta} dx \\
&= -\int \partial_{11}\overline{u_1} \partial_1 \widetilde{\theta} \Delta\widetilde{\theta} dx - \int \partial_{22}\overline{u_1} \partial_1 \widetilde{\theta} \partial_{11}\widetilde{\theta} dx - \int \partial_{22}\overline{u_1} \partial_1 \widetilde{\theta} \partial_{22}\widetilde{\theta} dx \\
&:= J_{2121} + J_{2122} + J_{2123}.
\end{aligned} \tag{4.3.38}$$

Due to Lemma 4.3.1,

$$J_{2121} := -\int \underbrace{\partial_{11}\overline{u_1}}_{=0} \partial_1 \widetilde{\theta} \Delta\widetilde{\theta} dx = 0. \tag{4.3.39}$$

Integrating by parts and using Lemma 4.3.1 yield

$$J_{2122} := -\int \partial_{22}\overline{u_1} \partial_1 \widetilde{\theta} \partial_{11}\widetilde{\theta} dx = -\frac{1}{2}\int \partial_{22}\overline{u_1} \partial_1 (\partial_1 \widetilde{\theta})^2 dx = \frac{1}{2}\int \partial_{22} \underbrace{\partial_1 \overline{u_1}}_{=0} (\partial_1 \widetilde{\theta})^2 dx = 0. \tag{4.3.40}$$

It follows from Lemma 4.3.4 and Young's inequality,

$$\begin{aligned}
J_{2123} &:= -\int \partial_{22}\overline{u_1} \partial_1 \widetilde{\theta} \partial_{22}\widetilde{\theta} dx \\
&\le c\|\partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_{22}\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_{22}\overline{u_1}\|_{L^2} \\
&\le c\|u\|_{H^2} \|\partial_2 \theta\|_{H^2}^{\frac{3}{2}} \|\partial_1 \theta\|_{L^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2} \Big(\|\partial_2 \theta\|_{H^2}^2 + \|\partial_1 \theta\|_{L^2}^2 \Big).
\end{aligned} \tag{4.3.41}$$

Combining the bounds in (4.3.39), (4.3.40), (4.3.41) and inserting them in (4.3.38) yield

$$J_{212} \leq c\|u\|_{H^2}\Big(\|\partial_2\theta\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2\Big). \tag{4.3.42}$$

By Lemma 4.3.1, Lemma 4.3.4 and Cauchy's inequality,

$$\begin{aligned}
J_{213} &:= -\int \Delta\widetilde{u_1}\partial_1\widetilde{\theta}\Delta\overline{\theta}dx \\
&= -\int \Delta\widetilde{u_1}\partial_1\widetilde{\theta}\partial_{22}\overline{\theta}dx \\
&\leq c\|\Delta\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\
&\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2} \\
&\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned} \tag{4.3.43}$$

Applying Lemma 4.3.4 and Cauchy's inequality, we have

$$\begin{aligned}
J_{214} &:= -\int \Delta\widetilde{u_1}\partial_1\widetilde{\theta}\Delta\widetilde{\theta}dx \\
&\leq c\|\Delta\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\
&\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2} \\
&\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1\theta\|_{L^2} \\
&\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big).
\end{aligned} \tag{4.3.44}$$

Collecting (4.3.37), (4.3.42), (4.3.43), (4.3.44) and inserting them in (4.3.36) we get

$$J_{21} \leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big). \tag{4.3.45}$$

Using the orthogonal decomposition of u and θ we can split J_{22} into four integrals as follows

$$\begin{aligned}
J_{22} &:= -\int \Delta\theta\Delta u_2\partial_2\theta dx \\
&= -\int \Delta\overline{\theta}\Delta\overline{u_2}\partial_2\theta dx - \int \Delta\overline{\theta}\Delta\widetilde{u_2}\partial_2\theta dx - \int \Delta\widetilde{\theta}\Delta\overline{u_2}\partial_2\theta dx - \int \Delta\widetilde{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\
&:= J_{221} + J_{222} + J_{223} + J_{224}.
\end{aligned} \tag{4.3.46}$$

We start with J_{221}. By the divergence free condition of u, Lemma 4.3.1 and Lemma 4.3.4,

$$\begin{aligned} J_{221} &:= -\int \Delta\overline{\theta}\Delta\overline{u_2}\partial_2\theta dx \\ &= -\int \partial_{22}\overline{\theta}\partial_{22}\overline{u_2}\partial_2\theta dx \\ &= \int \partial_{22}\overline{\theta}\partial_2 \underbrace{\partial_1\overline{u_1}}_{=0}\partial_2\theta dx = 0. \end{aligned} \tag{4.3.47}$$

Similarly,

$$J_{223} := -\int \Delta\widetilde{\theta}\Delta\overline{u_2}\partial_2\theta dx = 0. \tag{4.3.48}$$

According to Lemma 4.3.1, Lemma 4.3.4 and Young's inequality

$$\begin{aligned} J_{222} &:= -\int \Delta\overline{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\ &= -\int \partial_2\partial_2\overline{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\ &\leq c\|\Delta\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2} \\ &\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\ &\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{4.3.49}$$

By Lemma 4.3.4 and Young's inequality,

$$\begin{aligned} J_{224} &:= -\int \Delta\widetilde{\theta}\Delta\widetilde{u_2}\partial_2\theta dx \\ &\leq c\|\Delta\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{L^2} \\ &\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\ &\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \end{aligned} \tag{4.3.50}$$

Combining (4.3.47), (4.3.48), (4.3.49), (4.3.50) and inserting them in (4.3.46) we obtain,

$$J_{22} \leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{4.3.51}$$

To bound J_{23}, we start by writing it into a summation of four integrals,

$$\begin{aligned}
J_{23} &:= -2\int \Delta\theta\nabla u_1 \cdot \partial_1\nabla\theta dx \\
&= -2\int \Delta\theta\partial_1 u_1\partial_1\partial_1\theta dx - 2\int \Delta\theta\partial_2 u_1\partial_1\partial_2\theta dx \\
&= -2\int \Delta\widetilde{\theta}\partial_1 u_1\partial_1\partial_1\theta dx - 2\int \Delta\overline{\theta}\partial_1 u_1\partial_1\partial_1\theta dx \\
&\quad - 2\int \Delta\widetilde{\theta}\partial_2 u_1\partial_1\partial_2\theta dx - 2\int \Delta\overline{\theta}\partial_2 u_1\partial_1\partial_2\theta dx \\
&:= J_{231} + J_{232} + J_{233} + J_{234}. \qquad (4.3.52)
\end{aligned}$$

Using Lemma 4.3.1, we can write J_{231} as,

$$\begin{aligned}
J_{231} &:= -2\int \Delta\widetilde{\theta}\partial_1 u_1\partial_1\partial_1\theta dx \\
&= -2\int \Delta\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta} dx \\
&= -2\int \partial_1\partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta} dx - 2\int \partial_2\partial_2\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta} dx \\
&:= J_{2311} + J_{2312}. \qquad (4.3.53)
\end{aligned}$$

Due to the divergence free condition of u, integration by parts, Lemma 4.3.4 and Young's inequality,

$$\begin{aligned}
J_{2311} &:= -2\int \partial_1\widetilde{u_1}(\partial_1\partial_1\widetilde{\theta})^2 dx \\
&= 2\int \partial_2\widetilde{u_2}(\partial_1\partial_1\widetilde{\theta})^2 dx \\
&= -4\int \widetilde{u_2}\partial_1\partial_1\widetilde{\theta}\partial_2\partial_1\partial_1\widetilde{\theta} dx \\
&\le c\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.54)
\end{aligned}$$

By Lemma 4.3.4 and Young's inequality,

$$
\begin{aligned}
J_{2312} &:= -2\int \partial_2\partial_2\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta}dx \\
&\le c\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.55)
\end{aligned}
$$

Collecting (4.3.54) and (4.3.55) and inserting them into (4.3.53), we get

$$
J_{231} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.56)
$$

To bound J_{232}, we use Lemma 4.3.1, Lemma 4.3.4 and Young's inequality,

$$
\begin{aligned}
J_{232} &:= -2\int \Delta\overline{\theta}\partial_1 u_1\partial_1\partial_1\theta dx \\
&= -2\int \partial_2\partial_2\overline{\theta}\partial_1\widetilde{u_1}\partial_1\partial_1\widetilde{\theta}dx \\
&\le c\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.57)
\end{aligned}
$$

To deal with J_{233}, we invoke the decompositions $u=\overline{u}+\widetilde{u}$ and $\theta=\overline{\theta}+\widetilde{\theta}$ to write it into three terms,

$$
\begin{aligned}
J_{233} &:= -2\int \Delta\widetilde{\theta}\partial_2 u_1\partial_1\partial_2\theta dx \\
&= -2\int \Delta\widetilde{\theta}\partial_2\widetilde{u_1}\partial_1\partial_2\widetilde{\theta}dx - 2\int \Delta\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&= -2\int \Delta\widetilde{\theta}\partial_2\widetilde{u_1}\partial_1\partial_2\widetilde{\theta}dx - 2\int \partial_1\partial_1\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx - 2\int \partial_2\partial_2\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&= J_{2331}+J_{2332}+J_{2333}. \qquad (4.3.58)
\end{aligned}
$$

According to Lemma 4.3.4 and Young's inequality,

$$
\begin{aligned}
J_{2331} &:= -2\int \Delta\widetilde{\theta}\partial_2\widetilde{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.59)
\end{aligned}
$$

Using respectively integration by parts, Lemma 4.3.1, Hölder's inequality, Lemma 4.3.2 and Cauchy's inequality, we have

$$\begin{aligned}
J_{2332} &:= -2\int \partial_1\partial_1\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&= 2\int \partial_1\widetilde{\theta}\underbrace{\partial_1\partial_2\overline{u_1}}_{=0}\partial_1\partial_2\widetilde{\theta}dx + 2\int \partial_1\widetilde{\theta}(\partial_2\overline{u_1}\partial_1\partial_1\partial_2\widetilde{\theta})dx \\
&= 2\int \partial_1\widetilde{\theta}(\partial_2\overline{u_1}\partial_1\partial_1\partial_2\widetilde{\theta})dx \\
&= 2\int_{\mathbb{R}} \partial_2\overline{u_1}\Big(\int_{\mathbb{T}} \partial_1\widetilde{\theta}(\partial_1\partial_1\partial_2\widetilde{\theta})dx_1\Big)dx_2 \\
&\le 2\int_{\mathbb{R}} |\partial_2\overline{u_1}|\|\partial_1\widetilde{\theta}\|_{L^2_{x_1}}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_1}}dx_2 \\
&\le 2\|\partial_2\overline{u_1}\|_{L^\infty_{x_2}}\|\partial_1\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|_{L^2_{x_2}L^2_{x_1}} \\
&\le c\|\partial_2\overline{u_1}\|_{H^1}\|\partial_1\widetilde{\theta}\|_{L^2}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|_{L^2} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1\theta\|^2_{L^2} + \|\partial_2\theta\|^2_{H^2}\Big). \qquad (4.3.60)
\end{aligned}$$

Due to Lemma 4.3.4,

$$\begin{aligned}
J_{2333} &:= -2\int \partial_2\partial_2\widetilde{\theta}\partial_2\overline{u_1}\partial_1\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_1\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_2\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\overline{u_1}\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_2\theta\|^2_{H^2}. \qquad (4.3.61)
\end{aligned}$$

Combining (4.3.59), (4.3.60), (4.3.61) and inserting them in (4.3.58) we get

$$J_{233} \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1 u\|^2_{H^2} + \|\partial_2\theta\|^2_{H^2} + \|\partial_1\theta\|^2_{L^2}\Big). \qquad (4.3.62)$$

From Lemma 4.3.1 and Lemma 4.3.4,

$$\begin{aligned}
J_{234} &:= -2\int \Delta\overline{\theta}\partial_2 u_1\partial_1\partial_2\theta dx \\
&= -2\int \partial_2\partial_2\overline{\theta}\partial_2 u_1\partial_1\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_1\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_1\partial_2\widetilde{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_2\overline{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2\partial_2\partial_2\overline{\theta}\|^{\frac{1}{2}}_{L^2}\|\partial_2 u_1\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_2\theta\|^2_{H^2}. \qquad (4.3.63)
\end{aligned}$$

Collecting the estimates (4.3.56), (4.3.57), (4.3.62) and (4.3.63) and inserting them in (4.3.52) we obtain,

$$J_{23} \leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{4.3.64}$$

Invoking the decomposition $u = \overline{u} + \widetilde{u}$ and $\theta = \overline{\theta} + \widetilde{\theta}$ and applying Lemma 4.3.1, we can write J_{24} as,

$$\begin{aligned}
J_{24} &:= -2\int \Delta\theta\nabla u_2 \cdot \partial_2\nabla\theta dx \\
&= -2\int (\partial_1 u_2\partial_1\partial_2\theta\Delta\theta + \partial_2 u_2\partial_2\partial_2\theta\Delta\theta)dx \\
&= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta - 2\int \partial_2 u_2\partial_2\partial_2\theta\Delta\theta dx \\
&= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta dx - 2\int \partial_2\overline{u_2}\partial_2\partial_2\theta\Delta\theta dx - 2\int \partial_2\widetilde{u_2}\partial_2\partial_2\theta\Delta\theta dx \\
&:= J_{241} + J_{242} + J_{243}.
\end{aligned} \tag{4.3.65}$$

We start with J_{241}. By Lemma 4.3.4 and Young's inequality we have

$$\begin{aligned}
J_{241} &:= -2\int \partial_1\widetilde{u_2}\partial_1\partial_2\widetilde{\theta}\Delta\theta dx \\
&\leq c\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2} \\
&\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\
&\leq c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned} \tag{4.3.66}$$

Next, using the divergence free condition of u and Lemma 4.3.1,

$$\begin{aligned}
J_{242} &:= -2\int \partial_2\overline{u_2}\partial_2\partial_2\theta\Delta\theta dx \\
&= 2\int \partial_1\overline{u_1}\partial_2\partial_2\theta\Delta\theta dx = 0.
\end{aligned} \tag{4.3.67}$$

According to Lemma 4.3.4 and Young's inequality,

$$
\begin{aligned}
J_{244} &:= -2\int \partial_2\widetilde{u}_2\partial_2\partial_2\theta\Delta\theta dx \\
&\le c\|\partial_2\widetilde{u}_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u}_2\|_{L^2}^{\frac{1}{2}}\|\Delta\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\theta\|_{L^2} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{3}{2}} \\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.68)
\end{aligned}
$$

Collecting (4.3.66), (4.3.67), and (4.3.68) and inserting them in (4.3.65), we obtain

$$
J_{24} \le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.69)
$$

Thus, by (4.3.45), (4.3.51), (4.3.64), (4.3.69), and (4.3.35),

$$
J_2 \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.70)
$$

It remains to bound J_1. To do so, we split it into four integrals

$$
\begin{aligned}
J_1 &:= -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\, dx \\
&= -\int \partial_1 u_1(\partial_1\omega)^2\, dx - \int \partial_1 u_2\partial_1\omega\partial_2\omega\, dx - \int \partial_2 u_1\partial_1\omega\partial_2\omega\, dx - \int \partial_2 u_2(\partial_2\omega)^2\, dx \\
&:= J_{11}+J_{12}+J_{13}+J_{14}. \qquad (4.3.71)
\end{aligned}
$$

Due to Lemma 4.3.1 and Lemma 4.3.4,

$$
\begin{aligned}
J_{11} &:= -\int \partial_1 u_1(\partial_1\omega)^2\, dx \\
&= -\int \partial_1\widetilde{u}_1(\partial_1\widetilde{\omega})(\partial_1\widetilde{\omega})\, dx \\
&\le c\|\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\omega}\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \qquad (4.3.72)
\end{aligned}
$$

According to Lemma 4.3.1 and Lemma 4.3.4,

$$\begin{aligned}
J_{12} &:= -\int \partial_1 u_2 \partial_1 \omega \partial_2 \omega \, dx \\
&= -\int \partial_1 \widetilde{u_2} \partial_1 \widetilde{\omega} \partial_2 \omega \, dx \\
&\leq c\|\partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \omega\|_{L^2} \\
&\leq c\|u\|_{H^2} \|\partial_1 u\|_{H^2}^2. \qquad (4.3.73)
\end{aligned}$$

Making use of the orthogonal decomposition of u_1 and ω and Lemma 4.3.1, we can write J_{13} as

$$\begin{aligned}
J_{13} &:= -\int \partial_2 u_1 \partial_1 \omega \partial_2 \omega \, dx \\
&= -\int \partial_2 u_1 \partial_1 \widetilde{\omega} \partial_2 \omega \, dx \\
&= -\int \partial_2 \overline{u_1} \partial_1 \widetilde{\omega} \partial_2 \overline{\omega} \, dx - \int \partial_2 \overline{u_1} \partial_1 \widetilde{\omega} \partial_2 \widetilde{\omega} \, dx - \int \partial_2 \widetilde{u_1} \partial_1 \widetilde{\omega} \partial_2 \omega \, dx \\
&= J_{131} + J_{132} + J_{133}. \qquad (4.3.74)
\end{aligned}$$

According to Lemma 4.3.1, it is easy to see that

$$J_{131} := -\int \partial_2 \overline{u_1} \partial_1 \widetilde{\omega} \partial_2 \overline{\omega} \, dx = 0. \qquad (4.3.75)$$

To bound J_{132} we use Lemma 4.3.4

$$\begin{aligned}
J_{132} &:= -\int \partial_2 \overline{u_1} \partial_1 \widetilde{\omega} \partial_2 \widetilde{\omega} \, dx \\
&\leq c\|\partial_2 \overline{u_1}\|_{L^2} \|\partial_2 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \\
&\leq c\|u\|_{H^2} \|\partial_1 \partial_2 \widetilde{\omega}\|_{L^2}^{\frac{3}{2}} \|\partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \\
&\leq c\|u\|_{H^2} \|\partial_1 u\|_{H^2}^2. \qquad (4.3.76)
\end{aligned}$$

Similarly,

$$\begin{aligned}J_{133} &:= -\int \partial_2\widetilde{u_1}\partial_1\widetilde{\omega}\partial_2\omega\,dx\\
&\le c\|\partial_2\omega\|_{L^2}\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\\
&\le c\|\nabla\omega\|_{L^2}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\\
&\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \qquad (4.3.77)\end{aligned}$$

Thus, by (4.3.75), (4.3.76), (4.3.77), and (4.3.74),

$$J_{13} \le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \qquad (4.3.78)$$

Due to $\nabla\cdot u = 0$, Lemma 4.3.1, and the inequality (4.3.11)

$$\begin{aligned}J_{14} &:= -\int \partial_2 u_2(\partial_2\omega)^2\,dx\\
&= \int \partial_1\widetilde{u_1}(\partial_2\overline{\omega}+\partial_2\widetilde{\omega})^2\,dx\\
&= 2\int \partial_1\widetilde{u_1}\partial_2\overline{\omega}\partial_2\widetilde{\omega}\,dx + 2\int \partial_1\widetilde{u_1}(\partial_2\widetilde{\omega})^2\,dx\\
&\le c\Big(\|\partial_2\overline{\omega}\|_{L^2}+\|\partial_2\widetilde{\omega}\|_{L^2}\Big)\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}\\
&\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2. \qquad (4.3.79)\end{aligned}$$

Collecting the results obtained in (4.3.72), (4.3.73), (4.3.78), (4.3.79) and inserting them in (4.3.71) we obtain

$$J_1 \le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}^2\,. \qquad (4.3.80)$$

Combining the upper bounds in (4.3.70), (4.3.80) and inserting them in (4.3.33), we get

$$\begin{aligned}&\frac12\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2)+\eta\|\partial_2\nabla\theta\|_{L^2}^2+\nu\|\partial_1\nabla\omega\|_{L^2}^2\\
&\qquad\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2+\|\partial_1\theta\|_{L^2}^2+\|\partial_2\theta\|_{H^2}^2\Big). \qquad (4.3.81)\end{aligned}$$

Integrating (4.3.81) over [0,t], we get

$$\begin{aligned}
&\|\nabla\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2 + 2\eta\int_0^t \|\partial_2\nabla\theta\|_{L^2}^2\,d\tau + 2\nu\int_0^t \|\partial_1\nabla\omega\|_{L^2}^2\,d\tau \\
&\quad\le c\int_0^t \|(u,\theta)\|_{H^2}\Big(\|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\Big)\,d\tau + \|\Delta u_0\|_{L^2}^2 + \|\Delta\theta_0\|_{L^2}^2 \\
&\quad\le \|\Delta u_0\|_{L^2}^2 + \|\Delta\theta_0\|_{L^2}^2 + c\,E_1(t)^{\frac{3}{2}} + c\,E_2(t)^{\frac{3}{2}}. \qquad (4.3.82)
\end{aligned}$$

It follows from (4.3.16), (4.3.32) and (4.3.82)

$$\begin{aligned}
E_1(t) :&= \max_{0\le\tau\le t}(\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2) + 2\nu\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + 2\eta\int_0^t \|\partial_2\theta\|_{H^2}^2 d\tau \\
&\le E_1(0) + c_3\,E_1(t)^{\frac{3}{2}} + c_4\,E_2(t)^{\frac{3}{2}}, \qquad (4.3.83)
\end{aligned}$$

for some constants $c_3, c_4 > 0$.

In the final step, we bound the extra term $E_2(t) := \int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau$ of the energy $E(t)$ defined in (4.3.12). To do so, we use the equations of the vorticity $\omega := \nabla\times u$ and the temperature θ,

$$\begin{cases}
\partial_t\omega + u\cdot\nabla\omega = \nu\partial_{11}\omega + \partial_1\theta, \\
\partial_t\theta + u\cdot\nabla\theta + u_2 = \eta\partial_{22}\theta.
\end{cases} \qquad (4.3.84)$$

Dotting the first equation of (4.3.84) by $\partial_1\theta$ and then integrating in space, we get

$$\begin{aligned}
\|\partial_1\theta\|_{L^2}^2 &= \int \partial_1\theta(\partial_t\omega - \nu\partial_{11}\omega + u\cdot\nabla\omega)dx \\
&= \frac{d}{dt}\int \partial_1\theta\omega dx - \int \omega\partial_1\partial_t\theta dx - \nu\int \partial_1\theta\partial_{11}\omega dx + \int \partial_1\theta(u\cdot\nabla\omega)dx \\
&:= A + B + C + D.
\end{aligned}$$

Due to Hölder inequality and Cauchy's inequality, we have

$$\begin{aligned}
\int_0^t A d\tau &:= \int_0^t \frac{d}{dt}\int \partial_1\theta\omega dx\,d\tau \\
&= \Big(\int \partial_1\theta(t)\omega(t)dx - \int \partial_1\theta_0\omega_0 dx\Big) \\
&\le \|\partial_1\theta\|_{L^2}\|\omega\|_{L^2} + \|\partial_1\theta_0\|_{L^2}\|\omega_0\|_{L^2} \\
&\le \frac{1}{2}\Big(\|\theta\|_{H^2}^2 + \|u\|_{H^2}^2\Big) + \frac{1}{2}\Big(\|\theta_0\|_{H^2}^2 + \|u_0\|_{H^2}^2\Big). \qquad (4.3.85)
\end{aligned}$$

Integrating by parts and using the second equation in (4.3.84), we can decompose the integral B as follows,

$$
\begin{aligned}
B &:= -\int \omega\, \partial_1 \partial_t \theta\, dx = \int \partial_1 \omega\, \partial_t \theta\, dx \\
&= \int \partial_1 \omega \Big(\eta \partial_{22} \theta - u \cdot \nabla \theta - u_2 \Big) dx \\
&= \eta \int \partial_1 \omega\, \partial_2 \partial_2 \theta dx - \int \partial_1 \omega\, u_2 dx - \int \partial_1 \omega\, u \cdot \nabla \theta\, dx \\
&:= B_1 + B_2 + B_3. \qquad (4.3.86)
\end{aligned}
$$

By Hölder's inequality and the Cauchy's inequality with epsilon,

$$
B_1 := \eta \int \partial_1 \omega\, \partial_2 \partial_2 \theta dx \leq \eta \|\partial_1 \omega\|_{L^2} \|\partial_2 \partial_2 \theta\|_{L^2} \leq \|\partial_1 u\|_{H^2}^2 + \frac{\eta^2}{4} \|\partial_2 \theta\|_{H^2}^2. \qquad (4.3.87)
$$

Integrating by parts and making use of Lemma 4.3.1, Lemma 4.3.5 and Hölder's inequality

$$
\begin{aligned}
B_2 &:= -\int \partial_1 \omega u_2 dx = -\int \partial_1 \widetilde{\omega} u_2 dx = \int \widetilde{\omega} \partial_1 u_2 dx \\
&\leq \|\widetilde{\omega}\|_{L^2} \|\partial_1 u_2\|_{L^2} \leq \|\partial_1 \widetilde{\omega}\|_{L^2} \|\partial_1 u_2\|_{L^2} \leq \|\partial_1 u\|_{H^2}^2. \qquad (4.3.88)
\end{aligned}
$$

Further, by Lemma 4.3.1 one can decompose B_3 as follows,

$$
\begin{aligned}
B_3 &:= -\int \partial_1 \omega\, u \cdot \nabla \theta dx \\
&= -\int \partial_1 \widetilde{\omega} u_1 \partial_1 \widetilde{\theta} dx - \int \partial_1 \widetilde{\omega} u_2 \partial_2 \theta dx \\
&:= B_{31} + B_{32}. \qquad (4.3.89)
\end{aligned}
$$

Due to Lemma 4.3.4 and Cauchy's inequality,

$$
\begin{aligned}
B_{31} &:= -\int \partial_1 \widetilde{\omega} u_1 \partial_1 \widetilde{\theta} dx \\
&\leq c \|\partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|u_1\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 u\|_{H^2}^{\frac{1}{2}} \|\partial_1 u\|_{H^2}^{\frac{1}{2}} \|\partial_1 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_2 \theta\|_{H^2}^{\frac{1}{2}} \\
&\leq c \|u\|_{H^2} \|\partial_1 u\|_{H^2} \|\partial_1 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_2 \theta\|_{H^2}^{\frac{1}{2}} \\
&\leq c \|u\|_{H^2} \Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1 \theta\|_{L^2}^2 + \|\partial_2 \theta\|_{H^2}^2 \Big). \qquad (4.3.90)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
B_{32} &:= -\int \partial_1 \widetilde{\omega} u_2 \partial_2 \theta dx \\
&\leq c\|\partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1 \widetilde{\omega}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_2 \theta\|_{L^2}^{\frac{1}{2}} \|u_2\|_{L^2} \\
&\leq c\|u\|_{H^2} \|\partial_1 u\|_{H^2} \|\partial_2 \theta\|_{H^2} \\
&\leq c\|u\|_{H^2} \Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2 \theta\|_{H^2}^2 \Big).
\end{aligned}
\tag{4.3.91}
$$

In view of (4.3.90), (4.3.91) and (4.3.89) we have

$$
B_3 \leq c\|u\|_{H^2} \Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1 \theta\|_{L^2}^2 + \|\partial_2 \theta\|_{H^2}^2 \Big). \tag{4.3.92}
$$

Combining (4.3.87), (4.3.88), (4.3.92) and inserting them in (4.3.86) yield

$$
B \leq 2\|\partial_1 u\|_{H^2}^2 + \frac{\eta^2}{4} \|\partial_2 \theta\|_{H^2}^2 + c\|u\|_{H^2} \Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2 \theta\|_{H^2}^2 + \|\partial_1 \theta\|_{L^2}^2 \Big).
$$

Hence,

$$
\begin{aligned}
\int_0^t B d\tau \leq & 2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + \frac{\eta^2}{4} \int_0^t \|\partial_2 \theta\|_{H^2}^2 d\tau \\
& + c\int_0^t \|u\|_{H^2} \Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_2 \theta\|_{H^2}^2 + \|\partial_1 \theta\|_{L^2}^2 \Big) d\tau.
\end{aligned}
\tag{4.3.93}
$$

To bound the integral C, we use both Hölder's inequality and Young's inequality

$$
C := -\nu \int \partial_1 \theta \partial_{11} w dx \leq \nu \|\partial_1 \theta\|_{L^2} \|\partial_{11} w\|_{L^2} \leq \frac{1}{4} \|\partial_1 \theta\|_{L^2}^2 + \nu^2 \|\partial_1 u\|_{H^2}^2.
$$

Hence,

$$
\int_0^t C d\tau \leq \frac{1}{4} \int_0^t \|\partial_1 \theta\|_{L^2}^2 d\tau + \nu^2 \int_0^t \|\partial_1 u\|_{H^2}^2 d\tau. \tag{4.3.94}
$$

Due to Lemma 4.3.1, D can be written as,

$$
\begin{aligned}
D &:= \int \partial_1 \theta (u \cdot \nabla \omega) dx \\
&= \int \partial_1 \widetilde{\theta} (u \cdot \nabla \omega) dx \\
&= \int \partial_1 \widetilde{\theta} u_1 \partial_1 \partial_1 u_2 dx - \int \partial_1 \widetilde{\theta} u_1 \partial_1 \partial_2 u_1 dx + \int \partial_1 \widetilde{\theta} u_2 \partial_2 \partial_1 u_2 dx - \int \partial_1 \widetilde{\theta} u_2 \partial_2 \partial_2 u_1 dx \\
&:= D_1 + D_2 + D_3 + D_4.
\end{aligned}
\tag{4.3.95}
$$

The integrals D_1 up to D_3 can be bounded as follows by using Lemma 4.3.1, Lemma 4.3.4 and Cauchy's inequality,

$$\begin{aligned}
D_1 &:= \int \partial_1\widetilde{\theta}u_1\partial_1\partial_1 u_2 dx \\
&= \int \partial_1\widetilde{\theta}u_1\partial_1\partial_1 \widetilde{u_2} dx \\
&\le c\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big),
\end{aligned} \tag{4.3.96}$$

$$\begin{aligned}
D_2 &:= -\int \partial_1\widetilde{\theta}u_1\partial_1\partial_2 u_1 dx \\
&= -\int \partial_1\widetilde{\theta}u_1\partial_1\partial_2 \widetilde{u_1} dx \\
&\le c\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big),
\end{aligned} \tag{4.3.97}$$

$$\begin{aligned}
D_3 &:= \int \partial_1\widetilde{\theta}u_2\partial_2\partial_1 u_2 dx \\
&= \int \partial_1\widetilde{\theta}u_2\partial_2\partial_1 \widetilde{u_2} dx \\
&\le c\|\partial_2\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|u_2\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned} \tag{4.3.98}$$

Using the fact that $\overline{u_2} = 0$, the inequality (4.3.11) and Cauchy's inequality, D_4 can be bounded by

$$
\begin{aligned}
D_4 &:= -\int \partial_1\widetilde{\theta} u_2 \partial_2\partial_2 u_1 dx \\
&= -\int \partial_1\widetilde{\theta}\widetilde{u_2}\partial_2\partial_2 u_1 dx \\
&\le c\|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2 u_1\|_{L^2} \\
&\le c\|u\|_{H^2}\|\partial_1 u\|_{H^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}\|\partial_2\theta\|_{H^2}^{\frac{1}{2}} \\
&\le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big).
\end{aligned} \tag{4.3.99}
$$

In view of (4.3.95), collecting the bounds in (4.3.96), (4.3.97), (4.3.98) and (4.3.99) we get

$$
D \le c\|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big). \tag{4.3.100}
$$

Hence,

$$
\int_0^t D d\tau \le c\int_0^t \|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big)d\tau. \tag{4.3.101}
$$

Therefore, combining the estimates (4.3.85), (4.3.93), (4.3.94) and (4.3.101), we obtain

$$
\begin{aligned}
E_2(t) &:= \int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau \\
&\le \frac{1}{2}\Big(\|\theta\|_{H^2}^2 + \|u\|_{H^2}^2\Big) + \frac{1}{2}\Big(\|\theta_0\|_{H^2}^2 + \|u_0\|_{H^2}^2\Big) \\
&\quad + 2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau + \frac{\eta^2}{4}\int_0^t \|\partial_2\theta\|_{H^2}^2 d\tau \\
&\quad + \frac{1}{4}\int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau + \nu^2\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau \\
&\quad + c\int_0^t \|u\|_{H^2}\Big(\|\partial_1 u\|_{H^2}^2 + \|\partial_1\theta\|_{L^2}^2 + \|\partial_2\theta\|_{H^2}^2\Big)d\tau \\
&\le \frac{1}{4}\int_0^t \|\partial_1\theta\|_{L^2}^2 d\tau + cE_1(0) + cE_1(t) + cE_1(t)^{\frac{3}{2}} + cE_2(t)^{\frac{3}{2}}.
\end{aligned} \tag{4.3.102}
$$

It results from (4.3.102),

$$
E_2(t) \le c_5 E_1(0) + c_6\, E_1(t) + c_7\, E_1(t)^{\frac{3}{2}} + c_8\, E_2(t)^{\frac{3}{2}}, \tag{4.3.103}
$$

where c_5 through c_8 are all positive constants.

Adding (4.3.83) with $1/(2c_6)$ of (4.3.103) yields,

$$E(t) := E_1(t) + E_2(t) \le c_1 E(0) + c_2 E(t)^{\frac{3}{2}}. \tag{4.3.104}$$

where c_1 and c_2 are positive constants. Finally, applying the bootstrapping argument to the inequality (4.3.104) leads to the desired stability result. Indeed, if the initial data (u_0, θ_0) is sufficiently small,

$$\|(u_0, \theta_0)\|_{H^2} \le \varepsilon := \frac{1}{4\sqrt{c_1 c_2}},$$

then (4.3.104) allows us to show that

$$\|(u(t), \theta(t))\|_{H^2} \le \sqrt{2c_1}\, \varepsilon.$$

In the rest of the proof, we show the uniqueness part of Theorem 4.3.6. We show that two solutions $(u^{(1)}, p^{(1)}, \theta^{(1)})$ and $(u^{(2)}, p^{(2)}, \theta^{(2)})$ of (4.3.1) with one of them in the H^2-regularity class say $(u^{(1)}, \theta^{(1)}) \in L^\infty(0, T, H^2(\Omega))$ must coincide. Their difference (u^*, p^*, θ^*) with $u^* = u^{(1)} - u^{(2)}$, $p^* = p^{(1)} - p^{(2)}$, $\theta^* = \theta^{(1)} - \theta^{(2)}$ satisfies according to (4.3.1)

$$\begin{cases} \partial_t u^* + u^{(2)} \cdot \nabla u^* + u^* \cdot \nabla u^{(1)} + \nabla p^* = \nu\, \partial_{11} u^* + \theta^* \mathbf{e}_2, \\ \partial_t \theta^* + u^{(2)} \cdot \nabla \theta^* + u^* \cdot \nabla \theta^{(1)} + u_2^* = \eta\, \partial_{22} \theta^*, \\ \nabla \cdot u^* = 0, \\ u^*(x, 0) = 0, \quad \theta^*(x, 0) = 0. \end{cases} \tag{4.3.105}$$

We estimate the difference (u^*, p^*, θ^*) in $L^2(\Omega)$. Dotting (4.3.105) by (u^*, θ^*) and applying the divergence free condition of u^*, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(u^*, \theta^*)\|_{L^2}^2 + \nu\|\partial_1 u^*\|_{L^2}^2 + \eta\|\partial_2 \theta^*\|_{L^2}^2 & \\ = - \int u^* \cdot \nabla u^{(1)} \cdot u^*\, dx - \int u^* \cdot \nabla \theta^{(1)} \cdot \theta^*\, dx & \\ := I_1 + I_2. & \end{aligned} \tag{4.3.106}$$

Due to Lemma 4.3.3, Cauchy's inequality, Cauchy's inequality with epsilon and the uniformly global bound for $\|u^{(1)}\|_{H^2}$,

$$\begin{aligned}
I_1 &:= -\int u^* \cdot \nabla u^{(1)} \cdot u^* \, dx \\
&\leq c\|u^*\|_{L^2}^{\frac{1}{2}} \Big(\|u^*\|_{L^2} + \|\partial_1 u^*\|_{L^2} \Big)^{\frac{1}{2}} \underbrace{\|\nabla u^{(1)}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla u^{(1)}\|_{L^2}^{\frac{1}{2}}}_{\leq c} \|u^*\|_{L^2} \\
&\leq c\|u^*\|_{L^2} \Big(\|u^*\|_{L^2} + \|\partial_1 u^*\|_{L^2} \Big) + \|u^*\|_{L^2}^2 \\
&\leq c\|u^*\|_{L^2}^2 + \frac{\nu}{2}\|\partial_1 u^*\|_{L^2}^2.
\end{aligned} \tag{4.3.107}$$

By Lemma 4.3.3, Cauchy's inequality, Cauchy's inequality with epsilon and the uniformly bound for $\|\theta^{(1)}\|_{H^2}$,

$$\begin{aligned}
I_2 &:= -\int u^* \cdot \nabla \theta^{(1)} \cdot \theta^* \, dx \\
&\leq c \underbrace{\|\nabla \theta^{(1)}\|_{L^2}^{\frac{1}{2}} \Big(\|\nabla \theta^{(1)}\|_{L^2} + \|\partial_1 \nabla \theta^{(1)}\|_{L^2} \Big)^{\frac{1}{2}}}_{\leq c} \|\theta^*\|_{L^2}^{\frac{1}{2}} \|\partial_2 \theta^*\|_{L^2}^{\frac{1}{2}} \|u^*\|_{L^2} \\
&\leq c\|\theta^*\|_{L^2}^{\frac{1}{2}} \|\partial_2 \theta^*\|_{L^2}^{\frac{1}{2}} \|u^*\|_{L^2} \\
&\leq c\|u^*\|_{L^2} \Big(\|\theta^*\|_{L^2} + \|\partial_2 \theta^*\|_{L^2} \Big) \\
&\leq c\|u^*\|_{L^2}^2 + c\|\theta^*\|_{L^2}^2 + \frac{\eta}{2}\|\partial_2 \theta^*\|_{L^2}^2.
\end{aligned} \tag{4.3.108}$$

Now, inserting the estimates (4.3.107) and (4.3.108) in (4.3.106) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u^*, \theta^*)\|_{L^2}^2 + \nu\|\partial_1 u^*\|_{L^2}^2 + \eta\|\partial_2 \theta^*\|_{L^2}^2 \\
&\qquad \leq c\Big(\|u^*\|_{L^2}^2 + \|\theta^*\|_{L^2}^2 \Big) + \frac{\nu}{2}\|\partial_1 u^*\|_{L^2}^2 + \frac{\eta}{2}\|\partial_2 \theta^*\|_{L^2}^2.
\end{aligned}$$

Hence,

$$\frac{d}{dt}\|(u^*, \theta^*)\|_{L^2}^2 + \nu\|\partial_1 u^*\|_{L^2}^2 + \eta\|\partial_2 \theta^*\|_{L^2}^2 \leq c\|(u^*, \theta^*)\|_{L^2}^2. \tag{4.3.109}$$

Gronwall's inequality applied to (4.3.109) implies that $\|u^*\|_{L^2}^2 = \|\theta^*\|_{L^2}^2 = 0$. This completes the proof of Theorem 4.3.6. ∎

4.3.3 Decay Rates Result

The next theorem rigorously establishes what we have observed in numerical simulations of buoyancy-driven stratified fluids (see, e.g., [24]). Perturbations governed by the Boussinesq systems near the hydrostatic equilibrium are observed to stratify and eventually approach their horizontal averages while the oscillation parts of both u and θ are observed to decay to zero. The following theorem verifies that indeed the oscillation part $(\widetilde{u}, \widetilde{\theta})$ corresponding to the solution of the system (4.3.1) decays to zero at algebraic rates.

Theorem 4.3.7 *Let $u_0, \theta_0 \in H^2(\Omega)$ with $\nabla \cdot u_0 = 0$. Assume that (u_0, θ_0) satisfies*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon,$$

for sufficiently small $\varepsilon > 0$. Let (u, θ) be the corresponding solution of (4.3.1). Then the oscillation part $(\widetilde{u}, \widetilde{\theta})$ satisfies the following algebraic decay in time,

$$\|\widetilde{u}\|_{H^1} + \|\widetilde{\theta}\|_{H^1} \leq c(1+t)^{-\frac{1}{2}}, \tag{4.3.110}$$

for some constant $c > 0$ and for all $t > 0$. In addition, $(\widetilde{u}, \widetilde{\theta})$ has the asymptotic behavior, as $t \to \infty$,

$$t\,(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

As a consequence of Theorem 4.3.7, the solution (u, θ) of (4.3.1) approaches the horizontal average $(\overline{u}, \overline{\theta})$ asymptotically, and the 2D Boussinesq system (4.3.1) evolves to the following 1D system

$$\begin{cases} \partial_t \overline{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \overline{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\theta} \end{pmatrix}, \\ \partial_t \overline{\theta} + \overline{u \cdot \nabla \widetilde{\theta}} = \eta \partial_2^2 \overline{\theta}. \end{cases}$$

Proof. We start the proof of Theorem 4.3.7 by writing the system governing the horizontal average $(\overline{u}, \overline{\theta})$, namely,

$$\begin{cases} \partial_t \overline{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \overline{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\theta} \end{pmatrix}, \\ \partial_t \overline{\theta} + \overline{u \cdot \nabla \widetilde{\theta}} = \eta \partial_2^2 \overline{\theta}. \end{cases} \tag{4.3.111}$$

Taking the difference of (4.3.1) and (4.3.111), we get

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + \widetilde{u}_2 \partial_2 \overline{u} - \nu \partial_1^2 \widetilde{u} + \nabla \widetilde{p} = \widetilde{\theta} e_2, \\ \partial_t \widetilde{\theta} + \widetilde{u \cdot \nabla \widetilde{\theta}} + \widetilde{u}_2 \partial_2 \overline{\theta} - \eta \partial_2^2 \widetilde{\theta} + \widetilde{u}_2 = 0. \end{cases} \tag{4.3.112}$$

Dotting the system (4.3.112) by $(\widetilde{u}, \widetilde{\theta})$ yields,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big) + \nu\|\partial_1 \widetilde{u}\|_{L^2}^2 + \eta\|\partial_2 \widetilde{\theta}\|_{L^2}^2 \\ &\qquad = -\int \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx - \int \widetilde{u}_2 \partial_2 \overline{u} \cdot \widetilde{u} dx - \int \widetilde{u \cdot \nabla \widetilde{\theta}} \cdot \widetilde{\theta} dx - \int \widetilde{u}_2 \partial_2 \overline{\theta} \cdot \widetilde{\theta} dx \\ &\qquad := A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{4.3.113}$$

Using the divergence-free condition of u and Lemma 4.3.1,

$$A_1 := -\int \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx = \underbrace{-\int u \cdot \nabla \widetilde{u} \cdot \widetilde{u} dx}_{=0} + \underbrace{\int \overline{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u} dx}_{=0} = 0. \tag{4.3.114}$$

Similarly,

$$A_3 := \int \widetilde{u \cdot \nabla \widetilde{\theta}} \cdot \widetilde{\theta} dx = 0. \tag{4.3.115}$$

By Lemma 4.3.4, the divergence free condition of u, Lemma 4.3.1, and Lemma 4.3.5,

$$\begin{aligned} A_2 &:= -\int \widetilde{u}_2 \partial_2 \overline{u} \cdot \widetilde{u} dx \\ &\le c\|\partial_2 \overline{u}\|_{L^2} \|\widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le c\|\partial_2 \overline{u}\|_{L^2} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le c\|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{L^2}^2. \end{aligned} \tag{4.3.116}$$

Then, we estimate A_4 using Hölder's inequality, Lemma 4.3.2, Lemma 4.3.5 and Cauchy's inequality

$$\begin{aligned}
A_4 &:= -\int \widetilde{u_2}\partial_2\overline{\theta}\cdot\widetilde{\theta}dx \\
&\leq c\|\partial_2\overline{\theta}\|_{L^\infty_{x_2}}\|\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\
&\leq c\|\partial_2\overline{\theta}\|_{H^1}\|\partial_1\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\
&\leq c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\widetilde{\theta}\|_{L^2} \\
&\leq c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big).
\end{aligned} \tag{4.3.117}$$

Collecting the estimates (4.3.114), (4.3.115), (4.3.116), (4.3.117) and (4.3.113) we get,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)+\nu\|\partial_1\widetilde{u}\|_{L^2}^2+\eta\|\partial_2\widetilde{\theta}\|_{L^2}^2 \\
&\qquad\leq c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\Big).
\end{aligned} \tag{4.3.118}$$

Further, applying ∇ to (4.3.112) yields,

$$\begin{cases}
\partial_t\nabla\widetilde{u}+\nabla(\widetilde{u\cdot\nabla\widetilde{u}})+\nabla(\widetilde{u_2}\partial_2\overline{u})-\nu\partial_1^2\nabla\widetilde{u}+\nabla\nabla\widetilde{p}=\nabla(\widetilde{\theta}e_2), \\
\partial_t\nabla\widetilde{\theta}+\nabla(\widetilde{u\cdot\nabla\widetilde{\theta}})+\nabla(\widetilde{u_2}\partial_2\overline{\theta})-\eta\partial_2^2\nabla\widetilde{\theta}+\nabla\widetilde{u_2}=0.
\end{cases} \tag{4.3.119}$$

Taking the L^2-inner product of (4.3.119) with $(\nabla\widetilde{u},\nabla\widetilde{\theta})$ we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2+\|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big)+\nu\|\partial_1\nabla\widetilde{u}\|_{L^2}^2+\eta\|\partial_2\nabla\widetilde{\theta}\|_{L^2}^2 \\
&\qquad= -\int\nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}dx-\int\nabla(\widetilde{u_2}\partial_2\overline{u})\cdot\nabla\widetilde{u}dx \\
&\qquad\quad -\int\nabla(\widetilde{u\cdot\nabla\widetilde{\theta}})\cdot\nabla\widetilde{\theta}dx-\int\nabla(\widetilde{u_2}\partial_2\overline{\theta})\cdot\nabla\widetilde{\theta}dx \\
&\qquad:= B_1+B_2+B_3+B_4.
\end{aligned} \tag{4.3.120}$$

According to Lemma 4.3.1, we write B_1 explicitly into the following four integrals,

$$
\begin{aligned}
B_1 &:= -\int \nabla(\widetilde{u \cdot \nabla \widetilde{u}}) \cdot \nabla \widetilde{u} dx \\
&= -\int \nabla(u \cdot \nabla \widetilde{u}) \cdot \nabla \widetilde{u}\, dx + \underbrace{\int \nabla(\overline{u \cdot \nabla \widetilde{u}}) \cdot \nabla \widetilde{u} dx}_{=0} \\
&= -\int \partial_1 u_1 \partial_1 \widetilde{u} \partial_1 \widetilde{u} dx - \int \partial_1 u_2 \partial_2 \widetilde{u} \partial_1 \widetilde{u} dx - \int \partial_2 u_1 \partial_1 \widetilde{u} \partial_2 \widetilde{u} dx - \int \partial_2 u_2 \partial_2 \widetilde{u} \partial_2 \widetilde{u} dx \\
&:= B_{11} + B_{12} + B_{13} + B_{14}.
\end{aligned} \tag{4.3.121}
$$

We start with B_{11}. Due to Lemma 4.3.1 and the inequality (4.3.11),

$$
\begin{aligned}
B_{11} &:= -\int \partial_1 u_1 \partial_1 \widetilde{u} \partial_1 \widetilde{u} dx = -\int \partial_1 \widetilde{u_1} \partial_1 \widetilde{u} \partial_1 \widetilde{u} dx \\
&\leq c \|\partial_1 \partial_1 \widetilde{u}\|_{L^2} \|\partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2.
\end{aligned} \tag{4.3.122}
$$

Similarly,

$$
\begin{aligned}
B_{12} &:= -\int \partial_1 u_2 \partial_2 \widetilde{u} \partial_1 \widetilde{u} dx = -\int \partial_1 \widetilde{u_2} \partial_2 \widetilde{u} \partial_1 \widetilde{u} dx \\
&\leq c \|\partial_1 \partial_1 \widetilde{u_2}\|_{L^2} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2.
\end{aligned} \tag{4.3.123}
$$

Using the inequality (4.3.11),

$$
\begin{aligned}
B_{13} &:= -\int \partial_2 u_1 \partial_1 \widetilde{u} \partial_2 \widetilde{u} dx \\
&\leq c \|\partial_1 \partial_2 \widetilde{u}\|_{L^2} \|\partial_2 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2.
\end{aligned} \tag{4.3.124}
$$

According to the divergence-free condition of u, Lemma 4.3.1 and the inequality

(4.3.11),

$$\begin{aligned}
B_{14} &:= -\int \partial_2 u_2 \partial_2 \widetilde{u} \partial_2 \widetilde{u} dx = \int \partial_1 u_1 \partial_2 \widetilde{u} \partial_2 \widetilde{u} dx \\
&= \int \partial_1 \widetilde{u_1} \partial_2 \widetilde{u} \partial_2 \widetilde{u} dx \\
&\leq c \|\partial_1 \partial_2 \widetilde{u}\|_{L^2} \|\partial_2 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u_1}\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2. \qquad (4.3.125)
\end{aligned}$$

In view (4.3.121), collecting the estimates (4.3.122), (4.3.123), (4.3.124) and (4.3.125) we get,

$$B_1 \leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2. \qquad (4.3.126)$$

Now, we write B_2 explicitly,

$$\begin{aligned}
B_2 &:= -\int \nabla(\widetilde{u_2} \partial_2 \overline{u}) \cdot \nabla \widetilde{u} dx \\
&= -\int \partial_1 \widetilde{u_2} \partial_2 \overline{u} \partial_1 \widetilde{u} dx - \int \partial_2 \widetilde{u_2} \partial_2 \overline{u} \partial_2 \widetilde{u} dx \\
&\quad - \int \widetilde{u_2} \partial_1 \partial_2 \overline{u} \partial_1 \widetilde{u} dx - \int \widetilde{u_2} \partial_2 \partial_2 \overline{u} \partial_2 \widetilde{u} dx \\
&:= B_{21} + B_{22} + B_{23} + B_{24}. \qquad (4.3.127)
\end{aligned}$$

We start with B_{21}. By the inequality (4.3.11) and Lemma 4.3.1,

$$\begin{aligned}
B_{21} &:= -\int \partial_1 \widetilde{u_2} \partial_2 \overline{u} \partial_1 \widetilde{u} dx \\
&\leq c \|\partial_1 \partial_1 \widetilde{u}\|_{L^2} \|\partial_2 \overline{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \overline{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u_2}\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2. \qquad (4.3.128)
\end{aligned}$$

For B_{22}, we use the divergence-free condition of u, the inequality (4.3.11) and Lemma

4.3.1,

$$
\begin{aligned}
B_{22} &:= -\int \partial_2 \widetilde{u_2} \partial_2 \overline{u} \partial_2 \widetilde{u} dx \\
&= \int \partial_1 \widetilde{u_1} \partial_2 \overline{u} \partial_2 \widetilde{u} dx \\
&\leq c \|\partial_1 \partial_1 \widetilde{u_1}\|_{L^2} \|\partial_2 \overline{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \overline{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2. \qquad (4.3.129)
\end{aligned}
$$

Due to the definition of $\overline{u}$,

$$
B_{23} := -\int \widetilde{u_2} \partial_1 \partial_2 \overline{u} \partial_1 \widetilde{u} dx = 0. \qquad (4.3.130)
$$

To estimate B_{24}, we make use of the inequality (4.3.11) and the divergence- free condition of u

$$
\begin{aligned}
B_{24} &:= -\int \widetilde{u_2} \partial_2 \partial_2 \overline{u} \partial_2 \widetilde{u} dx \\
&\leq c \|\partial_1 \partial_2 \widetilde{u}\|_{L^2} \|\partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \overline{u}\|_{L^2} \\
&\leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2. \qquad (4.3.131)
\end{aligned}
$$

Combining (4.3.128), (4.3.129), (4.3.130), (4.3.131) and (4.3.127) we obtain,

$$
B_2 \leq c \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1}^2. \qquad (4.3.132)
$$

Further, by the definition of $\overline{u}$, we can split B_3 into four integrals,

$$
\begin{aligned}
B_3 &:= -\int \nabla (\widetilde{u \cdot \nabla \widetilde{\theta}}) \cdot \nabla \widetilde{\theta} dx \\
&= -\int \nabla (u \cdot \nabla \widetilde{\theta}) \cdot \nabla \widetilde{\theta} dx + \underbrace{\int \overline{\nabla (u \cdot \nabla \widetilde{\theta})} \cdot \nabla \widetilde{\theta} dx}_{=0} \\
&= -\int \partial_1 \widetilde{\theta} \partial_1 \widetilde{u_1} \partial_1 \widetilde{\theta} dx - \int \partial_2 \widetilde{\theta} \partial_1 \widetilde{u_2} \partial_1 \widetilde{\theta} dx \\
&\quad - \int \partial_1 \widetilde{\theta} \partial_2 u_1 \partial_2 \widetilde{\theta} dx - \int \partial_2 \widetilde{\theta} \partial_2 \widetilde{u_2} \partial_2 \widetilde{\theta} dx \\
&:= B_{31} + B_{32} + B_{33} + B_{34}. \qquad (4.3.133)
\end{aligned}
$$

Using the divergence-free condition of u, integration by parts, the inequality (4.3.11) and Young's inequality, we bound B_{31} as follows,

$$\begin{aligned}
B_{31} &:= -\int \partial_1\widetilde{\theta}\partial_1\widetilde{u_1}\partial_1\widetilde{\theta}dx = \int \partial_2\widetilde{u_2}(\partial_1\widetilde{\theta})^2dx = -2\int \widetilde{u_2}\partial_2\partial_1\widetilde{\theta}\partial_1\widetilde{\theta}dx\\
&\le c\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}\|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\\
&\le c\|\partial_2\widetilde{\theta}\|_{H^1}^{\frac{3}{2}}\|\partial_1\widetilde{u}\|_{H^1}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\\
&\le c\|u\|_{H^2}^{\frac{1}{2}}\|\theta\|_{H^2}^{\frac{1}{2}}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \qquad (4.3.134)
\end{aligned}$$

To deal with B_{32}, we use Lemma 4.3.4 and Cauchy's inequality,

$$\begin{aligned}
B_{32} &:= -\int \partial_2\widetilde{\theta}\partial_1\widetilde{u_2}\partial_1\widetilde{\theta}dx\\
&\le c\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2}\\
&\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\partial_2\widetilde{\theta}\|_{H^1}\\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \qquad (4.3.135)
\end{aligned}$$

For B_{33}, we invoke the decomposition $u_1 = \overline{u_1} + \widetilde{u_1}$ to write it into two integrals

$$\begin{aligned}
B_{33} &:= -\int \partial_1\widetilde{\theta}\partial_2 u_1\partial_2\widetilde{\theta}dx\\
&= -\int \partial_1\widetilde{\theta}\partial_2\widetilde{u_1}\partial_2\widetilde{\theta}dx - \int \partial_1\widetilde{\theta}\partial_2\overline{u_1}\partial_2\widetilde{\theta}dx\\
&:= B_{331} + B_{332}. \qquad (4.3.136)
\end{aligned}$$

By integration by parts, Hölder's inequality, Lemma 4.3.1, Lemma 4.3.4 and Cauchy's inequality, we get

$$\begin{aligned}
B_{331} &:= -\int \partial_1\widetilde{\theta}\,\partial_2\widetilde{u_1}\,\partial_2\widetilde{\theta}\,dx\\
&= \int \widetilde{\theta}\,\partial_1\partial_2\widetilde{u_1}\,\partial_2\widetilde{\theta}dx + \int \widetilde{\theta}\,\partial_2\widetilde{u_1}\,\partial_1\partial_2\widetilde{\theta}dx\\
&\le c\|\partial_2\widetilde{\theta}\|_{L^2}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}\|\widetilde{\theta}\|_{L^\infty} + c\|\widetilde{\theta}\|_{L^\infty}\|\partial_2\widetilde{u_1}\|_{L^2}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}\\
&\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\partial_2\widetilde{\theta}\|_{H^1} + c\|\theta\|_{H^2}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}\\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \qquad (4.3.137)
\end{aligned}$$

Due to Lemma 4.3.2, Hölder's inequality and Cauchy's inequaliy,

$$\begin{aligned}
B_{332} &:= -\int \widetilde{\theta}\, \partial_2 \overline{u_1}\, \partial_1 \partial_2 \widetilde{\theta}\, dx = \int_{\mathbb{R}} \partial_2 \overline{u_1} \int_{\mathbb{T}} \widetilde{\theta}\, \partial_1 \partial_2 \widetilde{\theta}\, dx_1 dx_2 \\
&\le c \int_{\mathbb{R}} |\partial_2 \overline{u_1}|\, \|\widetilde{\theta}\|_{L^2_{x_1}} \|\partial_1 \partial_2 \widetilde{\theta}\|_{L^2_{x_1}} dx_2 \\
&\le c \|\partial_2 \overline{u_1}\|_{L^\infty_{x_2}} \|\widetilde{\theta}\|_{L^2_{x_1} L^2_{x_2}} \|\partial_1 \partial_2 \widetilde{\theta}\|_{L^2_{x_1} L^2_{x_2}} \\
&\le c \|\partial_2 \overline{u_1}\|_{H^1} \|\widetilde{\theta}\|_{L^2} \|\partial_1 \partial_2 \widetilde{\theta}\|_{L^2} \\
&\le c \|u\|_{H^2} \Big(\|\widetilde{\theta}\|_{L^2}^2 + \|\partial_2 \widetilde{\theta}\|_{H^1}^2 \Big). \qquad (4.3.138)
\end{aligned}$$

Combining (4.3.137), (4.3.138) and (4.3.136) we obtain,

$$B_{33} \le c \|(u,\theta)\|_{H^2} \Big(\|\widetilde{\theta}\|_{L^2}^2 + \|\partial_2 \widetilde{\theta}\|_{H^1}^2 + \|\partial_1 \widetilde{u}\|_{H^1}^2 \Big). \qquad (4.3.139)$$

According to the divergence-free condition of u, Lemma 4.3.4 and Cauchy's inequality,

$$\begin{aligned}
B_{34} &:= -\int \partial_2 \widetilde{\theta} \partial_2 \widetilde{u_2} \partial_2 \widetilde{\theta} dx = \int \partial_2 \widetilde{\theta} \partial_1 \widetilde{u_1} \partial_2 \widetilde{\theta} dx \\
&\le c \|\partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{\theta}\|_{L^2} \\
&\le c \|\theta\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1} \|\partial_2 \widetilde{\theta}\|_{H^1} \\
&\le c \|\theta\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\partial_2 \widetilde{\theta}\|_{H^1}^2 \Big). \qquad (4.3.140)
\end{aligned}$$

Combining the estimates (4.3.134), (4.3.135), (4.3.139), (4.3.140) and inserting them in (4.3.133) we get

$$B_3 \le c \|(u,\theta)\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\partial_2 \widetilde{\theta}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big). \qquad (4.3.141)$$

It remains to bound B_4. By integration by parts, we write B_4 into four terms as follows,

$$
\begin{aligned}
B_4 &:= -\int \nabla(\widetilde{u_2}\partial_2\overline{\theta})\cdot\nabla\widetilde{\theta}dx \\
&= -\int \partial_1(\widetilde{u_2}\partial_2\overline{\theta})\cdot\partial_1\widetilde{\theta}dx - \int \partial_2(\widetilde{u_2}\partial_2\overline{\theta})\cdot\partial_2\widetilde{\theta}dx \\
&= -\int \partial_1\widetilde{u_2}\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx - \int \widetilde{u_2}\partial_1\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx \\
&\quad -\int \partial_2\widetilde{u_2}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx - \int \widetilde{u_2}\partial_2\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx \\
&:= B_{41}+B_{42}+B_{43}+B_{44}. \qquad (4.3.142)
\end{aligned}
$$

We start with B_{41}. To bound B_{41}, we use integration by parts, Hölder's inequality, Lemma 4.3.2 and Cauchy's inequality

$$
\begin{aligned}
B_{41} &:= -\int \partial_1\widetilde{u_2}\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx \\
&= \int \partial_1\partial_1\widetilde{u_2}\ \partial_2\overline{\theta}\ \widetilde{\theta}dx \\
&\le c\|\partial_2\overline{\theta}\|_{L^\infty}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\
&\le c\|\partial_2\overline{\theta}\|_{H^1}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\
&\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\widetilde{\theta}\|_{L^2} \\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big). \qquad (4.3.143)
\end{aligned}
$$

Due to the definition of the horizontal average $\overline{\theta}$,

$$
B_{42} := -\int \widetilde{u_2}\partial_1\partial_2\overline{\theta}\partial_1\widetilde{\theta}dx = 0. \qquad (4.3.144)
$$

To estimate B_{43}, we use the divergence-free condition of u, Lemma 4.3.4 and Cauchy's inequality,

$$
\begin{aligned}
B_{43} &:= -\int \partial_2\widetilde{u_2}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx = \int \partial_1\widetilde{u_1}\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx \\
&\le c\|\partial_1\widetilde{u_1}\|_{L^2}\|\partial_1\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\overline{\theta}\|_{L^2} \\
&\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\partial_2\widetilde{\theta}\|_{H^1} \\
&\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \qquad (4.3.145)
\end{aligned}
$$

For B_{44}, we use Lemma 4.3.1, the inequality (4.3.11) and Cauchy's inequality,

$$\begin{aligned} B_{44} &:= -\int \widetilde{u_2}\partial_2\partial_2\overline{\theta}\partial_2\widetilde{\theta}dx \\ &\le c\|\partial_1\widetilde{u_2}\|_{L^2}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{\theta}\|_{L^2} \\ &\le c\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\partial_2\widetilde{\theta}\|_{H^1} \\ &\le c\|\theta\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \end{aligned} \tag{4.3.146}$$

Combining the estimates (4.3.143), (4.3.144), (4.3.145), (4.3.146) and inserting them in (4.3.142) we get

$$B_4 \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big). \tag{4.3.147}$$

Collecting the estimates (4.3.126), (4.3.132), (4.3.141), (4.3.147) and inserting them in (4.3.120) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\widetilde{u}(t)\|_{L^2}^2 + \|\nabla\widetilde{\theta}(t)\|_{L^2}^2\Big) + \nu\|\partial_1\nabla\widetilde{u}\|_{L^2}^2 + \eta\|\partial_2\nabla\widetilde{\theta}\|_{L^2}^2 \\ &\qquad\qquad \le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}(t)\|_{H^1}^2 + \|\partial_2\widetilde{\theta}(t)\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big). \end{aligned} \tag{4.3.148}$$

In order to control the norm $\|\widetilde{\theta}\|_{L^2}$ appearing in (4.3.118) and (4.3.148) we need to add the following term,

$$-\frac{d}{dt}\Big(\delta(\widetilde{u_2},\widetilde{\theta})\Big) = -\delta(\partial_t\widetilde{u_2},\widetilde{\theta}) - \delta(\widetilde{u_2},\partial_t\widetilde{\theta}).$$

where $\delta > 0$ is a small constant to be fixed in the end of the proof. The inclusion of this term will generate an extra regularization term to help bound $\|\widetilde{\theta}\|_{L^2}$. Clearly this stabilizing term comes from the interaction between $\widetilde{u}$ and $\widetilde{\theta}$. By Hölder's inequality and Cauchy's inequality, one can easily sees that, for sufficiently small $\delta > 0$,

$$\|(\widetilde{u},\widetilde{\theta})\|_{H^1}^2 - \delta(\widetilde{u_2},\widetilde{\theta}) \ge 0.$$

Due to the first equation of (4.3.112) and the fact that $\overline{u_2} = 0$, we have

$$\partial_t\widetilde{u_2} + \widetilde{u\cdot\nabla\widetilde{u_2}} + \underbrace{\widetilde{u_2}\partial_2\overline{u_2}}_{=0} - \nu\partial_1^2\widetilde{u_2} + \partial_2\widetilde{p} = \widetilde{\theta}. \tag{4.3.149}$$

On the other hand, applying $\nabla\cdot$ to the first equation of (4.3.112), we get

$$\nabla\cdot(\widetilde{u\cdot\nabla\widetilde{u}})+\nabla\cdot(\widetilde{u}_2\partial_2\overline{u})+\Delta\widetilde{p}=\partial_2\widetilde{\theta}. \tag{4.3.150}$$

By (4.3.150), we can write

$$\widetilde{p}=-\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla\widetilde{u}})-\Delta^{-1}\nabla\cdot(\widetilde{u}_2\partial_2\overline{u})+\Delta^{-1}\partial_2\widetilde{\theta}. \tag{4.3.151}$$

Hence,

$$\partial_2\widetilde{p}=-\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla\widetilde{u}})-\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u}_2\partial_2\overline{u})+\partial_2\partial_2\Delta^{-1}\widetilde{\theta}. \tag{4.3.152}$$

Using (4.3.149) and the second equation of (4.3.112) we get,

$$\begin{aligned}
-\delta\frac{d}{dt}(\widetilde{u}_2,\widetilde{\theta}) &= -\delta(\partial_t\widetilde{u}_2,\widetilde{\theta})-\delta(\widetilde{u}_2,\partial_t\widetilde{\theta})\\
&= -\delta(\widetilde{\theta}-\partial_2\widetilde{p}+\nu\partial_1^2\widetilde{u}_2-\widetilde{u\cdot\nabla\widetilde{u}_2},\widetilde{\theta})-\delta(\widetilde{u}_2,-\widetilde{u}_2+\eta\partial_2^2\widetilde{\theta}-\widetilde{u}_2\partial_2\overline{\theta}-\widetilde{u\cdot\nabla\widetilde{\theta}})\\
&= -\delta\|\widetilde{\theta}\|_{L^2}^2+\int\partial_2\widetilde{p}\widetilde{\theta}dx-\delta\nu\int\partial_1^2\widetilde{u}_2\widetilde{\theta}dx+\delta\int\widetilde{u\cdot\nabla\widetilde{u}_2}\widetilde{\theta}dx\\
&\quad+\delta\|\widetilde{u}_2\|_{L^2}^2-\delta\eta\int\partial_2^2\widetilde{\theta}\widetilde{u}_2dx+\delta\int\widetilde{u}_2\widetilde{u}_2\partial_2\overline{\theta}dx+\delta\int\widetilde{u\cdot\nabla\widetilde{\theta}}\widetilde{u}_2dx\\
&:= N_1+\cdots+N_8.
\end{aligned} \tag{4.3.153}$$

We start with N_2. By (4.3.152), we have

$$\begin{aligned}
N_2 &:= \delta\int\partial_2\widetilde{p}\,\widetilde{\theta}dx\\
&= -\delta\int\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\widetilde{\theta}dx-\delta\int\partial_2\Delta^{-1}\nabla\cdot(\widetilde{u}_2\partial_2\overline{u})\cdot\widetilde{\theta}dx\\
&\quad+\delta\int\partial_2\partial_2\Delta^{-1}\widetilde{\theta}\cdot\widetilde{\theta}dx\\
&:= N_{21}+N_{22}+N_{23}.
\end{aligned} \tag{4.3.154}$$

Using respectively Hölder's inequality, the boundedness of the Riesz transform, Lemma 4.3.4, Lemma 4.3.5 and Cauchy's inequality we get

$$
\begin{aligned}
N_{21} &:= -\delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u \cdot \nabla \widetilde{u}}) \cdot \widetilde{\theta} dx \\
&\leq c\delta \|\partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u \cdot \nabla \widetilde{u}})\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|\widetilde{u \cdot \nabla \widetilde{u}}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|u \cdot \nabla \widetilde{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|u\|_{L^\infty} \|\nabla \widetilde{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|u\|_{H^2} \|\partial_1 \nabla \widetilde{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{H^1} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|u\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big). \qquad (4.3.155)
\end{aligned}
$$

By Hölder's inequality, the boundedness of the Riesz transform, Lemma 4.3.2 and Cauchy's inequality

$$
\begin{aligned}
N_{22} &:= -\delta \int \partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u}) \cdot \widetilde{\theta} dx \\
&\leq c\delta \|\partial_2 \Delta^{-1} \nabla \cdot (\widetilde{u_2} \partial_2 \overline{u})\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|\widetilde{u_2} \partial_2 \overline{u}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|\partial_2 \overline{u}\|_{L^\infty_{x_2}} \|\widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|\partial_2 \overline{u}\|_{H^1} \|\widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|u\|_{H^2} \|\partial_1 \widetilde{u_2}\|_{L^2} \|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta \|u\|_{H^2} \Big(\|\partial_1 \widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2 \Big). \qquad (4.3.156)
\end{aligned}
$$

For N_{23}, we use respectively integration by parts and Plancherel's theorem

$$\begin{aligned}
N_{23} &:= \delta \int \partial_2\partial_2\Delta^{-1}\widetilde{\theta}\cdot\widetilde{\theta}dx \\
&= \delta \int \partial_2\Delta^{-\frac{1}{2}}\widetilde{\theta}\cdot\partial_2\Delta^{-\frac{1}{2}}\widetilde{\theta}dx \\
&= \delta\|\partial_2\Lambda^{-1}\widetilde{\theta}\|_{L^2}^2 \\
&= \delta\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\int_{\mathbb{R}}\frac{\xi_2^2}{k^2+\xi_2^2}|\widehat{\widetilde{\theta}}(k,\xi_2)|^2 d\xi_2 \\
&\leq \delta\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\int_{\mathbb{R}}\xi_2^2|\widehat{\widetilde{\theta}}(k,\xi_2)|^2 d\xi_2 = \delta\|\partial_2\widetilde{\theta}\|_{L^2}^2,
\end{aligned} \tag{4.3.157}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and we have used the fact that the oscillation part has the horizontal mode equal to 0, or $\widehat{\widetilde{\theta}}(0,\xi_2) = 0$.

Combining (4.3.155), (4.3.156), (4.3.157) and (4.3.154) we get

$$N_2 \leq c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big) + \delta\|\partial_2\widetilde{\theta}\|_{L^2}^2. \tag{4.3.158}$$

Due to Hölder's inequality and Cauchy's inequality with epsilon,

$$N_3 := -\delta\nu\int\partial_1^2\widetilde{u_2}\widetilde{\theta}dx \leq \delta\nu\|\partial_1^2\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \leq \delta\nu^2\|\partial_1\widetilde{u}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2. \tag{4.3.159}$$

To bound N_4, we use respectively, Lemma 4.3.1, Hölder's inequality, Lemma 4.3.3, Lemma 4.3.5 and Cauchy's inequality

$$\begin{aligned}
N_4 &:= \delta\int\widetilde{u\cdot\nabla\widetilde{u_2}}\widetilde{\theta}dx \\
&= \delta\int u\cdot\nabla\widetilde{u_2}\widetilde{\theta}dx - \underbrace{\delta\int\overline{u\cdot\nabla\widetilde{u_2}}\widetilde{\theta}dx}_{=0} \\
&\leq c\delta\|u\cdot\nabla\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta\|u\|_{L^\infty}\|\nabla\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta\|u\|_{H^2}\|\partial_1\nabla\widetilde{u_2}\|_{L^2}\|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta\|u\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}\|\widetilde{\theta}\|_{L^2} \\
&\leq c\delta\|u\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big).
\end{aligned} \tag{4.3.160}$$

By Lemma 4.3.5,

$$N_5 := \delta\|\widetilde{u_2}\|_{L^2}^2 \leq c\delta\|\partial_1\widetilde{u_2}\|_{L^2}^2 \leq c\delta\|\partial_1\widetilde{u}\|_{H^1}^2. \tag{4.3.161}$$

Due to Hölder's inequality, Lemma 4.3.5 and Cauchy's inequality,

$$\begin{aligned} N_6 &:= -\delta\eta\int \partial_2^2\widetilde{\theta}\widetilde{u_2}dx \\ &\leq c\delta\|\partial_2^2\widetilde{\theta}\|_{L^2}\|\widetilde{u_2}\|_{L^2} \\ &\leq c\delta\|\partial_2\widetilde{\theta}\|_{H^1}\|\partial_1\widetilde{u_2}\|_{L^2} \\ &\leq c\delta\|\partial_2\widetilde{\theta}\|_{H^1}\|\partial_1\widetilde{u}\|_{L^2} \\ &\leq c\delta\Big(\|\partial_2\widetilde{\theta}\|_{H^1}^2 + \|\partial_1\widetilde{u}\|_{H^1}^2\Big). \end{aligned} \tag{4.3.162}$$

Using Lemma 4.3.4 and Lemma 4.3.5, we get

$$\begin{aligned} N_7 &:= \delta\int \widetilde{u_2}\widetilde{u_2}\partial_2\overline{\theta}dx \\ &\leq c\delta\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\overline{\theta}\|_{L^2}^{\frac{1}{2}}\|\widetilde{u_2}\|_{L^2} \\ &\leq c\delta\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \end{aligned} \tag{4.3.163}$$

To deal with N_8, we split it first into three terms using Lemma 4.3.1,

$$\begin{aligned} N_8 &:= \delta\int \widetilde{u\cdot\nabla\widetilde{\theta}}\widetilde{u_2}dx \\ &= \delta\int u\cdot\nabla\widetilde{\theta}\widetilde{u_2}dx - \underbrace{\delta\int \overline{u\cdot\nabla\widetilde{\theta}}\widetilde{u_2}dx}_{=0} \\ &= \delta\int \widetilde{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx + \delta\int \overline{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx + \delta\int u_2\partial_2\widetilde{\theta}\widetilde{u_2}dx \\ &:= N_{81} + N_{82} + N_{83}. \end{aligned} \tag{4.3.164}$$

By the inequality (4.3.11) and divergence free condition of u, we have

$$\begin{aligned} N_{81} &:= \delta\int \widetilde{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx \\ &\leq c\delta\|\partial_1\widetilde{u_1}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{\theta}\|_{L^2} \\ &\leq c\delta\|\theta\|_{H^2}\|\partial_1\widetilde{u}\|_{H^1}^2. \end{aligned} \tag{4.3.165}$$

By integration by parts, Hölder's inequality, Lemma 4.3.2 and Cauchy's inequality

$$
\begin{aligned}
N_{82} := \delta \int \overline{u_1}\partial_1\widetilde{\theta}\widetilde{u_2}dx
&= -\delta \int \overline{u_1}\widetilde{\theta}\partial_1\widetilde{u_2}dx \\
&\leq \delta\|\overline{u_1}\|_{L^\infty_{x_2}}\|\widetilde{\theta}\partial_1\widetilde{u_2}\|_{L^1} \\
&\leq c\delta\|\overline{u_1}\|_{L^\infty_{x_2}}\|\widetilde{\theta}\|_{L^2}\|\partial_1\widetilde{u_2}\|_{L^2} \\
&\leq c\delta\|u\|_{H^1}\|\widetilde{\theta}\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2} \\
&\leq c\delta\|u\|_{H^2}\Big(\|\widetilde{\theta}\|_{L^2}^2 + \|\partial_1\widetilde{u}\|_{H^1}^2\Big). \qquad (4.3.166)
\end{aligned}
$$

Due to the inequality (4.3.11), Lemma 4.3.5 and the divergence-free condition of u

$$
\begin{aligned}
N_{83} := \delta \int u_2\partial_2\widetilde{\theta}\widetilde{u_2}dx
&= \delta \int \widetilde{u_2}\partial_2\widetilde{\theta}\widetilde{u_2}dx \\
&\leq c\delta\|\partial_1\widetilde{u_2}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2} \\
&\leq c\delta\|\partial_1\widetilde{u}\|_{H^1}^2\|\theta\|_{H^2}. \qquad (4.3.167)
\end{aligned}
$$

In view of (4.3.164), combining (4.3.165), (4.3.166) and (4.3.167) we get

$$
N_8 \leq c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big). \qquad (4.3.168)
$$

Inserting (4.3.158), (4.3.159), (4.3.160), (4.3.161), (4.3.162), (4.3.163) and (4.3.168) in (4.3.153) leads to

$$
\begin{aligned}
-\delta\frac{d}{dt}(\widetilde{u_2},\widetilde{\theta}) \leq& -\delta\|\widetilde{\theta}\|_{L^2}^2 + c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{L^2}^2\Big) \\
&+ \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2 + c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}\|_{H^1}^2\Big). \qquad (4.3.169)
\end{aligned}
$$

Putting (4.3.118), (4.3.148) and (4.3.169) together, we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)&+2\nu\|\partial_1\widetilde{u}\|_{H^1}^2+2\eta\|\partial_2\widetilde{\theta}\|_{H^1}^2\\ &\le c\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)\\ &\quad-\frac{3\delta}{4}\|\widetilde{\theta}\|_{L^2}^2+c\delta\|(u,\theta)\|_{H^2}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)\\ &\quad+c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).\end{aligned}\tag{4.3.170}$$

Now, by Theorem 4.3.6, if $\varepsilon>0$ is sufficiently small and $\|u_0\|_{L^2}+\|\theta_0\|_{L^2}\le\varepsilon$, then $\|(u(t),\theta(t))\|_{H^2}\le c\varepsilon$. Hence we get

$$\begin{aligned}\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)&+2\nu\|\partial_1\widetilde{u}\|_{H^1}^2+2\eta\|\partial_2\widetilde{\theta}\|_{H^1}^2\\ &\le c\epsilon\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)\\ &\quad-\frac{3\delta}{4}\|\widetilde{\theta}\|_{L^2}^2+c\delta\epsilon\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)\\ &\quad+c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).\end{aligned}\tag{4.3.171}$$

Choosing $\epsilon>0$ such that $c\epsilon\le\min(\frac{1}{4},\frac{\delta}{4})$ we get

$$\begin{aligned}\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)&+2\nu\|\partial_1\widetilde{u}\|_{H^1}^2+2\eta\|\partial_2\widetilde{\theta}\|_{H^1}^2\\ &\le\frac{\delta}{4}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big)+\frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2\\ &\quad-\frac{3\delta}{4}\|\widetilde{\theta}\|_{L^2}^2+\frac{\delta}{4}\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{L^2}^2\Big)\\ &\quad+c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big)\\ &\le-\frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2+c\delta\Big(\|\partial_1\widetilde{u}\|_{H^1}^2+\|\partial_2\widetilde{\theta}\|_{H^1}^2\Big).\end{aligned}\tag{4.3.172}$$

Choosing $\delta>0$ such that $c\delta\le\min(\nu,\eta,\frac{c}{2})$, we obtain

$$\frac{d}{dt}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2-\delta(\widetilde{u_2},\widetilde{\theta})\Big)+\nu\|\partial_1\widetilde{u}\|_{H^1}^2+\eta\|\partial_2\widetilde{\theta}\|_{H^1}^2+\frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2\le 0.\tag{4.3.173}$$

Due to the choice of δ, we have

$$\frac{1}{2}\Big(\|\widetilde{u}\|_{H^1}^2+\|\widetilde{\theta}\|_{H^1}^2\Big)-\delta(\widetilde{u_2},\widetilde{\theta})\ge 0,$$

or

$$\frac{1}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2) \le \|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2 - \delta(\widetilde{u_2}, \widetilde{\theta}) \le \frac{3}{2}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{\theta}\|_{H^1}^2).$$

For any $0 \le s \le t$, integrating (4.3.173) in time yields

$$\frac{1}{2}(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) + \int_s^t (\nu\|\partial_1\widetilde{u}\|_{H^1}^2 + \eta\|\partial_2\widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2)\, d\tau$$
$$\le \frac{3}{2}(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2).$$

Especially, for any $0 \le s \le t$,

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \le 3(\|\widetilde{u}(s)\|_{H^1}^2 + \|\widetilde{\theta}(s)\|_{H^1}^2) \tag{4.3.174}$$

and

$$\int_0^\infty (\nu\|\partial_1\widetilde{u}\|_{H^1}^2 + \eta\|\partial_2\widetilde{\theta}\|_{H^1}^2 + \frac{\delta}{4}\|\widetilde{\theta}\|_{L^2}^2)\, d\tau \le C < \infty.$$

Combining with the time integral bounds from Theorem 4.3.6,

$$\int_0^\infty \|\partial_1 u\|_{H^2}^2\, dt < \infty, \quad \int_0^\infty \|\partial_1\theta\|_{L^2}^2\, dt < \infty \quad \textit{and} \quad \int_0^\infty \|\partial_2\theta\|_{H^2}^2\, dt < \infty,$$

we obtain

$$\int_0^\infty (\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2)\, dt < \infty. \tag{4.3.175}$$

Applying Lemma A.1.3 to (4.3.174) and (4.3.175) yields

$$\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2 \le c(1+t)^{-1},$$

and the asymptotic behavior, as $t \to \infty$,

$$t\,(\|\widetilde{u}(t)\|_{H^1}^2 + \|\widetilde{\theta}(t)\|_{H^1}^2) \to 0.$$

This completes the proof of Theorem 4.3.7. ■

APPENDIX

A.1 Sobolev Spaces and Preliminary Inequalities

Throughout this Appendix, Ω denotes an open subset of the n-dimensional space $\mathbb{R}^n$.

Definition A.1.1 *Lebesgue space $L^p(\Omega)$ $(1 \leq p < \infty)$ is the vector space of the functions $u : \Omega \to \mathbb{R}$ for which $|u|^p$ is Lebesgue integrable on Ω (i.e. $\int_\Omega |u(x)|^p\,dx < \infty$). It is a Banach space with respect to the norm*

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|^p\,dx\Big)^{\frac{1}{p}}.$$

$L^\infty(\Omega)$ is the Banach space of the measurable functions which are defined on Ω and bounded outside a set of measure zero. It is equiped with the norm

$$\|u\|_{L^\infty(\Omega)} = ess \sup_{x\in\Omega} |u(x)|.$$

In what follows, We list several fundamental inequalities associated with Lebesgue spaces.

Theorem A.1.1 *(Hölder's inequality) Assume $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\int_\Omega |uv|\,dx \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)} \quad \forall u \in L^p(\Omega), \forall v \in L^q(\Omega).$$

Definition A.1.2 *Given a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ of non-negative integers, let $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The differential operator, D^α, of order $|\alpha|$ is given by*

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}.$$

Definition A.1.3 *Suppose u, $D^\alpha u \in L^1(\Omega)$. If*

$$\int_\Omega u(x)D^\alpha\phi(x)\,dx = (-1)^{|\alpha|}\int_\Omega D^\alpha u(x)\phi(x)\,dx,$$

for all test functions $\phi \in C_0^\infty$, then we say that $D^\alpha u$ is the week partial derivative of u of order α.

Definition A.1.4 *(Sobolev Space) For a given integer $m \geq 0$ and a real number $1 \leq p \leq \infty$, the Sobolev space $W^{m,p}(\Omega)$ is defined as*

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \quad \text{for all} \quad 0 \leq |\alpha| \leq m\}$$

which is equipped with the norm

$$\|u\|_{W^{m,p}} := \begin{cases} \Big(\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_{L^p}^p\Big)^{\frac{1}{p}} & \text{if} \quad 1 \leq p < +\infty, \\ \sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_{L^\infty} & \text{if} \quad p = +\infty. \end{cases}$$

Note that $L^p(\Omega) = W^{0,p}(\Omega)$, and for the special case $p = 2$ we denote $H^m(\Omega) = W^{m,p}(\Omega)$. More generally, for any $s \geq 0$,

$$\|u\|_{H^s} := \Big(\int (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} < \infty.$$

The space H^s is called the inhomogeneous Sobolev space.

In the rest of the present Appendix, we recall some basic inequalities used throughout the dissertation.

Proposition A.1.1 *Let a, b be any real numbers and let p, q be real numbers connected by the relationship $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{1}{2}(a^2 + b^2), \quad \textit{(Cauchy's inequality)}$$

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \quad \forall \epsilon > 0, \quad \textit{(Cauchy's inequality with epsilon)}$$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \quad \textit{(Young's inequality)}$$

Lemma A.1.2 *(Gronwall's inequality) Assume* $\rho(t) \geq 0$*,* $c(t) \geq 0$ *and* $\gamma(t) \geq 0$ *are continuous functions on* $[0,T]$*. Assume that* $c'(t)$ *exists and* $\gamma(t)$ *is integrable on* $[0,T]$*. If*

$$\rho(t) \leq c(t) + \int_0^t \gamma(\tau)\rho(\tau)d\tau \quad \textit{for any } t \in [0,T],$$

then

$$\rho(t) \leq c(0)e^{\int_0^t \gamma(\tau)d\tau} + \int_0^t c'(t)e^{\int_\tau^t \gamma(s)ds}d\tau \quad \textit{for any } t \in [0,T].$$

Especially, when $c(t) \equiv 0$ *would imply that* $\rho(t) \equiv 0$ *for any* $t \in [0,T]$*.*

Lemma A.1.3 *Let* $f = f(t)$ *be a nonnegative function satisfying , for two constants* $C_0 > 0$ *and* $C_1 > 0$*,*

$$\int_0^\infty f(\tau)d\tau < C_0 \quad \textit{and} \quad f(t) \leq C_1 f(s) \quad \textit{for any} \quad 0 \leq s < t. \tag{A.1.1}$$

Then, for $C_2 = \max\{2C_1 f(0), 4C_0C_1\}$ *and for any* $t > 0$*,*

$$f(t) \leq C_2(1+t)^{-1}. \tag{A.1.2}$$

Furthermore, $f(t)$ *has the following large-time asymptotic behavior,*

$$\lim_{t\to\infty} t\, f(t) = 0.$$

Proof. For all $0 \leq t \leq 1$, we have by (A.1.1),

$$f(t) \leq C_1 f(0) \leq 2C_1 f(0)\,(1+t)^{-1}. \tag{A.1.3}$$

Due to (A.1.1), for any $t \geq 1$,

$$C_0 \geq \int_{\frac{t}{2}}^t f(\tau)d\tau \geq \int_{\frac{t}{2}}^t C_1^{-1} f(t)d\tau = C_1^{-1} f(t)\frac{t}{2}$$

or

$$f(t) \leq 2C_0C_1t^{-1} \leq 4C_0C_1(1+t)^{-1}. \tag{A.1.4}$$

Hence (A.1.3) and (A.1.4) imply (A.1.2) for $C_2 = \max\{2C_1 f(0), 4C_0C_1\}$.

It remains to prove that $\lim\limits_{t\to\infty} t\, f(t) = 0$. By the second property in (A.1.1), we have

$$2C_1 \int_{t/2}^{t} f(s)ds \geq 2 \int_{t/2}^{t} f(t)ds = tf(t).$$

By the first property in (A.1.1),

$$\lim_{t\to+\infty} \int_{t}^{+\infty} f(s)ds = 0,$$

hence

$$\lim_{t\to+\infty} \int_{t/2}^{t} f(s)ds = \lim_{t\to+\infty} \int_{t/2}^{+\infty} f(s)ds - \lim_{t\to+\infty} \int_{t}^{+\infty} f(s)ds = 0.$$

Thus,

$$0 = 2C_1 \lim_{t\to+\infty} \int_{t/2}^{t} f(s)ds \geq \lim_{t\to+\infty} tf(t) \geq 0.$$

This completes the proof of Lemma A.1.3. ■

A.2 Basic Functional Analysis Results

In this appendix, we recall some essential results in functional analysis which are necessary in our work.

The following Proposition is taken from [[58],p.21].

Proposition A.2.1 *(Abstract bootstrap principle). Let I be a time interval, and for each $t \in I$ suppose we have two statements, a "hypothesis" $\boldsymbol{H}(t)$ and a "conclusion" $\boldsymbol{C}(t)$. Suppose we can verify the following four assertions:*

(a) (Hypothesis implies conclusion) If $\boldsymbol{H}(t)$ is true for some time $t \in I$, then $\boldsymbol{C}(t)$ is also true for that time t.

(b) (Conclusion is stronger than hypothesis) If $\boldsymbol{C}(t)$ is true for some $t \in I$, then $\boldsymbol{H}(t')$ is true for all $t' \in I$ in a neighbourhood of t.

(c) (Conclusion is closed) If $t_1, t_2, ...$ is a sequence of times in I which converges to another time $t \in I$, and $\boldsymbol{C}(t_n)$ is true for all t_n, then $\boldsymbol{C}(t)$ is true.

(d) *(Base case)* $\boldsymbol{H}(t)$ *is true for at least one time* $t \in I$. *Then* $\boldsymbol{C}(t)$ *is true for all* $t \in I$.

Definition A.2.1 *Let* X *and* Y *be two normed vector spaces, with norms* $\|\cdot\|_X$ *and* $\|\cdot\|_Y$ *respectively. We say that* X *is continuously embedded in* Y *and we write* $X \hookrightarrow Y$, *if the following conditions hold*

(a) $X \subset Y$

(b) *There exists an* $M > 0$ *such that* $\|x\|_Y \leq M\|x\|_X \quad$ *for all* $x \in X$.

Definition A.2.2 *Let* X *and* Y *be two normed vector spaces, with norms* $\|\cdot\|_X$ *and* $\|\cdot\|_Y$ *respectively and suppose that* $X \subseteq Y$. *We say that* X *is compactly embedded in* Y, *and write* $X \subset\subset Y$, *if the following conditions hold*

(a) X *is continuously embedded in* Y.

(b) *The embedding of* X *into* Y *is a compact operator: any bounded set in* X *is totally bounded in* Y, *i.e. every sequence in such a bounded set has a subsequence that is Cauchy in the norm* $\|\cdot\|_Y$.

Lemma A.2.2 *(Aubin-Lions).Let* $X_1 \hookrightarrow X_2 \hookrightarrow X_3$ *be three Banach spaces with the first embedding being compact and the second being continuous. Let* $T > 0$. *For* $1 \leq p, q \leq +\infty$, *let*

$$W = \{u \in L^p(0,T;X_1), \partial_t u \in L^q(0,T;X_3)\}.$$

Then

(i) *If* $p < \infty$, *then the embedding of* W *into* $L^p(0,T;X_2)$ *is compact;*

(ii) *If* $p = \infty$ *and* $q > 1$, *then the embedding of* W *into* $C(0,T;X_2)$ *is compact.*

Lemma A.2.2 states that any bounded sequence in W has a convergent subsequence in $L^p(0,T;X_2)$.

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