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Spectral Analysis of Complex Dynamical Systems

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by

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2020

Approval of the Dissertation Committee

This dissertation has been duly read, reviewed, and critiqued by the committee listed below, which hereby approves the manuscript of Casey Johnson as fulfilling the scope and quality requirements for meriting the degree of Doctor of Philosophy in Mathematics.

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Abstract

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The spectrum of any differential equation or a system of differential equations is related to several important properties about the problem and its subsequent solution. So much information is held within the spectrum of a problem that there is an entire field devoted to it; spectral analysis. In this thesis, we perform spectral analysis on two separate complex dynamical systems.

The vibrations along a continuous string or a string with beads on it are the governed by the continuous or discrete wave equation. We derive a small-vibrations model for multi-connected continuous strings that lie in a plane. We show that lateral vibrations of such strings can be decoupled from their in-plane vibrations. We then study the eigenvalue problem originating from the lateral vibrations. We show that, unlike the well-known one string vibrations case, the eigenvalues in a multi-string vibrating system do not have to be simple. Moreover we prove that the multiplicities of the eigenvalues depend on the symmetry of the model and on the total number of the connected strings [50]. We also apply Nevanlinna functions theory to characterize the spectra and to solve the inverse problem for a discrete multi-string system in a more general setting than it was done in [71],[73], [22], [69]-[72]. We also represent multi-string vibrating systems using a coupling of non-densely defined symmetric operators acting in the infinite dimensional Hilbert space. This coupling is defined by a special set of boundary operators acting in finite dimensional Krein space (the space with indefinite inner product). The main results of this research are published in [50].

The Hypothalamic Pituitary Adrenal (HPA) axis responds to physical and mental challenge to maintain homeostasis in part by controlling the body's cortisol level. Dysregulation of the HPA axis is implicated in numerous stress-related diseases. For a structured model of the HPA axis that includes the glucocorticoid receptor but does not take into account the system response delay, we first perform rigorous stability analysis of all multi-parametric steady states and secondly,

by construction of a Lyapunov functional, we prove nonlinear asymptotic stability for some of multi-parametric steady states. We then take into account the additional effects of the time delay parameter on the stability of the HPA axis system. Finally we prove the existence of periodic solutions for the HPA axis system. The main results of this research are published in [51].

Dedication

I would like to dedicate my thesis to Isabel and Olivia whose constant curiosity in what I was learning was always an inspiration to me.

Acknowledgements

I would very much like to thank my advisor Professor Marina Chugunova for not only guiding me in the research and writing of this thesis but also throughout my graduate career. I would also like to thank the members of my dissertation committee, Professors Ali Nadim and Asuman Aksoy for answering my various questions and reviewing my thesis.

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Modeling of string vibrations and Sturm-Liouville problems

1.1 Introduction

According to Jacques Hadamard, a well-posed problem has three properties:

1. (Existence) A solution to the problem exists.
2. (Uniqueness) The problem has only one solution.
3. (Stability) The solution depends in a continuous fashion on the data associated with the problem.

Unfortunately, almost all inverse problems in physics or biology are ill-posed. However, the problems of existence and uniqueness can often be addressed by considering a generalized solution and then placing constraints on it. Stability is often lacking in inverse problems.

Stability is a property of solutions that describes the extent to which they can be expected to exist. If it is known that a problem is well-posed, then we know that unique solutions exist possibly under certain conditions. Once we know that unique solutions exist, we can then focus on the resulting behavior of the solutions as time evolves. When describing the dynamics of a problem, we often begin by first identifying a stationary or time-periodic solution. Then we study the conditions for which this solution would exist or how it will behave as time goes on.

This falls into two categories of stability; sensitivity to perturbations in the system parameters and sensitivity to perturbations to the initial condition or in the current state of the system. The first category analyzes the robustness of the problem itself: does a unique solution still exist if a parameter in the problem is changed? The second category analyzes how dependent the solution is to the initial condition: that is, does the solution behave the same if the initial condition is changed by a small increment [9].

In this chapter, we derive a small-vibrations model for multi-connected continuous strings that lie in a plane. We show that lateral vibrations of multi-connected strings can be decoupled from in-plane vibrations. We then study the eigenvalue problem originating from the lateral vibrations. We show that, unlike the well-known one-string vibrations case, the eigenvalues in a multi-string vibrating system do not have to be simple. Moreover we prove that the multiplicities of the eigenvalues depend on the symmetry of the model and on the total number of the connected strings [50].

1.2 Sturm Liouville Problems

A very important and widely studied class of differential equations are Sturm-Liouville equations.

Definition 1.2.1. *A* **Sturm-Liouville problem** *is a real second-order linear ordinary differential equation of the form*

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = -\lambda w(x)y \tag{1.1}$$

for given coefficient functions $p(x)$, $q(x)$, and $w(x) > 0$. The function $w(x)$ is sometimes called the weight/density function.

All second-order linear ordinary differential equations can be reduced to this form. A Sturm-Liouville problem is said to be **regular** if $p(x)$, $w(x) > 0$, and $p(x)$, $p'(x)$, $q(x)$, and $w(x)$ are continuous functions over the finite interval $[a, b]$ with the separated boundary conditions

$$c_1 y(a) + c_2 y'(a) = 0 \qquad c_1^2 + c_2^2 > 0$$
$$d_1 y(b) + d_2 y'(b) = 0 \qquad d_1^2 + d_2^2 > 0$$

The main result of Sturm-Liouville theory says [6]

Theorem 1. *For a regular Sturm-Liouville problem*

- *The eigenvalues are real and form an increasing sequence such that*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \to \infty$$

- *For each eigenvalue λ_n, there is a unique (up to constant multiple) eigenfunction $y_n(x)$ with exactly $n-1$ zeros in (a,b)*

- *These normalized eigenfunctions form an orthonormal basis under the w-weighted inner product in the Hilbert space $L^2_{[a,b],w(x)}$.*

$$(y_n, y_m) = \int_a^b y_n(x) y_m(x) w(x) dx = \delta_{mn}$$

In ([15]), Binding, Browne, and Watson analyzed regular Sturm-Liouville problems on the interval $[0,1]$ subject to various types of boundary conditions defined by the ratio $\rho = y'/y$. They analyze $\rho(0) = \alpha$ where $\alpha = \infty$ and α is finite. They analyze the boundary condition at $x = 1$ for different cases: $\rho(1) = \beta$, $\rho(1) = a\lambda + b$ where $a > 0$, $\rho(1) = \frac{a\lambda+b}{c\lambda+d}$ where $ad - bc > 0$ and $c \neq 0$. They resolve preceeding issues by extending the analysis of the spectra to the norming constants $v_n = ||y_n||^2$ where y_n is an eigenfunction corresponding to λ_n. They produce an isomorphism between different type problems, preserving both spectrum and norming constants. They discover that there is precisely one map with desired preservation property. Moreover, given sequences,

$$\lambda_n^B = (n-1)^2\pi^2 + k + o(1)$$
$$v_n^B = \frac{1}{2} + o(\frac{1}{n})$$

as $n \to \infty$, with k independent of n, there is precisely one Neumann bilinear problem with spectrum and norming constants given by λ_n^B and v_n^B, respectively.

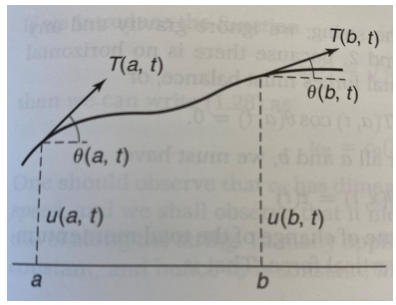


Figure 1.1: String segment with tension forces shown.

1.3 Modeling Single String Vibrations

Vibrations, or wave motion, occur almost everywhere in nature. We begin with the simple model of the small, transverse vibrations of a flexible string. First consider a taut string of length l fastened at the ends to something. Let $u = u(x,t)$ describe the vertical displacement of each point of the string x at time t. For our purposes, we will assume that the string can only move vertically and that there will be no horizontal displacement (see Figure 1.2). Let us assume that the string has density $\rho(x,t)$, with units mass per unit length, at each point in time. Since mass in neither created nor destroyed over time, we can just consider $\rho_0(x)$. The tension in the string is given by $T(x,t)$, with force units. Note that $T(x,t)$ is the force to the left of x caused by the portion of the string to the right of x and always directed along the tangent at x. A segment of the string between the positions $x = a$ and $x = b$ is illustrated in Figure 1.1. Let $\theta(x,t)$ denote the angle that the tangent makes with the horizontal and that $\tan\theta(x,t) = u_x(x,t)$.

Now let us apply Newton's second law which says that the rate of change with respect to time of the total momentum must equal the net external force. For our purposes, we will assume that tension caused by the string is the only external force. We will ignore gravity and damping forces. Balancing the horizontal forces yields

$$T(a,t)\cos\theta(a,t) = T(b,t)\cos\theta(b,t) = \tau(t) \tag{1.2}$$

but since this must be true for any segment, we will call it a function τ [62]. Then the rate of

change with respect to time of the total momentum must equal the net vertical force,

$$\frac{d}{dt}\int_a^b \rho_0(x)u_t(x,t)dx = T(b,t)\sin\theta(b,t) - T(a,t)\sin\theta(a,t)$$

Bringing the derivative inside and applying (1.2) gives

$$\begin{aligned}\int_a^b \rho_0(x)u_{tt}(x,t)dx &= T(b,t)\sin\theta(b,t) - T(a,t)\sin\theta(a,t)\\ &= T(b,t)\cos\theta(b,t)(\tan\theta(b,t) - \tan\theta(a,t))\\ &= \tau(t)(u_x(b,t) - u_x(a,t)).\end{aligned}$$

Then by the fundamental theorem of calculus, we can obtain

$$\int_a^b \rho_0(x)u_{tt}(x,t)dx = \tau(t)\int_a^b u_{xx}(x,t)dx.$$

Since this must be true for any segment, we have

$$\rho_0(x)u_{tt}(x,t) = \tau(t)u_{xx}(x,t),\ \ 0 < x < l,\ \ t > 0.$$

If we assume that tension is constant, $\tau(t) = \tau_0$, then we can further simplify the equation to

$$u_{tt} = c_0(x)^2 u_{xx} \tag{1.3}$$

where $c_0(x) = \sqrt{\frac{\tau_0}{\rho_0}}$ is called the wave speed and has units of speed. This is the wave equation and if c_0 is constant, it is easy to show that $u(x,t) = F(x - c_0 t)$ and $u(x,t) = G(x + c_0 t)$ are solutions for $F, G \in \mathbb{C}^2$. The last thing we need to consider are the end points of the string, *boundary conditions*, and initial position $\phi(x)$ and velocity $\psi(x)$ of the string, *initial conditions*. There are different types of boundary conditions; Dirichlet, Neumann, and mixed. The wave equation with homogeneous Dirichlet boundary conditions is

Figure 1.2: A continuous string of length l that is showing some vibrations/waves.

$$\begin{aligned} u_{tt} &= c_0^2 u_{xx} \qquad && \text{for } 0 < x < l \\ u(0,t) &= 0 && u(l,t) = 0 \\ u(x,0) &= \phi(x) && u_t(x,0) = \psi(x) \end{aligned}$$

The wave equation with Neumann boundary conditions is

$$\begin{aligned} u_{tt} &= c_0^2 u_{xx} \qquad && \text{for } 0 < x < l \\ u_x(0,t) &= 0 && u_x(l,t) = 0 \\ u(x,0) &= \phi(x) && u_t(x,0) = \psi(x) \end{aligned}$$

We focus on the spectrum of the wave equation; it's eigenvalues. These are found by utilizing separation of variables, $u(x,t) = X(x)T(t)$, and focusing solely on the solutions of $X(x)$,

$$\begin{aligned} X'' &= -\lambda X \qquad && 0 < x < l \\ X(0) &= 0 && X(l) = 0 \end{aligned}$$

which yields

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad X_n(x) = \sin\frac{n\pi x}{l}, \;\; n = 1, 2, 3, \ldots$$

or

$$\begin{aligned} X'' &= -\lambda X \qquad && 0 < x < l \\ X'(0) &= 0 && X'(l) = 0 \end{aligned}$$

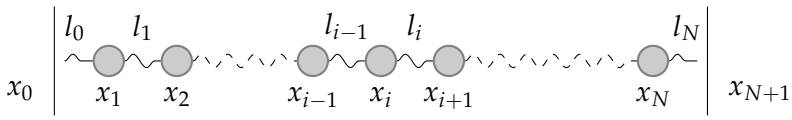


Figure 1.3: Single discrete string model illustrating the notation used.

which yields

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad X_n(x) = \cos\frac{n\pi x}{l}, \;\; n = 0, 1, 2, \ldots$$

Instead of a continuous string, we can also consider its discrete equivalent. A discrete string is a very fine thread with beads spread out along the thread and is called a Stieltjes string. Again, we will consider u_i to be the vertical displacement of the ith bead. We are going to assume uniform tension and that tension T is strong enough to neglect gravity. Let the ith have mass m_i and the beads spread out uniformly with a distance of l between two beads as shown in Figure 1.3. Let there be N beads, so then the boundary conditions will be at u_0 and u_{N+1}.The force for the ith bead is

$$F_i = -T\left(\frac{u_i - u_{i-1}}{l_{i-1}}\right) + T\left(\frac{u_{i+1} - u_i}{l_i}\right), \;\; i = 1, 2, \ldots, N$$

Force is also equal to mass times acceleration, $\ddot{u}_i$. Thus,

$$m_i\ddot{u}_i = -T\left(\frac{u_i - u_{i-1}}{l_{i-1}}\right) + T\left(\frac{u_{i+1} - u_i}{l_i}\right), \;\; i = 1, 2, \ldots, N$$

Now if we assume uniform length between each bead, then the model simplifies to

$$\begin{aligned} m_i\ddot{u}_i &= -\left(\frac{2T}{l}\right)u_i + \left(\frac{T}{l}\right)(u_{i-1} + u_{i+1}) \\ \ddot{u}_i &= -2\omega_i^2 u_i + \omega_i^2(u_{i-1} + u_{i+1}), \;\; i = 1, 2, \ldots, N \end{aligned}$$

where $\omega_i = \sqrt{\frac{T}{m_i l}}$.

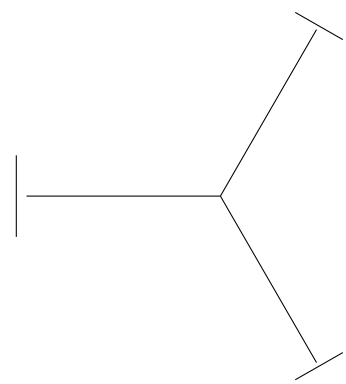


Figure 1.4: Configuration of multiple strings.

1.4 Modeling Multi-String Vibrations

What if we want to connect multiple strings together? If we connect, say three strings, in a configuration similar to Figure 1.4.

Using the laws of physics to generate a model that governs the lateral displacement of the strings that lie in a plane. We first need to represent the strings in parametric form,

$$\vec{x} = \vec{x_n}(t, s)$$

where $n = 1, 2, 3$ is the string number, $\vec{x}$ is the position, t is time, and $s \in [0, l]$ is the Lagrangian marker where l is the length of the non-stretched string. To begin, we will assume that all the strings have the same density, ρ.

The tension T is a function of stretching, $T = T\left(\left|\frac{\partial \vec{x_n}}{\partial s}\right|\right)$. According to Newton's Second Law of Motion, the acceleration of an object is directly proportional to the magnitude of the net force in the same direction and inversely proportional to the mass of the object. Therefore the law for an arbitrary string segment $(s_0 - \Delta s, s_0 + \Delta s)$ gives us

$$\frac{\partial}{\partial t}\left(\int_{s_0-\delta s}^{s_0+\delta s} \rho \frac{\partial \vec{x_n}}{\partial t} ds\right) = \left[\frac{\partial \vec{x_n}}{\partial s}\left|\frac{\partial \vec{x_n}}{\partial s}\right|^{-1} T\left(\left|\frac{\partial \vec{x_n}}{\partial s}\right|\right)\right]_{s=s_0+\Delta s} - \left[\frac{\partial \vec{x_n}}{\partial s}\left|\frac{\partial \vec{x_n}}{\partial s}\right|^{-1} T\left(\left|\frac{\partial \vec{x_n}}{\partial s}\right|\right)\right]_{s=s_0-\Delta s}$$

If we then take the limit $\Delta s \to 0$, we get

$$\rho \frac{\partial^2 \vec{x_n}}{\partial t^2} = \frac{\partial}{\partial s}\left[\frac{\partial \vec{x_n}}{\partial s}\left|\frac{\partial \vec{x_n}}{\partial s}\right|^{-1} T\left(\left|\frac{\partial \vec{x_n}}{\partial s}\right|\right)\right]$$

Then, we need to state the boundary conditions. If we define $s = 0$ to be the point where the strings join together, we want their positions to be the same and the sum of their derivatives to be 0.

$$\vec{x_1} = \vec{x_2} = \vec{x_3} \quad \text{at } s = 0$$

$$\frac{\partial \vec{x_1}}{\partial s} \left| \frac{\partial \vec{x_1}}{\partial s} \right|^{-1} T\left(\left| \frac{\partial \vec{x_1}}{\partial s} \right| \right) + \frac{\partial \vec{x_2}}{\partial s} \left| \frac{\partial \vec{x_2}}{\partial s} \right|^{-1} T\left(\left| \frac{\partial \vec{x_2}}{\partial s} \right| \right) + \frac{\partial \vec{x_3}}{\partial s} \left| \frac{\partial \vec{x_3}}{\partial s} \right|^{-1} T\left(\left| \frac{\partial \vec{x_3}}{\partial s} \right| \right) = \vec{0} \quad \text{at } s = 0$$

And at the other boundary, $s = l$,

$$\vec{x_n} = R\vec{e_n} \quad \text{at } s = l$$

where

$$\vec{e_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e_2} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \quad \vec{e_3} = \begin{bmatrix} \frac{-\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

Solving for the steady state solution,

$$\begin{aligned} \frac{\partial}{\partial s} \left[\frac{\partial \vec{x_n}}{\partial s} \left| \frac{\partial \vec{x_n}}{\partial s} \right|^{-1} T\left(\left| \frac{\partial \vec{x_n}}{\partial s} \right| \right) \right] &= 0 \\ \frac{\partial \vec{x_n}}{\partial s} \left| \frac{\partial \vec{x_n}}{\partial s} \right|^{-1} T\left(\left| \frac{\partial \vec{x_n}}{\partial s} \right| \right) &= k_1 \\ x_n = Re_n \frac{s}{l} & \end{aligned}$$

If we then consider a perturbation of the solution,

$$x_n = Re_n \frac{s}{l} + \tilde{x_n}$$

If we substitute the perturbation into the problem and linearize it, we obtain

$$\rho\frac{\partial^2(Re_n\frac{s}{l}+\tilde{x_n})}{\partial t^2}=\frac{\partial}{\partial s}\left[\frac{\partial(Re_n\frac{s}{l}+\tilde{x_n})}{\partial s}\left|\frac{\partial(Re_n\frac{s}{l}+\tilde{x_n})}{\partial s}\right|^{-1}T\left(\left|\frac{\partial(Re_n\frac{s}{l}+\tilde{x_n})}{\partial s}\right|\right)\right]$$

$$\rho\frac{\partial^2\tilde{x_n}}{\partial t^2}=\frac{\partial}{\partial s}\left[\left(\left(\frac{R}{l}e_n\right)+\frac{\partial\tilde{x_n}}{\partial s}\right)\left|\left(\frac{R}{l}e_n\right)+\frac{\partial\tilde{x_n}}{\partial s}\right|^{-1}T\left(\left|\left(\frac{R}{l}e_n\right)+\frac{\partial\tilde{x_n}}{\partial s}\right|\right)\right]$$

$$\rho\frac{\partial^2\tilde{x_n}}{\partial t^2}=\frac{\partial}{\partial s}\left[\frac{\frac{R}{l}e_nT\left(\left|\frac{R}{l}e_n+\frac{\partial\tilde{x_n}}{\partial s}\right|\right)+\frac{\partial\tilde{x_n}}{\partial s}T\left(\left|\frac{R}{l}e_n+\frac{\partial\tilde{x_n}}{\partial s}\right|\right)}{\left|\frac{R}{l}e_n+\frac{\partial\tilde{x_n}}{\partial s}\right|}\right]$$

$$\rho\frac{\partial^2x_n}{\partial t^2}=P\frac{\partial^2x_n}{\partial s^2}+Qe_n\left(\frac{\partial^2x_n}{\partial s^2}\cdot e_n\right)$$

where the tildes where dropped and

$$P(\alpha)=\alpha^{-1}T(\alpha),\quad Q(\alpha)=T'(\alpha)-\alpha^{-1}T(\alpha),\quad \alpha=\frac{R}{l}$$

Observe that, if $T(\alpha)=T_0\alpha$, then $Q=0$.

So right now, the model is

$$\rho\frac{\partial^2x_n}{\partial t^2}=P\frac{\partial^2x_n}{\partial s^2}+Qe_n\left(\frac{\partial^2x_n}{\partial s^2}\cdot e_n\right) \tag{1.4}$$

$$(x_1)_{s=0}=(x_2)_{s=0}=(x_3)_{s=0} \tag{1.5}$$

$$\left[P\frac{\partial^2x_1}{\partial s^2}+Qe_1\left(\frac{\partial^2x_1}{\partial s^2}\cdot e_1\right)\right]_{s=0}+\left[P\frac{\partial^2x_2}{\partial s^2}+Qe_2\left(\frac{\partial^2x_2}{\partial s^2}\cdot e_2\right)\right]_{s=0}+\left[P\frac{\partial^2x_3}{\partial s^2}+Qe_3\left(\frac{\partial^2x_3}{\partial s^2}\cdot e_3\right)\right]_{s=0}=0 \tag{1.6}$$

$$(x_n)_{s=l}=0 \tag{1.7}$$

Let

$$\mathrm{x}_n = a_n \mathrm{e}_n + b_n \mathrm{e}_n \times \mathrm{e}_z + c_n \mathrm{e}_z$$

where

$$\mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and plugging it into (1.4-1.7) yields

$$\rho \frac{\partial^2 a_n}{\partial t^2} = (P+Q)\frac{\partial^2 a_n}{\partial s^2}, \quad \rho \frac{\partial^2 b_n}{\partial t^2} = P\frac{\partial^2 b_n}{\partial s^2}, \quad \rho \frac{\partial^2 c_n}{\partial t^2} = P\frac{\partial^2 c_n}{\partial s^2} \tag{1.8}$$

$$a_n = b_n = c_n = 0 \quad \text{at } s = l \tag{1.9}$$

$$\begin{aligned} a_1\vec{e_1} + b_1(\vec{e_1} \times \vec{e_z}) + c_1\vec{e_z} = a_2\vec{e_2} + b_2(\vec{e_2} \times \vec{e_z}) + c_2\vec{e_z} \\ = a_3\vec{e_3} + b_3(\vec{e_3} \times \vec{e_z}) + c_3\vec{e_z} \text{at } s = 0 \end{aligned} \tag{1.10}$$

$$\begin{aligned} (P+Q)\frac{\partial a_1}{\partial s}\vec{e_1} + P\frac{\partial b_1}{\partial s}\vec{e_1} \times \vec{e_z} + P\frac{\partial c_1}{\partial s}\vec{e_z} + (P+Q)\frac{\partial a_2}{\partial s}\vec{e_2} + P\frac{\partial b_2}{\partial s}\vec{e_2} \times \vec{e_z} + P\frac{\partial c_2}{\partial s}\vec{e_z} \\ + (P+Q)\frac{\partial a_3}{\partial s}\vec{e_3} + P\frac{\partial b_3}{\partial s}\vec{e_3} \times \vec{e_z} + P\frac{\partial c_3}{\partial s}\vec{e_z} = 0 \text{at } s = 0 \end{aligned} \tag{1.11}$$

Let's note that

$$\vec{e_1} \times \vec{e_z} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e_2} \times \vec{e_z} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \quad \vec{e_3} \times \vec{e_z} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

Obtain

$$
\begin{aligned}
a_1\vec{e_1} + b_1(\vec{e_1} \times \vec{e_z}) + c_1\vec{e_z} &= a_2\vec{e_2} + b_2(\vec{e_2} \times \vec{e_z}) + c_2\vec{e_z} \\
&= a_3\vec{e_3} + b_3(\vec{e_3} \times \vec{e_z}) + c_3\vec{e_z} \quad \text{at } s = 0
\end{aligned}
$$

$$
\begin{aligned}
a_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= a_2 \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= a_3 \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{at } s = 0
\end{aligned}
$$

$$
\begin{aligned}
&(P+Q)\frac{\partial a_1}{\partial s} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + P\frac{\partial b_1}{\partial s} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + P\frac{\partial c_1}{\partial s} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&+(P+Q)\frac{\partial a_2}{\partial s} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + P\frac{\partial b_2}{\partial s} \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} + P\frac{\partial c_2}{\partial s} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&+(P+Q)\frac{\partial a_3}{\partial s} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + P\frac{\partial b_3}{\partial s} \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} + P\frac{\partial c_3}{\partial s} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \quad \text{at } s = 0
\end{aligned}
$$

Therefore, condensing the above information yields the following

$$
\begin{bmatrix} b_1 \\ a_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2}a_2 - \frac{1}{2}b_2 \\ -\frac{1}{2}a_2 - \frac{\sqrt{3}}{2}b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2}a_3 - \frac{1}{2}b_3 \\ -\frac{1}{2}a_3 + \frac{\sqrt{3}}{2}b_3 \\ c_3 \end{bmatrix} = 0 \quad \text{at } s = 0
$$

and

$$
\begin{bmatrix} P\frac{\partial b_1}{\partial s} \\ (P+Q)\frac{\partial a_1}{\partial s} \\ P\frac{\partial c_1}{\partial s} \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{3}}{2}(P+Q)\frac{\partial a_2}{\partial s} - \frac{1}{2}P\frac{\partial b_2}{\partial s} \\ -\frac{1}{2}(P+Q)\frac{\partial a_2}{\partial s} - \frac{\sqrt{3}}{2}P\frac{\partial b_2}{\partial s} \\ P\frac{\partial b_2}{\partial s} \end{bmatrix}
$$

$$
+ \begin{bmatrix} -\frac{\sqrt{3}}{2}(P+Q)\frac{\partial a_3}{\partial s} - \frac{1}{2}P\frac{\partial b_3}{\partial s} \\ -\frac{1}{2}(P+Q)\frac{\partial a_3}{\partial s} + \frac{\sqrt{3}}{2}P\frac{\partial b_3}{\partial s} \\ P\frac{\partial c_3}{\partial s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{at } s = 0
$$

which combines to

$$
\begin{bmatrix} P\left(\frac{\partial b_1}{\partial s} - \frac{1}{2}\frac{\partial b_2}{\partial s} - \frac{1}{2}\frac{\partial b_3}{\partial s}\right) + \frac{\sqrt{3}}{2}(P+Q)\left(\frac{\partial a_2}{\partial s} - \frac{\partial a_3}{\partial s}\right) \\ (P+Q)\left(\frac{\partial a_1}{\partial s} - \frac{1}{2}\frac{\partial a_2}{\partial s} - \frac{1}{2}\frac{\partial a_3}{\partial s}\right) - \frac{\sqrt{3}}{2}P\left(\frac{\partial b_2}{\partial s} - \frac{\partial b_3}{\partial s}\right) \\ P\left(\frac{\partial c_1}{\partial s} + \frac{\partial c_2}{\partial s} + \frac{\partial c_3}{\partial s}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{at } s = 0
$$

Uncoupling these into two groups, one for c_n and one for (a_n, b_n)

$$
\rho\frac{\partial^2 c_n}{\partial t^2} = P\frac{\partial^2 c_n}{\partial s^2}
$$

$$
c_n = 0 \quad \text{at } s = l
$$

$$
c_1 = c_2 = c_3 \quad \text{at } s = 0
$$

$$
\frac{\partial c_1}{\partial s} + \frac{\partial c_2}{\partial s} + \frac{\partial c_3}{\partial s} = 0 \quad \text{at } s = 0
$$

$$
\rho\frac{\partial^2 a_n}{\partial t^2} = (P+Q)\frac{\partial^2 a_n}{\partial s^2}, \quad \rho\frac{\partial^2 b_n}{\partial t^2} = P\frac{\partial^2 b_n}{\partial s^2}
$$

$$
a_n = b_n = 0 \quad \text{at } s = l
$$

$$
\begin{bmatrix} 2b_1 \\ 2a_1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}a_2 - b_2 \\ -a_2 - \sqrt{3}b_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}a_3 - b_3 \\ -a_3 + \sqrt{3}b_3 \end{bmatrix} \quad \text{at } s = 0
$$

$$
\begin{bmatrix} P\left(2\frac{\partial b_1}{\partial s} - \frac{\partial b_2}{\partial s} - \frac{\partial b_3}{\partial s}\right) + \sqrt{3}(P+Q)\left(\frac{\partial a_2}{\partial s} - \frac{\partial a_3}{\partial s}\right) \\ (P+Q)\left(2\frac{\partial a_1}{\partial s} - \frac{\partial a_2}{\partial s} - \frac{\partial a_3}{\partial s}\right) + \sqrt{3}P\left(\frac{\partial b_2}{\partial s} - \frac{\partial b_3}{\partial s}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{at } s = 0
$$

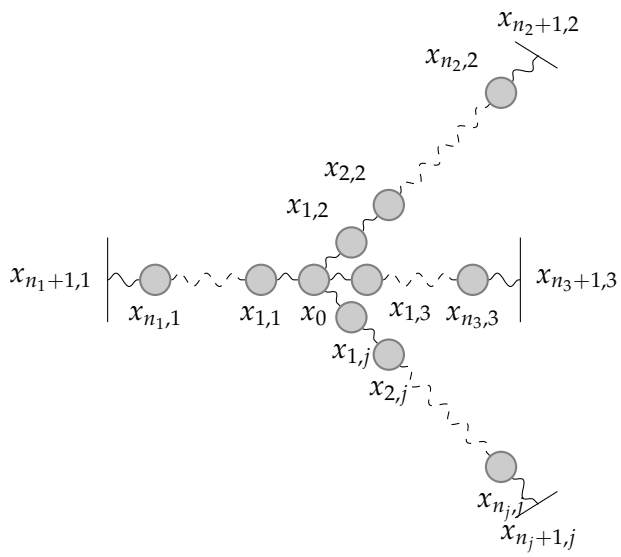


Figure 1.5: Stieltjes multi-string model with a middle mass. Note that while only 4 strings are shown for simplicity, the notation corresponds to the notation for N strings.

If we have multiple Stieltjes strings, there could be two situations: one where there is a bead at the middle joining point or one where there is no middle bead at the joining point.

The model very closely resembles that of a single Stieltjes string except for the middle mass. The middle bead now has the forces from each strand affecting its vibrations. Consider a multistring system that has N strings where the jth string has n_j beads on it. We will denote the placement of the ith bead on the jth string as $x_{i,j}$ which will have mass $m_{i,j}$. The beads on each strand will be counted from the middle out to the edge. The position of the middle bead is denoted x_0 and the edges of each string are denoted $x_{n_j+1,j}$. When considering multiple strings, we will assume the length between each bead is uniform. Refer to Figure 1.5 for the corresponding notation.

$$\begin{cases} \ddot{u_{i,j}} = -\frac{2T}{lm_{i,j}} u_{i,j} + \frac{T}{lm_{i,j}}(u_{i-1,j} + u_{i+1,j}), \quad i = 1, 2, \ldots, n_j, j = 1, 2, \ldots, N \\ \ddot{u_0} = -N\frac{T}{lm_0} u_0 + \frac{T}{lm_0}(\sum_{j=1}^{N} u_{1,j}) \\ u_{n_j+1,j} = 0, \quad \text{Dirichlet on an end} \\ u_{n_j+1,j} = u_{n_j,j}, \quad \text{Neumann on an end} \end{cases} \tag{1.12}$$

Now if instead, there is no middle bead at the adjoining point as shown in Figure 1.6, the only

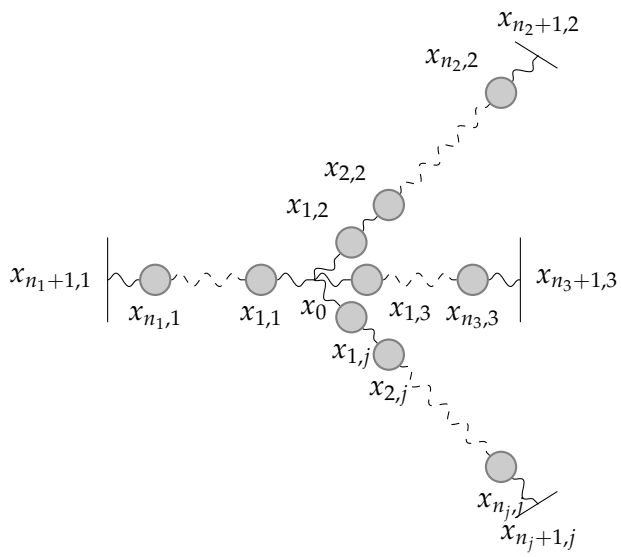


Figure 1.6: Stieltjes multi-string model with no middle mass. Note that while only 4 strings shown for simplicity,the notation corresponds to the notation for N strings.

difference is how the adjoining position in the middle is taken care of

$$\begin{cases} \ddot{u_{i,j}} = -\frac{2T}{lm_{i,j}} u_{i,j} + \frac{T}{lm_{i,j}}(u_{i-1,j} + u_{i+1,j}), \quad i = 1, 2, \ldots, n_j, j = 1, 2, \ldots, N \\ 0 = -N\frac{T}{l}u_0 + \frac{T}{l}(\sum_{j=1}^{N} u_{1,j}) \\ u_{n_j+1,j} = 0, \quad \text{Dirichlet on an end} \\ u_{n_j+1,j} = u_{n_j,j}, \quad \text{Neumann on an end} \end{cases} \tag{1.13}$$

1.5 Numerical Simulations

We can further analyze and visualize the dynamics of a multistring system by utilizing numerical simulations.

We will utilize finite differences to solve (1.3). Using standard second order central difference for the each individual string, we get

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\Delta t^2} = c_0^2 \left(\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2} \right) + O(\Delta x^2, \Delta t^2)$$

$$u_n^{k+1} = 2u_n^k - u_n^{k-1} + \left(\frac{c_0 \Delta t}{\Delta x} \right)^2 \left(u_{n+1}^k - 2u_n^k + u_{n-1}^k \right)$$

$$u_n^{k+1} = \alpha^2 u_{n+1}^k + 2(1 - \alpha^2) u_n^k + \alpha^2 u_{n-1}^k - u_n^{k-1}$$

where $\alpha = \frac{c_0 \Delta t}{\Delta x}$. Now since we need the previous time step, we will have to derive a starting condition to get this scheme started. We will use the initial velocity condition which we approximate using central difference:

$$\begin{aligned}
\frac{u_n^1 - u_n^{-1}}{2\Delta t} &= g_n(x) = g(x_n) \\
u_n^{-1} &= u_n^1 - 2\Delta t g(x_n) \\
u_n^1 &= \alpha^2 u_{n+1}^0 + 2(1-\alpha^2)u_n^0 + \alpha^2 u_{n-1}^0 - u_n^{-1} \\
u_n^1 &= \frac{1}{2}\left(\alpha^2 u_{n+1}^0 + 2(1-\alpha^2)u_n^0 + \alpha^2 u_{n-1}^0\right) + \Delta t g(x_n)
\end{aligned}$$

A visual representation of the continuous 10 string model can be seen in Figure 1.7. Refer to Appendix A for the code.

Now for the continuous model, the eigenvalues of the system depend on the length and number of strings in the system. However if we consider the eigenvalues of the discrete system, we can look at the dependence of the eigenvalues on the number of strings, the number of masses on each string, the weight of each mass, etc.

Consider a system with three strings, no middle mass, setting $m = T = l = 1$, Dirichlet conditions on the end of the strings, and 10 masses on the constant string. We start with having 19 masses on what we will call the top string and only 1 mass on what we will call the bottom string. Tracking the eigenvalues as a mass moves from the top string to the bottom string shows the following behavior illustrated in Figures 1.8-1.12.

If we increase the spring constant with Dirichlet conditions on the end of the strands, the eigenvalues are affected as illustrated in Figures 1.13-1.17.

If we increase the weight of a single mass at a time with Dirichlet conditions on the end of the strands, the eigenvalues are affected as illustrated in Figures 1.18-1.22.

If instead we impose Neumann conditions on the ends of the strands, the effect on the eigenvalues is illustrated in Figures 1.23-1.27.

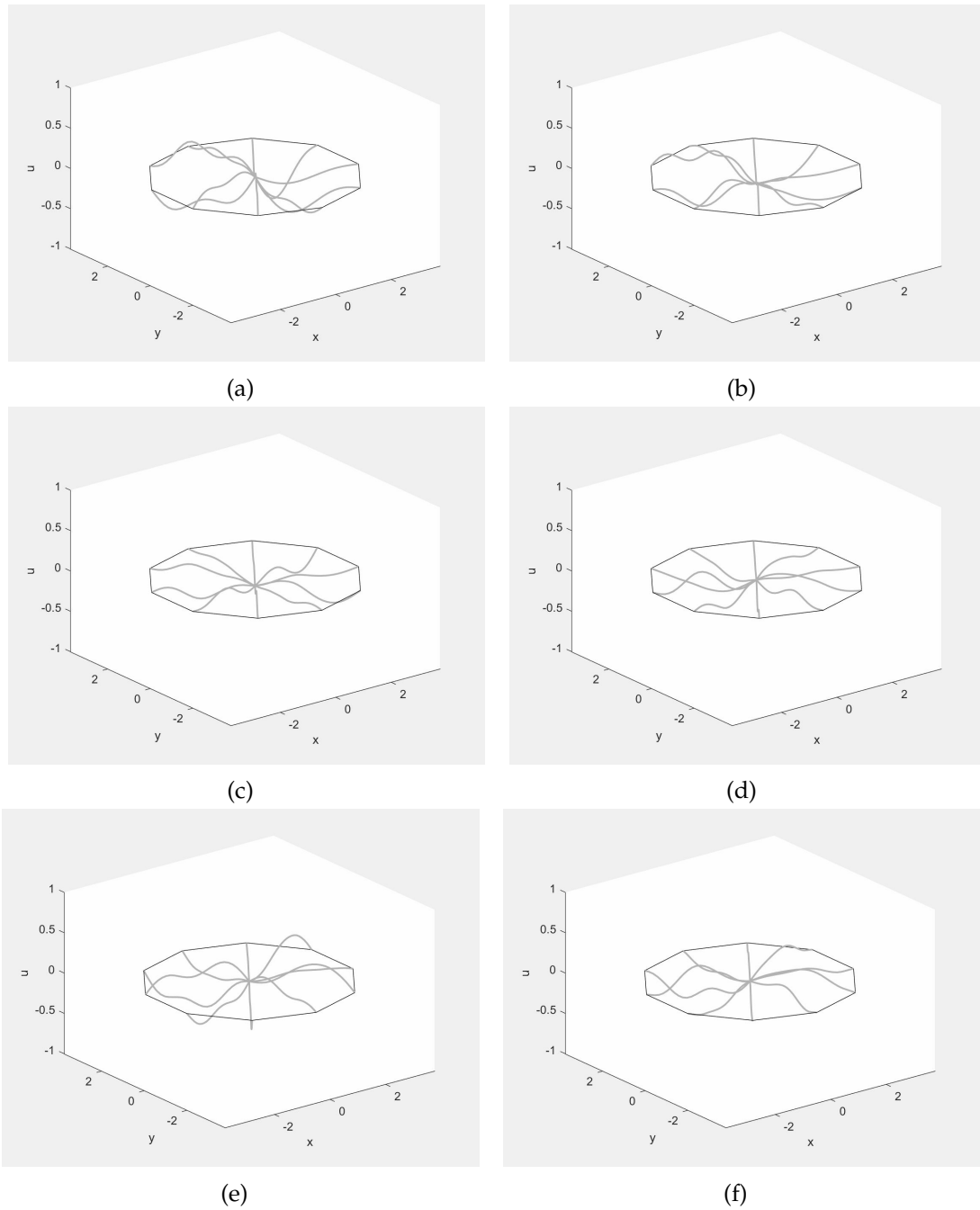


Figure 1.7: Numerically simulated model of 10 continuous strings at different time steps.

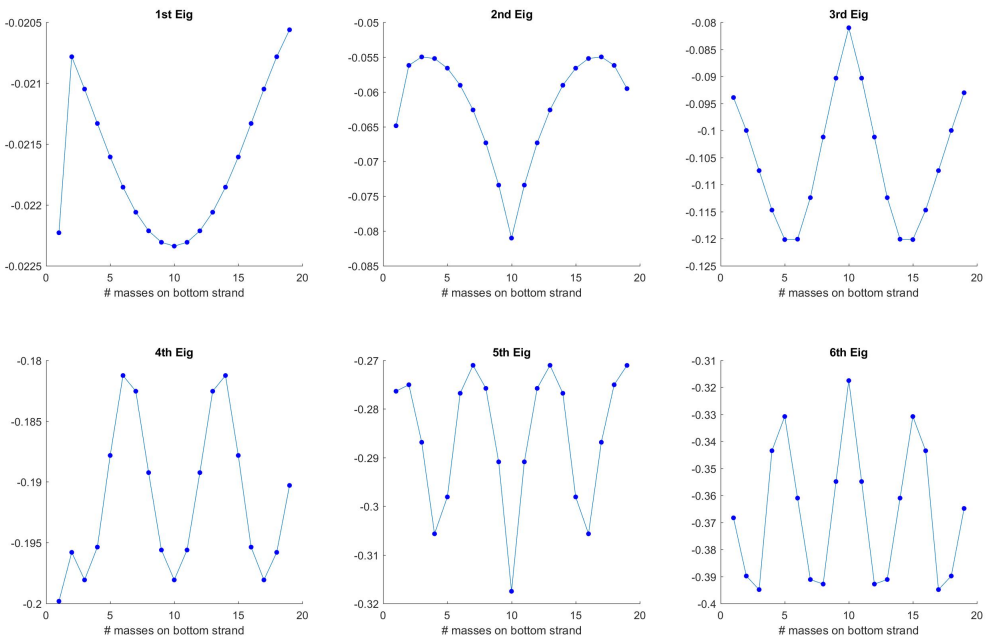


Figure 1.8: As masses move from one strand to another, the effect on the eigenvalues is shown. Note that the eigenvalues are listed in increasing order. Part 1

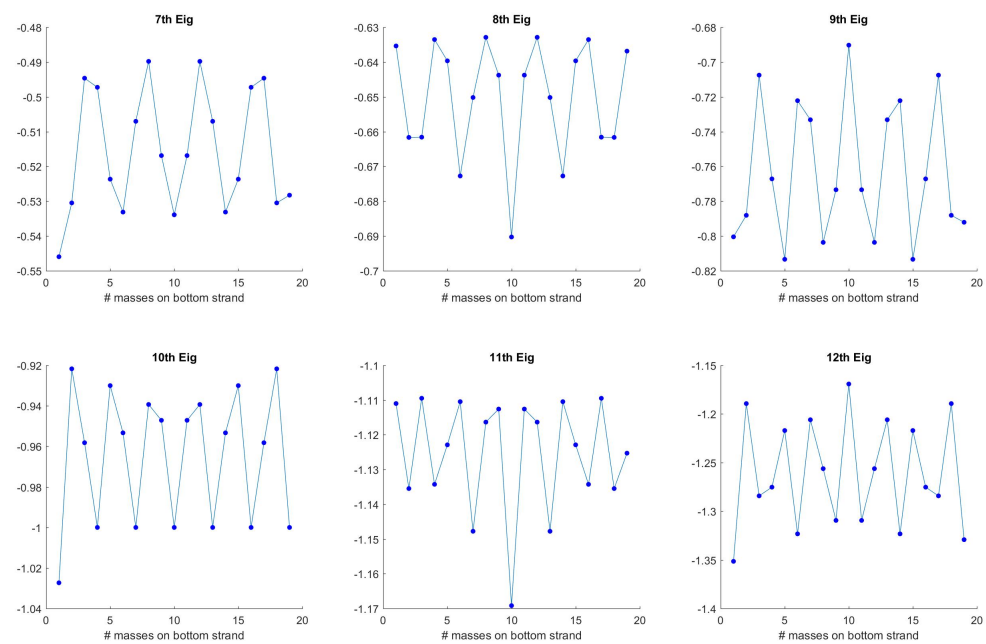


Figure 1.9: As masses move from one strand to another, the effect on the eigenvalues is shown. Note that the eigenvalues are listed in increasing order. Part 2

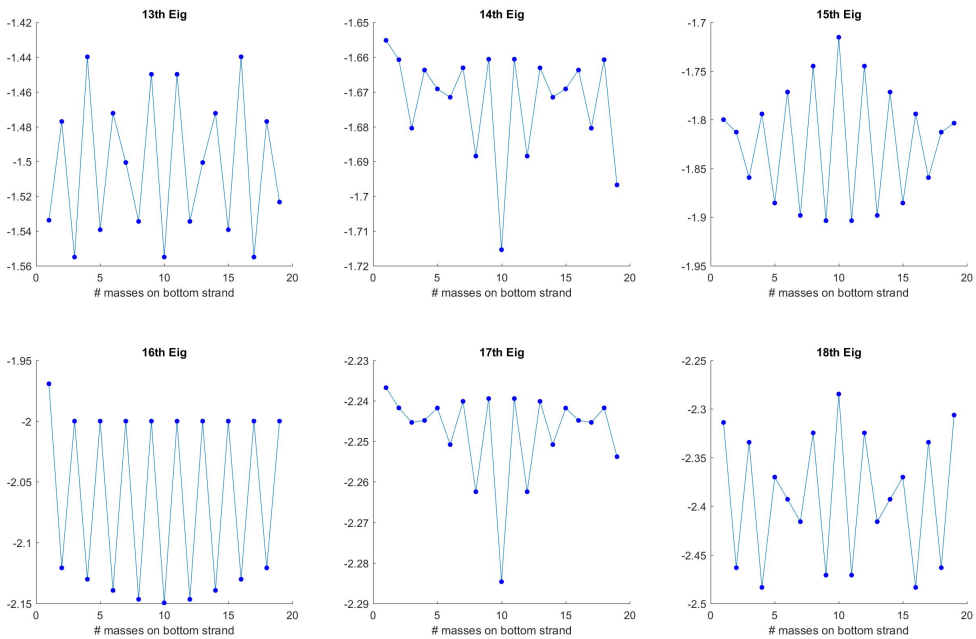


Figure 1.10: As masses move from one strand to another, the effect on the eigenvalues is shown. Note that the eigenvalues are listed in increasing order. Part 3

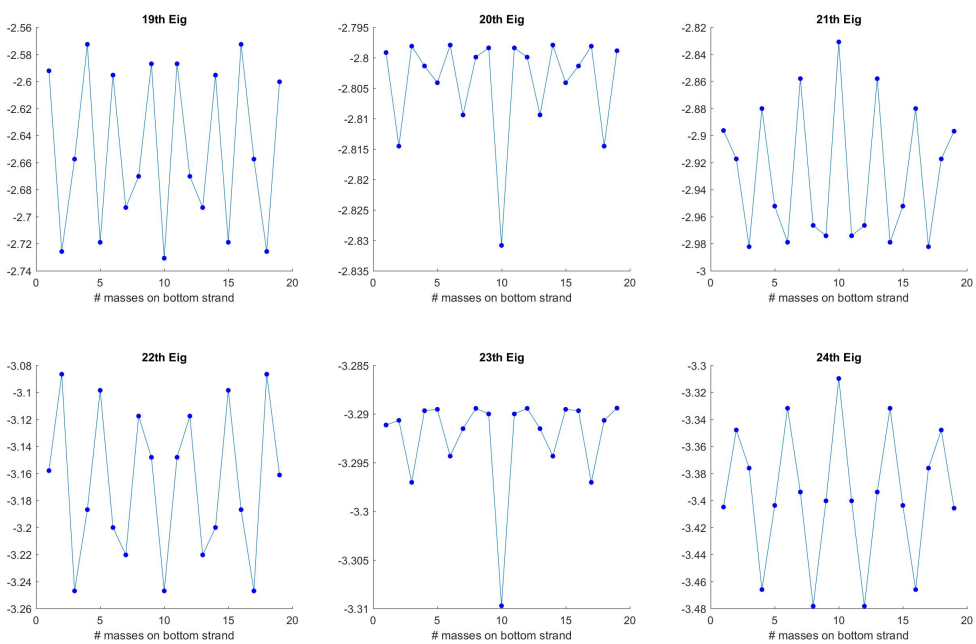


Figure 1.11: As masses move from one strand to another, the effect on the eigenvalues is shown. Note that the eigenvalues are listed in increasing order. Part 4

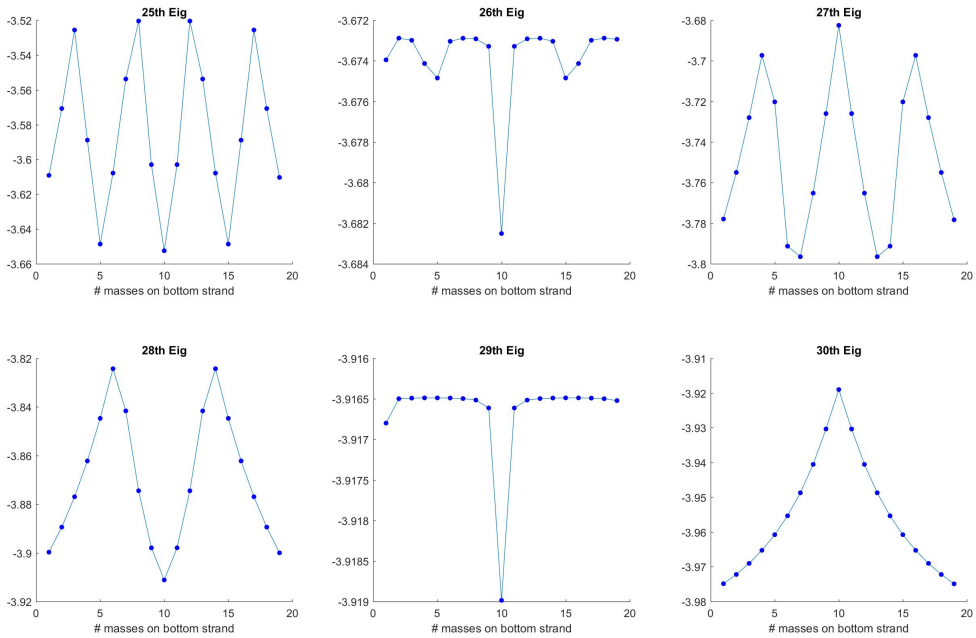


Figure 1.12: As masses move from one strand to another, the effect on the eigenvalues is shown. Note that the eigenvalues are listed in increasing order. Part 5

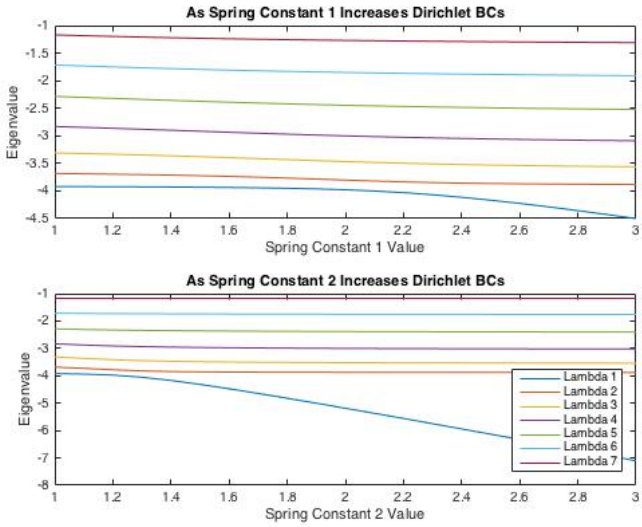


Figure 1.13: As we increase a single spring constant at a time from 1 to 3, the effect on the eigenvalues is shown Part 1

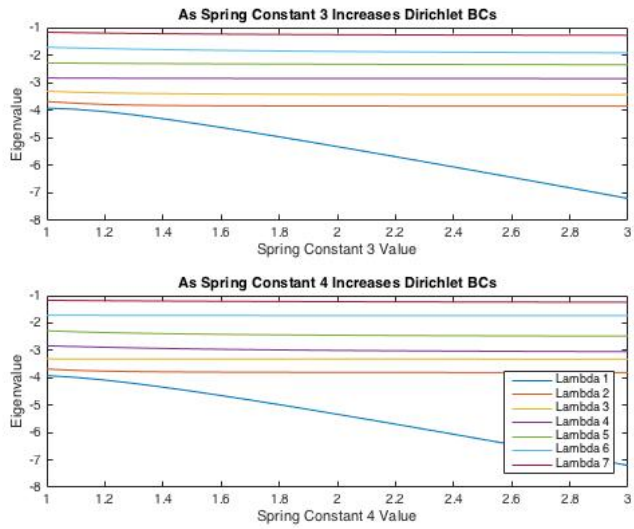


Figure 1.14: As we increase a single spring constant at a time from 1 to 3, the effect on the eigenvalues is shown Part 2

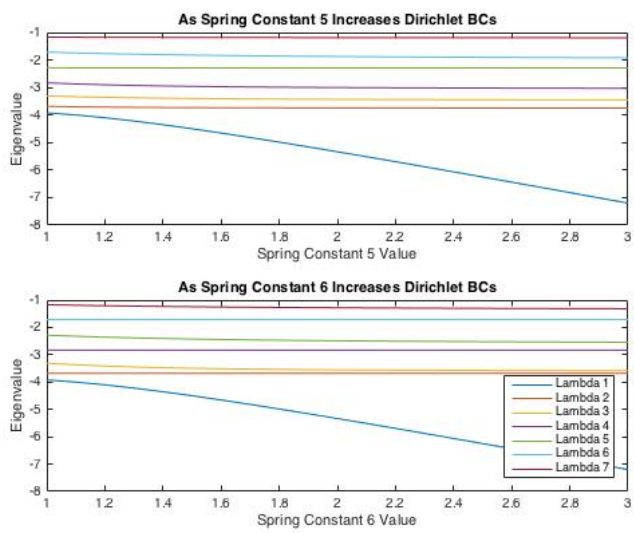


Figure 1.15: As we increase a single spring constant at a time from 1 to 3, the effect on the eigenvalues is shown Part 3

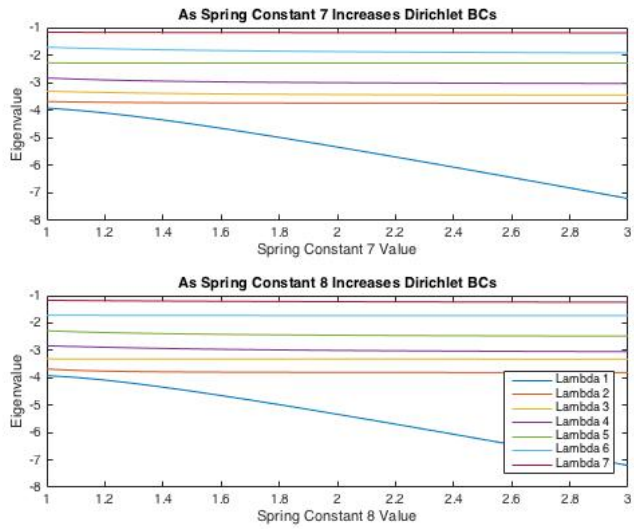


Figure 1.16: As we increase a single spring constant at a time from 1 to 3, the effect on the eigenvalues is shown Part 4

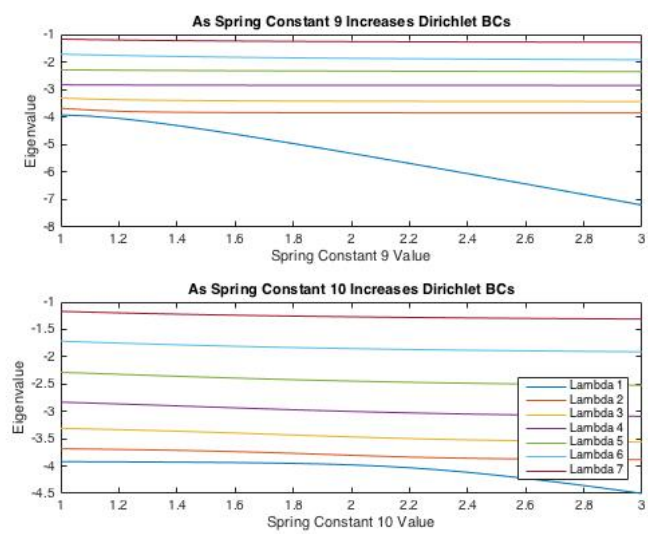


Figure 1.17: As we increase a single spring constant at a time from 1 to 3, the effect on the eigenvalues is shown Part 5

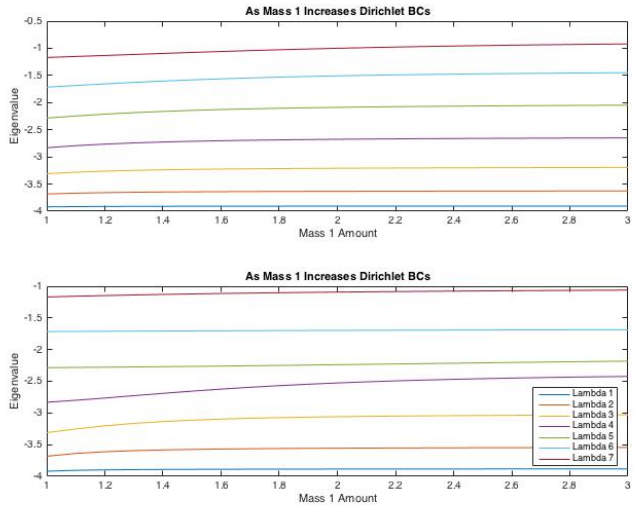


Figure 1.18: As we increase the mass of a single bead from 1 to 3 on a string with Dirichlet boundary conditions, the effect on the eigenvalues is shown Part 1

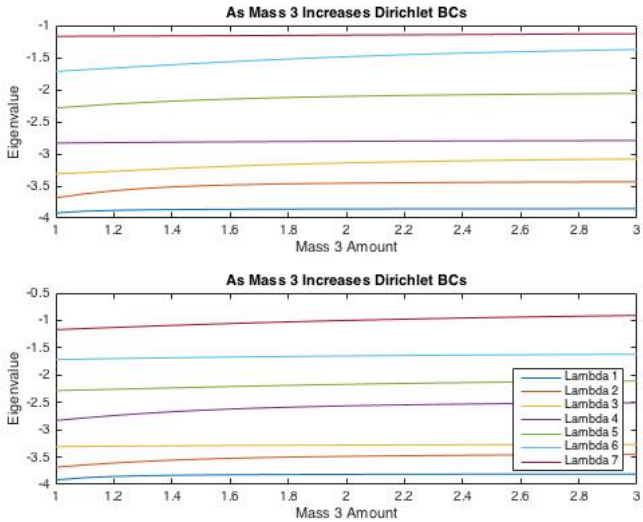


Figure 1.19: As we increase the mass of a single bead from 1 to 3 on a string with Dirichlet boundary conditions, the effect on the eigenvalues is shown Part 2

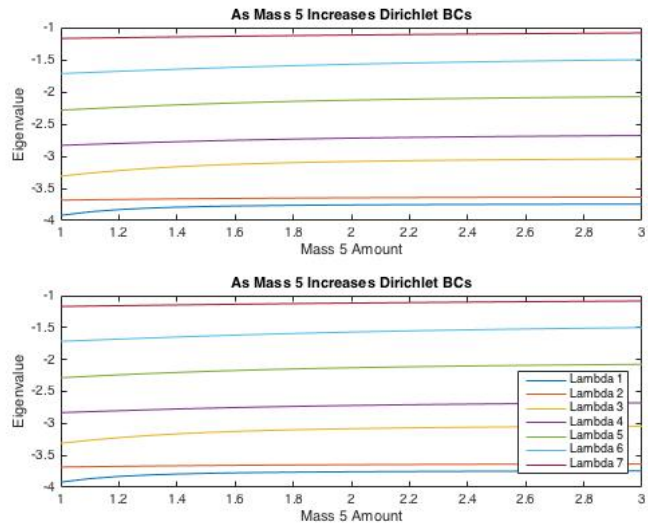


Figure 1.20: As we increase the mass of a single bead from 1 to 3 on a string with Dirichlet boundary conditions, the effect on the eigenvalues is shown Part 3

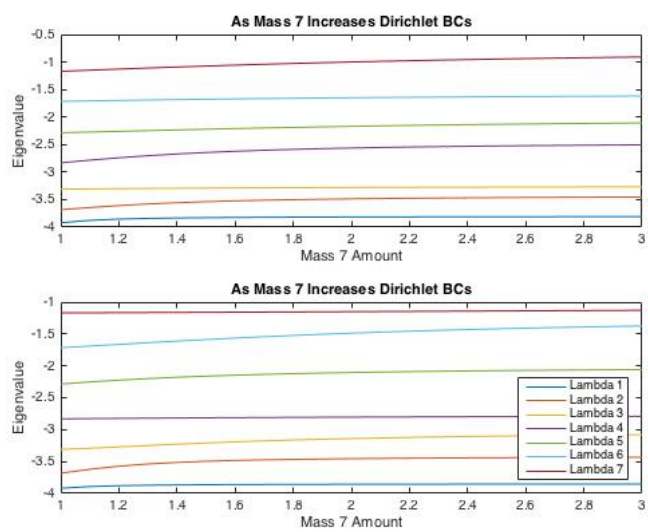


Figure 1.21: As we increase the mass of a single bead from 1 to 3 on a string with Dirichlet boundary conditions, the effect on the eigenvalues is shown Part 4

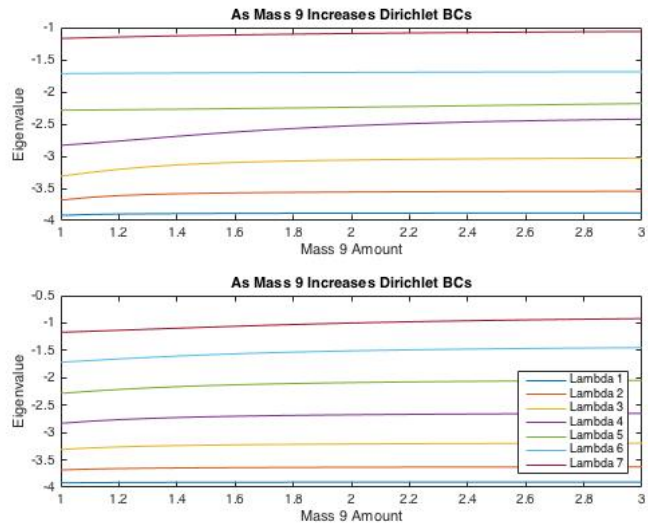


Figure 1.22: As we increase the mass of a single bead from 1 to 3 on a string with Dirichlet boundary conditions, the effect on the eigenvalues is shown Part 5

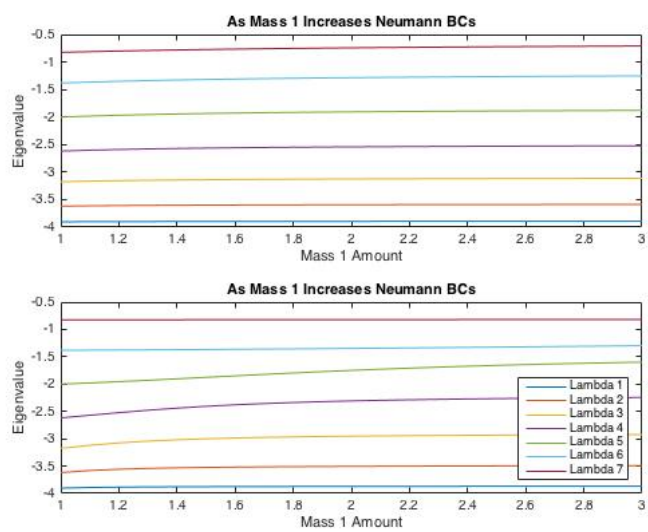


Figure 1.23: As we increase the mass of a single bead from 1 to 3 on a string with Neumann boundary conditions, the effect on the eigenvalues is shown Part 1

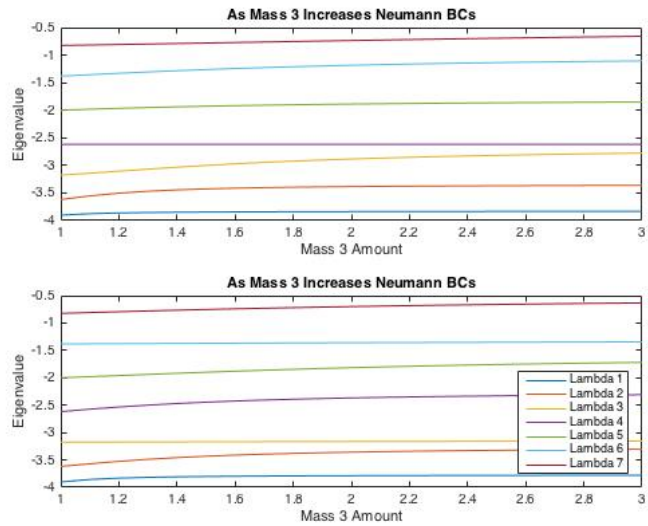


Figure 1.24: As we increase the mass of a single bead from 1 to 3 on a string with Neumann boundary conditions, the effect on the eigenvalues is shown Part 2

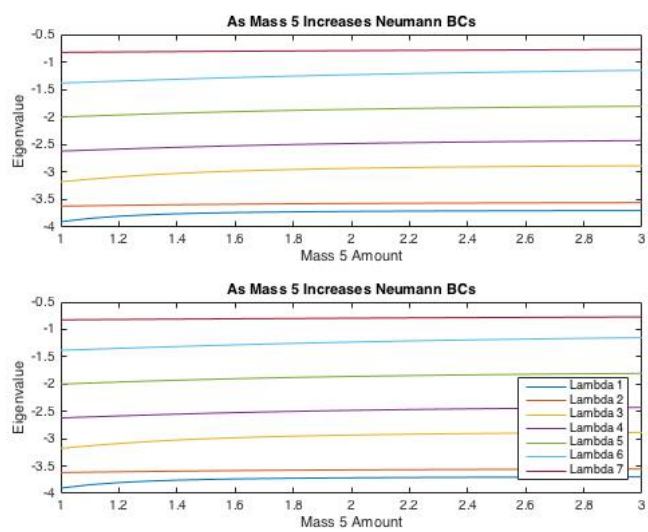


Figure 1.25: As we increase the mass of a single bead from 1 to 3 on a string with Neumann boundary conditions, the effect on the eigenvalues is shown Part 3

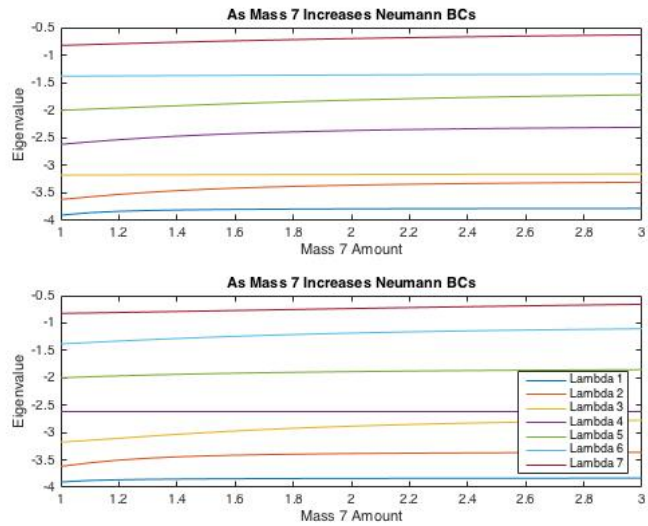


Figure 1.26: As we increase the mass of a single bead from 1 to 3 on a string with Neumann boundary conditions, the effect on the eigenvalues is shown Part 4

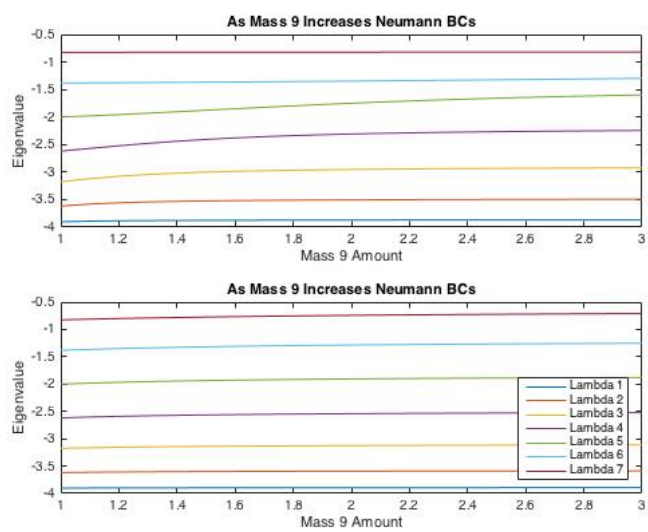


Figure 1.27: As we increase the mass of a single bead from 1 to 3 on a string with Neumann boundary conditions, the effect on the eigenvalues is shown Part 5

Chapter 2

Nevanlinna Functions in Spectral Analysis

2.1 Introduction

Continuing on with the multi-string problem, we will now discuss its spectral properties. In order to discuss the spectral properties, Nevanlinna functions will be used in the results. In the mathematical field of complex analysis, Nevanlinna theory is part of the theory of meromorphic functions. The theory describes the asymptotic distribution of solutions of the equation $f(z) = a$, as a varies. In its original form, Nevanlinna theory deals with meromorphic functions of one complex variable defined in a disc or in the whole complex plane ([80]).

Definition 2.1.1. *We say that the function $f(z)$ belongs to the* **Nevanlinna class** *if the following conditions are satisfied:*

a. $\mathbb{C} \setminus \mathbb{R} \subseteq dom(f)$

b. $\mathbb{C} \setminus \mathbb{R} \subseteq hol(f)$ *(that is, f is holomorphic away from the real axis)*

c. if $z \in hol(f)$ *then* $f(\overline{z}) = \overline{f(z)}$

d. for any $z \in \mathbb{C} \setminus \mathbb{R}$, $\frac{Im\, f(z)}{Im\, z} \geq 0$

Note that if instead, for any $z \in \mathbb{C} \setminus \mathbb{R}$, $\frac{Im\, f(z)}{Im\, z} \leq 0$, *then the function is said to be anti-Nevanlinna.*

Example 2.1.1. *Show that $f(z) = \frac{z}{1-z^2}$ is a Nevanlinna function.*

Since it is obvious that conditions a and b are satisfied, only show that conditions c and d are satisfied. So

$$f(\overline{z}) = \frac{\overline{z}}{1-\overline{z}^2} = \frac{x-iy}{1-(x-iy)^2} = \frac{x-iy}{1-x^2+2ixy+y^2}$$

$$\overline{f(z)} = \overline{\frac{z}{1-z^2}} = \frac{\overline{z}}{1-z^2} = \frac{x-iy}{1-(x+iy)^2} = \frac{x-iy}{1-x^2-2ixy+y^2} = \frac{x-iy}{1-x^2+2ixy+y^2}$$

These two statements are equal so c is satisfied.

And

$$\begin{aligned}
Im\, f(z) &= Im\left(\frac{x+iy}{1-x^2+y^2-2ixy}\right) \\
&= Im\left(\frac{(x+iy)(1-x^2+y^2+2ixy)}{(1-x^2+y^2-2ixy)(1-x^2+y^2+2ixy)}\right) \\
&= Im\left(\frac{x-x^3+xy^2+2ix^2y+iy-iyx^2+iy^3-2xy^2}{(1-x^2+y^2)^2+(2xy)^2}\right) \\
&= \frac{2x^2y+y-yx^2}{(1-x^2+y^2)^2+(2xy)^2} = \frac{x^2y+y}{(1-x^2+y^2)^2+(2xy)^2} \\
Im\, z &= y \\
\frac{Im\, f(z)}{Im\, z} &= \frac{\frac{x^2y+y}{(1-x^2+y^2)^2+(2xy)^2}}{y} \\
&= \frac{x^2+1}{(1-x^2+y^2)^2+(2xy)^2} \geq 0
\end{aligned}$$

Thus, condition d is satisfied. Therefore $f(z) = \frac{z}{1-z^2}$ is a Nevanlinna function.

Now let's investigate several properties of Nevanlinna functions.

Lemma 2.1.1. *Let $f(z)$ be a Nevanlinna function. Then $-\frac{1}{f(z)}$ is a Nevanlinna function.*

Proof. Let $f(z)$ be a Nevanlinna function. Then $f(z)$ satisfies all the conditions for a Nevanlinna function. From simple inspection, we can see that a and b are satisfied. We just need to show that c and d are satisfied.

c)

$$-\frac{1}{f(\overline{z})} = -\frac{1}{\overline{f(z)}} = \overline{-\frac{1}{f(z)}}$$

d)

$$\begin{aligned}
Im\left(-\frac{1}{f(z)}\right) &= Im\left(-\frac{\overline{f(z)}}{|f(z)|^2}\right) \\
&= \frac{Im\left(-\overline{f(z)}\right)}{|f(z)|^2} \\
&= \frac{Im\left[-(Re(f) - i\,Im(f))\right]}{|f(z)|^2} \\
&= \frac{Im[-Re(f) + iIm(f)]}{|f(z)|^2} = \frac{Im(f)}{|f(z)|^2} \\
\frac{Im\left(-\frac{1}{f(z)}\right)}{Im(z)} &= \frac{\frac{Im(f)}{|f(z)|^2}}{Im(z)} = \frac{1}{|f(z)|^2}\frac{Im(f)}{Im(z)} \geq 0
\end{aligned}$$

Thus, $-\frac{1}{f(z)}$ is a Nevanlinna function. □

We can then say that $-\frac{1}{f(z)}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Therefore, the singularities of $-\frac{1}{f(z)}$ are located on the real axis. So then the singularities become the zeros of $f(z)$ whose locations do not change. Thus, the zeros of $f(z)$ are real. this is formalized in the Lemma 2.1.2 below.

Lemma 2.1.2. *If $f(z)$ is a Nevanlinna function and $f(z_0) = 0$, then the z_0's are real.*

Lemma 2.1.3. *If $\frac{h(z)}{g(z)}$ is a rational Nevanlinna function where*

$$g(z) = z^n + \cdots + a_1 z + a_0,$$

$$h(z) = b_m z^m + \cdots + b_1 z + b_0$$

then all $a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_m \in \mathbb{R}$.

Proof. Let us note that $\overline{\frac{g(z)}{h(z)}} = \frac{\overline{g(z)}}{\overline{h(z)}}$. To be clear, note that we can write both $h(z)$ and $g(z)$ in its

factored form. Since they are Nevalinna functions, the roots are real numbers.

$$\begin{aligned}
g(z) &= (z-z_1^*)(z-z_2^*)\cdots(z-z_n^*) \\
h(z) &= b_m(z-z_1)(z-z_2)\cdots(z-z_m) \\
\overline{g(z)} &= (\overline{z}-z_1^*)(\overline{z}-z_2^*)\cdots(\overline{z}-z_n^*) \\
\overline{h(z)} &= \overline{b_m}(\overline{z}-z_1)(\overline{z}-z_2)\cdots(\overline{z}-z_m) \\
g(\overline{z}) &= (\overline{z}-z_1^*)(\overline{z}-z_2^*)\cdots(\overline{z}-z_n^*) \\
h(\overline{z}) &= b_m(\overline{z}-z_1)(\overline{z}-z_2)\cdots(\overline{z}-z_m)
\end{aligned}$$

We will use the fact $\overline{\frac{g(z)}{h(z)}} = \frac{\overline{g(z)}}{\overline{h(z)}}$:

$$\begin{aligned}
\overline{\frac{g(z)}{h(z)}} &= \frac{\overline{g(z)}}{\overline{h(z)}} \\
\frac{(\overline{z}-z_1^*)(\overline{z}-z_2^*)\cdots(\overline{z}-z_n^*)}{\overline{b_m}(\overline{z}-z_1)(\overline{z}-z_2)\cdots(\overline{z}-z_m)} &= \frac{(\overline{z}-z_1^*)(\overline{z}-z_2^*)\cdots(\overline{z}-z_n^*)}{b_m(\overline{z}-z_1)(\overline{z}-z_2)\cdots(\overline{z}-z_m)} \\
\frac{1}{\overline{b_m}}\frac{(\overline{z}-z_1^*)(\overline{z}-z_2^*)\cdots(\overline{z}-z_n^*)}{(\overline{z}-z_1)(\overline{z}-z_2)\cdots(\overline{z}-z_m)} &= \frac{1}{b_m}\frac{(\overline{z}-z_1^*)(\overline{z}-z_2^*)\cdots(\overline{z}-z_n^*)}{(\overline{z}-z_1)(\overline{z}-z_2)\cdots(\overline{z}-z_m)} \\
\frac{1}{\overline{b_m}} &= \frac{1}{b_m}
\end{aligned}$$

Therefore, b_m must be real. Since we know the zeros are real, it forces all the coefficients to also be real. □

Lemma 2.1.4. *$f(z) = \frac{p(z)}{q(z)}$ is a rational Nevanlinna function if and only if*

$$f(z) = c_1 z + c_0 + \sum_{l=1}^{r} \frac{c_{1l}}{z_l - z}$$

where $c_1 \geq 0$, $c_0 \in \mathbb{R}$, $c_{1l} > 0$, $z_l \in \mathbb{R}\ \forall l = 1, \ldots, r$.

Note that this does not contradict previous lemmas since the summation is showing poles of order 1 which are real.

Proof. $\Rightarrow$ By definition, we can rewrite $f(z)$ as a general function with poles of various order. We

will proceed to show that these poles must be of order 1, that is $m_1 = m_2 = \cdots = m_k = 1$.

$$f(z) = p(z) + \sum_{k=1}^{m_1} \frac{c_{k1}}{(z_1 - z)^k} + \cdots + \sum_{k=1}^{m_r} \frac{c_{kr}}{(z_r - z)^k}$$

The $z_1, z_2, \ldots, z_k \in \mathbb{R}$. Consider

$$s_1(z) = \sum_{k=1}^{m_1} \frac{c_{k1}}{(z_1 - z)^k}$$

We have already shown that the $c_{k1} \in \mathbb{R}$.

$$\begin{aligned} 2i\, Im(s_1(z)) = s_1(z) - \overline{s_1(z)} &= \sum_{k=1}^{m_1} \frac{c_{k1}}{(z_1 - z)^k} - \sum_{k=1}^{m_1} \frac{c_{k1}}{(z_1 - \overline{z})^k} \\ &= \sum_{k=1}^{m_1} c_{k1} \frac{(z_1 - \overline{z})^k - (z_1 - z)^k}{|z_1 - z|^{2k}} \end{aligned}$$

We now want to rewrite the above in polar notation to make computations easier. Let $\varphi_1 = arg(z_1 - z)$.

$$\begin{aligned} &= \sum_{k=1}^{m_1} c_{k1} \frac{|z_1 - z|^k (\cos(-\varphi_1) + i \sin(-\varphi_1))^k - |z_1 - \overline{z}|^k (\cos(\varphi_1) + i \sin(\varphi_1))^k}{|z_1 - z|^{2k}} \\ &= \sum_{k=1}^{m_1} c_{k1} \frac{|z_1 - z|^k (\cos(k\varphi_1) - i \sin(k\varphi_1)) - |z_1 - \overline{z}|^k (\cos(k\varphi_1) + i \sin(k\varphi_1))}{|z_1 - z|^{2k}} \\ &= \sum_{k=1}^{m_1} 2c_{k1} \frac{-i \sin(k\varphi_1)}{|z_1 - z|^k} \end{aligned}$$

Therefore, we have

$$\begin{aligned} 2i\, Im(s_1(z)) &= \sum_{k=1}^{m_1} 2c_{k1} \frac{-i \sin(k\varphi_1)}{|z_1 - z|^k} \\ Im(s_1(z)) &= \sum_{k=1}^{m_1} c_{k1} \frac{-\sin(k\varphi_1)}{|z_1 - z|^k} \\ Im(f(z)) &= \frac{p(z) - \overline{p(z)}}{2i} + \sum_{k=1}^{m_1} c_{k1} \frac{-\sin(k\varphi_1)}{|z_1 - z|^k} + \cdots + \sum_{k=1}^{m_r} c_{kr} \frac{-\sin(k\varphi_r)}{|z_r - z|^k} \end{aligned}$$

Consider $\lim_{z \to z_1}$, we can then determine the sign of $Imf(z)$ because it depends on

$$\sum_{k=1}^{m_1} c_{k1} \frac{-\sin(k\varphi_1)}{|z_1 - z|^k}$$

which blows up and is the dominant term of the function. It dictates the sign of the function in the neighborhood of z_1. Recall $\varphi_1(z) = arg(z_1 - z)$. Let z run through the entire upper half-plane. Thus $Im\, z > 0$, and therefore $z_1 - z$ runs through the lower half-plane, $arg(z_1 - z) \in (-\pi, 0)$. So $m_1\varphi_1 \in (-m_1\pi, 0)$.

We will then take 2 complex numbers z' and z'' such that

$$\varphi_1(z') = arg(z_1 - z') \in (-\frac{2\pi}{m_1}, -\frac{\pi}{m_1})$$

$$\varphi_1(z'') = arg(z_1 - z'') \in (-\frac{\pi}{m_1}, 0)$$

So we now have

$$-\sin(m_1\varphi_1(z')) < 0 \quad -\sin(m_1\varphi_1(z'')) > 0$$

Thus we have two points in the neighborhood of z_1 such that $Im(f(z))$ is changing signs contradicts the definition of the Nevanlinna function. We assumed that the multiplicity of the pole was ≥ 2 but we got a contradiction. Only if $m_1 \geq 2$ is it possible to find the two intervals which lead to our contradiction. The contradiction is not possible if $m_1 = 1$, therefore $m_1 = 1$. We repeat this argument for any z_t.

Now we need to deal with $p(z)$. $p(z)$ has all real coefficients.

$$\begin{aligned} Im(p(z)) &= \frac{p(z) - \overline{p(z)}}{2i} = \frac{p(z) - p(\overline{z})}{2i} \\ &= \frac{1}{2i} \sum_{k=0}^{n} c_k (z^k - \overline{z}^k) \\ &= \frac{1}{2i} \sum_{k=0}^{n} c_k |z|^k \sin(k\, arg(z)) \end{aligned}$$

We got this because we can group by the coefficients since they are all real. We are interested in the sign of $Im(p(z))$. We will apply similar logic and look at $p(z)$ close to infinity. Looking close

to infinity allows this term to dominate the fractions. The term $c_n \sin(n\, arg(z))$ dictates the sign of $Im(p(z))$. Assume $n \geq 2$ and look at two complex numbers z' and z'' near infinity such that $Im(z') \geq 0$ and $Im(z'') > 0$ but $\sin(n\, arg(z'))$ and $\sin(n\, arg(z''))$ are different.

We need $n \geq 2$ to get the desired contradiction, otherwise the interval would just be $(0, \pi)$.

Since we got a contradiction for $n \geq 2$, we have shown that $p(z)$ must be linear. All that is left is to show the coefficient conditions. Let us do this by computing $Im(f(z))$.

$$\begin{aligned}
Im(f(z)) &= \frac{c_1 z + c_0 - \overline{c_1 z + c_0}}{2i} + \frac{c_{11}}{2i}\left(\frac{1}{z_1 - z} - \overline{\left(\frac{1}{z_1 - z}\right)}\right) \\
&\quad + \cdots + \frac{c_{1r}}{2i}\left(\frac{1}{z_r - z} - \overline{\left(\frac{1}{z_r - z}\right)}\right) \\
\frac{Im(f(z))}{Im(z)} &= c_1 + \frac{c_1 1}{|z_1 - z|^2} + \cdots + \frac{c_{1r}}{|z_r - z|^2} \geq 0
\end{aligned}$$

Looking individually at neighborhoods of z_i (and $z = \infty$) to get dominant term, see that that coefficient has to be ≥ 0 and continue until all coefficients are ≥ 0.

$\Leftarrow$ Clearly conditions a, b, and c are satisfied. Let's investigate condition d.

$$\frac{Im(f(z))}{Im(z)} = c_1 + \frac{c_1 1}{|z_1 - z|^2} + \cdots + \frac{c_{1r}}{|z_r - z|^2}$$

We know that all the coefficients are positive and therefore can say that

$$\frac{Im(f(z))}{Im(z)} \geq 0$$

□

Theorem 2. *Let*

$$\begin{aligned}
h(z) &= a_m z^m + \cdots + a_0 \\
g(z) &= z^n + \cdots + b_0.
\end{aligned}$$

$\frac{h(z)}{g(z)}$ *is a Nevanlinna function if and only if* $\frac{h(z)}{g(z)} = m_1 z + m_0 + \frac{h_1(z)}{g(z)}$ *where* $m_1 \geq 0$, $m_0 \in \mathbb{R}$, *and*

$\lim_{z\to+\infty} \frac{h_1(z)}{g(z)}$ *and roots of g and roots of h_1 are interlacing.*

$$nul(g) = \{z_1, z_2, \ldots, z_n\} \qquad nul(h_1) = \{z_1^*, z_2^*, \ldots, z_n^*\}$$
$$z_1 < z_1^* < z_2 < z_2^* < \cdots < z_n$$

Proof. $\Leftarrow$ Assume that all conditions are satisfied. We want to show that $\frac{h(z)}{g(z)}$ is a Nevanlinna function.

$$\frac{h(z)}{g(z)} = m_1 z + m_0 + \frac{c_1}{z - z_1} + \frac{c_2}{z - z_2} + \cdots + \frac{c_n}{z - z_n}$$

The function breaks down like this because of the interlacing property. The strict inequalities require that each pole has multiplicity 1. Using the Lemma 2.1.4 we simply need to show the conditions for the coefficients are satisfied. We want to show that for any coefficients c_i and c_{i+1} have the same sign. We will assume the contrary, $c_i < 0$ and $c_{i+1} > 0$. Therefore,

$$\begin{aligned}
\lim_{z\to z_i+0} \frac{h_1(z)}{g(z)} &= \lim_{z\to z_i+0} \left(\cdots + \frac{c_i}{z - z_i} + \cdots \right) = -\infty \\
\lim_{z\to z_i-0} \frac{h_1(z)}{g(z)} &= \lim_{z\to z_i+0} \left(\cdots + \frac{c_{i+1}}{z - z_{i+1}} + \cdots \right) = -\infty
\end{aligned}$$

Therefore, there is either no $h(z)$ zeros or an even number between z_i and z_{i+1}. This is a contradiction because we can only have 1 zero of $h(z)$ between 2 zeros of $g(z)$.

$$\lim_{z\to+\infty} \frac{h_1(z)}{g(z)} = -0$$

Thus, the fraction piece goes to -0. Therefore, they cannot change sign. Thus, all coefficients are negative. This is okay because the previous theorem was $z_1 - z$ and we have $z - z_1$

$\Rightarrow$ Note $\frac{h(z)}{g(z)} = m_1 z + m_0 + \frac{c_1}{z-z_1} + \frac{c_2}{z-z_2} + \cdots + \frac{c_n}{z-z_n}$. We simply need to show the interlacing property since everything else we get for free or simply by utilizing Lemma 2.1.4.

We will rewrite $\frac{h(z)}{g(z)}$ to get a common denominator.

$$\begin{aligned}\frac{h(z)}{g(z)} &= m_1 z + m_0 \\ &+ \frac{c_1(z-z_2)\cdots(z-z_n) + c_2(z-z_1)(z-z_3)\cdots(z-z_n) + \cdots + c_n(z-z_1)\cdots(z-z_{n-1})}{(z-z_1)\cdots(z-z_n)}\end{aligned}$$

So $h_1(z) = c_1(z-z_2)\cdots(z-z_n) + c_2(z-z_1)(z-z_3)\cdots(z-z_n) + \cdots + c_n(z-z_1)\cdots(z-z_{n-1})$ and $sign(h_1(z_n)) = -1$ and $sign(h_1(z_{n-1})) = 1$. Note that $c_{n-1} = -1$ but $(z - z_n) < 0$.
The function is changing sign between zeros of g and there is only $n-1$ zeros of h_1 since h has $n-1$ roots, it is inside of g. □

Theorem 3. *Let $f(z) = c_1 z + c_0 + \frac{h_1(z)}{g(z)}$ be a Nevanlinna function. Then $f(z) = \frac{P(z)}{Q(z)}$ has the interlacing property.*

Example 2.1.2. *Show that*

$$\theta(\lambda) = \frac{a - b\lambda}{c - d\lambda} \tag{2.1}$$

is Nevanlinna function if and only if $ad - bc \geq 1$. (If $ad - bc = 1$, then $\theta(\lambda)$ is a Nevanlinna function)

Proof. $\Rightarrow$ Let $\lambda = m + in$. Clearly Condition a and b are satisfied for $\theta(\lambda)$ to be a Nevanlinna function. Now to evaluate condition c and d.

$$\begin{aligned}\theta(\overline{\lambda}) &= \frac{a - b\overline{\lambda}}{c - d\overline{\lambda}} \\ \overline{\theta(\lambda)} &= \frac{\overline{a - b\lambda}}{\overline{c - d\lambda}} = \frac{a - b\overline{\lambda}}{c - d\overline{\lambda}}\end{aligned}$$

Thus $\theta(\overline{\lambda}) = \overline{\theta(\lambda)}$ and condition c is satisfied. Now to evaluate condition d. Note that $Im(\lambda) = n$.

$$\begin{aligned}
\theta(\lambda) &= \frac{a - bm - ibn}{c - dm - idn} \\
&= \frac{a - bm - ibn}{c - dm - idn} \cdot \frac{(c - dm + idn)}{(c - dm + idn)} \\
&= \frac{ac - adm + iadn - bcm + bdm^2 - ibdmn - ibcn + ibdmn + bdn^2}{c^2 - cdm + icdn - cdm + d^2m^2 - id^2mn - icdn + id^2mn + d^2n^2} \\
&= \frac{ac - adm - bcm + bdm^2 + bdn^2 + i(adn - bcn)}{c^2 + d^2m^2 + d^2n^2} \\
Im\left(\theta(\lambda)\right) &= \frac{n(ad - bc)}{c^2 + d^2m^2 + d^2n^2} \\
\frac{Im(\theta(\lambda))}{Im(\lambda)} &= \frac{ad - bc}{c^2 + d^2m^2 + d^2n^2} \geq 0
\end{aligned}$$

Therefore $ad - bd \geq 0$ for $\theta(\lambda)$ to be a Nevanlinna function.

$\Leftarrow$ Now, assume that $ad - bc = 1$ and show that $\theta(\lambda) = \frac{a-b\lambda}{c-d\lambda}$ is a Nevanlinna function. We will proceed by cases

Case 1: $a = 0$ and $b \neq 0$ Without loss of generality, let $b = 1$. So $c = -1$. Therefore $\theta(\lambda) = \frac{-\lambda}{-1-d\lambda}$. Now to check condition d of being Nevanlinna,

$$\begin{aligned}
\theta(\lambda) &= \frac{-m - in}{-1 - dm - idn} \\
&= \frac{-m - in}{-1 - dm - idn} \cdot \frac{(-1 - dm + idn)}{(-1 - dm + idn)} \\
&= \frac{m + dm^2 - idmn + in + idmn + dn^2}{1 + dm - idn + dm + d^2m^2 - id^2mn + idn + id^2mn + d^2n^2} \\
&= \frac{m + dm^2 + dn^2 + in}{1 + 2dm + d^2(n^2 + m^2)} \\
Im(\theta(\lambda)) &= \frac{n}{1 + 2dm + d^2(n^2 + m^2)} \\
\frac{Im(\theta(\lambda))}{Im(\lambda)} &= \frac{1}{1 + 2dm + d^2(n^2 + m^2)} \geq 0
\end{aligned}$$

Case 2: $a \neq 0$ and $b = 0$ Without loss of generality, let $a = 1$. So $d = 1$. Therefore $\theta(\lambda) = \frac{1}{c-\lambda}$. Now

to check condition d of being Nevanlinna,

$$\begin{aligned}
\theta(\lambda) &= \frac{1}{c-m-in} \\
&= \frac{1}{c-m-in} \cdot \frac{(c-m+in)}{(c-m+in)} \\
&= \frac{c-m+in}{c^2-cm+icn-cm+m^2+imn-icn-imn+n^2} \\
&= \frac{c-m+in}{c^2-2cm+m^2+n^2} \\
Im(\theta(\lambda)) &= \frac{n}{c^2-2cm+m^2+n^2} \\
\frac{Im(\theta(\lambda))}{Im(\lambda)} &= \frac{1}{c^2-2cm+m^2+n^2} \geq 0
\end{aligned}$$

Case 3: $a = 0$ and $c \neq 0$ Without loss of generality, let $c = 1$. So $b = -1$. Therefore $\theta(\lambda) = \frac{\lambda}{1-d\lambda}$. Now to check condition d of being Nevanlinna,

$$\begin{aligned}
\theta(\lambda) &= \frac{m+in}{1-dm-idn} \\
&= \frac{m+in}{1-dm-idn} \cdot \frac{(1-dm+idn)}{(1-dm+idn)} \\
&= \frac{m+in-dm^2-idmn+idmn-dn^2}{1-dm+idn-dm+d^2m^2-id^2mn-idn+id^2mn+d^2n^2} \\
&= \frac{m-dm^2-dn^2+in}{1-2dm+d^2n^2+d^2m^2} \\
Im(\theta(\lambda)) &= \frac{n}{1-2dm+d^2n^2+d^2m^2} \\
\frac{Im(\theta(\lambda))}{Im(\lambda)} &= \frac{1}{1-2dm+d^2n^2+d^2m^2} \geq 0
\end{aligned}$$

Case 4: $a \neq 0$ and $b \neq 0$ Without loss of generality, let $a = 1$ and $b = 1$. So $d - c = 1$ or $d = 1 + c$.

Therefore $\theta(\lambda) = \frac{1-\lambda}{c-(1+c)\lambda}$. Now to check condition d of being Nevanlinna,

$$\begin{aligned}
\theta(\lambda) &= \frac{1-m-in}{c-(1+c)(m+in)} \\
&= \frac{1-m-in}{c-m-in-cm-icn} \\
&= \frac{1-m-in}{c-m-in-cm-icn} \cdot \frac{(c-m+in-cm+icn)}{(c-m+in-cm+icn)} \\
&= \frac{c-m+in-cm+icn-cm+m^2-imn+cm^2-icmn-icn+imn+n^2+icmn+cn^2}{c^2-cm-c^2m-cm+m^2+cm^2+n^2+cn^2-c^2m+cm^2+c^2m^2+cn^2+c^2n^2} \\
&= \frac{c-m-2cm+m^2+n^2+cm^2+cn^2+in}{c^2-2cm-2c^2m+m^2+2cm^2+n^2+2cn^2+c^2m^2+c^2n^2} \\
Im(\theta(\lambda)) &= \frac{n}{c^2-2cm-2c^2m+m^2+2cm^2+n^2+2cn^2+c^2m^2+c^2n^2} \\
\frac{Im(\theta(\lambda))}{Im(\lambda)} &= \frac{1}{c^2-2cm-2c^2m+m^2+2cm^2+n^2+2cn^2+c^2m^2+c^2n^2} \geq 0
\end{aligned}$$

□

Lemma 2.1.5. *Let* $\dfrac{A(\lambda)}{B(\lambda)}$ *be a rational Nevanlinna function such that* $A(\lambda) = A_0 \prod_{j=1}^{n}(\lambda+\alpha_j)$ *and* $B(\lambda) = B_0 \prod_{j=1}^{n}(\lambda+\beta_j)$ *such that* $\beta_0 < \alpha_0 < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_n < \alpha_n$ *and* $A_0 > 0$ *and* $B_0 > 0$. *Then there is a unique* a *and* b *such that*

$$\frac{A(\lambda)}{B(\lambda)} = a + \frac{1}{b\lambda + \frac{B^{(1)}(\lambda)}{A^{(1)}(\lambda)}}$$

where $a > 0$ *and* $b > 0$. *Also where* $\dfrac{B^{(1)}(\lambda)}{A^{(1)}(\lambda)}$ *is also a rational Nevanlinna function such that* $A^{(1)}(\lambda)$ *and* $B^{(1)}(\lambda)$ *are of degree* $n-1$.

Proof. Let's begin by dividing and writing it as

$$\frac{A(\lambda)}{B(\lambda)} = a + \frac{A^{(1)}(\lambda)}{B^{(1)}(\lambda) + b\lambda A^{(1)}(\lambda)}$$

Clearly, $a = \dfrac{A_0}{B_0} > 0$ and

$$A^{(1)}(\lambda) = A(\lambda) - aB(\lambda) \tag{2.2}$$

$$B(\lambda) = B^{(1)}(\lambda) + b\lambda A^{(1)}(\lambda) \tag{2.3}$$

Let's also define as follows

$$A^{(1)}(\lambda) = A_0^{(1)}\lambda^{n-1} + A_1^{(1)}\lambda^{n-2} + \cdots + A_{n-1}^{(1)}$$
$$B^{(1)}(\lambda) = B_0^{(1)}\lambda^{n-1} + B_1^{(1)}\lambda^{n-2} + \cdots + B_{n-1}^{(1)}$$

Equating the λ^n coefficients of (2.3) and $B(\lambda)$, yields $bA_0^{(1)} = B_0 > 0$. Equating the λ^{n-1} coefficients of (2.2) yields

$$\begin{aligned} A_0^{(1)} &= A_1 + aB_1 \\ &= A_0 \sum_{i=1}^{n} \alpha_i - aB_0 \sum_{i=1}^{n} \beta_i \\ &= A_0 \sum_{i=1}^{n} \alpha_i - A_0 \sum_{i=1}^{n} \beta_i \\ &= A_0 \sum_{i-1}^{n} (\alpha_i - \beta_i) > 0 \end{aligned}$$

We know that this is greater than zero using the interlacing property of the zeros. Now since $A_0^{(1)} > 0$, then $b > 0$.

Then dividing (2.2) by $B(\lambda)$ we get

$$\frac{A^{(1)}(\lambda)}{B(\lambda)} = \frac{A(\lambda)}{B(\lambda)} - a$$

Assuming that λ goes from $-\infty$ to 0 and using the interlacing properties,

$$\lim_{\lambda \to -\beta_j^+} \frac{A(\lambda)}{B(\lambda)} = +\infty \qquad \lim_{\lambda \to -\beta_j^-} \frac{A(\lambda)}{B(\lambda)} = -\infty$$

These limits also imply that

$$\lim_{\lambda \to -\beta_j^+} \frac{A^{(1)}(\lambda)}{B(\lambda)} = +\infty \qquad \lim_{\lambda \to -\beta_j^-} \frac{A^{(1)}(\lambda)}{B(\lambda)} = -\infty$$

Using the value of the limits and the Intermediate Value Theorem for continuous functions over the interval $(-\beta_{j+1}, -\beta_j)$ for $j = 1, 2, \ldots, n-1$, we know that there must be a zero of $A^{(1)}(\lambda)$ in

the interval. Let's call it $-\alpha'_j$ for $j = 1, 2, \ldots, n-1$. Thus

$$A^{(1)}(\lambda) = A_0^{(1)} \prod_{j=1}^{n-1} (\lambda - \alpha'_j)$$

such that $\beta_j < \alpha'_j < \beta_{j+1}$ for $j = 1, 2, \ldots, n-1$. This is all the zeros that exist.

Now if we divide (2.3) by $A^{(1)}(\lambda)$ and using similar logic

$$\frac{B^{(1)}(\lambda)}{A^{(1)}(\lambda)} = -b\lambda + \frac{B(\lambda)}{A^{(1)}(\lambda)}$$

$$\lim_{\lambda \to -\alpha'^{+}_j} \frac{B(\lambda)}{A^{(1)}(\lambda)} = -\infty \qquad \lim_{\lambda \to -\alpha'^{-}_j} \frac{B(\lambda)}{A^{(1)}(\lambda)} = +\infty$$

Using the value of the limits and the Intermediate Value Theorem for continuous functions over the interval $(-\alpha'_{j+1}, -\alpha'_j)$ for $j = 1, 2, \ldots, n-2$, we know that there must be a zero of $B^{(1)}(\lambda)$ in the interval. Let's call it $-\beta'_j$ for $j = 1, 2, \ldots, n-2$. To recover the last root, we use the Intermediate Value Theorem over $(\alpha'_1, 0)$. Since

$$\frac{B^{(1)}(0)}{A^{(1)}(0)} = \frac{B(0)}{A^{(1)}(0)} > 0$$

Thus, we recover the last root in the interval $(\alpha'_1, 0)$. Thus we have

$$B^{(1)}(\lambda) = B_0^{(1)} \prod_{j=1}^{n-1} (\lambda - \beta'_j)$$

where $0 < \beta'_1 < \alpha'_1 < \beta'_2 < \alpha'_2 < \cdots < \beta'_{n-1} < \alpha'_{n-1}$. Therefore $\dfrac{B^{(1)}(\lambda)}{A^{(1)}(\lambda)}$ is a rational Nevanlinna function itself. We can continue this logic and receive the following continued fraction unique

representation

$$\frac{A(\lambda)}{B(\lambda)} = a_n + \cfrac{1}{b_n\lambda + \cfrac{1}{a_{n-1} + \cfrac{1}{b_{n-1}\lambda + \ddots + \cfrac{1}{a_1 + \cfrac{1}{b_1\lambda + \cfrac{1}{a_0}}}}}} \tag{2.4}$$

□

Lemma 2.1.6. *A scalar function Q is said to be Nevalinna function if it has the integral representation of the form*

$$Q(\lambda) = \alpha + \beta\lambda + \int_{\mathbb{R}} (\frac{1}{s-\lambda} - \frac{s}{s^2+1}) d\sigma(s), \ \ \int_{\mathbb{R}} \frac{d\sigma(s)}{s^2+1} < \infty, \ \ \lambda \in \mathbb{C} \setminus \mathbb{R}$$

where $\alpha \in \mathbb{R}$ and $\beta \geq 0$ and σ is a nondecreasing function on $\mathbb{R}$([10]).

Also, there is a Stieltjes class of functions such that a Nevanlinna function Q belongs to the Stieltjes class if and only if Q is holomorphic and nonnegative on $(-\infty, 0)$.

Definition 2.1.2. *The function θ is said to be an S-function if θ is defined and analytic on $\mathbb{C} \setminus [0, \infty)$ and if $\theta(z) > 0$ for $z \in (-\infty, 0)$.*

Definition 2.1.3. *A meromorphic S-function θ is said to be an S_0-function if 0 is not a pole of θ.*

Nevanlinna functions have been used as part of spectral-parameter dependent boundary conditions. We will see how they have been used in Sturm-Liouville problems and then more specifically in single string problems. Then we derive a formula using Nevanlinna functions to represent the spectral properties of each string in the multi string case. This function will depend on the number of beads on the strand and the boundary condition on the end of the strand. Finally we will look at the spectra of specific examples that will illustrate the phenomenon that occurs in

multistring systems. That is, we will prove that the multiplicities of the eigenvalues depend on the symmetry of the model and on the total number of strings.

2.2 Application of Nevanlinna Functions to Sturm-Liouville problem with spectral-parameter dependent boundary conditions

Let us begin by considering the nonhomogeneous Sturm-Liouville problem

$$-y''(x) + q(x)y(x) = \lambda y(x) \tag{2.5}$$

with boundary conditions that will vary depending on the author and scope. In [61], Levitan considers (2.5) with the following boundary conditions:

$$y(a)\cos\alpha + y'(a)\sin\alpha = 0$$
$$y(b)\cos\beta + y'(b)\sin\beta = 0.$$

Levitan shows that for the eigenvalues λ_i and their corresponding eigenfunctions $y(x, \lambda_i)$, the eigenfunctions are orthogonal and the eigenvalues are real.

Zhamel looks at (2.5) with

$$y'(0) = 0$$
$$y'(\pi) - m\lambda y(\pi) = 0$$

where λ is the spectral parameter and m is a physical parameter in [86] while Amara and Shkalikov study the spectral properties and dynamics of the eigenvalues and eigenfunctions in [4]. In [86], he considers the case when $m < 0$ and the behavior as $m \to 0^-$, specifically that the first negative eigenvalue tends to $-\infty$ as $m \to 0^-$. Regular Sturm-Liouville problems that involve the eigenvalue parameter in the boundary condition at one end-point is using Walters ([85]) operator theoretic formulation in ([38]). Fulton extends this work to singular problems with the eigenvalue parameter linearly in a regular or a limit-circle boundary condition at the left endpoint

([37]). For regular Sturm-Liouville problems with discontinuous boundary value problems with eigendependent boundary conditions, some of the fundamental spectral properties are extended in ([3]).

Computations of the eigenvalues of a regular Sturm-Liouville problem with eigendependent boundary conditions can be cumbersome. Chanane discusses methods to for computing the eigenvalues of regular Sturm-Liouville problems with Dirichlet boundary conditions in ([20]). In ([19]) and ([21]), they extended the idea to singular problems. Then in ([28]), they expanded the scope to include regular SL problems with general separated boundary conditions. In ([30]), ([31]), ([29]) they extended the scope to those with coupled self-adjoint boundary conditions and even regular fourth order SL problems. Finally in ([27]), they demonstrate a method to compute the eigenvalues for SL problems with general separated boundary conditions that are nonlinear in the eigenvalue parameter.

Binding considers (2.5) with boundary conditions

$$y(0)\cos\alpha = y'(0)\sin\alpha \quad \alpha \in [0,\pi) \tag{2.6}$$

$$\frac{y'}{y}(1) = f(\lambda) \tag{2.7}$$

where f belongs to the Nevanlinna class of functions, $\mathcal{R}_N$, with the form

$$f(\lambda) = a\lambda + b - \sum_{k=1}^{N} \frac{b_k}{\lambda - c_k}. \tag{2.8}$$

He shows that the (see [12])

- Eigenvalues are real, simple, and form an increasing sequence accumulating at ∞ with $\lambda_0 < c_1$
- If b is decreased while c_k, q are increased then each λ_j is increased
- If $a > 0$ is decreased and b_k is increased then each positive $\lambda_j > c_k$ is increased.
- The eigenvalues of (2.5),(2.6) with the Dirichlet condition $y(1) = 0$ will be denoted λ_i^D for $i = 0, 1, \ldots$ and λ_i^{cD} to be the sequence of all c_j and λ_k^D in non-decreasing order. Then eigenvalues interlace as follows $\lambda_0 < \lambda_0^{cD} \leq \lambda_1 \leq \lambda_1^{cD} \leq \cdots$.

Binding, Browne, and Watson considered (2.5), (2.6-2.7) in [17]. These f, (2.8) will also belong to a class usually associated with the names of Herglotz or Nevanlinna. They use differential equation techniques to derive properties of the eigenvalues and eigenfunctions generalizing classic Sturm theory. They modified a transformation in which they ensured regularity. They note that non-Dirichlet conditions transform to Dirichlet and repeated transformations will be needed. Each class $\mathcal{R}_N$ is the union of two subclasses $\mathcal{R}_N^{\dagger}$ and $\mathcal{R}_N^0$ and their transformation will provide direct links between these subclasses for various values of N. They then analyze existence, oscillation and comparison theory. Their analysis determines that rather than one eigenvalue per oscillation count, N 'extra' eigenvalues appear with arbitrary oscillation counts. They also prove that if their transformation is applied to 'old' (original) problem, then the new spectrum contains the old eigenvalues (except possibly the first one). Using oscillation theory, they show that these are the only eigenvalues of the new problem. Therefore the transformation is isospectral. They investigate the spectral properties of the nonlinear Sturm-Liouville boundary problem

$$-(py')' + qy = \lambda(1-f)ry \text{ on } [0,1]$$
$$(a_j\lambda + b_j)y(j) = (c_j\lambda + d_j)(py')(j), \;\; j = 0,1$$

where $a_0 = 0 = c_0$ and $p, r > 0$ and q are functions depending on x while f depends on x, y, y' in ([16], [13]).

In ([2]), Altinisik, Kadakal, and Mukhtarov consider a discontinuous eigenvalue problem that consists of the differential equation

$$\tau u := -a(x)u'' + q(x)u = \lambda u$$

to hold in $[-1,1]$ except at the point $x = 0$, with the boundary conditions

$$\alpha_1 u(-1) + \alpha_2 u'(-1) = 0$$
$$(\beta_1'\lambda + \beta_1)u(1) = (\beta_2/\lambda + \beta_2)u'(1)$$

and at the point of discontinuity

$$\gamma_1 u(0-) = \delta_1 u(0+)$$
$$\gamma_2 u'(0-) = \delta_2 u'(0+)$$

where $a(x) = a_1^2$ for $x \in [-1, 0)$, $a(x) = a_2^2$ for $x \in (0, 1]$; $a_1 > 0$ and $a_2 > 0$ are given real numbers; $q(x)$ is a given real valued function continuous in $[-1, 0]$ and $[0, 1]$. the coefficients of the boundary and transmission conditions are real numbers. We assume $|\alpha_1| + |\alpha_2| \neq 0$, $\beta_1'\beta_2 - \beta_2'\beta_1 \neq 0$, $|\gamma_i| + |\delta_i| \neq 0$ $(i = 1, 2)$ and we write $\rho := \beta_1'\beta_2 - \beta_2'\beta_1 > 0$.

Tretter considers the eigenvalue problem for ordinary differential equations of the form $N\eta = \lambda P\eta$ on a compact interval with λ-polynomial boundary conditions ([82]). This leads to the nonclassical spectral problem since P doesn't need to be invertible. After linearizing the problems, Tretter arranges the system to correspond to assumptions of previous work which yields an asymptotic fundamental matrix. Tretter then introduces a particular asymptotic fundamental matrix and defines a notion of regularity of an nth order boundary eigenvalue problem. Then they prove completeness in certain finite codimensional subspaces and they study the minimality of the eigenfunctions and associated functions in the Sobolev spaces $W_2^l(a, b)$ for $l \geq p$. The basic properties of the eigenfunctions and associated functions are investigated and are shown that under certain conditions, the canonical system of eigenfunctions and associated functions even form a Riesz basis. Finally, they apply these new theory to the equation of motion of a clamped-free elastic beam, with a mass-spring system attached at its free end and show that the eigenfunctions and associated functions are complete in a set of spaces, form a minimal system of with defect, and form a Riesz basis with defect.

When multiple Stieltjes strings are connected to form trees, like in Figure 2.1, then Pivovarchik shows that an eigenvalue may have multiplicity greater than 1 ([70]). He also introduces the concept of listing the eigenvalues for a multistring problem in decreasing multiplicity order. If instead of Stieltjes strings connected to form trees we have continuous strings, then the eigenvalues of the Sturm-Liouville problem exhibit multiplicity and interlacing properties in [58].

In order to solve the non-homogeneous problem, we will need to find a Green's function that satisfies the homogeneous problem.

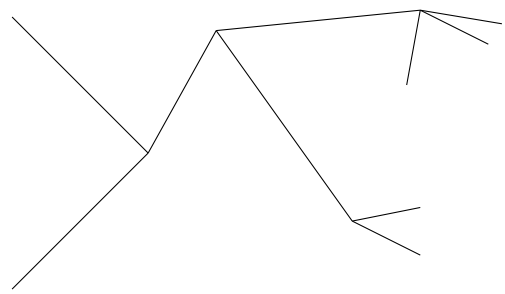


Figure 2.1: Multiple Stieltjes strings connected to form trees.

Definition 2.2.1. *We say that a function of two variables $G(t,\tau)$ is a Green's function for the differential equation of order n subject to the boundary conditions on the interval $[a,b]$ if $G(t,\tau)$ satisfies the following conditions:*

1. *$G(t,\tau)$ is a continuous function in $t,\tau \in [a,b]$ and it has continuous derivatives of the order up to and including $[n-2]$ with respect to t for any fixed τ, $t,\tau \in [a,b]$.*

2. *On subintervals $[a,\tau)$, $(\tau,b]$, the function $G(t,\tau)$ is considered a function of t (τ is fixed) and has continuous derivatives with respect to t of the order $[n-1]$, $[n]$. At the same time, $[n-1]$ order derivatives have a discontinuity jump at $t=\tau$ which is equal to -1*

$$\frac{\partial^{n-1}}{\partial t^{n-1}}G(\tau^{0+},\tau) - \frac{\partial^{n-1}}{\partial t^{n-1}}G(\tau^{0-},\tau) = -1$$

3. *For any fixed τ, $G(t,\tau)$ as a function of t on both subintervals from the second condition and solves the differential equation and satisfies the boundary conditions.*

Note also that for all $\tau \in [a,b]$, $G(t,\tau)$ solves the differential equation, $LG = -G_{tt} + q(t)G = 0$, $t \neq \tau$.

Let

$$G(t,\tau,\lambda) = \begin{cases} G_1(t,\tau,\lambda) = a_1(\tau,\lambda)u_1(t,\lambda) + a_2(\tau,\lambda)u_2(t,\lambda) & 0 \le t < \tau < l \\ G_2(t,\tau,\lambda) = b_1(\tau,\lambda)u_1(t,\lambda) + b_2(\tau,\lambda)u_2(t,\lambda) & 0 < \tau < t \le l \end{cases}$$

Now, we need G_1 to satisfy the left boundary condition: $G_1(0,\tau,\lambda) = \theta(\lambda)G_{1,t}(0,\tau,\lambda)$

$$
\begin{aligned}
& a_1(\tau,\lambda)u_1(0,\lambda) + a_2(\tau,\lambda)u_2(0,\lambda) = \theta(\lambda)\left[a_1(\tau,\lambda)u_{1,t}(0,\lambda) + a_2(\tau,\lambda)u_{2,t}(0,\lambda)\right] \\
\rightarrow \quad & a_1(\tau,\lambda) = \theta(\lambda)a_2(\tau,\lambda)(-1) \\
\rightarrow \quad & \frac{a_1(\tau,\lambda)}{a_2(\tau,\lambda)} = -\theta(\lambda)
\end{aligned}
$$

Now, we need G_2 to satisfy the right boundary condition: $G_2(l,\tau,\lambda) = HG_{2,t}(l,\tau,\lambda)$. We also know that $\psi(l,\lambda) = H\psi_t(l,\lambda)$ which will help eliminate H from our final function.

$$
\begin{aligned}
u_2(l,\lambda) + \alpha(\lambda)u_1(l,\lambda) &= H\left[u_{2,t}(l,\lambda) + \alpha(\lambda)u_{1,t}(l,\lambda)\right] \\
\alpha(\lambda)\left[u_1(l,\lambda) - Hu_{1,t}(l,\lambda)\right] &= Hu_{2,t}(l,\lambda) - u_2(l,\lambda) \\
\rightarrow \alpha(\lambda) &= \frac{Hu_{2,t}(l,\lambda) - u_2(l,\lambda)}{u_1(l,\lambda) - Hu_{1,t}(l,\lambda)}
\end{aligned}
$$

$$
\begin{aligned}
b_1(\tau,\lambda)u_1(l,\lambda) + b_2(\tau,\lambda)u_2(l,\lambda) &= H\left[b_1(\tau,\lambda)u_{1,t}(l,\lambda) + b_2(\tau,\lambda)u_{2,t}(l,\lambda)\right] \\
b_1(\tau,\lambda)\left[u_1(l,\lambda) - Hu_{1,t}(l,\lambda)\right] &= b_2(\tau,\lambda)\left[Hu_{2,t}(l,\lambda) - u_2(l,\lambda)\right] \\
\rightarrow \frac{b_1(\tau,\lambda)}{b_2(\tau,\lambda)} &= \frac{Hu_{2,t}(l,\lambda) - u_2(l,\lambda)}{u_1(l,\lambda) - Hu_{1,t}(l,\lambda)} = \alpha(\lambda)
\end{aligned}
$$

Therefore, so far we have

$$
G(t,\tau,\lambda) = \begin{cases} -\theta(\lambda)a_2(\tau,\lambda)u_1(t,\lambda) + a_2(\tau,\lambda)u_2(t,\lambda) = a_2(\tau,\lambda)\phi(t,\lambda) & 0 \le t < \tau < l \\ \alpha(\lambda)b_2(\tau,\lambda)u_1(t,\lambda) + b_2(\tau,\lambda)u_2(t,\lambda) = b_2(\tau,\lambda)\psi(t,\lambda) & 0 < \tau < t \le l \end{cases}
$$

Now let's use continuity and the jump condition to solve for $a_2(\tau,\lambda)$ and $b_2(\tau,\lambda)$. Notice that our

differential equation is of order 2 so $n = 2$ and therefore $n - 1 = 1$. We need

$$G(\tau^{0+}, \tau, \lambda) - G(\tau^{0-}, \tau, \lambda) = 0$$
$$\frac{\partial}{\partial t} G(\tau^{0+}, \tau, \lambda) - \frac{\partial}{\partial t} G(\tau^{0-}, \tau, \lambda) = -1$$

$$\begin{cases} -\theta(\lambda) a_2(\tau,\lambda) u_1(\tau^{0+},\lambda) + a_2(\tau,\lambda) u_2(\tau^{0+},\lambda) - \alpha(\lambda) b_2(\tau,\lambda) u_1(\tau^{0-},\lambda) \\ \qquad\qquad -b_2(\tau,\lambda) u_2(\tau^{0-},\lambda) &= 0 \\ -\theta(\lambda) a_2(\tau,\lambda) u_{1,t}(\tau^{0+},\lambda) + a_2(\tau,\lambda) u_{2,t}(\tau^{0+},\lambda) - \alpha(\lambda) b_2(\tau,\lambda) u_{1,t}(\tau^{0-},\lambda) \\ \qquad\qquad -b_2(\tau,\lambda) u_{2,t}(\tau^{0-},\lambda) &= -1 \end{cases}$$

Solving this system:

$$a_2(\tau,\lambda)\left[u_2(\tau,\lambda) - \theta(\lambda) u_1(\tau,\lambda)\right] = b_2(\tau,\lambda)\left[u_2(\tau,\lambda) + \alpha(\lambda) u_1(\tau,\lambda)\right]$$
$$a_2(\tau,\lambda) = b_2(\tau,\lambda)\left[\frac{u_2(\tau,\lambda) + \alpha(\lambda) u_1(\tau,\lambda)}{u_2(\tau,\lambda) - \theta(\lambda) u_1(\tau,\lambda)}\right]$$
$$\rightarrow a_2(\tau,\lambda) = b_2(\tau,\lambda)\left[\frac{\psi(\tau,\lambda)}{\phi(\tau,\lambda)}\right]$$

$$-\theta(\lambda) b_2(\tau,\lambda)\left[\frac{\psi(\tau,\lambda)}{\phi(\tau,\lambda)}\right] u_{1,t}(\tau,\lambda) + b_2(\tau,\lambda)\left[\frac{\psi(\tau,\lambda)}{\phi(\tau,\lambda)}\right] u_{2,t}(\tau,\lambda) - \alpha(\lambda) b_2(\tau,\lambda) u_{1,t}(\tau,\lambda)$$
$$-b_2(\tau,\lambda) u_{2,t}(\tau,\lambda) = -1$$
$$b_2(\tau,\lambda)\left[\frac{\psi(\tau,\lambda)}{\phi(\tau,\lambda)}\right]\left[u_{2,t}(\tau,\lambda) - \theta(\lambda) u_{1,t}(\tau,\lambda)\right] - b_2(\tau,\lambda)\left[\alpha(\lambda) u_{1,t}(\tau,\lambda) + u_{2,t}(\tau,\lambda)\right] = -1$$
$$b_2(\tau,\lambda)\left[\psi(\tau,\lambda)\left(u_{2,t}(\tau,\lambda) - \theta(\lambda) u_{1,t}(\tau,\lambda)\right) - \phi(\tau,\lambda)\left(\alpha(\lambda) u_{1,t}(\tau,\lambda) + u_{2,t}(\tau,\lambda)\right)\right] = -\phi(\tau,\lambda)$$
$$b_2(\tau,\lambda) = \frac{-\phi(\tau,\lambda)}{\psi(\tau,\lambda)\left(u_{2,t}(\tau,\lambda) - \theta(\lambda) u_{1,t}(\tau,\lambda)\right) - \phi(\tau,\lambda)\left(\alpha(\lambda) u_{1,t}(\tau,\lambda) + u_{2,t}(\tau,\lambda)\right)}$$
$$b_2(\tau,\lambda) = \frac{-\phi(\tau,\lambda)}{\psi(\tau,\lambda)\phi_t(\tau,\lambda) - \phi(\tau,\lambda)\psi_t(\tau,\lambda)}$$

Simplifying the denominator separately

$$
\begin{aligned}
\psi(\tau,\lambda)\phi_t(\tau,\lambda)-\phi(\tau,\lambda)\psi_t(\tau,\lambda) &= [u_2(\tau,\lambda)+\alpha(\lambda)u_1(\tau,\lambda)]\,[u_{2,t}(\tau,\lambda)-\theta(\lambda)u_{1,t}(\tau,\lambda)] \\
&\quad -[u_2(\tau,\lambda)-\theta(\lambda)u_1(\tau,\lambda)]\,[u_{2,t}(\tau,\lambda)+\alpha(\lambda)u_{1,t}(\tau,\lambda)] \\
&= u_2(\tau,\lambda)u_{2,t}(\tau,\lambda)-\theta(\lambda)u_{1,t}(\tau,\lambda)u_2(\tau,\lambda) \\
&\quad +\alpha(\lambda)u_1(\tau,\lambda)u_{2,t}(\tau,\lambda)-\alpha(\lambda)\theta(\lambda)u_1(\tau,\lambda)u_{1,t}(\tau,\lambda) \\
&\quad -u_2(\tau,\lambda)u_{2,t}(\tau,\lambda)-\alpha(\lambda)u_{1,t}(\tau,\lambda)u_2(\tau,\lambda) \\
&\quad +\theta(\lambda)u_1(\tau,\lambda)u_{2,t}(\tau,\lambda)+\alpha(\lambda)\theta(\lambda)u_1(\tau,\lambda)u_{1,t}(\tau,\lambda) \\
&= [\theta(\lambda)+\alpha(\lambda)][u_1(\tau,\lambda)u_{2,t}(\tau,\lambda)-u_{1,t}(\tau,\lambda)u_2(\tau,\lambda)] \\
&= [\theta(\lambda)+\alpha(\lambda)](-1)
\end{aligned}
$$

Therefore,

$$
b_2(\tau,\lambda)=\frac{\phi(\tau,\lambda)}{\theta(\lambda)+\alpha(\lambda)}
$$

So now we have

$$
\begin{aligned}
G(t,\tau,\lambda) &= \begin{cases} a_2(\tau,\lambda)\phi(t,\lambda) & 0\le t<\tau<l \\ b_2(\tau,\lambda)\psi(t,\lambda) & 0<\tau<t\le l \end{cases} \\
&= \begin{cases} \frac{\phi(\tau,\lambda)}{\theta(\lambda)+\alpha(\lambda)}\left(\frac{\psi(\tau,\lambda)}{\phi(\tau,\lambda)}\right)\phi(t,\lambda) & 0\le t<\tau<l \\ \frac{\phi(\tau,\lambda)}{\theta(\lambda)+\alpha(\lambda)}\psi(t,\lambda) & 0<\tau<t\le l \end{cases} \\
&= \frac{1}{\theta(\lambda)+\alpha(\lambda)}\begin{cases} \psi(\tau,\lambda)\phi(t,\lambda) & 0\le t<\tau<l \\ \psi(t,\lambda)\phi(\tau,\lambda) & 0<\tau<t\le l \end{cases}
\end{aligned}
$$

Let's return to the nonhomogeneous problem

$$
\begin{cases} -y''(t)+q(t)y(t)=f(t), \quad t\in(0,l) \\ y(0)-\theta(\lambda)y'(0)=0 \\ y'(l)-Hy'(l)=0 \end{cases}
$$

From Lagrange's Identity, where $Ly(t,\lambda) = f(t)$,

$$G(t,\tau,\lambda)Ly(t,\lambda) - y(t,\lambda)LG(t,\tau,\lambda) = \frac{d}{dt}\left[G(t,\tau,\lambda)y_t(t,\lambda) - y(t,\lambda)G_t(t,\tau,\lambda)\right]$$

Due to jump discontinuity, integrating both sides, and the fact that $G(t,\tau,\lambda)$ is a solution to homogeneous problem,

$$\begin{aligned}
\int_0^{\tau^-} G(t,\tau,\lambda)Ly(t,\lambda)dt &= [G(t,\tau,\lambda)y_t(t,\lambda) - y(t,\lambda)G_t(t,\tau,\lambda)]_0^{\tau^-} \\
\int_{\tau^+}^{l} G(t,\tau,\lambda)Ly(t,\lambda)dt &= [G(t,\tau,\lambda)y_t(t,\lambda) - y(t,\lambda)G_t(t,\tau,\lambda)]_{\tau^+}^{l} \\
\rightarrow \int_0^{l} G(t,\tau,\lambda)Ly(t,\lambda)dt &= [G(t,\tau,\lambda)y_t(t,\lambda) - y(t,\lambda)G_t(t,\tau,\lambda)]_0^{l} \\
&\quad - [G(t,\tau,\lambda)y_t(t,\lambda) - y(t,\lambda)G_t(t,\tau,\lambda)]_{\tau^-}^{\tau^+} \\
\rightarrow \int_0^{l} G(t,\tau,\lambda)Ly(t,\lambda)dt &= [G(t,\tau,\lambda)y_t(t,\lambda) - y(t,\lambda)G_t(t,\tau,\lambda)]_{\tau^-}^{\tau^+} \\
&= -[G(\tau^+,\tau,\lambda)y_t(\tau,\lambda) - G_t(\tau^+,\tau,\lambda)y(\tau,\lambda) \\
&\quad -G(\tau^-,\tau,\lambda)y_t(\tau,\lambda) + G_t(\tau^-,\tau,\lambda)y(\tau,\lambda)] \\
&= -\left[\frac{\partial}{\partial t}G(\tau^-,\tau,\lambda) - \frac{\partial}{\partial t}G(\tau^+,\tau,\lambda)\right]y(\tau) \\
&= -(-1)y(\tau) \\
&= y(\tau) \\
\rightarrow \int_0^{l} G(t,\tau,\lambda)Ly(t,\lambda)dt &= y(\tau) \\
\rightarrow \int_0^{l} G(t,\tau,\lambda)f(t)dt &= y(\tau) \\
\rightarrow \int_0^{l} G(t,\tau,\lambda)f(\tau)d\tau &= y(t)
\end{aligned}$$

We have found Green's function for a single string with eigendependent boundary conditions. Let us now analyze a concrete example of a Sturm-Liouville problem with eigendependent boundary conditions.

Example 2.2.1. *We will begin by considering the vibrations of a single continuous string of length π (see Figure 2.2). In subsequent examples, we want to move half of the string into a boundary condition using the Nevanlinna function $\theta(\lambda)$ without changing the eigenvalues and eigenfunctions of the entire problem.*

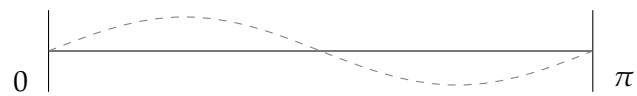


Figure 2.2: A continuous string of length π that is showing some vibrations/waves.

The vibrations of the entire string are governed by

$$-y'' = \lambda y \tag{2.9}$$

$$y(0) = y(\pi) = 0 \tag{2.10}$$

We have the ansatz $y(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$. Applying the boundary conditions gives

$$\begin{aligned} y(\pi) &= A\sin(\sqrt{\lambda}\pi) + B\cos(\sqrt{\lambda}\pi) = 0 \\ y(0) &= B = 0 \end{aligned}$$

Then $A\sin(\sqrt{\lambda}\pi) = 0$. Solving this gives the eigenvalues and eigenfunctions:

$$\lambda_n = n^2$$

$$y_\lambda(x) = \sin(\sqrt{\lambda}x)$$

Example 2.2.2. *For the same single string of length π, if we hide the right half of the string (see Figure 2.3), what does $\theta_1(\lambda)$ have to be in order for the problem to have the same eigenvalues and eigenfunctions as (2.9) or (2.2.1).*

$$\begin{aligned} -y'' &= \lambda y \\ y(0) &= 0 \\ y'\left(\frac{\pi}{2}\right) &= \theta_1(\lambda) y\left(\frac{\pi}{2}\right) \end{aligned}$$

We have the ansatz $y(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$. Applying the boundary conditions

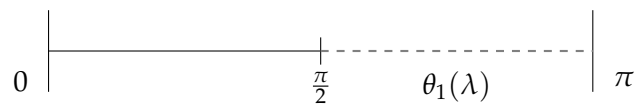


Figure 2.3: A continuous string of length π where the right half is going to be 'hidden'.

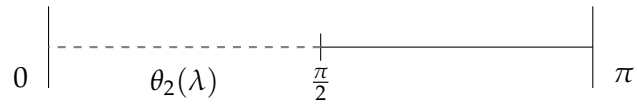


Figure 2.4: A continuous string of length π where the left half is going to be 'hidden'.

gives

$$\begin{aligned} y(0) &= B = 0 \\ y\left(\frac{\pi}{2}\right) &= A\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) \\ y'\left(\frac{\pi}{2}\right) &= A\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) \end{aligned}$$

So then

$$\theta_1(\lambda) = \frac{y'\left(\frac{\pi}{2}\right)}{y\left(\frac{\pi}{2}\right)} = \sqrt{\lambda}\cot\left(\sqrt{\lambda}\frac{\pi}{2}\right)$$

Example 2.2.3. *For the same single string of length π, if we hide the left half of the string (see Figure 2.4), what does $\theta_2(\lambda)$ have to be in order for the problem to have the same eigenvalues and eigenfunctions as (2.9) or (2.2.1).*

$$\begin{aligned} -y'' &= \lambda y \\ y(\pi) &= 0 \\ y'\left(\frac{\pi}{2}\right) &= \theta_2(\lambda) y\left(\frac{\pi}{2}\right) \end{aligned}$$

We start with the ansatz $y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$$\begin{aligned}
y(\pi) = 0 \quad &\rightarrow \quad A\cos(\sqrt{\lambda}\pi) + B\sin(\sqrt{\lambda}\pi) = 0 \\
A \quad &= \quad \frac{-B\sin(\sqrt{\lambda}\pi)}{\cos(\sqrt{\lambda}\pi)} \\
y\left(\frac{\pi}{2}\right) \quad &= \quad A\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) + B\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) \\
y'\left(\frac{\pi}{2}\right) \quad &= \quad -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right)
\end{aligned}$$

Thus

$$-A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) = \theta_2(\lambda)\left[A\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) + B\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right)\right]$$

After some simplification, we get

$$\theta_2(\lambda) = \frac{-B\sqrt{\lambda}\sin^2\left(\sqrt{\lambda}\frac{\pi}{2}\right)\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) + B\sqrt{\lambda}\cos^3\left(\sqrt{\lambda}\frac{\pi}{2}\right)}{-B\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right)\cos^2\left(\sqrt{\lambda}\frac{\pi}{2}\right) - B\sin^3\left(\sqrt{\lambda}\frac{\pi}{2}\right)} = \sqrt{\lambda}\cot\left(\sqrt{\lambda}\frac{\pi}{2}\right)$$

We can to check that $y(\pi) = 0$ and $y'\left(\frac{\pi}{2}\right) = \theta_2(\lambda)y\left(\frac{\pi}{2}\right)$ give the expected eigenvalues and eigenfunctions.

$$\begin{aligned}
y(\pi) \quad &= \quad A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right) = 0 \\
y\left(\frac{\pi}{2}\right) \quad &= \quad A\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) + B\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) \\
y'\left(\frac{\pi}{2}\right) \quad &= \quad -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) \\
\rightarrow -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) \quad &= \quad \frac{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right)}{\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right)}\left(A\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) + B\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right)\right) \\
\rightarrow B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right)\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) \quad &= \quad A\sqrt{\lambda}\left[\cos^2\left(\sqrt{\lambda}\frac{\pi}{2}\right) + \sin^2\left(\sqrt{\lambda}\frac{\pi}{2}\right)\right] \\
&\quad\quad +B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right)\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) \\
\rightarrow \frac{B}{2}\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) \quad &= \quad A\sqrt{\lambda} + \frac{B}{2}\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right)
\end{aligned}$$

Therefore the system that will yield our eigenvalues and eigenfunctions is

$$\begin{cases} A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right) = 0 \\ \frac{B}{2}\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) = A\sqrt{\lambda} + \frac{B}{2}\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) \end{cases}$$

$$\begin{cases} A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right) = 0 \\ \frac{B}{2}\sin\left(\sqrt{\lambda}\pi\right) = A\sqrt{\lambda} + \frac{B}{2}\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) \end{cases}$$

$$\cos\left(\sqrt{\lambda}\pi\right) \neq 0$$

$$\begin{cases} A\cos^2\left(\sqrt{\lambda}\pi\right) + \frac{B}{2}\sin\left(2\sqrt{\lambda}\pi\right) = 0 \\ \frac{B}{4}\sin\left(2\sqrt{\lambda}\pi\right) = A\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) + \frac{B}{4}\sqrt{\lambda}\sin\left(2\sqrt{\lambda}\pi\right) \end{cases}$$

$$\Rightarrow \cos\left(\sqrt{\lambda}\pi\right) = \frac{\left(\frac{B}{4} - \frac{B}{4}\sqrt{\lambda}\right)\sin\left(2\sqrt{\lambda}\pi\right)}{A\sqrt{\lambda}}$$

Plug back into the first equation of the system

$$\Rightarrow A\frac{\left(\frac{B}{4} - \frac{B}{4}\sqrt{\lambda}\right)^2}{A^2\lambda}\sin^2\left(2\sqrt{\lambda}\pi\right) + \frac{B}{2}\sin\left(2\sqrt{\lambda}\pi\right) = 0$$

factor

$$\sin(2\sqrt{\lambda}\pi) = 2\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\pi) = 0$$

$$\Rightarrow \sin(\sqrt{\lambda}\pi) = 0$$

$$\Rightarrow \lambda = n^2$$

Check the second solution for the equation A times complicated fraction.

$$Ly = \lambda y$$

$$y_\lambda(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

$$y(0) = 0 \to y_\lambda(x) = \sin(\sqrt{\lambda}x)$$

Now we need to define $\theta(\lambda)$ so that $y'_\lambda\left(\frac{\pi}{2}\right) = \theta(\lambda)y_\lambda\left(\frac{\pi}{2}\right)$ in such a way that it has the same

eigenvalues.

$$
\begin{aligned}
y_\lambda\left(\frac{\pi}{2}\right) &= \sin(\sqrt{\lambda}\frac{\pi}{2}) \\
y'_\lambda\left(\frac{\pi}{2}\right) &= \sqrt{\lambda}\cos(\sqrt{\lambda}\frac{\pi}{2}) \\
\rightarrow y'_\lambda\left(\frac{\pi}{2}\right) &= \frac{\sqrt{\lambda}\cos(\sqrt{\lambda}\frac{\pi}{2})}{\sin(\sqrt{\lambda}\frac{\pi}{2})} y_\lambda\left(\frac{\pi}{2}\right)
\end{aligned}
$$

Therefore, we can say that

$$
\begin{cases} -y'' = \lambda y \\ y(0) = y(\pi) = 0 \end{cases}
$$

and

$$
\begin{cases} -y'' = \lambda y \\ y(\pi) = 0 \\ y'\left(\frac{\pi}{2}\right) = \theta(\lambda) y\left(\frac{\pi}{2}\right) \end{cases}
$$

where $\theta(\lambda) = \sqrt{\lambda}\cot\left(\sqrt{\lambda}\frac{\pi}{2}\right)$

have the same spectral properties.

The idea presented in Examples 2.2.1-2.2.3 is similar to the system studied in [47] except two strings are joined together by a single mass. However, rather than studying the spectrum, they examine the boundary control and stabilization.

Example 2.2.4. *Now if we look at three strings (see Figure 1.4), each of length π where the outer end is fixed at $x = 0$ and has continuity in the middle at the joining point for $x = \pi$.*

$$\begin{cases} y_i'' = \lambda y \quad i = 1,2,3 \\ y_i(0) = 0 \\ y_1'(\pi) + y_2'(\pi) + y_3'(\pi) = 0 \\ y_1(\pi) = y_2(\pi) = y_3(\pi) \end{cases}$$

$$y_{1\lambda} = a_1 \sin(\sqrt{\lambda} x)$$

$$y_{2\lambda} = a_2 \sin(\sqrt{\lambda} x)$$

$$y_{3\lambda} = a_3 \sin(\sqrt{\lambda} x)$$

Applying the boundary condition at π yields

$$a_1 \sin(\sqrt{\lambda}\pi) = a_2 \sin(\sqrt{\lambda}\pi)$$

$$a_2 \sin(\sqrt{\lambda}\pi) = a_3 \sin(\sqrt{\lambda}\pi)$$

Now, there yields two different cases: either $\sin(\sqrt{\lambda}\pi) \neq 0$ or $\sin(\sqrt{\lambda}\pi) = 0$.

Case1: $\sin(\sqrt{\lambda}\pi) \neq 0$

$$\sin(\sqrt{\lambda}\pi) \neq 0$$

$$a_1 = a_2 = a_3 = a$$

$$3a\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$$

$$\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$$

$$\lambda_n = \frac{(2n+1)^2}{4}, \quad n = 1,2,3,\ldots$$

Case 2: $\sin(\sqrt{\lambda}\pi) = 0$

$$\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) \neq 0$$

$$\lambda = n^2, \quad n = 1,2,3,\ldots$$

$$a_1 + a_2 + a_3 = 0$$

then $a_1 = 1, a_2 = 0, a_3 = 0$, or $a_1 = 0, a_2 = 1, a_3 = 0$, or $a_1 = 0, a_2 = 0, a_3 = 1$.

2.3 Application of Nevanlinna Functions in Multi-String Case Dirichlet BC

For the multiple Stieltjes string model, let each edge have Dirichlet boundary conditions and set $m = T = l = 1$. Then (1.12) simplifies to

$$\ddot{u}_{i,j} = -2u_{i,j} + (u_{i-1,j} + u_{i+1,j}), \quad i = 1, 2, \ldots, n_j, \;\; j = 1, 2, \ldots, N \tag{2.11}$$

$$\ddot{u}_0 = -Nu_0 + (\sum_{j=1}^{N} u_{1,j}) \tag{2.12}$$

$$u_{n_j+1,j} = 0 \tag{2.13}$$

We can represent $N-1$ strings as part of a boundary condition utilizing Nevanlinna functions. This representation preserves the spectrum of the model. Without loss of generality, let's represent strings $j = 2, 3, \ldots, N$ into the boundary condition. Thus

$$\ddot{u}_{i,1} = -2u_{i,1} + (u_{i-1,1} + u_{i+1,1}) \;\; i = 1, 2, \ldots, n_1 \tag{2.14}$$

$$u_{n_1+1,1} = 0 \tag{2.15}$$

$$u_{1,1} = \theta(\lambda)u_0 \tag{2.16}$$

where $\theta(\lambda) = (\lambda + N) - \sum_{j=2}^{N} \theta_{n_j}(\lambda)$ and n_i is the number of masses on the jth strand. Several $\theta_{n_j}(\lambda)$ are calculated below.

Number of Masses $= n_j$	$\theta_{n_j}(\lambda)$ where $k = \lambda + 2$
1	$\frac{1}{k}$
2	$\frac{k}{k^2-1}$
3	$\frac{k^2-1}{k^3-2k}$
n	$\frac{1}{k-\theta_{n-1}}$

Lemma 2.3.1. *For any n_j, $\theta_{n_j}(\lambda)$ is an anti-Nevanlinna function, and $\theta(\lambda)$ is Nevanlinna.*

Proof. In order to show that $\theta_{n_j}(\lambda)$ is an anti-Nevalinna function, we proceed by induction. To see that $\theta_1(\lambda) = f(\lambda) = \frac{1}{\lambda+2}$ is anti-Nevanlinna, let $\lambda = x + iy$ and

$$\begin{aligned}
f(\overline{\lambda}) &= \frac{1}{x+iy+2} \\
&= \frac{x+iy+2}{(x-iy+2)(x+iy+2)} \\
&= \frac{x+iy+2}{x^2+4x+y^2+4} \\
\overline{f(\lambda)} &= \overline{\frac{x+iy+2}{}} \\
&= \frac{(x-iy+2)}{(x+iy+2)(x-iy+2)} \\
&= \frac{(x+iy+2)}{x^2+4x+y^2+4}
\end{aligned}$$

Therefore, $f(\overline{\lambda}) = \overline{f(\lambda)}$. Also,

$$\begin{aligned}
Im(f(\lambda)) &= \frac{-y}{x^2+4x+y^2+4} \\
Im(\lambda) &= y \\
\frac{Im(f(\lambda))}{Im(\lambda)} &= \frac{-1}{x^2+4x+y^2+4} \leq 0
\end{aligned}$$

Therefore, $f(\lambda) = \frac{1}{\lambda+2}$ is an anti Nevanlinna function.

Now let's use the general form to classify if $n_j = 2$. $\lambda + 2$ is Nevanlinna and since θ_1 is anti-Nevanlinna, then $-\theta_1$ is then Nevanlinna. We also know that the sum of Nevanlinna functions is Nevanlinna, so $\lambda + 2 + (-\theta_1)$ is Nevanlinna. Based on one of the properties $\frac{-1}{\lambda+2+(-\theta_1)}$ is Nevanlinna. Therefore, $\frac{1}{\lambda+(-\theta_1)}$ is anti Nevanlinna.

Now assume that $\theta_{n_j-1}(\lambda)$ is anti Nevanlinna function. Clearly, $\lambda + 2$ is a Nevanlinna function as well as $-\theta_{n_j-1}$. The sum of two Nevanlinna functions is also a Nevanlinna function. Based on previous Lemma, if $f(x)$ is a Nevanlinna function, then $\frac{-1}{f(x)}$ is a Nevanlinna function. Since $\theta_{n_j}(\lambda) = \frac{1}{\lambda+2+(-\theta_{n_j-1}(\lambda))}$, then $\theta_d(\lambda)$ is an anti Nevanlinna function for any n_j.

Since if $\theta_{n_j}(\lambda)$ is anti Nevanlinna, then $-\theta_d(\lambda)$ is Nevanlinna. The sum of Nevanlinna functions are Nevanlinna. Therefore $\theta(\lambda)$ is Nevanlinna. □

2.4 Application of Nevanlinna Functions in Multi-String Case Neumann BC

Now if we consider the model (2.11-2.13) but with Neumann conditions on the edges gives

$$\ddot{u_{i,j}} = -2u_{i,j} + (u_{i-1,j} + u_{i+1,j}), \quad i = 1, 2, \ldots, n_j, j = 1, 2, \ldots, N \tag{2.17}$$

$$\ddot{u_0} = -Nu_0 + (\sum_{j=1}^{N} u_{1,j)} \tag{2.18}$$

$$u_{n_j+1,j} = u_{n_j,j} \tag{2.19}$$

We can represent $N-1$ strings as part of a boundary condition utilizing Nevanlinna functions. This representation preserves the spectrum of the model. Without loss of generality, let's represent strings $j = 2, 3, \ldots, N$ into the boundary condition. Thus

$$\ddot{u}_{i,1} = -2u_{i,1} + (u_{i-1,1} + u_{i+1,1}) \quad i = 1, 2, \ldots, n_1 \tag{2.20}$$

$$u_{n_1+1,1} = u_{n_1,1} \tag{2.21}$$

$$u_{1,1} = \tilde{\theta}(\lambda)u_0 \tag{2.22}$$

where $\tilde{\theta}(\lambda) = (\lambda + N) - \sum_{j=2}^{N} \tilde{\theta}_{n_j}(\lambda)$ and n_j is the number of masses on the jth strand. Several $\tilde{\theta}_{n_j}(\lambda)$ are calculated below.

Number of Masses $= n_j$	$\tilde{\theta}_{n_j}(\lambda)$ where $k = \lambda + 2$
1	$\frac{1}{k-1}$
2	$\frac{k-1}{k(k-1)-1}$
n	$\frac{1}{k-\tilde{\theta}_{n-1}}$

Lemma 2.4.1. *For any n_j, $\tilde{\theta}_{n_j}(\lambda)$ is an anti-Nevanlinna function, and $\tilde{\theta}(\lambda)$ is Nevanlinna.*

Proof. Note that the only difference between θ_{n_j} and $\tilde{\theta}_{n_j}$ is when $n_j = 1$. To see that $\theta_1(\lambda) = f(\lambda) =$

$\frac{1}{\lambda+1}$ is anti-Nevanlinna, let $\lambda = x + iy$ and

$$\begin{aligned}
f(\overline{\lambda}) &= \frac{1}{x - iy + 1} \\
&= \frac{x + iy + 1}{(x - iy + 1)(x + iy + 1)} \\
&= \frac{x + iy + 1}{x^2 + 2x + y^2 + 1} \\
\overline{f(\lambda)} &= \overline{\frac{1}{x + iy + 1}} \\
&= \overline{\frac{(x - iy + 1)}{(x + iy + 1)(x - iy + 1)}} \\
&= \frac{(x + iy + 1)}{x^2 + 2x + y^2 + 1}
\end{aligned}$$

Therefore, $f(\overline{\lambda}) = \overline{f(\lambda)}$. Also,

$$\begin{aligned}
Im(f(\lambda)) &= \frac{-y}{x^2 + 2x + y^2 + 1} \\
Im(\lambda) &= y \\
\frac{Im(f(\lambda))}{Im(\lambda)} &= \frac{-1}{x^2 + 2x + y^2 + 1} \leq 0
\end{aligned}$$

Therefore, $f(\lambda) = \frac{1}{\lambda+1}$ is an anti Nevanlinna function. Since we have shown that $f(\lambda) = \frac{1}{\lambda+1}$ is anti Nevanlinna, we can follow the same induction argument to show that $\tilde{\theta}_{n_j}(\lambda)$ is anti Nevanlinna for any number of masses. □

2.5 Numerical Simulations

While investigating whether the eigenvalues of the same systems with only difference being presence or lack there of a middle mass, it was observed that the eigenvalues did in fact 'interlace' but there was overlap on certain eigenvalues and that they often occur in multiplicity. The following calculations are investigating the occurrence of eigenvalues with multiplicity.

Example 2.5.1. *Consider 3 Stieltjes strings with two beads on each string, a center bead, and $m = T = l = 1$ for the entire system.*

Representing the model as $\frac{d\vec{u}}{dt} = A\vec{u}$, where

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

The characteristic polynomial is $p(\lambda) = -\lambda^7 - 15\lambda^6 - 90\lambda^5 - 277\lambda^4 - 465\lambda^3 - 417\lambda^2 - 180\lambda - 27 = -(\lambda+3)^2(\lambda+1)^2(\lambda^3 + 7\lambda^2 + 12\lambda + 3)$ Calculating $\theta_1(\lambda)$

$$\begin{aligned} y_6 - 2y_7 &= \lambda y_7 & y_7 &= \frac{1}{\lambda+2} y_6 \\ y_3 - 2y_6 + y_7 &= \lambda y_6 & y_6 &= \frac{\lambda+2}{(\lambda+2)^2 - 1} y_3 \\ y_4 - 2y_5 &= \lambda y_5 & y_5 &= \frac{1}{\lambda+2} y_4 \\ y_3 - 2y_4 + y_5 &= \lambda y_4 & y_4 &= \frac{\lambda+2}{(\lambda+2)^2 - 1} y_3 \\ y_2 - 3y_3 + y_4 + y_6 &= \lambda y_3 \\ y_2 = \left[\lambda + 3 - \left(\frac{\lambda+2}{(\lambda+2)^2-1}\right) - \left(\frac{\lambda+2}{(\lambda+2)^2-1}\right)\right] y_3 & & y_2 &= \theta_1(\lambda) y_3 \end{aligned}$$

Thus, $\theta_1(\lambda) = \lambda + 3 - \left(\frac{\lambda+2}{(\lambda+2)^2-1}\right) - \left(\frac{\lambda+2}{(\lambda+2)^2-1}\right) = \lambda + 3 - 2\left(\frac{\lambda+2}{(\lambda+2)^2-1}\right) = \frac{\lambda^3+7\lambda^2+13\lambda+5}{\lambda^2+4\lambda+3}$ which is Nevanlinna.

Calculating $\theta_2(\lambda)$

$$\begin{aligned} -2y_1 + y_2 &= \lambda y_1 & y_1 &= \frac{1}{\lambda+2} y_2 \\ y_1 - 2y_2 + y_3 &= \lambda y_2 & y_3 &= \frac{(\lambda+2)^2 - 1}{\lambda+2} y_2 \end{aligned}$$

Thus, $\theta_2(\lambda) = \frac{(\lambda+2)^2-1}{\lambda+2} = \frac{\lambda^2+4\lambda+3}{\lambda+2}$ which is Nevanlinna.

Equating $y_2 = \left[\lambda + 3 - \left(\frac{\lambda+2}{(\lambda+2)^2-1}\right) - \left(\frac{\lambda+2}{(\lambda+2)^2-1}\right)\right] y_3$ and $y_3 = \frac{(\lambda+2)^2-1}{\lambda+2} y_2$ yields

$$\frac{\lambda+2}{\lambda^2+4\lambda+3} y_3 = \frac{\lambda^3+7\lambda^2+13\lambda+5}{\lambda^2+4\lambda+3} y_3 \qquad 0 = \frac{\lambda^3+7\lambda^2+12\lambda+3}{\lambda^2+4\lambda+3} y_3$$
$$\frac{\lambda^2+4\lambda+3}{\lambda^3+7\lambda^2+13\lambda+5} y_2 = \frac{\lambda^2+4\lambda+3}{\lambda+2} y_2 \qquad 0 = \frac{-(\lambda^2+4\lambda+3)(\lambda^3+7\lambda^2+12\lambda+3)}{(\lambda+2)(\lambda^3+7\lambda^2+13\lambda+5)} y_2$$

The solutions to these yield all the eigenvalues, but do not illustrate their multiplicity.

Example 2.5.2. *Consider the same configuration of Stieltjes strings as (2.5.1) but with no middle bead.*

The characteristic polynomial is $p(\lambda) = \frac{1}{27}(27\lambda^6 + 297\lambda^5 + 1269\lambda^4 + 2646\lambda^3 + 2781\lambda^2 + 1377\lambda + 243) = \frac{1}{27}(\lambda+3)^2(\lambda+1)^2(\lambda^2+3\lambda+1)$

$$\rightarrow y_6 = \frac{1}{\lambda+2} y_5$$
$$\rightarrow y_4 = \frac{1}{\lambda+2} y_3$$

$$\rightarrow \begin{cases} \frac{1}{3}y_2 + \frac{1}{3}y_3 - \frac{5}{3}y_5 + y_6 = \lambda y_5 \\ \frac{1}{3}y_2 - \frac{5}{3}y_2 + y_4 + \frac{1}{3}y_5 = \lambda y_3 \end{cases}$$
$$\begin{cases} y_2 + y_3 - 5y_5 + \frac{3}{\lambda+2}y_5 = 3\lambda y_5 \\ y_2 - 5y_2 + \frac{3}{\lambda+2}y_3 + y_5 = 3\lambda y_3 \end{cases}$$
$$y_5 = \frac{(3\lambda^2+12\lambda+9)(\lambda+2)}{(3\lambda^2+11\lambda+7)^2-(\lambda+2)^2} y_2$$
$$y_3 = \frac{(3\lambda^2+12\lambda+9)(\lambda+2)}{(3\lambda^2+11\lambda+7)^2-(\lambda+2)^2} y_2$$
$$\rightarrow y_1 - \frac{5}{3}y_2 + \frac{1}{3}y_3 + \frac{1}{3}y_5 = \lambda y_2$$
$$y_1 = \left(\lambda + \frac{5}{3}\right) - \frac{2}{3}\left(\frac{3\lambda^3+18\lambda^2+33\lambda+18}{9\lambda^4+66\lambda^3+162\lambda^2+150\lambda+45}\right) y_2$$
$$y_1 = \frac{9(\lambda^2+4\lambda+3)(3\lambda^3+15\lambda^2+21\lambda+7)}{9(\lambda^2+4\lambda+3)(3\lambda^2+10\lambda+5)} y_2$$

Thus, $\theta_1(\lambda) = \dfrac{9(\lambda^2+4\lambda+3)(3\lambda^3+15\lambda^2+21\lambda+7)}{9(\lambda^2+4\lambda+3)(3\lambda^2+10\lambda+5)}$. We also know that $y_1 = \frac{1}{\lambda+2}y_2$. Equat-

ing these and solving yields the following equation which yields the eigenvalues, but not multiplicity. Note though that the eigenvalues with multiplicity > 1 are the ones that can get cancelled from $\theta_1(\lambda)$.

$$\frac{3(\lambda^4 + 7\lambda^3 + 16\lambda^2 + 13\lambda + 3)}{(\lambda + 2)(3\lambda^2 + 10\lambda + 5)} = 0$$
$$\frac{3(\lambda + 1)(\lambda + 3)(\lambda^2 + 3\lambda + 1)}{(\lambda + 2)(3\lambda^2 + 10\lambda + 5)} = 0$$

Example 2.5.3. *Now if instead we have a strand with 2 beads, a strand with 3 beads, a strand with 1 bead, and a middle mass.*

Then

$$\theta(\lambda) = \lambda + 3 - \frac{1}{\lambda + 2} - \frac{\lambda^2 + 4\lambda + 3}{(\lambda + 2)^3 - 2(\lambda + 2)} = \frac{\lambda^4 + 9\lambda^3 + 26\lambda^2 + 26\lambda + 7}{(\lambda + 2)(\lambda^2 + 4\lambda + 2)}$$

which is Nevanlinna and has the eigenvalues
$\lambda = -4.4266, -3.3209, -2.7709, -2.0000, -1.4247, -0.7634, -0.2934$. Notice that all are simple eigenvalues.

Example 2.5.4. *Looking even further at (2.5.1), where*

$$\theta_2(\lambda) = \frac{(\lambda + 2)^2 - 1}{\lambda + 2} = \frac{\lambda^2 + 4\lambda + 3}{\lambda + 2}$$

.

The eigenvalues and non-normal eigenvectors are

$$\lambda = -4.4605, \quad v = \begin{bmatrix} 1 \\ -2.4605 \\ 5.0541 \\ -2.4605 \\ 1 \\ -2.4605 \\ 1 \end{bmatrix} \quad \lambda = -3, \quad v = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -0.6129 \\ 0.6129 \\ 1.6129 \\ -1.6129 \end{bmatrix} \quad \lambda = -3, \quad v = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1.1739 \\ -1.1739 \\ -0.1739 \\ 0.1739 \end{bmatrix}$$

$$\lambda = -2.2391, \quad v = \begin{bmatrix} 1 \\ -0.2391 \\ -0.9428 \\ -0.2391 \\ 1 \\ -0.2391 \\ 1 \end{bmatrix} \quad \lambda = -1, \quad v = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0.6121 \\ 0.6121 \\ -1.6121 \\ -1.6121 \end{bmatrix} \quad \lambda = -1, \quad v = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1.1744 \\ -1.1744 \\ 0.1744 \\ 0.1744 \end{bmatrix}$$

$$\lambda = -0.3004, \quad v = \begin{bmatrix} 1 \\ 1.6996 \\ 1.8887 \\ 1.6996 \\ 1 \\ 1.6996 \\ 1 \end{bmatrix}$$

It seems that the eigenvectors associated with the repeated eigenvalue can have strands swap places without really affecting anything. Since there are 3 strands to begin with, two swaps are only possible before returning back to original. This can be verified if looking at the eigenvectors

for $\lambda = -3$,

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1.1739 \\ -1.1739 \\ -0.1739 \\ 0.1739 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -0.6129 \\ 0.6129 \\ 1.6129 \\ -1.6129 \end{bmatrix} \text{ rotate each strand clockwise yielding}$$

$$v_{new} = \begin{bmatrix} 1.6129 \\ -1.6129 \\ 0 \\ 1 \\ -1 \\ -0.6129 \\ 0.6129 \end{bmatrix}$$

If this is the case, we should be able to find an α and β such that

$$\begin{cases} \alpha + \beta = 1.6129 \\ -0.6129\alpha + 1.1739\beta = 1 \end{cases}$$

In fact, $\alpha = 1.11291$ and $\beta = 0.499991$ but $\alpha v_1 + \beta v_2$ yields v_{new} with the signs switched on the last two components .

Lemma 2.5.1. *The maximal multiplicity of the eigenvalues for the system is equal to $N - 1$ and is achieved on the symmetric configuration.*

Example 2.5.5. *To see Lemma 2.5.1, consider 4 Stieltjes strings joined with a middle mass. To simplify computations, assume that all masses are 1, the uniform tension is 1, and the length between each mass is uniform at 1 unit length. Assuming Dirichlet boundary conditions on the ends of the Stieltjes string. Calculating the eigenvalues for different configurations with 4 strings:*

# beads per string	*Eigenvalues*
1,2,3,4	*-5.3, -3.5, -3.3,-2.8,-2.5,-2.0,-1.6,-1.2,-0.7, -0.4,-0.2*
1,2,3,3	*-5.3,-3.4,-3.2,-2.8,-2.0 (multiplicity 2),-1.4,-0.8,-0.5,-0.2*
1,3,3,3	*-5.3, -3.4142 (multiplicity 2), -3.1, -2.0 (multiplicity 3), -1.2, -0.5 (multiplicity 2), -0.2*
3,3,3,3	*-5.3,-3.4 (multiplicity 3),-3.0,-2.0 (multiplicity 3),-1.3, -0.5 (multiplicity 3), -0.1*

Chapter 3

Inverse Problem for Multi-String Systems

3.1 Introduction

For mathematicians, after knowing a problem has a unique solution, the next logical question is "If I know the solution, can I reproduce the problem?". These are inverse problems. The inverse problems we will consider are: given the spectrum, what other conditions must we know in order to uniquely identify the problem that yields that spectrum? The eigenvalues correspond to the physical manifestion of frequency which is measurable. While we will narrow our focus to the eigenvalues of strings, the inverse spectral problem for the 3D wave equation across a bounded domain is solved in [53] and the inverse spectral problem for a network is solved in [11]. We begin our investigation into inverse problems with the inverse regular Sturm-Liouville problem.

$$-y'' + q(x)y = \lambda y \qquad 0 \leq x \leq l \tag{3.1}$$

$$y'(0) - h_1 y(0) = 0 \qquad y(l) = \theta(\lambda) y'(l) \tag{3.2}$$

$$y'(0) - h_2 y(0) = 0, \qquad y(l) = \theta(\lambda) y'(l) \tag{3.3}$$

where $q(x)$ is a real-valued continuous function, h_1, h_2 are finite real numbers with $h_1 \neq h_2$, $\theta(\lambda)$ is a Nevanlinna function with $\theta(\lambda) = \frac{\theta_1(\lambda)}{\theta_2(\lambda)}$ with $\theta_1(\lambda)$ and $\theta_2(\lambda)$ are relatively prime polynomials, and (3.2),(3.3) are two boundary value problems. Note that if (3.2),(3.3) do not contain the

eigenvalue parameter λ, then the inverse problem is given in ([57], [56],[40]). Let the solutions of (3.1) be $u_1(x,\lambda)$ and $u_2(x,\lambda)$ such that they solve $u_1(0,\lambda) = 1$, $u_1'(0,\lambda) = h_1$, $u_2(0,\lambda) = 1$, and $u_2'(0,\lambda) = h_2$. The spectrum of (3.1),(3.2) are $\{\lambda_n\}_{n=0}^{\infty}$ of the entire function $\varphi_1(\lambda) = \theta_2(\lambda)u_1(l,\lambda) - \theta_1(\lambda)u_1'(l,\lambda)$. The spectrum of (3.1),(3.3) are $\{\mu_n\}_{n=0}^{\infty}$ of the entire function $\varphi_2(\lambda) = \theta_2(\lambda)u_2(l,\lambda) - \theta_1(\lambda)u_2'(l,\lambda)$ ([32]).

Theorem 4. *Let two spectra $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ be given so that $\lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \cdots$, equalities*

$$\sqrt{\lambda_{n+m}} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + o(\frac{1}{n^3})$$
$$\sqrt{\mu_{n+m}} = n + \frac{a_0'}{n} + \frac{a_1'}{n^3} + o(\frac{1}{n^3})$$

take place, moreover $a_0 \neq a_0'$. Then there exist an absolutely continuous function $q(x)$, real numbers h_1, h_2 and a rational function $\theta(\lambda)$ for which $\operatorname{Im}\theta(\lambda)\operatorname{Im}\lambda \leq 0$, such that λ_n is the spectrum of the problem (3.1),(3.2) μ_n is the spectrum of the problem (3.1),(3.3). [32]

Col considers the Sturm-Liouville boundary value problem with the boundary condition $(\alpha_0 + i\alpha_1\lambda - \alpha\lambda^2 - i\alpha_3\lambda^3)y'(0) - (\beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3)y(0) = 0$ where $q(x)$ is a real valued function with the condition $\int_0^{\infty}(1+x)|q(x)|dx < \infty$, and $\rho(x) = \begin{cases} \alpha^2 & 0 \leq x < a \\ 1 & a \leq x < \infty \end{cases}$ $1 \neq \alpha > 0$ in ([33]). They define the polynomials $p_1(\lambda) := \alpha_0 + i\alpha_1\lambda - \alpha_2\lambda^2 - i\alpha_3\lambda^3$ and $p_2(\lambda) := \beta_0 + i\beta_1\lambda - \beta_2\lambda^2 - i\beta_3\lambda^3$ with the relation $\alpha_{i+1}\beta_i - \alpha_i\beta_{i+1} > 0$, $\alpha_{i+2}\beta_i - \alpha_i\beta_{i+2} < 0$, $\alpha_{i+3}\beta_i - \alpha_i\beta_{i+3} = 0$. Col considers the case of the discontinuous coefficient $\rho(x)$. The main result is that the potential $q(x)$ can be uniquely recovered from the given scattering data. The result was obtained by using the Marchenko method. It is applied to solving the boundary value problem when the boundary conditions depend on spectral parameter as nonlinear. If $q(x) = 0$, the result is obtained

$$f_0(x,\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^+(x)} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu^-(x)}$$

where $u^{\pm}(x) = \pm x\sqrt{\rho(x)} + a(1 \mp \sqrt{\rho(x)})$ If the condition on $q(x)$ is satisfied, the boundary value problem has a unique solution which satisfies the asymptotic behavior $\lim_{x\to\infty} e^{i\lambda x}f(x,\lambda) = 1$ for $Im\lambda \geq 0$ and can be expressed as $f(x,\lambda) = f_0(x,\lambda) + \int_{\mu^+(x)}^{\infty} K(x,t)e^{i\lambda t}dt$ which is called the Jost

solution. For the kernel function $K(x,t)$,

$$\int_{\mu^+(x)}^{\infty} |K(x,t)|dt \leq c\left(exp\left(\int_x^{\infty} t|q(t)|dt\right)\right), \quad 0 < c = \text{constant}$$

is satisfied. This kernel satisfies a bunch of properties and yields the spectral condition.

For a regular Sturm-Liouville problem with eigendependent boundary conditions, the potential and the asymptotic boundary conditions are uniquely determined by a dense set of nodal points of eigenfunctions ([26]).

Binding, Browne, and Watson discuss three inverse problems for a Sturm-Liouville problem with the boundary conditions $y(0)\cos\alpha = y'(0)\sin\alpha$ and $y'(1) = f(\lambda)y(1)$ for rational f in [18]. They show that the Weyl m-function uniquely determines α, f, and q, and is in turn uniquely determined by either two spectra from different values of α or by Prüfer angle. In ([14]) they discuss the inverse problem for the nonlinear Sturm-Liouville problem. For (3.2) with $f(t) = \lambda y(t)$ and boundary conditions $y'(0) - hy(0) = 0$ and $\lambda(y'(\pi) + Hy(\pi)) = H_1y'(\pi) + H_2y(\pi)$ where h, H, H_1, $H_2 \in \mathbb{R}$ and $\rho := HH_1 - H_2 > 0$, Guliyev studies the inverse problem from the sequences of eigenvalues and norming constants as well as from two spectra in ([43]) and obtained the regularized trace formula where the first boundary condition is changed to $\lambda(y'(0) - hy(0) = h_1y'(0) - h_2y(0)$ where $h, h_1, h_2 \in \mathbb{R}$ and $\delta := hh_1 - h_2 > 0$ in ([44]).

If for (2.8) $f(\lambda) = \infty$, then the boundary condition is interpreted as a Dirichlet condition. They discuss the inverse problem of recovering q, α, and f from given spectral data. The data to be used consists of eigenvalues and norming constants, which are often observable quantities. It was found that the eigenvalues form a real sequence $\lambda_0 < \lambda_1 < \cdots$ accumulating at $+\infty$. The "norming" of these eigenfunctions involve a Hilbert space structure which they develop. It allows the problem to be viewed as a standard eigenvalue problem for a self-adjoint operator with compact resolvent. They construct a chain of problems that connect their problem with a "standard" Sturm-Liouville problem using transformations. They also need to invert the above transformations to complete the solution of their inverse problem. They prescribe necessary and sufficient conditions for given sequences λ_n, ρ_n to be generated by the problem and that such problem must be unique. In ([45]), he proves that if the spectra of two seemingly different problems of (3.2) with both of his boundary conditions, then the $q(x)$'s are equal a.e. and all the h and H's are equal. In ([63]), they

introduce a new method to recover the potential of the Sturm-Liouville equation on a half-interval using the spectrum of the Dirichlet on both sides problem and a known potential on the other half interval.

Pivovarchik solves the inverse problem for system of Sturm-Liouville equations in ([74]). Given $n \in \mathbb{N}$, real and square-integrable potentials q_j, $j = 1, \ldots, n$ which are defined on respective intervals $[0, a_j]$, $a_j \in (0, \infty)$, the numbers $\gamma_j \in (-\infty, \infty]$, $j = 1, \ldots, n$, and $\alpha > 0$ and $\beta \in \mathbb{R}$, then he can state the problem

$$\begin{aligned} y_j'' + \lambda^2 y_j - q(x) y_j = 0, \quad j = 1, \ldots, n, \quad x \in [0, a_j] \\ y_1(\lambda, 0) = y_2(\lambda, 0) = \cdots = y_n(\lambda, 0) \\ \sum_{j=1}^{n} y_j'(\lambda, 0) + (i\alpha\lambda + \beta) y_1(\lambda, 0) = 0 \\ \gamma_j y_j'(\lambda, a_j) + y_j(\lambda, a_j) = 0, \quad j = 1, \ldots, n. \end{aligned}$$

A Stieltjes string is a thread bearing a finite number of point masses. F.R. Gantmacher and M.G. Krein solved the inverse problem of identifying the location and mass of each bead just given two spectra, one corresponding to Dirichlet and one corresponding to Neumann boundary conditions, for the boundary value problem in [39]. Rather than needing the spectrum from both types of boundary conditions, it was shown that the inverse problem can be solved for a given spectrum of the Dirichlet problem generated by a Sturm-Liouville equation with a real potential on an interval $[0, a]$, and spectra of the Dirichlet problems generated by the same equation on the subintervals $[0, a/2]$ and $[a/2, 0]$ this is for continuous string [69], while [24] is for Stieltjes string.

The inverse spectral problem for star graphs of Stieltjes strings with Dirichlet and Neumann boundary conditions at either one pendant vertex of the star graph or the central vertex was solved in [71] uniquely for the varying conditions on the pendant vertex and non-uniquely if the varying conditions is on the central vertex. Necessary and sufficient conditions on the location and multiplicities of two finite sequences of numbers corresponding to the Dirichlet and Neumann eigenvalues are derived in [73] where Neumann and Dirichlet boundary conditions are imposed on the central vertex and Dirichlet boundary conditions on the pendant vertex. The corresponding inverse problem was studied where the spectrums, the number and length of edges is given,

and the number of masses on each edge is given in [73]. Boyko and Pivovarchik solved the inverse problem of recovering the masses and the lengths of the intervals between them given the spectrum of the entire graph of q Stieltjes strings and the spectra of its q strings obtained by clamping the graph at the interior vertex in [22]. In [59], they consider T to be a metric tree with q complementary subtrees T_i, $i = 1, \ldots, q$. If they are given $q+1$ sequences $\{\lambda_k\}_{k=1}^{n}$ and $\{v_{k,i}\}_{k=1}^{n_i}$ such that $n = \sum_1^q n_i$, they can determine the point masses on the edges of T such that the spectra of the corresponding Dirichlet problems on T and T_i are exactly $\{\lambda_k\}_{k=1}^{n}$ and $\{v_{k,i}\}_{k=1}^{n_i}$.

For continuous connected strings, Eckhardt solves the inverse problem for a star graph of connected Krein strings where the known spectral data comprises the spectrum associated with the whole graph and the spectra associated with the individual edges and these coupling matrices ([36]). This coupling matrix is defined as

$$\Gamma_\lambda = (\Gamma_{\lambda,ed})_{e,d\in\mathcal{E}} = \left(\frac{||\phi_e(\lambda,\cdot)||^2_{H_0^2(I_e)}}{||\phi_d(\lambda,\cdot)||^2_{H_0^1(I_d)}} \frac{\dot{\phi}_d(\lambda,0)^2}{\dot{\phi}_e(\lambda,0)^2} \right)_{e,d\in\mathcal{E}_\lambda}$$

for every $\lambda \in \sigma(S)$ for which the set $\mathcal{E}_\lambda$ is nonempty where $\mathcal{E}_\lambda = \{e \in \mathcal{E} | \phi_e(\lambda, 0) = 0\}$ and the rest of the notation is defined in ([36]).

Before we even begin to study the inverse spectral problem for a star graph, let us understand the inverse spectral problem for a single Stieltjes string. Let's assume that we have a string of length l with n beads located at $l_1, l_2, \ldots, l_n$ and that each have a mass of $m_1, m_2, \ldots, m_n$. Let σ be the uniform tension across the entire string. Let $u_i(t)$ be the displacement of the ith bead. The displacement can be modeled by

$$\frac{u_i - u_{i+1}}{l_i} + \frac{u_i - u_{i-1}}{l_{i-1}} - m_i p^2 u_i = 0 \qquad i = 1, 2, \ldots, n \tag{3.4}$$

Example 3.1.1 (Dirichlet on Left Hand Side)**.** *Assume you have (3.4) where the left hand side is fixed Dirichlet. If you know the frequencies of the string, $p_1, p_2, \ldots, p_n$, with Dirichlet boundary condition on the right and the frequencies, $p_1', p_2', \ldots, p_n'$, with Neumann boundary condition on the right, then we can determine the location and mass of each bead. [39]*

Following [39] let's first consider both ends to be fixed. That is, $u_0 = u_{n+1} = 0$. Solving the

recursive relation in (3.4) yields

$$u_i = R_{2i-2}(\lambda)u_1 \qquad i = 1, 2, \ldots, n$$

where $R_{2i-2}(\lambda)$ is of order $i-1$ for $i = 1, 2, \ldots, n+1$ and $\lambda = -p^2$. The sequence of frequencies, $p_1 < p_2 < \cdots < p_n$ are roots of

$$R_{2n}(\lambda) = 0$$

because of the right hand boundary condition. Now these take care of the even R's. The odd R's are found by

$$\begin{aligned} R_{2i-1}(\lambda) &= \frac{R_{2i}(\lambda) - R_{2i-2}(\lambda)}{l_i} \qquad i = 1, 2, \ldots, n \\ \frac{u_{i+1} - u_i}{l_i} &= R_{2i-1}(\lambda)u_1 \qquad i = 1, 2, \ldots, n+1 \end{aligned}$$

The relationship between various R's is as follows:

$$R_{2i-1}(\lambda) = \lambda m_i R_{2i-2}(\lambda) + R_{2i-3}(\lambda) \tag{3.5}$$

$$R_{2i}(\lambda) = l_i R_{2i-1}(\lambda) + R_{2i-2}(\lambda) \tag{3.6}$$

$$R_{-1}(\lambda) = \frac{1}{l_0} \qquad R_0(\lambda) = 1 \tag{3.7}$$

Now in application, when λ_i is a solution of an even $R_{even}(\lambda)$, it gives the amplitude of the beads up to a scalar multiple of u_1. When λ_i is a solution of the odd $R_{odd}(\lambda)$, its the tangent of the angle for threads between beads up to a scalar multiple of u_1.

Since there are explicit formulas for R's, we can uniquely write the rational function as

$$\frac{R_{2n}(\lambda)}{R_{2n-1}(\lambda)} = l_n + \frac{R_{2i-2}(\lambda)}{R_{2n-1}(\lambda)} \tag{3.8}$$

$$= l_n + \cfrac{1}{m_n\lambda + \cfrac{1}{\cfrac{R_{2n-2}(\lambda)}{R_{2i-3}(\lambda)}}} \tag{3.9}$$

$$= l_n + \cfrac{1}{m_n\lambda + \cfrac{1}{l_{n-1} + \cfrac{1}{m_{n-1}\lambda + + \ddots + \cfrac{1}{l_1 + \cfrac{1}{m_1\lambda + \cfrac{1}{l_0}}}}}} \tag{3.10}$$

Now since we can write the right hand side uniquely, then the left hand side is Nevanlinna rational function.

Now let's consider the right hand side have a Neumann boundary condition. That is, $u_{n+1} = u_n$ yields

$$R_{2n}(\lambda) = R_{2n-2}(\lambda)$$

If consecutive even R's are equal, then the odd R between them is equal to zero. That is

$$R_{2n-1}(\lambda) = 0$$

Denote solutions as the sequence $p'_1 < p'_2 < \cdots < p'_n$, where $\lambda'_j = -p'^2_j$ for $j = 1, 2, \ldots, n$. Now

we need to see the corresponding change to V

$$V = \frac{1}{2}\sum_{i=0}^{n}\frac{1}{l_i}(y_{i+1}-y_i)^2 \qquad y_0 = y_{n+1} = 0$$
$$V' = \frac{1}{2}\sum_{i=0}^{n}\frac{1}{l_i}(y_{i+1}-y_i)^2 \qquad y_0 = 0,\ y_n = y_{n+1}$$

Clearly V' is one degree less than V . Now since (3.10) is Nevanlinna and p_i are the zeros and p_i' are the poles, there interlace

$$p_1' < p_1 < p_2' < \cdots < p_n' < p_n$$

Let's rewrite the R's using their zeros.

$$R_{2n}(\lambda) = C\prod_{i=1}^{n}(\lambda - p_i^2) \qquad C > 0$$
$$R_{2n-1}(\lambda) = C'\prod_{i=1}^{n}(\lambda - p_i'^2) \qquad C' > 0$$

Now assume we are only given the sequence $p_1' < p_1 < p_2' < \cdots < p_n' < p_n$, we can formulate the following two polynomials

$$A(\lambda) = A_0\prod_{i=1}^{n}(\lambda - p_i^2)$$
$$B(\lambda) = B_0\prod_{i=1}^{n}(\lambda - p_i'^2)$$

where $A_0 > 0$ and $B_0 > 0$. Then

$$\frac{R_{2n}(\lambda)}{R_{2n-1}(\lambda)} = \rho + \frac{A(\lambda)}{B(\lambda)}$$

Now we can equate both sides and use (3.5, 3.6) and (2.4) to get

$$l_i = \rho a_i \qquad i = 0, 1, \ldots, n$$
$$m_i = \frac{1}{\rho}b_i \qquad i = 1, 2, \ldots, n$$

We also know the total length of the string l. That is, $l = l_0 + l_1 + \cdots + l_n$. Therefore, adding all the l_i's yields

$$\rho = \frac{l}{a_0 + a_1 + \cdots + a_n}$$

This implies that

$$\begin{aligned} l_i &= \frac{a_i}{a_0 + a_1 + \cdots + a_n} l \qquad i = 0, 1, \ldots, n \\ m_i &= \frac{\sigma}{l}(a_0 + a_1 + \cdots + a_n) b_i \qquad i = 1, 2, \ldots, n \end{aligned}$$

Now you get the choice of $A(\lambda)$ and $B(\lambda)$, but the optimal choice is

$$\begin{aligned} A(\lambda) &= \prod_{i=1}^{n} \left(1 + \frac{\lambda}{p_i^2}\right) \\ B(\lambda) &= \prod_{i=1}^{n} \left(1 + \frac{\lambda}{p_i'^2}\right) \end{aligned}$$

and it yields

$$\begin{aligned} l_i &= a_i l \qquad i = 0, 1, \ldots, n \\ m_i &= b_i \frac{\sigma}{l} \qquad i = 1, 2, \ldots, n \end{aligned}$$

Example 3.1.2 (Neumann on Left Hand Side)**.** *Assume you have (3.4) where the left hand side is Neumann. If you know the frequencies of the string, $p_1, p_2, \ldots, p_n$, with Dirichlet boundary condition on the right and the frequencies, $p_1', p_2', \ldots, p_n'$, with Neumann boundary condition on the right, we can determine the location and mass of each bead. [39]*

Following very similar logic and [39], if the right hand side is Dirichlet, then let $u_{n+1} = 0$ and if it is Neumann, let $u_n = u_{n+1}$. In order to differentiate the two problems, let's call the R's Q's. Therefore

$$u_i = Q_{2i-2}(\lambda) u_1 \qquad \lambda = -p^2 \quad Q_0 = 1 \quad i = 1, 2, \ldots, n+1$$

The exact same recursive relationships occur. We let $p_1 < p_2 < \cdots < p_n$ be the solutions of $Q_{2n}(\lambda) = 0$ and let $p'_1 < p'_2 < \ldots < p'_n$ be the solutions of $Q_{2n-1}(\lambda) = 0$. Also

$$\frac{Q_{2n}(\lambda)}{Q_{2n-1}(\lambda)} = l_n + \cfrac{1}{m_n\lambda + \cfrac{1}{l_{n-1} + \cfrac{1}{m_{n-1}\lambda + + \ddots + \cfrac{1}{l_1 + \cfrac{1}{m_1\lambda + \cfrac{1}{l_0}}}}}}$$

Now if we are given the sequence $p'_1 < p_1 < p'_2 < p_2 < \cdots < p'_n < p_n$, we can formulate the functions

$$C(\lambda) = C_0 \prod_{j=1}^{n} (\lambda + p_j^2) \qquad C_0 > 0$$

$$D(\lambda) = D_0 \prod_{j=1}^{n} (\lambda + p_j'^2) \qquad D_0 > 0$$

such that

$$\frac{C(\lambda)}{D(\lambda)} = a_n + \cfrac{1}{b_n\lambda + \cfrac{1}{a_{n-1} + \cfrac{1}{b_{n-1}\lambda + \ddots + \cfrac{1}{a_1 + \cfrac{1}{b_1\lambda + \cfrac{1}{a_0}}}}}}$$

Equating the fractions similar to before yields

$$
\begin{aligned}
l_i &= \frac{a_i}{a_1 + a_2 + \cdots + a_n} l \qquad i = 1, 2, \ldots, n \\
m_i &= \frac{\sigma}{l}(a_1 + a_2 + \cdots + a_n) b_i \qquad i = 1, 2, \ldots, n
\end{aligned}
$$

Now if instead of using the total length, we want to use that we are given the total mass, M, of the entire string, then we need to consider

$$
\frac{D(\lambda)}{\lambda C(\lambda)} = \cfrac{1}{c_n\lambda + \cfrac{1}{b_n + \cfrac{1}{a_{n-1}\lambda + \ddots + \cfrac{1}{b_2 + \cfrac{1}{a_1\lambda + \cfrac{1}{b_1}}}}}}
$$

taking the limit as $\lambda \to 0$ yields

$$
\frac{D_{n-1}}{C_n} = b_1 + b_2 + \cdots + b_n
$$

now the optimal choice is

$$
\begin{aligned}
C(\lambda) &= \prod_{j=1}^{n} \left(1 + \frac{\lambda}{p_j^2}\right) \\
D(\lambda) &= \prod_{j=1}^{n} \left(1 + \frac{\lambda}{p_j'^2}\right)
\end{aligned}
$$

and it yields

$$
\begin{aligned}
m_i &= b_i M \qquad i = 1, 2, \ldots, n \\
l_i &= \sigma \frac{a_i}{M} \qquad i = 1, 2, \ldots, n
\end{aligned}
$$

This work of Krein's was then extended to consider a Stieltjes string where every single mass could be damped. It is shown that we can recover masses, coefficients of damping and the lengths between masses using the two spectra in ([25]).

Throughout the rest of the chapter, we explain the approach of Pivovarchik to solve the inverse problem for connected Stieltjes strings and then we apply Nevanlinna functions theory to characterize the spectra and to solve the inverse problem for a discrete multi-string system in a more general setting.

3.2 Pivovarchik's Approach Inverse Spectral Problem for Multi-String Systems

In order to discuss previous work done by Pivovarchik, we need to introduce his notation ([71],[73], [22],[69],[24],[74],[70],[72],[23],[25]). Consider a plane star graph of q Stieltjes strings, $q \in \mathbb{N}$ and $q \geq 2$, where the strings are connected in the middle with a mass $M \geq 0$ and the outer ends are either fixed, Dirichlet problem, or free to move orthogonally to the equilibrium position, Neumann problem. We label the edges by $j = 1, 2, \ldots, q$ where the jth edge has $n_j > 0$ masses $m_{k,j}$, $k = 1, 2, \ldots, n_j$. Note that the central mass M is not considered to belong to an edge. The masses subdivide the jth edge into $n_j + 1$ intervals of length $l_{k,j}$, $k = 0, 1, \ldots, n_j$ where we count masses and intervals from the exterior towards the center. Note this is where the notation differs. If we want to talk about the entire length of an edge, we denote it $l_j := \sum_{k=0}^{n_j} l_{k,j}$. The total number of masses on the star graph, without the middle mass M, is denoted by $n := \sum_{j=1}^{q} n_j$. We want to talk about the position and displacement of each mass so we will denote the position of the mass $m_{k,j}$ as $x_{k,j}$ with the connection at the middle denoted $x_{n_j+1,j}$, where $j = 1, 2, \ldots, q$ and the outer ends denoted $x_{0,j}$, and it's vertical displacement denoted $v_{k,j}(t)$. Assume stretched by forces each equal to 1.

Then using this new notation, the model for the graph with a mass in the middle and Dirichlet conditions on the boundary is

$$
\begin{aligned}
m_{k,j}\ddot{v}_{k,j}(t) &= \frac{v_{k+1,j}(t) - v_{k,j}(t)}{l_k, j} - \frac{v_{k,j}(t) - v_{k-1,j}(t)}{l_{k-1,j}} \quad k = 1, 2, \ldots, n_j, \;\; j = 1, 2, \ldots, q \\
v_{n_1+1,1}(t) &= v_{n_2,2}(t) = \cdots = v_{n_q+1,q}(t) \\
M\ddot{v}_{n_1+1,1}(t) &= -\sum_{j=1}^{q} \frac{v_{n_1+1,j}(t) - v_{n_j,j}(t)}{l_{n_j,j}} \\
y_{0,j}(t) &= 0 \qquad\qquad j = 1, 2, \ldots, q
\end{aligned}
$$

Proceeding with separation of variables $v_{i,j}(t) = u_{i,j}e^{i\lambda t}$ yields the following difference equations for the displacement amplitudes $u_{i,j}$ for the Neumann and Dirichlet problem.

Neumann Problem (N1) If the central vertex carrying the mass M is allowed to move freely, then

$$
\lambda^2 m_{k,j}\ddot{u}_{k,j} = \frac{u_{k+1,j} - u_{k,j}}{l_k, j} - \frac{u_{k,j} - u_{k-1,j}}{l_{k-1,j}} \quad k = 1, 2, \ldots, n_j, \;\; j = 1, 2, \ldots, q \tag{3.11}
$$

$$
\lambda^2 M\ddot{u}_{n_1+1,1} = -\sum_{j=1}^{q} \left(\frac{u_{n_j+1,j} - u_{n_j,j}}{l_{n_j,j}} \right) \qquad j = 1, 2, \ldots, q \tag{3.12}
$$

$$
u_{n_1+1,1} = u_{n_2+1,2} = \cdots = u_{n_q+q,j} \tag{3.13}
$$

$$
u_{0,j} = 0 \qquad j = 1, 2, \ldots, q \tag{3.14}
$$

Dirichlet Problem (D1) If we clamp all the strings at the middle vertex, then the system decouples into q separate Stieltjes strings problems with Dirichlet conditions at both ends.

$$
\lambda^2 m_{k,j}\ddot{u}_{k,j} = \frac{u_{k+1,j} - u_{k,j}}{l_k, j} - \frac{u_{k,j} - u_{k-1,j}}{l_{k-1,j}} \quad k = 1, 2, \ldots, n_j, \tag{3.15}
$$

$$
u_{n_j+1,j} = 0 \quad j = 1, 2, \ldots, q \tag{3.16}
$$

$$
u_{0,j} = 0 \quad j = 1, 2, \ldots, q \tag{3.17}
$$

Note that if there is no middle mass, we just set $M = 0$ in the appropriate problem.

They also denote the following by

- $n = \sum_{j=1}^{q} n_j$ the number of masses on the star graph no counting the possible center mass M
- $\begin{cases} \{\lambda_k\}_{k=-(n+1),k\neq 0}^{n+1} & \text{if } M > 0 \\ \{\lambda_k\}_{k=-n,k\neq 0}^{n} & \text{if } M = 0 \end{cases}$ $\lambda_{-k} = -\lambda_k$, $\lambda_k \geq \lambda_{k'}$ for $k > k' > 0$, the eigenvalues of the Neumann problem (3.11 -3.14)

- $\{\nu_{k,j}\}_{\kappa=-n_j,\kappa\neq 0}^{n_j}$, $\nu_{-\kappa,j} = -\nu_{\kappa,j}$, $\nu_{\kappa,j} > \nu_{\kappa',j}$ for $\kappa > \kappa' > 0$, the eigenvalues of the Dirichlet problem (3.15-3.17)

- $\{\zeta_{k=-n,k\neq 0}^{n} = \cup_{j=1}^{q}\{\nu_{\kappa,j}\}_{\kappa=-n_j,\kappa\neq 0}^{n_j}$, $\zeta_{-k} = -\zeta_k$,$\zeta_k \geq \zeta_{k'}$ for $k > k' > 0$, the eigenvalues of the Dirichlet problem (3.15-3.17)

We will now investigate the interlacing properties and multiplicities of the eigenvalues of the Dirichlet problem (D1) and Neumann problem (N1). Following [71] and[22] who use the same notation as [39] for each $j = 1, 2, \ldots, q$, we may obtain the solutions $u_{i,j}$, $i = 1, 2, \ldots, n_j + 1$ of (3.15) with the boundary condition (3.17) successively in the form

$$u_{i,j} = R_{2i-2,j}(l_0, \lambda^2)u_{1,j} \quad i = 1, 2, \ldots, n_j + 1 \tag{3.18}$$

where $R_{2i-2,j}(l_0, \lambda^2)$ are polynomials of degree $2i - 2$ which is found by solving (3.15) and

$$u_{i,j} = R_{2i-2,j}(\infty, \lambda^2)u_{1,j} \quad i = 1, 2, \ldots, n_j + 1 \tag{3.19}$$

where $R_{2i-2,j}(l_0\infty, \lambda^2)$ are polynomials of degree $2i - 2$ which is found by solving (3.11). We then set

$$R_{2i-1,j}(\cdot, \lambda^2) := \frac{R_{2i,j}(\cdot, \lambda^2) - R_{2i-2,j}(\cdot, \lambda^2)}{l_i, j} \quad i = 1, 2, \ldots, n_j$$

These polynomials also satisfy the recurrence relations

$$R_{2i-1,j}(\cdot, \lambda^2) = -\lambda^2 m_{i,j} R_{2i-2,j}(\cdot, \lambda^2) + R_{2i-3,j}(\cdot, \lambda^2) \quad i = 1, 2, \ldots, n_j \tag{3.20}$$

$$R_{2i,j}(\cdot, \lambda^2) = l_{i,j} R_{2i-1,j}(\cdot, \lambda^2) + R_{2i-2,j}(\cdot, \lambda^2) \quad i = 1, 2, \ldots, n_j \tag{3.21}$$

$$R_{0,j}(\cdot, \lambda^2) = 1 \quad R_{-1,j}(\cdot, \lambda^2) = \begin{cases} \frac{1}{l_{0,j}} & \text{if } l_0 \in (0, \infty) \\ 0 & \text{if } l_0 = \infty \end{cases} \tag{3.22}$$

Plugging these into the boundary conditions (3.16)-(3.17) (or(3.12-(3.13)) at the central vertex

yields the following system of linear equations for $u_{1,j}$, $j = 1, 2, \ldots, q$

$$R_{2n_1,1}(\cdot,\lambda^2)u_{1,1} = R_{2n_2,2}(\cdot,\lambda^2)u_{1,2} = \cdots = R_{2n_q,q}(\cdot,\lambda^2)u_{1,q} \tag{3.23}$$

$$\sum_{j=1}^{q} R_{2n_j-1,j}(\cdot,\lambda^2)u_{1,j} = M\lambda^2 R_{2n_1,1}(\cdot,\lambda^2)u_{1,q} \tag{3.24}$$

Thus, the spectrum of the Dirichlet problem (3.15)-(3.17) coincides with the zeros of the polynomial

$$\phi_{D,q}(\lambda^2) := \sum_{j=1}^{q}\left[\left(R_{2n_j-1,j}(l_0,\lambda^2) - \frac{M}{q}\lambda^2 R_{2n_j,j}(l_0,\lambda^2)\right)\prod_{k=1,k\neq j}^{q} R_{2n_k,k}(l_0,\lambda^2)\right] \tag{3.25}$$

and the spectrum of the Neumann problem (3.11)-(3.14) coincides with the zeros of the polynomial

$$\phi_{N,q}(\lambda^2) := \sum_{j=1}^{q}\left[\left(R_{2n_j-1,j}(\infty,\lambda^2) - \frac{M}{q}\lambda^2 R_{2n_j,j}(\infty,\lambda^2)\right)\prod_{k=1,k\neq j}^{q} R_{2n_k,k}(\infty,\lambda^2)\right] \tag{3.26}$$

The degree of the polynomials $\phi_{D,q}(z)$ and $\phi_{N,q}(z)$ are both n where n is the total number of masses on the star graph.

Lemma 3.2.1. *There is the following continued fraction expansions ([22])*

$$\frac{R_{2n,j}(l_0,z)}{R_{2n-1,j}(l_0,z)} = l_{n,j} + \cfrac{1}{-m_{n_j,j}z + \cfrac{1}{l_{n_j-1,j} + \cfrac{1}{-m_{n_j-1,j}z + \cdots + \cfrac{1}{l_{1,j} + \cfrac{1}{-m_{1,j}z + \cfrac{1}{l_{0,j}}}}}}} \tag{3.27}$$

They then prove the following theorem

Theorem 5. *The sequences $\{\lambda_k\}_{-n,k\neq 0}^{n}$ and $\{\zeta_k\}_{-n,k\neq 0}^{n}$ interlace as follows:*

- $\zeta_- n \leq \lambda_{-n} \leq \zeta_{-n+1} \leq \cdots \leq \zeta_{-1} < \lambda_{-1} < 0 < \lambda_1 < \zeta_1 \leq \lambda_2 \leq \cdots \leq \zeta_n$
- $\zeta_{k-1} = \lambda_k$ *if and only if* $\lambda_k = \zeta_k$
- *the multiplicity of* ζ_k *does not exceed* q.

After they show the properties of the spectra, they are able to solve the inverse problem

Theorem 6. *Let $l_j > 0 (j = 1, 2, \ldots, q)$ be given. Let the sequences of real numbers $\{\lambda_k\}_{k=-n,k\neq 0}^{n}$, $\left\{\nu_k^{(j)}\right\}_{k=-n_j,k\neq 0}^{n_j}$ $\left(j = 1, 2, \ldots, q, n = \sum_{j=1}^{q} n_j\right)$ satisfy the conditions*

1. *$\lambda_{-k} = -\lambda_k$ for each $k; \lambda_k < \lambda_{k'}$ if $k < k', \nu_{-k}^{(j)} = -\nu_k^{(j)}$ for each k and each j; and for each $j = 1, 2, \ldots, n_j, \nu_k^{(j)} < \nu_{k'}^{(j)}$ if $k < k'$*

2. *$\{\lambda_k\}_{k=-n,k\neq 0}^{n} \cap \left\{\nu_k^{(j)}\right\}_{k=-n_j,k\neq 0}^{n_j} = \emptyset$ for $j = 1, 2, \ldots, q$, and $\left\{\nu_k^{(j)}\right\}_{k=-n_j,k\neq 0}^{n_j} \cap \left\{\nu_k^{(s)}\right\}_{k=-n_s,k\neq 0}^{n_s} = \emptyset$ for $j, s \in \{1, 2, \ldots, q\}$ and $j \neq s$*

3. *Elements of the set $\{\zeta_k\}_{k=-n}^{n} = \text{def}\{0\} \cup_{j=1}^{q} \left\{\nu_k^{(j)}\right\}_{k=-n_j,k\neq 0}^{n_j}$ are indexed in such a way that $\zeta_{-k} = -\zeta_k$ for each $k; \zeta_k < \zeta_{k'}$ if $k < k'$ interlace with elements of $\{\lambda_k\}_{k=-n}^{n}, k \neq 0$ (3.1) $\zeta_{-n} < \lambda_{-n} < \zeta_{-n+1} < \cdots < \lambda_{-1} < 0 < \lambda_1 < \zeta_1 < \cdots < \zeta_n$*

Then there exist a unique collection of sets $\left\{m_k^{(j)}\right\}_{k=1}^{n_j}, \quad (j = 1, 2, \ldots, q), \left\{l_k^{(j)}\right\}_{k=0}^{n_j}$ $(j = 0, 1, 2, \ldots, q)$ such that $\sum_{k=0}^{n_j} l_k^{(j)} = l_j$, which generate problems $(2.2) - (2.5)$ and $(2.8) - (2.10)$ with the spectra $\{\lambda_k\}_{k=-n,k\neq 0}^{n}, \left\{\nu_k^{(j)}\right\}_{k=-n_j,k\neq 0}^{n_j}$, respectively. [22], [71]

Möller and Pivovarchik give necessary and sufficient conditions on a sequence of complex number to be the spectrum of a problem consisting of multiple Stieltjes strings damped in the middle at the joining point ([66]). They followed the procedure of clamping the middle and considering the spectrum of the N strings individually. Then they consider the spectrum with the middle not damped and finally discuss the changes to the spectrum if there is damping present.

3.3 Nevaninlinna Approach to Inverse Spectral Problem for Multi-String Systems

Now we know that the Nevanlinna function $\theta(\lambda)$ yields the entire spectrum of the problem not counting multiplicity. Let's consider Dirichlet conditions on each edge with a middle bead so we can say that

$$u_{1,j} = \theta_{n_j} u_0 \qquad j = 1, 2, \ldots, N$$

Then the spectrum solves

$$\lambda u_0 = -Nu_0 + \sum_{j=1}^{N} \theta_{n_j} u_0$$
$$\rightarrow 0 = (\lambda + N) - \sum_{j=1}^{N} \theta_{n_j} = \Theta(\lambda)$$

The structure of this $\Theta(\lambda)$ allows us to solve the inverse problem.

Theorem 7. *Assuming $m = l = T = 1$, given the spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ for N connected Stieltjes strings with a middle mass, we can determine the number of masses on each strand.*

Note that the model cannot distinguish between the swapping strings.

Proof. Let's take a step back and recall that the eigenvalue problem for the middle mass is

$$\lambda + N - \sum_{j=1}^{N} \theta_{n_j}(\lambda) = 0 \tag{3.28}$$

Since each $\theta_{n_j}(\lambda)$ is anti-Nevanlinna function, (3.28) is a Nevanlinna function. Let us represent it as

$$\Theta(\lambda) = \frac{f(\lambda)}{g(\lambda)} := \lambda + N - \sum_{j=1}^{N} \theta_{n_j}(\lambda) = 0 \tag{3.29}$$

Because of the nature of the θ's, the degree of $f(\lambda)$ will be one larger than the degree of $g(\lambda)$. We also know that the eigenvalues are the zeros and poles of (3.29) and that they interlace such that if you put them in increasing order they follow the pattern zero, pole, zero, pole, ..., zero.

Now assume we are given $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ and N. Write the λ's in increasing order and renumber so that

$$\lambda_{1,z} < \lambda_{2,p} < \lambda_{3,z} < \lambda_{4,p} < \cdots < \lambda_{k,z}$$

The subscript z means that eigenvalue is a zero of the desired rational function while the subscript p means that eigenvalue is a pole. Let $p(x) = (x - \lambda_{1,z})(x - \lambda_{3,z}) \cdots (x - \lambda_{k,z})$ and $q(x) = (x -$

$\lambda_{2,p})(x - \lambda_{4,p}) \cdots (x - \lambda_{k-1,p})$. We know that

$$\frac{p(x)}{q(x)} = x + N - \frac{p_1(x)}{q(x)}. \tag{3.30}$$

Now we just need to break $\frac{p_1(x)}{q(x)}$ into N θ functions. The denominators of the θ's are distinct so we can factor $q(x)$ such that it's factors are denominators of θ's. Using a modified partial fraction decomposition in that we only want denominators of the form of θ's denominators, decompose $\frac{p_1(x)}{q(x)}$ into the sum of θ's. Therefore

$$\frac{p(x)}{q(x)} = x + N - \theta_{1,a_1}(x) - \theta_{2,a_2}(x) - \cdots - \theta_{N,a_N}(x) \tag{3.31}$$

where $\theta_{i,a_i}(x)$ means its the ith θ function with a_i beads on that string. Thus, we have N a_i's and since each a_i is the number of beads on that strand, we have the number of beads on each strand. Remember that the numbering of the strands is arbitrary. $\square$

Chapter 4

Application of Operator Couplings in Spectral Analysis

4.1 Introduction

A **Hilbert space** H is an inner product space that is a complete metric space with respect to the metric generated by the inner product. Let us first consider $L^2(a,b)$ where (a,b) is any measurable set on the real axis. $L^2(a,b)$ is the set of all complex valued Lebesgue measurable functions f defined on (a,b) such that $|f|^2$ is Lebesgue integrable on (a,b). [1]

Example 4.1.1. *$L^2(a,b)$ is a Hilbert space with the inner product defined as $(f,g) = \int_a^b f(t)\overline{g(t)}dt$.*

Definition 4.1.1. *Let D denote a subset of the space H. A function T which relates to each element $f \in D$ a particular element $Tf = g \in H$ is called an* **operator** *in H with domain D. The set H' consisting of all $g = Tf$, where f runs through D is called the* **range** *of T. [1]*

Let us now look at the model in terms of operator theory. There are some properties of operators that we need to consider.

Definition 4.1.2. *An operator T is* **linear** *if its domain of definition D is a linear manifold and if*

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg$$

for any $f, g \in D$ and any complex numbers α, β.

Note that this definition does not require the operator to be bounded. Therefore, we say

Definition 4.1.3. *A linear operator T is* **bounded** *if*

$$\sup_{f \in D_T, ||f||_H \leq 1} ||Tf|| < \infty$$

.

We use this definition to define the norm of a bounded linear operator to be

$$||T|| = \sup_{||f||_H = 1} ||Tf||_H = \sup_{f \in D} \frac{||Tf||}{||f||}$$

where $x \in D_T$.

Thus we can make the following connections. [1]

- A bounded linear operator is continuous.
- If a linear operator is continuous at one point, then it is bounded.
- If S and T are linear operators, then $\alpha S + \beta T$, where $\alpha, \beta \in \mathbb{C}$, is a linear operator with the domain of definition $D_S \cap D_T$. Each product ST and TS is also a linear operator. If S and T are bounded linear operators defined everywhere in$H$, then the operators ST and TS are also bounded linear operators defined everywhere in$H$, and $||ST|| \leq ||S|| \cdot ||T||$ and $||TS|| \leq ||T|| \cdot ||S||$.

Definition 4.1.4. *We say that an operator T, is* **continuous** *at a point $f_0, f \in D_T$ if*

$$\lim_{f \to f_0} Tf = Tf_0$$

. Equivalently, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if f satisfies the inequality

$$||f - f_0|| < \delta$$

then

$$||Tf - Tf_0|| < \epsilon$$

.

Note that if a linear operator is continuous at one point, then it is bounded. By continuous at one point, x_0, we mean that if $x_1, x_2, \ldots \to x_0$.

Definition 4.1.5. *An operator T, not necessarily linear, is* **closed** *if the relations for $f_n \in D_T$*

1. $\lim_{n\to\infty} f_n = f$, *and*
2. $\lim_{n\to\infty} Tf_n = g$

imply that $f \in D_T$ and $Tf = g$.

Thus if an operator is continuous, then it is closed. An operator is linear and continuous if and only if it is bounded and linear. Also, if an operator that is linear and continuous at one point, then it is continous at any point. Since differential operators are unbounded, they are not continous but are closed. If T is a bounded linear operator defined on H, the expression

$$(f, Tg)$$

defines a bilinear functional on H with the norm $||T||$. There exists a unique bounded linear operator T^* defined on H with norm $||T^*|| = ||T||$ such that

$$(f, Tg) = (T^* f, g)$$

for $f, g \in H$. The operator T^* is then called the **adjoint** of T. If T is bounded and $T^* = T$, then T is said to be **self-adjoint**. Similar to matrices, it is said to be **normal** if the operator commutes with its adjoint, i.e. $TT^* = T^*T$.

Theorem 8. *If T is a bounded self-adjoint operator, then*

$$\sup_{||f||=||g||=1} |(Tf, g)| = \sup_{||f||=1} |(Tf, f)|$$

In other words,

$$||A|| = \max\{|\gamma|, |\lambda|\}$$

where

$$\gamma = \sup_{||f||=1} (Tf, f), \qquad \lambda = \inf_{||f||=1} (T, f)$$

Theorem 9. *Let T and T^* be linear operators defined on H and assume that*

$$(Tf, g) = (f, T^* g)$$

for $f, g \in H$. Then T is bounded and T^* is the adjoint of T.

Now let's define the more general definition of the adjoint operator.

Definition 4.1.6. *Now if T is an unbounded linear operator, we can say that*

$$(Tf, g) = (f, g^*) \tag{4.1}$$

for $f \in D_T$. If D_T is dense in H, then the operator T has an adjoint operator T^*. The domain, D_{T^*} is defined: $g \in D_{T^*}$ if and only if there exists a vector g^* such that (4.1) is satisfied for $f \in D_{T^*}$. That is,

$$T^* g = g^*$$

Now there are several properties that follow direction from the definition.

- The operator T^* is linear.
- If $S \subset T$, then $S^* \supset T^*$.
- The operator T^* is closed whether or not T is closed.
- If the operator T has a closure $\overline{T}$, then $(\overline{T})^* = T^*$.
- If the operator T^{**} exists, then $T \subset T^{**}$.

Definition 4.1.7. *A linear operator T is said to be* **symmetric/Hermitian** *if*

- *its domain D_T is dense in H, and*
- *for $f, g \in D_T$, $(Tf, g) = (f, Tg)$*

If T is a symmetric operator, then $T \subset T^*$. Therefore a symmetric operator always has a closure. An operator T for which $T = T^*$ is said to be **self-adjoint**.

Theorem 10. *A symmetric operator T such that its range is all of H is called self-adjoint.*

Proof. It is sufficient to verify that every element $g \in D_{T^*}$ also belongs to D_T. Let $g \in D_{T^*}$ and $T^*g = g^*$. Since range is H, there exists an element $h \in D_T$ such that $Th = g^*$. Consequently, for each $f \in D_T$

$$(Tf, g) = (f, g^*) = (f, Th) = (Af, h)$$

Since range of T is H, we have $g = h$. Therefore $g \in D_T$. □

Example 4.1.2. *$L[y] = -y''$ with $y(0) = y(1) = y'(1) = y'(0) = 0$ is not self adjoint since $dom(L) \neq dom(L^*)$*

Example 4.1.3. *$L[y] = -y''$ with $y(0) = y(1) = 0$ is self adjoint since $dom(L) = dom(L^*)$*

Definition 4.1.8. *A complex number λ is called an* **eigenvalue** *of the linear operator T if there exists an $f \in D_T$, $f \neq 0$ such that*

$$Tf = \lambda f.$$

[1]

Theorem 11. *The eigenvalues of a symmetric operator are real.*

Proof. If $Tf = \lambda f$ for $f \neq 0$, then

$$\lambda(f, f) = (\lambda f, f) = (Tf, f) = (f, Tf) = (f, \lambda f) = \overline{\lambda}(f, f)$$

So $\lambda = \overline{\lambda}$ which is only true if $\lambda \in \mathbb{R}$. □

4.1.1 Boundary Spaces

Since we are interested in connecting strings, we need to consider coupling the corresponding differential operators.

Definition 4.1.9. *A linear space L with the inner product defined as $[\varphi_1, \varphi_2]$ is called a* **boundary space of the operator** *A if there exists a linear operator $\Gamma : dom(A) \to L$ such that for any function $f \in dom(A)$*

$$[\Gamma f, \Gamma f] = \frac{1}{i}\left([Af, f] - [f, Af]\right)$$

It is important to note that the domain includes the boundary conditions. If you can construct Γ, then you have L. Let's look at some examples to get some intuition for constructing boundary spaces.

Example 4.1.4. *Working in the Hilbert space, $H = L^2(a,b)$ with the operator A given by the differential operator $Ay = i\frac{dy}{dx}$ with the boundary condition $y(a) = 0$. The domain of A consists of absolutely continuous functions, $f(x)$, on $[a,b]$ such that $f' \in L^2(a,b)$ and $f(a) = 0$.*

Then for any function from the domain of A,

$$\begin{aligned}
\frac{1}{i}[(Af, f) - (f, Af)] &= \frac{1}{i}\left[\int_a^b if'\overline{f}ds - \int_a^b f\overline{if}'dx\right] \\
&= \frac{1}{i}\left[if\overline{f}|_a^b - i\int_a^b f\overline{f}'dx + i\int_a^b f\overline{f}'dx\right] \\
&= f\overline{f}|_a^b \\
&= f(b)\overline{f}(b) - f(a)\overline{f}(a) \\
&= f(b)\overline{f}(b)
\end{aligned}$$

Therefore we can see that the boundary space of the operator A is 1-dimensional with the inner product $[x_1, x_2] = x_1\overline{x_2}$.

The boundary operator Γ acts as $\Gamma_A f = \alpha f(b)$ where α is any complex number such that $|\alpha| = 1$.

Example 4.1.5. *Working in the Hilbert space, $H = L^2(a,b)$ with the adjoint operator A^* given by the differential operator $-A^*g = -i\frac{dg}{dx}$ with the boundary condition $g(b) = 0$. The domain of A^* consists of absolutely continuous functions, $f(x)$, on $[a,b]$ such that $f' \in L^2(a,b)$ and $f(b) = 0$.*

Then for any function from the domain of A^*, following almost identical computations as

Example 4.1.4

$$\begin{aligned}\frac{1}{i}[(-A^*g,g)-(g,-A^*g)] &= \frac{1}{i}\left[\int_a^b -ig'\overline{g}ds - \int_a^b g(\overline{-ig'})dx\right] \\ &= g(a)\overline{g}(a)\end{aligned}$$

Thus we have the same one dimensional boundary space as for A with the inner product $[x_1, x_2] = x_1\overline{x_2}$.

The boundary operator Γ acts as $\Gamma_{-A^*}g = \beta g(a)$ where β is any complex number such that $|\beta| = 1$.

Notice that Example 4.1.4 and Example 4.1.5 are related in that one is the adjoint of the other. They have the same boundary space and very similar boundary operators which aligns with our intuition.

Example 4.1.6. *Working in the Hilbert space, $H = L^2_{(a,b)}$ with the operator A given by the differential operator $Af = -f''$ with the boundary condition $f(a) = f'(a) = 0$.*

Then for any function from the domain of A,

$$\begin{aligned}\frac{1}{i}[(Af,f)-(f,Af)] &= \frac{1}{i}\left[\int_a^b -f''\overline{f}ds - \int_a^b -f\overline{f}''dx\right] \\ &= \frac{1}{i}\left[-f'\overline{f}|_a^b + \int_a^b f'\overline{f}'dx + f\overline{f}'|_a^b - \int_a^b f'\overline{f'}dx\right] \\ &= \frac{1}{i}\left[f\overline{f'}|_a^b - f'\overline{f}|_a^b\right] \\ &= \frac{1}{i}\left[f(b)\overline{f'(b)} - f(a)\overline{f'(a)} - f'(b)\overline{f(b)} + f'(a)\overline{f(a)}\right] \\ &= \frac{1}{i}\left[f(b)\overline{f'(b)} - f'(b)\overline{f(b)}\right] \\ &= if'(b)\overline{f(b)} - if(b)\overline{f'(b)}\end{aligned}$$

The boundary space for this operator is two dimensional. Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ such that it has the inner product $[\vec{x}, \vec{y}] = (J\vec{x}, \vec{y}) = \vec{y}^* J\vec{x}$.

The boundary operator Γ act as $\Gamma_A f = M \begin{pmatrix} f(b) \\ f'(b) \end{pmatrix}$. Now since we are two dimensional, we don't need the magnitude of M to be 1. Instead, we need $(J\vec{x}, \vec{y})$

$$\begin{aligned} [\vec{x}, \vec{y}] &= (J\vec{x}, \vec{y}) \\ [\Gamma\vec{x}, \Gamma\vec{y}] &= (JM\vec{x}, M\vec{y}) \\ &= (M^* JM\vec{x}, \vec{y}) \end{aligned}$$

This tells us that instead of magnitude being 1, we need $M^* JM = J$.

Example 4.1.7. *Working in the Hilbert space, $H = L^2(a,b)$ with the operator A^* given by the differential operator $-A^* g = g''$ with the boundary condition $g(b) = g'(b) = 0$.*

Then for any function from the domain of A^*,

$$\begin{aligned} \frac{1}{i}[(-A^* g, g) - (g, -A^* g)] &= \frac{1}{i}\left[\int_a^b g''\overline{g} ds - \int_a^b g\overline{g}'' dx\right] \\ &= \frac{1}{i}\left[g'\overline{g}|_a^b - \int_a^b g'\overline{g}' dx - g\overline{g}'|_a^b + \int_a^b g'\overline{g'} dx\right] \\ &= ig'(a)\overline{g(a)} - ig(a)\overline{g'(a)} \end{aligned}$$

The boundary space for this operator is two dimensional. Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ such that it has the inner product $[\vec{x}, \vec{y}] = (J\vec{x}, \vec{y}) = \vec{y}^* J\vec{x}$. Notice that this is the same J as in Example 4.1.6.

The boundary operator Γ act as $\Gamma_{A^*} g = \tilde{M} \begin{pmatrix} g(a) \\ g'(a) \end{pmatrix}$. Now since we are two dimensional, we need $\tilde{M}^* J\tilde{M} = J$.

So what does this M and $\tilde{M}$ look like? From Example 4.1.6 we have $M^* JM = J$ and from Example 4.1.7 we have $\tilde{M}^* J\tilde{M} = J$. They will be part of the same class of matrices, so let's just

consider $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We want to know what the conditions are for M such that $M^* J M = J$.

$$\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

For the above to be true, one of the following situations must be satisfied

- if $a, b, c, d \in \mathbb{R}$, then $ad - bc = 1$
- if $a, b, c, d \in \mathbb{C}$, then $\frac{\overline{a}}{a} = \frac{\overline{c}}{c}$, $\frac{\overline{b}}{b} = \frac{\overline{d}}{d}$, $\overline{a}d - b\overline{c} = 1$, and $a\overline{d} - \overline{b}c = 1$

These conditions represent the entire class of boundary operators that satisfy the operator A and A^*. Therefore, Example 4.1.6 and 4.1.7 can have any matrix M that satisfies the above conditions.

Example 4.1.8. *Working in the Hilbert space, $H = L^2\left(0, \frac{\pi}{2}\right)$ with the operator A_1 given by the differential operator $A_1 f_1 = -f_1''$ with the boundary condition $f_1(0) = 0$.*

Then for any function from the domain of A_1,

$$\begin{aligned} \frac{1}{i}[(A_1 f, f) - (f, A_1 f)] &= \frac{1}{i}\left[\int_0^{\pi/2} -f''\overline{f} ds - \int_0^{\pi/2} -f\overline{f}'' dx\right] \\ &= if'\left(\frac{\pi}{2}\right)\overline{f\left(\frac{\pi}{2}\right)} - if\left(\frac{\pi}{2}\right)\overline{f'\left(\frac{\pi}{2}\right)} \end{aligned}$$

The boundary space for this operator is two dimensional. Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ such that it has the inner product $[\vec{x}, \vec{y}] = (J\vec{x}, \vec{y}) = \vec{y}^* J \vec{x}$. Again, this is the same reoccurring J.

The boundary operator Γ acts as $\Gamma_{A_1} f_1 = M_1 \begin{pmatrix} f_1\left(\frac{\pi}{2}\right) \\ f_1'\left(\frac{\pi}{2}\right) \end{pmatrix}$. Now since we are two dimensional, we need $M_1^* J M_1 = J$.

Example 4.1.9. *Working in the Hilbert space, $H = L^2_{\left(\frac{\pi}{2}, \pi\right)}$ with the operator A_2 given by the differential operator $A_2 f_2 = f_2''$ with the boundary condition $f_2(\pi) = 0$.*

Then for any function from the domain of A_2,

$$\begin{aligned}\frac{1}{i}[(A_2 f, f) - (f, A_2 f)] &= \frac{1}{i}\left[\int_{\pi/2}^{\pi} f''\overline{f}ds \int_{\pi/2}^{\pi} f\overline{f}''dx\right] \\ &= if'\left(\frac{\pi}{2}\right)\overline{f\left(\frac{\pi}{2}\right)} - if\left(\frac{\pi}{2}\right)\overline{f'\left(\frac{\pi}{2}\right)}\end{aligned}$$

The boundary space for this operator is two dimensional. Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ such that it has the inner product $[\vec{x}, \vec{y}] = (J\vec{x}, \vec{y}) = \vec{y}^* J\vec{x}$.

The boundary operator Γ act as $\Gamma_{A_2} f_2 = M_2 \begin{pmatrix} f_2\left(\frac{\pi}{2}\right) \\ f_2'\left(\frac{\pi}{2}\right) \end{pmatrix}$. Now since we are two dimensional, we need $M_2^* J M_2 = J$.

The boundary spaces are the same for A_1 and A_2 (two dimensional with indefinite metric) $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ such that it has the inner product $[\vec{x}, \vec{y}] = (J\vec{x}, \vec{y}) = \vec{y}^* J\vec{x}$. Therefore we have a class of boundary operators such that we need we need $M_1^* J M_1 = J$ and $M_2^* J M_2 = J$. Since J is the same as previous examples, it has the same conditions.

In this chapter, we will represent multi-string vibrating systems using a coupling of non-densely defined symmetric operators acting in the infinite dimensional Hilbert space. This coupling is defined by a special set of boundary operators acting in finite dimensional Krein space (the space with indefinite inner product). The coupling of operators are coupling of two continuous operators, two discrete operators, and a continuous and discrete operator.

4.2 Operator Couplings

Definition 4.2.1. *The linear operator A acting in the orthogonal sum of Hilbert spaces $H = H_1 \oplus H_2$ is called a* **simple coupling of the linear operators** A_1 **and** A_2*, acting in H_1 and H_2 respectively, if $P_k\,(dom(A)) = dom(A_k)$, $k = 1, 2, \ldots$ and $P_k A f_{k=1,2} = A_k P_k f$, where P_k is the projecting operator from H to H_k.*

Let A_1, A_2 be symmetric operators in H_1 and H_2. Let A_1^*, $-A_2^*$ have the same boundary space

then the simple coupling of A_1^* and A_2^* defined by the condition $\Gamma_{A_1^*} f_1 = \Gamma_{-A_2^*} f_2$ will be a self-adjoint operator in $H = H_1 \oplus H_2$ according to Strauss.

We will now build these coupling operators and the corresponding boundary space. We will do this by being given A_1 and A_2 then determining the boundary space and then build a simple coupling that is self-adjoint.

Example 4.2.1. *Coupling of Example 4.1.8 & 4.1.9 Let $A_1 f_1 = -f_1''$ with $dom(A_1)$ is $f_1(0) = 0$ and $f_1(\frac{\pi}{2}) = f_1'(\frac{\pi}{2}) = 0$. Let $A_2 f_2 = -f_2''$ with $dom(A_2)$ is $f_2(\pi) = 0$ and $f_2(\frac{\pi}{2}) = f'(\frac{\pi}{2}) = 0$. Both A_1 and A_2 are symmetric but not self-adjoint.*

We need to first determine A_1^*, $-A_2^*$ and their corresponding domains.

$$\begin{aligned}
\langle A_1 f_1, g_1 \rangle &= \int_0^{\frac{\pi}{2}} -f_1'' \overline{g_1} dx = -f_1' \overline{g_1}|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f_1' \overline{g_1}' dx \\
\langle f_1, A_1^* g_1 \rangle &= \int_0^{\frac{\pi}{2}} -f_1 \overline{g_1}//dx = -f_1 \overline{g_1}'|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f_1 \overline{g_1}' dx \\
\langle A_1 f_1, g_1 \rangle = \langle f_1, A_1^* g_1 \rangle &\Rightarrow f_1'(0) \overline{g_1(0)} = 0
\end{aligned}$$

Therefore, $A_1^* f_1 = -f_1''$ where $dom(A_1^*)$ is $\overline{g_1(0) = 0}$.

$$\begin{aligned}
\langle A_2 f_2, g_2 \rangle &= \int_{\frac{\pi}{2}}^{\pi} -f_2'' \overline{g_2} dx = -f_2'' \overline{g_2}|_{\frac{\pi}{2}}^{\pi} + \int_{\frac{\pi}{2}}^{\pi} f_2' \overline{g_2}' dx \\
\langle f_2, A_2^* f_2 \rangle &= \int_{\frac{\pi}{2}}^{\pi} -f_2 \overline{g_2}'' dx = -f_2 \overline{g_2}'|_{\frac{\pi}{2}}^{\pi} + \int_{\frac{\pi}{2}}^{\pi} f_2' \overline{g_2}' dx \\
\langle A_2 f_2, g_2 \rangle = \langle f_2, A_2^* g_2 \rangle &\Rightarrow -f_2'(\pi) \overline{g_2(\pi)} = 0
\end{aligned}$$

Therefore $A_2^* g_2 = -g_2''$ or $-A_2^* g_2 = g_2''$ with $dom(A_2^*)$ is $g_2(\pi) = 0$.
Now we need to determine $\Gamma_{A_1^*}$ and $\Gamma_{-A_2^*}$ using the definition of Boundary Space of a Differential Operator. We use the equation,

$$[\Gamma_A f, \Gamma_A f] = \frac{1}{i} \left([Af, f] - [f, Af] \right)$$

for both A_1^* and $-A_2^*$.

$$
\begin{aligned}
[\Gamma_{A_1^*}f, \Gamma_{A_1^*}f] &= \frac{1}{i}\left[(A_1^*f, f) - (f, A_1^*f)\right] \\
&= \frac{1}{i}\left[-\int_0^{\frac{\pi}{2}} f''\overline{f}dx + \int_{\frac{\pi}{2}}^{\pi} f\overline{f}''dx\right] \\
&= \frac{1}{i}\left[-f'\overline{f}|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} f'\overline{f}'dx + f\overline{f}'|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} f'\overline{f}'dx\right] \\
&= \frac{1}{i}\left[f\left(\frac{\pi}{2}\right)\overline{f'}\left(\frac{\pi}{2}\right) - f'\left(\frac{\pi}{2}\right)\overline{f}\left(\frac{\pi}{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
[\Gamma_{-A_2^*}f, \Gamma_{-A_2^*}f] &= \frac{1}{i}\left[(-A_2^*f, f) - (f, -A_2^*f)\right] \\
&= \frac{1}{i}\left[\int_{\frac{\pi}{2}}^{\pi} f''\overline{f}dx - \int_{\frac{\pi}{2}}^{\pi} f\overline{f''}dx\right] \\
&= \frac{1}{i}\left[f'\overline{f}|_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} f'\overline{f'}dx - f\overline{f'}|_{\frac{\pi}{2}}^{\pi} + \int_{\frac{\pi}{2}}^{\pi} f'\overline{f'}dx\right] \\
&= \frac{1}{i}\left[f\left(\frac{\pi}{2}\right)\overline{f'}\left(\frac{\pi}{2}\right) - f'\left(\frac{\pi}{2}\right)\overline{f}\left(\frac{\pi}{2}\right)\right]
\end{aligned}
$$

So now we need to determine $\Gamma_{A_1^*}$ and $\Gamma_{-A_2^*}$ which is our choice. Since we are in a continuous space, we need $\Gamma_{A_1^*}f_1 = \Gamma_{-A_2^*}f_2$.

Let's define $\Gamma_{A_1^*}f = \begin{pmatrix} f\left(\frac{\pi}{2}\right) \\ f'\left(\frac{\pi}{2}\right) \end{pmatrix}$ with the indefinite inner product $\langle x, y\rangle_{L_1} = \langle x, M_1 y\rangle$ with $M_1 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. Let's define $\Gamma_{-A_2^*}f = \begin{pmatrix} f\left(\frac{\pi}{2}\right) \\ f'\left(\frac{\pi}{2}\right) \end{pmatrix}$ with the indefinite inner product space $\langle x, y\rangle_{L_2} = \langle x, M_2 y\rangle$ with $M_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. Both M_1 and M_2 are self-adjoint. Therefore, we have a simple coupling of the operators A_1^* and A_2^* in $H = H_1 \oplus H_2$. For $k = 1, 2$, clearly $P_k(dom(A)) = dom(A_k)$, $P_1 Af = A_1^* P_1 f$, and $P_2 Af = A_2^* P_2 f$.

All that is left is to show that A, which is a simple coupling of A_1^* and A_2^* defined by the condition $\Gamma_{A_1^*}f_1 = \Gamma_{-A_2^*}f_2$, is self-adjoint. Note that A is $Af = -f''$ with $f(0) = 0$ and $f(\pi) = 0$ as well as

$\Gamma_{A_1^*} f_1 = \Gamma_{-A_2^*} f_2$.

$$
\begin{aligned}
\langle Af, g\rangle &= \int_0^{\frac{\pi}{2}-} -f''\overline{g}dx + \int_{\frac{\pi}{2}+}^{\pi} -f''\overline{g}dx \\
&= -f'\overline{g}|_0^{\frac{\pi}{2}-} - f'\overline{g}|_{\frac{\pi}{2}+}^{\pi} + \int_0^{\pi} f'\overline{g}'dx \\
&= -f'\left(\frac{\pi}{2}-\right)\overline{g\left(\frac{\pi}{2}-\right)} + f'(0)\overline{g(0)} - f'(\pi)\overline{g(\pi)} \\
&\quad + f'\left(\frac{\pi}{2}+\right)\overline{g\left(\frac{\pi}{2}+\right)} + \int_0^{\pi} f'\overline{g}'dx \\
&= f'(0)\overline{g(0)} - f'(\pi)\overline{g(\pi)} + \int_0^{\pi} f'\overline{g}'dx \\
\langle f, A^*g\rangle &= \int_0^{\frac{\pi}{2}-} -f\overline{g}''dx + \int_{\frac{\pi}{2}+}^{\pi} -f\overline{g}''dx \\
&= -f\overline{g}'|_0^{\frac{\pi}{2}-} - f\overline{g}'|_{\frac{\pi}{2}+}^{\pi} + \int_0^{\pi} f'\overline{g}'dx \\
&= -f\left(\frac{\pi}{2}-\right)\overline{g'\left(\frac{\pi}{2}-\right)} + f(0)\overline{g'(0)} - f(\pi)\overline{g'(\pi)} \\
&\quad + f\left(\frac{\pi}{2}+\right)\overline{g'\left(\frac{\pi}{2}+\right)} + \int_0^{\pi} f'\overline{g}'dx \\
&= \int_0^{\pi} f'\overline{g}'dx \\
\langle Af, g\rangle = \langle f, A^*g\rangle &\rightarrow f'(0)\overline{g(0)} - f'(\pi)\overline{g(\pi)} = 0
\end{aligned}
$$

Therefore, $dom(A^*)$ is $g(0) = 0$ and $g(\pi) = 0$. This yields $dom(A) = dom(A^*)$ and therefore A is self-adjoint.

Example 4.2.2. *Operator T_1 is acting in $\mathbb{C}^{m+2}$ such that $(T_1u)_j = a_{j-1}^1 u_{j-1} + b_j^1 u_j + a_j^1 u_{j+1}$ $j = 0, 1, 2, \ldots, m$ with the boundary condition $u_{-1} = 0$. Note that for $\mathbb{C}^{m+2}$, $(u_0, u_1, \ldots, u_m, u_{m+1})$.*

Let operator T_2 act in $\mathbb{C}^{n+2}$ such that $(T_2u)_j = a_{j-1}^2 u_{j-1} + b_j^2 u_j + a_j^2 u_{j+1}$, $j = 0, 1, \ldots, n$ with the boundary condition $u_{n+1} = 0$. Note that for $\mathbb{C}^{n+2}$, $(u_{-1}, u_0, \ldots, u_n)$.

Then for any vector from the domain of T_1

$$
\begin{aligned}
\left[\Gamma_{T_1}u^1, \Gamma_{T_1}u^1\right] &= \frac{1}{i}\left[(T_1u^1, u^1) - (u^1, T_1u^1)\right] \\
&= ia_m^1\left(u_{m+1}^1\overline{u}_m^1 - u_m^1\overline{u}_{m+1}^1\right)
\end{aligned}
$$

The boundary space for this operator is two dimensional and thus has the following inner

product in the two dimensional boundary space $[\vec{x},\vec{u}] = (J\vec{x},\vec{y}) = \vec{y}^* J\vec{x}$ where $J = \begin{pmatrix} 0 & ia^1_m \\ -ia^1_m & 0 \end{pmatrix}$.

Then the boundary operator Γ acts as $\Gamma_{T_1} u^1 = M \begin{pmatrix} u^1_m \\ u^1_{m+1} \end{pmatrix}$ with the condition that $M^* J M = J$.

Then for any vector from the domain of $-T_2$

$$\begin{aligned} \left[\Gamma_{-T_2} u^2, \Gamma_{-T_2} u^2\right] &= \frac{1}{i}\left[(-T_2 u^2, u^2) - (u^2, -T_2 u^2)\right] \\ &= ia^2_{-1}\left(u^2_0 \overline{u}^2_{-1} - u^2_{-1}\overline{u}^2_0\right) \end{aligned}$$

The boundary space for this operator is two dimensional and thus has the following inner product in the two dimensional boundary space $[\vec{x},\vec{u}] = (J\vec{x},\vec{y}) = \vec{y}^* J\vec{x}$ where $J = \begin{pmatrix} 0 & ia^2_{-1} \\ -ia^2_{-1} & 0 \end{pmatrix}$.

Then the boundary operator Γ acts as $\Gamma_{-T_2} u^2 = M \begin{pmatrix} u^2_{-1} \\ u^2_0 \end{pmatrix}$ with the condition that $M^* J M = J$.

Clearly, $M = I$ satisfies this, but let's look at one more similar to the continuous case.

If $M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$,

$$\begin{aligned} M^* J M &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & ia^1_m \\ -ia^1_m & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} ia^1_m & ia^1_m \\ -ia^1_m & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & ia^1_m \\ -ia^1_m & 0 \end{pmatrix} \\ &= J \end{aligned}$$

So this M satisfies the necessary condition and therefore we can define $\Gamma_{T_1} u^1 = \begin{pmatrix} u^1_m \\ u^1_{m+1} - u^1_m \end{pmatrix}$.

This choice of M for the last row looks like the finite difference for the derivative and therefore more closely resembles our continuous case.

So then we have a simple coupling of T_1 and T_2 in $\mathbb{C}^{m+2} \oplus \mathbb{C}^{n+2}$ defined by the condition $\Gamma_{T_1} u^1 = \frac{a^2_{-1}}{a^1_m} \Gamma_{T_2} u^2$. For $k = 1, 2$, clearly $P_k(dom(T)) = dom(T_k)$, $P_1 Tu = T_1 P_1 u$, and $P_2 Tu = T_2 P_2 u$.

All that is left is to show that T, which is a simple coupling of T_1 and T_2 defined by the condition $\Gamma_{T_1} u^1 = \frac{a^2_{-1}}{a^1_m} \Gamma_{T_2} u^2$ is self-adjoint. Note that $(Tu)_j = a_{j-1} u_{j-1} + b_j u_j + a_j u_{j+1}$ with $u_{-1} = 0$ and $u_{n+1} = 0$

$$
\begin{aligned}
< Tu, v > &= (T_1 u, v) + (T_2 u, v) \\
&= \left(\begin{bmatrix} b^1_0 u_0 + a^1_0 u_1 \\ a^1_0 u_0 + b^1_1 u_1 + a^1_1 u_2 \\ a^1_1 u_1 + b^1_2 u_2 + a^1_2 u_3 \\ \vdots \\ a^1_{m-1} u_{m-1} + b^1_m u_m + a^1_m u_{m+1} \end{bmatrix}, \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \right) \\
&\quad + \left(\begin{bmatrix} a^2_{-1} u_{-1} + b^2_0 u_0 + a^2_0 u_1 \\ a^2_0 u_0 + b^2_1 u_1 + a^2_1 u_2 \\ a^2_1 u_1 + b^2_2 u_2 + a^2_2 u_3 \\ \vdots \\ a^2_{n-1} u_{n-1} + b^2_n u_n \end{bmatrix}, \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) \\
&= b^1_0 u_0 \overline{v_0} + a^1_0 u_1 \overline{v_0} + a^1_0 u_0 \overline{v_1} + b^1_1 u_1 \overline{v_1} + a^1_1 u_2 \overline{v_1} + a^1_1 u_1 \overline{v_2} + b^1_2 u_2 \overline{v_2} + a^1_2 u_3 \overline{v_2} + \cdots \\
&\quad + a^1_{m-1} u_{m-1} \overline{v_m} + b^1_m u_m \overline{v_m} + a^1_m u_{m+1} \overline{v_m} + a^2_{-1} u_{-1} \overline{v_0} + b^2_0 u_0 \overline{v_0} + a^2_0 u_1 \overline{v_0} + a^2_0 u_0 \overline{v_1} + \\
&\quad b^2_1 u_1 \overline{v_1} + a^2_1 u_2 \overline{v_1} + a^2_1 u_1 \overline{v_2} + b^2_2 u_2 \overline{v_2} + a^2_2 u_3 \overline{v_2} + \cdots + a^2_{n-1} u_{n-1} \overline{v_n} + b^2_n u_n \overline{v_n}
\end{aligned}
$$

$$
\begin{aligned}
< u, Tv > &= (u, T_1 v) + (u, T_2 v) \\
&= \left(\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \begin{bmatrix} b_0^1 v_0 + a_0^1 v_1 \\ a_0^1 v_0 + b_1^1 v_1 + a_1^1 v_2 \\ a_1^1 v_1 + b_2^1 v_2 + a_2^1 v_3 \\ \vdots \\ a_{m-1}^1 v_{m-1} + b_m^1 v_m + a_m^1 v_{m+1} \end{bmatrix} \right) \\
&\quad + \left(\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \begin{bmatrix} a_{-1}^2 v_{-1} + b_0^2 v_0 + a_0^2 v_1 \\ a_0^2 v_0 + b_1^2 v_1 + a_1^2 v_2 \\ a_1^2 v_1 + b_2^2 v_2 + a_2^2 v_3 \\ \vdots \\ a_{n-1}^2 v_{n-1} + b_n^2 v_n \end{bmatrix} \right) \\
&= b_0^1 u_0 \overline{v_0} + a_0^1 u_1 \overline{v_0} + a_0^1 u_0 \overline{v_1} + b_1^1 u_1 \overline{v_1} + a_1^1 u_2 \overline{v_1} + a_1^1 u_1 \overline{v_2} + b_2^1 u_2 \overline{v_2} + a_2^1 u_3 \overline{v_2} + \cdots \\
&\quad + a_{m-1}^1 u_{m-1} \overline{v_m} + b_m^1 u_m \overline{v_m} + a_m^1 u_{m+1} \overline{v_m} + a_{-1}^2 u_{-1} \overline{v_0} + b_0^2 u_0 \overline{v_0} + a_0^2 u_1 \overline{v_0} + a_0^2 u_0 \overline{v_1} + \\
&\quad b_1^2 u_1 \overline{v_1} + a_1^2 u_2 \overline{v_1} + a_1^2 u_1 \overline{v_2} + b_2^2 u_2 \overline{v_2} + a_2^2 u_3 \overline{v_2} + \cdots + a_{n-1}^2 u_{n-1} \overline{v_n} + b_n^2 u_n \overline{v_n}
\end{aligned}
$$

Clearly $< Tu, v > = < u, Tv >$ so T is self-adjoint.

Example 4.2.3. *Operators: T_1^1 with same finite difference system as T_1 and BC $u_{-1}^1 = u_0^1$ and $u_m^1 = 0$. T_1'' with same finite difference system as T_1 and BC $u_{-1}^1 = 0$ and $u_m^1 = 0$*

Operators: T_2^1 with same finite difference system as T_2 and BC $u_{-1}^2 = u_0^2$ and u_0^2. T_2'' with same finite difference system as T_2 and BC $u_{-1}^2 = 0$ and $u_n^2 = 0$.

Example 4.2.4 (Three continuous strings)**.** *Show that $-y_i'' = \lambda y_i$ for $i = 1, 2, 3$ on $x \in [0, l]$ is a self adjoint operator. The boundary conditions are $y_1(0) = y_2(0) = y_3(0) = 0$ and $y_1(l) = y_2(l) = y_3(l) = \tilde{y}(l)$ and $y_1'(l) + y_2'(l) + y_3'(l) = 0$.*

$$
\begin{aligned}
(L\vec{f},\vec{g}) &= (Lf_1,g_1)+(Lf_2,g_2)+(Lf_3,g_3)\\
&= \int_0^l -f_1''g_1dx+\int_0^l -f_2''g_2dx+\int_0^l -f_3''g_3dx\\
&= -f_1'g_1|_0^l+\int_0^l f_1'g_1'dx-f_2'g_2|_0^l+\int_0^l f_2'g_2'dx-f_3'g_3|_0^l+\int_0^l f_3'g_3'dx\\
&= -f_1'g_1|_0^l+f_1g_1'|_0^l+\int_0^l f_1(-g_1'')dx-f_2'g_2|_0^l+f_2g_2'|_0^l+\int_0^l f_2(-g_2'')dx\\
&\quad -f_3'g_3|_0^l+f_3g_3'|_0^l+\int_0^l f_3(-g_3'')dx\\
&= \tilde{f}(l)(g_1'(l)+g_2'(l)+g_3'(l))-\tilde{g}(l)(f_1'(l)+f_2'(l)+f_3'(l))+(f_1,L^*g_1)\\
&\quad +(f_2,L^*g_2)+(f_3,L^*g_3)\\
&= (\vec{f},L^*\vec{g})
\end{aligned}
$$

where the last line is only true if $g_1(0)=g_2(0)=g_3(0)=0$, $g_1(l)=g_2(l)=g_3(l)=\tilde{g}(l)$, and $g_1(l)+g_2(l)+g_3(l)=0$. Since the domain of definitions are equivalent, the operator is self adjoint.

Example 4.2.5 (Continuous and discrete)**.** *Selfadjoint extension to $L^2(0,l)\oplus\mathbb{C}$ of differential operator given by*

$$
\begin{aligned}
l[y] &= -y''+q(x)y\\
Dom\ \ y_1'(0) &= y_1(0)=0\\
y_1'(1) &= hy_1(1)
\end{aligned}
$$

A_1 symmetric but not self-adjoint.

One to one mapping with $\theta(\lambda)=\frac{a-b\lambda}{c-d\lambda}$, $ad-bc=1$.

$$
\begin{aligned}
H_1 = L^2(0,l) \qquad & H_2 = \mathbb{C}, \text{ complex numbers} \\
\tilde{H} = H_1 \oplus H_2 \qquad & \\
\text{element of } \tilde{H} \qquad & \tilde{f} =< f, c_1 >, \ \tilde{g} =< g, c_2 > \\
& < \tilde{f}, \tilde{g} >_{\tilde{H}} = \int_0^l f\overline{g} dx + c_1 c_2
\end{aligned}
$$

Introduce an operator $\tilde{A}$ in $\tilde{H}$.

$$
\begin{aligned}
D(\tilde{A}) &= \{< f, c_1 > \in \tilde{H} : f \in \mathrm{Dom}(A_1^*)\} \\
c_1 &= a_1 f(0) + b_2 f'(0) \\
\tilde{A} < f, c_1 > &= < -f'' + g(x) f, c_2 > \\
c_2 &= c f(0) + d f'(0)
\end{aligned}
$$

where $\tilde{A} = \tilde{A}^*$ is self adjoint. Find the condition on a, b, c, d for it to be self-adjoint.

We first need to determine the domain of A_1^* such that A_1 is symmetric but not self adjoint.

$$
\begin{aligned}
A_1 \sim l[y] &= -y'' + q(x) y \\
y_1'(0) &= y_1'(0) = 0 \\
y_1'(1) &= h y_1(1)
\end{aligned}
$$

$$
\begin{aligned}
< Lf, g > &= \int_0^1 (-f'' + q(x)f)\overline{g}dx \\
&= -\int_0^1 f''\overline{g}dx + \int_0^1 q(x)f\overline{g}dx \\
&= -\left[f'\overline{g} \mid_0^1 - \int_0^1 f'\overline{g}'dx \right] + \int_0^1 q(x)f\overline{g}dx \\
&= -f'(1)\overline{g}(1) + f'(0)\overline{g}(0) + \left[f\overline{g}' \mid_0^1 - \int_0^1 f\overline{g}''dx \right] + \int_0^1 q(x)f\overline{g}dx \\
&= -f'(1)\overline{g}(1) + f'(0)\overline{g}(0) + f(1)\overline{g}'(1) - f(0)\overline{g}'(0) + < f, L^*g > \\
&= -hf(1)\overline{g}(1) + f(1)\overline{g}'(1) + < f, Lg > \\
&= f(1)\left[\overline{g}'(1) - h\overline{g}(1) \right] + < f, Lg >
\end{aligned}
$$

Therefore, for A_1 to be symmetric, $\mathrm{Dom}(A_1^*) = \{g'(1) = hg(1)\}$.

Let $c_1 = af(0) + bf'(0)$, $c_2 = cf(0) + df'(0)$, $c_3 = ag(0) + bg'(0)$, $c_4 = cg(0) + dg'(0)$.

$$
\begin{aligned}
< \tilde{A}\tilde{f}, \tilde{g} > &= \left\langle < -f'' + h(x)f, c_2 >, < g, c_3 > \right\rangle \\
&= \int_0^1 (-f'' + h(x)f)\overline{g}dx + c_2\overline{c_3} \\
&= \int_0^1 (-f''\overline{g})dx + \int_0^1 h(x)f\overline{g}dx + c_2\overline{c_3} \\
&= -\left[f'\overline{g} \mid_0^1 - \int_0^1 f'\overline{g}'dx \right] + \int_0^1 h(x)f\overline{g}dx + c_2\overline{c_3} \\
&= -f'(1)\overline{g}(1) + f'(0)\overline{g}(0) + \int_0^1 f'\overline{g}'dx + \int_0^1 h(x)f\overline{g}dx + c_2\overline{c_3} \\
&= -f'(1)\overline{g}(1) + f'(0)\overline{g}(0) + \left[f\overline{g}' \mid_0^1 - \int_0^1 f\overline{g}''dx \right] + \int_0^1 h(x)f\overline{g}dx + c_2\overline{c_3} \\
&= -f'(1)\overline{g}(1) + f'(0)\overline{g}(0) + f(1)\overline{g}'(1) - f(0)\overline{g}'(0) - \int_0^1 f\overline{g}''dx + \int_0^1 h(x)f\overline{g}dx \\
&\quad + \left(cf(0) + df'(0)\right)\overline{(ag(0) + bg'(0))} \\
&= -f'(1)\overline{g}(1) + f'(0)\overline{g}(0) + f(1)\overline{g}'(1) - f(0)\overline{g}'(0) - \int_0^1 f\overline{g}''dx + \int_0^1 h(x)f\overline{g}dx \\
&\quad + acf(0)\overline{g}(0) + bcf(0)\overline{g}'(0) + adf'(0)\overline{g}(0) + bdf'(0)\overline{g}'(0) \\
&= -f'(1)\overline{g}(1) + f'(0)\overline{g}(0) + f(1)\overline{g}'(1) - f(0)\overline{g}'(0) - \int_0^1 f\overline{g}''dx + \int_0^1 h(x)f\overline{g}dx \\
&\quad + acf(0)\overline{g}(0) + bcf(0)\overline{g}'(0) + adf'(0)\overline{g}(0) + bdf'(0)\overline{g}'(0) \\
&= -f'(1)\overline{g}(1) + f'(0)\overline{g}(0) + f(1)\overline{g}'(1) - f(0)\overline{g}'(0) - \int_0^1 f\overline{g}''dx \\
&\quad + \int_0^1 h(x)f\overline{g}dx + acf(0)\overline{g}(0) + bcf(0)\overline{g}'(0) + adf'(0)\overline{g}(0) + bdf'(0)\overline{g}'(0) \\
&= f(1)[\overline{g}'(1) - h\overline{g}(1)] + (1 - ad)f'(0)\overline{g}(0) + (bc - 1)f(0)\overline{g}'(0) + acf(0)\overline{g}(0) \\
&\quad + bdf'(0)\overline{g}'(0) - \int_0^1 f\overline{g}''dx + \int_0^1 h(x)f\overline{g}dx \\
&= (1 - ad)f'(0)\overline{g}(0) + (bc - 1)f(0)\overline{g}'(0) + acf(0)\overline{g}(0) \\
&\quad + bdf'(0)\overline{g}'(0) - \int_0^1 f\overline{g}''dx + \int_0^1 h(x)f\overline{g}dx
\end{aligned}
$$

$$
\begin{aligned}
< \tilde{f}, \tilde{A}^*\tilde{g} > &= \int_0^1 f\overline{(-g'' + h(x)g)}dx + c_1\overline{c_4} \\
&= -\int_0^1 f\overline{g}''dx + \int_0^1 h(x)f\overline{g}dx + \left(af(0) + bf'(0)\right)\overline{(cg(0) + dg'(0))} \\
&= -\int_0^1 f\overline{g}''dx + \int_0^1 h(x)f\overline{g}dx + acf(0)\overline{g}(0) + adf(0)\overline{g}'(0) + bcf'(0)\overline{g}(0) \\
&\quad + bdf'(0)\overline{g}'(0)
\end{aligned}
$$

Since we want $\tilde{A} = \tilde{A}^*$ to be self adjoint, we need to equate both of the end lines.

$$1 - ad = bc$$

$$bc - 1 = ad$$

4.3 Spectral Properties of Operator Couplings

Let $\{\lambda_n^1\}$ be eigenvalues of T_1^1, let $\{\lambda_m^2\}$ be eigenvalues of T_2^1. If $\{\mu_k\} \in \{\lambda_n^1\} \cap \{\lambda_m^2\}$, then $\{\mu_k\} \subset \{\lambda_{n+m}\}$ where $\{\lambda_{n+1}\}$ are eigenvalues of the problem with eigenvalue dependent boundary condition defined by T_2^1 (if boundary condition is on the right or by T_1^1 if it is on the left)

Example 4.3.1. *Consider $a_{-1}^2 = a_m^1$. If a simple coupling of T_1 and T_2 is the operator T ($u_m^1 = u_{-1}^2$, $u_{m+1}^1 = u_0^2$) acting in $\mathbb{C}^{m+n+2}$*

T is given by

$$\begin{aligned} (Tu)_j &= a_{j-1}^1 u_{j-1} + b_j^1 u_j + a_j^1 u_{j+1} \qquad (j = 0, 1, 2, \ldots, m) \\ (Tu)_j &= a_{j-1}^2 u_{j-1} + b_j^2 + a_j^2 u_{j+1} \quad (j = m+1, \ldots, m+n) \\ u_{-1} &= 0, \qquad u_{m+n+1} = 0 \end{aligned}$$

We can consider T_1 as a boundary condition

$$\theta(\lambda) = \frac{u_{m+1}^1(\lambda)}{u_{m+1}^1(\lambda) - u_m^1(\lambda)}$$

The total number of eigenvalues of T is equal to m plus the number of poles of $\theta(\lambda)$.

Example 4.3.2. T

$$\begin{aligned} u_0 + 2u_1 + u_2 &= \lambda u_1 \\ u_1 + u_2 + u_3 &= \lambda u_2 \\ u_2 + 2u_3 + u_4 &= \lambda u_3 \\ u_0 = 0 \quad u_4 &= 0 \end{aligned}$$

Eigenvalues of T are zeros of the polynomial $P(\lambda) = \lambda(\lambda - 3)(\lambda - 2)$.

$$\begin{aligned} T_1^{'} \quad & u_0 + 2u_1 + u_2 = \lambda u_1 \\ BC \quad & u_1 = u_2, \quad u_0 = 0 \end{aligned}$$

with $P_1(\lambda) = \lambda - 3$ and eigenvalues $\{\lambda_n^1\} = \{3\}$.

$$\begin{aligned} T_1^{''} \quad & u_0 + 2u_1 + u_2 = \lambda u_1 \\ BC \quad & u_0 = 0, \quad u_2 = 0 \end{aligned}$$

with $Q_1(\lambda) = \lambda - 2$ and eigenvalue 2.

$$\begin{aligned} T_2^{'} \quad & u_1 + u_2 + u_3 = \lambda u_2 \\ & u_2 + 2u_3 + u_4 = \lambda u_3 \\ BC \quad & u_1 = u_2, \quad u_4 = 0 \end{aligned}$$

with $P_2(\lambda) = (\lambda - 3)(\lambda - 1)$ and eigenvalues $\{\lambda_m^2\} = \{1, 3\}$.

$$\begin{aligned} T_2^{''} \quad & u_1 + u_2 + u_3 = \lambda u_2 \\ & u_2 + 2u_3 + u_4 = \lambda u_3 \\ BC \quad & u_1 = 0 \quad u_4 = 0 \end{aligned}$$

with $Q_2(\lambda) = \lambda^2 - 3\lambda + 1$.

The eigenvalue boundary condition dependent problem

$$u_1 + u_2 + u_3 = \lambda u_2$$

$$u_2 + 2u_3 + u_4 = \lambda u_3$$

$$\frac{u_2 - u_1}{u_1} = \lambda - 3, \quad u_4 = 0$$

$$\theta(\lambda) = \lambda - 3$$

$$\{\lambda_{m+n}\} = \{0, 2, 3\}$$

$$\{3\} = \{3\} \cap \{1, 3\}$$

Also,

$$m(\lambda) = \frac{\lambda^2 - 4\lambda + 3}{\lambda^2 - 3\lambda + 1} = \frac{P_2(\lambda)}{Q_2(\lambda)}$$

where $P_2(\lambda)$ is the characteristic polynomial of $T_2^{'}$ and $Q_2(\lambda)$ is the characteristic polynomial of $T_2^{''}$. The eigenvalues of the boundary condition dependent problem are zeros of $\phi(\lambda) = \theta(\lambda) + m(\lambda)$.

Example 4.3.3. T

$$a_0u_0 + b_1u_1 + a_1u_2 = \lambda u_1$$

$$a_1u_1 + b_2u_2 + a_2u_3 = \lambda u_2$$

$$a_2u_2 + b_3u_3 + a_3u_4 = \lambda u_3$$

$$\hline$$

$$a_3u_3 + b_4u_4 + a_4u_5 = \lambda u_4$$

$$a_4u_4 + b_5u_5 + a_5u_6 = \lambda u_5$$

$$u_0 = 0 \quad u_6 = 0 =$$

We calculate the characteristic polynomial by

$$
\begin{aligned}
u_2 &= \frac{(\lambda - b_1)}{a_1} u_1 \\
u_3 &= \frac{(\lambda - b_2)(\lambda - b_1)}{a_2 a_1} u_1 - \frac{a_1}{a_2} u_1 = \left(\frac{(\lambda - b_2)(\lambda - b_1) - a_1^2}{a_2 a_1} \right) u_1 \\
u_4 &= \frac{(\lambda - b_3)(\lambda - b_2)(\lambda - b_1)}{a_3 a_2 a_1} u_1 - \frac{a_1(\lambda - b_3)}{a_3 a_2} u_1 - \frac{a_2(\lambda - b_1)}{a_3 a_1} u_1
\end{aligned}
$$

$$
\begin{aligned}
u_5 &= \frac{(\lambda - b_4)(\lambda - b_3)(\lambda - b_2)(\lambda - b_1)}{a_4 a_3 a_2 a_1} u_1 - \frac{a_1(\lambda - b_4)(\lambda - b_3)}{a_4 a_3 a_2} u_1 \\
&\quad - \frac{a_2(\lambda - b)_4)(\lambda - b_1)}{a_4 a_3 a_1} u_1 - \frac{a_3(\lambda - b_2)(\lambda - b_1)}{a_4 a_2 a_1} u_1 + \frac{a_3 a_1}{a_4 a_2} u_1 \\
u_6 &= \frac{(\lambda - b_5)(\lambda - b_4) \cdots (\lambda - b_1)}{a_5 a_4 a_3 a_2 a_1} u_1 - \frac{a_1(\lambda - b_5)(\lambda - b_4)(\lambda - b_3)}{a_5 a_4 a_3 a_2} u_1 \\
&\quad - \frac{a_2(\lambda - b_5)(\lambda - b_4)(\lambda - b_1)}{a_5 a_4 a_3 a_1} u_1 - \frac{a_3(\lambda - b_5)(\lambda - b_2)(\lambda - b_1)}{a_5 a_4 a_2 a_1} u_1 \\
&\quad + \frac{a_3 a_1(\lambda - b_5)}{a_5 a_4 a_2} u_1 - \frac{a_4(\lambda - b_3)(\lambda - b_2)(\lambda - b_1)}{a_5 a_3 a_2 a_1} u_1 \\
&\quad \frac{a_4 a_1(\lambda - b_3)}{a_5 a_3 a_2} u_1 + \frac{a_4 a_2(\lambda - b_1)}{a_5 a_3 a_1} u_1
\end{aligned}
$$

Since $u_6 = 0$,

$$
\begin{aligned}
u_6 &= \frac{(\lambda - b_5)(\lambda - b_4) \cdots (\lambda - b_1)}{a_5 a_4 a_3 a_2 a_1} u_1 - \frac{a_1(\lambda - b_5)(\lambda - b_4)(\lambda - b_3)}{a_5 a_4 a_3 a_2} u_1 \\
&\quad - \frac{a_2(\lambda - b_5)(\lambda - b_4)(\lambda - b_1)}{a_5 a_4 a_3 a_1} u_1 - \frac{a_3(\lambda - b_5)(\lambda - b_2)(\lambda - b_1)}{a_5 a_4 a_2 a_1} u_1 \\
&\quad + \frac{a_3 a_1(\lambda - b_5)}{a_5 a_4 a_2} u_1 - \frac{a_4(\lambda - b_3)(\lambda - b_2)(\lambda - b_1)}{a_5 a_3 a_2 a_1} u_1 \\
&\quad \frac{a_4 a_1(\lambda - b_3)}{a_5 a_3 a_2} u_1 + \frac{a_4 a_2(\lambda - b_1)}{a_5 a_3 a_1} u_1 = 0 \\
\to 0 &= \frac{(\lambda - b_5)(\lambda - b_4) \cdots (\lambda - b_1)}{a_5 a_4 a_3 a_2 a_1} - \frac{a_1(\lambda - b_5)(\lambda - b_4)(\lambda - b_3)}{a_5 a_4 a_3 a_2} \\
&\quad - \frac{a_2(\lambda - b_5)(\lambda - b_4)(\lambda - b_1)}{a_5 a_4 a_3 a_1} - \frac{a_3(\lambda - b_5)(\lambda - b_2)(\lambda - b_1)}{a_5 a_4 a_2 a_1} \\
&\quad + \frac{a_3 a_1(\lambda - b_5)}{a_5 a_4 a_2} - \frac{a_4(\lambda - b_3)(\lambda - b_2)(\lambda - b_1)}{a_5 a_3 a_2 a_1} \\
&\quad \frac{a_4 a_1(\lambda - b_3)}{a_5 a_3 a_2} + \frac{a_4 a_2(\lambda - b_1)}{a_5 a_3 a_1}
\end{aligned}
$$

Finding P_1

$$\frac{u_4}{u_3} = \frac{\frac{(\lambda-b_3)(\lambda-b_2)(\lambda-b_1)}{a_3a_2a_1}u_1 - \frac{a_1(\lambda-b_3)}{a_3a_2}u_1 - \frac{a_2(\lambda-b_1)}{a_3a_1}u_1}{\frac{(\lambda-b_2)(\lambda-b_1)-a_1^2}{a_2a_1}u_1}$$

Finding P_2

$$\begin{aligned}
u_5 &= \frac{-a_4}{(b_5-\lambda)}u_4 \\
u_4 &= \frac{-a_3}{(b_4-\lambda)}u_3 - \frac{a_4^2}{(b_4-\lambda)(b_5-\lambda)}u_4 \\
\left[1 + \frac{a_4^2}{(b_4-\lambda)(b_5-\lambda)}\right]u_4 &= -\frac{a_3}{(b_4-\lambda)}u_3 \\
\frac{u_4}{u_3} &= \frac{-a_3(b_4-\lambda)(b_5-\lambda)}{(b_4-\lambda)\left[(b_4-\lambda)(b_5-\lambda)+a_4^2\right]}
\end{aligned}$$

The eigenvalues are given by

$$\begin{aligned}
\frac{\frac{(\lambda-b_3)(\lambda-b_2)(\lambda-b_1)}{a_3a_2a_1} - \frac{a_1(\lambda-b_3)}{a_3a_2} - \frac{a_2(\lambda-b_1)}{a_3a_1}}{\frac{(\lambda-b_2)(\lambda-b_1)-a_1^2}{a_2a_1}} &= \frac{-a_3(b_4-\lambda)(b_5-\lambda)}{(b_4-\lambda)\left[(b_4-\lambda)(b_5-\lambda)+a_4^2\right]} \\
\frac{(\lambda-b_3)(\lambda-b_2)(\lambda-b_1) - a_1^2(\lambda-b_3) - a_2^2(\lambda-b_1)}{a_3\left[(\lambda-b_2)(\lambda-b_1)-a_1^2\right]} &= \frac{-a_3(b_4-\lambda)(b_5-\lambda)}{(b_4-\lambda)\left[(b_4-\lambda)(b_5-\lambda)+a_4^2\right]}
\end{aligned}$$

L_2'' where $u_6 = 0$ and $u_3 = 0$

$$\begin{aligned}
u_3 &= \left(\frac{b_4-\lambda}{a_3} + \frac{a_4^2}{a_3(b_5-\lambda)}\right)u_4 \\
\frac{b_4-\lambda}{a_3} + \frac{a_4^2}{a_3(b_5-\lambda)} &= 0
\end{aligned}$$

with $Q_2(\lambda) = (b_4-\lambda)(b_5-\lambda) + a_4^2$.(But if $\lambda = b_5$, then its undefined)

L_1'' where $u_0 = 0$ and $u_4 = 0$

4.4 Operator couplings of Continuous and Discrete Operators

Given the operator $A : L^2(0,l) \bigoplus \mathbb{C}^2 \to \mathbb{C}$ which acts according to

$$A \begin{pmatrix} f(x) \\ c_1 \end{pmatrix} = \begin{pmatrix} -f'' + pf \\ c_2 \end{pmatrix}$$

$$\text{where } c_1 = A_1 \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_1 \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}$$

$$c_2 = A_2 \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_2 \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}$$

A_1, A_2, B_1, and B_2 are 2×2 matrices. Now if we want to find the adjoint of A, A^*, let's define

$$A^* \begin{pmatrix} g(x) \\ m_1 \end{pmatrix} = \begin{pmatrix} -g'' + pg \\ m_2 \end{pmatrix}$$

$$\text{where } m_1 = D_1 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + H_1 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}$$

$$m_2 = D_2 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + H_2 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}$$

Also for notation later, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\left\langle A\begin{pmatrix} f(x) \\ c_1 \end{pmatrix}, \begin{pmatrix} g(x) \\ m_1 \end{pmatrix}\right\rangle = \left\langle \begin{pmatrix} f(x) \\ c_1 \end{pmatrix}, A^* \begin{pmatrix} g(x) \\ m_1 \end{pmatrix}\right\rangle$$

$$\to \left\langle \begin{pmatrix} -f''+qf \\ A_2\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_2\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix} \end{pmatrix}, \begin{pmatrix} g(x) \\ D_1\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + H_1\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \end{pmatrix}\right\rangle$$

$$= \left\langle \begin{pmatrix} f(x) \\ A_1\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_1\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix} \end{pmatrix}, \begin{pmatrix} -g''+qg \\ D_2\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + H_2\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \end{pmatrix}\right\rangle$$

$$\to \int_0^l (-f''+qf)\overline{g}dx + \left\langle A_2\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_2\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, D_1\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + H_1\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle$$

$$= \int_0^1 f(\overline{-g''+qg})dx + \left\langle A_1\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_1\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, D_2\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + H_2\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle$$

$$\begin{aligned}
&\to -f'(l)\overline{g(l)} + f'(0)\overline{g(0)} + f(l)\overline{g'(l)} - f(0)\overline{g'(0)} + \int_0^l -f\overline{g''}dx + \int_0^l qf\overline{g}dx \\
&\quad + \left\langle A_2\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, D_1\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}\right\rangle + \left\langle A_2\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, H_1\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle \\
&\quad + \left\langle B_2\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, D_1\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}\right\rangle + \left\langle B_2\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, H_1\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle \\
&= \int_0^l f(\overline{-g''+qg})dx + \left\langle A_1\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, D_2\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}\right\rangle + \left\langle A_1\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, H_2\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle \\
&\quad + \left\langle B_1\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, D_2\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}\right\rangle + \left\langle B_1\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, H_2\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\to -f'(l)\overline{g(l)} + f'(0)\overline{g(0)} + f(l)\overline{g'(l)} - f(0)\overline{g'(0)} + \int_0^l -f\overline{g''}dx + \int_0^l qf\overline{g}dx \\
&\quad + \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_2^* D_1 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_2^* H_1 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \right\rangle \\
&\quad + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_2^* D_1 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_2^* H_1 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \right\rangle \\
= \int_0^l f(\overline{-g'' + qg})dx &+ \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_1^* D_2 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_1^* H_2 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \right\rangle \\
&\quad + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_1^* D_2 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_1^* H_2 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\to \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, J \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, -J \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \right\rangle + \int_0^l -f\overline{g''}dx + \int_0^1 qf\overline{g}dx \\
&\quad + \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_2^* D_1 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_2^* H_1 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \right\rangle \\
&\quad + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_2^* D_1 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_2^* H_1 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \right\rangle \\
= \int_0^l f(\overline{-g'' + qg})dx &+ \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_1^* D_2 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_1^* H_2 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \right\rangle \\
&\quad + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_1^* D_2 \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_1^* H_2 \begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \right\rangle
\end{aligned}
$$

In order for these to be equal, the following must be true

$$
\begin{aligned}
J + A_2^* D_1 - A_1^* D_2 &= 0 \\
-J + B_2^* H_1 - B_1^* H_2 &= 0 \\
A_2^* H_1 - A_1^* H_2 &= 0 \\
B_2^* D_1 - B_1^* D_2 &= 0
\end{aligned}
$$

Now if we want A to be self-adjoint,

$$\left\langle A\begin{pmatrix} f(x) \\ c_1 \end{pmatrix}, \begin{pmatrix} g(x) \\ c_1 \end{pmatrix}\right\rangle = \left\langle \begin{pmatrix} f(x) \\ c_1 \end{pmatrix}, A^*\begin{pmatrix} g(x) \\ c_1 \end{pmatrix}\right\rangle$$

$$\to \left\langle \begin{pmatrix} -f''+qf \\ A_2\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_2\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix} \end{pmatrix}, \begin{pmatrix} g(x) \\ A_1\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + B_1\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \end{pmatrix}\right\rangle$$

$$= \left\langle \begin{pmatrix} f(x) \\ A_1\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_1\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix} \end{pmatrix}, \begin{pmatrix} -g''+qg \\ A_2\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + B_2\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix} \end{pmatrix}\right\rangle$$

$$\to \int_0^l (-f''+qf)\overline{g}dx + \left\langle A_2\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_2\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, A_1\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + B_1\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle$$

$$= \int_0^1 f(\overline{-g''+qg})dx + \left\langle A_1\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} + B_1\begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, A_2\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} + B_2\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle$$

$$\begin{aligned}
\to \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, J\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}\right\rangle + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, -J\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle + \int_0^l -f\overline{g''}dx + \int_0^1 qf\overline{g}dx \\
+ \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_2^*A_1\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}\right\rangle + \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_2^*B_1\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle \\
+ \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_2^*A_1\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}\right\rangle + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_2^*B_1\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle \\
= \int_0^l f(\overline{-g''+qg})dx + \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_1^*A_2\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}\right\rangle + \left\langle \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, A_1^*B_2\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle \\
+ \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_1^*A_2\begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}\right\rangle + \left\langle \begin{pmatrix} f(l) \\ f'(l) \end{pmatrix}, B_1^*B_2\begin{pmatrix} g(l) \\ g'(l) \end{pmatrix}\right\rangle
\end{aligned}$$

The conditions for A to be self-adjoint is as follows:

$$A_2^* A_1 - A_1^* A_2 = -J$$

$$B_2^* B_1 - B_1^* B_2 = J$$

$$A_2^* B_1 - A_1^* B_2 = 0$$

$$B_2^* A_1 - B_1^* A_2 = 0$$

Chapter 5

Modeling Hypothalamus Pituitary Adrenal Axis

Another biological area of interest is the system governing hormone production and action. Hormones control a vast array of bodily functions that include sexual reproduction and development, whole-body metabolism, blood glucose levels, and so on [84]. Hormones are produced and released in organs throughout the body such as the hypothalamus, pituitary, and adrenal gland. Depending on the distance between the production site and the site of action, hormones are capable of a diffusion whole-body or localized effect. The endocrine system governing hormones is an intercellular signaling system in which cells communicate via cellular secretions. The distance between the sites of hormone production and action and the complexities due to the mode of transportation make it extraordinarily difficult to construct quantitative models of hormonal control.

The hypothalamus pituitary adrenal axis is a central neuroendocrine system which consists of hypothalamus, pituitary, and adrenal glands. The hypothalamus pituitary adrenal (HPA) axis is a central neuroendocrine system, which consists of the hypothalamus, pituitary, and adrenal glands (See Figure 5.2). The paraventricular nucleus of the hypothalamus secrets corticotropin releasing hormone (CRH), which is transferred to the pituitary and stimulates the synthesis and release of adrenocorticotropic hormone (ACTH). ACTH moves through the bloodstream and reaches the adrenal gland in which it stimulates the secretion of cortisol. In response to stress, the concen-

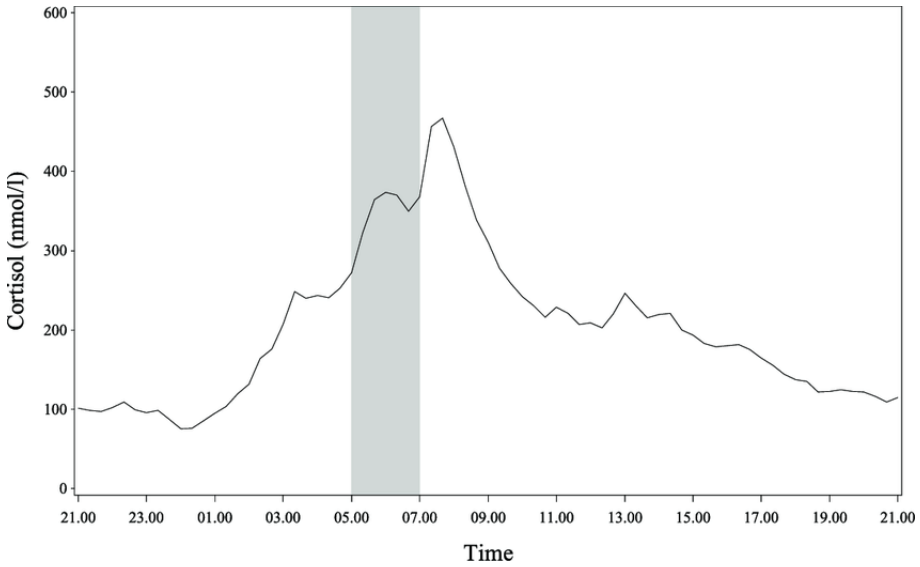


Figure 5.1: CORT's natural circadian rhythm

trations of the HPA axis hormones are increased. A brief review of the HPA axis and the various factors that regulate its functions are described in [68].

The most commonly known HPA axis hormone is cortisol (CORT). CORT is often known as the stress hormone and is involved with or responsible for body temperature regulation, digestion, the immune system, memory, mood, sexuality, etc. CORT levels in the body follow a natural circadian rhythm as illustrated in Figure 5.1. Disruption of the HPA axis regulation results in the disruption of circadian rhythms of CORT levels. It has been found that several medical disorders, diseases, and syndromes are related to abnormal levels of CORT. For example, increased cortisol has been shown in patients with major depressive disorder (see [52, 41]), and decreased cortisol has been observed in people with post-traumatic stress disorder (see [76]). It has been shown that those with Cushing's syndrome have excessive levels of CORT while those with Addison's disease and Nelson's syndrome have insufficient levels. Stress-related disorders such as major depressive disorder and post-traumatic stress disorder show increased and decreased levels of CORT respectively. Therefore, a deeper understanding of the dynamics of the HPA axis are important to the medical field.

There have been multiple models of the HPA axis derived in an attempt to characterize the oscillations seen in the hormone concentrations and to examine HPA axis dysfunction. Most of these models have been constructed using deterministic coupled ordinary differential equations

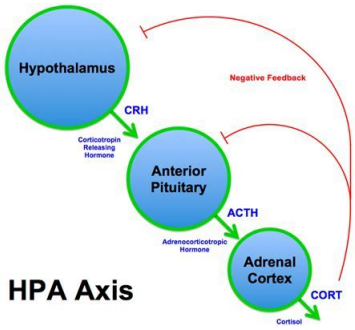


Figure 5.2: The HPA axis model

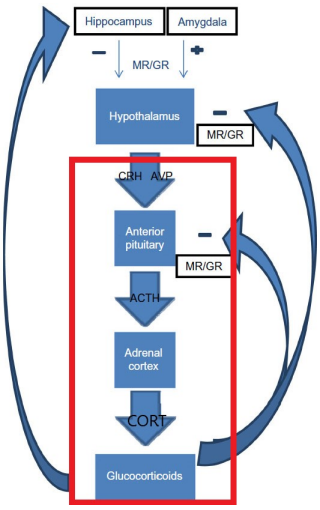


Figure 5.3: The HPA axis model with Glucocorticoids

(see [42]). A major inconsistency among different existing HPA models that was mentioned in [48] is related to their treatment of the circadian and ultradian oscillations (see Figure 5.1). For example, the authors of [79] and [7] assumed that both oscillations can be generated inside the HPA axis system by interaction of its elements; the authors of [83], [5], and [49] treat the circadian and ultradian oscillations differently assuming that only ultradian oscillations are HPA axis based but at the same time circadian rhythms are due to external input. Only one model made no explicit assumption about the origin of the oscillations and was developed to replicate the HPA axis response to CRH injection (see [34]). It has also been suggested that the ultradian rhythm arises from the introduction of a time delay (see [7]). Other models based on delay-differential equations include [60] and [46].

To determine if delay-differential equations could predict the general features of CORT production, the experimental data was compared to a simulated CORT curve in [46]. It was not possible to obtain any experimental fitting of ACTH for the model since hypothalamic derived CRH cannot be measured. Inclusion of the glucocorticoid receptor (GR), which is illustrated in Fig 5.3, in a HPA axis model reveals 'bi-stability' (see [46]). To be more concrete, there arises a nonlinear Gauss type function with compact support, which is characterized by the parameter p_4. This Hill function arises as a result of 'inner' nonlinearity in the physiological system which is produced by the stress impulse, which is activated by the outer impulse that is called by an acute stress. This situation is provided formally by the two parameters p_4 and CRH. The amplitude of the Hill function determines GR density in the pituitary, which is coupled nonlinearly in reaction with regulated levels of CORT which in turn mediate a wide range of physiological processes, including metabolic, immunological and cognitive function (see [75, 65]).

The stress response is subserved by the stress system which is located both in the central nervous system and the periphery. The principal effects to the stress system include the CRH. The secretion of CRH causes the anterior pituitary to synthesize ACTH which then stimulates the adrenal glands to release CORT that regulate the blood concentration of CRH and ACTH via different negative feedback mechanisms.A model has been developed that links the HPA axis and the memory system in the stress reaction (see [78]). The HPA axis is the subject of intensive research in endocrinology. A study has shown that CRH may have a positive very short loop feedback action that enhances stress-induced ACTH released (see [67]). This model is based on the feed-forward

and feedback interactions between the anterior pituitary and adrenal glands. Because responsiveness of the stress system to stressors is crucial for life, it is important to consider the simpler case when distributions of hormones in the system become unstable by action on stress, and further to consider influence on the delay time as response of the physiological system on action on stress.

Mathematically, it means that we can consider two mathematical models: the first one is described by a system of ordinary differential equations with initial distributions of hormones at a point $t = 0$, and the second one is based upon a system of differential equations with initial distributions of hormones on the interval $[-\tau, 0)$, where τ is a time delay. It turns out that bi-stability is present in both models, i.e. limit distributions of hormones may be stable or unstable depending on parameter values.

In this thesis, we study a system of delay differential equations (see [46, 75]):

$$\frac{da}{dt} = \frac{CRH}{1+p_2 or} - p_3 a =: f_1, \tag{5.1}$$

$$\frac{dr}{dt} = \frac{(or)^2}{p_4+(or)^2} + p_5 - p_6 r =: f_2, \tag{5.2}$$

$$\frac{do}{dt} = a(t-\tau) - o =: f_3, \tag{5.3}$$

with initial conditions

$$a(t) = a_\tau(t)\ \forall\, t \in [-\tau, 0],\ r(0) = r_0,\ o(0) = o_0, \tag{5.4}$$

where

$$0 \leqslant a_\tau(t) \in C^1[-\tau, 0],\ r_0 > 0,\ o_0 > 0. \tag{5.5}$$

Based on the principles of mass action kinetics, these equations describe the production and degradation of the hormones ACTH (a), and CORT (o), as well as GR density (r) in the pituitary. Here, the parameters p_{2-6} represent dimensionless forms of rate constants of the system, and the dimensionless parameter τ represents a discrete delay, which accounts for the delayed response of the adrenal gland to ACTH. The dimensionless time $t = 0$ corresponds to the maximal value of an

ACTH pulse.

We also study this system of nonlinear ODES without delay

$$\frac{da}{dt} = \frac{A}{1+p_2 or} - p_3 a, \tag{5.6}$$

$$\frac{dr}{dt} = \frac{(or)^2}{p_4+(or)^2} + p_5 - p_6 r, \tag{5.7}$$

$$\frac{do}{dt} = a - o, \tag{5.8}$$

with initial conditions

$$a(0) = a_0 \geqslant 0,\ r(0) = r_0 > 0,\ o(0) = o_0 > 0, \tag{5.9}$$

where $A :=$ CRH $\geqslant 0$, and $p_i \geqslant 0$.

For the model of the Hypothalamous Pituitary Adrenal (HPA) axis, we first perform rigorous stability analysis of all multi-parametric steady states and secondly, by construction of a Lyapunov functional, we prove nonlinear asymptotic stability for some of multi-parametric steady states. We then take into account the additional effects of the time delay parameter on the stability of the HPA axis system. Finally we prove the existence of periodic solutions for the HPA axis system. The main results of this research are published in [51].

Chapter 6

Spectral Analysis of HPA Model without Delay

6.1 Introduction

When presented with a system of differential equations, we first attempt to get an understanding of the behavior of the solution before trying to determine the solution itself. We begin by understanding all possible trajectories corresponding to different solutions depending on initial conditions. For a system $\frac{d}{dt}\vec{x} = f(\vec{x}, t)$, we first identify fixed points $\vec{x*}$, which are the points such that $\frac{d}{dt}\vec{x} = \vec{0}$. We can determine the long term behavior of the solution in relationship to the fixed point. We say that $\vec{x*}$ is **attracting** if there is a $\delta > 0$ such that $\lim_{t\to\infty} \vec{x}(t) = \vec{x*}$ whenever $||\vec{x(0)} - \vec{x*}|| < \delta$ [81]. This allows trajectories to stray from $\vec{x*}$ for a short time, but must return to $\vec{x*}$ in the long run. If the trajectories remain in the neighborhood of $\vec{x*}$, then we say that the fixed point is Liapunov stable. More precisely, we say that $\vec{x*}$ is **Liapunov stable** if for each $\epsilon > 0$, there is a $\delta > 0$ such that $||\vec{x(t)} - \vec{x*}|| < \epsilon$ whenever $t \geq 0$ and $||\vec{x(0)} - \vec{x*}|| < \delta$ [81]. The fixed points that are both attracting and Liapunov stable are **asymptotically stable**, while fixed points that are neither Liapunov stable or attracting is said to be **unstable** [81]. Now if the system is linear, it can

be represented as

$$\begin{aligned}
\dot{x_1} &= a_{1,1}x_1 + a_{1,2}x_2 + \cdots a_{1,n}x_n \\
\dot{x_2} &= a_{2,1}x_1 + a_{2,2}x_2 + \cdots a_{2,n}x_n \\
&\vdots \\
\dot{x_n} &= a_{n,1}x_1 + a_{n,2}x_2 + \cdots a_{n,n}x_n
\end{aligned}$$

or as $\frac{d}{dt}\vec{x} = A\vec{x}$, where the matrix A is the coefficient matrix. If the eigenvalues of A are all real and nonnegative, then the system is asymptotically stable.

However, if the system is nonlinear and represented as

$$\dot{x_1} = f_1(x_1, x_2, \ldots, x_n) \tag{6.1}$$

$$\dot{x_2} = f_2(x_1, x_2, \ldots, x_n) \tag{6.2}$$

$$\vdots \tag{6.3}$$

$$\dot{x_n} = f_n(x_1, x_2, \ldots, x_n) \tag{6.4}$$

then we do not have a coefficient matrix to calculate eigenvalues for. Instead, we linearize the system by calculating the **Jacobian matrix** by

$$J_{\vec{x*}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\vec{x*}}.$$

We then analyze stability by studying the system $\frac{d}{dt}\vec{x} = J\vec{x}$ for each fixed point $\vec{x*}$. It is important to note that different fixed points can have different stabilities. Now when the system has parameter values instead of fixed values as coefficients, stability is often greatly affected by the different values the parameter may take on.

Since the system (5.6-5.8) has three independent variables and several parameters, the characteristic equation will have degree 3 and it may not be immediately clear how to solve for the

eigenvalues. The following lemma will be helpful in determining the roots of the characteristic equation.

Lemma 6.1.1 (Routh Hurwitz Criteria for a Nonlinear System)**.** *Suppose*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{f}: \mathbb{R}^3 \to \mathbb{R}^3,\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{6.5}$$

Suppose $\boldsymbol{x}_s$ is a fixed point of (6.5) and the characteristic polynomial at the fixed point is

$$\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3 = 0,\ \ \alpha_i \in \mathbb{R}^1.$$

If $\alpha_1 > 0$, $\alpha_3 > 0$ and $\alpha_1\alpha_2 > \alpha_3$, then the fixed point is asymptotically stable. If $\alpha_1 < 0$, $\alpha_3 < 0$ or $\alpha_1\alpha_2 < \alpha_3$, then the fixed point is unstable.

For this chapter where we consider the HPA model without delay, we first perform rigorous stability analysis of all multi-parametric steady states and secondly, by construction of a Lyapunov functional, we prove nonlinear asymptotic stability for some of multi-parametric steady states.

6.2 Spectral Analysis Without Delay

Now let us consider the model or system without a time delay (5.6-5.8). Following general spectral analysis techniques for nonlinear ODEs, we obtain the following equations for the nullclines:

$$o = a,\ \ \tfrac{A}{1+p_2 or} - p_3 a = 0,\ \ \tfrac{(or)^2}{p_4+(or)^2} + p_5 - p_6 r = 0. \tag{6.6}$$

The algebraic system (6.6) has a nonnegative solution in the following domain:

$$D := \{(a, r, o) \in R^3_+ : a = o,\ 0 \leqslant a \leqslant \tfrac{A}{p_3},\ \tfrac{p_5}{p_6} \leqslant r \leqslant \tfrac{p_5+1}{p_6}\}.$$

From (6.6) we have

$$o = a = \tfrac{1}{2p_2 r}\left[\sqrt{1 + \tfrac{4p_2 A}{p_3} r} - 1\right],$$

$$\frac{1}{4p_2^2}\left[\sqrt{1+\frac{4p_2A}{p_3}r}-1\right]^2=\frac{p_4(p_6r-p_5)}{1+p_5-p_6r}\Rightarrow A=\frac{p_3}{r}\sqrt{\frac{p_4(p_6r-p_5)}{1+p_5-p_6r}}\left(1+p_2\sqrt{\frac{p_4(p_6r-p_5)}{1+p_5-p_6r}}\right). \tag{6.7}$$

Note that the equation (6.7) has a unique solution $r^* \in (\frac{p_5}{p_6}, \frac{p_5+1}{p_6})$ in D. Really, the function $f_1(r) := \frac{1}{4p_2^2}[\sqrt{1+\frac{4p_2A}{p_3}r}-1]^2$ is nonnegative and monotone increasing on $[0,+\infty)$ such that $f_1(0)=0$, $f_1(+\infty)=+\infty$, but the function $f_2(r) := \frac{p_4(p_6r-p_5)}{1+p_5-p_6r}$ is nonnegative and monotone increasing on $[\frac{p_5}{p_6}, \frac{p_5+1}{p_6}]$ such that $f_2''(r)>0$, $f_2(\frac{p_5}{p_6})=0$, $f_2(\frac{p_5+1}{p_6})=+\infty$. Therefore, there is only one intersection of $f_1(r)$ and $f_2(r)$ on the interval $[\frac{p_5}{p_6}, \frac{p_5+1}{p_6}]$. On the other hand, let us denote by

$$z := \sqrt{1+\frac{4p_2A}{p_3}r}-1 \geqslant 0 \Rightarrow r=\frac{p_3}{4p_2A}z(z+2).$$

Then (6.7) can be rewritten in the following form

$$z^4+2z^3+C_1z^2+C_2z-C_3=0, \tag{6.8}$$

where

$$C_1 := \frac{4p_2(p_2p_3p_4p_6-A(p_5+1))}{p_3p_6},\ C_2 := 8p_2^2p_4>0,\ C_3 := \frac{16Ap_2^3p_4p_5}{p_3p_6}>0.$$

So, we can find the explicit value of $r^* \in (\frac{p_5}{p_6}, \frac{p_5+1}{p_6})$ as a solution of (6.8).

As a result, the system (5.6)–(5.8) has only one fixed point (a^*, r^*, o^*) in D. Here,

$$a^*=o^*=\frac{1}{2p_2r^*}\left[\sqrt{1+\frac{4p_2A}{p_3}r^*}-1\right]=\frac{1}{r^*}\sqrt{\frac{p_4(p_6r^*-p_5)}{1+p_5-p_6r^*}},$$

and r^* is the solution of (6.7) or (6.8).

Next, we find the Jacobian matrix J^* for (5.6)–(5.8) at the fixed point (a^*, r^*, o^*).

$$J^*=\begin{pmatrix} -p_3 & -K_1 & -K_3 \\ 0 & -p_6+K_2 & K_4 \\ 1 & 0 & -1 \end{pmatrix},$$

where

$$K_1 = \frac{Ap_2a^*}{(1+p_2a^*r^*)^2} = \frac{2A(\sqrt{1+\frac{4p_2A}{p_3}r^*}-1)}{r^*(1+\sqrt{1+\frac{4p_2A}{p_3}r^*})^2} = \frac{Ap_2\sqrt{p_4(p_6r^*-p_5)(1+p_5-p_6r^*)}}{r^*(p_2\sqrt{p_4(p_6r^*-p_5)}+\sqrt{1+p_5-p_6r^*})^2} =$$

$$\frac{p_2p_3p_4(p_6r^*-p_5)}{(r^*)^2(p_2\sqrt{p_4(p_6r^*-p_5)}+\sqrt{1+p_5-p_6r^*})^2}\left(1+p_2\sqrt{\frac{p_4(p_6r^*-p_5)}{1+p_5-p_6r^*}}\right) \geqslant 0,$$

$$K_2 = \frac{2p_4r^*(a^*)^2}{(p_4+(a^*r^*)^2)^2} = \frac{1}{2p_2^2p_4r^*}\left[\sqrt{1+\frac{4p_2A}{p_3}r^*}-1\right]^2(1+p_5-p_6r^*)^2 =$$

$$\frac{2}{r^*}(p_6r^*-p_5)(1+p_5-p_6r^*) \geqslant 0 \text{ and } 0 \leqslant K_2 \leqslant 2p_6(\sqrt{1+p_5}-\sqrt{p_5})^2,$$

$$K_3 = \frac{r^*}{a^*}K_1 = \frac{p_2p_3\sqrt{p_4(p_6r^*-p_5)}}{\sqrt{1+p_5-p_6r^*}+p_2\sqrt{p_4(p_6r^*-p_5)}} \text{ and } 0 \leqslant K_3 \leqslant p_3,$$

$$K_4 = \frac{r^*}{a^*}K_2 = \frac{2r^*}{\sqrt{p_4}}(1+p_5-p_6r^*)^{\frac{3}{2}}\sqrt{(p_6r^*-p_5)} \geqslant 0.$$

Next, we will analyze the stability of the fixed point. First, we look for eigenvalues for J^*. So,

$$|J^*-\lambda I| = \begin{vmatrix} -p_3-\lambda & -K_1 & -K_3 \\ 0 & -p_6+K_2-\lambda & K_4 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0,$$

whence we obtain the characteristic equation:

$$(\lambda+1)(\lambda+p_3)(\lambda+p_6-K_2)+K_3(\lambda+p_6) = 0,$$

i. e.

$$\lambda^3+\alpha_1\lambda^2+\alpha_2\lambda+\alpha_3 = 0, \tag{6.9}$$

where

$$\alpha_1 = p_3+p_6-K_2+1,\ \alpha_2 = p_3+p_6-K_2+p_3(p_6-K_2)+K_3,\ \alpha_3 = p_3(p_6-K_2)+p_6K_3.$$

Let us denote by

$$\Delta := 18\alpha_1\alpha_2\alpha_3 - 4\alpha_1^3\alpha_3 + \alpha_1^2\alpha_2^2 - 4\alpha_2^3 - 27\alpha_3^2.$$

If $\Delta > 0$, then (6.9) has three distinct real roots. If $\Delta = 0$, then (6.9) has a multiple root and all of its roots are real. If $\Delta < 0$, then (6.9) has one real root and two complex roots.

To analyze stability, we will use Lemma 6.1.1. Let $x_1 := \frac{p_5}{p_6} \leqslant x := r^* \leqslant x_2 := \frac{p_5+1}{p_6}$. Then in our case,

$$\alpha_1 > 0 \Leftrightarrow 0 \leqslant K_2 < p_6 + p_3 + 1 \Leftrightarrow (x - x_1)(x_2 - x) < \frac{p_6+p_3+1}{2p_6^2}x,$$

is true provided

$$\frac{(\sqrt{p_5+1}-\sqrt{p_5})^2}{p_6} < \frac{p_6+p_3+1}{2p_6^2} \Rightarrow p_5 > 0 \text{ if } p_6 < p_3 + 1,$$

$$p_5 > \frac{(p_3-p_6+1)^2}{2p_6(p_3+p_6+1)} \text{ if } p_6 \geqslant p_3 + 1,$$

and

$$\alpha_3 > 0 \Leftrightarrow 0 \leqslant K_2 < p_6(1 + \frac{K_3}{p_3}) \Leftrightarrow (x - x_1)(x_2 - x) < \frac{x}{2p_6}\left[1 + \frac{p_2\sqrt{p_4(x-x_1)}}{\sqrt{x_2-x}+p_2\sqrt{p_4(x-x_1)}}\right].$$

Which is true for

$$\frac{(\sqrt{p_5+1}-\sqrt{p_5})^2}{p_6} < \frac{1}{2p_6} \Rightarrow p_5 > \frac{1}{8}.$$

As $\alpha_1 > 0$ and $\alpha_3 > 0$ then

$$\alpha_2 > 0 \Rightarrow 0 \leqslant K_2 \leqslant p_6 + \frac{K_3+p_3}{p_3+1}, \tag{6.10}$$

$$\alpha_1\alpha_2 > \alpha_3 \Leftrightarrow K_2^2 - K_2(p_3 + 1 + 2p_6 + \frac{K_3}{p_3+1}) + (p_3 + p_6)(p_6 + 1) + K_3 > 0. \tag{6.11}$$

From (6.10) and (6.11) it follows that

$$0 \leqslant K_2 \leqslant p_6 + \frac{K_3+p_3}{p_3+1} \Leftrightarrow (x - x_1)(x_2 - x) \leqslant \frac{x}{2p_6^2}\left[p_6 + \frac{p_3}{p_3+1} + \frac{1}{p_3+1}\frac{p_2p_3\sqrt{p_4(x-x_1)}}{\sqrt{x_2-x}+p_2\sqrt{p_4(x-x_1)}}\right].$$

This is true provided

$$\frac{(\sqrt{p_5+1}-\sqrt{p_5})^2}{p_6} < \frac{1}{2p_6^2}(p_6 + \frac{p_3}{p_3+1}), \quad \Leftrightarrow 2p_5 + \frac{1}{2} - \frac{p_3}{2p_6(p_3+1)} < 2\sqrt{p_5(p_5+1)},$$
$$\Rightarrow 0 < p_5 < \frac{p_3 - p_6(p_3+1)}{4p_6(p_3+1)} \text{ if } 0 < p_6 < \frac{p_3}{p_3+1},$$

and

$$p_5 \geqslant \frac{(p_6(p_3+1)-p_3)^2}{p_6(p_3+1)+p_3} \text{ if } p_6 \geqslant \frac{p_3}{p_3+1}.$$

Therefore, the fixed point is asymptotically stable if:

$$\frac{1}{8} < p_5 < \frac{p_3 - p_6(p_3+1)}{4p_6(p_3+1)} \text{ if } 0 < p_6 < \frac{p_3}{p_3+1},$$
$$p_5 > \max\{\frac{1}{8}, \frac{(p_6(p_3+1)-p_3)^2}{p_6(p_3+1)+p_3}\} \text{ if } \frac{p_3}{p_3+1} \leqslant p_6 < p_3 + 1,$$
$$p_5 > \max\{\frac{1}{8}, \frac{(p_6(p_3+1)-p_3)^2}{p_6(p_3+1)+p_3}, \frac{(p_3-p_6+1)^2}{2p_6(p_3+p_6+1)}\} \text{ if } p_6 \geqslant p_3 + 1.$$

Therefore, we can conclude

Lemma 6.2.1. *Assume that $A > 0$ and $p_i > 0$. Then the system (5.6)–(5.8) has a unique fixed point and this point is asymptotically stable if the following conditions are satisfied [51]:*

$$\frac{1}{8} < p_5 < \frac{p_3 - p_6(p_3+1)}{4p_6(p_3+1)} \textit{ if } 0 < p_6 < \frac{p_3}{p_3+1},$$
$$p_5 > \max\{\frac{1}{8}, \frac{(p_6(p_3+1)-p_3)^2}{p_6(p_3+1)+p_3}\} \textit{ if } \frac{p_3}{p_3+1} \leqslant p_6 < p_3 + 1,$$
$$p_5 > \max\{\frac{1}{8}, \frac{(p_6(p_3+1)-p_3)^2}{p_6(p_3+1)+p_3}, \frac{(p_3-p_6+1)^2}{2p_6(p_3+p_6+1)}\} \textit{ if } p_6 \geqslant p_3 + 1.$$

Example 6.2.1. *Let $A = 1$, $p_2 = 15$, $p_3 = 7.2$, $p_4 = 0.05$, $p_5 = 0.11$, and $p_6 = 2.9$. Then $r^* \approx 0.03$, $a^* = o^* \approx 0.12$,*

$$\alpha_1 = 11.1 - K_2 = 11.1 - \frac{1}{22.5r^*}\left[\sqrt{1 + \frac{25A}{3}r^*} - 1\right]^2 (1.11 - 2.9r^*)^2 \approx 11.07,$$

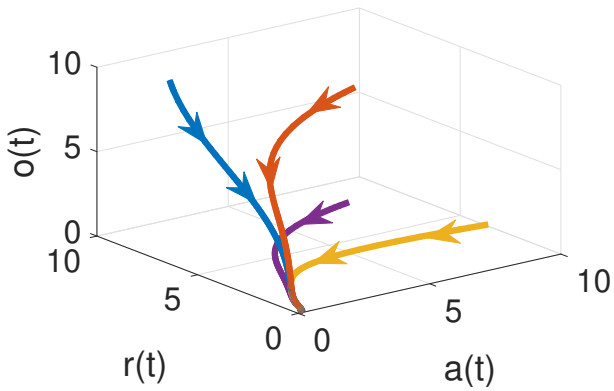


Figure 6.1: A plot of different trajectories illustrating the stable node associated with parameter values given in Example 6.2.1.

$$\alpha_2 = 30.98 - 8.2K_2 + K_3 = 30.98 - \frac{8.2}{22.5r^*}\left[\sqrt{1+\frac{25A}{3}r^*}-1\right]^2(1.11-2.9r^*)^2 + \frac{60A(r^*)^2}{\left[\sqrt{1+\frac{25A}{3}r^*}+1\right]^2} \approx 30.78,$$

$$\alpha_3 = 20.88 - 7.2K_2 + 2.9K_3 = 20.88 - \frac{7.2}{22.5r^*}\left[\sqrt{1+\frac{25A}{3}r^*}-1\right]^2(1.11-2.9r^*)^2 + \frac{174A(r^*)^2}{\left[\sqrt{1+\frac{25A}{3}r^*}+1\right]^2} \approx 20.75,$$

$\Delta \approx 2509.05 > 0$, $\alpha_1\alpha_2 > \alpha_3$. *As a result, all characteristic roots are negative real numbers and the fixed point is stable node.*

A visual representation of this stable node can be found in Figure 6.1. This plot was created using the Matlab ode45 solver [64] using various starting values and the parameter values given above. The starting values were selected so that $a_0 > 0$, $r_0 > 0$ and $o_0 > 0$ to imitate real initial hormone levels.

6.3 Parameter Analysis

Since the conditions for which the system is asymptotically stable involve numerous parameters, let's investigate what happens when certain parameters are set to 0.

Case 1: If $A = 0$ then the system (5.6)–(5.8) has the fixed point $\left(0, \frac{p_5}{p_6}, 0\right)$. The corresponding characteristic equation is

$$(\lambda+1)(\lambda+p_3)(\lambda+p_6) = 0,$$

whence $\lambda_i = -1, -p_3, -p_6 < 0$. As a result, $\left(0, \frac{p_5}{p_6}, 0\right)$ is stable node.

Case 2: If $p_2 = 0$ then the system (5.6)–(5.8) has the fixed point $\left(\frac{A}{p_3}, r^*, \frac{A}{p_3}\right)$, where r^* is a solution of the equation:

$$\frac{p_4}{p_4+(\frac{A}{p_3})^2 r^2} = 1 + p_5 - p_6 r.$$

This equation can also be written as

$$r\left[\frac{p_6}{p_4 p_5}\left(\frac{A}{p_3}\right)^2 r^2 - \frac{1+p_5}{p_4 p_5}\left(\frac{A}{p_3}\right)^2 r + \frac{p_6}{p_5}\right] = 1$$

which yields the following cases:

- if $p_4 > \left[\frac{A(1+p_5)}{p_3 p_6}\right]^2$ then there is one real root;
- if $p_4 = \left[\frac{A(1+p_5)}{p_3 p_6}\right]^2$ then $r\left(r - \frac{1+p_5}{2p_6}\right)^2 = \frac{p_4 p_5}{p_6}$, whence
  - if $\frac{p_5 A^2}{p_3^2 p_6} < \frac{1}{54}$ then we have 3 real roots,
  - if $\frac{p_5 A^2}{p_3^2 p_6} = \frac{1}{54}$ then we have 2 real roots,
  - if $\frac{p_5 A^2}{p_3^2 p_6} > \frac{1}{54}$ then we have 1 real root;
- if $p_4 < \left[\frac{A(1+p_5)}{p_3 p_6}\right]^2$ then $r(r-r_1)(r-r_2) = \frac{p_4 p_5}{p_6}\left(\frac{p_3}{A}\right)^2$, where $r_{1,2} = \frac{1}{2p_6}\left[1 + p_5 \pm \sqrt{(p_5+1)^2 - p_4 p_6^2\left(\frac{p_3}{A}\right)^2}\right]$ so
  - if $\frac{p_4 p_5}{p_6}(\frac{p_3}{A})^2 < \frac{(r_1+r_2-K)(r_2-2r_1-K)(r_1-2r_2-K)}{27}$ then we have 3 real roots,
  - if $\frac{p_4 p_5}{p_6}(\frac{p_3}{A})^2 = \frac{(r_1+r_2-K)(r_2-2r_1-K)(r_1-2r_2-K)}{27}$ then we have 2 real roots,
  - if $\frac{p_4 p_5}{p_6}(\frac{p_3}{A})^2 > \frac{(r_1+r_2-K)(r_2-2r_1-K)(r_1-2r_2-K)}{27}$ then we have 1 real root, where $K^2 = r_1^2 - r_1 r_2 + r_2^2$.

The corresponding characteristic equation is

$$(\lambda+1)(\lambda+p_3)(\lambda+p_6-K_2) = 0,$$

whence $\lambda_i = -1, -p_3, -p_6 + K_2$. If $K_2 < p_6$ then $(\frac{A}{p_3}, r^*, \frac{A}{p_3})$ is stable node. If $K_2 > p_6$ then $(\frac{A}{p_3}, r^*, \frac{A}{p_3})$ is saddle. If $K_2 = p_6$ then it is a non-hyperbolic fixed point.

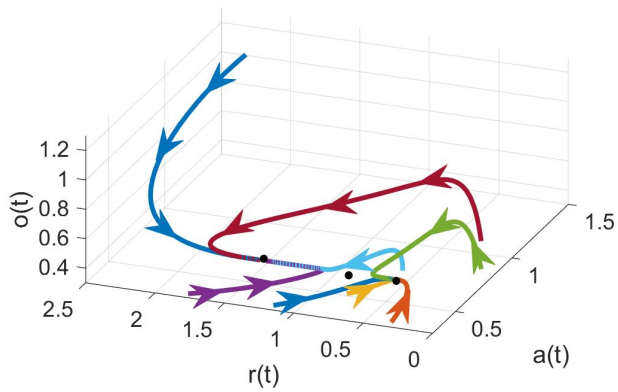


Figure 6.2: A plot of different trajectories illustrating the unstable saddle-node with only realistic initial conditions and the above parameter values stated in Example 6.3.1.

Example 6.3.1. *Let* $A = 0.106$, $p_2 = 0$, $p_3 = 0.222$, $p_4 = 0.464$, $p_5 = 0.094$, *and* $p_6 = 0.418$. *Then* $r^* \approx 0.39, 0.83, 1.38$ *and* $a^* = o^* \approx 0.47$. *Using similar calculations as above according to the defined values. If* $r^* \approx 0.39$ *then* $\alpha_1 \approx 1.30$, $\alpha_2 \approx 0.32$, $\alpha_3 \approx 0.01$, *and* $K_2 \approx 0.33 < p_6$ *which means this is a stable node. If* $r^* \approx 0.83$ *then* $\alpha_1 \approx 1.18$, $\alpha_2 \approx 0.17$, *and* $\alpha_3 \approx -0.008$, *and* K_2 $.45 > p_6$ *which means this is a saddle. If* $r^* \approx 1.38$ *then* $\alpha_1 \approx 1.28$, $\alpha_2 \approx 0.29$, $\alpha_3 \approx 0.01$, *and* $K_2 \approx 0.35 < p_6$ *which means this is a stable node.*

This is illustrated in Figure 6.2 using the stated above parameter values. The starting values were selected so that $a_0 > 0$, $r_0 > 0$ *and* $o_0 > 0$ *to imitate real initial hormone levels.*

Case 3: If $p_3 = 0$ then the system (5.6)–(5.8) has the fixed point $(+\infty, \frac{p_5+1}{p_6}, +\infty)$. The corresponding characteristic equation is

$$\lambda(\lambda+1)(\lambda+p_6) = 0,$$

whence $\lambda_i = -1, 0, -p_6$. As a result, $(+\infty, \frac{p_5+1}{p_6}, +\infty)$ is non-hyperbolic fixed point.
Case 4: If $p_4 = 0$ then the system (5.6)–(5.8) has the fixed point $\left(a^*, \frac{p_5+1}{p_6}, a^*\right)$, where

$$a^* = \frac{p_6}{2p_2(p_5+1)}[\sqrt{1+\frac{4Ap_2(p_5+1)}{p_3p_6}} - 1].$$

In this case, we have that $K_2 = K_3 = 0$ and

$$(\lambda+1)(\lambda+p_3)(\lambda+p_6) = 0,$$

whence $\lambda_i = -1, -p_3, -p_6$. Hence, the fixed point is a stable node.

Case 5: If $p_2 = p_4 = 0$ then we obtain the explicit solution

$$\begin{aligned} a(t) &= (a_0 - \tfrac{A}{p_3})e^{-p_3 t} + \tfrac{A}{p_3} \to \tfrac{A}{p_3} \text{ as } t \to +\infty, \\ r(t) &= (r_0 - \tfrac{p_5+1}{p_6})e^{-p_6 t} + \tfrac{1+p_5}{p_6} \to \tfrac{1+p_5}{p_6} \text{ as } t \to +\infty, \\ o(t) &= (o_0 - \tfrac{A}{p_3})e^{-t} + (a_0 - \tfrac{A}{p_3})e^{-t} \int_0^t e^{(1-p_3)s} ds + \tfrac{A}{p_3} \to \tfrac{A}{p_3} \text{ as } t \to +\infty. \end{aligned}$$

Case 6: If $p_5 = 0$ then the one of fixed points is $\left(\frac{A}{p_3}, 0, \frac{A}{p_3}\right)$ and

$$(\lambda + 1)(\lambda + p_3)(\lambda + p_6) = 0,$$

whence $\lambda_i = -1, -p_3, -p_6$. Hence, this fixed point is stable node. In this case, by (6.6) we obtain that

$$\tfrac{A}{1+p_2 ar} = p_3 a \Leftrightarrow r = \tfrac{1}{p_2 a}(\tfrac{A}{p_3 a} - 1) \quad \text{provided} \quad 0 < a < \tfrac{A}{p_3},$$

whence

$$\tfrac{(ar)^2}{p_4+(ar)^2} = p_6 r$$

implies that

$$r = 0 \text{ or } \tfrac{a^2 r}{p_4+(ar)^2} = p_6.$$

Hence, we find that

$$a^3 + (\tfrac{p_6(1+p_2^2 p_4)}{p_2} - \tfrac{A}{p_3})a^2 - 2\tfrac{Ap_6}{p_2 p_3}a = -\tfrac{p_6}{p_2}(\tfrac{A}{p_3})^2 \Leftrightarrow f(a) := a(a-a_1)(a-a_2) = -\tfrac{p_6}{p_2}(\tfrac{A}{p_3})^2,$$

where

$$a_{1,2} = \frac{1}{2}\left[-\left(\frac{p_6(1+p_2^2p_4)}{p_2} - \frac{A}{p_3}\right) \pm \sqrt{\left(\frac{p_6(1+p_2^2p_4)}{p_2} - \frac{A}{p_3}\right)^2 + 8\frac{Ap_6}{p_2p_3}}\right]$$

and $a_1 < 0 < a_2$. Note that $a_2 \leqslant \frac{A}{p_3}$ provided $p_2^2p_4 \geqslant 1$. As a result,

- if $p_2^2p_4 \geqslant 1$ then
  - if $f_{min} > -\frac{p_6}{p_2}(\frac{A}{p_3})^2$ then no real roots;
  - if $f_{min} = -\frac{p_6}{p_2}(\frac{A}{p_3})^2$ then 1 positive real root;
  - if $f_{min} < -\frac{p_6}{p_2}(\frac{A}{p_3})^2$ then 2 positive real roots;
- if $p_2^2p_4 < 1$ then
  - if $f_{min} \geqslant -\frac{p_6}{p_2}(\frac{A}{p_3})^2$ then no real roots;
  - if $f_{min} < -\frac{p_6}{p_2}(\frac{A}{p_3})^2$ then 1 positive real root.

Case 7: If $p_6 = p_5 = 0$ then we have the following system

$$\begin{aligned} a'(t) &= \frac{A}{1+p_2or} - p_3a, \\ r'(t) &= \frac{(or)^2}{p_4+(or)^2}, \\ o'(t) &= a - o. \end{aligned}$$

If $r_0 = 0$ then we find the explicit solution

$$\begin{aligned} a(t) &= (a_0 - \frac{A}{p_3})e^{-p_3t} + \frac{A}{p_3}, \\ r(t) &= 0, \\ o(t) &= (o_0 - \frac{A}{p_3})e^{-t} + (o_0 - \frac{A}{p_3})e^{-t}\int_0^t e^{(1-p_3)s}ds + \frac{A}{p_3}. \end{aligned}$$

If $r_0 \neq 0$ then we approximately have

$$
\begin{aligned}
a'(t) &\approx -p_3\left(a - \tfrac{A}{p_3}\right) - \tfrac{A^2 p_2}{p_3} r, \\
r'(t) &\approx \tfrac{1}{p_4}\left(\tfrac{A}{p_3}\right)^2 r^2, \\
o'(t) &= a - o,
\end{aligned}
$$

whence

$$
\begin{aligned}
r(t) &\approx \tfrac{r_0}{1-\frac{r_0}{p_4}(\frac{A}{p_3})^2 t} \to +\infty \text{ as } t \to T^* := \tfrac{p_4}{r_0}\left(\tfrac{p_3}{A}\right)^2, \\
a(t) &\approx a_0 e^{-p_3 t} + \tfrac{A}{p_3}(1 - e^{-p_3 t}) - \tfrac{A^2 p_2}{p_3} e^{-p_3 t} \int_0^t r(s) e^{p_3 s} ds, \\
o(t) &\approx o_0 e^{-t} + e^{-t} \int_0^t a(s) e^s ds.
\end{aligned}
$$

If $p_6 = 0$ but $p_5 \neq 0$ then $r(t)$ blows up in a finite time too.

Also, note that if $p_6 = 0$ and $p_5 = 0$ then the system (5.6)–(5.8) has the fixed point $(\frac{A}{p_3}, 0, \frac{A}{p_3})$. The corresponding characteristic equation is

$$
\lambda(\lambda + 1)(\lambda + p_3) = 0,
$$

whence $\lambda_i = -1, -p_3, 0$. As a result, $(\frac{A}{p_3}, 0, \frac{A}{p_3})$ is non-hyperbolic fixed point.

6.4 Nonlinear Stability Analysis

In this section, we show the stability of the fixed point by using the Lyapunov function approach. We consider the system (5.6)-(5.8) and denote

$$
W(t) := \tfrac{1}{2}[(a(t) - a^*)^2 + (r(t) - r^*)^2 + (o(t) - o^*)^2 + (o(t)r(t) - o^* r^*)^2],
$$

where (a^*, r^*, o^*) is the fixed point ($a^* = o^*$).

Lemma 6.4.1 (Stability). *Assume that*

$$A \geqslant 0,\ p_2 \geqslant 0,\ p_3 > \tfrac{1}{2},\ p_4 \geqslant 0,\ p_6 > \frac{1}{\min\{p_3 - \frac{1}{2}, p_6, 1\}},$$

and

$$0 \leqslant p_5 < p_6 \min\{p_3 - \tfrac{1}{2}, p_6, 1\} - 1.$$

Then there exist $W^* > 0$, $A_0 > 0$ and $p_4^* > 0$ such that

$$W(t) \to 0 \text{ as } t \to +\infty \tag{6.12}$$

provided $W(0) < W^*$, $0 \leqslant A < A_0$, $p_4 > p_4^*$, hence, the fixed point (a^*, r^*, o^*) is globally stable. If $p_4 \leqslant p_4^*$ then there exist $A_0 \leqslant A_1 < A_2$ such that (6.12) holds provided $W(0) < W^*$, $A_1 < A < A_2$ [51].

Proof of Lemma 6.4.1. Using the system (5.6)–(5.8), we have

$$\begin{aligned}\tfrac{d}{dt}W(t) - (or - o^*r^*)(o(t)r(t))' = -p_3(a-a^*)^2 - p_6(r-r^*)^2 - (o-o^*)^2 + \\ (a-a^*)\big[\tfrac{A}{1+p_2 or} - \tfrac{A}{1+p_2 o^* r^*}\big] + (r-r^*)\big[\tfrac{p_4}{p_4+(o^*r^*)^2} - \tfrac{p_4}{p_4+(or)^2}\big] + (a-o^*)(o-o^*).\end{aligned}$$

As

$$2(a-o^*)(o-o^*) \leqslant (a-a^*)^2 + (o-o^*)^2, \quad \big|\tfrac{1}{1+p_2 or} - \tfrac{1}{1+p_2 o^* r^*}\big| \leqslant p_2|or - o^*r^*|,$$

$$\begin{aligned}(o(t)r(t))' = r(a-o) + o[-\tfrac{p_4}{p_4+(or)^2} + 1 + p_5 - p_6 r] = (r-r^*)(a-a^*) + r^*(a-a^*) + a^*(r-r^*) \\ - (or - o^*r^*) - p_6(r-r^*)(o-o^*) - p_6 o^*(r-r^*) + (o-o^*)\big[\tfrac{p_4}{p_4+(o^*r^*)^2} - \tfrac{p_4}{p_4+(or)^2}\big] \\ + o^*\big[\tfrac{p_4}{p_4+(o^*r^*)^2} - \tfrac{p_4}{p_4+(or)^2}\big],\end{aligned}$$

$$\big|\tfrac{p_4}{p_4+(o^*r^*)^2} - \tfrac{p_4}{p_4+(or)^2}\big| \leqslant \tfrac{|or - o^*r^*|\cdot|or + o^*r^*|}{p_4+(o^*r^*)^2}, \quad |or + o^*r^*| \leqslant |or - o^*r^*| + 2o^*r^*,$$

then

$$\tfrac{d}{dt}W(t) \leqslant -\alpha W(t) + \beta W^{\frac{3}{2}}(t) + \gamma W^2(t),$$

i. e.

$$\frac{d}{dt}W(t) \leqslant \gamma W(t)\left[W^{\frac{1}{2}}(t) + \frac{\beta-\sqrt{\beta^2+4\alpha\gamma}}{2\gamma}\right]\left[W^{\frac{1}{2}}(t) + \frac{\beta+\sqrt{\beta^2+4\alpha\gamma}}{2\gamma}\right], \tag{6.13}$$

where

$$\alpha = 2[\min\{p_3 - \tfrac{1}{2}, p_6, 1\} - Ap_2 - r^* - (p_6+1)a^* - \tfrac{4o^*r^*}{p_4+(o^*r^*)^2}] > 0$$

$$\beta = 2^{\frac{3}{2}}[p_6 + 1 + \tfrac{3o^*r^*}{p_4+(o^*r^*)^2}] \leqslant 2^{\frac{3}{2}}[p_6 + 1 + 3\min\{\tfrac{1}{2p_4^{\frac{1}{2}}}, \tfrac{2p_2}{\sqrt{1+\frac{4p_2p_5A}{p_6}}-1}\}],$$

$$\gamma = \tfrac{4o^*r^*}{p_4+(o^*r^*)^2} \leqslant 4\min\{\tfrac{1}{2p_4^{\frac{1}{2}}}, \tfrac{2p_2}{\sqrt{1+\frac{4p_2p_5A}{p_6}}-1}\},$$

provided

$$0 < r^* + (p_6+1)a^* + \tfrac{4o^*r^*}{p_4+(o^*r^*)^2} < \min\{p_3 - \tfrac{1}{2}, p_6, 1\} - Ap_2. \tag{6.14}$$

As $0 \leqslant a^* = o^* \leqslant \frac{A}{p_3}$, $\frac{p_5}{p_6} \leqslant r^* \leqslant \frac{p_5+1}{p_6}$ and $o^*r^* \geqslant \frac{1}{2p_2}[\sqrt{1+\frac{4p_2p_5A}{p_6}}-1]$ then by (6.14) we get

$$\tfrac{p_6+1+p_2p_3}{p_3}A + \min\{\tfrac{2}{p_4^{\frac{1}{2}}}, \tfrac{8p_2}{\sqrt{1+\frac{4p_2p_5A}{p_6}}-1}\} < B := \min\{p_3 - \tfrac{1}{2}, p_6, 1\} - \tfrac{p_5+1}{p_6}.$$

Hence,

$$\tfrac{p_6+1+p_2p_3}{p_3}A + \tfrac{2}{p_4^{\frac{1}{2}}} < B \text{ and } A \leqslant \tfrac{2p_6p_4^{\frac{1}{2}}}{p_5}(1+2p_2p_4^{\frac{1}{2}}),$$

whence

$$0 \leqslant A < A_0 := \min\{\tfrac{2p_6p_4^{\frac{1}{2}}}{p_5}(1+2p_2p_4^{\frac{1}{2}}), \tfrac{p_3}{p_6+1+p_2p_3}(B - \tfrac{2}{p_4^{\frac{1}{2}}})\},$$

or

$$F(A) := \tfrac{p_6+1+p_2p_3}{p_3}A + \tfrac{8p_2}{\sqrt{1+\frac{4p_2p_5A}{p_6}}-1} < B \text{ and } A > \tfrac{2p_6p_4^{\frac{1}{2}}}{p_5}(1+2p_2p_4^{\frac{1}{2}}).$$

As the function $F(A)$ has a unique minimum for positive A, denote by $A_{\min}$, then there exist $0 < A_1 < A_{\min} < A_2$ such that $F(A) < B$ provided $F(A_{\min}) < B$.

So, if $W(0) < [\frac{\sqrt{\beta^2+4\alpha\gamma}-\beta}{2\gamma}]^2$ then by (6.13) we deduce that

$$W(t) \to 0 \text{ as } t \to +\infty.$$

□

Chapter 7

Spectral Analysis of HPA Model with Delay

7.1 Introduction

Delay-differential equations (DDEs) are a large and important class of dynamical systems. They arise in systems that have a component which makes adjustments to the system based on it's observations. In the HPA model, the CORT reacts based on how much ACTH there was at the previous time, not instantaneously since the ACTH needs to travel considerable lengths within the body. Therefore there is a delay between the amount made at time t_1 and how CORT responds to the amount of ACTH when it gets to CORT at time $t_1 + \tau$.

There are different kinds of delay-differential equations but we will focus on those of the form

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{x}\left(t-\tau_1\right), \mathbf{x}\left(t-\tau_2\right), \ldots, \mathbf{x}\left(t-\tau_n\right)\right)$$

where the quantities τ_i are positive constants. It is important to note that although we will not consider them, there are other forms of DDEs such as equations with state-dependent delays where the τ_i 's depend on $\mathbf{x}$ or with distributed delays where the right-hand side of the differential equation is a weighted integral over past states. [77]

When we give initial conditions for finite-dimensional dynamical systems, we only need to specify the initial values of the state variables. In order to solve a delay equation, we need more.

At every time step, we have to look back to earlier τ values of $\mathbf{x}$. So therefore we need to specify an initial function which gives the behavior of the system prior to time t_0 which is typically 0. This function has to cover a period at least as long as the longest delay since we will be looking back in time that far.

Let us narrow our focus to equations with a single delay, i.e.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau))$$

The initial function would be a function $\mathbf{x}(t)$ defined on the interval $[-\tau, 0]$. The solutions of this dynamical system can be thought of as a sequence of functions $\mathbf{f}_0(t), \mathbf{f}_1(t), \mathbf{f}_2(t), \ldots$ defined over a set of contiguous time intervals of length τ. The points $t = 0, \tau, 2\tau, \ldots$ where the solution segments meet are called knots. Existence and uniqueness theorems analogous to those for ordinary differential equations are much more easily proven in this conceptual framework than by trying to think of DDEs as an evolution over the state space $\mathbf{x}$.[77]

Example 7.1.1. *Consider the delay-differential equation*

$$\dot{x} = -x(t-1) \tag{7.1}$$

The equilibrium points satisfy

$$\mathbf{f}\left(\mathbf{x}^*, \mathbf{x}^*, \mathbf{x}^*, \ldots, \mathbf{x}^*\right) = 0$$

We proceed with DDEs, similarly to ODEs, except that the phase space is now our infinite-dimensional function space, so that we have to consider displacements from equilibrium in this space. In other words, our displacements will be time-dependent functions $\delta\mathbf{x}(t)$ over an interval of at least $\tau_{\max}$, the longest delay.

Since writing things like $\mathbf{x}(t-\tau)$ is tedious. It is a common convention for delayed variables to be indicated by the subscripted value of the delay $\mathbf{x}_\tau \equiv \mathbf{x}(t-\tau)$. So let $\mathbf{x}^*$ be an equilibrium of (7.1) and let the system be disturbed from equilibrium by a small perturbation which lasts from $t = t_0 - \tau_{\max}$ to t_0. Let $\delta\mathbf{x}(t)$ be the displacement from equilibrium, assumed small, at any time in

the open interval $[t_0 - \tau_{\max}, \infty)$. Accordingly,

$$\mathbf{x} = \mathbf{x}^* + \delta\mathbf{x}$$

and

$$\dot{\mathbf{x}} = \dot{\delta}\mathbf{x} = \mathbf{f}\left(\mathbf{x}^* + \delta\mathbf{x}, \mathbf{x}^* + \delta\mathbf{x}_{\tau_1}, \mathbf{x}^* + \delta\mathbf{x}_{\tau_2}, \ldots, \mathbf{x}^* + \delta\mathbf{x}_{\tau_n}\right) \tag{7.2}$$

since each of the quantities $\delta\mathbf{x}, \delta\mathbf{x}_{\tau_1}, \delta\mathbf{x}_{\tau_2}, \ldots, \delta\mathbf{x}_{\tau_n}$ is small, we can linearize

$$\dot{\delta\mathbf{x}} \approx \mathbf{J}_0\delta\mathbf{x} + \mathbf{J}_{\tau_1}\delta\mathbf{x}_{\tau_1} + \mathbf{J}_{\tau_2}\delta\mathbf{x}_{\tau_2} + \ldots + \mathbf{J}_{\tau_n}\delta\mathbf{x}_{\tau_n}$$

$\mathbf{J}_0$ is the usual Jacobian with respect to $\mathbf{x}$ evaluated at the equilibrium point, while the matrices $\mathbf{J}_{\tau_i}$ are the Jacobians with respect to the corresponding $\mathbf{x}_{\tau_i}$ again evaluated at $\mathbf{x} = \mathbf{x}_{\tau_1} = \mathbf{x}_{\tau_2} = \ldots = \mathbf{x}_{\tau_n} = \mathbf{x}^*$. Suppose that (7.2) also has exponential solutions, i.e. that we can write

$$\delta\mathbf{x}(t) = \mathbf{A}e^{\lambda t}$$

Substituting this ansatz into (7.2) and rearranging a bit, we get

$$\lambda\mathbf{A} = \left(\mathbf{J}_0 + e^{-\lambda\tau_1}\mathbf{J}_{\tau_1} + e^{-\lambda\tau_2}\mathbf{J}_{\tau_2} + \ldots + e^{-\lambda\tau_n}\mathbf{J}_{\tau_n}\right)\mathbf{A}$$

This equation can only be satisfied if

$$\left|\mathbf{J}_0 + e^{-\lambda\tau_1}\mathbf{J}_{\tau_1} + e^{-\lambda\tau_2}\mathbf{J}_{\tau_2} + \ldots + e^{-\lambda\tau_n}\mathbf{J}_{\tau_n} - \lambda\mathbf{I}\right| = 0 \tag{7.3}$$

where I is the identity matrix. (7.3) is called the characteristic equation of the equilibrium point which is a quasi-polynomial or transcendental equation[77]. Since the goal of stability analysis is still the same, having a transcendental equation makes it a lot harder to determine the roots.

In systems of equations with delay, the characteristic equation may be a transcendental equation of the form

$$P(z) + Q(z)e^{-Tz} = 0 \tag{7.4}$$

where P and Q are usually polynomials with real coefficients of degree n and m respectively, and T is a nonnegative constant. We call (7.4) stable if all zeros lie in $Re(z) < 0$, and unstable if at least one zero lies in the half-plane $Re(z) > 0$. If we allow T to vary, as a delay may do, it may happen that zeros cross the imaginary axis, and the equation may change from stable to unstable or vice versa. We call this a stability switch.

Theorem 12. *[35] Consider equation (7.4), where P and Q are analytic functions in a right half-plane $Re(z) > -\delta$, $\delta > 0$, which satisfy the following conditions:*

i) *$P(z)$ and $Q(z)$ have no common imaginary zeros.*

ii) *$\overline{P(-iy)} = P(iy)$ and $\overline{Q(-iy)} = Q(iy)$ for real y.*

iii) *$P(0) + Q(0) \neq 0$.*

iv) *There are at most a finite number of roots of (7.4) in the right half-plane when $T = 0$.*

v) *$F(y) \equiv |P(iy)|^2 - |Q(iy)|^2$ for real y, has at most a finite number of real zeros.*

Under these conditions, the following statements are true.

a) *Suppose that the equation $F(y) = 0$ has no positive roots. Then if (7.4) is stable at $T = 0$ it remains stable for all $T \geq 0$, whereas if it is unstable at $T = 0$ it remains unstable for all $T \geq 0$.*

b) *Suppose that the equation $F(y) = 0$ has at least one positive root and that each positive root is simple. As T increases, stability switches may occur. There exists a positive number T^* such that the equation (7.4) is unstable for all $T > T^*$. As T varies from 0 to T^*, at most a finite number of stability switches may occur.*

Proof. First note that restriction (i) is not an important restriction because if there is a common imaginary zero $z = iy$, then $P(z) + Q(z)e^{-Tz} = (iy)^k(P_1(z) + Q_1(z)e^{-Tz})$ where k is an integer and P_1 and Q_1 have no common zeros and the theorem can be applied to $(P_1(z) + Q_1(z)e^{-Tz}$. Secondly, restriction (iii) is the same as saying $z = 0$ is not a root. If $z = 0$ is a common root of P and Q, we may proceed by removing the common factor. If $P(0) + Q(0) = 0$ but $z = 0$ is not a common root of P and Q, then $z = 0$ is a zero for all $T \geq 0$, and thus the equation is not stable.

Thus, let's begin by looking at purely imaginary roots $z = iy \neq 0$ of (7.4). Assume that (i), (ii), and (iii) hold. Because of (ii) we may choose $y > 0$ WLOG. Then (7.4) implies that

$$\begin{aligned} P(iy) &= -Q(iy)e^{-Tiy} \\ |P(iy)| &= |-Q(iy)e^{-Tiy}| \\ |P(iy)| &= |Q(iy)| \end{aligned} \tag{7.5}$$

and this determines possible y. If we set

$$P(iy) = P_R(y) + iP_I(y) \quad Q(iy) = Q_R(y) + iQ_I(y)$$

where P_R, P_I, Q_R, and Q_I are real-valued functions of y. Thus, (7.4) implies that

$$\begin{aligned} (P_R(y) + iP_I(y)) + (P_R(y) + iP_I(y))e^{-Tiy} &= 0 \\ (P_R(y) + iP_I(y)) + (P_R(y) + iP_I(y))(\cos(y) + i\sin(y)) &= 0 \end{aligned}$$

Equating real and imaginary parts yields

$$\begin{cases} Q_R\cos(Ty) + Q_I\sin(Ty) = -P_R \\ Q_I\cos(Ty) - Q_R\sin(Ty) = -P_I \end{cases} \tag{7.6}$$

Thus

$$\sin(Ty) = \frac{-P_RQ_I + Q_RP_I}{Q_R^2 + Q_I^2} \quad \cos(Ty) = -\frac{P_RQ_R + P_IQ_I}{Q_R^2 + Q_I^2} \tag{7.7}$$

It is not possible that Q_R and Q_I are both zero, since then $Q_R + iQ_I = 0$. Then by (7.5) it would imply $P(iy) = Q(iy) = 0$ and thus they would have a common root which we assumed we do not have. So for each root y of (7.5), it may be possible to determine values of T that satisfy (7.7) with sine and cosine in $[-1, 1]$.

Assuming that we have found values of iy and T that satisfy (7.4), (7.5), (7.6), (7.7), we can assume that the root $z = x + iy$ of (7.4) as a function of T and try to determine the direction of

motion of z as T is varied. Thus, we determine

$$s = sign\left\{Re\left(\frac{dz}{dT}|_{z=iy}\right)\right\} = sign\left\{\frac{d}{dT}(Rez)|_{z=iy}\right\}.$$

Since the left side of (7.4) is an analytic function of z and T, a root z will be a differentiable function of T, except at points where the root is a multiple. At a multiple root, we have

$$P'(z) + [Q'(z) - TQ(z)]e^{-Tz} = 0 \tag{7.8}$$

or since $e^{-Tz} = -\dfrac{P(z)}{Q(z)}$,

$$P'(z)Q(z) - P(z)Q'(z) + TP(z)Q(z) = 0 \tag{7.9}$$

If we assume that (7.8) does not hold, we may consider $z = z(T)$ to be a differentiable function, and then differentiating (7.4) with respect to T, under the assumption that P and Q are independent of T, yields

$$P'(z)\frac{dz}{dT} + Q'(z)e^{-Tz}\frac{dz}{dT} - TQ(z)e^{-Tz}\frac{dz}{dT} - zQ(z)e^{-Tz} = 0$$
$$\frac{dz}{dT} = \frac{zQ(z)}{P'(z)e^{Tz} + Q'(z) - TQ(z)}$$

Taking the inverse and using (7.4),

$$\begin{aligned}\left(\frac{dz}{dT}\right)^{-1} &= \frac{P'(z)e^{Tz} + Q'(z) - TQ(z)}{zQ(z)} \\ &= \frac{P'(z)e^{Tz}}{-zP(z)e^{Tz}} + \frac{Q'(z)}{zQ(z)} - \frac{TQ(z)}{zQ(z)} \\ &= -\frac{P'(z)}{zP(z)} + \frac{Q'(z)}{zQ(z)} - \frac{T}{z}\end{aligned} \tag{7.10}$$

Now it is important to note that at a simple root (7.9) fails and therefore $(\frac{dz}{dT})^{-1}$ is not zero. At a root of (7.4), $P(z) = 0$ implies that $Q(z) = 0$ and vice versa, which contradicts our assumption that

P and Q are not simultaneously zero. Thus, (7.10) holds at any simple root iy of (7.4). Moreover,

$$s = \text{sign}\left\{Re\left(\frac{dz}{dT}\right)|_{z=iy}\right\} = \text{sign}\left[-\frac{P'(iy)}{iyP(iy)} + \frac{Q'(iy)}{iyQ(iy)} - \frac{T}{iy}\right] \tag{7.11}$$

$$= -\text{sign}Im\left[\frac{P'(iy)}{yP(iy)} - \frac{Q'(iy)}{yQ(iy)}\right] \tag{7.12}$$

By (7.5) and assuming that $y > 0$, we can further simplify to

$$s = -\text{sign}Im[P'(iy)\overline{P(iy)} - Q'(iy)\overline{Q(iy)}] \tag{7.13}$$

Now this equation tells us the direction in which a root $z(T)$ of (7.4) crosses the imaginary axis at any simple root iy of (7.4) if $s \neq 0$. It is important to note that the crossing direction at iy depends on y only and is independent of T.

There is another useful form of (7.13) that we get by noting

$$P'(iy) = P'(z)|_{z=iy} = \frac{1}{i}\frac{dP(iy)}{dy} = -i\frac{d}{dy}[P_R(y) + iP_I(y)]$$

Using the notation that $P'_R(y) = \frac{dP_R(y)}{dy}$ and so on, we have $P'(iy) = P'_I(y) - iP'_R(y)$ and $Q'(iy) = Q'_I(y) - iQ'_R(y)$. Thus,

$$-Im[P'(iy)\overline{P(iy)} - Q'(iy)\overline{Q(iy)}] = P_R P'_R + P_I P'_I - Q_R Q'_R - Q_I Q'_I \tag{7.14}$$

Also, we define

$$F(y) = |P(iy)|^2 - |Q(iy)|^2 = P_R^2(y) + P_I^2(y) - Q_R^2(y) - Q_I^2(y)$$

If iy is a root of (7.5), then $F(y) = 0$ and that y is a simple root of $F(y)$ if and only if it is a simple root of (7.5). We also know that

$$F'(y) = 2(P_R P'_R + P_I P'_I - Q_R Q'_R - Q_I Q'_I)$$

which yields that

$$s = \text{sign} F'(y). \tag{7.15}$$

Everything can be summarized as follows:

Assume that P and Q are as in (12), and that they satisfy (i), (ii) and (iii) of the theorem. The the following assertions hold.

i) If $iy(y > 0)$ and T satisfy (7.4) and if iy is a simple root and $s \neq 0$, then y is a simple root of $F(y) = 0$ and the root $z(T)$ of (7.4) crosses the imaginary axis (as T increases) in the direction given by $s = signF'(y)$.

ii) $iy(y > 0)$ is a simple root of (7.5) if and only if it is a simple root of $F(y) = 0$. Then for this y there are infinitely many values of T satisfying (7.7) and for each such value T, $z = iy$ is a simple root of (7.4) and $s \neq 0$.

□

This theorem and its proof served as the template used to analyze the effects of the time delay on the stability in the system (5.1)–(5.3).

For this chapter we take into account the additional effects of the time delay parameter on the stability of the HPA axis system. Then we prove the existence of periodic solutions for the HPA axis system.

7.2 Spectral Analysis with Delay

Note the fixed point (a^*, r^*, o^*) for (5.6)–(5.8) coincides with the one for (5.1)–(5.3). Let us denote by

$$J_\tau := \begin{pmatrix} \frac{\partial f_1}{\partial a_\tau} & \frac{\partial f_1}{\partial r_\tau} & \frac{\partial f_1}{\partial o_\tau} \\ \frac{\partial f_2}{\partial a_\tau} & \frac{\partial f_2}{\partial r_\tau} & \frac{\partial f_2}{\partial o_\tau} \\ \frac{\partial f_3}{\partial a_\tau} & \frac{\partial f_3}{\partial r_\tau} & \frac{\partial f_3}{\partial o_\tau} \end{pmatrix},$$

where $a_\tau = a(t-\tau)$, $r_\tau = r(t-\tau)$, and $o_\tau = o(t-\tau)$. Then J_τ at the point (a^*, r^*, a^*) is equal

$$J_\tau^* := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now we will look for eigenvalues for the matrix $J^* + e^{-\lambda\tau} J_\tau^*$. So,

$$|J^* + e^{-\lambda\tau} J_\tau^* - \lambda I| = \begin{vmatrix} -p_3 - \lambda & -K_1 & -K_3 \\ 0 & -p_6 + K_2 - \lambda & K_4 \\ 1 + e^{-\lambda\tau} & 0 & -1 - \lambda \end{vmatrix} = 0,$$

whence we obtain the characteristic equation:

$$(\lambda + 1)(\lambda + p_3)(\lambda + p_6 - K_2) = -(1 + e^{-\lambda\tau}) K_3 (\lambda + p_6).$$

Time delays are known to affect the stability of a fixed point. They can induce stability switches in which the zeros of the characteristic equation may cross the imaginary axis as the delay, τ, increases. Looking at the characteristic equation as a function of τ, and analyzing the location of the roots and the direction of motion as they cross the imaginary axis (see [35]). Destabilization will happen at critical values τ_c which is when there is a pair of purely imaginary characteristic values. Following the ideas of papers [35] and [8], let's rewrite the characteristic equation as

$$\begin{aligned} C(\lambda) &:= (\lambda + 1)(\lambda + p_3)(\lambda + p_6 - K_2) + (1 + e^{-\lambda\tau}) K_3 (\lambda + p_6) \\ &= ((\lambda + 1)(\lambda + p_3)(\lambda + p_6 - K_2) + K_3(\lambda + p_6)) + e^{-\lambda\tau} K_3 (\lambda + p_6) \\ &= P(\lambda) + Q(\lambda) e^{-\lambda\tau} = 0. \end{aligned} \tag{7.16}$$

Then define

$$\begin{aligned} F(y) &= |P(iy)|^2 - |Q(iy)|^2 \\ &= y^6 + (p_6^2 - 2K_2p_6 + p_3^2 - 2K_3 + K_2^2 + 1)y^4 + (K_2^2 - 2K_2K_3 + 2K_3p_3 \\ &\quad -2K_2K_3p_3 + p_3^2 + K_2^2p_3^2 - 2K_2p_6 + 2K_2K_3p_6 - 2K_2p_3^2p_6 + p_6^2 - 2K_3p_6^2 \\ &\quad +p_3^2p_6^2)y^2 + (K_2^2p_3^2 - 2K_2K_3p_3p_6 - 2K_2p_3^2p_6 + 2K_3p_3p_6^2 + p_3^2p_6^2). \end{aligned} \tag{7.17}$$

We want to use Theorem 12, so we need to check the following conditions:

1. $P(\lambda) = (\lambda+1)(\lambda+p_3)(\lambda+p_6-K_2) + K_3(\lambda+p_6)$ and $Q(\lambda) = K_3(\lambda+p_6)$ have no common imaginary zeros since each p_i are real values.

2. It is quick to see that $\overline{P(i\lambda)} = P(i\lambda)$ and $\overline{Q(i\lambda)} = Q(i\lambda)$ for real λ.

3. $P(0) + Q(0) = p_3(p_6 - K_2) + 2K_3p_6 \neq 0$ so this is an important restriction in order to use the Theorem 12.

4. Referring back to (6.9), we see that there are at most 3 roots of (7.16) if $\tau = 0$.

5. (7.17) has at most 6 real zeros for real y.

Therefore, by Theorem 12 from the Appendix, if $F(y)$ has no positive roots, the system is stable for all $\tau \geqslant 0$. If $F(y)$ has a simple positive root y_0, then there exists a pair of purely imaginary roots $\pm iv_0$ such that $v_0 = \sqrt{y_0}$. For this v_0, there is a countable sequence of $\{\tau_0^n\}$ of delays for which stability switches can occur. Also, there exists a positive τ_c such that the system is unstable for all $\tau > \tau_c$. Investigating this further, let $x = y^2$

$$F(x) = x^3 + b_1x^2 + b_2x + b_3, \tag{7.18}$$

where $b_1 = p_6^2 - 2K_2p_6 + p_3^2 - 2K_3 + K_2^2 + 1$, $b_2 = K_2^2 + -2K_2K_3 + 2K_3p_3 - 2K_2K_3p_3 + p_3^2 + K_2^2p_3^2 - 2K_2p_6 + 2K_2K_3p_6 - 2K_2p_3^2p_6 + p_6^2 - 2K_3p_6^2 + p_3^2p_6^2$, and $b_3 = K_2^2p_3^2 - 2K_2K_3p_3p_6 - 2K_2p_3^2p_6 + 2K_3p_3p_6^2 + p_3^2p_6^2$. Note that

$$F'(x) = 3x^2 + 2b_1x + b_2$$

and

$$\Delta_0 = b_1^2 - 3b_2. \tag{7.19}$$

Now analyzing the roots of (7.18),

- If $\Delta_0 \leqslant 0$, then $F'(0) \geqslant 0$ and $F(x)$ is monotonically non-decreasing. Further,
  - if $F(0) > 0$, then F has no positive roots and all the roots of the characteristic will remain to the left of the imaginary axis for all $\tau > 0$.
  - if $F(0) < 0$, then since $\lim_{x\to\infty} F(x) = \infty$, there is at least one positive root of F and thus the roots of the characteristic equation can cross the imaginary axis.
- If $\Delta_0 > 0$ then F has critical points

$$x_{c_1} = \tfrac{-b_1+\sqrt{\Delta_0}}{3}, \qquad x_{c_2} = \tfrac{-b_1-\sqrt{\Delta_0}}{3}$$

  and if $x_{c_1} > 0$ and $F(x_{c_1}) < 0$, then F has positive roots (see [35]).

Stability switches are possible for each positive simple root x_j of (7.18) and the cross is from left to right if $F'(v_0) > 0$, and from right to left if $F'(v_0) < 0$ according to Theorem 12 (see [35]). Now let's analyze the characteristic quasi-polynomial (7.16) for $\lambda = iv$:

$$C(iv) = A_1 - A_2\cos(v\tau) - A_3\sin(v\tau) + i[A_4 - A_3\cos(v\tau) + A_2\sin(v\tau)] = 0,$$

where

$$\begin{aligned} A_1(v) &= p_3p_6 - K_2p_3 + K_3p_6 - v^2(p_6 + p_3 - K_2 + 1), \qquad & A_2 &= -K_3p_6, \\ A_4(v) &= v(p_3 - K_2 - K_2p_3 + p_6 + p_3p_6 + K_3) - v^3, \qquad & A_3(v) &= -K_3v. \end{aligned}$$

So x_j $(j = 1, 2, 3)$ is a positive root of $F(x) = 0$ and $v_j = \sqrt{x_j}$. Then v_j satisfies (7.20) if its a solution to the system

$$\begin{cases} A_1(v) - A_2\cos(v\tau) - A_3(v)\sin(v\tau) = 0, \\ A_4(v) - A_3(v)\cos(v\tau) + A_2\sin(v\tau) = 0. \end{cases}$$

This yields

$$\sin(v\tau) = \frac{A_1(v)A_3(v)-A_2A_4(v)}{A_2^2+A_3^2(v)}, \qquad \cos(v\tau) = \frac{A_1(v)A_2+A_3(v)A_4(v)}{A_2^2+A_3^2(v)},$$

provided $\max\{|A_1(v)A_3(v) - A_2A_4(v)|, |A_1(v)A_2 - A_3(v)A_4(v)|\} \leqslant A_2^2 + A_3^2(v)$,

Therefore, for every positive root v_j, it yields the following sequence of delays $\{\tau_j^n\}$ for which there are pure imaginary roots (7.16):

$$\tau_j^n = \frac{1}{v_j}\left\{\arctan\left(\frac{A_1(v_j)A_3(v_j)-A_2A_4(v_j)}{A_1(v_j)A_2+A_3(v_j)A_4(v_j)} + \pi\, n\right)\right\} \qquad \text{for } n = 0, 1, 2, \ldots \tag{7.20}$$

As a result the following statement holds.

Lemma 7.2.1. *The system (5.6)–(5.8) with delay and $p_3(p_6 - K_2) + 2K_3p_6 \neq 0$ is stable for all $\tau \geqslant 0$ if $F(0) > 0$ and $\Delta_0 \leqslant 0$ (where Δ_0 is from (7.19)). The system has stability switches at some $\{\tau_j^n\}$ for every positive root v_j of (7.16). Furthermore, if $A = 0$, $p_2 = 0$ or $p_5 = p_6 = 0$ then the delay has no affect on the stability of the system. [51]*

7.3 Existence of Periodic Solutions

In this section, by Picard's method we prove the existence of solutions to problem (5.1)–(5.4).

Theorem 13. *[51] Let $A > 0$, $p_i > 0$, and assumption* (5.5) *hold. Let*

$$a'_\tau(0) + p_3a_\tau(0) = \frac{A}{1+p_2o_0r_0}, \tag{7.21}$$

then problem (5.1)–(5.4) has a unique non-negative solution $(a(t), r(t), o(t))$ in C^2 for all $t \geq 0$. Moreover,

for all time $t \geq 0$ there are estimates

$$\frac{Ap_6}{p_3p_6 + 12Ap_2(p_5+1)}[1-e^{-p_3\tau}][1+(p_3\tau)^{-1}] \leq a(t) \leq \frac{3A}{p_3} + a_\tau(0),$$
$$\frac{Ap_6}{p_3p_6 + 12Ap_2(p_5+1)}[1-e^{-2\tau}][1-e^{-p_3\tau}][1+(p_3\tau)^{-1}] \leq o(t) \leq \frac{3A}{p_3} + a_\tau(0) + o_0 + \max_{t\in[-\tau,0]} a_\tau(t),$$
$$\frac{p_5}{p_6}[1-e^{-p_6\tau}][1+(p_6\tau)^{-1}] \leq r(t) \leq \frac{p_5+1}{p_6} + r_0.$$

Remark 7.3.1. *It is apparent that for $a_\tau(t) = a_0 + \Lambda\, t^2 e^{-t}$, where Λ is a positive number, fitting condition* (7.21) *of this theorem is rewritten as*

$$a_0 = \tfrac{A}{p_3(1+p_2o_0r_0)},\ o_0 > 0,\ r_0 > 0.$$

Theorem 14. *Under the conditions of Theorem 13, the system (5.1)–(5.5) has at least one C^2-smooth T-periodic solution, where $T \neq \tau$. [51]*

Proof of Theorem 14. The main line of proof follows (see [55, pp. 278–280]) (see also [54, Theorem 5]). Rewrite the system of (5.1)–(5.3) in the following form

$$\mathbf{x}'(t) = M\,\mathbf{x}(t) + B\,\mathbf{x}(t-\tau) + \mathbf{f}(\mathbf{x}(t)), \tag{7.22}$$

where

$$\mathbf{x}(t) = \begin{pmatrix} a(t) \\ r(t) \\ o(t) \end{pmatrix},\ \mathbf{f}(\mathbf{x}(t)) = \begin{pmatrix} \frac{A}{1+p_2or} \\ p_5 + \frac{(or)^2}{p_4+(or)^2} \\ 0 \end{pmatrix},$$

$$M = \begin{pmatrix} -p_3 & 0 & 0 \\ 0 & -p_6 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Obviously, the right-hand side of (7.22) is T-periodic with respect to t as it does not depend on

time explicitly. Without loss of generality, we may assume that

$$0 \leqslant \tau < T.$$

This is true because otherwise we could represent the τ in the form

$$0 < \tau = nT + \tau_1, \text{ where } n \in \mathbb{Z}^+,\ \tau_1 \in [0, T).$$

Then shift to the auxiliary equation

$$\mathbf{x}'(t) = M\,\mathbf{x}(t) + B\,\mathbf{x}(t - \tau_1) + \mathbf{f}(\mathbf{x}(t))$$

the T-periodic solutions of which coincide with the T-periodic solutions of (7.22).

On the set of all vector-valued functions $\mathbf{x}(t)$ defined on $[0, T]$, let us define an operator S_τ by

$$S_\tau \mathbf{x}(t) := \begin{cases} \mathbf{x}(t - \tau) \text{ if } \tau \leqslant t \leqslant T, \\ \mathbf{x}(t - \tau + T) \text{ if } 0 \leqslant t < \tau. \end{cases}$$

Note that the T-periodic solutions of (7.22) coincides with the solutions of the following integral equations:

$$\mathbf{x}(t) = T(\tau, \mathbf{x}) := \mathbf{x}(0) + \int_0^t (M\,\mathbf{x}(s) + B\,S_\tau \mathbf{x}(s) + \mathbf{f}(\mathbf{x}(s)))\,ds.$$

The operator $T(\tau, \mathbf{x})$ maps every continuous vector-valued function $\mathbf{x}(t)$ into a continuous vector-valued function for $0 \leqslant t \leqslant T$, therefore $T(\tau, \mathbf{x})$ is compact in C. Next, we will show that for all T-periodic solutions $\mathbf{x}_p(t)$ there exists $R > 0$ such that

$$|\mathbf{x}_p(t)| \leqslant R < \infty. \tag{7.23}$$

Really, from (7.3) we deduce that

$$|\mathbf{x}_p(t)| \leqslant |\mathbf{x}_p(0)| + \int_0^t [|M|\,|\mathbf{x}_p(s)| + |B|\,|S_\tau \mathbf{x}_p(s)| + |\mathbf{f}(\mathbf{x}_p(s))|]\,ds \leqslant$$

$$|\mathbf{x}_p(0)| + [(p_3^2 + p_6^2 + 1)^{\frac{1}{2}} + 1] \int_0^t |\mathbf{x}_p(s)|\,ds + (A^2 + (p_5+1)^2)^{\frac{1}{2}} t.$$

From here, using Grönwall's Lemma, we arrive at

$$|\mathbf{x}_p(t)| \leqslant (|\mathbf{x}_p(0)| + a)e^{bT} - a,$$

where $a = \frac{(A^2+(p_5+1)^2)^{\frac{1}{2}}}{(p_3^2+p_6^2+1)^{\frac{1}{2}}+1}$, $b = (p_3^2 + p_6^2 + 1)^{\frac{1}{2}} + 1$. Hence, (7.23) holds with $R = (|\mathbf{x}_p(0)| + a)e^{bT} - a$. As a result, by the fixed point theorem the integral equation (7.3) has at least one solution, and consequently the equation (7.22) has at least one T-periodic solution. □

7.3.1 Periodic Solutions with the Period $T = \tau$

Lemma 7.3.1. *If*

$$o_0 = a_\tau(-\tau),\ a_\tau(0) = \frac{A}{p_3}\left[1 + p_2\sqrt{\frac{p_4(p_6 r_0 - p_5)}{p_5+1-p_6 r_0}}\right]^{-1},\ \frac{p_5}{p_6} \leqslant r_0 \leqslant \frac{p_5+1}{p_6}.$$

then the problem (5.1)–(5.4) has at least one τ-periodic solution.

Proof of Lemma 7.3.1. Note that if a function $\Phi(t) \in C^2[a,b]$ is periodic with a period $T > 0$ then $\Phi(t+T) = \Phi(t)$ and $\Phi'(t+T) = \Phi'(t)$. Let $(a(t), r(t), o(t))$ be a T-periodic solution of (5.1)–(5.3) from Theorem 14. Then for this solution we have

$$\frac{A}{1+p_2 o(t) r(t)} - p_3 a(t) = \frac{A}{1+p_2 o(t+T) r(t+T)} - p_3 a(t+T),$$

$$-\frac{p_4}{p_4+(o(t)r(t))^2} + 1 + p_5 - p_6 r(t) = -\frac{p_4}{p_4+(o(t+T)r(t+T))^2} + 1 + p_5 - p_6 r(t+T), \tag{7.24}$$

$$a(t-\tau) - o(t) = a(t+T-\tau) - o(t+T), \tag{7.25}$$

whence

$$o(t)r(t) = o(t+T)r(t+T), \;\; a(t-\tau) = a(t+T-\tau). \tag{7.26}$$

Let $T = \tau > 0$. Then from (7.26) for $t \in [0, \tau]$ we have

$$a_\tau(t) = a_1(t), \;\; o_1(t)r_1(t) = o_2(t+\tau)r_2(t+\tau). \tag{7.27}$$

By (7.27) at $t = 0$ we get

$$a_\tau(0) = a_1(0) = e^{-p_3\tau}a_\tau(0) + A\, e^{-p_3\tau} \int\limits_0^\tau \tfrac{e^{p_3 s}\, ds}{1+p_2 r_1(s)o_1(s)},$$

$$o_0 = o_1(\tau) = e^{-\tau}o_0 + \int\limits_{-\tau}^0 a_\tau(s)e^s\, ds,$$

$$r_0 = r_1(\tau) = e^{-p_6\tau}r_0 - p_4\, e^{-p_6\tau} \int\limits_0^\tau \tfrac{e^{p_6 s}\, ds}{p_4+r_1^2(s)o_1^2(s)} + \tfrac{p_5+1}{p_6}(1-e^{-p_6\tau}),$$

whence

$$a_\tau(0) = \tfrac{A\, e^{-p_3\tau}}{1-e^{-p_3\tau}} \int\limits_0^\tau \tfrac{e^{p_3 s}\, ds}{1+p_2 r_1(s)o_1(s)}, \; o_0 = \tfrac{1}{1-e^{-\tau}} \int\limits_{-\tau}^0 a_\tau(s)e^s\, ds,$$

$$r_0 = \tfrac{e^{-p_6\tau}}{1-e^{-p_6\tau}}\Big[-p_4 \int\limits_0^\tau \tfrac{e^{p_6 s}\, ds}{p_4+r_1^2(s)o_1^2(s)} + \tfrac{p_5+1}{p_6}(e^{p_6\tau}-1)\Big].$$

Hence

$$f_1(\tau) := (e^{p_3\tau}-1)\tfrac{a_\tau(0)}{A} - \int\limits_0^\tau \tfrac{e^{p_3 s}\, ds}{1+p_2 r_1(s)o_1(s)} = 0,$$

$$f_2(\tau) := (1-e^{-\tau})o_0 - \int\limits_{-\tau}^0 a_\tau(s)e^s\, ds = 0,$$

$$f_3(\tau) := \tfrac{1}{p_4}\Big[\tfrac{p_5+1}{p_6}(e^{p_6\tau}-1) - (e^{p_6\tau}-1)r_0\Big] - \int\limits_0^\tau \tfrac{e^{p_6 s}\, ds}{p_4+r_1^2(s)o_1^2(s)} = 0.$$

As $f_i(0) = 0$ and

$$f_1'(\tau) = e^{p_3\tau}\big[\tfrac{p_3 a_\tau(0)}{A} - \tfrac{1}{1+p_2 r_0 o_0}\big] \equiv 0 \text{ if } \tfrac{p_3 a_\tau(0)}{A} = \tfrac{1}{1+p_2 r_0 o_0},$$

$$f_2'(\tau) := e^{-\tau}(o_0 - a_\tau(-\tau)) \equiv 0 \text{ if } o_0 = a_\tau(-\tau),$$

$$f_3'(\tau) := e^{p_6\tau}\left[\frac{p_5+1-p_6r_0}{p_4} - \frac{1}{p_4+r_0^2o_0^2}\right] \equiv 0 \text{ if } \frac{p_5+1-p_6r_0}{p_4} = \frac{1}{p_4+r_0^2o_0^2},$$

then $f_i(\tau) = 0$ for all $\tau \geqslant 0$ provided

$$o_0 = a_\tau(-\tau),\ a_\tau(0) = \frac{A}{p_3}\left[1 + p_2\sqrt{\frac{p_4(p_6r_0-p_5)}{p_5+1-p_6r_0}}\right]^{-1},\ \frac{p_5}{p_6} \leqslant r_0 \leqslant \frac{p_5+1}{p_6}.$$

□

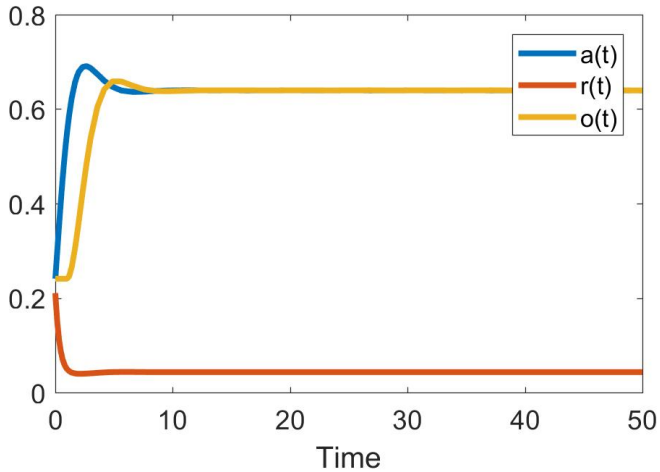
Example 7.3.1. *Let $A = 1$, $p_2 = 11$, $p_3 = 1.2$, $p_4 = 0.05$, $p_5 = 0.11$, and $p_6 = 2.9$. Then the initial conditions are $a_0 = \frac{A}{p_3}\left(1 + p_2\sqrt{\frac{p_4(p_6r_0-p_5)}{p_5+1-p_6r_0}}\right)^{-1}$, $r_0 = \frac{1}{2}\left(\frac{p_5}{p_6} + \frac{p_5+1}{p_6}\right)$, and $o_0 = a_0$. With these parameter values we solve (5.1)–(5.4) numerically using the Matlab solver dde23 [64]. The resulting periodic solutions can be seen in Figure 7.1.*

If we perturb the parameters by a bit, the periodicity changes. We illustrate a periodicity change by using the parameters $A = 1$, $p_2 = 7$, $p_3 = 1.2$, $p_4 = 0.05$, $p_5 = 0.51$, and $p_6 = 3.1$. Then the initial conditions are $a_0 = \frac{A}{p_3}\left(1 + p_2\sqrt{\frac{p_4(p_6r_0-p_5)}{p_5+1-p_6r_0}}\right)^{-1}$, $r_0 = \frac{1}{2}\left(\frac{p_5}{p_6} + \frac{p_5+1}{p_6}\right)$, and $o_0 = a_0$. The resulting periodic solutions can be seen in Figure 7.2.

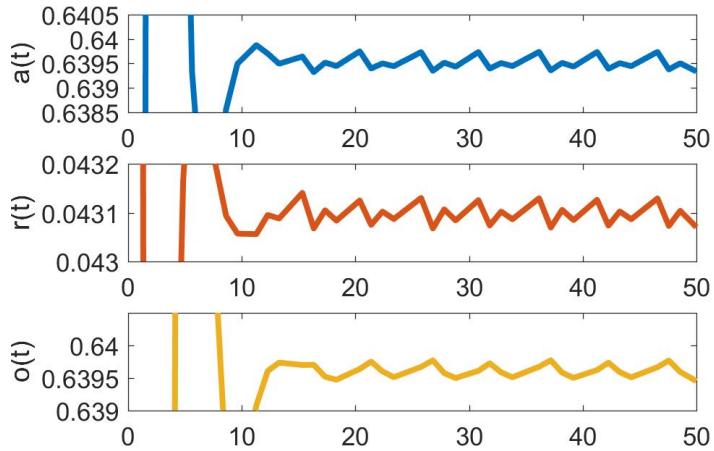
Periodicity of solutions can also be illustrated by plotting delayed function versus no delay function or function versus derivative as seen in Figure 7.3.

7.4 Numerical Analysis

In order to concretely understand how the delay is affecting stability, let's set the parameters in equation in the equation (7.16) to $p_3 = 0.41$, $p_6 = 0.91$, $K_2 = 0.81$, and $K_3 = 0.41$. We then proceed with our calculations using these values to illustrate the dynamics of eigenvalues with respect to

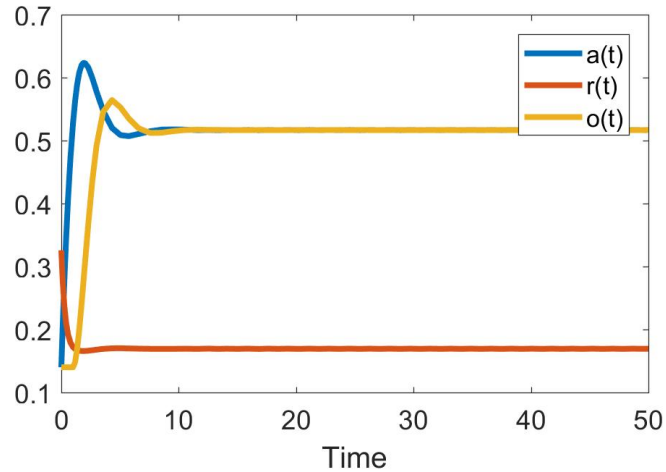


(a)

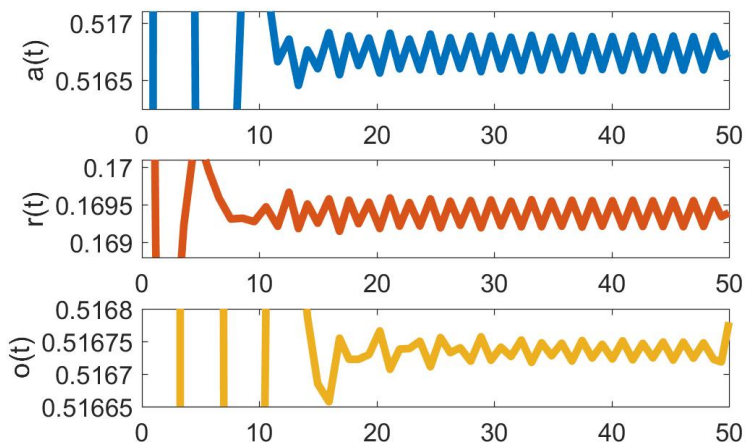


(b)

Figure 7.1: Plot of the solutions $a(t)$, $r(t)$, and $o(t)$ with the parameter values in Example 7.3.1 part a.

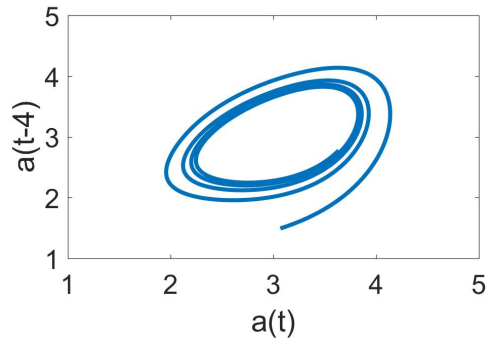


(a)

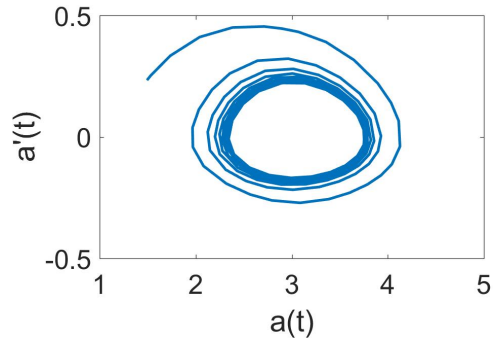


(b)

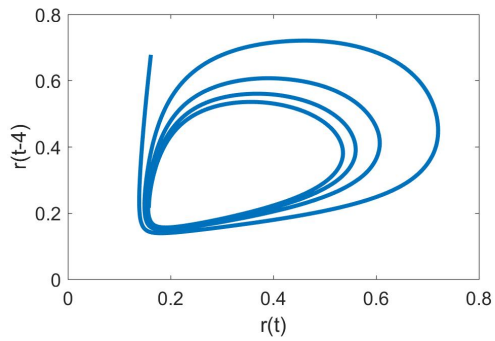
Figure 7.2: Plot of the solutions $a(t)$, $r(t)$, and $o(t)$ with the parameter values in Example 7.3.1 part b.



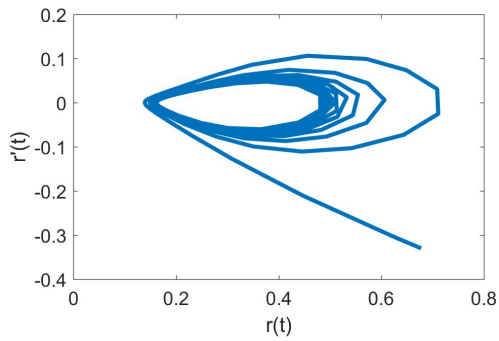
(a) $a(t)$ and $a(t-4)$



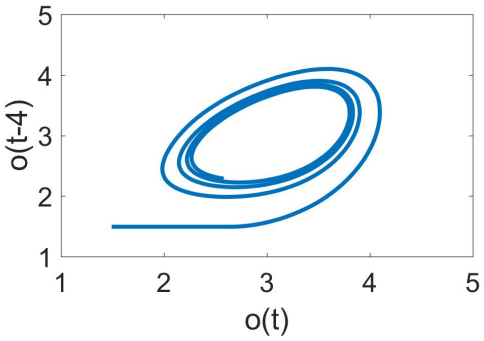
(b) $a(t)$ and $a'(t)$



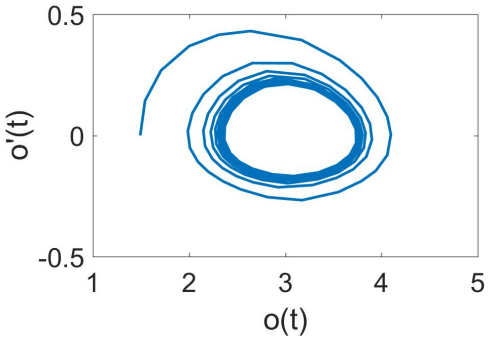
(c) $r(t)$ and $r(t-4)$



(d) $r(t)$ and $r'(t)$



(e) $o(t)$ and $o(t-4)$



(f) $o(t)$ and $o'(t)$

Figure 7.3: Plots using parameter values $A = 1.5$, $p_2 = 1.8$, $p_3 = 0.2$, $p_4 = 5$, $p_5 = 0.11$, $p_6 = 0.9$

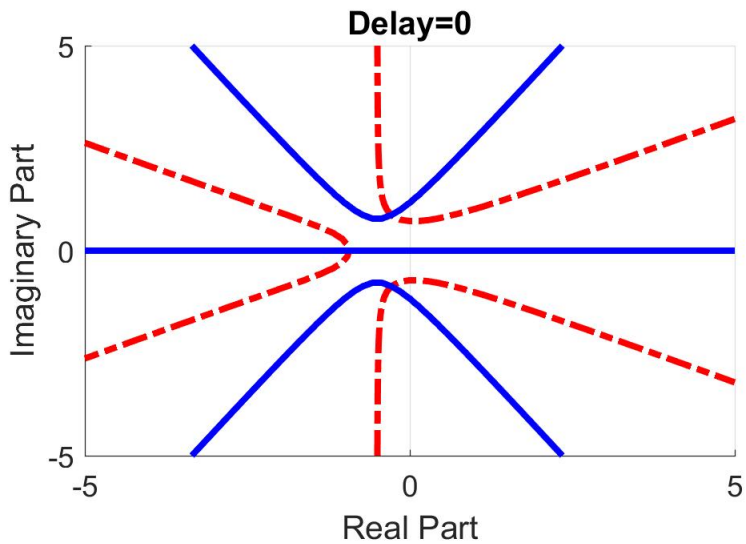


Figure 7.4: Contour plot for $\tau = 0$.

the time delay. Taking the real and imaginary parts, we rewrite equation (7.16) as a system

$$\begin{cases} -K_2p_3 & +K_3p_6 + p_3p_6 - K_2x + K_3x + p_3x - K_2p_3x + p_6x + p_3p_6x + x^2 \\ & -K_2x^2 + p_3x^2 + p_6x^2 + x^3 - y^2 + K_2y^2 - p_3y^2 - p_6y^2 - 3xy^2 \\ & +e^{-\tau x}K_3p_6\cos(\tau y) + e^{-\tau x}K_3x\cos(\tau y) + e^{-\tau x}K_3y\sin(\tau y) = 0, \\ -K_2y & +K_3y + p_3y - K_2p_3y + p_6y + p_3p_6y + 2xy - 2K_2xy + 2p_3xy \\ & +2p_6xy + 3x^2y - y^3 + e^{-\tau x}K_3y\cos(\tau y) - e^{-\tau x}K_3p_6\sin(\tau y) \\ & -e^{-\tau x}K_3x\sin(\tau y) = 0. \end{cases}$$

The red lines represent the solution curves for the first equation and the blue lines represent the solution curves for the second equation for different values of the delay τ in Figures 7.4-7.9. The eigenvalues for the system (5.6)–(5.8) are roots of (7.16) and correspond to the intersections between the red and blue lines.

When there is no delay, i.e. $\tau = 0$, we only have three eigenvalues $\lambda \approx -0.9, -0.2 \pm 0.8i$ (see Figure 7.4). Note that zooming out on the graph would not show additional crossings of the solution curves. If we were to classify these three eigenvalues, they would all be attractors, with one traditional attractor and one spiraling attractor.

When the delay is non-zero, i.e. $\tau = 1.3$, we notice that an additional pair of complex eigenvalues appears from $-\infty$. This is shown in Figure 7.5. It isn't as clear, but the original two complex

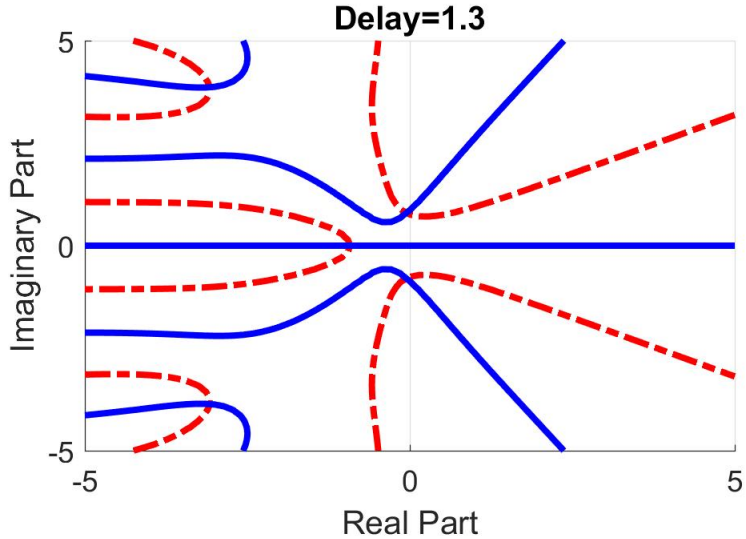


Figure 7.5: Contour plot for $\tau = 1.3$.

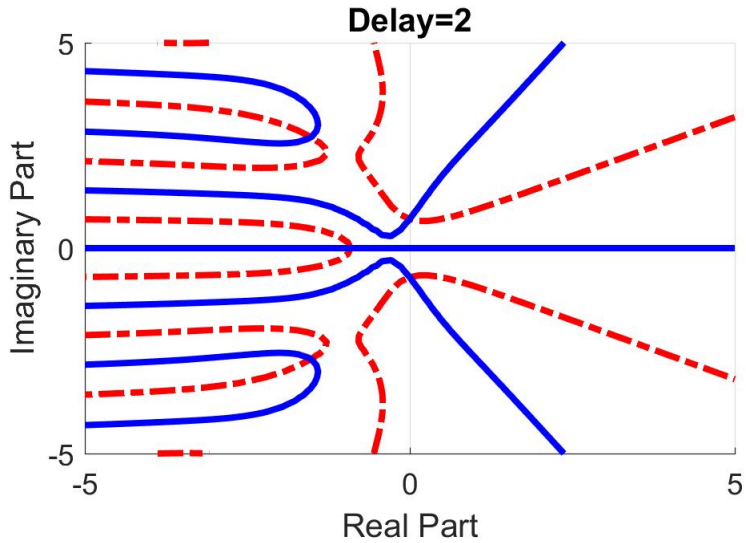


Figure 7.6: Contour plot for $\tau = 2$.

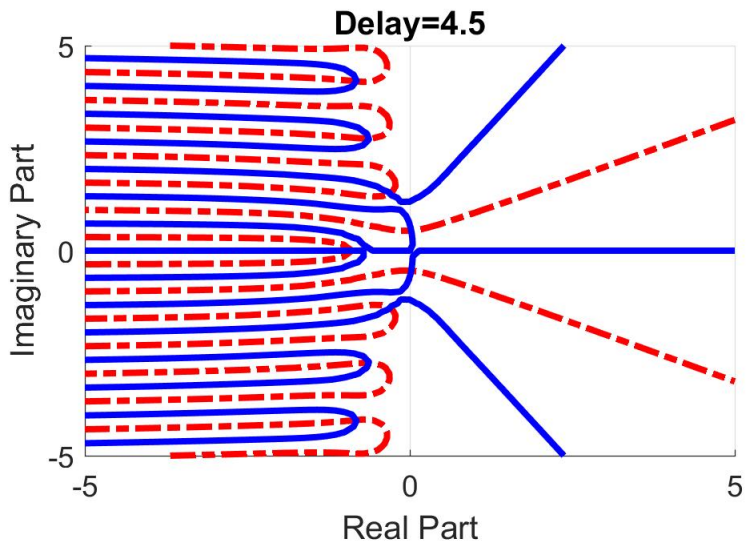


Figure 7.7: Contour plot for $\tau = 4.5$

pairs of eigenvalues have also moved slightly toward the imaginary axis. As τ increases to $\tau = 2$ (see Figure 7.6), the original pair of complex eigenvalues is about to cross the imaginary axis while the other pair of complex values are closer. When the pair of complex eigenvalues crosses the imaginary axis, the once stable system is no unstable.

When we skip to $\tau = 4.5$, it is obvious (see Figure 7.7) that countably many eigenvalues will originate from $-\infty$ and move toward the imaginary axis as τ increases. When $\tau = 8$ (see Figure 7.8), we can see that there looks to be two pairs of complex eigenvalues crossing the imaginary axis. It isn't as clear, but between $\tau = 8$ and $\tau = 10.8$ the first pair of complex eigenvalues that crossed the imaginary axis have crossed back into the stable region while another pair crossed into the unstable region.

The eigenvalues can cross the imaginary axis only at the points $y_1 \approx \pm 0.7$ and $y_2 \approx \pm 0.25$ which are real roots of the equation (7.17). It is easier to see these values in Figure 7.10 and 7.11. The density of complex eigenvalues around these crossing points y_1, y_2 is increasing as the τ gets larger.

When the delay $\tau < \tau^* \approx 2$ (where τ^* is a critical value found as a solution of (7.20) with $v_1 = \sqrt{|y_1|}$) all eigenvalues are stable. The first stability switch happens at $\tau^* \approx 2$ when two complex conjugate eigenvalues cross the imaginary axis at $y_1 \approx \pm 0.7$ changing the sign of the real part from negative to positive. At a later time $\tau^* \approx 11$ (where this τ^* is a critical value found as a

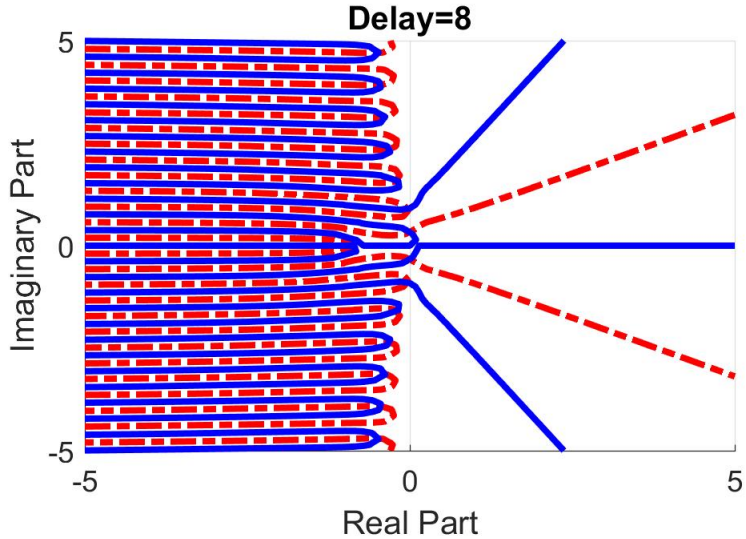


Figure 7.8: Contour plot for $\tau = 8$

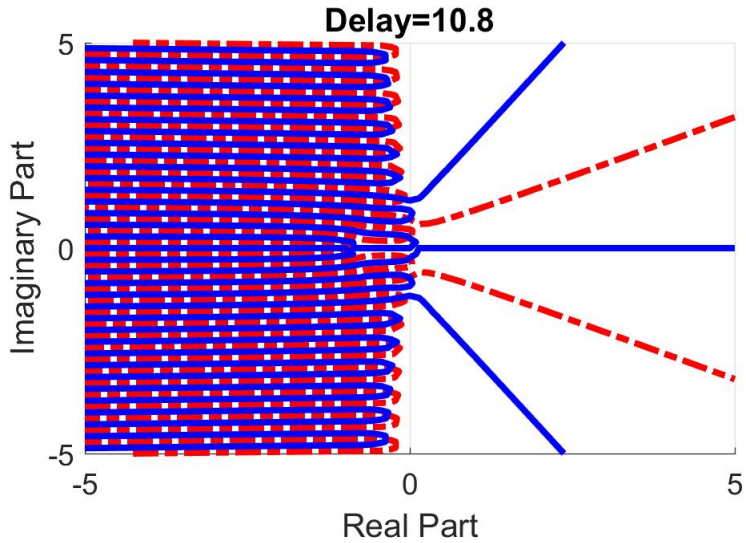
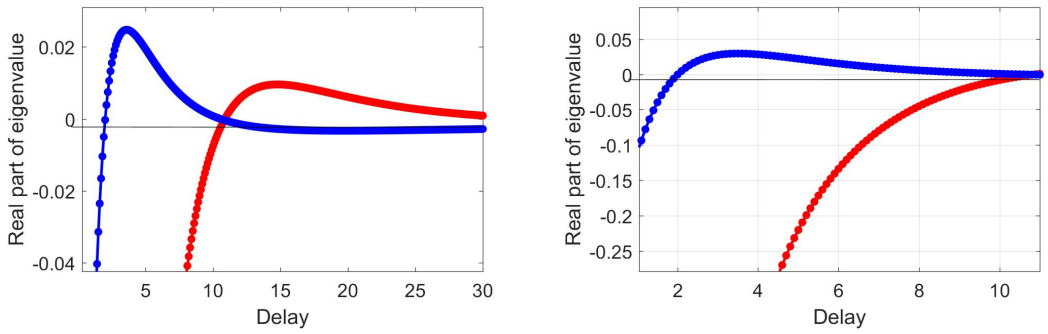
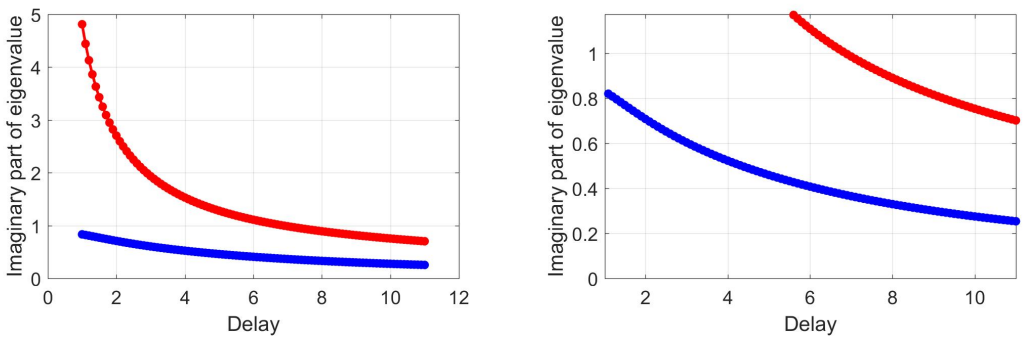


Figure 7.9: Contour plot for $\tau = 10.8$



(a) (b)

Figure 7.10: Tracking two complex eigenvalues to see how the value of their real part changes.



(a) (b)

Figure 7.11: Tracking two complex eigenvalues to see how the value of their imaginary part changes.

solution of (7.20) with $v_2 = \sqrt{|y_2|}$) this complex pair will cross the imaginary axis back changing the sign of the real part from positive to negative (see Figure 7.9).

Solving (7.20) and taking into account the periodicity of the arctangent function one can obtained the infinite sequences of delays associated with v_1 and another infinite sequence associated with v_2 at which stability switches may happen. At time delays associated with v_1 a complex conjugate pair of eigenvalues may cross the imaginary axis from left to right and for time delays associated with v_2 the pair may cross the imaginary axis from right to left. If the derivative of $F(y)$ (see (7.16)) does not change sign at the corresponding τ^* from either of the two sequences above, then the crossing of the imaginary axis does not happen.

Chapter 8

Conclusions

After our investigation into both the multistring systems and the HPA axis system, we have a thorough understanding of their respective spectral properties and the affect they have on the behavior of the solution. We looked at operator couplings to ensure the model would still be self adjoint if we coupled a discrete string with a continuous string. Then we utilized Nevanlinna functions to describe the full spectrum of connected Stieltjes strings for any number of strands with various number of beads on them. Utilizing these Nevanlinna functions, we showed an alternate/improved inverse solution to the connected Stieltjes string problem. Finally we looked at the HPA axis system. We investigated the stability of the solution without a time delay parameter and then looked at the affect the time delay parameter would have on the stability of the solution.

All of this work just touches the surface of work left to be done in this field. The ability to couple two different operators and have the result be self adjoint is great, but it would be even better to couple more than two operators. The current method for calculating the boundary space only allows for 2 operators. Coupling more than two operators will require developing a new method of calculating a boundary space which would ensure the coupled operators will be self adjoint.

While we performed a decently rigorous study of multistring systems, we can take it one step further. Rather than have the edges of the strings be fixed, how does the system behave if the edges of the strings are connected to their neighbor edge? We could also study a multistring system in which some strands are continuous while others are discrete.

Biological processes are mysterious compared to physical phenomena in that we don't have

concrete governing equations. There are mathematicians whose research field is deriving the best model to explain a biological process. Although it may not have been very clear, the mathematical model of the HPA axis we studied is just one of many that mathematicians claim accurately explain the HPA axis. For future work, we can perform similar analysis on the other models of the HPA axis. Depending on the results of the analysis of another model, we could also compare the findings with the model considered here.

Appendix A

Appendix Matlab Code

Modeling N Strings

```
clear all;
close all;
clc
% Wave Equation for N strings connected in the middle
% u_tt=c^2u_xx
% x in [0,xend] xend=length of individual string
% t in [0,tend]
% u(x,0)=f(x)      u_t(x,0)=g(x)=0

%SetUp Conditions
N=6;        %number of strings meeting in the middle
c=1;        %wave speed
tend=12;    %time endpoint
numt=100*tend;    %number of time steps

xpts=25;        %number of x grid points per string (not including
    joining pt)
```

```
numx=N*xpts+1;  %number of x grid points
xend=pi;        %length of individual string
xstart=0;       %always starts at 0
x=linspace(xstart,xend,(xpts+1)); %generates x values

dx=x(2)-x(1);
dt=tend/numt;
u=zeros(numx,numt);

%repeat x values in order to evaluate initial values
for j=2:N
    for i=((j-1)*xpts+2):(j*xpts+1)
        x(i)=x(i-((j-1)*xpts));
    end
end

%%%%INITIAL POSITION: Add as many loops necessary for each string's initial
%%%%position
for i=1:(xpts+1)
     u(i,1)=(1/10)*(-2*sin(x(i))-sin(2*x(i)))*(3/2); %String 1 Initial Position
end
for i=(xpts+2):(2*xpts+1)
    u(i,1)=(1/10)*(-2*sin(x(i))-sin(2*x(i)))*(1/2); %String 2 Initial Position
end
for i=(2*xpts+2):(3*xpts+1)
    u(i,1)=(1/10)*(1/4)*(4*sin(x(i))+sin(4*x(i))); %String 3 Initial
```

```
        Position
end
for i=(3*xpts+2):(4*xpts+1)
    u(i,1)=(1/10)*(3/4)*(4*sin(x(i))+sin(4*x(i))); %String 4 Initial
        Position
end
for i=(4*xpts+2):(5*xpts+1)
    u(i,1)=(1/10)*(1/2)*(-3*sin(x(i))-sin(3*x(i))); %String 5 Initial
        Position
end
for i=(5*xpts+2):(6*xpts+1)
    u(i,1)=(1/10)*(1/2)*(3*sin(x(i))+sin(3*x(i))); %String 6 Initial
        Position
end

%%%CHECK STABILITY
alpha=c*dt/dx;
alpha^2*sin(dx/2)^2

if( 1< alpha^2*sin(dx/2)^2 )
    fprintf( 'The stability condition fails.\n' )
end

%%%%Boundary Conditions
for j=1:N
    for i=1:numt
        u((j*xpts+1),i)=0;  %Dirichlet at the ends
    end
end
```

```
%%%%First Iteration
for k=1:N
    if k<N
        u(2,2)=(1-alpha^2)*u(2,1)+(1/2)*alpha^2*(u(3,1)+u(1,1));
        for i=((k-1)*xpts+3):(k*xpts)
            u(i,2)=(1-alpha^2)*u(i,1)+(1/2)*alpha^2*(u(i+1,1)+u(i-1,1))
                ;
        end
        u((k*xpts+2),2)=(1-alpha^2)*u((k*xpts+2),1)+(1/2)*alpha^2*(u((k*
            xpts+3),1)+u(1,1));
    else
        for i=((k-1)*xpts+3):(k*xpts)
            u(i,2)=(1-alpha^2)*u(i,1)+(1/2)*alpha^2*(u(i+1,1)+u(i-1,1))
                ;
        end
    end
end
u(1,2)=(1/(3*N))*(4*u(2,2)-u(3,2)+4*u(27,2)-u(28,2)+4*u(52,2)...
    -u(53,2)+4*u(77,2)-u(78,2));

%%%All Subsequent Iterations

for j=2:(numt-1)
        u(2,j+1)=-u(2,j-1)+2*(1-alpha^2)*u(2,j)...
            +alpha^2*(u(3,j)+u(1,j));
        for k=1:N
        if k<N
            for i=((k-1)*xpts+3):(k*xpts)
```

```
                u(i,j+1)=-u(i,j-1)+2*(1-alpha^2)*u(i,j)...
                +alpha^2*(u(i+1,j)+u(i-1,j));
           end
           u((k*xpts+2),j+1)=-u((k*xpts+2),j-1)...
               +2*(1-alpha^2)*u((k*xpts+2),j)+alpha^2*(u((k*xpts+3),j)+u(1,j));
        else
            for i=((k-1)*xpts+3):(k*xpts)
                u(i,j+1)=-u(i,j-1)+2*(1-alpha^2)*u(i,j)...
                +alpha^2*(u(i+1,j)+u(i-1,j));
            end
        end
     end
        for m=1:N
            u(1,(j+1))=u(1,(j+1))+(1/(3*N))*(4*u(((m-1)*xpts+2),(j+1))-u(((m-1)*xpts+3),(j+1)));
        end
end


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 str=xpts+1;
xnew=zeros(str,(2*N));
unew=zeros((N*str),numt);
unew(1,:)=u(1,:);

 for i=1:(N-1)
    unew((i*str+1),:)=u(1,:);
end
```

```
for i=1:N
    theta=((2*i)-1)*pi/N;
    th=[cos(theta) -sin(theta); sin(theta) cos(theta)];
    xnew(2:str,(2*(i-1)+1))=x(((i-1)*xpts+2):((i*xpts)+1));
    for k=1:str
        xnew(k,(2*(i-1)+1):(2*(i-1)+2))=th*xnew(k,(2*(i-1)+1):(2*(i-1)
            +2))';
    end
    unew(((i-1)*str+2):(i*str),:)=u(((i-1)*xpts+2):(i*xpts+1),:);
end

%Plotting
h=figure;
filename = 'NStrings.gif';
for j=1:10:numt
    plot3(xnew(:,1),xnew(:,2),unew(1:26,j),'Color',[.8 .8 .8])
    hold on
    for k=2:N
        plot3(xnew(:,(2*(k-1)+1)),xnew(:,(2*(k-1)+2)),unew(((k-1)*str
            +1):(k*str),j),'Color',[.8 .8 .8])
    end
    axis([-(6*pi/5) 6*pi/5 -6*pi/5 6*pi/5 -2 2])
    xlabel('x')
    ylabel('y')
    zlabel('u')
    hold off
   frame=getframe(h);
    im=frame2im(frame);
    [imind,cm]=rgb2ind(im,256);
```

```
    if j==1
        imwrite(imind,cm,filename,'gif','Loopcount',inf);
    else
        imwrite(imind,cm,filename,'gif','WriteMode','append');
    end
    pause(.05)
end
```

Modeling Delay Parameter

```
clear all;
close all;
clc

global A p2 p3 p4 p5 p6 %ca cv R r Vstr gammaH        ...
        %alpha0 alphas alphap alphaH ...
        %beta0  betas  betap  betaH

% A=1;
% p2=15;
% p3=7.2;
% p4=0.05;
% p5=0.11;
% p6=2.9;

A=1.5;
p2=1.8;
p3=.2;
p4=5;
p5=0.11;
```

```
p6=.9;

r0=.5*(p5/p6+(p5+1)/p6);
a0=(A/p3)*(1+p2*sqrt(p4*(p6*r0-p5)/(p5+1-p6*r0)))^(-1);
o0=a0;%+A*exp(-1);
history=[a0; r0; o0];
% P0       = 93;
% Paval    = P0;
% Pvval    = (1 / (1 + R/r)) * P0;
% Hval     = (1 / (R * Vstr)) * (1 / (1 + r/R)) * P0;
% history = [Paval; Pvval; Hval];

for tau = [4]
    sol = dde23('prob2f',tau,history,[0, 500]);
    t=linspace(tau,100,1000);
    o=deval(sol,t);
    olag=deval(sol,t-tau);
    figure
    plot(o(3,:),olag(3,:));
    xlabel('o(t)');
    ylabel('o(t-4)');
    figure
    plot(o(1,:),olag(1,:));
    xlabel('a(t)');
    ylabel('a(t-4)');
    figure
    plot(o(2,:),olag(2,:));
    xlabel('r(t)');
    ylabel('r(t-4)');
```

```
    figure
    plot(sol.y(1,:),sol.yp(1,:));
    xlabel('a(t)');
    ylabel('a''(t)');
    figure
    plot(sol.y(2,:),sol.yp(2,:));
    xlabel('r(t)');
    ylabel('r''(t)');
    figure
    plot(sol.y(3,:),sol.yp(3,:));
    xlabel('o(t)');
    ylabel('o''(t)');
end
```

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