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Banzhaf-Choquet-Copula-Based Aggregation Operators for Managing q-Rung Orthopair Fuzzy Information

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Abstract Information fusion of fuzzy numbers has played a vital role in the decision support systems under the environment of q-rung orthopair fuzzy set (q-ROFS), which is an effective extension of intuitionisitic fuzzy set (IFS) and fuzzy set (FS). The goals of the present work are to build a family of new aggregation operators (AOs) under q-ROF environment and apply them to MADM problems. First, the extended Archimedean coupla (EAC) and extended Archimedean co-coupla (EACC) are proposed to handle q-ROF information, consequently, the operational law of q-ROFNs are defined based on EAC and EACC. In order to comprehensively consider the relationship between attributes, the q-rung orthopair fuzzy Banzhaf Choquet-Copula AOs ($BCCA^q$) and q-rung orthopair fuzzy Banzhaf Choquet-Copula geometric operators ($BCCG^q$) are introduced on the basis of the operation of q-rung orthopair fuzzy numbers (q-ROFNs); Consequently, some special cases of $BCCA^q$ /$BCCG^q$ operators are investigated when the generators of coupla take different functions which satisfy the condition of the generators of copulas. In addition, to determine the fuzzy measure (FM) of attribute sets objectively, the improved maximum deviation method and Banzhaf function model are built. Finally, the corresponding decision-making approaches are constructed based on the proposed AOs and proposed models. Proposed approaches can overcome effectively the FMs of attribute sets are given by decision makers subjectively and can also effectively address the some decision making problems (DMPs), in which the weights of attributes incompletely unknown (completely unknown), whilst the correlation are also existed among all attribute sets.

Keywords q-rung orthopair fuzzy set (q-ROFS) · Extended Archimedean Copula and Extended Archimedean Co-copula · Banzhaf function · fuzzy measure · Aggregation operators · multiattribute decision making

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1 Introduction

1.1 Background and Motivations

With the development of social economy, decision-making problems (DMPs) become more and more complex. In order to make the best decision, it is often necessary to evaluate the candidate schemes in many aspects. As an important branch of management science and engineering, multi-attribute decision making (MADM) theory and model have been widely used in various fields of social economy and management. The so-called MADM is to evaluate the candidate scheme according to a series of criteria, and then select the best option. In recent years, the theory of MADM has been widely valued by scholars at home and abroad, and has been widely used in investment scheme selection, disease diagnosis, supplier selection and people's daily life. With the increasing complexity of DMPs and decision making envi-

ronment, how to improve the reliability and scientifically of decision is a fundamental and important problem in modern decision-making science. Because decision-making is driven by information, that is, information driven management decision-making, how to effectively express, manage and integrate expert decision-making information is a hot topic in the field of MADM hotspot issues. When using MADM theory to solve DMPs, it is necessary to accurately express the evaluation information of decision-makers. Due to the complexity and fuzziness of DMPs and the limitations of human cognitive process, it is difficult for decision-makers to express their decision-making opinions with accurate values.

In the actual decision-making problems (DMPs), there are a lot of uncertain, imprecise and fuzzy information. The representation and management of these information has always been the core of the DMPs. IFS [2], an important and effective extension of fuzzy set (FS) [1], is considered to be an appropriate tool for processing this information. An IFS Φ in a finite universe of discourse Z is written as $\Phi = \{\langle z, (\alpha_\Phi(z), \beta_\Phi(z))\rangle | z \in Z\}$, in which $\alpha_\Phi : Z \to [0,1]$ and $\beta_\Phi : Z \to [0,1]$ with condition $0 \leq \alpha_\Phi(z) + \beta_\Phi(z) \leq 1$. We denote $(\alpha_\Phi(z), \beta_\Phi(z))$ as $(\alpha_\Phi, \beta_\Phi)$, which is called an intuitionisitic fuzzy number (IFN). Some decision approaches based on IFS are very suitable to deal with DMPs [3,4,5,6,7,8,9,10,11] only owing to $0 \leq \alpha_\Phi + \beta_\Phi \leq 1$. However, in real DMPs, although the decision makers to express their preferences by employing the pair $(\alpha_\Phi, \beta_\Phi)$, but it maybe not fufill $0 \leq \alpha_\Phi + \beta_\Phi \leq 1$ and beyond the upper bound 1. At this point, these decision approaches will be invalid under these circumstances. However, Yager [12] proposed the Pythagorean fuzzy set (PFS) theory to handle with these circumstances. PFS is an extension of IFS by slackening the condition $0 \leq \alpha_\Phi^2 + \beta_\Phi^2 \leq 1$. What' more, Yager[13] introduced a new and more general concept q-rung orthopair fuzzy set (q-ROFS), which is an extension of IFS and PFS by further slackening the condition $0 \leq \alpha_\Phi^q + \beta_\Phi^q \leq 1$, where $q \geq 1$ and $(\alpha_\Phi(z), \beta_\Phi(z))$ is known as an q-rung orthopair fuzzy number (q-ROFN) and expressed by $(\alpha_\Phi, \beta_\Phi)$ for convenience. We must also note that q-ROFSs express more extensive fuzzy information; Whilst, q-ROFSs are more maneuverable and more appropriate for dealing with uncertainties information. Therefore, some decision making approaches based on q-ROF information are arousing the attentions of researchers.

According to the existing MADM approaches with q-ROF information, **the motivations of the current work are highlighted below**:

• MADM approaches not only consider the relationship between adjacent attribute combinations, but also capture the interaction among elements globally. However, some existing MADM approaches under q-ROF environment only consider the relationship between adjacent attribute combinations, but not capture the interaction among elements globally.

• In some MADM problems in which the attributes (or criteria) are independent, this assumption is too strong to be fulfilled in some MADM problems; In fact, some attributes maybe dependent in real decision problems, how to reflect the interaction among the attributes in process of decision information fusion needs to be explored in depth.

• Some existing MADM approaches for dealing with q-ROF information are valid under the hypothesis that experts have given the fuzzy measure (or weight information) of attribute sets in advance, and can not be directly used to MADM problems with unknown or partially unknown fuzzy measures. Therefore, it is necessary to build algorithm for determining the Banzhaf values related to fuzzy measure under completely unknown or partially unknown fuzzy measures.

1.2 Literature review

Since the q-ROFSs appearances in 2017, this new tool for dealing with uncertainty has attracted many scholar's attentions. So far, the researches of q-ROFSs are main focused on the following aspects: (1) Aggregation operators (AOs): Liu [14] introduced the q-ROF AOs and applied them to MAGDM, on this basis of q-ROF AOs, Liu et al [15,16] introduced the BMOs and ABOs under q-ROF environment, respecitvely, whislt, corresponding MAGDM methods are also constructed on the basis of these AOs; Liu et al [17] further introduced the power Bonferroni operators under q-ROF environment and applied these AOs to MAGDM problems; Yang [18] introduced the partitioned BMOs under q-ROF environment and applied to MADM problems. Some authors introduced the Heronian operators [19] in q-ROFS and its extensions [20] are also done. Du [21] introduced weighted power means and applied it to MADM problems, whilst, Ju [22] defined the a new AOs under q-ROF environment named generalized power weighted AOs and corresponding MAGDM approaches have been also constructed; Xing [23] also introduced a new kind of weighted AOs under q-ROF environment named point weighted AOs; Wang [24, 25] defined Hamy mean operators and Muirhead means under q-ROF environment, respectively, and corresponding decision making approached are also designed; Wei [26] introduced Maclaurin symmetric mean operator in q-ROFSs and applied to eemerging technology commercialization; Peng [27] introduced a new operation so-called exponetial operation and corresponding AOs are also defined; Jana [28] introduced novel AOs based on Dombi operation in q-ROFSs; Joshi [29] defined confidence level q-ROF AOs and applied to MCDM problems. (2) Some preference relations: Li [30] introduced the preference relations in q-ROFSs, Zhang[31] introduced additive consistency of q-ROF preference relation based on a new method, whilst, Zhang[32] also intro-

duced Multiplicative consistency of q-ROF preference relation. (3) Information measures: Peng [33] introduced some information measure of q-ROFS, Wang [34] introduced the similarity measure of q-ROFS and Liu [35] also introduced the cosine similarity measure, Du [36,37] investigated the distance measures and correlation coefficient of q-ROFSs, respectively. (4)Some studied related to analysis: Shu [38], Gao [39,40] investigated the integrations, derivatives, differentials and differential calculus in q-ROFSs, respectively. (5) Some extensions of q-ROFSs: the extensions of q-ROFSs are focused on interval-valued q-ROFS [41], linguistic q-ROFSs [42,43,44,45], uncertain linguistic q-ROFSs [46,47,63] and 2-tuple linguistic set under q-ROF environment [48].

As a basis of AOs under any fuzzy environment, the operations play a vital role in information fusion. In the above mentioned AOs under q-ROF environment, the operational law of any two q-ROFNs are built on the t-norms (TNs) and t-conorms (TCs). Commonly, TNs are applied to integrate MD of fuzzy sets, while copulas are tools to deal with probability distributions. Besides, there exist also TNs which are copulas and vice versa. Thus, it's of reality significance to investigate the application of copulas to fuzzy sets. Copulas [49] can not only reveal the dependence among attributes, but also prevent information losing in the midst of aggregation. Just as TNs and TCs, Couplas and Co-couplas are flexible because DMs can select different types of Couplas and Co-couplas to define the operations under fuzzy environment, and the results obtained from these operations are closed; Whilst, Coupla functions are flexible to capture the correlations among attributes in DMPs. Based upon thse significant advantages, Couplas have been applied to some DMPs [50,51,52,53]. In traditional MADM approaches, it is often assumed that attributes are independent of each other, that is, attribute weights are additive. However, in some practical MADM problems, attributes are not completely independent, but interrelated and interactive. The fuzzy measure (FM) proposed by Sugeno [54] provides an effective method for dealing with the related MADM problems whose attributes are interrelated and interactive. Since FM's appearance, FM becomes more and more attractive and has been applied in many fields. The Choquet integral w. r. t. FM is a very effective approach to measure the expected utility of uncertain events and can be used to describe the interaction of input parameters, and also has achieved in triumph in MADM [55,56,57,58,59,60,61,62,63,64]. These Choquet-based AOs can reflect the importance of various input data or their locations, and can be able to consider the relationship between among data itself or their locations.

1.3 Contributions and Structure

Based on the above motivations and literature analyses, the goal of the present work is to synthesize EACs (EACCs), Banzhaf-based Choquet integral and q-ROFS to develop a novel MADM approach with q-ROF information in which weight information partially known or completely unknown, the proposed decision approach can capture the interaction among elements globally. The main contributions of the present work listed as below:

(1) the extended Archimedean Copulas (EACs) and extended Archimedean Co-copulas (EACCs) are proposed to handle q-ROF information by extending the classical Copulas (Co-copulas) to q-ROF environment, and the novel operations of q-ROFNs based on EACs (EACCs) which can reflect the relevance are defined;

(2) to construct corresponding Banzhaf-based Choquet AOs and geometric operators regarding the proposed operations to fuse some decision making information with q-ROF environment along with their properties;

(3) a novel decision approach for MADM problems with q-ROF information under fuzzy measure information partially known or completely unknown, the algorithm for determinations of fuzzy measures is designed prior to MADM decision approach. In these three main aspects, constructing novel operations is the core issue.

For the above goals, the structure of this work is arranged as follows. Some notions on Choquet integral are reviewed firstly in Section 2. A new version of Copulas and Co-copulas (named EACs and EACCs) which can tackle the q-ROF information are given in Section 3. In Section 4, we introduce the q-rung orthopair fuzzy Banzhaf Choquet-Coupla AOs ($BCCA^q$) based on EACs and EACCs together with their properties, some special $BCCA^q$s have also been developed. In Section 5, the q-rung orthopair fuzzy Banzhaf Choquet-Coupla geometric operator ($BCCG^q$) based upon EACs and EACCs is proposed together with their properties, some special $BCCG^q$s have also been developed. The algorithm of MADM with q-ROF information based on $BCCA^q$ / $BCCG^q$ is constructed in Section 6. Case analysis will be carried out and some advantages of the proposed MADM approach based on $BCCA^q$ / $BCCG^q$ operators are analysed in Section 7 and the conclusion will be obtained in Section 8.

2 Choquet Integral and Banzhaf Values

We briefly review Chouqet integral and Banzhaf values in this part.

Sugeno[54] defined fuzzy measure (FM), which can be used to define a weight on each combination of criteria in Choquet integral model. In this section, the concepts of FM and Choquet integral are first reviewed.

Definition 1 *[54] Let X be a nonempty set and $\mathscr{P}(X)$ be the power set of X. A function $s:\mathscr{P}(X)\to[0,1]$ is called a fuzzy measure on X, if φ satisfies conditions:*
(1) $s(\emptyset)=0, s(X)=1$;
(2) For all $E,F\in\mathscr{P}(X)$ and $E\subset F$, then $s(E)\le s(F)$.

FM can not only express the weight of attribute and attribute set, but also the relationship between them. Let X be the criteria index set in a MADM problem, for all $E,F\in\mathscr{P}(X)$, $E\bigcap F=\emptyset$: (1) If $s(E\cup F)=s(E)+s(F)$, then it means that there is no interaction between E and F, which is independent; (2) If $s(E\cup F)>s(E)+s(F)$, then it means that there is an complementary relation between E and F; (3) If $s(E\cup F)<s(E)+s(F)$, then it means that there is a redundancy relation between E and F.

In associated MADM problems, the role of the attribute set $E\in\mathscr{P}(X)$ in decision-making is not only determined by $s(E)$ itself, but also related to other attribute sets. If $s(E)=0$, then the attribute set E is not important. However, for the attribute set $F\in\mathscr{P}(X)$, if $s(E\bigcup F)-s(F)>0$, then it means that the attribute set E is important. As an effective tool to handle with the some problems with associated properties, generalized Banzhaf value can be used as a comprehensive consideration of the importance of attribute set E.

Definition 2 *[65] Let $\mathscr{P}(X)$ be a power set of X and s be the FM on X. Then, for all $E\in\mathscr{P}(X)$, the generalized Banzhaf value is given as follows:*

$$\mathbf{B}(E)=\sum_{F\subset X\setminus E}\frac{s(E\bigcup F)-s(F)}{2^{|X|-|E|}}. \tag{1}$$

where $X\setminus E$ is the difference set of X and E, $|X|$ and $|E|$ are the cardinality of X and E, respectively.

Generalized Banzhaf value not only reflects the contribution value of single attribute or attribute set to the whole alliance, but also reflects the average contribution of single attribute or attribute set to the whole alliance. When $E=\{x_i\}$, $\mathbf{B}(E)$ reduces to Banzhaf function[66]

$$\mathbf{B}(E)=\sum_{F\subset X\setminus\{x_i\}}\frac{s(\{x_i\}\bigcup F)-s(F)}{2^{|X|-1}}. \tag{2}$$

A well-known AO is Choquet integral operator w. r. t. FM can be used to model the importance of criteria set X and aggregate fuzzy interactive information. In what follows, if no specific, S_n is always the set of all permutations of $(1,\cdots,n)$.

Definition 3 *[54] Let X be a nonempty set, $f:X\to\mathbf{R}^+$ be a function and s be a FM on X. The discrete Choquet integral operator of f w.r.t. s is defined as*

$$CI_\varphi(f)=\sum_{k=1}^{n}[f(x_{(l)})-f(x_{(l-1)})]s(\Delta_{(l)}) \tag{3}$$

where $(l)\in S_n$ s. t. $f(x_{(l)})\le f(x_{(l+1)})$ for all $l\in\{1,\cdots,n\}$ and $\Delta_{(l)}=\{x_{(j)}|j=l,l+1,\cdots,n\}$, $\Delta_{(0)}=\emptyset$.

The equivalent form of the Choquet integral can be described as follows:

$$CI_\varphi(f)=\sum_{k=1}^{n}[s(\Delta_{(l)})-s(\Delta_{(l+1)})]f(x_{(l)}) \tag{4}$$

where $(l)\in S_n$ s. t. $f(x_{(l)})\le f(x_{(l+1)})$ for all $k\in\{1,\cdots,n\}$ and $\Delta_{(l)}=\{x_{(j)}|j=l,l+1,\cdots,n\}$, $\Delta_{(n+1)}=\emptyset$.

It can be seen from the above-definition that the Choquet integral operator has a salient features: that is, it consider the interaction (correlation) between attribute and attribute index sets in real DMPs.

3 Copulas and Co-copulas for q-ROF information

3.1 Extended Archimedean Copula (Co-copula)

In existing Copula and Co-copula have been applied to some AOs under some fuzzy environment, but, it fails to manage q-ROF information (when $q>2$). In this section, we will extend the Copulas and Co-copula for the sake of handing with q-ROF information. The classical Copula [50] is defined as follows:

Definition 4 *[50] A mapping $\mathbb{C}$ from $[0,1]^2$ to $[0,1]$ is called a Copula if, for all $c,d,c^{'},d^{'}\in[0,t]$,*
(C1) $\mathbb{C}(c,d)+\mathbb{C}\left(c^{'},d^{'}\right)\ge\mathbb{C}\left(c,d^{'}\right)+\mathbb{C}\left(c^{'},d\right)$;
(C2) $\mathbb{C}(c,0)=\mathbb{C}(0,c)=0$;
(C3) $\mathbb{C}(c,1)=\mathbb{C}(1,c)=c$.

Definition 5 *[50] Let ε be a continuous and strictly decreasing function from $[0,1]$ to $[0,+\infty)$ with $\varepsilon(1)=0$, $\psi:[0,+\infty)\to[0,t]$. If ε,ψ satisfy the following condition,*

$$\psi(c)=\begin{cases}\varepsilon^{-1}(c), & c\in[0,\varepsilon(0)];\\ 0, & c\in[\varepsilon(0),+\infty).\end{cases}$$

and

$$\mathscr{C}(c,d)=\psi(\varepsilon(c)+\varepsilon(d)). \tag{5}$$

The Copula $\mathscr{C}$ is called called Archimedean Copula (AC).

The generator ε of an AC if a function from $[0,1]$ to $\mathbf{R}^+$ and ε^{-1} is the function from $\mathbf{R}^+$ to $[0,1]$ with $\varepsilon(0)=+\infty$ and $\varepsilon(1)=0$. In line with Genest et al [68], the $\mathscr{C}$ can be expressed as

$$\mathscr{C}(c,d)=\varepsilon^{-1}(\varepsilon(c)+\varepsilon(d)). \tag{6}$$

Remark 1 *We extend Archimedean copula to handle some q-ROF information, so we call the Archimedean Copula as extended Archimedean copula (EAC). In what follows, all copulas are EACs if no specific.*

In order to handle some DMPs with q-ROF information, we introduce the following extended Archimedean co-copula (EACC):

Definition 6 *Let $\mathscr{C}$ be a EAC, then extended Archimedean co-copula (EACC) is defined as*

$$\mathscr{C}^*(c,d) = \sqrt[q]{1-\mathscr{C}^q(\sqrt[q]{1-c^q},\sqrt[q]{1-d^q})}. \tag{7}$$

Prior to introducing the novel operations of q-ROFNs, following conclusion is given firstly.

Theorem 1 *For any $c_1,c_2,d_1,d_2 \in [0,1]$, $c_i^q + d_i^q \le 1 (i = 1,2)$, then $0 \le \mathscr{C}^q(c_1,c_2) + (\mathscr{C}^*(d_1,d_2))^q \le 1$.*

Proof. Obviously, $0 \le \mathscr{C}^q(c_1,c_2) + (\mathscr{C}^*(d_1,d_2))^q$. So we just need $\mathscr{C}^q(c_1,c_2) + (\mathscr{C}^*(d_1,d_2))^q \le 1$.

According to the definitions of EAC and EACC, we get

$$\begin{aligned}
&\mathscr{C}^q(c_1,c_2) + (\mathscr{C}^*(d_1,d_2))^q \\
&= \mathscr{C}^q(c_1,c_2) + \left(\left(1-\mathscr{C}^q\left(\sqrt[q]{1-d_1^q},\sqrt[q]{1-d_2^q}\right)\right)^{\frac{1}{q}}\right)^q \\
&= \left(\left(\varepsilon^{-1}(\varepsilon(c_1)+\varepsilon(c_2))\right)\right)^q \\
&+1-\left(\varepsilon^{-1}\left(\varepsilon\left(\sqrt[q]{1-d_1^q}\right)+\varepsilon\left(\sqrt[q]{1-d_2^q}\right)\right)\right)^q
\end{aligned}$$

As ε is strictly decreasing and $c_i^q + d_i^q \le 1 (i = 1,2)$, it follows that

$$\varepsilon(c_1)+\varepsilon(c_2) \ge \varepsilon\left(\sqrt[q]{1-d_1^q}\right)+\varepsilon\left(\sqrt[q]{1-d_2^q}\right).$$

Therefore

$$\varepsilon^{-1}(\varepsilon(c_1)+\varepsilon(c_2)) \le \varepsilon^{-1}\left(\varepsilon\left(\sqrt[q]{1-d_1^q}\right)+\varepsilon\left(\sqrt[q]{1-d_2^q}\right)\right).$$

So, we get

$$\begin{aligned}
&\mathscr{C}^q(d_1,d_2) + (\mathscr{C}^*(d_1,d_2))^q = \left(\left(\varepsilon^{-1}(\varepsilon(c_1)+\varepsilon(c_2))\right)\right)^q \\
&+1-\left(\varepsilon^{-1}\left(\varepsilon\left(\sqrt[q]{1-d_1^q}\right)+\varepsilon\left(\sqrt[q]{1-d_2^q}\right)\right)\right)^q \\
&\le \left(\left(\varepsilon^{-1}(\varepsilon(c_1)+\varepsilon(c_2))\right)\right)^q + 1 - \left(\left(\varepsilon^{-1}(\varepsilon(c_1)+\varepsilon(c_2))\right)\right)^q \\
&= 1.
\end{aligned}$$

3.2 Some EACs and EACCs for q-ROFNs

In this part, according to the generators, we introduce some special extended Archimedean Copulas (EACs) and extended Archimedean Co-copulas (EACCs) for q-ROFNs.

(**Case I**) Let generator of the EAC be $\varepsilon(a) = (-lna^q)^\theta$, where $\varepsilon^{-1}(a) = \sqrt[q]{e^{-a^{\frac{1}{\theta}}}}$ and $\theta \ge 1$.

It follows from the generator of EAC $\mathscr{C}_G$ that

$$\mathscr{C}_G(a,b) = \sqrt[q]{e^{-\left((-lna^q)^\theta+(-lnb^q)^\theta\right)^{\frac{1}{\theta}}}}.$$

According to the definition of EAC, we get

$$\mathscr{C}_G^*(a,b) = \sqrt[q]{1-\left(e^{-\left((-ln(1-a^q))^\theta+(-ln(1-b^q))^\theta\right)^{\frac{1}{\theta}}}\right)}.$$

When $\theta = 1$, $\mathscr{C}_G$ and $\mathscr{C}_G^*$ reduce to q-rung TN $\mathscr{C}_G(a,b) = ab$ and TC $\mathscr{C}_G^*(a,b) = \sqrt[q]{a^q+b^q-a^qb^q}$.

(**Case II**) Let generator of the EAC $\mathscr{C}_C$ be $\varepsilon(a) = a^{-q\theta} - 1$, where $\varepsilon^{-1}(a) = (a+1)^{-\frac{1}{q\theta}}$, where $\theta > 0$. According to the generator of $\mathscr{C}_C$, we get

$$\mathscr{C}_C(a,b) = \left(a^{-q\theta}+b^{-q\theta}-1\right)^{-\frac{1}{q\theta}}$$

and

$$\mathscr{C}_C^*(a,b) = \sqrt[q]{1-\left((1-a^q)^{-\theta}+(1-b^q)^{-\theta}-1\right)^{-\frac{1}{\theta}}}.$$

When $\theta = 1$, $\mathscr{C}_C$ reduces to TN T_κ and $\mathscr{C}_C^*$ reduces to Hamacher TC S_H under the q-ROF environment

$$T_\kappa(a,b) = \frac{ab}{\sqrt[q]{1+(1-a^q)(1-b^q)}},$$

$$S_H(a,b) = \sqrt[q]{\frac{a^q+b^q-2a^qb^q}{1-a^qb^q}}.$$

(**Case III**) Let generator of the EAC $\mathscr{C}_F$ be

$$\varepsilon(a) = ln\left(\frac{e^{-\theta a^q}-1}{e^{-\theta}-1}\right),$$

where $\varepsilon^{-1}(a) = \sqrt[q]{(-\frac{1}{\theta})ln(e^a(e^{-\theta}-1)+1)}$ and $\theta \ne 0$.

According to the generator of the $\mathscr{C}_F$, we get

$$\mathscr{C}_F(a,b) = \sqrt[q]{(-\frac{1}{\theta})ln\left[\frac{(e^{-\theta a^q}-1)(e^{-\theta b^q}-1)}{e^{-\theta}-1}+1\right]}.$$

and

$$\mathscr{C}_F^*(a,b) = \sqrt[q]{1+\frac{1}{\theta}ln\left[\frac{(e^{-\theta(1-a^q)}-1)(e^{-\theta(1-b^q)}-1)}{e^{-\theta}-1}+1\right]}.$$

(**Case IV**) Let generator of the EAC $\mathscr{C}_A$ be

$$\varepsilon(a) = ln\left(\frac{1-\theta(1-a^q)}{a^q}\right),$$

where $\varepsilon^{-1}(a) = \sqrt[q]{\frac{1-\theta}{e^a-\theta}}$, $\theta \in [-1,1]$. According to the generator of $\mathscr{C}_A$, we get

$$\mathscr{C}_A(a,b) = \frac{ab}{\sqrt[q]{1-\theta(1-a^q)(1-b^q)}}.$$

$$\mathscr{C}_A^*(a,b) = \sqrt[q]{1-\frac{(1-a^q)(1-b^q)}{1-\theta a^qb^q}}.$$

When $\theta = -1$, $\mathscr{C}_C$ reduce to TN T_κ and $\mathscr{C}_C^*$ reduce to Einstein TC S_κ under the q-ROF environment

$$T_\kappa(a,b) = \frac{ab}{\sqrt[q]{1+(1-a^q)(1-b^q)}},$$

$$S_\kappa(a,b) = \sqrt[q]{\frac{a^q+b^q}{1+a^qb^q}}.$$

(**Case V**) Let generator of the EAC $\mathscr{C}_J$ be

$$\varepsilon(a) = -ln\left(1-(1-t^q)^\theta\right),$$

where $\varepsilon^{-1}(a) = \sqrt[q]{1-(1-e^{-t})^{-\frac{1}{\theta}}}$, where $\theta \geq 1$. According to the generator of $\mathscr{C}_J$, we get

$$\mathscr{C}_J(a,b) = \sqrt[q]{1-\left((1-a^q)^\theta + (1-b^q)^\theta - (1-a^q)^\theta(1-b^q)^\theta\right)^{\frac{1}{\theta}}}.$$

$$\mathscr{C}_J^*(a,b) = \sqrt[q]{(a^{q\theta}+b^{q\theta}-(ab)^{q\theta})^{\frac{1}{\theta}}}.$$

4 q-Rung Orthopair Fuzzy Banzhaf Choquet-Copula Aggregation Operators

4.1 $BCCA^q$ operator

In this part, we will give the q-Rung orthopair fuzzy Banzhaf Choquet-Copula aggregation operators ($BCCA^q$) based on the q-ROFNs and EACs introduced in Section 3. In what follows, denote

$$\mathbb{A} = \{\kappa_i = (u_i, v_i)\,|i = 1,\cdots,n\}$$
$$\mathbb{B} = \left\{\kappa_i^{'} = \left(u_i^{'}, v_i^{'}\right)|i = 1,\cdots,n\right\}$$

as two collections of q-ROFNs if no specific. Prior to introducing the $BCCA^q$ operator, the novel operations of q-ROFNs based on EACs (EACCs) are defined as follows:

Definition 7 *Suppose $A = (u_1, v_1)$ and $B = (u_2, v_2)$ are two q-ROFNs, the operations of A and B are defined as follows:*

$(L1) A \oplus_{\mathscr{C}} B = (\mathscr{C}^*(u_1,u_2), \mathscr{C}(v_1,v_2))$;

$(L2) A \otimes_{\mathscr{C}} B = (\mathscr{C}(u_1,u_2), \mathscr{C}^*(v_1,v_2))$;

$$(L3) kA = \left(\left(1-\left(\varepsilon^{-1}\left(k\varepsilon\left(\sqrt[q]{1-u^q}\right)\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}(k\varepsilon(v))\right),$$

$$(L4) A^k = \left(\varepsilon^{-1}(k\varepsilon(u)), \left(1-\left(\varepsilon^{-1}\left(k\varepsilon\left(\sqrt[q]{1-v^q}\right)\right)\right)^q\right)^{\frac{1}{q}}\right).$$

It is easy to verify that $\oplus_{\mathscr{C}}, \otimes_{\mathscr{C}}$ satisfy commutative law and associative law. According Theorem 1 and above definitions, it is easy obtain the following theorem.

Theorem 2 *Let A and B be two q-ROFNs, then for any $k > 0$, $A \oplus_{\mathscr{C}} B$, $A \otimes_{\mathscr{C}} B$, kA, A^k are all q-ROFNs.*

Definition 8 *Let $\kappa_l \in \mathbb{A}$. A $BCCA^q$ is a function from $\mathbb{A}^n$ to $\mathbb{A}$ and*

$$BCCA^q(\kappa_1,\kappa_2,\cdots,\kappa_n) = (\oplus_{\mathscr{C}})_{l=1}^n \left(\mathbf{B}\left(\Delta_{(l)}\right) - \mathbf{B}\left(\Delta_{(l+1)}\right)\right)\kappa_{(l)}. \tag{8}$$

where $(l) \in S_n$ such that $\kappa_{(1)} \leq \cdots \leq \kappa_{(n)}$ and $\Delta_{(l)} = \{(j)|j = l,\cdots,n\}$, $\Delta_{(n+1)} = \emptyset$, $\mathbf{B}(\Delta_{(l)})$ is the generalized Banzhaf value w. r. t. FM $s(\Delta_{(l)}), l = 1,\cdots,n$.

For the sake of convenience, in what follows, we denote $\mathbf{W}_l$ as $\mathbf{B}\left(\Delta_{(l)}\right) - \mathbf{B}\left(\Delta_{(l+1)}\right)$ if no specific.

Theorem 3 *Let $\kappa_l \in \mathbb{A}$. Then*

$$\begin{aligned} BCCA^q\ & (\kappa_1,\kappa_2,\cdots,\kappa_n) \\ &= \left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^n \mathbf{W}_l\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}, \right. \\ &\quad \left.\varepsilon^{-1}\left(\sum_{l=1}^n \left(\mathbf{W}_l\varepsilon(v_{(l)})\right)\right)\right), \end{aligned} \tag{9}$$

where where $(l) \in S_n$ such that $\kappa_{(1)} \leq \cdots \leq \kappa_{(n)}$ and $\Delta_{(l)} = \{(j)|j = l,\cdots,n\}$, $\Delta_{(n+1)} = \emptyset$, $\mathbf{B}(\Delta_{(l)})$ is the generalized Banzhaf value w. r. t. FM $s(\Delta_{(l)}), l = 1,\cdots,n$.

Proof Mathematical induction will be adopt to prove Theorem 3.

(1) It is obvious that Theorem 3 holds when $n = 1$.

(2) Assume that theorem 3 holds when $n = k$, that is,

$$\begin{aligned} BCCA^q\ & (\kappa_1,\kappa_2,\cdots,\kappa_k) \\ &= \left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^k \mathbf{W}_l\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}, \right. \\ &\quad \left.\varepsilon^{-1}\left(\sum_{l=1}^k \left(\mathbf{W}_l\varepsilon(v_{(l)})\right)\right)\right). \end{aligned}$$

Then, when $n = k+1$, we get

$$\begin{aligned} BCCA^q\ & (\kappa_1,\kappa_2,\cdots,\kappa_{k+1}) \\ &= (\oplus_{\mathscr{C}})_{l=1}^k \mathbf{W}_l\kappa_{(l)} \oplus_{\mathscr{C}} \mathbf{W}_{k+1}\kappa_{(k+1)} \\ &= \left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^k \mathbf{W}_l)\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}, \\ &\quad \varepsilon^{-1}\left(\sum_{l=1}^k \left(\mathbf{W}_l\varepsilon(v_{(l)})\right)\right)\Bigg) \\ &\quad + \left(\left(1-\left(\varepsilon^{-1}\left(\mathbf{W}_l\varepsilon\left(\sqrt[q]{1-u^q}\right)\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}(\mathbf{W}_l\varepsilon(v))\right) \\ &= \left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{k+1} \mathbf{W}_l\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}, \right. \\ &\quad \left.\varepsilon^{-1}\left(\sum_{l=1}^{k+1} \left(\mathbf{W}_l\varepsilon(v_{(l)})\right)\right)\right). \end{aligned}$$

Therefore, theorem 3 holds for all $n \in \mathbf{N}^+$.

Theorem 4 *Let $\kappa_l \in \mathbb{A}$, $(l) \in S_n$ such that $\kappa_{(1)} \leq \cdots \leq \kappa_{(n)}$.*

$$BCCA^q(\gamma\kappa_1,\gamma\kappa_2,\cdots,\gamma\kappa_n) = \gamma(BCCA^q(\kappa_1,\kappa_2,\cdots,\kappa_n)).$$

where $l = 1,\cdots,n$.

Proof By the Theorem 1, 2, we get

$$\gamma a_l = \left(\left(1-\left(\varepsilon^{-1}\left(\gamma\varepsilon\left(\sqrt[q]{1-u_l^q}\right)\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}(\gamma\varepsilon(v_l))\right),$$

therefore,

$$BCCA^q\ (\gamma\kappa_1,\gamma\kappa_2,\cdots,\gamma\kappa_n) = \left(\left(1-\left(\varepsilon^{-1}\left(\gamma\left(\sum_{l=1}^{n}\mathbf{W}_l\right)\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\gamma\left(\sum_{l=1}^{n}\mathbf{W}_l\right)\varepsilon\left(v_{(l)}\right)\right)\right).$$

Since

$$\begin{aligned}
&\gamma\ (BCCA^q(\kappa_1,\kappa_2,\cdots,\kappa_n))\\
=&\ \gamma\left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\sum_{l=1}^{n}(\mathbf{W}_l\varepsilon(v_{(l)}))\right)\right)\\
=&\ \left(\left(1-\left(\varepsilon^{-1}\left(\gamma\left(\sum_{l=1}^{n}\mathbf{W}_l\right)\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\gamma\left(\sum_{l=1}^{n}\mathbf{W}_l\right)\varepsilon\left(v_{(l)}\right)\right)\right)\\
=&\ BCCA^q\left(\gamma\kappa_{(1)},\gamma\kappa_{(2)},\cdots,\gamma\kappa_{(n)}\right).
\end{aligned}$$

Theorem 5 *Let $\kappa_l\in\mathbb{A}$ and $\kappa=(u,v)$ be a q-ROFN, $(l)\in S_n$ such that $\kappa_{(1)}\le\cdots\le\kappa_{(n)}$, then*

$$BCCA^q\left(\kappa_1\oplus_{\mathscr{C}}\kappa,\kappa_2\oplus_{\mathscr{C}}\kappa,\cdots,\kappa_n\oplus_{\mathscr{C}}\kappa\right) = BCCA^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\oplus_{\mathscr{C}}\kappa.$$

where $l=1,\cdots,n$.

Proof As

$$\kappa_l\oplus_{\mathscr{C}}\kappa = \left(\left(1-\left(\varepsilon^{-1}\left(\varepsilon\left(\sqrt[q]{1-u_l^q}\right)+\varepsilon\left(\sqrt[q]{1-u^q}\right)\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\varepsilon\left(v_l\right)+\varepsilon\left(v\right)\right)\right),$$

then

$$BCCA^q\left(\kappa_1\oplus_{\mathscr{C}}\kappa,\kappa_2\oplus_{\mathscr{C}}\kappa,\cdots,\kappa_n\oplus_{\mathscr{C}}\kappa\right) = \left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)+\varepsilon(\sqrt[q]{1-u^q})\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon\left(v_{(l)}\right)+\varepsilon\left(v\right)\right)\right)\right).$$

Since $\sum_{l=1}^{n}\mathbf{W}_l=1$, so we get

$$BCCA^q\left(\kappa_1\oplus_{\mathscr{C}}\kappa,\kappa_2\oplus_{\mathscr{C}}\kappa,\cdots,\kappa_n\oplus_{\mathscr{C}}\kappa\right) = \left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)+\varepsilon(\sqrt[q]{1-u^q})\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon\left(v_{(l)}\right)+\varepsilon\left(v\right)\right)\right)\right).$$

And

$$BCCA^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\oplus_{\mathscr{C}}\kappa = \left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)+\varepsilon(\sqrt[q]{1-u^q})\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon\left(v_{(l)}\right)+\varepsilon\left(v\right)\right)\right)\right).$$

Therefore,

$$BCCA^q\left(\kappa_1\oplus_{\mathscr{C}}\kappa,\kappa_2\oplus_{\mathscr{C}}\kappa,\cdots,\kappa_n\oplus_{\mathscr{C}}\kappa\right) = BCCA^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\oplus_{\mathscr{C}}\kappa.$$

It is easy to obtain the following theorems from Theorem 4 and Theorem 5.

Proposition 1 *Let $\kappa_i\in\mathbb{A}$ and $\kappa=(u,v)$ be a q-ROFN, $(l)\in S_n$ such that $\kappa_{(1)}\le\cdots\le\kappa_{(n)}$, then*

$$BCCA^q\left(\gamma\kappa_1\oplus_{\mathscr{C}}\kappa,\gamma\kappa_2\oplus_{\mathscr{C}}\kappa,\cdots,\gamma\kappa_n\oplus_{\mathscr{C}}\kappa\right) = \gamma BCCA^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\oplus_{\mathscr{C}}\kappa.$$

where $l=1,\cdots,n$.

Proposition 2 *Let $\kappa_i\in\mathbb{A}$ and $\kappa_i^{'}\in\mathbb{B}$, $(l)\in S_n$ such that $\kappa_{(1)}\le\cdots\le\kappa_{(n)}$ and $\kappa_{(1)}^{'}\le\cdots\le\kappa_{(n)}^{'}$, then*

$$BCCA^q\left(\kappa_1\oplus_{\mathscr{C}}\kappa_1^{'},\kappa_2\oplus_{\mathscr{C}}\kappa_2^{'},\cdots,\kappa_n\oplus_{\mathscr{C}}\kappa_n^{'}\right) = BCCA^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\oplus_{\mathscr{C}}BCCA^q\left(\kappa_1^{'},\kappa_2^{'},\cdots,\kappa_n^{'}\right).$$

where $l=1,\cdots,n$.

Proposition 3 *Let $\kappa_i\in\mathbb{A}$, $(l)\in S_n$ such that $\kappa_{(1)}\le\cdots\le\kappa_{(n)}$, If $\kappa_i=\kappa=(u,v)$, then*

$$BCCA^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\kappa=(u,v).$$

where $l=1,\cdots,n$.

Proof If $\kappa_i=\kappa=(u,v)$ for $l=1,\cdots,n$, then

$$\begin{aligned}
BCCA^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)
&=\left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\sum_{l=1}^{n}(\mathbf{W}_l\varepsilon(v_{(l)}))\right)\right).\\
&=\left(\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(\sqrt[q]{1-u^q})\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\sum_{l=1}^{n}(\mathbf{W}_l\varepsilon(v))\right)\right).\\
&=\left(\left(1-\left(\varepsilon^{-1}\left(\varepsilon(\sqrt[q]{1-u^q})\right)\right)^q\right)^{\frac{1}{q}}, \varepsilon^{-1}\left(\varepsilon(v)\right)\right).\\
&=(u,v)\\
&=\kappa.
\end{aligned}$$

Definition 9 *Let $\kappa_i \in \mathbb{A}$, $\kappa_i^{'} \in \mathbb{B}$, $\mathbb{A}$ and $\mathbb{B}$ are said to be comonotonic if*

$\kappa_{(1)} \leq \cdots \leq \kappa_{(n)}$ *if and only if* $\kappa_{(1)}^{'} \leq \kappa_{(2)}^{'} \leq \cdots \leq \kappa_{(n)}^{'}$,

where $(l) \in S_n$ *such that* $u_{(l)} \leq u_{(l)}^{'}, v_{(l)} \geq v_{(l)}^{'}$ *and* $l = 1, \cdots, n$.

Proposition 4 *Let $\kappa_i \in \mathbb{A}, \kappa_i^{'} \in \mathbb{B}$, $(l) \in S_n$ such that $\kappa_{(1)} \leq \cdots \leq \kappa_{(n)}$, if κ_i and $\kappa_i^{'}$ are comonotonic, then*

$$BCCA^q(\kappa_1, \kappa_2, \cdots, \kappa_n) \leq BCCA^q\left(\kappa_1^{'}, \kappa_2^{'}, \cdots, \kappa_n^{'}\right).$$

where $l = 1, \cdots, n$.

Proof On the one hand, since $u_i \leq u_i^{'}$, we get $\sqrt[q]{1-u_{(l)}^q} \geq \sqrt[q]{1-(u_{(l)}^{'})^q}$. And $\varepsilon(x)$ and $\varepsilon^{-1}(x)$ are monotonicity decreasing function, thus $\varepsilon\left(\sqrt[q]{1-u_{(l)}^q}\right) \leq \varepsilon\left(\sqrt[q]{1-(u_{(l)}^{'})^q}\right)$, furthermore,

$$\sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon\left(\sqrt[q]{1-u_{(l)}^q}\right)\right) \leq \sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon\left(\sqrt[q]{1-(u_{(l)}^{'})^q}\right)\right).$$

And so

$$\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon\left(\sqrt[q]{1-u_{(l)}^q}\right)\right)\right)$$
$$\geq \varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon\left(\sqrt[q]{1-(u_{(l)}^{'})^q}\right)\right)\right),$$

and

$$\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon\left(\sqrt[q]{1-u_{(l)}^q}\right)\right)\right)\right)^q\right)^{\frac{1}{q}}$$
$$\leq \left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon\left(\sqrt[q]{1-(u_{(l)}^{'})^q}\right)\right)\right)\right)^q\right)^{\frac{1}{q}}.$$

On the other hand, since $v_l \geq v_l^{'}$, we get $\varepsilon(v_l) \leq \varepsilon(v_l^{'})$, and so

$$\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(v_l)\right) \leq \left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(v_l^{'})\right).$$

Then

$$\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(v_l)\right) \leq \varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(v_l^{'})\right).$$

that is, $BCCA^q(\kappa_1, \kappa_2, \cdots, \kappa_n) \geq BCCA^q\left(\kappa_1^{'}, \kappa_2^{'}, \cdots, \kappa_n^{'}\right)$.

According to Proposition 3 and Proposition 4, it is easy to obtain the following property.

Proposition 5 *Let $\kappa_i \in \mathbb{A}$, $(l) \in S_n$ such that $\kappa_{(1)} \leq \cdots \leq \kappa_{(n)}$, then*

$$\begin{aligned}(min_i(u_i), max_i(v_l)) &\leq BCCA^q(\kappa_1, \kappa_2, \cdots, \kappa_n)\\ &\leq (max_i(u_i), min_i(v_l)),\end{aligned}$$

where $l = 1, \cdots, n$.

4.2 Some Special aggregation operators via different Copulas

In this part, some special cases of $BCCA^q$s based on different copulas and co-copulas will be given as follows.

(**Case 4.2-1**) If the generator of $\mathscr{C}$ is $\varepsilon(a) = (-lna^q)^\theta$, where $\varepsilon^{-1}(a) = \sqrt[q]{e^{-a^{\frac{1}{\theta}}}}$ and $\theta \geq 1$. The $BCCA^q$ reduces to the following:

$$\begin{aligned}BCCA_G^q\ &(\kappa_1, \kappa_2, \cdots, \kappa_n)\\ &= \left(\sqrt[q]{1-e^{-\left(\sum_{l=1}^{n}\mathbf{W}_l\left(-ln(1-u_{(l)}^q)^\theta\right)\right)^{\frac{1}{\theta}}}},\right.\\ &\qquad \left.\sqrt[q]{e^{-\left(\sum_{l=1}^{n}\mathbf{W}_l\left(-ln(v_{(l)}^q)^\theta\right)\right)^{\frac{1}{\theta}}}}\right).\end{aligned}$$

(**Case 4.2-2**) When the generator of $\mathscr{C}$ is $\varepsilon(a) = a^{-q\theta} - 1$, where $\varepsilon^{-1}(a) = (a+1)^{-\frac{1}{q\theta}}$, where $\theta > 0$. The $BCCA^q$ reduces to the following:

$$BCCA_C^q(\kappa_1, \kappa_2, \cdots, \kappa_n) = (u, v)$$

where

$$u = \sqrt[q]{1-\left(\sum_{l=1}^{n}\mathbf{W}_l\left((1-u_{(l)}^q)^{-\theta}-1\right)+1\right)^{-\frac{1}{\theta}}},$$
$$v = \sqrt[q]{\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\left(v_{(l)}^{-q\theta}-1\right)+1\right)\right)^{-\frac{1}{\theta}}}$$

(**Case 4.2-3**) If the generator of $\mathscr{C}$ is

$$\varepsilon(a) = ln\left(\frac{e^{-\theta a^q}-1}{e^{-\theta}-1}\right),$$

where $\varepsilon^{-1}(a) = \sqrt[q]{(-\frac{1}{\theta})ln(e^a(e^{-\theta}-1)+1)}$ and $\theta \neq 0$. It follows that $BCCA^q$ reduces to

$$BCCA_F^q(\kappa_1, \kappa_2, \cdots, \kappa_n) = (u, v).$$

where

$$u = \sqrt[q]{1+\frac{1}{\theta}ln\left(1+(e^{-\theta}-1)\left(\prod_{l=1}^{n}\left(\frac{e^{-\theta(1-u_{(l)}^q)}}{e^{-\theta}-1}\right)^{\mathbf{W}_l}\right)\right)},$$
$$v = \sqrt[q]{-\frac{1}{\theta}ln\left(1+(e^{-\theta}-1)\prod_{l=1}^{n}\left(\frac{e^{-\theta v_{(l)}^q}}{e^{-\theta}-1}\right)^{\mathbf{W}_l}\right)}.$$

(**Case 4.2-4**) If the generator of $\mathscr{C}$ is

$$\varepsilon(a) = ln\left(\frac{1-\theta(1-a^q)}{a^q}\right),$$

where $\varepsilon^{-1}(a)=\sqrt[q]{\frac{1-\theta}{e^{a}-\theta}}$, $\theta\in[-1,1]$. The $BCCA^q$ reduces to the following:

$$BCCA_A^q(\kappa_1,\kappa_2,\cdots,\kappa_n)=(u,v),$$

where

$$u=\sqrt[q]{\frac{\prod_{l=1}^{n}\left(1-\theta u_{(l)}^q\right)^{\mathbf{W}_l}-\prod_{l=1}^{n}\left(1-u_{(l)}^q\right)^{\mathbf{W}_l}}{\prod_{l=1}^{n}\left(1-\theta u_{(l)}^q\right)^{\mathbf{W}_l}-\theta\prod_{l=1}^{n}\left(1-u_{(l)}^q\right)^{\mathbf{W}_l}}},$$

$$v=\sqrt[q]{\frac{(1-\theta)\prod_{l=1}^{n}\left(v_{(l)}^q\right)^{\mathbf{W}_l}}{\prod_{l=1}^{n}\left(1-\theta(1-v_{(l)}^q)\right)^{\mathbf{W}_l}-\theta\prod_{l=1}^{n}\left(v_{(l)}^q\right)^{\mathbf{W}_l}}}$$

(**Case 4.2-5**) When the generator of $\mathscr{C}_J$ is

$$\varepsilon(a)=-ln\left(1-(1-t^q)^{\theta}\right),$$

where $\varepsilon^{-1}(a)=\sqrt[q]{1-(1-e^{-t})^{-\frac{1}{\theta}}}$, where $\theta\geq 1$. It follows that $BCCA^q$ reduces to

$$BCCA_J^q(\kappa_1,\kappa_2,\cdots,\kappa_n)=(u,v).$$

where,

$$u=\sqrt[q]{\left(1-\prod_{l=1}^{n}\left(1-u_{(l)}^{q\theta}\right)^{\mathbf{W}_l}\right)^{\frac{1}{\theta}}},$$

$$v=\sqrt[q]{1-\left(1-\prod_{l=1}^{n}\left(1-(1-v_{(l)}^q)^{\theta}\right)^{\mathbf{W}_l}\right)^{\frac{1}{\theta}}}.$$

5 q-Rung orthopair Fuzzy Banzhaf Choquet-Copula geometric operators

5.1 $BCCG^q$ operators

In this section, a novel aggregation operator named q-Rung orthopair fuzzy Banzhaf Choquet-Copula geometric operator ($BCCG^q$) will be given along with their properties.

Definition 10 *Let $\kappa_l\in\mathbb{A}$. A $BCCG^q$ is a function from $\mathbb{A}^n$ to $\mathbb{A}$ and*

$$BCCG^q(\kappa_1,\kappa_2,\cdots,\kappa_n)=(\otimes_{\mathscr{C}})_{i=1}^{n}\left(\left(\kappa_{(l)}\right)^{\left(\mathbf{B}(\Delta_{(l)})-\mathbf{B}(\Delta_{(l+1)})\right)}\right). \quad (10)$$

where $(l)\in S_n$ such that $\kappa_{(1)}\leq\cdots\leq\kappa_{(n)}$ and $\Delta_{(l)}=\{(j)|j=l,\cdots,n\}$, $\Delta_{(n+1)}=\emptyset$, $\mathbf{B}(\Delta_{(l)})$ is a generalized Banzhaf values w.r.t. FM $s(\Delta_{(l)}),l=1,\cdots,n$.

For the sake of convenience, in what follows, we denote $\mathbf{W}_l$ as $\mathbf{B}\left(\Delta_{(l)}\right)-\mathbf{B}\left(\Delta_{(l+1)}\right)$ if no specific.

Theorem 6 *Let $\kappa_l\in\mathbb{A}$, $(l)\in S_n$ such that $\kappa_{(1)}\leq\cdots\leq\kappa_{(n)}$. Then*

$$\begin{aligned}BCCG^q&(\kappa_1,\kappa_2,\cdots,\kappa_n)\\&=\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon(u_{(l)})\right)\right),\right.\\&\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}\right). \quad (11)\end{aligned}$$

where $\Delta_{(l)}=\{(j)|j=l,\cdots,n\}$, $\Delta_{(n+1)}=\emptyset$.

Proof Theorem 6 will be proved by mathematical induction method.

(1) It is obvious that Theorem 6 holds when $n=1$.

(2) Suppose theorem 6 holds when $n=k$, that is,

$$\begin{aligned}BCCG^q&(\kappa_1,\kappa_2,\cdots,\kappa_k)\\&=\left(\varepsilon^{-1}\left(\sum_{l=1}^{k}\left(\mathbf{W}_l\varepsilon(u_{(l)})\right)\right),\right.\\&\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{k}\mathbf{W}_l\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}\right).\end{aligned}$$

Then, when $n=k+1$, we get

$$\begin{aligned}&BCCG^q(\kappa_1,\kappa_2,\cdots,\kappa_{k+1})\\&=(\oplus_{\mathscr{C}})_{l=1}^{k}\left(\left(\kappa_{(l)}\right)^{\mathbf{W}_l}\right)\oplus_{\mathscr{C}}\left(\left(\kappa_{(k+1)}\right)^{\mathbf{W}_{k+1}}\right)\\&=\left(\varepsilon^{-1}\left(\sum_{l=1}^{k}\left(\mathbf{W}_l\varepsilon(u_{(l)})\right)\right),\right.\\&\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{k}\mathbf{W}_l\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}\right)\\&+\left(\varepsilon^{-1}\left(\mathbf{W}_l\varepsilon(u)\right),\left(1-\left(\varepsilon^{-1}\left(\mathbf{W}_l\varepsilon\left(\sqrt[q]{1-v^q}\right)\right)\right)^q\right)^{\frac{1}{q}}\right)\\&=\left(\varepsilon^{-1}\left(\sum_{l=1}^{k+1}\left(\mathbf{W}_l\varepsilon(u_{(l)})\right)\right),\right.\\&\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{k+1}\mathbf{W}_l\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}\right).\end{aligned}$$

Therefore, theorem 6 holds for all $n\in\mathbf{N}^+$.

Proposition 6 *Suppose $\kappa_l\in\mathbb{A}$ and $\gamma>0$, $(l)\in S_n$ such that $\kappa_{(1)}\leq\cdots\leq\kappa_{(n)}$.*

$$BCCG^q\left(\kappa_1^{\gamma},\kappa_2^{\gamma},\cdots,\kappa_n^{\gamma}\right)=\left(BCCA^q(\kappa_1,\kappa_2,\cdots,\kappa_n)\right)^{\gamma}. \quad (12)$$

where $l=1,\cdots,n$.

Proof By the Theorem 1, 2, we get

$$\kappa_{(l)}^{\gamma}=\left(\varepsilon^{-1}\left(\gamma\varepsilon\left(u_{(l)}\right)\right),\left(1-\left(\varepsilon^{-1}\left(\gamma\varepsilon\left(\sqrt[q]{1-v_{(l)}^q}\right)\right)\right)^q\right)^{\frac{1}{q}}\right),$$

therefore,

$$BCCG^q\left(\kappa_1^\gamma,\kappa_2^\gamma,\cdots,\kappa_n^\gamma\right)$$
$$=\left(\varepsilon^{-1}\left(\gamma\left(\sum_{l=1}^{n}\mathbf{W}_l\right)\varepsilon\left(u_{(l)}\right)\right),\right.$$
$$\left.\left(1-\left(\varepsilon^{-1}\left(\gamma\left(\sum_{l=1}^{n}\mathbf{W}_l\right)\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}\right).$$

And so,

$$\left(BCCG^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\right)^\gamma$$
$$=\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon(v_{(l)})\right)\right),\right.$$
$$\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(\sqrt[q]{1-u_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}\right)^\gamma$$
$$=\left(\varepsilon^{-1}\left(\gamma\left(\sum_{l=1}^{n}\mathbf{W}_l\right)\varepsilon\left(u_{(l)}\right)\right),\right.$$
$$\left.\left(1-\left(\varepsilon^{-1}\left(\gamma\left(\sum_{l=1}^{n}\mathbf{W}_l\right)\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}\right)$$
$$=BCCG^q\left(\kappa_1^\gamma,\kappa_2^\gamma,\cdots,\kappa_n^\gamma\right).$$

Proposition 7 *Let $\kappa_l\in\mathbb{A}$ and $\kappa=(u,v)$ be a q-ROFN, $(l)\in S_n$ such that $\kappa_{(1)}\le\cdots\le\kappa_{(n)}$, then*

$$BCCG^q\left(\kappa_1\otimes_{\mathscr{C}}\kappa,\kappa_2\otimes_{\mathscr{C}}\kappa,\cdots,\kappa_n\otimes_{\mathscr{C}}\kappa\right)$$
$$=BCCG^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\otimes_{\mathscr{C}}\kappa.$$

where $l=1,2,\cdots,n$.

Proof As

$$\kappa_l\otimes_{\mathscr{C}}\kappa=\left(\varepsilon^{-1}\left(\varepsilon\left(u_l\right)+\varepsilon\left(u\right)\right),\right.$$
$$\left.\left(1-\left(\varepsilon^{-1}\left(\varepsilon\left(\sqrt[q]{1-v_l^q}\right)+\varepsilon\left(\sqrt[q]{1-v^q}\right)\right)\right)^q\right)^{\frac{1}{q}}\right),$$

then

$$BCCG^q\left(\kappa_1\otimes_{\mathscr{C}}\kappa,\kappa_2\otimes_{\mathscr{C}}\kappa,\cdots,\kappa_n\otimes_{\mathscr{C}}\kappa\right)$$
$$=\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon\left(u_{(l)}\right)+\varepsilon\left(v\right)\right)\right),\right.$$
$$\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)+\varepsilon(\sqrt[q]{1-v^q})\right)\right)^q\right)^{\frac{1}{q}}\right).$$

Since $\sum_{l=1}^{n}\mathbf{W}_l=1$, so we get

$$BCCG^q\left(\kappa_1\otimes_{\mathscr{C}}\kappa,\kappa_2\otimes_{\mathscr{C}}\kappa,\cdots,\kappa_n\otimes_{\mathscr{C}}\kappa\right)$$
$$=\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon\left(u_{(l)}\right)\right)+\varepsilon\left(v\right)\right),\right.$$
$$\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)+\varepsilon(\sqrt[q]{1-v^q})\right)\right)^q\right)^{\frac{1}{q}}\right).$$

And

$$BCCG^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\otimes_{\mathscr{C}}\kappa$$
$$=\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon\left(u_{(l)}\right)\right)+\varepsilon\left(u\right)\right),\right.$$
$$\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\left(\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)+\varepsilon(\sqrt[q]{1-v^q})\right)\right)^q\right)^{\frac{1}{q}}\right).$$

Therefore,

$$BCCG^q\left(\kappa_1\otimes_{\mathscr{C}}\kappa,\kappa_2\otimes_{\mathscr{C}}\kappa,\cdots,\kappa_n\otimes_{\mathscr{C}}\kappa\right)$$
$$=BCCG^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\otimes_{\mathscr{C}}\kappa.$$

According to Proposition 6 and Proposition 7, it is easy to obtain the following property.

Proposition 8 *Suppose $\kappa_l\in\mathbb{A}$ and $\kappa=(u,v)$ is a q-ROFN, $(l)\in S_n$ such that $\kappa_{(1)}\le\cdots\le\kappa_{(n)}$, then*

$$BCCG^q\left(\kappa_1^\gamma\otimes_{\mathscr{C}}\kappa,\kappa_2^\gamma\otimes_{\mathscr{C}}\kappa,\cdots,\kappa_n^\gamma\otimes_{\mathscr{C}}\kappa\right)$$
$$=\left(BCCG^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\right)^\gamma\otimes_{\mathscr{C}}\kappa.$$

where $l=1,\cdots,n$.

Proposition 9 *Let $\kappa_l\in\mathbb{A},\kappa_i^{'}\in\mathbb{B}$, $(l)\in S_n$ such that $\kappa_{(1)}\le\cdots\le\kappa_{(n)}$ and $\kappa_1^{'}\le\kappa_2^{'}\le\cdots\le\kappa_n^{'}$, then*

$$BCCG^q\left(\kappa_1\otimes_{\mathscr{C}}\kappa_1^{'},\kappa_2\otimes_{\mathscr{C}}\kappa_2^{'},\cdots,\kappa_n\otimes_{\mathscr{C}}\kappa_n^{'}\right)$$
$$=\left(BCCG^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)\right)\otimes_{\mathscr{C}}\left(BCCG^q\left(\kappa_1^{'},\kappa_2^{'},\cdots,\kappa_n^{'}\right)\right).$$

where $l=1,\cdots,n$.

Proposition 10 *Let $\kappa_l\in\mathbb{A}$, $(l)\in S_n$ such that $\kappa_{(1)}\le\cdots\le\kappa_{(n)}$. If $\kappa_l=\kappa=(u,v)$ for $l=1,\cdots,n$, then*

$$BCCG^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)=\kappa=(u,v).$$

where $l=1,\cdots,n$.

Proof If $\kappa_l=\kappa=(u,v)$ for $l=1,\cdots,n$, then

$$BCCG^q\left(\kappa_1,\kappa_2,\cdots,\kappa_n\right)$$
$$=\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon(u_{(l)})\right)\right),\right.$$
$$\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(\sqrt[q]{1-v_{(l)}^q})\right)\right)^q\right)^{\frac{1}{q}}\right)$$
$$=\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\varepsilon(u)\right)\right),\right.$$
$$\left.\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\mathbf{W}_l\varepsilon(\sqrt[q]{1-v^q})\right)\right)^q\right)^{\frac{1}{q}}\right)$$
$$=\left(\varepsilon^{-1}\left(\varepsilon(u)\right),\sqrt[q]{1-\left(\varepsilon^{-1}\left(\varepsilon(\sqrt[q]{1-v^q})\right)\right)^q}\right)$$
$$=(u,v)=\kappa.$$

Proposition 11 *Let $\kappa_l \in \mathbb{A}, \kappa_l^{'} \in \mathbb{B}$. If κ_l and $\kappa_l^{'}$ are comonotonic, then*

$$BCCG^q(\kappa_1, \kappa_2, \cdots, \kappa_n) \leq BCCG^q\left(\kappa_1^{'}, \kappa_2^{'}, \cdots, \kappa_n^{'}\right).$$

Proof On the one hand, since $u_{(l)} \leq u_{(l)}^{'}$, we get $\varepsilon(u_{(l)}) \geq \varepsilon(u_{(l)}^{'})$, and so

$$\left(\sum_{l=1}^{n} \mathbf{W}_l \varepsilon(u_{(l)})\right) \geq \left(\sum_{l=1}^{n} \mathbf{W}_l \varepsilon(v_{(l)}^{'})\right).$$

Then

$$\varepsilon^{-1}\left(\sum_{l=1}^{n} \mathbf{W}_l \varepsilon(u_{(l)})\right) \leq \varepsilon^{-1}\left(\sum_{l=1}^{n} \mathbf{W}_l \varepsilon(u_{(l)}^{'})\right).$$

On the other hand, since $v_{(l)} \geq v_{(l)}^{'}$, we get $\sqrt[q]{1-v_{(l)}^q} \leq \sqrt[q]{1-(v_{(l)}^{'})^q}$. And $\varepsilon(x)$ and $\varepsilon^{-1}(x)$ are monotonicity decreasing function, thus $\varepsilon\left(\sqrt[q]{1-v_{(l)}^q}\right) \geq \varepsilon\left(\sqrt[q]{1-(v_{(l)}^{'})^q}\right)$, furthermore,

$$\sum_{l=1}^{n}\left(\mathbf{W}_l \varepsilon\left(\sqrt[q]{1-v_{(l)}^q}\right)\right) \geq \sum_{l=1}^{n}\left(\mathbf{W}_l \varepsilon\left(\sqrt[q]{1-(v_{(l)}^{'})^q}\right)\right).$$

And so

$$\begin{aligned}&\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l \varepsilon\left(\sqrt[q]{1-v_{(l)}^q}\right)\right)\right)\\ &\leq \varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l \varepsilon\left(\sqrt[q]{1-(v_{(l)}^{'})^q}\right)\right)\right).\end{aligned}$$

and

$$\begin{aligned}&\left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l \varepsilon\left(\sqrt[q]{1-v_{(l)}^q}\right)\right)\right)\right)^q\right)^{\frac{1}{q}}\\ &\geq \left(1-\left(\varepsilon^{-1}\left(\sum_{l=1}^{n}\left(\mathbf{W}_l \varepsilon\left(\sqrt[q]{1-(v_{(l)}^{'})^q}\right)\right)\right)\right)^q\right)^{\frac{1}{q}}.\end{aligned}$$

that is, $BCCG^q(\kappa_1, \kappa_2, \cdots, \kappa_n) \geq BCCG^q\left(\kappa_1^{'}, \kappa_2^{'}, \cdots, \kappa_n^{'}\right)$.

The following property can be obtained easily from Proposition 10 and Proposition 11.

Proposition 12 *Let $\kappa_i \in \mathbb{A}$, $(l) \in S_n$ such that $\kappa_{(1)} \leq \cdots \leq \kappa_{(n)}$. Then*

$$\begin{aligned}(min_i(u_i), max_i(v_l)) &\leq BCCG^q(\kappa_1, \kappa_2, \cdots, \kappa_n)\\ &\leq (max_i(u_i), min_i(v_l)).\end{aligned}$$

where $l = 1, \cdots, n$.

5.2 Family of $BCCG^q$

In this part, some special cases of $BCCG^q$s based on different EACs and EACCs will be given as follows:

(**Case 5.2-1**) When generator of the EAC $\mathscr{C}_G$ is $\varepsilon(a) = (-lna^q)^\theta$, where $\varepsilon^{-1}(a) = \sqrt[q]{e^{-a^{\frac{1}{\theta}}}}$ and $\theta \geq 1$. The $BCCG^q$ reduces to the following:

$$\begin{aligned}&BCCG_G^q\ (\kappa_1, \kappa_2, \cdots, \kappa_n)\\ &= \left(\sqrt[q]{e^{-\left(\sum_{l=1}^{n}\mathbf{W}_l\left(-ln(u_{(l)}^q)^\theta\right)\right)^{\frac{1}{\theta}}}},\right.\\ &\qquad \left.\sqrt[q]{1-e^{-\left(\sum_{l=1}^{n}\mathbf{W}_l\left(-ln(1-v_{(l)}^q)^\theta\right)\right)^{\frac{1}{\theta}}}}\right).\end{aligned}$$

(**Case 5.2-2**) If the generator of the EAC is

$$\varepsilon(a) = a^{-q\theta} - 1,$$

where $\varepsilon^{-1}(a) = (a+1)^{-\frac{1}{q\theta}}$, where $\theta > 0$. The $BCCG^q$ reduces to the following:

$$BCCG_C^q(\kappa_1, \kappa_2, \cdots, \kappa_n) = (u, v).$$

where

$$u = \sqrt[q]{\left(\sum_{l=1}^{n}\left(\mathbf{W}_l\left(u_{(l)}^{-q\theta}-1\right)+1\right)\right)^{-\frac{1}{\theta}}},$$

$$v = \sqrt[q]{1-\left(\sum_{l=1}^{n}\mathbf{W}_l\left((1-v_{(l)}^q)^{-\theta}-1\right)+1\right)^{-\frac{1}{\theta}}}.$$

(**Case 5.2-3**) When the generator of the EAC $\mathscr{C}_F$ is

$$\varepsilon(a) = ln\left(\frac{e^{-\theta a^q}-1}{e^{-\theta}-1}\right),$$

where $\varepsilon^{-1}(a) = \sqrt[q]{(-\frac{1}{\theta})ln(e^a(e^{-\theta}-1)+1)}$ and $\theta \neq 0$. It follows that $BCCG^q$ reduces to

$$BCCG_F^q(\kappa_1, \kappa_2, \cdots, \kappa_n) = (u, v).$$

where

$$u = \sqrt[q]{-\frac{1}{\theta}ln\left(1+(e^{-\theta}-1)\prod_{l=1}^{n}\left(\frac{e^{-\theta u_{(l)}^q}-1}{e^{-\theta}-1}\right)^{\mathbf{W}_l}\right)},$$

$$v = \sqrt[q]{1+\frac{1}{\theta}ln\left(1+(e^{-\theta}-1)\prod_{l=1}^{n}\left(\frac{e^{-\theta(1-v_{(l)}^q)}-1}{e^{-\theta}-1}\right)^{\mathbf{W}_l}+1\right)}.$$

(**Case 5.2-4**) When generator of the EAC $\mathscr{C}_A$ be $\varepsilon(a) = ln\left(\frac{1-\theta(1-a^q)}{a^q}\right)$, where $\varepsilon^{-1}(a) = \sqrt[q]{\frac{1-\theta}{e^a-\theta}}$, $\theta \in [-1, 1]$. The $BCCG^q$ reduces to the following:

$$BCCG_A^q(\kappa_1, \kappa_2, \cdots, \kappa_n) = (u, v)$$

where,

$$u=\sqrt[q]{\frac{(1-\theta)\prod_{l=1}^{n}\left(u_{(l)}^{q}\right)^{\mathbf{w}_l}}{\prod_{l=1}^{n}\left(1-\theta(1-v_{(l)}^{q})\right)^{\mathbf{w}_l}-\theta\prod_{l=1}^{n}\left(u_{(l)}^{q}\right)^{\mathbf{w}_l}}},$$

$$v=\sqrt[q]{\frac{\prod_{l=1}^{n}\left(1-\theta u_{(l)}^{q}\right)^{\mathbf{w}_l}-\prod_{l=1}^{n}\left(1-v_{(l)}^{q}\right)^{\mathbf{w}_l}}{\prod_{l=1}^{n}\left(1-\theta u_{(l)}^{q}\right)^{\mathbf{w}_l}-\theta\prod_{l=1}^{n}\left(1-v_{(l)}^{q}\right)^{\mathbf{w}_l}}}.$$

(**Case 5.2-5**) When generator of the EAC $\mathscr{C}_J$ is

$$\varepsilon(a)=-ln\left(1-(1-t^{q})^{\theta}\right),$$

where $\varepsilon^{-1}(a)=\sqrt[q]{1-(1-e^{-t})^{-\frac{1}{\theta}}}$ with $\theta\geq 1$. The $BCCG^{q}$ reduces to the following:

$$BCCG_{J}^{q}(\kappa_1,\kappa_2,\cdots,\kappa_n)=(u,v).$$

where

$$u=\sqrt[q]{1-\left(1-\prod_{l=1}^{n}\left(1-(1-u_{(l)}^{q})^{\theta}\right)^{\mathbf{w}_l}\right)^{\frac{1}{\theta}}},$$

$$v=\sqrt[q]{\left(1-\prod_{l=1}^{n}\left(1-v_{(l)}^{q\theta}\right)^{\mathbf{w}_l}\right)^{\frac{1}{\theta}}}.$$

6 Approach for MADM with q-ROF information

In this part, we will give an approach for MADM with q-ROF information. Generally speaking, MADM problem is devoted to seek the best one from finite alternatives. MADM approach with the intent of managing the MADM problems with q-ROF information, especially the MADM problems. In general, a MADM problem consists of the following parts: (1) Alternatives set: $\Xi=\{\Psi_1,\cdots,\Psi_m\}$; (2) Attributes (Criteria) set: $A=\{\kappa_1,\cdots,\kappa_n\}$ with weight vector $(w_1,\cdots,w_n)$ with $\sum_{i=1}^{n}w_i=1$. Decision maker evaluate alternative $\Psi_i\in\Xi\,(i=1,2,\cdots,m)$ by applying q-ROFNs, the attribute values of alternative Ψ_i under the attribute κ_j can be presented by a q-ROFN $\gamma_{ij}=(u_{ij},v_{ij})$, where $u_{ij}^{q}+v_{ij}^{q}\leq 1$. The decision matrix is listed as follows

$$R=(\gamma_{ij})_{m\times n}=\begin{pmatrix}(u_{11},v_{11}) & (u_{12},v_{12}) & \cdots & (u_{1n},v_{1n})\\ (u_{21},v_{21}) & (u_{22},v_{22}) & \cdots & (u_{2n},v_{2n})\\ \cdots & \cdots & \cdots & \cdots\\ (u_{m1},v_{m1}) & (u_{m2},v_{m2}) & \cdots & (u_{mn},v_{mn})\end{pmatrix}$$

6.1 Determination of Attributes (sets)' FM

Maximizing Deviations Method has been widely used in MADM model to ordain the weight of attribute. In this section, we use this model to ordain the attributes weight in MADM problems with q-ROF information. Firstly, we give the distance measure between two q-ROFNs:

Let $A=(u_1,v_1)$ and $B=(u_2,v_2)$ be two q-ROFNs. Then Euclidean distance (d(A, B)) of A and B is given as:

$$d(A,B)=\frac{1}{2}\left[\left|(u_1)^{q}-(u_2)^{q}\right|^{2}+\left|(v_1)^{q}-(v_2)^{q}\right|^{2}\right]^{\frac{1}{2}},\qquad(13)$$

Based on the above defined distance measures of two q-ROFNs, we establish the maximizing deviation model to determine the FM of attribute or attribute set. There are two cases need to be discussed.

(1) **Fuzzy measure incompletely unknown**

When the FM of attribute is incompletely unknown, the FMs of each attribute and power sets can obtained by the following non-linear programme model

$$maxD(\mathbf{B})=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}d\left(\gamma_{ij},\gamma_{kj}\right)\mathbf{B}(\{\kappa_j\}).$$

$$s.t\begin{cases}s(\{\kappa_j\})\in R_{\kappa_j},j=1,\cdots,n;\\ s(\emptyset)=0,s(C)=1;\\ s(E)\leq s(F),\forall E,F\subseteq A,E\subseteq F.\end{cases}\qquad(14)$$

where $d\left(\gamma_{ij},\gamma_{kj}\right)$ is the distance of q-ROFNs γ_{ij} and γ_{kj}, $s(\{\kappa_j\})$ is FM of the attribute κ_j, $\mathbf{B}(\{\kappa_j\})$ is the Banzhaf function of attribute κ_j and R_{κ_j} is the range of FM of attribute κ_j.

(2) **Fuzzy measure completely unknown**

When the FM of attribute is completely unknown, we can determine the FM of each attribute and power set by the following algorithm:

Step 1. Building non-linear programme model to obtain optimal weight vector $(w_1,w_2,\cdots,w_n)$ of attributes;

$$maxD(\mathbf{B})=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}d\left(\gamma_{ij},\gamma_{kj}\right)w_j.$$

$$s.t\begin{cases}\sum_{j=1}^{n}(w_j)^{2}=1,\\ w_j\geq 0,j=1,\cdots,n.\end{cases}\qquad(15)$$

where $d\left(\gamma_{ij},\gamma_{kj}\right)$ is the distance of q-ROFNs γ_{ij} and γ_{kj}.

Step 2. Let $s(a_i)=s_i$. We need to determine the value of λ by the following equation:

$$\lambda+1=\prod_{l=1}^{n}(1+\lambda s_i).\qquad(16)$$

Step 3. Calculate the FM of all attributes and attribute sets by the following equation:

$$s(E\bigcup F)=s(E)+s(F)+\lambda s(E)s(F).\qquad(17)$$

Step 4. Calculate the (generalized) Banzhaf values of attributes and attribute sets by the Eq. (1).

6.2 An algorithm for MADM with q-ROF information

In this subsection, an algorithm for MADM problems with q-ROF information will be devised and is stated as follows:

Algorithm for MADM with q-ROF information

Step 1. A modified decision matrix $\hat{R} = (\hat{\gamma}_{ij})_{m\times n}$ is obtained by in the light of Eq.(17):

$$\hat{\gamma}_{ij} = (\hat{u}_{ij}, \hat{v}_{ij}) = \begin{cases} \gamma_{ij} = (u_{ij}, v_{ij}), \kappa_j \text{ is benefit type} \\ (\gamma_{ij})^c = (v_{ij}, u_{ij}), \kappa_j \text{ is cost type.} \end{cases} \quad (18)$$

Step 2. Reorder all q-ROFNs in term of the following comparison rule [15] of q-ROFNs.

Let $a \in \mathbb{A}$, score function $Sc(a) = u^q - v^q$ and accuracy function $Ac(a) = u^q + v^q$. For any $a, b \in \mathbb{A}$,

If $Sc(a) > Sc(b)$, then $a > b$;
If $Sc(a) < Sc(b)$, then $a < b$;
If $Sc(a) = Sc(b)$ and $Ac(a) > Ac(b)$, then $a > b$;
If $Sc(a) = Sc(b)$ and $Ac(a) < Ac(b)$, then $a < b$;
If $Sc(a) = Sc(b)$ and $Ac(a) = Ac(b)$, then $a = b$.

Step 3. Determining the FM of attribute set A by the following Steps:

Step 3-1: Building non-linear programme model to obtain optimal weight vector of attribute by model (14) or model (15);

Step 3-2: Determining the value of λ by the Eq. (16);

Step 3-3: Determining the FM of all attributes and attribute sets by Eq. (1).

Step 4. Determining the generalized Banzhaf values of attribute set A.

Step 5. Aggregating all $\hat{\gamma}_{ij} = (u_{ij}, v_{ij})(j = 1, \cdots, n)$ in the ith row of the decision matrix $(\hat{\gamma}_{ij})_{m\times n}$ into collective values $\gamma_i = (u_i, v_l)(i = 1, \cdots, m)$ of the alternatives A_i by proposed $BCCA^q$ operators or $BCCG^q$ operators.

Step 6. Calculating the score values of alternatives by score function and accuracy function.

Step 7. Sorting the alternatives according to the principle of comparison, and gaining the best alternative.

According to algorithm of MADM with q-ROF mentioned above, the flowchart of decision process can be designed as Fig.1.

7 Case Study

In this section, we will give a MADM problem in which three cases will be considered. Consequently, the influence of parameters change on the ranking order will be conducted; Finally, the comparison analysis will be carried out with existing MADM methods and merits of the proposed approaches also summarized in this section.

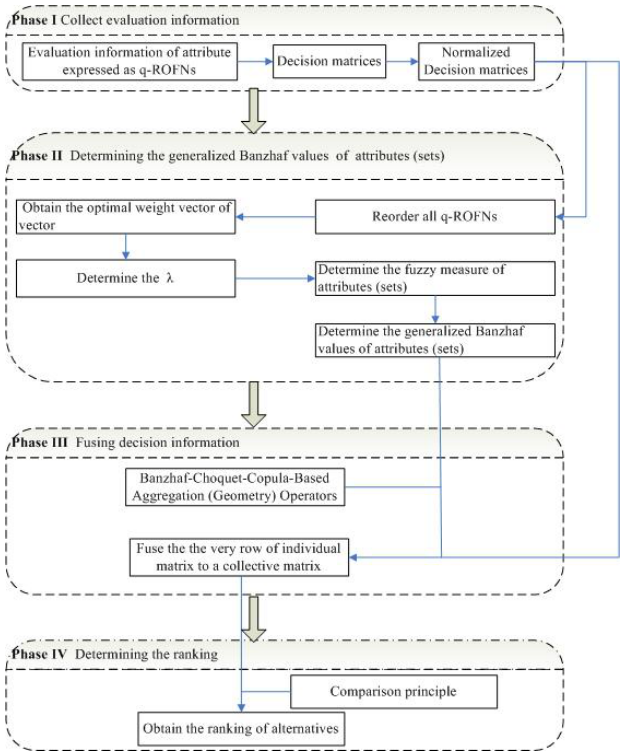

Fig. 1 The flowchart of MADM approach with q-ROF information

7.1 Illustrative Examples

Renewable energy is an important components of the energy system. It has the characteristics of wide distribution of resources, great development potential, small environmental impact and sustainable utilization. It is conducive to the harmonious development of human and nature. At present, the development and utilization of renewable energy has proved to be a significant measure to ensure energy security, enhance environmental protection and tackle climate change. With the development of economy and society, the global energy demand continues to grow, and the energy resources and environmental problems become increasingly salient. Accelerating the development and utilization of renewable energy has become the only way for the world to deal with the increasingly serious energy and environmental problems. In 2019, China's consumption of renewable energy increased by more than any other country, to 25% percent of the total. Solar energy generation consumption accounted for 50% of China's total renewable energy growth, followed by wind power, which accounted for about 40%. There are five clean energy generation projects in Sichuan Province of China, they are: Ψ_1 : Solar energy; Ψ_2: Natural gas; Ψ_3 Hydroelectricity; Ψ_4: Non-fossil energy; Ψ_5: Wind energy. Among these projects, four attributes should be considered: κ_1: technical capability; κ_2: Environment; κ_3: Policy envi-

ronment, that is, consistency of the project with current national policies; κ_4: Economic. The original decision matrix with q-ROF information is listed in Table 1.

Table 1 Original q-ROF decision matrix

Alternatives	κ_1	κ_2	κ_3	κ_4
Ψ_1	(0.5, 0.4)	(0.5, 0.3)	(0.2, 0.6)	(0.4, 0.4)
Ψ_2	(0.7, 0.3)	(0.7, 0.2)	(0.6, 0.2)	(0.6, 0.2)
Ψ_3	(0.5, 0.4)	(0.6, 0.4)	(0.6, 0.2)	(0.5, 0.3)
Ψ_4	(0.8, 0.2)	(0.7, 0.2)	(0.4, 0.2)	(0.5, 0.2)
Ψ_5	(0.4, 0.3)	(0.4, 0.2)	(0.4, 0.5)	(0.4, 0.6)

Example 1 *In above decision making problem, if the attributes are mutual independence, and the weight vector of attributes is* $(0.3, 0.2, 0.4, 0.1)$*, we can solve the DMP by the following decision process.*

Decision-making Process for Example 1

Step 1. Because all attributes are benefit-type, so the normalized decision matrix is the same with original decision making matrix;

Step 2. Calculating the score values of q-ROFNs and reorder the decision making matrix is listed in Table 2.

Table 2 Modified q-ROF decision matrix

Alternatives	$\kappa_{(1)}$	$\kappa_{(2)}$	$\kappa_{(3)}$	$\kappa_{(4)}$
Ψ_1	(0.2, 0.6)	(0.4, 0.4)	(0.5, 0.4)	(0.5, 0.3)
Ψ_2	(0.6, 0.3)	(0.6, 0.2)	(0.7, 0.3)	(0.7, 0.2)
Ψ_3	(0.5, 0.4)	(0.5, 0.3)	(0.6, 0.4)	(0.6, 0.2)
Ψ_4	(0.4, 0.2)	(0.5, 0.2)	(0.7, 0.2)	(0.8, 0.2)
Ψ_5	(0.4, 0.6)	(0.4, 0.5)	(0.4, 0.3)	(0.4, 0.2)

Step 3. Determining the FM of each attribute (or attribute sets). As all attributes are mutual independence and the weight vector of the attribute is $(0.3, 0.2, 0.4, 0.1)$, therefore

$$
\begin{aligned}
&s(\kappa_1) = 0.3, s(\kappa_2) = 0.2, s(\kappa_3) = 0.4, s(\kappa_4) = 0.1,\\
&s(\kappa_1, \kappa_2) = 0.5, s(\kappa_1, \kappa_3) = 0.7, s(\kappa_1, \kappa_4) = 0.4,\\
&s(\kappa_2, \kappa_3) = 0.6, s(\kappa_2, \kappa_4) = 0.3, s(\kappa_3, \kappa_4) = 0.5,\\
&s(\kappa_1, \kappa_2, \kappa_3) = 0.9, s(\kappa_1, \kappa_2, \kappa_4) = 0.6, s(\kappa_1, \kappa_3, \kappa_4) = 0.8,\\
&s(\kappa_2, \kappa_3, \kappa_4) = 0.7, s(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = 1.
\end{aligned}
$$

Step 4. Calculating the generalized Banzhaf value of attribute sets, the results are

$$
\begin{aligned}
&\mathbf{B}(\kappa_1) = 0.3, \mathbf{B}(\kappa_2) = 0.2, \mathbf{B}(\kappa_3) = 0.4, \mathbf{B}(\kappa_4) = 0.1,\\
&\mathbf{B}(\kappa_1, \kappa_2) = 0.5, \mathbf{B}(\kappa_1, \kappa_3) = 0.7, \mathbf{B}(\kappa_1, \kappa_4) = 0.4,\\
&\mathbf{B}(\kappa_2, \kappa_3) = 0.6, \mathbf{B}(\kappa_2, \kappa_4) = 0.3, \mathbf{B}(\kappa_3, \kappa_4) = 0.5,\\
&\mathbf{B}(\kappa_1, \kappa_2, \kappa_3) = 0.9, \mathbf{B}(\kappa_1, \kappa_2, \kappa_4) = 0.6,\\
&\mathbf{B}(\kappa_1, \kappa_3, \kappa_4) = 0.8, \mathbf{B}(\kappa_2, \kappa_3, \kappa_4) = 0.7,\\
&\mathbf{B}(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = 1.
\end{aligned}
$$

Step 5. Aggregating individual decision matrix $\tilde{R} = (\hat{\gamma}_{ij})_{5\times 4}$ into a collective decision matrix $\hat{R} = (\hat{\gamma}_i)_{5\times 1}$ by Eq.(21)(We choose Gumbel type aggregation operator when $\theta = 1$ and $q = 2$) and listed in Table 3.

Table 3 Overall LPF Decision matrix

Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5
(0.4504, 0.3046)	(0.7297, 0.1421)	(0.7386, 0.0732)	(0.7286, 0.0648)	(0.519, 0.2544)

Step 6. Calculating the score function:

$Sc(\Psi_1) = 0.1100$, $Sc(\Psi_2) = 0.5171$, $Sc(\Psi_3) = 0.5401$, $Sc(\Psi_4) = 0.5267$, $Sc(\Psi_5) = 0.2047$.

Step 7. Sorting the alternatives on the basis of the comparison principle, the rank of alternatives is $\Psi_3 > \Psi_4 > \Psi_2 > \Psi_5 > \Psi_1$, and so Ψ_3 is the best alternative, which means that Hydroelectricity is the best choice in Sichuan Province.

Example 2 *In Example 1, if the attributes are not mutual independence, but there are certain relationships among the attributes, and the FM of attributes is*

$$
\begin{aligned}
&s(\{\kappa_1\}) \in [0.3, 0.4], s(\{\kappa_2\}) \in [0.15, 0.25],\\
&s(\{\kappa_3\}) \in [0.2, 0.25], s(\{\kappa_4\}) \in [0.25, 0.5]
\end{aligned}
$$

we can solve the decision making problem by the following decision process.

Decision-making Process for Example 2

Step 1-Step 2 are the same as Example 1.

Step 3. Construct the nonlinear program model to determine the fuzzy measure.

Step 3.1. Calculate the distance of two q-ROFNs and construct the following nonlinear program model according

to Eq. (35)

$$\begin{aligned} maxD(\mathbf{B}) = & -0.505(s(\kappa_1)-s(\kappa_2,\kappa_3,\kappa_4)) \\ & -0.655(s(\kappa_2)-s(\kappa_1,\kappa_3,\kappa_4)) \\ & -0.665(s(\kappa_3)-s(\kappa_1,\kappa_2,\kappa_4)) \\ & -0.695(s(\kappa_4)-s(\kappa_1,\kappa_2,\kappa_3)) \\ & -0.1(s(\kappa_3,\kappa_4)-s(\kappa_1,\kappa_2)) \\ & -0.09(s(\kappa_2,\kappa_4)-s(\kappa_1,\kappa_3)) \\ & -0.06(s(\kappa_2,\kappa_3)-s(\kappa_1,\kappa_4))+1.26. \end{aligned}$$

$$s.t \begin{cases} s(\{\kappa_1\}) \in [0.3,0.4], s(\{\kappa_2\}) \in [0.15,0.25], \\ s(\{\kappa_3\}) \in [0.2,0.25], s(\{\kappa_4\}) \in [0.25,0.5]; \\ s(\emptyset)=0, s(\kappa_1,\kappa_2,\kappa_3,\kappa_4)=1; \\ s(E) \le s(F), \forall E,F \subseteq A, E \subseteq F. \end{cases}$$

Step 3.2. Solve above nonlinear programme model and get the FM:

$$\begin{aligned} & s(\kappa_1)=0.3, s(\kappa_2)=0.15, s(\kappa_3)=0.2, s(\kappa_4)=0.25, \\ & s(\kappa_1,\kappa_2)=s(\kappa_1,\kappa_3)=s(\kappa_1,\kappa_4)=1, s(\kappa_2,\kappa_3)=0.2, \\ & s(\kappa_2,\kappa_4)=s(\kappa_3,\kappa_4)=0.25, s(\kappa_1,\kappa_2,\kappa_3) \\ & =s(\kappa_1,\kappa_2,\kappa_4)=s(\kappa_1,\kappa_3,\kappa_4)=s(\kappa_2,\kappa_3,\kappa_4) \\ & =s(\kappa_1,\kappa_2,\kappa_3,\kappa_4)=1. \end{aligned}$$

Step 4. Determining the generalized Banzhaf values and list as follows:

$$\begin{aligned} & \mathbf{B}(\kappa_1)=0.625, \mathbf{B}(\kappa_2)=0.2, \mathbf{B}(\kappa_3)=0.2125, \\ & \mathbf{B}(\kappa_4)=0.2375, \mathbf{B}(\kappa_1,\kappa_2)=0.825, \\ & \mathbf{B}(\kappa_1,\kappa_3)=0.8375, \mathbf{B}(\kappa_1,\kappa_4)=0.8625, \\ & \mathbf{B}(\kappa_2,\kappa_3)=0.4125, \mathbf{B}(\kappa_2,\kappa_4)=0.4375, \\ & \mathbf{B}(\kappa_3,\kappa_4)=0.45, \mathbf{B}(\kappa_1,\kappa_2,\kappa_3)=0.875, \\ & \mathbf{B}(\kappa_1,\kappa_2,\kappa_4)=0.9, \mathbf{B}(\kappa_1,\kappa_3,\kappa_4)=0.925, \\ & \mathbf{B}(\kappa_2,\kappa_3,\kappa_4)=0.85, \mathbf{B}(\kappa_1,\kappa_2,\kappa_3,\kappa_4)=1. \end{aligned}$$

Step 5. Aggregating individual decision matrix $\tilde{R}=(\hat{\gamma}_{ij})_{5\times 4}$ into a collective decision matrix $\hat{R}=(\hat{\gamma}_i)_{5\times 1}$ by Eq.(21)(We choose Gumbel type aggregation operator when $\theta=1$ and $q=2$) and listed in Table 4.

Table 4 Overall q-ROF Decision matrix

Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5
(0.4807, 0.2385)	(0.7746, 0.0873)	(0.6143, 0.1983)	(0.8662, 0.0129)	(0.5221, 0.2438)

Step 6. Calculating the score values of $\Psi_i(i=1,\cdots,5)$, $Sc(\Psi_1)=0.1742$, $Sc(\Psi_2)=0.5943$, $Sc(\Psi_3)=0.3380$, $Sc(\Psi_4)=0.7501$, $Sc(\Psi_5)=0.2131$.

Step 7. Sorting the alternatives, the rank of alternatives is $\Psi_4>\Psi_2>\Psi_3>\Psi_5>\Psi_1$, and so Ψ_4 is the best alternative.

As can be seen from the above results that the ranking order is not the same with results obtained by other decision-making approach [12,14,69,70]. The reasons for the different order are as follows: In Example 2, $s(\kappa_1,\kappa_2)=1>s(\kappa_1)+s(\kappa_2)=0.45$, $s(\kappa_1,\kappa_3)=1>s(\kappa_1)+s(\kappa_3)=0.5$, $s(\kappa_1,\kappa_4)=1>s(\kappa_1)+s(\kappa_4)=0.55$, that is, there are complementary relationships between attribute C_1 and C_2, C_1 and C_3, C_1 and C_4. However, these approach [12,14,69,70] just tackle some MADM problems in which all attributes are mutually independent, that is, all attribute should satisfy the condition: $s(E\bigcup F)=s(E)+s(F)$. Obviously, the result obtained by these approach [12,14,69,70] is unreasonable under the condition that the attributes are not mutual independence.

Example 3 *In Example 1, if the attributes are not mutual independence and the FM is completely unknown, we can solve the DMP by the following decision process.*

Decision-making Process for Example 3.

Step 1-Step 2 are the same as Example 1.

Step 3. Constructing the nonlinear program model to determine the FM.

Step 3-1. Calculating the distance of two q-ROFNs and construct the following nonlinear program model according to Eq. (36)

$$maxD(\mathbf{B}) = 3.02w_1+2.42w_2+2.38w_3+2.26w_4.$$

$$s.t \begin{cases} \sum_{j=1}^{4}(w_j)^2=1, \\ w_j \ge 0, j=1,2,3,4. \end{cases}$$

to obtain the optimal weight vector of attributes $(w_1,w_2,w_3,w_4)=(0.9512,0.7622,0.7496,0.7118)$.

Step 3-2. Let $s(\kappa_1)=0.9512, s(\kappa_2)=0.7622, s(\kappa_3)=0.7496, s(\kappa_4)=0.7118$. We can obtain the $\lambda=-0.9991$.

Step 3-3. Calculating the FMs of attributes and attribute sets and listed as follows:

$$\begin{aligned} & s(\kappa_1)=0.9512, s(\kappa_2)=0.7622, s(\kappa_3)=0.7496, \\ & s(\kappa_4)=0.7118, s(\kappa_1,\kappa_2)=0.9891, s(\kappa_1,\kappa_3)=0.9884, \\ & s(\kappa_1,\kappa_4)=0.9866, s(\kappa_2,\kappa_3)=0.9409, s(\kappa_2,\kappa_4)=0.9320, \\ & s(\kappa_3,\kappa_4)=0.9283, s(\kappa_1,\kappa_2,\kappa_3)=0.9979, \\ & s(\kappa_1,\kappa_2,\kappa_4)=0.9975, s(\kappa_2,\kappa_3,\kappa_4)=0.9836, \\ & s(\kappa_1,\kappa_2,\kappa_3,\kappa_4)=1. \end{aligned}$$

Step 3-4. Determining the generalized Banzhaf values and list as follows:

$$\begin{aligned} & \mathbf{B}(\kappa_1)=0.9512, \mathbf{B}(\kappa_2)=0.7622, \mathbf{B}(\kappa_3)=0.7496, \\ & \mathbf{B}(\kappa_4)=0.7118, \mathbf{B}(\kappa_1,\kappa_2)=0.3987, \\ & \mathbf{B}(\kappa_1,\kappa_3)=0.3944, \mathbf{B}(\kappa_1,\kappa_4)=0.3821, \\ & \mathbf{B}(\kappa_2,\kappa_3)=0.3182, \mathbf{B}(\kappa_2,\kappa_4)=0.3094, \\ & \mathbf{B}(\kappa_3,\kappa_4)=0.3017, \mathbf{B}(\kappa_1,\kappa_2,\kappa_3)=0.6431, \\ & \mathbf{B}(\kappa_1,\kappa_2,\kappa_4)=0.6239, \mathbf{B}(\kappa_1,\kappa_3,\kappa_4)=0.6176, \\ & \mathbf{B}(\kappa_2,\kappa_3,\kappa_4)=0.5162, \mathbf{B}(\kappa_1,\kappa_2,\kappa_3,\kappa_4)=1. \end{aligned}$$

Step 5. Aggregating individual decision matrix $\tilde{R}=(\hat{\gamma}_{ij})_{5\times 4}$ into a collective decision matrix $\hat{R}=(\hat{\gamma}_i)_{5\times 1}$ by Eq.(17)(We

Table 5 Overall q-ROF Decision matrix

Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5
(0.4359, 0.2977)	(0.7443, 0.1191)	(0.6105, 0.2138)	(0.7050, 0.0554)	(0.5018, 0.2930)

choose Gumbel type aggregation operator when $\theta = 1$ and $q = 2$) and listed in Table 5.

Step 6. Calculating the score values of $\Psi_i (i = 1, \cdots, 5)$:

$Sc(\Psi_1) = 0.1014$, $Sc(\Psi_2) = 0.5426$, $Sc(\Psi_3) = 0.3270$, $Sc(\Psi_4) = 0.4940$, $Sc(\Psi_5) = 0.1659$.

Step 7. Ranking the alternatives on the basis of the comparison rules, the order of alternatives is $\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$, and so Ψ_2 is the best alternative.

7.2 Discussions of Parameters

In this subsection, the effect of parameter changes on the results will be discussed. We take Example 3 as an example to analysis the effect of parameter under the FM completely unknown.

(1) When $\theta = 1$ in Gumbel type AOs, the influences of parameter q on the ranking of alternatives, the sorting results are listed in Table 6.

Table 6 Overall assessment of alternatives by $BCCA^q$ when q changes and ranking order

Parameter	$S(\Psi_i)(i = 1, 2, \cdots, 5)$					Ranking order
	$Sc(\Psi_1)$	$Sc(\Psi_2)$	$Sc(\Psi_3)$	$Sc(\Psi_4)$	$Sc(\Psi_5)$	
$q = 2$	0.1014	0.5426	0.3270	0.4940	0.1659	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
$q = 5$	0.0129	0.1533	0.0591	0.1494	0.0148	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
$q = 10$	0.0004	0.0171	0.0030	0.0337	0.0002	$\Psi_4 > \Psi_2 > \Psi_3 > \Psi_1 > \Psi_5$
$q = 20$	3.1279E-07	0.0003	1.25612E-05	0.0029	1.82727E-08	$\Psi_4 > \Psi_2 > \Psi_3 > \Psi_1 > \Psi_5$
$q = 100$	-2.39251E-53	1.11022E-16	-1.0198E-67	4.83646E-11	-4.90573E-54	$\Psi_4 > \Psi_2 > \Psi_3 > \Psi_5 > \Psi_1$

It is showed from Table 6 that the desirable one will be slightly changed between Ψ_2 and Ψ_4.

(2) We fix $q = 2$ in the Gumbel type AOs, the influences of parameter θ on the ranking of alternatives, the sorting results are listed in Table 7.

Table 7 Overall assessment of alternatives by $BCCA^q$ when θ changes and ranking order

Parameter	$S(\Psi_i)(i = 1, 2, \cdots, 5)$					Ranking order
	$Sc(\Psi_1)$	$Sc(\Psi_2)$	$Sc(\Psi_3)$	$Sc(\Psi_4)$	$Sc(\Psi_5)$	
$\theta = 1$	0.1014	0.5426	0.3270	0.4940	0.1659	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
$\theta = 2$	0.3669	0.6692	0.5359	0.6571	0.4241	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
$\theta = 5$	0.4433	0.5781	0.5165	0.5811	0.4676	$\Psi_4 > \Psi_2 > \Psi_3 > \Psi_5 > \Psi_1$
$\theta = 10$	0.4067	0.4767	0.4442	0.4795	0.4189	$\Psi_4 > \Psi_2 > \Psi_3 > \Psi_1 > \Psi_5$
$\theta = 100$	0.2956	0.3028	0.2994	0.3032	0.2969	$\Psi_4 > \Psi_2 > \Psi_3 > \Psi_5 > \Psi_1$

It is showed from Table 7 that the desirable one will be changed slightly still between Ψ_2 and Ψ_4 when θ changes. The ranking results also can be showed in following Fig. 2.

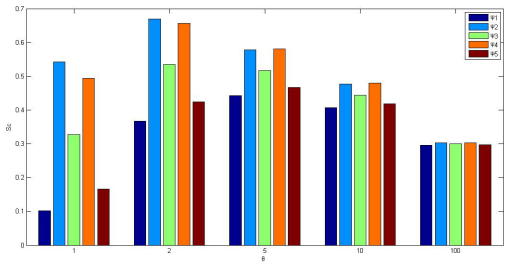

Fig. 2 Overall assessment of alternatives by $BCCA^q$ when θ changes and ranking order

Remark 2 *(1) It follows from Table 6 that the value of the parameter q should not be too large when a type of AOs is chosen to fusion decision information, because the bigger the q, the smaller the difference in scores of alternatives.*

(2) It follows from Table 7 and Fig. 2 that the value of the parameter θ should not be too large when a type of AOs is chosen to fusion decision information, because the bigger the θ, the smaller the difference in scores of alternatives. It is seen from Fig.2 that there are better discrimination of score of alternatives when $\theta \in [1, 10]$. This merely means that degree of differentiation of the scores of alternatives, but does not mean that the orders of alternatives will change.

7.3 Analyses and Comparisons

In the upcoming contents, proposed MADM approach will be analyzed and comparisons with existing approaches also be investigated.

Firstly, In our proposed AOs, which are a family of AOs with five different types. Therefore, we can use some special cases of the proposed five AOs to sort alternatives when $q = 2$ and θ takes different value.

(1) There always be an accompanied parameter in different $BCCA^q$s. We use different types of the proposed $BCCA^q$ AOs, the ranking results are listed in the following Table 8.

Table 8 Overall assessment of alternatives by $BCCA^q$ and ranking order

AOs Type	Parameters	$S(\Psi_i)(i = 1, 2, \cdots, 5)$					Ranking order
		$Sc(\Psi_1)$	$Sc(\Psi_2)$	$Sc(\Psi_3)$	$Sc(\Psi_4)$	$Sc(\Psi_5)$	
Gumbel	$\theta = 1$	0.1014	0.5426	0.3270	0.4940	0.1659	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
Clayton	$\theta = 1$	0.0561	0.4754	0.2819	0.4699	0.1352	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
Frank	$\theta = -1$	0.1996	0.6800	0.4611	0.6239	0.2853	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
Ali-Mikhai-Hap	$\theta = 1$	0.1223	0.5781	0.3505	0.5104	0.1953	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
Joe	$\theta = 1$	0.1014	0.5426	0.3270	0.4940	0.1659	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$

It is seen from Table 8 that the order of alternatives remains unchanged when different types EACs and EACCs are chosen. The ranking results obtained by different types of AOs, the results can be showed in Fig.3.

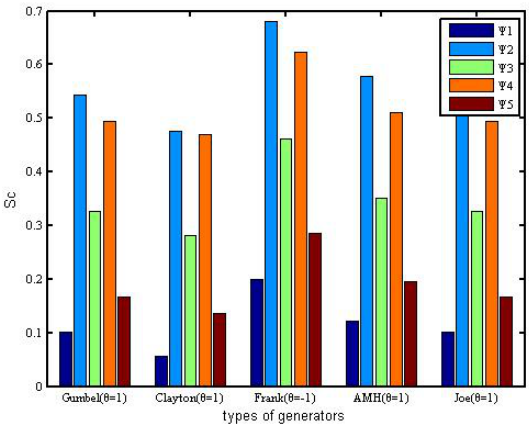

Fig. 3 Overall assessment of alternatives by different $BCCA^q$ and ranking order

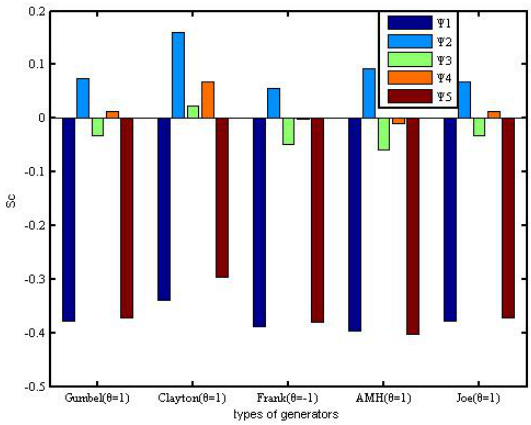

Fig. 4 Overall assessment of alternatives by different $BCCG^q$ and ranking order

(2) In Section 6, we introduced the $BCCG^q$ aggregation operators. We use different types of $BCCG^q$ operators in Step 5, all ranking results are listed in the following Table 9 under $q = 2$.

Table 9 Overall assessment of alternatives by $BCCG^q$ and ranking order

AOs Type	Parameters	$S(\Psi_i)(i=1,2,\cdots,5)$					Ranking order
		$Sc(\Psi_1)$	$Sc(\Psi_2)$	$Sc(\Psi_3)$	$Sc(\Psi_4)$	$Sc(\Psi_5)$	
Gumbel	$\theta=1$	-0.3777	0.0731	-0.03209	0.0126	-0.3717	$\Psi_2>\Psi_4>\Psi_3>\Psi_5>\Psi_1$
Clayton	$\theta=1$	-0.3392	0.1594	0.0223	0.0684	-0.2969	$\Psi_2>\Psi_4>\Psi_3>\Psi_5>\Psi_1$
Frank	$\theta=-1$	-0.3888	0.0556	-0.0489	-0.0026	-0.3805	$\Psi_2>\Psi_4>\Psi_3>\Psi_5>\Psi_1$
Ali-Mikhai-Hap	$\theta=1$	-0.3976	0.0919	-0.058	-0.0095	-0.4023	$\Psi_2>\Psi_4>\Psi_3>\Psi_5>\Psi_1$
Joe	$\theta=1$	-0.3777	0.0688	-0.03209	0.0126	-0.3717	$\Psi_2>\Psi_4>\Psi_3>\Psi_5>\Psi_1$

It is seen from Table 9 that the order of alternatives remains unchanged when different types EACs and EACCs are chosen. The ranking results obtained by different types of AOs, the results can be showed in Fig.4.

Secondly, compared with some existing methods.

(1) Xing et. al [70] introduced Pythagorean Choquet-Frank AOs and built related decision making approach, the weight of all attributes are known in this approach. However, Tao et al [51] put forward to a new decision model based intuitionistic fuzzy Coupla AOs and the weight information completely unknown in this DMP. In order to compare with Xing's method and Tao's approach, respectively, we analyze the efficiency of our proposed method from the following two aspects, that is, FM known and FM completely unknown:

(**Case I. FM information known**.) We use the proposed MADM approach to address the following DMP which is from reference [70]. All original data is from this work. We also use the FMs of all attributes or attribute sets. And the Banzhaf values of attributes and attribute sets are calculated and listed as follows:

$$\mathbf{B}(\kappa_1)=0.1795,\mathbf{B}(\kappa_2)=0.2725,\mathbf{B}(\kappa_3)=0.1798,$$
$$\mathbf{B}(\kappa_4)=0.3675,\mathbf{B}(\kappa_1,\kappa_2)=0.452,\mathbf{B}(\kappa_1,\kappa_3)=0.3593,$$
$$\mathbf{B}(\kappa_1,\kappa_4)=0.547,\mathbf{B}(\kappa_2,\kappa_3)=0.4523,$$
$$\mathbf{B}(\kappa_2,\kappa_4)=0.64,\mathbf{B}(\kappa_3,\kappa_4)=0.5473,$$
$$\mathbf{B}(\kappa_1,\kappa_2,\kappa_3)=0.6315,\mathbf{B}(\kappa_1,\kappa_2,\kappa_4)=0.82,$$
$$\mathbf{B}(\kappa_1,\kappa_3,\kappa_4)=0.727,\mathbf{B}(\kappa_2,\kappa_3,\kappa_4)=0.82,$$
$$\mathbf{B}(\kappa_1,\kappa_2,\kappa_3,\kappa_4)=1.$$

we use Gumbel type AO to address this MADM problem when $\theta = 1$ and $q = 2$, the aggregation results are listed in Table 10.

Table 10 Aggregation Results

Ψ_1	Ψ_2	Ψ_3	Ψ_4
(0.7910,0.4086)	(0.7960, 0.1490)	(0.8037, 0.1801)	(0.8254, 0.1757)

And so the score values are $Sc(\Psi_1)=0.4587, Sc(\Psi_2)=0.6114, Sc(\Psi_3)=0.6135, Sc(\Psi_4)=0.6504$.

Therefore, we can obtain the rank of all alternatives is $\Psi_4>\Psi_3>\Psi_2>\Psi_1$, which is the same with [70]'s method. It also shows the effectiveness and feasibility of our method under weight known.

(**Case II. FM information completely unknown.**) We use the proposed MADM approach to address the following DMP which is from the reference [51]. All original data is from this work and the weight information is completely unknown. The decision processes of this DMPs are the same with **Exampl 3**, so, the processes of the DMPs are omitted here. The Banzhaf values of attributes and attribute sets are

calculated and listed as follows:

$\mathbf{B}(\kappa_1) = 0.1617, \mathbf{B}(\kappa_2) = 0.2188, \mathbf{B}(\kappa_3) = 0.1409,$
$\mathbf{B}(\kappa_4) = 0.1821, \mathbf{B}(\kappa_5) = 0.1333, \mathbf{B}(\kappa_1, \kappa_2) = 0.3744,$
$\mathbf{B}(\kappa_1, \kappa_3) = 0.3027, \mathbf{B}(\kappa_1, \kappa_4) = 0.3439,$
$\mathbf{B}(\kappa_1, \kappa_5) = 0.2951, \mathbf{B}(\kappa_2, \kappa_3) = 0.3598,$
$\mathbf{B}(\kappa_2, \kappa_4) = 0.4009, \mathbf{B}(\kappa_2, \kappa_5) = 0.3512,$
$\mathbf{B}(\kappa_3, \kappa_4) = 0.3230, \mathbf{B}(\kappa_3, \kappa_5) = 0.2806,$
$\mathbf{B}(\kappa_4, \kappa_5) = 0.3154, \mathbf{B}(\kappa_1, \kappa_2, \kappa_3) = 0.5505,$
$\mathbf{B}(\kappa_1, \kappa_2, \kappa_4) = 0.5854, \mathbf{B}(\kappa_1, \kappa_2, \kappa_5) = 0.5471,$
$\mathbf{B}(\kappa_1, \kappa_3, \kappa_4) = 0.4995, \mathbf{B}(\kappa_1, \kappa_3, \kappa_5) = 0.4894,$
$\mathbf{B}(\kappa_1, \kappa_4, \kappa_5) = 0.4910, \mathbf{B}(\kappa_2, \kappa_3, \kappa_4) = 0.5617,$
$\mathbf{B}(\kappa_2, \kappa_3, \kappa_5) = 0.5076, \mathbf{B}(\kappa_2, \kappa_4, \kappa_5) = 0.5530,$
$\mathbf{B}(\kappa_3, \kappa_4, \kappa_5) = 0.4685, \mathbf{B}(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = 0.7785,$
$\mathbf{B}(\kappa_1, \kappa_2, \kappa_3, \kappa_5) = 0.7144, \mathbf{B}(\kappa_1, \kappa_2, \kappa_4, \kappa_5) = 0.7680,$
$\mathbf{B}(\kappa_1, \kappa_3, \kappa_4, \kappa_5) = 0.6695, \mathbf{B}(\kappa_2, \kappa_3, \kappa_4, \kappa_5) = 0.7404,$
$\mathbf{B}(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5) = 1.$

we use Gumbel type AO when $\theta = 1$ and $q = 2$, the aggregation results are list in the following Table 11.

Table 11 Aggregation Results

Ψ_1	Ψ_2	Ψ_3	Ψ_4
(0.7003,0.1337)	(0.5969, 0.0663)	(0.7787, 0.0802)	(0.7708, 0.0963)

And so the score values are $Sc(\Psi_1) = 0.4725, Sc(\Psi_2) = 0.3519, Sc(\Psi_3) = 0.6000, Sc(\Psi_4) = 0.5849$.

Therefore, we can obtain the rank of all alternatives are $\Psi_3 > \Psi_4 > \Psi_1 > \Psi_2$, which is the same with [51]'s method. It also shows the effectiveness and feasibility of our method under the weight information (fuzzy measure information) completely unknown.

(2) **Our proposed approaches can also be used to address the MAGDM problems.** We take Chen's [71] as an example to show this view. We use our proposed $BCCA^q$ operators to fusion decision information. In this MAGDM problems, there are three experts E_1, E_2, E_3 whose weight vector is $(0.33, 0.34, 0.33)$, four alternatives $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ and three attributes $\kappa_1, \kappa_2, \kappa_3$ with weight $(0.4, 0.2, 0.4)$. In this MAGDM problems, all attributes and experts are independent. We use Gumbel Type AO to fuse decision information under $q = 2$ and $\theta = 1$. The procedure can be summarized as follows:

Firstly, we need to calculate the Banzhaf values of the attribute sets and list as follows:

$\mathbf{B}(\kappa_1) = 0.4$, $\mathbf{B}(\kappa_2) = 0.2$, $\mathbf{B}(\kappa_3) = 0.4$, $\mathbf{B}(\kappa_1, \kappa_2) = 0.6$, $\mathbf{B}(\kappa_1, \kappa_3) = 0.8$, $\mathbf{B}(\kappa_2, \kappa_3) = 0.6$, $\mathbf{B}(\kappa_1, \kappa_2, \kappa_3) = 1$.

Secondly, we use the Gumbel type $BCCA^q$ operator to synthesize the individual decision matrix to a collective individual decision matrix and list as the Table 12.

Table 12 Collective decision matrix

Alternatives	E_1	E_2	E_3
Ψ_1	(0.5576, 0.14956)	(0.6676, 0.2091)	(0.8443, 0.10506)
Ψ_2	(0.7559, 0.1855)	(0.7738, 0.2209)	(0.9378, 0.0394)
Ψ_3	(0.7624, 0.1630)	(0.8478, 0.0798)	(0.8516, 0.0805)
Ψ_4	(0.7906, 0.1587)	(0.8013, 0.0816)	(0.7739, 0.1077)

Thirdly, we determine the Banzhaf values of experts and expert sets,

$\mathbf{B}(E_1) = 0.33$, $\mathbf{B}(E_2) = 0.34$, $\mathbf{B}(E_3) = 0.33$, $\mathbf{B}(E_1, E_2) = 0.67$, $\mathbf{B}(E_1, E_3) = 0.68$, $\mathbf{B}(E_2, E_3) = 0.67$, $\mathbf{B}(E_1, E_2, E_3) = 1$.

Fourthly, synthesize all collective decision matric into the overall decision matrix by using the Gumbel type $BCCA^q$ operators under $q = 2$ and $\theta = 1$, the results are listed in Table 13.

Table 13 Overall Decision matrix

Ψ_1	Ψ_2	Ψ_3	Ψ_4
(0.7583, 0.0794)	(0.8896, 0.0676)	(0.8719, 0.0561)	(0.8522, 0.0513)

Finally, we calculate the score value of all alternatives and listed as follows:

$Sc(\Psi_1) = 0.5688$, $Sc(\Psi_2) = 0.7868$, $Sc(\Psi_3) = 0.7571$, $Sc(\Psi_4) = 0.7235$.

Therefore, the order is $\Psi_2 > \Psi_3 > \Psi_4 > \Psi_1$, which is the same with Chen's method [71]. This example is also show that our proposed method can also be applied to MAGDM problems.

(3) **Our proposed method can be easily extended to interval-valued q-ROFS.** As we all know, the operations are the basis of the fuzzy AOs. However, the operations of q-ROFNs can be easily extended to interval-valued q-ROFSs (IVq-ROFSs) and the EACs and EACCs can be carried out in IVq-ROFSs by some feasible transforming technology, therefore, the proposed AOs also can be easily to apply in IVq-ROFSs or other fuzzy sets. Besides the above methods, there are some existing MADM approaches. We use the following existing MADM approach to address the same MADM problems in **Example 1**, the results are list as follows:

Thirdly. The merits of proposed methods.

• (1) Compared with some decision making approaches [12, 14, 69, 72], these decision making approaches can deal with some DMPs in which attribute values are represented by IFNs (or PFNs) and mutually independent of each other. However, these mentioned methods can not deal with

Table 14 The comparisons of different operators

	Methods	Weights	Independence of attributes	Ranking order
Existing Methods	IFWA operator [72]	K	Yes	$\Psi_3 > \Psi_4 > \Psi_2 > \Psi_5 > \Psi_1$
	PFWA operator [12]	K	Yes	$\Psi_3 > \Psi_4 > \Psi_2 > \Psi_5 > \Psi_1$
	PFEWA operator [69]	K	Yes	$\Psi_3 > \Psi_4 > \Psi_2 > \Psi_5 > \Psi_1$
	PFEWG operator [69]	K	Yes	$\Psi_3 > \Psi_4 > \Psi_2 > \Psi_5 > \Psi_1$
	q-ROFWA operator [14]	K	Yes	$\Psi_3 > \Psi_4 > \Psi_2 > \Psi_5 > \Psi_1$
	q-ROFWG operator [14]	K	Yes	$\Psi_3 > \Psi_4 > \Psi_2 > \Psi_5 > \Psi_1$
	PFCFA [70]	K	Yes	$\Psi_3 > \Psi_4 > \Psi_2 > \Psi_5 > \Psi_1$
	q-ROF Neural AOs [73]	PK	No	$\Psi_4 > \Psi_2 > \Psi_3 > \Psi_5 > \Psi_1$
	IFCAA operator [51]	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
Proposed Methods	Gumbel type $BCCA^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
	Clayton type $BCCA^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
	Frank type $BCCA^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
	Ali-Mikhail-Hap type $BCCA^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
	Joe type $BCCA^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
	Gumbel type $BCCG^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
	Clayton type $BCCG^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
	Frank type $BCCG^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
	Ali-Mikhail-Hap type $BCCG^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$
	Joe type $BCCG^q$	CU	No	$\Psi_2 > \Psi_4 > \Psi_3 > \Psi_5 > \Psi_1$

In this table, K means Known, PK means partially Known, CU means Completely Unknown.

the MADM problem with complementary or redundant attributes. In addition, these mentioned methods can only deal with the MADM problem with known attribute weight information known. Therefore, these methods can solve the MADM with known attribute weight and independent attributes (such as **example 1**), but can not deal with the problem with weight information partially known (such as **example 2**) and weight information completely unknown (such as **Example 3**). Our proposed methods not only can deal with some DMPs under weight information known, but also can handle DMPs where weight information partially known and completely unknown, the details comparison information can be founded in Table 14.

• (2) Our proposed $BCCA^q$ operator can not only consider the relationship between adjacent attribute combinations, but also comprehensively consider the relationship between attributes. In our Gumbel type $BCCA^q$ operators, when $q = 2$, $\theta = 1$ and all attributes are mutually independent, Gumbel type $BCCA^q$ operators will reduce to PFWA [12]. When $\theta = 1$ and all attributes are mutually independent in our Gumbel type $BCCA^q$ operators, Gumbel type $BCCA^q$ and Gumbel type $BCCG^q$ operators will reduce to q-ROFWA [14] and q-ROFWG[14], respectively. Moreover, the model based on maximum deviation and Banzhaf function constructed in this work can objectively solve the FM of attribute sets.

• (3) We can seen from the above discussions that our proposed methods can not only handle MADM problems with q-ROF information, but also effectively solve MAGDM problems under q-ROF environment. Whilst, the proposed $BCCA^q$ / $BCCG^q$ operators can be easily extended to IVq-ROF environment and other types of fuzzy information.

8 Conclusions

In real MADM problems, the relationships between adjacent attribute combinations not only be considered, the interaction among elements globally should also be captured. However, some existing MADM approaches under q-ROF environment seems to be invalid on this point, they only consider the relationship between adjacent attribute combinations, but not capture the interaction among elements globally. Whilst, some attributes maybe dependent in real decision problems, how to reflect the interaction among the attributes in process of decision information fusion needs to be explored in depth. Some existing MADM approaches for dealing with q-ROF information are valid under the hypothesis that experts have given the fuzzy measure (or weight information) of attribute sets in advance, and can not be directly used to MADM problems with unknown or partially unknown fuzzy measures. Therefore, the goal of the present work is to synthesize EACs (EACCs), Banzhaf-based Choquet integral and q-ROFS to develop a novel MADM approach with q-ROF information in which weight information partially known or completely unknown, the proposed decision approach can effectively address above mentioned drawbacks. Firstly, the EAC and EACC are extended to handle q-ROFNs and the operations of q-ROFNs based on EAC and EACCs are given. In order to comprehensively consider the relationship between attributes, the $BCCA^q$ / $BCCG^q$ are introduced on the basis of the operations of q-ROFNs; Consequently, some special cases of $BCCA^q$ / $BCCG^q$ are investigated when the generators of EACs take different types function which satisfied the condition of the generators of copulas. In addition, to determine the FM of attribute sets objectively, the improved maximum deviation method and Banzhaf function model are built. Finally, the corresponding decision-making approaches are constructed based on the proposed AOs and proposed models. Proposed approaches can overcome effectively the fuzzy measures of attribute sets are given by decision makers subjectively and can also effectively address the some DMPs, in which the weight information are incompletely unknown or completely unknown and the relationship are existed among all attribute sets.

In line with the developed q-ROFN's operation laws, one of the merit advantages is that, on the one hand, the proposed operations can contribute more choices for decision makers and the mutuality among attributes can also be determined, on the the other hand, the family of AOs are more flexible to reflex the decision makers' attitude by adding a parameter θ. Therefore, the proposed operators are constructed by combining traditional mean with any copulas and corresponding co-copulas, and the relationships among all attribute are comprehensively considered, so a broader ability for modeling practical issues can be available. In multigranularity fuzzy linguistic modeling, it is allowed to use multiple LTSs in fuzzy linguistic modeling, because it allows each expert to use his LTS to express his/her preferences, so

it has been widely used in the field of GDM. A new linguistic computational model [74] is introduced to manage multigranular linguistic distribution assessments for its application to large-scale MAGDM problems. Large-scale group decision is an important branch in modern decision theory In large-scale group decision making process, there is a mass of heterogeneous information. Therefore, in our future study, with the help of academic thought of multigranularity group consensus decision making, we will investigate group consensus decision model of heterogeneous linguistic information.

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Conflict of interest

The authors declare that they have no conflict of interest.

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