[1].

An Extended WASPAS Approach for Teaching Quality Evaluation Based on Pythagorean Fuzzy Reducible Weighted Maclaurin Symmetric Mean

Dongmei Wei[a], Yuan Rong[b], Harish Garg[c], Jun Liu[d]

[a]*School of Computer and Software Engineering, Xihua University, Chengdu 610039, Sichuan, P.R China*
Email: happydongyu@163.com
[b]*School of Management, Shanghai University, Baoshan district 200444, Shanghai, P.R. China*
Email: rongyuanry@163.com
[c]*School of Mathematics, Thapar Institute of Engineering and Technology(Deemed University), Patiala, Punjab 147004, India*
Email: harishg58iitr@gmail.com
[d]*School of Computing, Ulster University at Jordanstown Campus, Belfast, Northern Ireland, UK*
Email: j.liu@ulster.ac.uk

Abstract

Teaching quality evaluation (TQE) can not only improve teachers' teaching skills, but also provide an important reference for school teaching management departments to formulate teaching reform measures and strengthen teaching management. TQE is a process of grading and ranking a given teachers based on the comprehensive consideration of multiple evaluation criteria by expert. The Maclaurin symmetric mean (MSM), as a powerful aggregation function, can capture the correlation among multiple input data more efficient. Although multitude weighted MSM operators have been developed to handle the Pythagorean fuzzy decision issues, these above operators do not possess the idempotency and reducibility during the procedure of information fusion. To conquer these defects, we present the Pythagorean fuzzy reducible weighted MSM (PFRWMSM) operator and Pythagorean fuzzy reducible weighted geometric MSM (PFRWGMSM) operator to fuse Pythagorean fuzzy assessment information. Meanwhile, several worthwhile properties and especial cases of the developed operators are explored at length. Afterwards, we develop a novel Pythagorean fuzzy entropy based upon knowledge measure to ascertain the weights of attribute. Furthermore, an extended weighted aggregated sum product assessment (WASPAS) method is developed by combining the PFRWMSM operator, PFRWGMSM operator and entropy to settle the decision problems of unknown weight information. The efficiency of the proffered method is demonstrated by a teaching quality evaluation issue, as well as the discussion of sensitivity analysis for decision outcomes. Consequently, a comparative study of the presented method with the extant Pythagorean fuzzy approaches is conducted to display the superiority of the propounded approach.

Keywords: Teaching quality evaluation, Pythagorean fuzzy set, information fusion, Reducible weighted MSM, WASPAS

1. Introduction

With the vigorous development and continuous reform of China's higher education, China's higher education has stepped into the connotative development road with the core concept of improving quality. Based on the new situation of China's economic development, paying attention to the development of high-quality connotative education has gradually become an inevitable requirement of economic and social development. Therefore, it is necessary for colleges and universities to think actively and constantly innovate the way of education and teaching reform to further improve the quality of education and teaching. As an important measure to measure the quality of higher education, the level of education quality greatly influences the comprehensive development level of schools. At present, centralized curriculum teaching is still the main development mode of education in our country. One of the core of improving

[1]Corresponding author: Yuan Rong. E-mail: rongyuanry@163.com

education and teaching quality is to improve teachers' teaching quality. Through the effective evaluation of teachers' teaching quality, we can find the problems in the teaching process of teachers and lay a solid foundation for constantly improving the teaching quality and promoting the reform and development of education and teaching quality. Therefore, scientific and reasonable evaluation of education and teaching is of great significance for improving the quality of higher education. In the process of education and teaching quality assessment, schools determine different attribute value indicators by analyzing the differences of different disciplines and teaching environment, and make a scientific and reasonable assessment of the teaching quality of teachers according to these attributes. This process can be regarded as an assessment process for coping with multi-attribute decision-making (MADM) problems. Nevertheless, because of the sophisticated decision environment and experts' cognitive psychology, experts usually can not provide an appropriate number to effectively express their assessment opinions during the procedure of the aforementioned MADM issues. This greatly leads to the irrationality of the final evaluation results. In view of this, it is worthwhile for the education department or expert to come up with an innovative approach to evaluate the teaching quality efficiently.

In light of the complexity of decision setting and evaluation information, the fuzzy set (FS) [1] is originally developed to express uncertain and ambiguous information through a membership grade from the interval $[0, 1]$. Since its introduction, FS has been employed to various aspects by multitude investigators and gained a series of research achievements [2, 3, 4, 5, 6, 7, 8, 9, 10]. However, the FS only utilizes a membership grade to portray the ambiguous and ill-defined information, which cannot satisfy the demand of decision experts to fully express their assessment viewpoint. For making up this defect of FS, the intuitionistic fuzzy set (IFS)[11], as an efficient extension of FS, is exhibited through adding a nonmembership grade and the sum of them is restricted in $[0, 1]$. Since the outstanding ability of IFS for describing indeterminacy and ambiguous information, multitude researchers have successfully investigated it and attained achievement in various aspects including intuitionistic fuzzy logic [12], intuitionistic fuzzy control [13, 14], decision analysis [15, 16] and so forth. Among them, decision analysis, as an important investigation direction in decision science and management engineering, has received numerous attentions through constructing different decision approaches. Xu and Yager [17] contemplated a series of aggregation operators of IFS to fuse intuitionistic fuzzy preference information. In addition, to take into consideration the correlation of criterions, several operators generated by some special functions are propounded to efficient aggregate intuitionistic fuzzy evaluation information, such as generalized Bonferroni mean (BM) [18], MSM [19] Hamy mean [20], Muirhead mean [21] and so on. Apart from these, some decision methodologies are contemplated to deal with decision issues in diverse situations. Rani et al. [22] proposed an extended TODIM (an acronym in Portuguese for Interactive Multi-Criteria Decision Making) method on the basis of shapley weighted divergence measure. Mishra et al. [23] presented the divergence measure of IFS and propounded additive ratio assessment method to select the IT personnel. Mishra et al. [24] proffered an integration decision technique by combining the complex proportional assessment and step-wise weight assessment ratio analysis method for evaluating the bioenergy production process.

Although IFS has been successfully employed to different domains, the range of information expression also possess limitations. When the sum of membership and nonmembership grade is greater than one, IFS is invalid to portray this kind of information. In view of this shortcoming, Yager [25, 26] built up the theory of Pythagorean fuzzy set (PFS) through changing the limitation condition of membership and nonmembership grade, which makes the sum of squares of membership garde and nonmembership grade is less than 1. It is obvious that PFS provides more space and selections for experts to give their assessment viewpoint than IFS. Hence, various scholars have devoted to the research of decision approaches under Pythagorean fuzzy circumstances and achieved a series of important investigation outcomes. Among these achievements, the information aggregation, as a straightforward and significant decision method, has been paid close attention to fix decision issues. The presented works of PFS on aggregation operator can be divided as the following two categories: (1) Suppose that the fused data are independent of each other. For this category, Zhang [27] developed several frequent aggregation operators to aggregae preference information including the weighted averaging(WA)operator, weighted geometric (WG) operator and their ordered weighted forms. Furthermore, based on different operational laws, the Einstein WA operator [28], Einstein WG operator [29], Choquet-Frank operators [30], logarithmic WA and WG operator [31], Dombi WA and WG operator [32] and neutrality geometric operator [33] of PFS are propounded to rich Pythagorean fuzzy information fusion theory. The more research for this type can be studied in [34, 35, 36, 37, 38, 39, 40]. (2) Suppose that the fused data possesses interactive and interdependent of each other. For this category, Liang et al.[41] developed the geometric BM operator and combined it with projection model to set up group decision approach. Li et al. [42] advanced some Pythagorean fuzzy Hamy mean operators to select optimal supplier. Because the BM and HM operator can only consider the relevance between two input parameters, it fails to settle the situation that considers the relevance among multiple input parameters. For this, Li et al. [43] brought forward the power Muirhead mean operator of PFS to fully ponder the interrelationship and support degree of input data. Wei et al. [44] introduced the Pythagorean fuzzy MSM (PFMSM) operators and its application in decision analysis. Further, Qin [45] propounded the generalized PFMSM operator and combined the SIR method to construct group decision model. Yang and Pang [46] developed several novel PFMSM operator on the basis of interactive operations. To date, a series of MSM operators and its extension are investigated under diverse vague settings [47, 48, 49, 50, 51].

The WASPAS approaches, as an efficient generalization of weighted product model (WPM) and weighted sum model(WSM), was initially propounded by Zavadskas et al [52]. The WASPAS method showed better accuracy in dealing with MADM problems than the application of WPM or WSM. In light of its merits, the increasingly research on WASPAS method is investigated in various utilizations. Zavadskas et al. [53] extended the WASPAS method to interval-valued intuitionistic fuzzy context and applied it to MADM. Zavadskas et al. [54] presented the single-valued neutrosophic WASPAS method and utilized it to choose the construction of a waste incineration Plant address. Ghorabaee et al. [55] built up a novel decision model with the aid of WASPAS method and combinative weight method to select a satisfied green supplier under interval-2 fuzzy environment. Pend and Dai [56] counseled hesitant fuzzy soft WASPAS method and used it to MADM. Moreover, the WASPAS method is integrated with other traditional approaches to better develop decision analysis [57, 58, 59, 60]. The above-mentioned investigations illustrate that the WASPAS method has the powerful capability in settling decision issues under uncertainty environments.

Based upon the aforementioned investigation of weighted MSM operator and WASPAS method in different fuzzy circumstances, we find several defects in previous works: (1)The extant weighted MSM operators [44, 45, 46, 47, 48, 49, 50, 51] fail to degenerate into their correspond MSM operators when the important degree of fused data is equal. (2) The extant weighted MSM operators [44, 45, 46, 47, 48, 49, 50, 51] do not have the characteristic of idempotency, which will produce an irrational fusion outcomes. (3) The traditional WASPAS method fails to take into the interrelationship of diverse criterions consideration and produce much effect from the awkward data. (4)In most extensions of WASPAS method, the weight information of attributes are provided through decision experts. Nevertheless, it is difficult for experts to directly ascertain the importance of attributes. Accordingly, most practical problems can not gain weight in advance. Encouraged by the reducible weighted MSM (RWMSM) operator propounded by Shi and Xiao [63], by considering the merit of PFS and WASPAS method, we design an innovative MADM methodology through combining the PFS, PFRWMSM operator and PFRWGMSM operator and WASPAS method for handling decision problems with unknown weight information. Accordingly, the innovations and contribution of this article can be summarized as below:

(1) To present two novel integration operators including PFRWMSM operator and PFRWGMSM operator and prove several valuable properties of them;

(2) To present an entropy measure based on knowledge measure of PFS for ascertaining the attribute weights information;

(3) To design a novel MADM methodology through combining the WASPAS method and the advanced operators to deal with the decision issues with unknown weight information;

(4) To build a comprehensive assessment model to develop teacher TQE;

(5) To explicitly expound the feasibility and superiority of the created approach through an example and comparison studying, severally.

To accomplish the aforementioned objectives, the overall structure of the essay is allocated as below. In section 2, we succinctly retrospect several fundamental concepts including PFS and MSM operators. Section 3 propounds the notion of PFRWMSM and PFRWGMSM operator and also studies several worthwhile features and particular cases of them. Section 4 presents an innovative knowledge measure and entropy of PFS to determine the weight information. Section 5 is concerned with the novel MADM methodology on the basis of PFRWMSM, PFRWGMSM operator and WASPAS method. In section 6, a teaching quality assessment problem is utilized to show the efficiency and a contrastive study is performed to highlight the merits of the developed method. Several conclusion remarks are listed in the end.

2. Preliminaries

Several necessary definitions involving notion and comparison approach of PFS are briefly retrospect. In addition, the MSM operator and it extension formations are concrete introduced.

2.1. PFS

The PFS propounded by Yager [25] is a more powerful information expression technique than FS and IFS, which provides more space for experts to portray their viewpoint under uncertain and ambiguous setting. The detailed definition of PFS is exhibited as below.

Definition 1. *[26] Assume X is a domain of discourse. A PFS $\bar{\mathcal{P}}$ on X is represented as*

$$\bar{\mathcal{P}} = \{\langle x, \bar{\mu}_{\bar{\mathcal{P}}}(x), \bar{\nu}_{\bar{\mathcal{P}}}(x)\rangle | x \in X\} \tag{1}$$

where $\bar{\mu}_{\bar{\mathcal{P}}}(x) : X \to [0,1]$ and $\bar{\nu}_{\bar{\mathcal{P}}}(x) : X \to [0,1]$ severally signify the grade of membership and non-membership of the element x to $\bar{\mathcal{P}}$ with the restriction that $(\bar{\mu}_{\bar{\mathcal{P}}}(x))^2 + (\bar{\nu}_{\bar{\mathcal{P}}}(x))^2 \leq 1$. The hesitancy grade $\bar{\pi}_{\bar{\mathcal{P}}}(x) = \sqrt{1 - (\bar{\mu}_{\bar{\mathcal{P}}}(x))^2 - (\bar{\nu}_{\bar{\mathcal{P}}}(x))^2}$. The pair $\bar{\mathcal{P}} = (\bar{\mu}_{\bar{\mathcal{P}}}(x), \bar{\nu}_{\bar{\mathcal{P}}}(x))$ is usually utilized to signified a Pythagorean fuzzy number (PFN)[64] and simplified as $(\bar{\mu}, \bar{\nu})$ with $0 \leq \bar{\mu}^2 + \bar{\nu}^2 \leq 1$.

Yager and Abbasov [25] also utilized another geometric manner to represent the PFN, namely, $\bar{\mathcal{P}} = (r_{\bar{\mathcal{P}}}(x), d_{\bar{\mathcal{P}}}(x))$, in which $r_{\bar{\mathcal{P}}}(x)$ signifies the strength of commitment and $d_{\bar{\mathcal{P}}}(x)$ signifies the strength of the direction of commitment, respectively. The relations between $(\bar{\mu}_{\bar{\mathcal{P}}}(x), \bar{\nu}_{\bar{\mathcal{P}}}(x))$ and $\left(\bar{r}_{\bar{\mathcal{P}}}(x), \bar{d}_{\bar{\mathcal{P}}}(x)\right)$ are described as $\bar{\mu}_{\bar{\mathcal{P}}}(x) = \bar{r}_{\bar{\mathcal{P}}}(x) \cdot \cos\left(\bar{\theta}_{\bar{\mathcal{P}}}(x)\right)$, $\bar{\nu}_{\bar{\mathcal{P}}}(x) = \bar{r}_{\bar{\mathcal{P}}}(x) \cdot \sin\left(\bar{\theta}_{\bar{\mathcal{P}}}(x)\right)$, $\bar{r}_{\bar{\mathcal{P}}}(x) = \sqrt{(\bar{\mu}_{\bar{\mathcal{P}}}(x))^2 + (\bar{\nu}_{\bar{\mathcal{P}}}(x))^2}$, $d_{\bar{\mathcal{P}}}(x) = 1 - 2\left(\bar{\theta}_{\bar{\mathcal{P}}}(x)/\pi\right)$, where $\bar{\theta}_{\bar{\mathcal{P}}}(x)$ indicates the radian with the range $[0, \pi/2]$.

Definition 2. *[25, 64] Suppose $\bar{\mathcal{P}} = (\bar{\mu}, \bar{\nu})$, $\bar{\mathcal{P}}_1 = (\bar{\mu}_1, \bar{\mu}_1)$ and $\bar{\mathcal{P}}_2 = (\bar{\mu}_2, \bar{\mu}_2)$ be three PFNs. The associated operations are defined as:*

$$(1)\ \bar{\mathcal{P}}_1 \oplus \bar{\mathcal{P}}_2 = \left(\sqrt{\left(1 - \left(1 - (\bar{\mu}_1)^2\right)\left(1 - (\bar{\mu}_2)^2\right)\right)}, \bar{\nu}_1\bar{\nu}_2\right);$$

$$(2)\ \bar{\mathcal{P}}_1 \otimes \bar{\mathcal{P}}_2 = \left(\bar{\mu}_1\bar{\mu}_2, \sqrt{\left(1 - \left(1 - (\bar{\nu}_1)^2\right)\left(1 - (\bar{\nu}_2)^2\right)\right)}\right);$$

$$(3)\ \lambda\bar{\mathcal{P}} = \left(\sqrt{1 - \left(1 - (\bar{\mu})^2\right)^{\lambda}}, (\bar{\nu})^{\lambda}\right), \lambda > 0;$$

$$(4)\ \bar{\mathcal{P}}^{\lambda} = \left((\bar{\mu})^{\lambda}, \sqrt{1 - \left(1 - (\bar{\nu})^2\right)^{\lambda}}\right), \lambda > 0;$$

$$(5)\ \bar{\mathcal{P}}^{c} = (\bar{\nu}, \bar{\mu});$$

$$(6)\ \bar{\mathcal{P}}_1 \cup \bar{\mathcal{P}}_2 = (\max\{\bar{\mu}_1, \bar{\mu}_2\}, \min\{\bar{\nu}_1, \bar{\nu}_2\});$$

$$(7)\ \bar{\mathcal{P}}_1 \cap \bar{\mathcal{P}}_2 = (\min\{\bar{\mu}_1, \bar{\mu}_2\}, \max\{\bar{\nu}_1, \bar{\nu}_2\}).$$

To compare and rank PFNs, Zhang and Xu [64] firstly developed the score function. However, the score function is invalid to differentiate two PFNs for the situation that the the membership degree is equal to nonmembership degree. For this, Peng and Yang [65] presented the accuracy function of PFNs and given the comparison method to rank PFNs.

Definition 3. *[64, 65] Suppose that $\bar{\mathcal{P}} = (\bar{\mu}, \bar{\nu})$ be a PFN. Then the score and accuracy function are defined as*

$$\mathbb{S}\left(\bar{\mathcal{P}}\right) = \bar{\mu}^2 - \bar{\nu}^2,\ \mathbb{S}\left(\bar{\mathcal{P}}\right) \in [-1, 1]. \tag{2}$$

$$\mathbb{H}\left(\bar{\mathcal{P}}\right) = \bar{\mu}^2 + \bar{\nu}^2,\ \mathbb{H}\left(\bar{\mathcal{P}}\right) \in [0, 1]. \tag{3}$$

Definition 4. *[65] Given two PFNs $\bar{\mathcal{P}}_1 = (\bar{\mu}_1, \bar{\nu}_1)$ and $\bar{\mathcal{P}}_2 = (\bar{\mu}_2, \bar{\nu}_2)$, then the comparison algorithm of $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$ is given as:*

1. *If* $\mathbb{S}(\bar{\mathcal{P}}_1) > \mathbb{S}(\bar{\mathcal{P}}_2)$*, then* $\bar{\mathcal{P}}_1 > \bar{\mathcal{P}}_2$
2. *If* $\mathbb{S}(\bar{\mathcal{P}}_1) = \mathbb{S}(\bar{\mathcal{P}}_2)$*, then*
   - *If* $\mathbb{H}(\bar{\mathcal{P}}_1) < \mathbb{H}(\bar{\mathcal{P}}_2)$*, then* $\bar{\mathcal{P}}_1 < \bar{\mathcal{P}}_2$
   - *If* $\mathbb{H}(\bar{\mathcal{P}}_1 = \mathbb{H}(\bar{\mathcal{P}}_2)$*, then* $\bar{\mathcal{P}}_1 = \bar{\mathcal{P}}_2$.

2.2. MSM

The MSM operator, as a significant aggregation function, can valid integrate the input data and take into account the correlation among the input data. The definition of MSM is stated as follows.

Definition 5. *[61] Let* $\varrho_i (i = 1, 2, \cdots, n)$ *be a family of positive real-number, and set* $\kappa = 1, 2, \cdots, n$*. Then the MSM operator is stated as*

$$\mathrm{MSM}^{(\kappa)}(\varrho_1, \varrho_2, \cdots, \varrho_n) = \left(\frac{\sum_{1 \le i_1 < \cdots < i_\kappa \le n} \left(\prod_{j=1}^{\kappa} \varrho_{i_j} \right)}{C_n^\kappa} \right)^{\frac{1}{\kappa}} \tag{4}$$

where $i_1, i_2, \cdots, i_\kappa$ *traverses all the* κ*-permutations of* $\{1, 2, \cdots, n\}$*,* C_n^κ *stands for the binomial coefficient whose expression is* $C_n^\kappa = \frac{n!}{\kappa!(n-\kappa)!}$.

Based upon the definition of MSM operator, the dual form of MSM operator is propounded by Qin and Liu [62], which is stated as below.

Definition 6. *[62] Let* $\varrho_i (i = 1, 2, \cdots, n)$ *be a family of positive real-number, and set* $\kappa = 1, 2, \cdots, n$*. Then the DMSM operator is stated as*

$$\mathrm{DMSM}^{(k)}(\varrho_1, \varrho_2, \cdots, \varrho_n) = \frac{1}{\kappa} \left(\prod_{1 \le i_1 < \cdots < i_\kappa \le n} \left(\sum_{j=1}^{\kappa} \varrho_{i_j} \right)^{\frac{1}{C_n^\kappa}} \right). \tag{5}$$

where $i_1, i_2, \cdots, i_\kappa$ *traverses all the* κ*-permutations of* $\{1, 2, \cdots, n\}$*,* C_n^κ *stands for the binomial coefficient whose expression is* $C_n^\kappa = \frac{n!}{\kappa!(n-\kappa)!}$.

Because the extant weighted MSM operators fail to deal with the problem of idempotency and reducibility. For this, Shi and Xiao [63] proffered the reducible weighted MSM (RWMSM) and the reducible weighted geometric MSM(RWGMSM)operator as follows.

Definition 7. *[63] Let* $\varrho_i (i = 1, 2, \cdots, n)$ *be a family of positive real-number, and* $\kappa = 1, 2, \cdots, n$*.* $\mathcal{W} = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *with* $\omega_i \in [0, 1]$ *and* $\sum_{i=1}^{n} \omega_i = 1$*. Then reducible weighted MSM (RWMSM)operator is given as below:*

$$\mathrm{RWMSM}^{(\kappa)}(\varrho_1, \varrho_2, \cdots, \varrho_n) = \left(\frac{\sum_{1 \le i_1 < \cdots < i_\kappa \le n} \left(\prod_{j=1}^{\kappa} \omega_{i_j} \right) \left(\prod_{j=1}^{\kappa} \varrho_{i_j} \right)}{\sum_{1 \le i_1 < \cdots < i_\kappa \le n} \prod_{j=1}^{\kappa} \omega_{i_j}} \right)^{\frac{1}{\kappa}}. \tag{6}$$

Definition 8. *[63] Let* $\varrho_i (i = 1, 2, \cdots, n)$ *be a family of positive real-number, and* $\kappa = 1, 2, \cdots, n$*.* $\mathcal{W} = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *with* $\omega_i \in [0, 1]$ *and* $\sum_{i=1}^{n} \omega_i = 1$*. Then reducible weighted MSM (RWMSM)operator is given as below:*

$$\mathrm{RWGMSM}^{(\kappa)}(\varrho_1, \varrho_2, \cdots, \varrho_n) = \frac{\prod_{1 \le i_1 < \cdots < i_\kappa \le n} \left(\sum_{j=1}^{\kappa} \varrho_{i_j} \right)^{\frac{\sum_{j=1}^{k} \omega_{i_j}}{\sum_{1 \le i_1 < \cdots < i_\kappa \le n} \sum_{j=1}^{k} \omega_{i_j}}}}{\kappa}. \tag{7}$$

3. Pythagorean fuzzy reducible weighted Maclaurin symmetric means

In this part, on the basis of the RWMSM and RWGMSM operator and operational laws of PFNs, the Pythagorean fuzzy RWMSM (PFRWMSM) operator and Pythagorean fuzzy RWGMSM (PFRWGMSM) operator are propounded to fuse Pythagorean fuzzy information. In addition, several worthwhile properties and especial instances of the PFRWMSM and PFRWGMSM operator are investigated at length.

3.1. Pythagorean fuzzy reducible weighted Maclaurin symmetric mean operator

Definition 9. *Assume $\bar{\mathcal{P}}_i = (\bar{\mu}_i, \bar{\nu}_i)$ be a family of "n" PFNs, and let $\mathcal{W} = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weights vector with $\omega_i \in [0,1]$ and $\sum_{i=1}^n \omega_i = 1$. The PFRWMSM operator is a mapping PFRWMSM: $\Gamma^n \to \Gamma$ stated as*

$$\mathrm{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) = \left(\frac{\sum\limits_{1\le i_1<\cdots<i_\kappa\le n} \left(\prod\limits_{\jmath=1}^{\kappa} \omega_{i_\jmath}\right)\left(\prod\limits_{\jmath=1}^{\kappa} \bar{\mathcal{P}}_{i_\jmath}\right)}{\sum\limits_{1\le i_1<\cdots<i_\kappa\le n} \prod\limits_{\jmath=1}^{\kappa} \omega_{i_\jmath}} \right)^{\frac{1}{\kappa}} \tag{8}$$

where Γ is the collection of PFNs. Then PFRWMSM is called Pythagorean fuzzy reducible weighted MSM operator.

With the assistance of the operational rules of PFNs depicted in Definition 2, based on Eq. (8), we can attain the fusion outcome displayed in Theorem 1.

Theorem 1. *Let $\bar{\mathcal{P}}_i = (\bar{\mu}_i, \bar{\nu}_i)$ be a family of "n" PFNs, and let $\mathcal{W} = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weights vector with $\omega_i \in [0,1]$ and $\sum_{i=1}^n \omega_i = 1$. The fusion outcome via the PFRWMSM operator is also a PFN and*

$$\begin{aligned}
&\mathrm{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) \\
&= \left(\left(\sqrt{1 - \left(\prod_{1\le i_1<\cdots<i_\kappa\le n} \left(1 - \prod_{\jmath=1}^{\kappa} \left(\bar{\mu}_{i_\jmath}\right)^2\right)^{\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}} \right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}}}} \right)^{\frac{1}{\kappa}}, \sqrt{1 - \left(1 - \left(\prod_{1\le i_1<\cdots<i_\kappa\le n} \left(1 - \prod_{\jmath=1}^{\kappa} \left(1 - \left(\bar{\nu}_{i_\jmath}\right)^2\right)\right)^{\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}} \right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}}}\right)^{\frac{1}{\kappa}}} \right).
\end{aligned} \tag{9}$$

Proof. In light of the operational rules of PFNs depicted in Definition 2, one has

$$\prod_{\jmath=1}^{\kappa} \bar{\mathcal{P}}_{i_\jmath} = \left(\prod_{\jmath=1}^{\kappa} \bar{\mu}_{i_\jmath}, \sqrt{1 - \prod_{\jmath=1}^{\kappa}\left(1 - \left(\bar{\nu}_{i_\jmath}\right)^2\right)} \right)$$

and

$$\left(\prod_{\jmath=1}^{\kappa} \omega_{i_\jmath}\right)\left(\prod_{\jmath=1}^{\kappa} \bar{\mathcal{P}}_{i_\jmath}\right) = \left(\sqrt{1 - \left(1 - \prod_{\jmath=1}^{\kappa}\left(\bar{\mu}_{i_\jmath}\right)^2\right)^{\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}}}, \left(\sqrt{1 - \prod_{\jmath=1}^{\kappa}\left(1 - \left(\bar{\nu}_{i_\jmath}\right)^2\right)}\right)^{\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}} \right).$$

Then

$$\sum_{1\le i_1<\cdots<i_\kappa\le n} \left(\prod_{\jmath=1}^{\kappa} \omega_{i_\jmath}\right)\left(\prod_{\jmath=1}^{\kappa} \bar{\mathcal{P}}_{i_\jmath}\right) = \left(\sqrt{1 - \prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1 - \prod_{\jmath=1}^{\kappa}\left(\bar{\mu}_{i_\jmath}\right)^2\right)^{\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}}}, \prod_{1\le i_1<\cdots<i_\kappa\le n}\left(\sqrt{1 - \prod_{\jmath=1}^{\kappa}\left(1 - \left(\bar{\nu}_{i_\jmath}\right)^2\right)}\right)^{\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}} \right).$$

Thus

$$\frac{\sum\limits_{1\le i_1<\cdots<i_\kappa\le n}\left(\prod\limits_{j=1}^{\kappa}\omega_{i_j}\right)\left(\prod\limits_{j=1}^{\kappa}\bar{\mathcal{P}}_{i_j}\right)}{\sum\limits_{1\le i_1<\cdots<i_\kappa\le n}\prod\limits_{j=1}^{\kappa}\omega_{i_j}} = \left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}},\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(\sqrt{1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)}\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right).$$

Accordingly,

$$\left(\frac{\sum\limits_{1\le i_1<\cdots<i_\kappa\le n}\left(\prod\limits_{j=1}^{\kappa}\omega_{i_j}\right)\left(\prod\limits_{j=1}^{\kappa}\bar{\mathcal{P}}_{i_j}\right)}{\sum\limits_{1\le i_1<\cdots<i_\kappa\le n}\prod\limits_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\kappa}} = \left(\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}},\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}}\right).$$

Consequently, the Eq. (9) is correct. Next, we shall testify the integrated outcome is also a PFN.

Let

$$\eta = \left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{I_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}},$$

$$\phi = \sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(\sqrt{1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)}\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}}.$$

Then, we need to prove the fusion outcome meeting the following conditions:

(i) $0\le\eta\le 1, 0\le\phi\le 1$;

(ii) $0\le\eta^2+\phi^2\le 1$.

Since $0\le\bar{\mu}_{i_j}\le 1$, then one has

$$0\le\left(\bar{\mu}_{i_j}\right)^2\le 1,\ 0\le 1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\le 1,$$

and

$$0\le\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\le 1,\ 0\le\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\le 1.$$

Furthermore, based upon $0 \le \frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}} \le 1$ and $0 < \frac{1}{\kappa} \le 1$. we have

$$0 \le 1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}} \le 1.$$

Then,

$$0 \le \left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}} \le 1.$$

Accordingly, $0 \le \eta \le 1$.
We can homologous acquired $0 \le \phi \le 1$.
Hence, the condition (i) is valid.
Since $0 \le \left(\bar{\mu}_{i_j}\right)^2 + \left(\bar{\nu}_{i_j}\right)^2 \le 1$, then $\left(\bar{\mu}_{i_j}\right)^2 \le 1-\left(\bar{\nu}_{i_j}\right)^2$

$$\begin{aligned}
&0 \le \eta^2+\phi^2 \\
&=\left(\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}}\right)^2+\left(\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}}\right)^2 \\
&=\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}+1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}} \\
&\le\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}+1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}} \\
&=1
\end{aligned}$$

Therefor, condition the condition (i) is true. Accordingly, Theorem 1 is proved.

Theorem 2. ***(Idempotency)*** *Let $\bar{\mathcal{P}} = (\bar{\mu}, \bar{\nu})$ be a family of "n" PFNs. If $\bar{\mathcal{P}}_1 = \bar{\mathcal{P}}_2 = \cdots = \bar{\mathcal{P}}_n = \bar{\mathcal{P}}$ for each i. Then*

$$\mathrm{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) = \bar{\mathcal{P}}. \tag{10}$$

PROOF.

$$
\begin{aligned}
&\mathrm{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)\\
&=\left(\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}},\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}}\right)\\
&=\left(\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}(\bar{\mu})^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}},\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-(\bar{\nu})^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}}\right)\\
&=\left(\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-(\bar{\mu})^{2\times\kappa}\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}},\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\left(1-(\bar{\nu})^2\right)^{\kappa}\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}}\right)\\
&=\left(\left(\sqrt{1-\left(\left(1-(\bar{\mu})^{2\times\kappa}\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}},\sqrt{1-\left(1-\left(\left(1-\left(1-(\bar{\nu})^2\right)^{\kappa}\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}}\right)\\
&=\left(\left(\sqrt{1-\left(1-(\bar{\mu})^{2\times\kappa}\right)}\right)^{\frac{1}{\kappa}},\sqrt{1-\left(1-\left(1-\left(1-(\bar{\nu})^2\right)^{\kappa}\right)\right)^{\frac{1}{\kappa}}}\right)=\left(\bar{\mu},\sqrt{1-\left(1-(\bar{\nu})^2\right)}\right)=(\bar{\mu},\bar{\nu}).
\end{aligned}
$$

Theorem 3. ***(Monotonicity)*** *Let $\bar{\mathcal{P}}_i=(\bar{\mu}_i,\bar{\nu}_i)$ and $\bar{\mathcal{P}}_i=\left(\bar{\mu}_i^{'},\bar{\nu}_i^{'}\right)$ be two families of "n" PFNs. If $\bar{\mu}_i\le\bar{\mu}_i^{'},\bar{\nu}_i\ge\bar{\nu}_i^{'}$ fir all i. Then*

$$
\mathrm{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)\le\mathrm{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1^{'},\bar{\mathcal{P}}_2^{'},\cdots,\bar{\mathcal{P}}_n^{'}\right). \tag{11}
$$

PROOF. Assume that $\mathrm{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)=\bar{\mathcal{P}}=(\bar{\mu},\bar{\nu})$ and $\mathrm{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1^{'},\bar{\mathcal{P}}_2^{'},\cdots,\bar{\mathcal{P}}_n^{'}\right)=\bar{\mathcal{P}}^{'}=\left(\bar{\mu}^{'},\bar{\nu}^{'}\right)$. Then

$$
\begin{aligned}
\bar{\mu}&=\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}},\ \bar{\mu}^{'}=\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}^{'}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}},\\
\bar{\nu}&=\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}},\ \bar{\nu}^{'}=\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}^{'}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}}.
\end{aligned}
$$

For testifying the monotonicity, we shall prove it through computing their score values $\mathbb{S}\left(\bar{\mathcal{P}}\right)$ and $\mathbb{S}\left(\bar{\mathcal{P}}^{'}\right)$ and further attain $\bar{\mathcal{P}}\le\bar{\mathcal{P}}^{'}$. Next, we divide two steps to achieve the goal, namely, (1) derive the comparison outcomes of their degree of membership and nonmembership; (2) compute and discuss their score values.

(1) In view of the known conditions $\bar{\mu}_i\le\bar{\mu}_i^{'},\bar{\nu}_i\ge\bar{\nu}_i^{'}$, we can deduce the following comparison relation:

For $\bar{\mu}$ and $\bar{\mu}^{'}$, one has

$$
\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}}\le\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\mu}_{i_j}^{'}\right)^2\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}},
$$

which is $\bar{\mu}\le\bar{\mu}^{'}$.

For $\bar{\nu}$ and $\bar{\nu}^{'}$, one has

$$\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}} \ge \sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\nu}^{'}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}},$$

which is $\bar{\nu} \ge \bar{\nu}^{'}$.

(2) With the help of the above comparison outcomes, we can easily get $\mathbb{S}(\Xi) = (\bar{\mu})^2 - (\bar{\nu})^2 \le \mathbb{S}\left(\bar{\mathcal{P}}^{'}\right) = \left(\bar{\mu}^{'}\right)^2 - \left(\bar{\nu}^{'}\right)^2$. Afterward, we discuss the following circumstances.

(a) When $\mathbb{S}\left(\bar{\mathcal{P}}\right) < \mathbb{S}\left(\bar{\mathcal{P}}^{'}\right)$, then $\bar{\mathcal{P}} < \bar{\mathcal{P}}^{'}$ via the Definition 2;

(b) When $\mathbb{S}\left(\bar{\mathcal{P}}\right) = \mathbb{S}\left(\bar{\mathcal{P}}^{'}\right)$, then $\bar{\mathcal{P}} = \bar{\mathcal{P}}^{'}$, i.e., $\bar{\mu} = \bar{\mu}^{'}$ and $\bar{\nu} = \bar{\nu}^{'}$, by the accuracy function defined in Definition 3, we can derive $\bar{\mathcal{P}} = \bar{\mathcal{P}}^{'}$.

Accordingly, we can attain $\bar{\mathcal{P}} \le \bar{\mathcal{P}}^{'}$, namely,

$$\text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) \le \text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}^{'}_1, \bar{\mathcal{P}}^{'}_2, \cdots, \bar{\mathcal{P}}^{'}_n\right).$$

Theorem 4. ***(Boundedness)*** *Let $\bar{\mathcal{P}}_i = (\bar{\mu}_i, \bar{\nu}_i)$ be a family of "n" PFN, and $\bar{\mathcal{P}}_i^{+} = \left(\max_i^n \bar{\mu}_i, \min_i^n \bar{\nu}_i\right)$, $\bar{\mathcal{P}}_i^{-} = \left(\min_i^n \bar{\mu}_i, \max_i^n \bar{\nu}_i\right)$. Then*

$$\bar{\mathcal{P}}_i^{+} \le \text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) \le \bar{\mathcal{P}}_i^{-}. \tag{12}$$

Proof. With the aid of the monotonicity the PFRWMSM operator, we have

$$\text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) \le \text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1^{+}, \bar{\mathcal{P}}_2^{+}, \cdots, \bar{\mathcal{P}}_n^{+}\right);$$
$$\text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) \ge \text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1^{-}, \bar{\mathcal{P}}_2^{-}, \cdots, \bar{\mathcal{P}}_n^{-}\right).$$

In light of the idempotency of the PFRWMSM operator, we have

$$\text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1^{+}, \bar{\mathcal{P}}_2^{+}, \cdots, \bar{\mathcal{P}}_n^{+}\right) = \bar{\mathcal{P}}_i^{+};$$
$$\text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1^{-}, \bar{\mathcal{P}}_2^{-}, \cdots, \bar{\mathcal{P}}_n^{-}\right) = \bar{\mathcal{P}}_i^{-}.$$

Accordingly, we can attain $\bar{\mathcal{P}}_i^{+} \le \text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) \le \bar{\mathcal{P}}_i^{-}$.

Theorem 5. ***(Commutativity)*** *Let $\left(\bar{\mathcal{P}}^{'}_1, \bar{\mathcal{P}}^{'}_2, \cdots, \bar{\mathcal{P}}^{'}_n\right)$ is any permutation of $\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right)$. Then*

$$\text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}^{'}_1, \bar{\mathcal{P}}^{'}_2, \cdots, \bar{\mathcal{P}}^{'}_n\right) = \text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right). \tag{13}$$

Proof. Since the known condition, we have

$$\left(\frac{\sum_{1\le i_1<\cdots<i_\kappa\le n}\left(\prod_{j=1}^{\kappa}\omega_{i_j}\right)\left(\prod_{j=1}^{\kappa}\bar{\mathcal{P}}_{i_j}\right)}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\kappa}} = \left(\frac{\sum_{1\le i_1<\cdots<i_\kappa\le n}\left(\prod_{j=1}^{\kappa}\omega^{'}_{i_j}\right)\left(\prod_{j=1}^{\kappa}\bar{\mathcal{P}}^{'}_{i_j}\right)}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{j=1}^{\kappa}\omega^{'}_{i_j}}\right)^{\frac{1}{\kappa}}.$$

Consequently, we can acquire $\text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}^{'}_1, \bar{\mathcal{P}}^{'}_2, \cdots, \bar{\mathcal{P}}^{'}_n\right) = \text{PFRWMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right)$.

In the next, several peculiar instances of PFRWMSM operator are explored through assigning diverse parameter κ.

Case 1. *When $\kappa = 1$, the presented PFRWMSM operator is degenerated into Pythagorean fuzzy weighted averaging*

(PFWA) operator, exhibited as below:

$$
\begin{aligned}
&\mathrm{PFRWMSM}^{(1)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)\\
&=\left(\left(\sqrt{1-\left(\prod_{1\le i_1\le n}\left(1-\left(\bar{\mu}_{i_j}\right)^2\right)^{\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1\le n}\omega_{i_j}}}}\right)^{\frac{1}{1}},\sqrt{1-\left(1-\left(\prod_{1\le i_1\le n}\left(1-\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1\le n}\omega_{i_j}}}\right)^{\frac{1}{1}}}\right)\\
&=\left(\sqrt{1-\prod_{1\le i_1\le n}\left(1-\left(\bar{\mu}_{i_j}\right)^2\right)^{\omega_{i_j}}},\prod_{1\le i_1\le n}\left(\bar{\nu}_{i_j}\right)^{\omega_{i_j}}\right)\\
&=\mathrm{PFWA}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right).
\end{aligned}
\tag{14}
$$

Case 2. *When $\kappa = 2$, the developed PFRWMSM operator is degraded into Pythagorean generalized weighted Heronian mean (PFGWHM) operator, exhibited as below:*

$$
\begin{aligned}
&\mathrm{PFRWMSM}^{(2)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)\\
&=\left(\left(\sqrt{1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{2}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<i_2\le n}\prod_{j=1}^{2}\omega_{i_j}}}}\right)^{\frac{1}{2}},\sqrt{1-\left(1-\left(\prod_{1\le i_1<i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{2}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<i_2\le n}\prod_{j=1}^{2}\omega_{i_j}}}\right)^{\frac{1}{2}}}\right)\\
&=\mathrm{PFGWHM}^{(1,1)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right).
\end{aligned}
\tag{15}
$$

Case 3. *When $\kappa = n$, the presented PFRWMSM operator is transformed into Pythagorean fuzzy geometric (PFWG) operator, exhibited as below:*

$$
\begin{aligned}
&\mathrm{PFRWMSM}^{(1)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)\\
&=\left(\left(\sqrt{1-\left(\left(1-\prod_{j=1}^{n}\left(\bar{\mu}_{i_j}\right)^2\right)^{\prod_{j=1}^{n}\omega_{i_j}}\right)^{\frac{1}{\prod_{j=1}^{n}\omega_{i_j}}}}\right)^{\frac{1}{n}},\sqrt{1-\left(1-\left(\left(1-\prod_{j=1}^{n}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)\right)^{\prod_{j=1}^{n}\omega_{i_j}}\right)^{\frac{1}{\prod_{j=1}^{n}\omega_{i_j}}}\right)^{\frac{1}{n}}}\right)\\
&=\left(\left(\prod_{j=1}^{n}\bar{\mu}_{i_j}\right)^{\frac{1}{n}},\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\bar{\nu}_{i_j}\right)^2\right)^{\frac{1}{n}}}\right)\\
&=\mathrm{PFG}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right).
\end{aligned}
\tag{16}
$$

3.2. Pythagorean fuzzy reducible weighted geometric Maclaurin symmetric mean operator

Definition 10. *Assume $\bar{\mathcal{P}}_i = (\bar{\mu}_i, \bar{\nu}_i)$ be a family of "n" PFNs, and let $\mathcal{W} = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weights vector with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n}\omega_i = 1$. The PFRWGMSM operator is a mapping PFRWGMSM: $\Gamma^n \to \Gamma$ stated as*

$$
\mathrm{PFRWGMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)=\frac{\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(\sum_{j=1}^{\kappa}\bar{\mathcal{P}}_{i_j}\right)^{\frac{\sum_{j=1}^{k}\omega_{i_j}}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{k}\omega_{i_j}}}}{\kappa},
\tag{17}
$$

where Γ is the collection of PFNs. Then PFRWGMSM is called Pythagorean fuzzy reducible weighted geometric MSM operator.

Theorem 6. *Let $\bar{\mathcal{P}}_i = (\bar{\mu}_i, \bar{\nu}_i)$ be a family of "n" PFNs, and let $\mathcal{W} = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weights vector with*

$\omega_i \in [0,1]$ *and* $\sum_{i=1}^{n}\omega_i = 1$. *The fusion outcome via the PFRWGMSM operator is also a PFN and*

$$
\begin{aligned}
&\mathrm{PFRWGMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)\\
&=\left(\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\mu}_{i_j}\right)^2\right)\right)^{\sum_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}},\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\nu}_{i_j}\right)^2\right)^{\sum_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}}\right).
\end{aligned}
\tag{18}
$$

Proof.

$$
\sum_{j=1}^{\kappa}\bar{\mathcal{P}}_{i_j}=\left(\sqrt{1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\mu}_{i_j}\right)^2\right)},\prod_{j=1}^{\kappa}\bar{\nu}_{i_j}\right)
$$

and

$$
\left(\sum_{j=1}^{\kappa}\bar{\mathcal{P}}_{i_j}\right)^{\frac{\sum_{j=1}^{k}\omega_{i_j}}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{k}\omega_{i_j}}}=\left(\left(\sqrt{1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\mu}_{i_j}\right)^2\right)}\right)^{\frac{\sum_{j=1}^{k}\omega_{i_j}}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{k}\omega_{i_j}}},\sqrt{1-\left(1-\prod_{j=1}^{\kappa}\left(\bar{\nu}_{i_j}\right)^2\right)^{\frac{\sum_{j=1}^{k}\omega_{i_j}}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{k}\omega_{i_j}}}}\right)
$$

$$
\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(\sum_{j=1}^{\kappa}\bar{\mathcal{P}}_{i_j}\right)^{\frac{\sum_{j=1}^{k}\omega_{i_j}}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{k}\omega_{i_j}}}=\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(\sqrt{1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\mu}_{i_j}\right)^2\right)}\right)^{\frac{\sum_{j=1}^{k}\omega_{i_j}}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{k}\omega_{i_j}}},\sqrt{1-\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(\left(1-\prod_{j=1}^{\kappa}\left(\bar{\nu}_{i_j}\right)^2\right)^{\frac{\sum_{j=1}^{k}\omega_{i_j}}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{k}\omega_{i_j}}}\right)}\right)
$$

$$
\begin{aligned}
&\frac{\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(\sum_{j=1}^{\kappa}\bar{\mathcal{P}}_{i_j}\right)^{\frac{\sum_{j=1}^{k}\omega_{i_j}}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{k}\omega_{i_j}}}}{\kappa}\\
&=\left(\sqrt{1-\left(1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(1-\left(\bar{\mu}_{i_j}\right)^2\right)\right)^{\sum_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{\kappa}\omega_{i_j}}}\right)^{\frac{1}{\kappa}}},\left(\sqrt{1-\left(\prod_{1\le i_1<\cdots<i_\kappa\le n}\left(1-\prod_{j=1}^{\kappa}\left(\bar{\nu}_{i_j}\right)^2\right)^{\sum_{j=1}^{\kappa}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}}\right).
\end{aligned}
$$

Accordingly, Theorem 6 is proved.

Analogous to the PFRWMSM operator, we expound the following characteristics of the propounded PFRWGMSM operator.

Theorem 7. ***(Idempotency)*** *Let* $\bar{\mathcal{P}} = (\bar{\mu},\bar{\nu})$ *be a family of "n" PFNs. If* $\bar{\mathcal{P}}_1 = \Xi_2 = \cdots = \bar{\mathcal{P}}_n = \bar{\mathcal{P}}$ *for each i. Then*

$$
\mathrm{PFRWGMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)=\bar{\mathcal{P}}. \tag{19}
$$

Theorem 8. ***(Monotonicity)*** *If* $\bar{\mu}_i \ge \bar{\mu}_i', \bar{\nu}_i \ge \bar{\nu}_i'$ *for all i. Then*

$$
\mathrm{PFRWGMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1,\bar{\mathcal{P}}_2,\cdots,\bar{\mathcal{P}}_n\right)\ge\mathrm{PFRWGMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1',\bar{\mathcal{P}}_2',\cdots,\bar{\mathcal{P}}_n'\right). \tag{20}
$$

Theorem 9. ***(Boundedness)*** *Let* $\bar{\mathcal{P}}_i = (\bar{\mu}_i, \bar{\nu}_i)$ *be a family of "n" PFNs, and* $\bar{\mathcal{P}}_i^+ = \left(\max\limits_i^n \bar{\mu}_i, \min\limits_i^n \bar{\nu}_i\right)$*,* $\bar{\mathcal{P}}_i^+ = \left(\min\limits_i^n \bar{\mu}_i, \max\limits_i^n \bar{\nu}_i\right)$*. Then*

$$\bar{\mathcal{P}}_i^+ \leq \mathrm{PFRWGMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) \leq \bar{\mathcal{P}}_i^-. \tag{21}$$

Theorem 10. ***(Commutativity)*** *Let* $\left(\bar{\mathcal{P}}_1', \bar{\mathcal{P}}_2', \cdots, \bar{\mathcal{P}}_n'\right)$ *is any permutation of* $\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right)$*. Then*

$$\mathrm{PFRWGMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1', \bar{\mathcal{P}}_2', \cdots, \bar{\mathcal{P}}_n'\right) = \mathrm{PFRWGMSM}^{(\kappa)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right). \tag{22}$$

Since the process of proof monotonicity, boundedness and commutativity of the developed PFRWGMSM operator are analogous with procedure of PFRWMSM operator, we are not going to repeat it.

Analogously, we possess the following several peculiar instances of PFRWMSM operator through assigning diverse values of parameter κ.

Case 4. *When* $\kappa = 1$*, the presented PFRWGMSM operator is degenerated into Pythagorean fuzzy weighted geometric (PFWG) operator, exhibited as below:*

$$\begin{aligned}
&\mathrm{PFRWGMSM}^{(1)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) \\
&= \left(\sqrt{1 - \left(1 - \left(\prod_{1\leq i_1\leq n}\left(1 - \prod_{j=1}^{1}\left(1 - \left(\bar{\mu}_{i_j}\right)^2\right)\right)^{\sum_{j=1}^{1}\omega_{i_1}}\right)^{\frac{1}{\sum_{1\leq i_1\leq n}\sum_{j=1}^{1}\omega_{i_1}}}\right)}^{\frac{1}{1}}, \left(\sqrt{1 - \left(\prod_{1\leq\leq n}\left(1 - \prod_{j=1}^{1}\left(\bar{\nu}_{\Xi_{i_1}}\right)^2\right)^{\sum_{j=1}^{\kappa}\omega_{i_1}}\right)^{\frac{1}{\sum_{1\leq\leq n}\sum_{j=1}^{1}\omega_{i_1}}}}\right)^{\frac{1}{1}} \right) \\
&= \left(\prod_{1\leq i_1\leq n}\left(\bar{\mu}_{i_1}\right)^{\omega_{i_1}}, \sqrt{1 - \left(1 - \prod_{1\leq i_1\leq n}\left(\bar{\nu}_{i_1}\right)^2\right)^{\omega_{i_1}}} \right) \\
&= \mathrm{PFWG}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right).
\end{aligned} \tag{23}$$

Case 5. *When* $\kappa = n$*, the presented PFRWGMSM operator is transformed into Pythagorean fuzzy averaging operator, exhibited as below:*

$$\begin{aligned}
&\mathrm{PFRWGMSM}^{(n)}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right) \\
&= \left(\sqrt{1 - \left(1 - \left(\prod_{1\leq i_1<\cdots<i_\kappa\leq n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(\bar{\mu}_{i_j}\right)^2\right)\right)^{\sum_{j=1}^{n}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\leq i_1<\cdots<i_\kappa\leq n}\sum_{j=1}^{n}\omega_{i_j}}}\right)}^{\frac{1}{\kappa}}, \left(\sqrt{1 - \left(\prod_{1\leq i_1<\cdots<i_\kappa\leq n}\left(1 - \prod_{j=1}^{n}\left(\bar{\nu}_{i_j}\right)^2\right)^{\sum_{j=1}^{n}\omega_{i_j}}\right)^{\frac{1}{\sum_{1\leq i_1<\cdots<i_\kappa\leq n}\sum_{j=1}^{\kappa}\omega_{i_j}}}}\right)^{\frac{1}{\kappa}} \right) \\
&= \left(\sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(\bar{\mu}_{i_j}\right)^2\right)^{\frac{1}{n}}}, \prod_{j=1}^{n}\left(\bar{\nu}_{i_j}\right)^{\frac{1}{n}} \right) \\
&= \mathrm{PFA}\left(\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2, \cdots, \bar{\mathcal{P}}_n\right).
\end{aligned} \tag{24}$$

4. Novel entropy-based knowledge measure for PFS

Knowledge measure originated by Szmidt et al.[66] is a efficient technique for depicting the amount of information of fuzzy set. It is of importance tool measure the fuzziness of a fuzzy set. Motivated by the think of the knowledge measure of IFS propounded by Szmidt et al.[67], we present a novel Pythagorean fuzzy knowledge measure (PFKM) and further define a entropy of PFS based upon the knowledge measure.

Definition 11. *Let* X *is a universe of discourse and* $\bar{\mathcal{P}} = \{\langle x, \bar{\mu}(x), \bar{\nu}(x)\}$ *be a PFS. The knowledge measure of* $\bar{\mathcal{P}}$ *is defined as follows:*

$$\mathrm{PFKM}\left(\bar{\mathcal{P}}\right) = 1 - \frac{1}{2n}\sum_{i=1}^{n}\left(1 - \left|\bar{\mu}^2(x_i) - \bar{\nu}^2(x_i)\right|\right)\left(1 + \bar{\pi}^2(x_i)\right). \tag{25}$$

As we see, the proposed knowledge can valid measure the amount of knowledge through taking into account the fuzziness and intuitionism of PFS. In what follows, we shall prove that the presented PFKM $\mathrm{PFKM}\left(\bar{\mathcal{P}}\right)$ is a valid knowledge measure if it fulfills the following axiomatic properties.

Theorem 11. *Suppose $\bar{\mathcal{P}} = \{\langle x, \bar{\mu}(x), \bar{\nu}(x)\}$ be a PFS. The PFKM $\mathrm{PFKM}(\Xi)$ defined in Eq.(25) fulfills the following axiomatic properties.*

(P1) $\mathrm{PFKM}\left(\bar{\mathcal{P}}\right) = 1$ *iff* $\bar{\mu}(x_i) = 1$ *or* $\bar{\nu}(x_i) = 1$*;*

(P2) $\mathrm{PFKM}\left(\bar{\mathcal{P}}\right) = 0$ *iff* $\bar{\pi}(x_i) = 1$*;*

(P3) $\mathrm{PFKM}\left(\bar{\mathcal{P}}_1\right) \geq \mathrm{PFKM}\left(\bar{\mathcal{P}}_2\right)$ *iff* $\bar{\mathcal{P}}_1$ *is less fuzzy than* $\bar{\mathcal{P}}_2$*, i.e.,* $\bar{\mathcal{P}}_1 \subseteq \bar{\mathcal{P}}_2$ *for* $\bar{\mu}_1(x_i) \leq \bar{\mu}_2(x_i)$ *or ;*

(P4) $\mathrm{PFKM}\left(\bar{\mathcal{P}}\right) = \mathrm{PFKM}\left(\bar{\mathcal{P}}^c\right)$.

Proof. **(P1)** Since $\mathrm{PFKM}\left(\bar{\mathcal{P}}\right) = 1$. Then $\frac{1}{2n}\sum_{i=1}^{n}\left(1 - \left|\bar{\mu}^2(x_i) - \bar{\nu}^2(x_i)\right|\right)\left(1 + \bar{\pi}^2(x_i)\right) = 0$, namely, $\left(1 - \left|\bar{\mu}^2(x_i) - \bar{\nu}^2(x_i)\right|\right)\left(1 + \bar{\pi}^2(x_i)\right) = 0$. Since, $1 + \bar{\pi}^2 \in [1, 2]$. Then $\left(1 - \left|\bar{\mu}^2(x_i) - \bar{\nu}^2(x_i)\right|\right) = 0$ for each $x_i \in X$. Hence, we have $\bar{\mu}(x_i) = 1$ or $\bar{\nu}(x_i) = 1$. In addition, if $\bar{\mu}(x_i) = 1$ or $\bar{\nu}(x_i) = 1$, it is easy to obtain $\mathrm{PFKM}\left(\bar{\mathcal{P}}\right) = 1$.

(P2) If $\bar{\pi}(x_i) = 1$, then $\bar{\mu}(x_i) = \bar{\nu}(x_i) = 0$. Thus, $\mathrm{PFKM}(\bar{\mathcal{P}}) = 0$. On the other hand, if $\mathrm{PFKM}(\bar{\mathcal{P}}) = 0$, then

$$\begin{aligned}
&\frac{1}{2}\left(\left(1 - \left|\bar{\mu}^2(x_i) - \bar{\nu}^2(x_i)\right|\right)\right)\left(1 + \bar{\pi}^2(x_i)\right) = 0\\
&\Rightarrow \left(1 + \bar{\pi}^2(x_i)\right)\left(1 - \left|\bar{\mu}^2(x_i) - \bar{\nu}^2(x_i)\right|\right) = 2\\
&\Rightarrow \bar{\pi}^2(x_i) - \left(1 + \bar{\pi}^2(x_i)\right)\left(1 - \left|\bar{\mu}^2(x_i) - \bar{\nu}^2(x_i)\right|\right) = 1\\
&\Rightarrow \left(1 + \bar{\pi}^2(x_i)\right)\left(1 - \left|\bar{\mu}^2(x_i) - \bar{\nu}^2(x_i)\right|\right) = \bar{\pi}^2(x_i) - 1.
\end{aligned}$$

Since the range of $\bar{\mu}^2(x_i), \bar{\nu}^2(x_i), \bar{\pi}(x_i)$, then we can attain $\bar{\pi}(x_i) = 1$.

(P3) Aiming at the proposed knowledge measure $\mathrm{PFKM}\left(\bar{\mathcal{P}}\right)$, we ponder the following function:

$$f(\bar{\mu}, \bar{\nu}) = 1 - \frac{1}{2}\left(2 - \left(\bar{\mu}^2 + \bar{\nu}^2\right)\right)\left(1 - \left|\bar{\mu}^2 - \bar{\nu}^2\right|\right).$$

in which $\bar{\mu}, \bar{\nu} \in [0, 1]$ and $0 \leq \bar{\mu}^2(x_i) + \bar{\nu}^2(x_i) \leq 1$. Then we have

$$\mathrm{PFKM}(\Xi) = \frac{1}{n}\sum_{i=1}^{n} f\left(\bar{\mu}(x_i), \bar{\nu}(x_i)\right).$$

For the first situation that $\bar{\mathcal{P}}_1 \subseteq \bar{\mathcal{P}}_2$ for $\bar{\mu}_1(x_i) \leq \bar{\mu}_2(x_i)$, it is obvious that $\bar{\mu}_1(x_i) \leq \bar{\mu}_2(x_i) \leq \bar{\nu}_2(x_i) \leq \bar{\mu}_1(x_i)$. Simply write as $\bar{\mu} \leq \bar{\nu}$. The the function can be rewritten as:

$$f(\bar{\mu}, \bar{\nu}) = 1 - \frac{1}{2}\left(2 - \left(\bar{\mu}^2 + \bar{\nu}^2\right)\right)\left(1 + \mid \bar{\mu}^2 - \bar{\nu}^2 \mid\right).$$

In the following, we research the partial derivatives of the function $f(\bar{\mu}, \bar{\nu})$. Then for $\bar{\mu}$, we have

$$\frac{\partial f(\bar{\mu}, \bar{\nu})}{\partial \bar{\mu}} = \bar{\mu}\left(2\bar{\mu}^3 - 1\right),$$

since $0 \leq \bar{\mu} \leq \bar{\nu} \leq 1$ and $0 \leq \bar{\mu}^2 + \bar{\nu}^2 \leq 1$, then $\bar{\mu} \leq \frac{\sqrt{2}}{2}$. Then

$$\frac{\partial f(\bar{\mu}, \bar{\nu})}{\partial \bar{\mu}} \leq 0.$$

which signifies the function $f(\bar{\mu}, \bar{\nu})$ is monotonically decreasing in term of $\bar{\mu}$. Furthermore,

$$\frac{\partial f(\bar{\mu}, \bar{\nu})}{\partial \bar{\mu}} = \bar{\nu}\left(3 - 2\bar{\nu}^3\right) \geq 0.$$

which signifies the function $f(\bar{\mu}, \bar{\nu})$ is monotonically increasing in term of $\bar{\nu}$.

In light of the above results and the relation $\bar{\mu}_1(x_i) \leq \bar{\mu}_2(x_i) \leq \bar{\nu}_2(x_i) \leq \bar{\mu}_1(x_i)$. It is known that $f(\bar{\mu}_1(x_i), \bar{\nu}_1(x_i)) \geq f(\bar{\mu}_2(x_i), \bar{\nu}_2(x_i))$. for all $x_i \in X$.

That further indicates

$$\frac{1}{n}\sum_{i=1}^{n} f(\bar{\mu}_1(x_i), \bar{\nu}_1(x_i)) \geq \frac{1}{n}\sum_{i=1}^{n} f(\bar{\mu}_2(x_i), \bar{\nu}_2(x_i)),$$

which means $\mathrm{PFKM}\left(\bar{\mathcal{P}}_1\right) \geq \mathrm{PFKM}\left(\bar{\mathcal{P}}_2\right)$ holds.

The second situation is similar to the first one, the illustration process is omitted here.

(P4) Since $\bar{\mathcal{P}}^c = \{\langle x, \bar{\nu}(x), \bar{\mu}(x)\}$. we can get

$$\mathrm{PFKM}\left(\bar{\mathcal{P}}\right) = 1 - \frac{1}{2n}\sum_{i=1}^{n}\left(1 - \left|\bar{\mu}^2(x_i) - \bar{\nu}^2(x_i)\right|\right)\left(1 + \bar{\pi}^2(x_i)\right),$$

$$\mathrm{PFKM}\left(\bar{\mathcal{P}}^c\right) = 1 - \frac{1}{2n}\sum_{i=1}^{n}\left(1 - \left|\bar{\nu}^2(x_i) - \bar{\mu}^2(x_i)\right|\right)\left(1 + \bar{\pi}^2(x_i)\right).$$

Accordingly, $\mathrm{PFKM}\left(\bar{\mathcal{P}}\right) = \mathrm{PFKM}\left(\bar{\mathcal{P}}^c\right)$.

Definition 12. *Assume that $\bar{\mathcal{P}} = (\bar{\mu}, \bar{\nu})$ be a PFN. The entropy measure of $\bar{\mathcal{P}}$ on the basis of the knowledge measure is depicted as:*

$$\mathrm{PFE}\left(\bar{\mathcal{P}}\right) = 1 - \mathrm{PFKM}\left(\bar{\mathcal{P}}\right) = \frac{1}{2}\left(1 + \bar{\pi}^2\right)\left(1 - \left|\bar{\mu}^2 - \bar{\nu}^2\right|\right). \tag{26}$$

The propounded Pythagorean fuzzy entropy in Definition 11 fulfills the conditions in Theorem 12.

Theorem 12. *Assume that $\bar{\mathcal{P}} = (\bar{\mu}, \bar{\nu})$ be a PFN. Then*

(P1) $0 \leq \mathrm{PFE}\left(\bar{\mathcal{P}}\right) \leq 1$*;*

(P2) $\mathrm{PFE}\left(\bar{\mathcal{P}}\right) = 1$ *iff* $\bar{\pi}(x_i) = 1$*;*

(P3) $\mathrm{PFE}\left(\bar{\mathcal{P}}\right) = 0$ *iff* $\bar{\mu} = 1$ *or* $\bar{\nu} = 1$.

(P4) $\mathrm{PFE}\left(\bar{\mathcal{P}}\right) = \mathrm{PFE}\left(\bar{\mathcal{P}}^c\right)$.

The proof of Theorem 12 is analogous Theorem 11, so it is omitted.

5. The MADM approach based upon the PFRWMSM and PFRWGMSM operator with unknown weight information

In this part, we employ the PFRWMSM and PFRWGMSM operator to fuse the Pythagorean fuzzy information and further design a sorting approach on the basis of the extended WASPAS method. Firstly, we give the general statement of PF-MADM issue. Secondly, with the assistance of the PFRWMSM and PFRWGMSM operator to integrate the expert opinions and devise an expanded WASPAS method for handling PFMADM issues. Ultimately, we sketch the devised decision algorithm of PFMADM.

5.1. *Depiction of the MADM issues*

Aiming at a traditional MADM issue, it possess the following elementary elements including the set of attributes and alternatives and attributes weight information. Suppose that $T = \{T_1, T_2, \cdots, T_n\}$ be family of alternatives, $\mathfrak{I} = \{\mathfrak{I}_1, \mathfrak{I}_2, \cdots, \mathfrak{I}_m\}$ be a set of attributes, and $\mathcal{W} = \{\omega_1, \omega_1, \cdots, \omega_m\}^T$ is the weight information of attributes with $\omega_J \in [0, 1], \sum_{J=1}^{m} \omega_J = 1$. We suppose that the specialist provides his(her) assessment viewpoint for the alternatives in their of diverse attributes in the form of PFNs and further build-up the Pythagorean fuzzy evaluation matrix $\mathcal{D} = \left(\mathcal{P}_{iJ}\right)_{n\times m} = \left(\bar{\mu}_{iJ}, \bar{\nu}_{iJ}\right)_{n\times m}$.

Table 1: Pythagorean fuzzy evaluation matrix $\mathcal{P} = \left(\Xi_{iJ}\right)_{n\times m}$.

	$\mathfrak{I}_1$	$\mathfrak{I}_2$	$\cdots$	$\mathfrak{I}_J$	$\cdots$	$\mathfrak{I}_n$
T_1	$(\bar{\mu}_{11}, \bar{\nu}_{11})$	$(\bar{\mu}_{12}, \bar{\nu}_{12})$	$\cdots$	$\left(\bar{\mu}_{1J}, \bar{\nu}_{1J}\right)$	$\cdots$	$(\bar{\mu}_{1m}, \bar{\nu}_{1m})$
T_2	$(\bar{\mu}_{21}, \bar{\nu}_{21})$	$(\bar{\mu}_{22}, \bar{\nu}_{22})$	$\cdots$	$\left(\bar{\mu}_{2J}, \bar{\nu}_{2J}\right)$	$\cdots$	$(\bar{\mu}_{2m}, \bar{\nu}_{2m})$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
T_i	$(\bar{\mu}_{i1}, \bar{\nu}_{i1})$	$(\bar{\mu}_{i2}, \bar{\nu}_{i2})$	$\cdots$	$\left(\bar{\mu}_{iJ}, \bar{\nu}_{iJ}\right)$	$\cdots$	$(\bar{\mu}_{im}, \bar{\nu}_{im})$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
T_m	$(\bar{\mu}_{n1}, \bar{\nu}_{m1})$	$(\bar{\mu}_{n2}, \bar{\nu}_{m2})$	$\cdots$	$\left(\bar{\mu}_{mJ}, \bar{\nu}_{nJ}\right)$	$\cdots$	$(\bar{\mu}_{nm}, \bar{\nu}_{nm})$

Table 2: Linguistic terms for experts to assess the alternatives.

Linguistic term	Abbreviation	Pythagorean fuzzy element
Extremely Low	EL	(0.15, 0.95)
Very Low	VL	(0.25, 0.85)
Low	L	(0.35, 0.75)
Middle low	ML	(0.45, 0.65)
Below middle	BL	(0.50, 0.60)
Middle	M	(0.55, 0.55)
Above middle	AM	(0.60, 0.50)
Middle hight	MH	(0.65, 0.45)
Hight	H	(0.75, 0.35)
Very hight	VH	(0.85, 0.25)
Extremely hight	EH	(0.95, 0.15)

5.2. *The propounded PF-WASPAS decision approach*

The WASPAS is an effectual assessment method propounded by Zavadskas et al. [52], which can more exact deal with actual issues by combines the weighted product and weighted sum model. To efficient process the aforementioned PF-MADM problem, we device an PF-WASPAS approach by combining the the PFRWMSM operator, PFRWGMSM operator and WASPAS method to cope with the above PF-MADM problem with unknown attribute weight information.

Determining assessment information. Based upon the depiction of decision problems, decision experts provide their opinion for the alternatives with respect to the pondered attributes. In light of the complexity of decision expert's cognition, they usually give the linguistic terms to express their judgement rather than utilize the Pythagorean fuzzy information directly. Hence, the assessment matrix can be ascertained through decision experts based on the linguistic terms listed in Table 2.

Transforming assessment information. The Pythagorean fuzzy assessment matrix $\mathcal{D} = \left(\mathcal{P}_{iJ}\right)_{m\times n}$ can be derived through the Table 2.

Standardized the Pythagorean fuzzy assessment matrix. In light of the diverse types of the attributes, we need to normalize the assessment matrix $\mathcal{D} = \left(\mathcal{P}_{i\jmath}\right)_{n\times m}$ to standardized Pythagorean fuzzy assessment matrix $\overline{\mathcal{D}} = \left(\overline{\mathcal{P}}_{i\jmath}\right)_{n\times m}$ through the Eq. (27):

$$\overline{\mathcal{P}}_{i\jmath} = \left(\bar{\bar{\mu}}_{i\jmath}, \bar{\bar{\nu}}_{i\jmath}\right) = \begin{cases} \left(\bar{\mu}_{i\jmath}, \bar{\nu}_{i\jmath}\right), & \mathfrak{I}_i \text{ is benefit attribute;} \\ \left(\bar{\nu}_{i\jmath}, \bar{\mu}_{i\jmath}\right), & \mathfrak{I}_i \text{ is cost attribute.} \end{cases} \tag{27}$$

Ascertaining the weights of attributes. The weight information of criterion is an crucial index during the decision process. In this essay, the weight vector of attribute is determined on the basis the developed Pythagorean fuzzy entropy. The concrete computation steps ar shown as below:

(1): The Pythagorean fuzzy entropy matrix $\text{PFEM} = \left(\text{PFE}_{i\jmath}\right)_{n\times m}$ is built-up through the Eq. (26).

$$\text{PFEM} = \left(\text{PFE}_{i\jmath}\right)_{n\times m} = \begin{pmatrix} \text{PFE}_{11} & \text{PFE}_{12} & \cdots & \text{PFE}_{1m} \\ \text{PFE}_{21} & \text{PFE}_{22} & \cdots & \text{PFE}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \text{PFE}_{n1} & \text{PFE}_{n2} & \cdots & \text{PFE}_{nm} \end{pmatrix}$$

(2): Calculating the standardized Pythagorean fuzzy entropy matrix $\text{PFENM} = \left(\text{PFNE}_{i\jmath}\right)_{n\times m}$ with the aid of the following formulation:

$$\text{PFNE}_{\jmath} = \frac{\sum_{i=1}^{n} \text{PFE}_{i\jmath}}{\sum_{i=1}^{n} \sum_{\jmath=1}^{m} \text{PFE}_{i\jmath}}.$$

(3): Ascertaining the weights of attribute through the following formulation,

$$\omega_{\jmath} = \frac{1 - \text{PFNE}_{\jmath}}{\sum_{\jmath=1}^{m} \left(1 - \text{PFNE}_{\jmath}\right)}. \tag{28}$$

The Computation of the WSM and WPM model. As mentioned before, the WASPAS method is made up of the weighted sum model (WSM) and weighted product model (WSM). The final value the the optimal alternative is calculated by combining the aggregated value of WSM and WPM. In this essay, we employ the PFRWMSM operator and PFRWGMSM operator to determine the WSM and WPM, severally.

Ascertaining weighted sum of alternative T_i through the developed PFRWMSM operator, the value Q_i^1 of weighted sum is computed by the Eq. (29)

$$\begin{aligned} Q_i^1 &= \text{PFRWMSM}^{(\kappa)}\left(\mathcal{P}_{i1}, \mathcal{P}_{i2}, \cdots, \mathcal{P}_{im}\right) \\ &= \left(\left(\sqrt{1 - \left(\prod_{1\le i_1<\cdots<i_\kappa\le n} \left(1 - \prod_{\jmath=1}^{\kappa} \left(\bar{\bar{\mu}}_{i_\jmath}\right)^2\right)^{\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}} \right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}}}} \right)^{\frac{1}{\kappa}}, \sqrt{1 - \left(1 - \left(\prod_{1\le i_1<\cdots<i_\kappa\le n} \left(1 - \prod_{\jmath=1}^{\kappa} \left(1 - \left(\bar{\bar{\nu}}_{i_\jmath}\right)^2\right)\right)^{\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}} \right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\prod_{\jmath=1}^{\kappa}\omega_{i_\jmath}}} \right)^{\frac{1}{\kappa}}} \right). \end{aligned} \tag{29}$$

Ascertaining weighted prod of alternative T_i through the developed PFRWGMSM operator, the value Q_i^2 of weighted prod is computed by the Eq. (30),

$$\begin{aligned} Q_i^2 &= \text{PFRWGMSM}^{(\kappa)}\left(\mathcal{P}_{i1}, \mathcal{P}_{i2}, \cdots, \mathcal{P}_{im}\right) \\ &= \left(\sqrt{1 - \left(1 - \left(\prod_{1\le i_1<\cdots<i_\kappa\le n} \left(1 - \prod_{\jmath=1}^{\kappa} \left(1 - \left(\bar{\bar{\mu}}_{i_\jmath}\right)^2\right)\right)^{\sum_{\jmath=1}^{\kappa}\omega_{i_\jmath}} \right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{\jmath=1}^{\kappa}\omega_{i_\jmath}}} \right)^{\frac{1}{\kappa}}}, \left(\sqrt{1 - \left(\prod_{1\le i_1<\cdots<i_\kappa\le n} \left(1 - \prod_{\jmath=1}^{\kappa} \left(\bar{\bar{\nu}}_{I_\jmath}\right)^2\right)^{\sum_{\jmath=1}^{\kappa}\omega_{i_\jmath}} \right)^{\frac{1}{\sum_{1\le i_1<\cdots<i_\kappa\le n}\sum_{\jmath=1}^{\kappa}\omega_{i_\jmath}}}} \right)^{\frac{1}{\kappa}} \right). \end{aligned} \tag{30}$$

Calculating the comprehensive assessment value. The comprehensive assessment value K_i of alternative is calculated through the Eq. (31),

$$K_i = \sigma Q_i^{(1)} \oplus (1 - \sigma)\, Q_i^{(2)}, \tag{31}$$

where $\sigma Q_i^1 = \left(\sqrt{1 - \left(1 - \left(\bar{\mu}_{Q_i^1}\right)^2\right)^{\sigma}}, \left(\bar{\nu}_{Q_i^1}\right)^{\sigma} \right)$, $(1 - \sigma)\, Q_i^2 = \left(\sqrt{1 - \left(1 - \left(\bar{\mu}_{Q_i^2}\right)^2\right)^{(1-\sigma)}}, \left(\bar{\nu}_{Q_i^2}\right)^{(1-\sigma)} \right)$. The coefficient σ takes from the interval $[0, 1]$, which can be ascertain according to risk preference of experts.

Ascertaining the order relation of alternatives. The alternative ranks with the help of the score values of $\mathbb{S}(K_i)$, in which the optimal alternative possess the most value of $\mathbb{S}(K_i)$.

5.3. The developed PF-WASPAS decision procedure

With the help of the aforementioned statement, we summarize the steps of the designed PF-WASPAS decision to resolve the actual problems with Pythagorean fuzzy information.

Pondering an empirical decision problem, let $T = \{T_1, T_2, \cdots, T_n\}$ be set of alternatives, $\mathfrak{I} = \{\mathfrak{I}_1, \mathfrak{I}_2, \cdots, \mathfrak{I}_m\}$ be a set of attributes, and $\mathcal{W} = \{\omega_1, \omega_1, \cdots, \omega_m\}^T$ is the weight information of attributes with $\omega_J \in [0, 1]$, $\sum_{J=1}^{m} \omega_J = 1$. The experts provide their assessment information by the linguistic terms displayed in Table 2. The following decision procedure will determine the optimal alternative and rank the alternatives in a declining order.

Step 1: Determining assessment information for alternatives with the aid of Table 2.

Step 2: Transforming the linguistic assessment matrix to Pythagorean fuzzy assessment matrix $\mathcal{D} = \left(\mathcal{P}_{iJ}\right)_{n\times m}$.

Step 3: Obtaining the standardized Pythagorean assessment matrix $\overline{\mathcal{D}} = \left(\overline{\mathcal{P}}_{iJ}\right)_{n\times m}$ through the Eq. (27).

Step 4: Ascertaining the weight vector $\mathcal{W} = \{\omega_1, \omega_2, \cdots, \omega_m\}^T$ with the aid of the Eq. (28).

Step 5: Working out the value of weighted sum $Q_i^{(1)}$ and weighted product $Q_i^{(2)}$ for every alternative T_i through the Eq. (29) and Eq. (30).

Step 6: Calculating the comprehensive assessment value K_i for each alternative T_i through Eq. (31).

Step 7: Computing the score values $\mathbb{S}(K_i)$ of alternative T_i.

Step 8: Finally, the order relation of alternatives is fixed by the score values of $\mathbb{S}(K_i)$.

By means of the above-mentioned progresses, we create a diagrammatic sketch to outline the diverse steps. The diagrammatic sketch is portrayed in Figure 1.

In what follows, we shall employ an real-life example to illustrate the feasibility and practicability of the designed decision algorithm.

6. Numerical example

This section first utilizes the presented decision algorithm to assess the teaching quality of teachers in university. Then, we implement the sensitivity analysing of the propounded approach. Finally, we conduct a comprehensive comparative study to highlight the merits and validity of the developed approach in this essay.

6.1. Background introduction

In the report of the 19th National Congress of the Communist Party of China, Chinese leaders clearly put forward that improving the quality of education should be placed in an important strategic position of the country. The report points out that we should always make the construction of a powerful education country the party's basic project and always take precedence to develop the business of education. In addition, it sets clear requirements for the key improvement of the quality of higher education, insists on deepening educational reform and modernization, speeds up the construction of first-class universities and disciplines, and realizes the connotative development of higher education. At the same time, the state has implemented a number of measures to comprehensively improve the quality of education, such as the quality of teachers and curriculum reform. Among them, the teaching quality assessment project will be the key to improving and guiding the quality of school education. Hence, the construction of teaching quality assessment model is a crucial research topic for enhancing the education quality of university. After investigating several teaching indexes and evaluation numerical examples, we summed up several important evaluation indicators as decision-making attributes to effectively carry out teaching quality evaluation. The concrete judgement illustrations are depicted as follows.

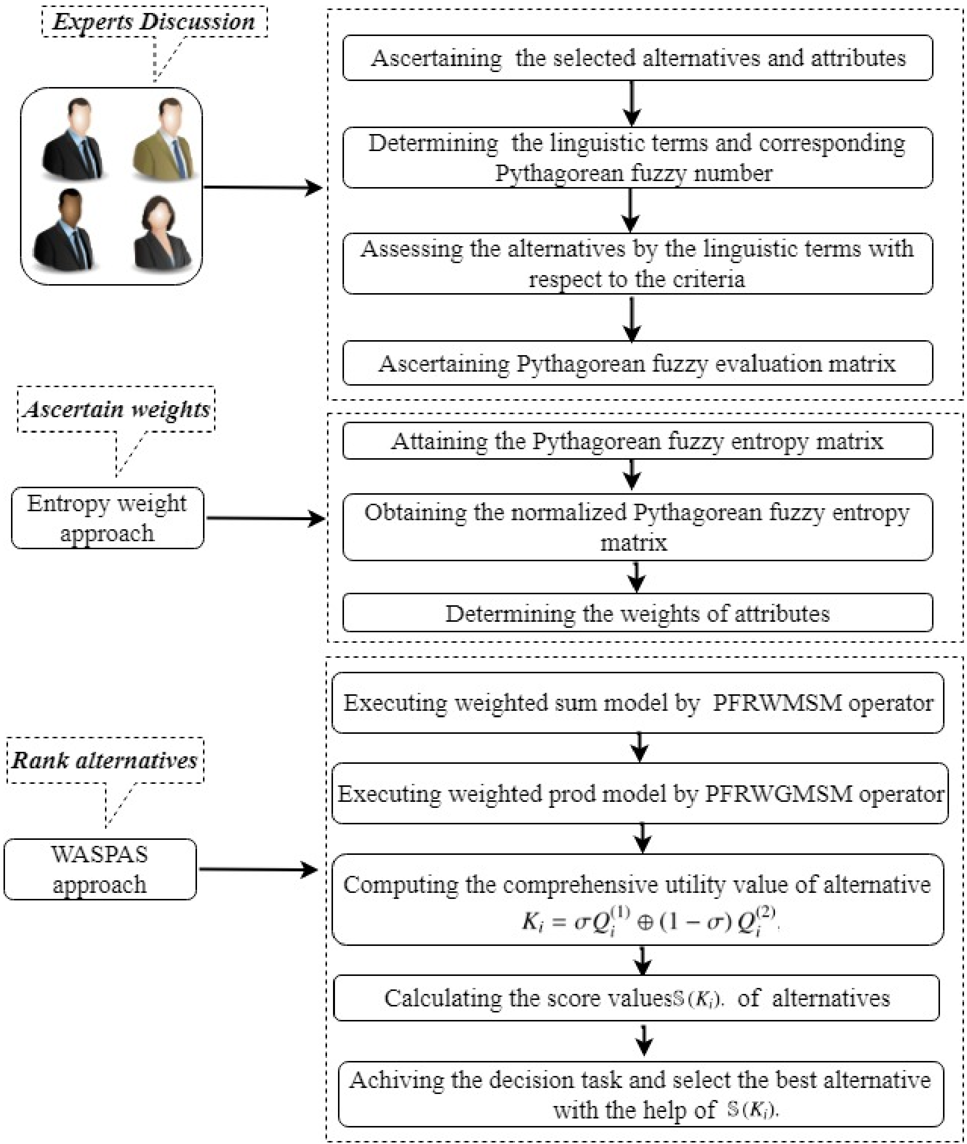


Figure 1: The framework of the propounded approach.

Teaching attitude $\Im_1$**.**Teachers' teaching attitude is an extremely important factor to ensure students' learning. A correct teaching attitude should include the following aspects: preparation before class, teachers' appearance and understanding of the current scientific research situation of the subject. The preparation before class is mainly reflected in the formulation of a clear teaching plan, the preparation of a complete course plan and the preparation of teaching course-ware. Teachers' dignified appearance is determined by their proper clothes, strict adherence to teachers' professional quality and rigorous teaching style. The degree of scientific research understanding of the discipline is reflected in whether the knowledge learned can be combined with the latest scientific research knowledge, so as to improve students' innovative thinking.

Teaching content $\Im_2$**.**Teaching content refers to the dynamically generated materials and information that interact with teachers and students in the teaching process and serve the purpose of teaching. How to make reasonable teaching content and make students quickly grasp knowledge is an important factor to measure the teaching quality of teachers. It mainly involves three aspects: (a) whether the selection of teaching content is reasonable; (b) whether the teaching content highlights the cultivation of practice and application skills; (c) whether to add relevant extension content to expand students' horizons.

Teaching method and technique $\Im_3$**.**Teaching method is the general term of the methods and means used by teachers and students in the teaching process in order to achieve the common teaching objectives and complete the common teaching tasks. It includes teachers' teaching methods, students' learning methods, teaching and learning methods. It is mainly reflected in the following aspects: (a) whether to innovate unique teaching methods to improve students' initiative and enthusiasm and to use the learned knowledge to mine new knowledge; (b) whether to adopt the teaching method of combining seminar, example, heuristic and theme teaching method; (c) whether to design unique teaching course-ware according to the teaching content and combine with practical cases to improve the cognition of students of teaching content.

Teaching effect $\Im_4$**.**Teaching effect is the result of teaching, it is the concentrated embodiment of teaching quality. It is mainly reflected in the following aspects: (a) whether the expected teaching objectives are achieved and the content of knowledge acquired by students is detected through classroom testing; (b) whether the feedback and suggestions of students on classroom teaching are rectified; (c) whether students are committed to student management to improve students' enthusiasm and thus enhance their interest in the subject.

Teaching characteristics $\Im_5$**.**Teaching characteristics are the concrete manifestation of teachers' teaching ability and a deep-seated element of improving teaching quality. It mainly includes the following aspects: (a) whether to make full use of body language to organize the classroom; (b) whether to design novel and interesting ways of introduction to stimulate learning desire; (c) whether the teaching process is progressive and effective.

Evaluation experts provide a comprehensive assessment viewpoint with the aid of the aforementioned decision attributes to conduct assessment activity efficiently.

6.2. Decision analysis

Example 1. *Assume that the colleges and universities will develop the teaching quality assessment for a group of full-time teachers. The group possesses six teachers to take part in the assessment. The assessment criterion set is* $\Im = \{\Im_1, \Im_2, \Im_3, \Im_4, \Im_5\}$*, which* $\Im_1$ *(signified as teaching attitude),* $\Im_2$ *(signified as teaching content),* $\Im_3$ *(signified as teaching method and technique),* $\Im_4$ *(signified as teaching effect),* $\Im_5$ *(signified as teaching characteristics). The above criterions are all deemed as benefit criteria and their weight information is entirely unknown. The assessment expert gives his(her) judgement for the six teachers with respect to the aforementioned criterions after listening their course carefully. The assessment information provided through the linguistics terms is displayed in Table 3.*

In what follows, the optimal teacher is selected through employing the designed approach under Pythagorean fuzzy context.

Step 1: The assessment information for teachers by expert is determined in Table 3.

Step 2: Transforming the linguistic assessment matrix to Pythagorean fuzzy assessment matrix $\mathcal{D} = \left(\mathcal{P}_{ij}\right)_{n\times m}$.

Table 3: Pythagorean fuzzy evaluation matrix.

	$\mathfrak{I}_1$	$\mathfrak{I}_2$	$\mathfrak{I}_3$	$\mathfrak{I}_4$	$\mathfrak{I}_5$
T_1	VH	M	H	MH	AM
T_2	VH	AM	MH	H	M
T_3	MH	AM	VH	H	M
T_4	VH	M	H	AM	MH
T_5	VH	AM	MH	H	H
T_6	H	MH	VH	MH	AM

$$\mathcal{D} = \left(\mathcal{P}_{iȷ}\right)_{n\times m} = \begin{pmatrix} (0.85, 0.25) & (0.55, 0.55) & (0.75, 0.35) & (0.65, 0.45) & (0.60, 0.50) \\ (0.85, 0.25) & (0.60, 0.50) & (0.65, 0.45) & (0.75, 0.35) & (0.55, 0.55) \\ (0.65, 0.45) & (0.60, 0.50) & (0.85, 0.25) & (0.75, 0.35) & (0.55, 0.55) \\ (0.85, 0.25) & (0.55, 0.55) & (0.75, 0.35) & (0.60, 0.50) & (0.65, 0.45) \\ (0.85, 0.25) & (0.60, 0.50) & (0.65, 0.45) & (0.75, 0.35) & (0.75, 0.35) \\ (0.75, 0.35) & (0.65, 0.45) & (0.85, 0.25) & (0.65, 0.45) & (0.60, 0.50) \end{pmatrix}$$

Step 3: Obtaining the standardized Pythagorean assessment matrix $\overline{\mathcal{D}} = \left(\overline{\mathcal{P}}_{iȷ}\right)_{n\times m}$ through the Eq. (27). Because all attributes are considered benefit type, thus the procedure of normalization is omitted, i.e., $\overline{\mathcal{D}} = \mathcal{P}$.
Step 4: Ascertaining the weight vector $\mathcal{W} = \{\omega_1, \omega_1, \cdots, \omega_m\}^T$ with the aid of the Eq. (28).
(1) The Pythagorean fuzzy entropy matrix is calculated as below:

$$\mathrm{PFEM} = \left(\mathrm{PFE}_{iȷ}\right)_{6\times 5} = \begin{pmatrix} 0.7935 & 0.3025 & 0.6318 & 0.4638 & 0.3815 \\ 0.7935 & 0.3815 & 0.4638 & 0.6318 & 0.3025 \\ 0.4638 & 0.3815 & 0.7935 & 0.6318 & 0.3025 \\ 0.7935 & 0.3025 & 0.6318 & 0.3815 & 0.4638 \\ 0.7935 & 0.3815 & 0.4638 & 0.6318 & 0.6318 \\ 0.6318 & 0.4638 & 0.7935 & 0.4638 & 0.3815 \end{pmatrix}$$

(2) The normalized Pythagorean fuzzy entropy values are obtained as follows:

$$\mathrm{PFNE}_1 = 0.2680,\ \mathrm{PFNE}_2 = 0.1389,\ \mathrm{PFNE}_3 = 0.2372,\ \mathrm{PFNE}_4 = 0.2012,\ \mathrm{PFNE}_5 = 0.1547.$$

(3) By utilizing the Eq. (28), we can acquire the weights of all attributes displayed as :

$$\omega_1 = 0.1830,\ \omega_2 = 0.2153,\ \omega_3 = 0.1907,\ \omega_4 = 0.1997,\ \omega_5 = 0.2113.$$

Step 5: By using the PFRWMSM operator and PFRWGMSM operator, the values of weighted sum $Q_i^{(1)}$ and weighted product $Q_i^{(2)}$ for each alternative (for simplicity, let $\kappa = 2$) are worked out shown as:

$$\begin{aligned} &Q_1^{(1)} = (0.6832, 0.4306),\ Q_2^{(1)} = (0.6841, 0.4298),\ Q_3^{(1)} = (0.6853, 0.4287),\\ &Q_4^{(1)} = (0.6837, 0.4302),\ Q_5^{(1)} = (0.7238, 0.3854),\ Q_6^{(1)} = (0.7026, 0.4074);\\ &Q_1^{(2)} = (0.6843, 0.4311),\ Q_2^{(2)} = (0.6848, 0.4306),\ Q_3^{(2)} = (0.6855, 0.4300),\\ &Q_4^{(2)} = (0.6845, 0.4308),\ Q_5^{(2)} = (0.7241, 0.3867),\ Q_6^{(2)} = (0.7033, 0.4079). \end{aligned}$$

Step 6: The comprehensive assessment value K_i for each alternative T_i (for convenience $\sigma = 0.5$) is computed as:

$$K_1 = (0.6838, 0.4308),\ K_2 = (0.6844, 0.4302),\ K_3 = (0.6854, 0.4294),$$
$$K_4 = (0.6841, 0.4305),\ K_5 = (0.7239, 0.3860),\ K_6 = (0.7029, 0.4077).$$

Step 7: The score $\mathbb{S}(K_i)$ of alternative T_i are computed as below:

$$\mathbb{S}(K_1) = 0.2819,\ \mathbb{S}(K_2) = 0.2834,\ \mathbb{S}(K_3) = 0.2854,$$
$$\mathbb{S}(K_4) = 0.2826,\ \mathbb{S}(K_5) = 0.3751,\ \mathbb{S}(K_6) = 0.3279.$$

Step 8: The order relation of alternatives is fixed based upon the values of K_i, which shown as below: $T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$. Accordingly, the teaching quality of the fifth teacher is optimal.

6.3. Parameter discussions

The subsection 6.2 shows the decision analysis procedure of the developed approach with a precondition $\kappa = 2$ and $\sigma = 0.5$, which fails to illustrate the flexibility and variation trend of our method. For this, this subsection shall conduct a comprehensive analysis about the parameter κ and σ and explore the effect of final decision outcomes by utilizing diverse parameter values.

First of all, we analyze the changes of the ultimate order relation of teachers from two versions. One is the changes of final orders based on the diverse values of parameter $\kappa = 2$, the scores and ranks of different teachers are computed in Table 4. From it, we can know that the final order relationship of the six teachers attained by utilizing the presented approach based upon diverse values of parameter σ are all the same, the optimal option is the fifth teacher T_5, which demonstrates that the developed methodology is stable for parameter σ. In addition, we can find that the score of each teacher decreases monotonically with the increase of parameter σ. The parameter σ can be deemed as a preference index of decision expert to control the information fusion manner. When σ is taken the maximum value, the expert shall use the PFRWMSM operator to integrate preference information. When σ is taken the minimum value, experts utilize the PFRWGMSM operator.

Table 4: The impact of σ for the ultimate decision results ($\kappa = 2$).

Parameter σ	Score values						Sorting
	$\mathbb{S}(K_1)$	$\mathbb{S}(K_2)$	$\mathbb{S}(K_3)$	$\mathbb{S}(K_4)$	$\mathbb{S}(K_5)$	$\mathbb{S}(K_6)$	
0	0.2825	0.2835	0.2849	0.2829	0.3747	0.3282	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
0.1	0.2824	0.2835	0.2850	0.2829	0.3748	0.3281	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
0.2	0.2822	0.2835	0.2851	0.2828	0.3749	0.3281	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
0.3	0.2821	0.2834	0.2852	0.2827	0.3749	0.3280	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
0.4	0.2820	0.2834	0.2853	0.2827	0.3750	0.3280	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
0.5	0.2819	0.2834	0.2854	0.2826	0.3751	0.3279	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
0.6	0.2818	0.2834	0.2855	0.2826	0.3751	0.3278	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
0.7	0.2817	0.2834	0.2856	0.2825	0.3752	0.3278	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
0.8	0.2816	0.2833	0.2857	0.2825	0.3753	0.3277	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
0.9	0.2814	0.2833	0.2857	0.2824	0.3753	0.3277	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
1.0	0.2813	0.2833	0.2858	0.2824	0.3754	0.3276	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$

Secondly, we explore the influence of parameter κ for the decision procedure. For this, we first fix the parameter $\sigma = 0.5$ and work out the scores and orders of the selected six teachers on the basis of dissimilar values of parameter κ, the outcomes are displayed in Table 5 and Figure 2. With the assistance of Table 5 and Figure 2, we can attain the following summarizations:

(1) The order relationship of the selected six teachers is slightly diverse, but the optimal choice is all the fifth teacher. The main reason is that the value of parameter κ signifies the relationship between different numbers of attributes. When $\kappa = 1$, the PFRWMSM and PFRWGMSM operator degenerate into the PFWA and PFWG operator severally, thus the PFRWMSM and PFRWGMSM operator fail to ponder the mutuality of diverse attributes in such

situation. Furthermore, the PFRWMSM and PFRWGMSM operator also ignore the correlation of attribute because they simplify to the PFG and PFA operator when $\kappa = 5$. In particular, we give the following illustration for the ranking obtained by the propounded method $\kappa = 5$. The first reason is that PFG and PFA operator fail to consider the importance of aggregated information. Another reason is that the selections of the linguistic term are relatively lesser. This makes experts have certain limitations in providing evaluation information through the given linguistic terms.

(2) For the proffered method, the score values of six teachers are first monotonically increasing when $\kappa \in [1, 2]$ and then decreasing monotonically when $\kappa \in [2, 5]$. The PFRWMSM and PFRWGMSM operator undergo the transformation from averaging operator to geometric operator and geometric operator to averaging operator when parameter κ changes from 1 to 5, severally.

(3) The parameter κ can provide more preference selections for experts, an expert possessing optimistic attitude can select a larger value of κ and an expert possessing pessimistic attitude can select a smaller value of κ. Moreover, experts also select appropriate value of parameter κ to fully take into the consideration the mutuality of criterions through practical situation.

Table 5: The impact of κ for the ultimate decision results($\sigma = 0.5$).

Parameter κ	Score values						Order relation
	$\mathbb{S}(K_1)$	$\mathbb{S}(K_2)$	$\mathbb{S}(K_3)$	$\mathbb{S}(K_4)$	$\mathbb{S}(K_5)$	$\mathbb{S}(K_6)$	
1	0.3355	0.4021	0.4052	0.3075	0.4743	0.3753	$T_5 \succ T_3 \succ T_2 \succ T_6 \succ T_1 \succ T_4$
2	0.2819	0.2834	0.2854	0.2826	0.3751	0.3279	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
3	0.2861	0.2869	0.2880	0.2865	0.3765	0.3308	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
4	0.2911	0.2915	0.2920	0.2913	0.3785	0.3343	$T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$
5	0.2986	0.2986	0.2986	0.2986	0.3826	0.3402	$T_5 \succ T_6 \succ T_1 \sim T_2 \sim T_3 \sim T_4$

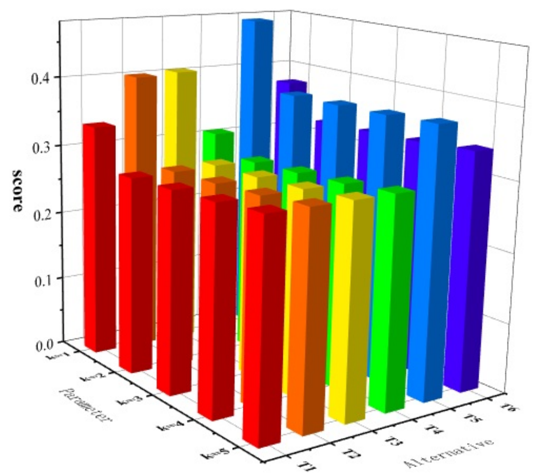
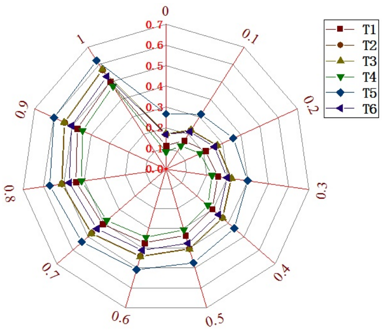


Figure 2: The change of scores obtained through diverse values of parameter κ.

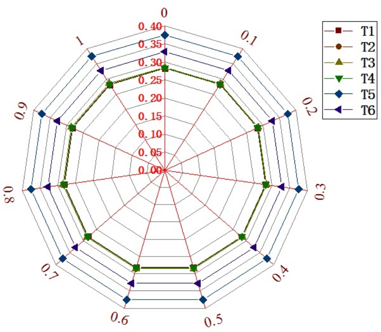
Furthermore, to better show the changes of parameter κ and σ during the decision procedure, we compute the eventual decision results on the basis of different combination of parameter κ and σ, which are displayed in Figure 3.

6.4. Validation test

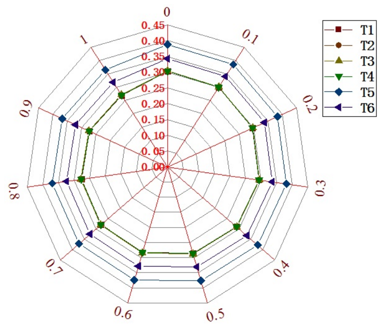
Since it is impossible to ascertain which methodology is most satisfied with provided decision-making issues, the diverse methodologies may produce different preference ordering of alternatives for the same decision issue. Accordingly, we utilize the following test criteria set up by Wang and Triantaphyllou [68] to examine the availability and dependability of our approaches.



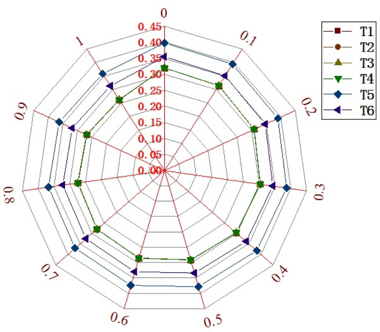
(a) $\kappa = 1$



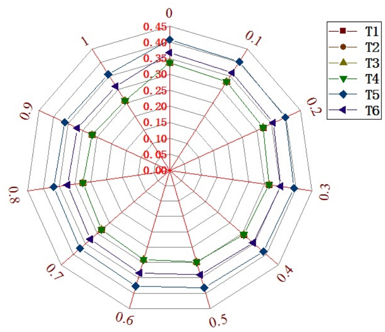
(b) $\kappa = 2$



(c) $\kappa = 3$



(d) $\kappa = 4$



(e) $\kappa = 5$

Figure 3: The change of scores obtained through diverse values of parameter κ.

Test Standard 1: If the decision value of the non optimal scheme is replaced by the decision value of the worse scheme, the decision value of the best scheme should not be changed.

Test Standard 2: An efficient MCDM method should meet transitivity.

Test Standard 3: When an MCDM issue is decomposed into several sub-issues, and the same MCDM method is applied to these sub-issues to sort the alternatives, the various ranks of the alternative should be the same as the rank of original decision issue.

In what follows, we shall execute the aforementioned three text standards to validate the proposed extended WASPAS approach under Pythagorean fuzzy circumstance.

6.4.1. The validity text on standard 1

Aiming at the test standard 1, we interchange the membership degree and nonmembership degree of attribute $\{T_6, T_3, T_2, T_4\}$ (non-optimal scheme) and T_1 (worse scheme)in assessment matrix $\mathcal{D}$, then shifted assessment matrix $\overline{\overline{\mathcal{D}'}}$ is formed and shown as

$$\overline{\overline{\mathcal{D}'}} = \begin{pmatrix} (0.25, 0.85) & (0.55, 0.55) & (0.35, 0.75) & (0.45, 0.65) & (0.50, 0.60) \\ (0.25, 0.85) & (0.50, 0.60) & (0.45, 0.65) & (0.75, 0.35) & (0.55, 0.55) \\ (0.45, 0.65) & (0.50, 0.60) & (0.25, 0.85) & (0.35, 0.75) & (0.55, 0.55) \\ (0.25, 0.85) & (0.55, 0.55) & (0.35, 0.75) & (0.50, 0.60) & (0.45, 0.65) \\ (0.85, 0.25) & (0.60, 0.50) & (0.65, 0.45) & (0.75, 0.35) & (0.75, 0.35) \\ (0.35, 0.75) & (0.45, 0.65) & (0.25, 0.85) & (0.45, 0.65) & (0.50, 0.60) \end{pmatrix}$$

Based upon the shifted assessment matrix $\overline{\overline{\mathcal{D}'}}$ and the presented approach, we obtain the novel order relation of teachers, the optimal selection is also the fifth teacher T_5, which is consistent with the initial ranking of teachers. Accordingly, the designed method is practicable for the test standard 1.

6.4.2. Validation via utilizing the Examine Standard 2 and Examine Standard 3

With the help of test standard 2 and standard 3, the decision issues can be disintegrated as the following sub-issues $\{T_1, T_2, T_3, T_4\}$, $\{T_2, T_3, T_4, T_5\}$ and $\{T_3, T_4, T_5, T_6\}$. Then we utilize the designed Pythagorean fuzzy WASPAS method to resolve the above three sub-problems, the corresponding decision outcomes are $T_3 \succ T_2 \succ T_4 \succ T_1$, $T_5 \succ T_3 \succ T_2 \succ T_4$ and $T_5 \succ T_6 \succ T_3 \succ T_4$, severally. Hence, through combining the test standard 2 and standard 3, we can acquire the entire order relation of alternatives as $T_5 \succ T_6 \succ T_3 \succ T_2 \succ T_4 \succ T_1$, which is same as the original decision result. Consequently, the proposed Pythagorean fuzzy WASPAS method is practicable for the test standard 2 and standard 3.

6.5. Comparison study

In this subsection, to validate the availability and the superiority of the designed innovative approach, we will execute a detailed comparison analysing between the designed method with the extant decision methodologies.

Firstly, we employ the previous methods involving PFWA operator [27], Pythagorean fuzzy Einstein (PFEWA) operator [28], Pythagorean fuzzy geometric BM (PFGBM) operator [41] Pythagorean fuzzy TOPSIS (PF-TOPSIS) method proposed by Zhang and Xu [64] to handle the issue of this manuscript and attain the corresponding decision outcomes, which are displayed in Table 6. From it, we can see that the order relations acquired from diverse Pythagorean fuzzy decision approaches are basically the same as the developed WASPAS method. The optimal selections are all T_5 which further validate the efficiency of the designed approach. In what follows, we ulteriorly expound the superiority with the aid of a detailed comparison analysis.

✓ *Compared with the TOPSIS method Zhang and Xu [64].* The PF-TOPSIS is presented by Zhang and Xu [64] based upon the classical TOPSIS method and distance measure, which ascertains the optimal selection through the closeness degree of alternative. It is obvious that the PF-TOPSIS method lacks the flexibility and the ability to ponder correlation of attributes. What's more, the weight information of criterions in PF-TOPSIS method is provided by expert committee according to their knowledge and experience, which has a certain subjectivity. Nevertheless, the presented Pythagorean fuzzy WASPAS method based upon the PFRWMSM and PFRWGMSM operator can efficiently conquer the mentioned shortcomings in the aspect of flexibility and attribute relevance. Further, the developed method can deal with the decision issues with unknown weight information and determine weights through the weight entropy approach. Consequently, the propounded Pythagorean fuzzy WASPAS approach is more workable and applicability for work out decision problems.

✓ *Compared with the PFWA operator presented by Zhang and Xu. [27].* As the most frequently used operator in fusing information, the advantage of PFWA operator that the computation procedure is simple and its defects is that it is invalid when the real situation needs to consider the relevance of dissimilar data and it fails to reflect the decision preference of experts. By comparison the developed method in this essay, the presented WASPAS method based upon the PFRWMSM and PFRWGMSM operator can not only ponder the correlation of different attributes but also adjust the risk preference of experts with the aid of changeable parameter. Moreover, the propounded approach can resolve the issues with complete unknown weight information which is more rational than other methods in which the weights of attributes are straightforward given from experts. Hence, the presented WASPAS method is more flexible and general for developing decision analysis.

✓ *Compared with the PFWGBM operator proposed by Liang et al. [41].* The PFWGBM operator possess several defects in aggregating fuzzy information: (1) it only takes into the correlation between any two attributes consideration and produces the redundancy phenomenon during the fusion process; (2) it has two adjustable parameters to control the preference of experts, although it can make the aggregation procedure more flexible, it is hard for experts to ascertain appropriate combination of parameter p and q and it shall increase the complexity of integration procedure; (3) it does not have the reducibility, namely, when the wights of PFWGBM operators are all equal, the PFWGBM operator fails to degenerate to the PFGBM operator. Nevertheless, the proposed WASPAS method based upon the PFRWMSM operator can ponder the relevance of multiple attributes in MADM process and it only possess an alterable parameter to be ascertained according to expert's preference. Further, the PFRWMSM and PFWGMSM operator have the property of reducibility, which can more comprehensive resolve the actual decision issues, especially the weights of aggregated data are equal. Accordingly, the developed method is more validity and powerful to settle real-life issues.

Table 6: A comparison with the previous approaches

Approaches	Score values						Sorting
	$\mathbb{S}(K_1)$	$\mathbb{S}(K_2)$	$\mathbb{S}(K_3)$	$\mathbb{S}(K_4)$	$\mathbb{S}(K_5)$	$\mathbb{S}(K_6)$	
The method based on PFWA operator [27]	0.5018	0.5688	0.5718	0.4745	0.6227	0.5308	$T_5 > T_3 > T_2 > T_6 > T_1 > T_4$
The method based on PFEWA operator [28]	0.3104	0.3128	0.3170	0.3115	0.3939	0.3502	$T_5 > T_6 > T_3 > T_2 > T_4 > T_1$
The method based on PFGBM operator [41]	0.8151	0.8121	0.8166	0.8320	0.8424	0.8332	$T_5 > T_6 > T_4 > T_3 > T_1 > T_2$
PF-TOPSIS method [64]	-0.2603	-0.2563	-0.2507	-0.1804	-0.1037	-0.1797	$T_5 > T_6 > T_4 > T_3 > T_2 > T_1$
The proposed Pythagorean fuzzy WASPAS method	0.2819	0.2834	0.2854	0.2826	0.3751	0.3279	$T_5 > T_6 > T_3 > T_2 > T_4 > T_1$

Secondly, by the above-mentioned analysis, we summarize the distinct features of the extant methods and the presented WASPAS approach. These characteristics are outlined in Table 7. In light of Table 7, the outstanding merits of the presented method are highlighted, which can further help decision experts to choose an appropriate approach to more effective cope with diverse practical issues.

Thirdly, for highlighting the dominant advantage of the propounded MSM operators with the previous MSM operators under dissimilar fuzzy environment, we implement a summarization about several MSM operators, which are exhibited in Table 8. From it, we can find that the existing weighted MSM operators do not have the feature of idempotency in information fusion. In addition, another defect of extant weighted MSM operators is that they fail to yield to the MSM operators when they possess an identical weight information, which implies the previous weighted MSM operators do not possess the reducibility. Nevertheless, the presented MSM operators in this research have the idempotency and reducibility, which illustrates that the developed operators can better fusion assessment information and further attain a reasonable decision result.

7. Conclusion

Teaching quality evaluation plays a vital role during the process of improving the quality of education. This research propounds a novel evaluation model based on the WASPSA method, PFRWMSM and PFRWGMSM operator

Table 7: Characteristic comparison with existing approaches

Approaches	**Capture correlation between two attributes**	**Capture correlation among multiple attributes**	**Flexibility of decision procedure**	**Solving the issues of unknown weight information**
The method based on PFWA operator [27]	×	×	×	×
The method based on PFWG operator [27]	×	×	×	×
The method based on PFEWA operator [28]	×	×	×	×
The method based on PFEWG operator [29]	×	×	×	×
The method based on PFDWA operator [32]	×	×	✓	×
The method based on PFDWG operator [32]	×	×	✓	×
The method based on geometric BM operator [41]	✓	×	✓	×
Pythagorean fuzzy TOPSIS method [64]	×	×	×	×
The proposed Pythagorean fuzzy WASPAS method	✓	✓	✓	✓

Table 8: Comparison with existing MSM operator based upon dissimilar fuzzy settings.

Settings	Integration operators	Whether has the feature of idempotency	Whether has the feature of reducibility
PFS	PFWMSM operator [44]	×	×
PFS	PFWGMSM operator [44]	×	×
PFS	PFIWMSM operator [46]	×	×
PFS	PFIWGMSM operator [46]	×	×
HPFS	HPFWMSM operator [47]	×	×
HPFS	HPFWGMSM operator [47]	×	×
IVPFS	IVPFWMSM operator [48]	×	×
IVPFS	IVPFWGMSM operator [48]	×	×
PFLS	PFLWPMSM operator [49]	×	×
PFLS	PFLWPGMSM operator [49]	×	×
LNS	LNWPGMSM operator [50]	×	×
DHFS	DHFWMSM operator [51]	×	×
DHFS	DHFWGMSM operator [51]	×	×
PFS	PFRWMSM operator in this essay	✓	✓
PFS	PFRWGMSM operator in this essay	✓	✓

and knowledge measure under Pythagorean fuzzy setting. Firstly, we present two innovative types of integration operators including PFRWMSM operator and PFRWGMSM operator and to investigate several worthwhile properties and especial instances of the developed operators. Then we present a knowledge measure and entropy measure of PFS for ascertaining the attribute weight information. Hereafter, we design a novel MADM methodology through combining the WASPAS method and the advanced operators to deal with the decision issues with unknown weight information. In addition, we build-up a comprehensive assessment model for teachers teaching quality evaluation and apply the example to testify applicability and effectiveness. Consequently, we execute the parameter analysing and comparison study to show the flexibility and significant merits of the developed approach, respectively. The apparent advantage of the developed approach is that (a) it can handle flexible ponder the correlation of any number of attributes during the process of information integration; (b) it can do with decision issues with complete unknown weight information and objective ascertain the weights of attributes; (c) it can deal with decision problems more accurately with the aid of combining the weighted sum and weighted product model method than a single information aggregation method. Furthermore, the essay also possesses several limitations that evaluator ignores subjective weight information and the preference information expressed by linguistic assessment words maybe more in line with experts' cognition and expression.

In future, we will focus on the following three aspects of research: (1) the designed approaches can be applied to settle other decision or assessment issues such as risk investment, big data assessment and project management; (2) the RWMSM and RWGMSM operator can be efficient generalized to other fuzzy contexts, for instance, Complex q-rung orthopair fuzzy 2-tuple linguistic setting [69], picture fuzzy set [70] and so forth; (3) the novel decision techniques will go on developing with the aid of combining the classic approaches and powerful integration operators.

Author Contributions

This paper is a result of the common work of the authors in all aspects. All authors read and agreed to the published version of the manuscript.

Data availability

The data to sustain the application of this investigation are included within the essay.

Conflicts of Interest

The authors declare that there are no conflicts of interest about the manuscript "An extended WASPAS approach for evaluating of higher education teaching quality based upon Pythagorean fuzzy Reducible Weighted Maclaurin symmetric mean".

Acknowledgments

This work was supported by the Education Department of Sichuan province under Grant 18ZB0562.

References

[1] Zadeh, L. A. (1965) Fuzzy sets. Information and Control, 8, 338-353
[2] Pei, Z., Liu, J., Hao, F., & Zhou, B. (2019). FLM-TOPSIS: The fuzzy linguistic multiset TOPSIS method and its application in linguistic decision making. Information Fusion, 45, 266-281.
[3] Kong, M., Pei, Z., Ren, F., & Hao, F. (2019). New Operations on Generalized Hesitant Fuzzy Linguistic Term Sets for Linguistic Decision Making. International Journal of Fuzzy Systems, 21(1), 243-262.
[4] Liu, Y., Liu, J., & Qin, Y. (2020). Pythagorean fuzzy linguistic Muirhead mean operators and their applications to multiattribute decision-making. International Journal of Intelligent Systems, 35(2), 300-332.
[5] Rong, Y., Liu, Y., & Pei, Z. (2020). Generalized Single-Valued Neutrosophic Power Aggregation Operators Based on Archimedean Copula and Co-Copula and Their Application to Multi-Attribute Decision-Making. IEEE Access, 8, 35496-35519.
[6] Liang, D., Wang, M., Xu, Z., & Liu, D. (2020). Risk appetite dual hesitant fuzzy three-way decisions with TODIM. Information Sciences,, 585-605.

[7] Krishankumar, R., Ravichandran, K. S., & Tyagi, S. K. (2020). Solving cloud vendor selection problem using intuitionistic fuzzy decision framework. Neural Computing and Applications, 32(2), 589-602.

[8] Yaqoob, N., Ali, G., Akram, M., & Chammam, W. (2020). Extensions of dombi aggregation operators for decision making under m-polar fuzzy information. Journal of Mathematics, https://doi.org/10.1155/2020/4739567.

[9] Ali, G., & Akram, M. (2020). Decision-making method based on fuzzy n-soft expert sets. Arabian Journal for Science and Engineering, 45(12), 10381-10400.

[10] Akram, M., Ali, G., Butt, M. A., & Alcantud, J. (2021). Novel MCGDM analysis under m-polar fuzzy soft expert sets. Neural Computing and Applications, https://doi.org/10.1007/s00521-021-05850-w.

[11] Atanassov, K. T. (1986) Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20(1), 87-96.

[12] Jemal, H., Kechaou, Z., & Ben Ayed, M. (2019). Multi-agent based intuitionistic fuzzy logic healthcare decision support system. Journal of Intelligent & Fuzzy Systems, 37(2), 2697-2712.

[13] Mishra, A. R. , & Rani, P. . (2018). Interval-valued intuitionistic fuzzy waspas method: application in reservoir flood control management policy. Group Decision and Negotiation, 27(6), 1047-1078.

[14] Zhang, Q. L., Liu, F., Fan, C. Q., & Xie, W. H. (2018). Fuzzy numbers intuitionistic fuzzy descriptor systems. Information Sciences, 469, 44-59.

[15] Liu, Y., Liu, J., & Qin, Y.(2019) Dynamic intuitionistic fuzzy multiattribute decision making based on evidential reasoning and MDIFWG operator. Journal of Intelligent & Fuzzy Systems, 36, 2161-2172.

[16] Yuan, J., & Luo, X. (2019). Approach for multi-attribute decision making based on novel intuitionistic fuzzy entropy and evidential reasoning. Computers & Industrial Engineering. https://doi.org/10.1016/j.cie.2019.06.031.

[17] Xu, Z., & Yager, R. R. (2006). Some geometric aggregation operators based on intuitionistic fuzzy sets. International journal of general systems, 35(4), 417-433.

[18] Xia, M. , Xu, Z. , & Zhu, B. . (2011). Generalized intuitionistic fuzzy bonferroni means. International Journal of Intelligent Systems, 27(1), 23-47.

[19] Wang, P. , & Liu, P. . (2019). Some maclaurin symmetric mean aggregation operators based on schweizer-sklar operations for intuitionistic fuzzy numbers and their application to decision making. Journal of Intelligent and Fuzzy Systems, 36(4), 3801-3824.

[20] Liu, P. , & Wang, Y. . (2019). Intuitionistic fuzzy interaction hamy mean operators and their application to multi-attribute group decision making. Group decision and negotiation, 28(1), 197-232.

[21] Liu, P., & Li, D. (2017). Some Muirhead Mean Operators for Intuitionistic Fuzzy Numbers and Their Applications to Group Decision Making. PLOS ONE, 12(1), e0168767. https://doi.org/10.1371/journal.pone.0168767.

[22] Rani, P. , Jain, D. , & Hooda, D. S. . (2018). Extension of intuitionistic fuzzy TODIM technique for multi-criteria decision making method based on shapley weighted divergence measure. Granular Computing, 4(3), 407-420.

[23] Mishra, A. R., Sisodia, G., Pardasani, K. R., & Sharma, K. (2020). Multi-criteria IT personnel selection on intuitionistic fuzzy information measures and ARAS methodology. Iranian Journal of Fuzzy Systems, https://doi.org/10.22111/ijfs.2020.5161.

[24] Mishra, A. R., Rani, P., Pandey, K., Mardani, A., Streimikis, J., Streimikiene, D., & Alrasheedi, M. (2020). Novel Multi-Criteria Intuitionistic Fuzzy SWARA-COPRAS Approach for Sustainability Evaluation of the Bioenergy Production Process. Sustainability, 12(10), 4155.

[25] Yager, R. R., & Abbasov, A. M. (2013). Pythagorean membership grades, complex numbers, and decision making. International Journal of Intelligent Systems, 28(5), 436-452.

[26] Yager, R. R. (2014). Pythagorean membership grades in multicriteria decision making. IEEE Transactions on Fuzzy Systems, 22(4), 958-965.

[27] Zhang, X. (2016). A Novel Approach Based on Similarity Measure for Pythagorean Fuzzy Multiple Criteria Group Decision Making. Journal of intelligent systems, 31(6), 593-611.

[28] Garg, H. (2016). A New Generalized Pythagorean Fuzzy Information Aggregation Using Einstein Operations and Its Application to Decision Making. Journal of intelligent systems, 31(9), 886-920.

[29] Garg, H. (2017). Generalized Pythagorean Fuzzy Geometric Aggregation Operators Using Einstein t-Norm and t-Conorm for Multicriteria Decision-Making Process. International Journal of Intelligent Systems, 32(6), 597-630.

[30] Xing, Y., Zhang, R., Wang, J., & Zhu, X. (2018). Some new Pythagorean fuzzy ChoquetCFrank aggregation operators for multi-attribute decision making. International Journal of Intelligent Systems, 33(11), 2189-2215.

[31] Garg, H. (2019). New logarithmic operational laws and their aggregation operators for Pythagorean fuzzy set and their applications. International Journal of Intelligent Systems, 34(1), 82-106.

[32] Khan, A., Ashraf, S., Abdullah, S., Qiyas, M., Luo, J., & Khan, S. U. (2019). Pythagorean Fuzzy Dombi Aggregation Operators and Their Application in Decision Support System. Symmetry, 11(3).

[33] Garg, H. (2019). Novel neutrality operationCbased Pythagorean fuzzy geometric aggregation operators for multiple attribute group decision analysis. International Journal of Intelligent Systems, 34(10), 2459-2489.

[34] Ma, Z. M., Xu, Z. (2016). Symmetric Pythagorean Fuzzy Weighted Geometric/Averaging Operators and Their Application in Multicriteria Decision-Making Problems. International Journal of intelligent systems, 31(12), 1198-1219.

[35] Wei, G., & Lu, M. (2018). Pythagorean fuzzy power aggregation operators in multiple attribute decision making. International Journal of Intelligent Systems, 33(1), 169-186.

[36] Peng, X., & Yang, Y. (2016). Pythagorean Fuzzy Choquet Integral Based MABAC Method for Multiple Attribute Group Decision Making. International Journal of intelligent systems, 31(10), 989-1020.

[37] Garg, H. (2018). Generalised Pythagorean fuzzy geometric interactive aggregation operators using Einstein operations and their application to decision making. Journal of Experimental and Theoretical Artificial Intelligence, 30(6), 763-794.

[38] Khan, M. S., Abdullah, S., & Ali, A. (2019). Multiattribute group decision-making based on Pythagorean fuzzy Einstein prioritized aggrega-

tion operators. International Journal of Intelligent Systems, 34(5), 1001-1033.
[39] Asif, M., Akram, M., & Ali, G. (2020). Pythagorean fuzzy matroids with application. Symmetry, 12(3).

[40] Bilal, M. A., & Shabir, M. (2021). Approximations of pythagorean fuzzy sets over dual universes by soft binary relations. Journal of Intelligent and Fuzzy Systems, https://doi.org/10.3233/JIFS-202725.
[41] Liang, D., Xu, Z., & Darko, A. P. (2017). Projection Model for Fusing the Information of Pythagorean Fuzzy Multicriteria Group Decision Making Based on Geometric Bonferroni Mean. International Journal of Intelligent Systems, 32(9), 966-987.
[42] Li, Z., Wei, G., & Lu, M. (2018). Pythagorean Fuzzy Hamy Mean Operators in Multiple Attribute Group Decision Making and Their Application to Supplier Selection. Symmetry, 10(10).
[43] Li, L., Zhang, R., Wang, J., Zhu, X., & Xing, Y. (2018). Pythagorean fuzzy power Muirhead mean operators with their application to multi-attribute decision making. Journal of Intelligent and Fuzzy Systems, 35(2), 2035-2050.
[44] Wei, G., & Lu, M. (2018). Pythagorean Fuzzy Maclaurin Symmetric Mean Operators in Multiple Attribute Decision Making. International Journal of Intelligent Systems, 33(5), 1043-1070.
[45] Qin, J. (2018). Generalized Pythagorean Fuzzy Maclaurin Symmetric Means and Its Application to Multiple Attribute SIR Group Decision Model. International Journal of Fuzzy Systems, 20(3), 943-957.
[46] Yang, W., & Pang, Y. (2018). New Pythagorean Fuzzy Interaction Maclaurin Symmetric Mean Operators and Their Application in Multiple Attribute Decision Making. IEEE Access,, 39241-39260.
[47] Garg, H. (2019). Hesitant Pythagorean fuzzy Maclaurin symmetric mean operators and its applications to multiattribute decision-making process. International Journal of Intelligent Systems, 34(4), 601-626.
[48] Wei, G., Garg, H., Gao, H., & Wei, C. (2018). Interval-Valued Pythagorean Fuzzy Maclaurin Symmetric Mean Operators in Multiple Attribute Decision Making. IEEE Access, https://doi.org/10.1109/ACCESS.2018.2877725.
[49] Teng, F., Liu, Z., & Liu, P. (2018). Some power Maclaurin symmetric mean aggregation operators based on Pythagorean fuzzy linguistic numbers and their application to group decision making. International Journal of Intelligent Systems, 33(9), 1949-1985.
[50] Liu, P., & You, X. (2020). Linguistic neutrosophic partitioned Maclaurin symmetric mean operators based on clustering algorithm and their application to multi-criteria group decision-making. Artificial Intelligence Review, 53(3), 2131-2170.
[51] Darko, A. P., & Liang, D. (2020). An extended COPRAS method for multiattribute group decision making based on dual hesitant fuzzy Maclaurin symmetric mean. International Journal of Intelligent Systems, 35(6), 1021-1068.
[52] Zavadskas, E. K., Turskis, Z., Antucheviciene, J., & Zakarevicius, A. (2012). Optimization of Weighted Aggregated Sum Product Assessment. Elektronika Ir Elektrotechnika, 122(6), 3-6.
[53] Zavadskas, E. K., Antucheviciene, J., Hajiagha, S. H., & Hashemi, S. S. (2014). Extension of weighted aggregated sum product assessment with interval-valued intuitionistic fuzzy numbers (WASPAS-IVIF). Applied Soft Computing, 24, 1013-1021.
[54] Zavadskas, E. K., Bausys, R., & Lazauskas, M. (2015). Sustainable Assessment of Alternative Sites for the Construction of a Waste Incineration Plant by Applying WASPAS Method with Single-Valued Neutrosophic Set. Sustainability, 7(12), 15923-15936.
[55] Ghorabaee, M. K., Zavadskas, E. K., Amiri, M., & Esmaeili, A. (2016). Multi-criteria evaluation of green suppliers using an extended WASPAS method with interval type-2 fuzzy sets. Journal of Cleaner Production,, 213-229.
[56] Peng, X., & Dai, J. (2017). Hesitant fuzzy soft decision making methods based on WASPAS, MABAC and COPRAS with combined weights. Journal of Intelligent and Fuzzy Systems, 33(2), 1313-1325.
[57] Ghorabaee, M. K., Amiri, M., Zavadskas, E. K., & Antuchevičienė, J. (2017). Assessment of third-party logistics providers using a CRITIC-WASPAS approach with interval type-2 fuzzy sets. Transport, 32(1), 66–78.
[58] Ilbahar, E., & Kahraman, C. (2018). Retail store performance measurement using a novel interval-valued Pythagorean fuzzy WASPAS method. Journal of Intelligent and Fuzzy Systems, 35(3), 3835-3846.
[59] Mohagheghi, V., & Mousavi, S. M. (2019). A new framework for high-technology project evaluation and project portfolio selection based on Pythagorean fuzzy WASPAS, MOORA and mathematical modeling. Iranian Journal of Fuzzy Systems, 16(6), 89-106.
[60] Davoudabadi, R., Mousavi, S. M., & Mohagheghi, V. (2020). A new last aggregation method of multi-attributes group decision making based on concepts of TODIM, WASPAS and TOPSIS under interval-valued intuitionistic fuzzy uncertainty. Knowledge and Information Systems, 62(4), 1371-1391.
[61] Maclaurin, C. (1729). A second letter to Martin Folkes, Esq.; concerning the roots of equations, with demonstration of other rules of algebra. Philos. Trans. Roy. Soc. London Ser. A, 36, 59-96.
[62] Qin, J., & Liu, X. (2015). Approaches to uncertain linguistic multiple attribute decision making based on dual Maclaurin symmetric mean. Journal of Intelligent & Fuzzy Systems, 29(1), 171-186.
[63] Shi, M., & Xiao, Q. (2019). Intuitionistic fuzzy reducible weighted Maclaurin symmetric means and their application in multiple-attribute decision making. Soft Computing, 23(20), 10029-10043.
[64] Zhang, X., & Xu, Z. (2014). Extension of TOPSIS to Multiple Criteria Decision Making with Pythagorean Fuzzy Sets. International Journal of intelligent systems, 29(12), 1061-1078.
[65] Peng, X., & Yang, Y. (2015). Some results for Pythagorean fuzzy sets. International Journal of Intelligent Systems, 30(11), 1133-1160.

[66] Szmidt E, Kacprzyk, J., Bujnowski P (2011). Measuring the amount of knowledge for Atanassovs intuitionistic fuzzy sets. In: International conference on fuzzy logic and applications. Springer, Berlin, pp 17-24.
[67] Szmidt E, Kacprzyk, J., Bujnowski, P. (2014). How to measure the amount of knowledge conveyed by Atanassovs intuitionistic fuzzy sets. Information Science 257:276-285
[68] Wang, X., & Triantaphyllou, E. (2008). Ranking irregularities when evaluating alternatives by using some ELECTRE methods. Omega-international Journal of Management Science, 36(1), 45-63.

[69] Rong, Y., Liu, Y., & Pei, Z. (2020). Complex q-rung orthopairfuzzy 2-tuple linguistic maclaurin symmetric mean operators and its application to emergency program selection. International Journal of Intelligent Systems, 35(11), 1749-1790.

[70] Rong, Y., Liu, Y., & Pei, Z. (2021). A novel multiple attribute decision-making approach for evaluation of emergency management schemes under picture fuzzy environment. International Journal of Machine Learning and Cybernetics, https://doi.org/10.1007/s13042-021-01280-1.