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Stochastic order and generalized weighted mean invariance

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Abstract

In this paper, we study the invariance of quasi-arithmetic means in monotonic series of numbers. We show that if two distributions of weights are ordered under first stochastic order then, for any monotonic series of numbers, all their weighted quasi-arithmetic means share the same order direction. This implies for instance that the arithmetic and harmonic means for two different distributions of weights always have to be aligned if the weights are stochastic ordered, i.e., either both means increase or both decrease. Moreover, we explore the invariance properties when convex (concave) functions define both the quasi-arithmetic mean and the series of numbers. We situate our work with existing previous invariance results and present an example that illustrate the usefulness and transversality of our approach.

Keywords: stochastic order, weighted mean, arithmetic mean, harmonic mean, Kolmogorov mean, quasi-arithmetic mean

1. Introduction and motivation

Stochastic orders [1], [2] are defined in probability theory and statistics, to quantify the concept of one random variable being *bigger* or *smaller* than another one. Discrete probability distributions on unidimensional values are sequences of n-tuples of non-negative values that add up to 1, and can be thus interpreted as weights for averaging the set of possible values taken by the random variable, or even as equivalence classes of compositional data [3]. Stochastic orders have found application in decision and risk theory, and in economics in general, among many other fields [2]. Some stochastic orders have been usually defined based on the order invariance, when we consider any possible increasing sequence of results, of the expected values of the random variables they compare, and consequently of their weighted arithmetic mean. This raises the question whether this invariance might also hold for other kind of means beyond the arithmetic mean. To our knowledge, only the works [4] [5] in the economics literature, have studied the invariance under generalized means defined by strictly increasing utility functions. In our recent work on mathematical inequalities [6] we have found order invariance properties that hold for *all* generalized or quasi-arithmetic means, defined by strictly monotonic, increasing or decreasing, functions. We generalize here these invariance properties and study its relationship to several stochastic orders.

For a long time, economists have discussed which mean should be used for a given problem [7], e.g., some problems are represented by harmonic mean rather than arithmetic mean, like the price earning ratio, P/E. Also, the power mean, depending on a parameter $\eta > 0$ which gives the elasticity of substitution among different types of labor, is implicitly[1] used for the aggregate labor demand (power mean with power equal to $\frac{\eta-1}{\eta}$), and for its corresponding wage (power mean with power equal to $1-\eta$) [8]. For instance, when $\eta = 2$ we have the square root mean for the aggregate labor and the harmonic mean for its wage. Interesting examples of two mean representations, arithmetic and harmonic, of a problem come from physics, where springs (helical metal coils) added in series combine harmonically, and in parallel arithmetically, while resistors in parallel combine harmonically, and in series arithmetically. These are examples that show that combining the same utilities in different ways will result in different means. Another example is traffic flow [9] where two different means for speeds are meaningful for the problem at hand, the arithmetic mean, or time mean speed, and harmonic mean, or space mean speed. And in [10] both geometric and harmonic mean are used in addition to arithmetic mean to improve noise source maps.

In this paper, we show that the order invariance is a necessary and sufficient condition for first stochastic order. We also show that invariance holds under any kind of quasi-arithmetic means, defined by strictly monotonic functions, and we also explore invariance under other stochastic orders. We also show the applicability of these results beyond the pure theoretical interest by

[1] By implicitly, here we refer to the fact that it is used without any reference to being a power mean

introducing an example in Section 7 where multiple quasi-arithmetic means are used to characterize a problem.

The rest of the paper is organized as follows. We introduce the stochastic order in Section 2, and the arithmetic mean and its relationship with stochastic order in Section 3. In Section 4, we present the invariance theorems discussing them in Section 5 in the case of concave (convex) functions, and in Section 6 in their application to stochastic orders. In Section 7, we show an example based on the linear combination of Monte Carlo estimators, and in Section 8 we discuss the relationship of our work to the work in [4] [5]. Finally conclusions and future work are given in Section 9.

2. Stochastic orders

Stochastic orders are pre-orders (i.e, binary relationships holding symmetric and transitive properties) defined on probability distributions with finite support. Note that equivalently, one can think of sequences (i.e., ordered sets) of non-negative weights/values that sum up to one. Observe that any sequence $\{\alpha_k\}$ of M positive numbers such that $\sum_{k=1}^{M} \alpha_k = 1$ can be considered a probability distribution. While several interpretations hold and hence increase the range of applicability, in the remaining of this paper, we will talk of sequence without any loss of generality.

Notation. We use the symbols $\prec, \succ$ to represent orders between two sequences $\{\alpha_k\}$ and $\{\alpha'_k\}$ of size M, e.g., and we write $\{\alpha_k\} \succ \{\alpha'_k\}$ or equivalently $\{\alpha'_k\} \prec \{\alpha_k\}$. We will denote the elements of the sequences without the curly brackets, e.g., the first element of the sequence $\{\alpha_k\}$ is denoted as α_1, while the last one is α_M. Moreover, the first and last elements of the sequence receive special attention, since when $\alpha_1 \leq \alpha'_1$, we can write this order as $\{\alpha_k\} \succ_F \{\alpha'_k\}$ (where $\succ_F$ stands for *first*) and whenever $\alpha_M \geq \alpha'_M$, we can write $\{\alpha_k\} \succ_L \{\alpha'_k\}$ (where $\succ_L$ stands for *last*). The case with both $\alpha_1 \leq \alpha'_1$ and $\alpha_M \geq \alpha'_M$, can we denote as $\{\alpha_k\} \succ_{FL} \{\alpha'_k\}$. The orders $\succ_F, \succ_L, \succ_{FL}$ are superorders of first stochastic dominance order, $\{\alpha_k\} \succ_{FSD} \{\alpha'_k\}$, that will be studied in Section 6. We denote as $\{\alpha_{M-k+1}\}$ the sequence with the same elements of $\{\alpha_k\}$ but in reversed order.

Toy example. Given sequences $\{\alpha_k\} = \{0.5, 0.1, 0.2, 0.2\}$, and $\{\alpha'_k\} = \{0.4, 0.1, 0.2, 0.3\}$, we have both $\{\alpha_k\} \prec_F \{\alpha'_k\}$, $\{\alpha_k\} \prec_L \{\alpha'_k\}$, and thus $\{\alpha_k\} \prec_{FL} \{\alpha'_k\}$. On the other hand for sequences $\{\alpha_k\}$ and $\{\alpha''_k\} = \{0.4, 0.3, 0.2, 0.1\}$, we have $\{\alpha_k\} \prec_F \{\alpha''_k\}$, $\{\alpha_k\} \not\prec_L \{\alpha''_k\}$, and thus $\{\alpha_k\} \not\prec_{FL} \{\alpha''_k\}$.

Mirror property. One desirable property of stochastic orders is such that when reversing the ordering of the sequence elements, the stochastic order should be reversed as well, i.e., if $\{\alpha_k\} \succ \{\alpha'_k\}$ then $\{\alpha'_{M-k+1}\} \succ \{\alpha_{M-k+1}\}$. This is similar to the invariance of physical laws to right and left hand. We call this property *mirror* property.

Definition 1 (Mirror property)**.** *We say that a stochastic order has the mirror property if*

$$\{\alpha'_k\} \prec \{\alpha_k\} \Rightarrow \{\alpha_{M-k+1}\} \prec \{\alpha'_{M-k+1}\}. \tag{1}$$

Observe that the simple orders defined before, $\succ_F$, $\succ_L$, do not hold this property, but $\succ_{FL}$ holds it. We will see in Section 6 that usual stochastic orders do hold the mirror property. But an order that is insensitive to the permutation of the elements of a sequence, like majorization or Lorentz orders, does not hold the mirror property.

3. Quasi-arithmetic mean

As we have told before, many stochastic orders are justified by invariance to arithmetic mean, and we want to investigate in this paper invariance to more general means. We define here the kind of means we are interested in.

Definition 2 (Quasi-arithmetic or Kolmogorov mean)**.** *A quasi-arithmetic generalized weighted mean (or Kolmogorov mean)* $\mathcal{M}(\{b_k\},\{\alpha_k\})$ *of sequence* $\{b_k\}_{k=1}^M$ *is of the form* $g^{-1}\Big(\sum_{k=1}^M \alpha_k g(b_k)\Big)$*, where* $g(x)$ *is an invertible strictly monotonic function, with inverse function* $g^{-1}(x)$ *and* $\{\alpha_k\}_{k=1}^M$ *are positive weights such that* $\sum_{k=1}^M \alpha_k = 1$*.*

Examples of such mean are arithmetic weighted mean ($g(x) = x$), harmonic weighted mean ($g(x) = 1/x$), geometric weighted mean ($g(x) = \log x$) and in general the weighted power means ($g(x) = x^r$).

Lemma 1. *Consider the sequences of* M *positive weights* $\{\alpha_k\}_{k=1}^M$ *and* $\{\alpha'_k\}_{k=1}^M$*,* $\sum_{k=1}^M \alpha_k = 1, \sum_{k=1}^M \alpha'_k = 1$*,* $g(x)$ *a strictly monotonic function and* $f(x)$ *a function such that the sequence* $\{f(k)\}_{k=1}^M$ *is increasing. The following conditions a),b),a'),b'),c) and d) are equivalent. If* $\{f(k)\}_{k=1}^M$ *is decreasing, the inequalities in a) through b') are reversed.*

a) for $g(x)$ increasing, for any function $f(x)$ such that the sequence $\{f(k)\}_{k=1}^{M}$ is increasing the following inequality holds:

$$\sum_{k=1}^{M} \alpha'_k g(f(k)) \leq \sum_{k=1}^{M} \alpha_k g(f(k)) \tag{2}$$

b) for $g(x)$ increasing, for any function $f(x)$ such that the sequence $\{f(k)\}_{k=1}^{M}$ is increasing the following inequality holds:

$$\sum_{k=1}^{M} \alpha_{M-k+1} g(f(k)) \leq \sum_{k=1}^{M} \alpha'_{M-k+1} g(f(k)) \tag{3}$$

a') for $g(x)$ decreasing, for any function $f(x)$ such that the sequence $\{f(k)\}_{k=1}^{M}$ is increasing the following inequality holds:

$$\sum_{k=1}^{M} \alpha_k g(f(k)) \leq \sum_{k=1}^{M} \alpha'_k g(f(k)) \tag{4}$$

b') for $g(x)$ decreasing, for any function $f(x)$ such that the sequence $\{f(k)\}_{k=1}^{M}$ is increasing the following inequality holds:

$$\sum_{k=1}^{M} \alpha'_{M-k+1} g(f(k)) \leq \sum_{k=1}^{M} \alpha_{M-k+1} g(f(k)) \tag{5}$$

c) the following inequalities hold:

$$\begin{aligned} \alpha'_1 &\geq \alpha_1 \\ \alpha'_1 + \alpha'_2 &\geq \alpha_1 + \alpha_2 \\ &\vdots \\ \alpha'_1 + \ldots + \alpha'_{M-1} &\geq \alpha_1 + \ldots + \alpha_{M-1} \\ \alpha'_1 + \ldots + \alpha'_{M-1} + \alpha'_M &= \alpha_1 + \ldots + \alpha_{M-1} + \alpha_M \end{aligned} \tag{6}$$

d) the following inequalities hold:

$$\begin{aligned} \alpha_M &\geq \alpha'_M \\ \alpha_M + \alpha_{M-1} &\geq \alpha'_M + \alpha'_{M-1} \\ &\vdots \\ \alpha_M + \ldots + \alpha_2 &\geq \alpha'_M + \ldots + \alpha'_2 \\ \alpha'_1 + \ldots + \alpha'_{M-1} + \alpha'_M &= \alpha_1 + \ldots + \alpha_{M-1} + \alpha_M \end{aligned} \tag{7}$$

Proof. The proof is in the Appendix. □

Note 1. *Lemma 1 can be extended to any real sequences* $\{y_k\}$ *and* $\{x_k\}$*, such that* $\sum_{k=1}^{M} y_k = \sum_{k=1}^{M} x_k$*. It is enough to observe that the order of all the inequalities are unchanged by adding a positive constant, so that* $\{y_k\}$ *and* $\{x_k\}$ *can be brought to be positive, and also they are unchanged by the multiplication of a positive constant, so that the resulting* $\{y_k\}$ *and* $\{x_k\}$ *sequences can be normalized.*

Theorem 1. *Given a mean* $\mathcal{M}$ *with strictly monotonic function* $g(x)$ *and two distributions* $\{\alpha'_k\}, \{\alpha_k\}$*, the following propositions are equivalent:*
a) for all increasing functions $f(x)$

$$\mathcal{M}(\{f(k)\}, \{\alpha'_k\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha_k\}) \tag{8}$$

b) for all decreasing functions $f(x)$

$$\mathcal{M}(\{f(k)\}, \{\alpha_k\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha'_k\}) \tag{9}$$

c) for all increasing functions $f(x)$

$$\mathcal{M}(\{f(k)\}, \{\alpha_{M-k+1}\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha'_{M-k+1}\}) \tag{10}$$

d) for all decreasing functions $f(x)$

$$\mathcal{M}(\{f(k)\}, \{\alpha'_{M-k+1}\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha_{M-k+1}\}) \tag{11}$$

Proof. It is a direct consequence of Lemma 1 and the definition of quasi-arithmetic mean. □

4. Invariance

Theorem 2 (Invariance)**.** *Given two distributions* $\{\alpha'_k\}, \{\alpha_k\}$*, and two quasi-arithmetic means* $\mathcal{M}$*,* $\mathcal{M}^\star$*, the following propositions are equivalent:*
a) for all increasing functions $f(x)$

$$\mathcal{M}(\{f(k)\}, \{\alpha'_k\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha_k\}) \tag{12}$$

b) for all increasing functions $f(x)$

$$\mathcal{M}^\star(\{f(k)\}, \{\alpha'_k\}) \leq \mathcal{M}^\star(\{f(k)\}, \{\alpha_k\}) \tag{13}$$

Proof. It is a direct consequence of the observation that conditions c) and d) in Lemma 1 do not depend on any particular $g(x)$ function considered, and thus the order of inequalities does not change with the mean considered as long as $\{\alpha_k\}$ and $\{\alpha'_k\}$ are kept fixed. □

Observe that Theorem 2 can be generalized to consider all possible cases by simple application of Theorem 1.
The following property relates stochastic order with weighted mean. Be $\mathcal{I}$ a set of increasing functions.

Definition 3 (preserve mean order)**.** *We say that a stochastic order preserves mean order for a given mean $\mathcal{M}$ and a set $\mathcal{I}$ of increasing functions when, for all functions $f(x) \in \mathcal{I}$ and any distributions $\{\alpha'_k\}, \{\alpha_k\}$,*

$$\{\alpha_k\} \prec \{\alpha'_k\} \Rightarrow \mathcal{M}(\{f(k)\}, \{\alpha'_k\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha_k\}). \tag{14}$$

The following proposition and theorem show that if a stochastic order holds the preserve mean order property for a given mean and for the set of all increasing functions than it holds too the antisymmetric property, and it is thus a partial order.

Proposition 1. *Given distributions $\{\alpha'_k\}, \{\alpha_k\}$ and a mean M, suppose that for all increasing (respectively decreasing) functions $f(x)$ it holds that $\mathcal{M}(\{f(k)\}, \{\alpha'_k\}) = \mathcal{M}(\{f(k)\}, \{\alpha_k\})$, then $\{\alpha'_k\} = \{\alpha_k\}$.*

Proof. If $g(x)$ is the strictly monotonic function that defines the mean,

$$\sum_{k=1}^{M} \alpha_k g(f(k)) = \sum_{k=1}^{M} \alpha'_k g(f(k)) \Rightarrow \sum_{k=1}^{M} (\alpha_k - \alpha'_k) g(f(k)) = 0. \tag{15}$$

Suppose first the case of $f(x)$ increasing. Taking as $f(x)$ successively in Eq. 15 $\{f(k)\} = \{1, 1, \ldots, 1\}$, $\{f(k)\} = \{1, 1, \ldots, 2\}$, ..., $\{f(k)\} = \{1, 2, \ldots, 2\}$ and subtracting we obtain that for all k, $\alpha_k = \alpha'_k$. For $f(x)$ decreasing we consider instead the decreasing sequences $\{f(k)\} = \{1, 1, \ldots, 1\}$, $\{f(k)\} = \{2, 1, \ldots, 1\}$, ..., $\{f(k)\} = \{2, 2, \ldots, 1\}$. □

Theorem 3 (Antisimmetry)**.** *A stochastic order that preserves mean order for the set of all increasing functions is antisymmetric.*

Proof. We have to show that

$$\{\alpha_k\} \succ \{\alpha'_k\} \wedge \{\alpha'_k\} \succ \{\alpha_k\} \Rightarrow \{\alpha_k\} = \{\alpha'_k\},$$

which is immediate by the preserve mean order hypothesis and Proposition 1. □

Observe that holding antisymmetric property makes the stochastic order a partial order.

Theorem 2 together with the preserve mean order property allows us to state the following invariance property:

Theorem 4 (preserve mean order invariance)**.** *For any stochastic order that preserves mean order for a given mean and for the set of all increasing functions, then it preserves mean order for any mean. In other words, the preserve mean order property is invariant with respect to the mean considered.*

The following proposition and theorem relate the mirror and preserve mean order properties.

Proposition 2. *Given an order that preserves mean order for the set of all increasing functions and any two ordered distributions,* $\{\alpha_k\} \succ \{\alpha'_k\}$*, then* $\{\alpha_{M-k+1}\} \succ \{\alpha'_{M-k+1}\} \Rightarrow \{\alpha_k\} = \{\alpha'_k\}$.

Proof. As mean order is preserved for any mean by Theorem 4, we can take a mean function $g(x)$ increasing, and for the mean order hypothesis, for any increasing function $f(x)$ we have that

$$\sum_{k=1}^{M} \alpha'_k g(f(k)) \leq \sum_{k=1}^{M} \alpha_k g(f(k)) \tag{16}$$

and applying Lemma 1

$$\sum_{k=1}^{M} \alpha_{M-k+1} g(f(k)) \leq \sum_{k=1}^{M} \alpha'_{M-k+1} g(f(k)). \tag{17}$$

As the order preserves mean order, $\{\alpha_{M-k+1}\} \succ \{\alpha'_{M-k+1}\}$ means that

$$\sum_{k=1}^{M} \alpha'_{M-k+1} g(f(k)) \leq \sum_{k=1}^{M} \alpha_{M-k+1} g(f(k)), \tag{18}$$

thus from 17 and 18, both means are equal for all increasing functions $f(x)$, and by Proposition 1 $\{\alpha_k\} = \{\alpha'_k\}$. □

Theorem 5. *Given an order that preserves mean order for the set of all increasing functions, and suppose that whenever two distributions* $\{\alpha_k\}, \{\alpha'_k\}$ *are ordered, the mirror distributions* $\{\alpha_{M-k+1}\}, \{\alpha'_{M-k+1}\}$ *are also ordered, then the order holds the mirror property.*

Proof. It is an immediate consequence from Proposition 2. □

5. Concavity and convexity

Lemma 2. *Given two distributions* $\{\alpha'_k\}, \{\alpha_k\}$*, and a quasi-arithmetic mean* $\mathcal{M}(\{f(k)\}, \{\alpha_k\})$ *with convex (respectively concave) function* $g(x)$*, we have that if for all increasing concave (respectively convex) function* $f(x)$*, it holds that*

$$\mathcal{M}(\{f(k)\}, \{\alpha'_k\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha_k\}), \tag{19}$$

and

$$\mathcal{M}(\{f(k)\}, \{\alpha_{M-k+1}\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha'_{M-k+1}\}) \tag{20}$$

then both inequalities hold too for any other mean with concave (respectively convex) $g(x)$ *function.*

The proof is in the Appendix.

Theorem 6. *Given an order that holds the mirror property, and a quasi-arithmetic mean* $\mathcal{M}(\{f(k)\}, \{\alpha_k\})$ *with convex (respectively concave) function* $g(x)$*, we have that, if for any increasing concave (respectively convex) function* $f(x)$*, if* $\{\alpha'_k\} \prec \{\alpha_k\}$ *it holds that*

$$\mathcal{M}(\{f(k)\}, \{\alpha'_k\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha_k\}), \tag{21}$$

then it holds too for any other mean with concave (respectively convex) $g(x)$ *function.*

Proof. By the mirror property and by the hypothesis we can state that

$$\mathcal{M}(\{f(k)\}, \{\alpha_{M-k+1}\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha'_{M-k+1}\}) \tag{22}$$

and then we apply Lemma 2. □

What Theorem 6 tells us is that if we are able to find a mean with $g(x)$ convex that holds Eq. 21 for all concave $f(x)$ then it holds for any other mean with concave $g(x)$. And if we are able to find a mean with $g(x)$ concave that holds Eq. 21 for all convex $f(x)$ then it holds for any other mean with convex $g(x)$.

Corollary 1. *Consider the weighted arithmetic mean $\mathcal{A}(\{\alpha_k f(k)\})$. Given an order that holds the mirror property, we have that if for any increasing concave (respectively convex) function $f(x)$, given $\{\alpha'_k\} \prec \{\alpha_k\}$ it holds that*

$$\mathcal{A}(\{\alpha'_k f(k)\}) \leq \mathcal{A}(\{\alpha_k f(k)\}), \tag{23}$$

then for any other mean $\mathcal{M}$ with concave (respectively convex) function $g(x)$ it holds

$$\mathcal{M}(\{f(k)\}, \{\alpha'_k\}) \leq \mathcal{M}(\{f(k)\}, \{\alpha_k\}). \tag{24}$$

Proof. Arithmetic mean is a quasi-arithmetic mean with function $g(x) = x$, which is concave and convex, and thus it is enough to apply Theorem 6. □

Functions x^p, for $p \geq 1$ or $p < 0$, are convex over $\mathcal{R}_{++}$, e^x over $\mathcal{R}$, while $\log x$, x^p, for $0 < p \leq 1$ are concave over $\mathcal{R}_{++}$. Affine functions are both concave and convex over $\mathcal{R}$. If $g(x)$ is convex $-g(x)$ is concave and viceversa, and the composition of concave and concave and increasing is concave, and convex and convex and increasing is convex.

6. Application to stochastic order

6.1. First order stochastic dominance

Originating in economics risk literature [11], first order stochastic dominance [2], [1], FSD, between two probability distributions, $\{\alpha_k\}$, $\{\alpha'_k\}$, $\{\alpha_k\} \succ_{FSD} \{\alpha'_k\}$, is defined when for any increasing function $f(x)$,

$$E[f(k)]_{\alpha'_k} \leq E[f(k)]_{\alpha_k}.$$

Remember that expected values are the arithmetic weighted mean. A necessary and sufficient condition for first order stochastic dominance is that the following inequalities hold:

$$
\begin{aligned}
\alpha'_1 &\geq \alpha_1 \\
\alpha'_1 + \alpha'_2 &\geq \alpha_1 + \alpha_2 \\
&\vdots \\
\alpha'_1 + \ldots + \alpha'_{M-1} &\geq \alpha_1 + \ldots + \alpha_{M-1} \\
\alpha'_1 + \ldots + \alpha'_{M-1} + \alpha'_M &= \alpha_1 + \ldots + \alpha_{M-1} + \alpha_M. \qquad (25)
\end{aligned}
$$

Observe that Eq. 25 is just the condition c) of Lemma 1, which is equivalent to condition d) of Lemma 1, thus $\{\alpha_k\} \succ_{FSD} \{\alpha'_k\} \Rightarrow \{\alpha'_{M-k+1}\} \succ_{FSD} \{\alpha_{M-k+1}\}$, i.e., the mirror property holds. From the definition of FSD, and from Theorem 4, we can redefine FSD, $\{\alpha_k\} \succ_{FSD} \{\alpha'_k\}$ as there exists a mean $\mathcal{M}$ such that, for any increasing function $f(x)$, Eq. 14 holds. This is, the definition of FSD is independent of the mean considered, while the original definition relied on the expected value (arithmetic mean). The mean considered can be arithmetic, harmonic, geometric or any other quasi-arithmetic mean.

6.2. Second order stochastic dominance and increase convex ordering

Second order stochastic dominance between two probability distributions, $\{\alpha_k\}$, $\{\alpha'_k\}$, $\{\alpha_k\} \succ_{SSD} \{\alpha'_k\}$, occurs when for any increasing concave function $f(x)$,

$$E[f(k)]_{\alpha'_k} \leq E[f(k)]_{\alpha_k}.$$

Trivially

$$\{\alpha_k\} \succ_{FSD} \{\alpha'_k\} \Rightarrow \{\alpha_k\} \succ_{SSD} \{\alpha'_k\}.$$

Let us consider the cumulative distribution function, $F_i = \alpha_1 + \alpha_2 + \ldots + \alpha_i$, and the survival function $\bar{F}_i = 1 - F_i = \alpha_M + \alpha_{M-1} + \ldots + \alpha_{i+1}$. A necessary and sufficient condition for second order stochastic dominance [2], [1] is the following:

$$
\begin{aligned}
F'_1 &\geq F_1 \\
F'_1 + F'_2 &\geq F_1 + F_2 \\
&\vdots \\
F'_1 + \ldots + F'_{M-1} &\geq F_1 + \ldots + F_{M-1} \\
F'_1 + \ldots + F'_{M-1} + F'_M &\geq F_1 + \ldots + F_{M-1} + F_M
\end{aligned}
\qquad (26)
$$

or equivalently,

$$\begin{aligned} \bar{F}'_1 &\leq \bar{F}_1 \\ \bar{F}'_1 + \bar{F}'_2 &\leq \bar{F}_1 + \bar{F}_2 \\ &\vdots \\ \bar{F}'_1 + \ldots + \bar{F}'_{M-1} &\leq \bar{F}_1 + \ldots + \bar{F}_{M-1} \\ \bar{F}'_1 + \ldots + \bar{F}'_{M-1} + \bar{F}'_M &\leq \bar{F}_1 + \ldots + \bar{F}_{M-1} + \bar{F}_M \end{aligned} \tag{27}$$

Observe first that $\{\alpha_k\} \succ_{SSD} \{\alpha'_k\} \Rightarrow \{\alpha'_{M-k+1}\} \succ_{SSD} \{\alpha_{M-k+1}\}$, i.e., SSD holds the mirror property. Define $G_i = \alpha_M + \alpha_{M-1} + \ldots + \alpha_{i+1}$, and the survival function $\bar{G}_i = 1 - G_i = \alpha_1 + \alpha_2 + \ldots + \alpha_i$. But $G_i = \bar{F}_i$, $\bar{G}_i = F_i$, $G'_i = \bar{F}'_i$, $\bar{G}'_i = F'_i$, and substituting in Eq.26 or Eq.27 we obtain the desired result.
From Corollary 1 second order stochastic dominance preserves mean order for all means $\mathcal{M}$ defined by a concave function $g(x)$ and for the set of all increasing concave functions $\mathcal{I}^{\mathcal{CV}}$. For instance the geometric mean, or any mean with $g(x) = x^p$, $0 < p \leq 1$.

Second order stochastic dominance is also called increasing concave order, ICV. When we consider $f(x)$ a convex instead of a concave function we talk of increasing convex order, ICX. $\{\alpha_k\}$ is greater in increasing convex order than $\{\alpha'_k\}$, $\{\alpha_k\} \succ_{ICX} \{\alpha'_k\}$, if and only if the following inequalities hold:

$$\begin{aligned} F'_M &\geq F_M \\ F'_M + F'_{M-1} &\geq F_M + F_{M-1} \\ &\vdots \\ F'_M + \ldots + F'_2 &\geq F_M + \ldots + F_2 \\ F'_M + \ldots + F'_2 + F'_1 &\geq F_M + \ldots + F_2 + F_1 \end{aligned} \tag{28}$$

or equivalently,

$$\begin{aligned} \bar{F}'_M &\leq \bar{F}_M \\ \bar{F}'_M + \bar{F}'_{M-1} &\leq \bar{F}_M + \bar{F}_{M-1} \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
\bar{F}'_M + \ldots + \bar{F}'_2 &\leq \bar{F}_M + \ldots + \bar{F}_2 \\
\bar{F}'_M + \ldots + \bar{F}'_2 + \bar{F}'_1 &\leq \bar{F}_M + \ldots + \bar{F}_2 + \bar{F}_1
\end{aligned}
\tag{29}
$$

Trivially

$$\{\alpha_k\} \succ_{FSD} \{\alpha'_k\} \Rightarrow \{\alpha_k\} \succ_{ICX} \{\alpha'_k\}.$$

Mirror property is also immediate. From Corollary 1 second order stochastic dominance preserves mean order for all means $\mathcal{M}$ defined by a convex function $g(x)$ and for the set of all increasing concave functions $\mathcal{I}^{\mathcal{CX}}$. For instance any mean with $g(x) = x^p$, $p \geq 1$, as the weighted quadratic mean and, with $p < 0$, as the weighted harmonic mean.

6.3. Likelihood ratio dominance

Likelihood ratio dominance, LR, $\{\alpha_k\} \succ_{LR} \{\alpha'_k\}$, is defined as

$$\forall (i, j), 1 \leq i, j \leq M, i \leq j \Rightarrow \alpha'_i/\alpha'_j \geq \alpha_i/\alpha_j. \tag{30}$$

as $\alpha'_i/\alpha'_j \geq \alpha_i/\alpha_j \Rightarrow \alpha_j/\alpha_i \geq \alpha'_j/\alpha'_i$, we have that LR holds the mirror property.
It can be shown that condition 30 implies condition 25, [6], thus

$$\{\alpha_k\} \succ_{LR} \{\alpha'_k\} \Rightarrow \{\alpha_k\} \succ_{FSD} \{\alpha'_k\},$$

and then LR order holds Theorem 4 and Eq. 14 holds for any mean $\mathcal{M}$.

6.4. Hazard rate

A probability distribution $\{\alpha_k\}$ is greater in hazard rate, HR, to $\{\alpha'_k\}$, $\{\alpha_k\} \succ_{HR} \{\alpha'_k\}$, if and only if for all $j \geq i$ the following condition is filled

$$\frac{\bar{F'}_i}{\bar{F}_i} \geq \frac{\bar{F'}_j}{\bar{F}_j}.$$

Observe that this can be written as

$$\frac{\bar{F}_j}{\bar{F'}_j} \geq \frac{\bar{F}_i}{\bar{F'}_i},$$

or

$$\frac{1 - F_j}{1 - F'_j} \geq \frac{1 - F_i}{1 - F'_i},$$

If we consider now the sequences written in inverse order, i.e., $\alpha_{M-i+1}, \alpha'_{M-i+1}$, it is clear that if $i \leq j$ then $M-j+1 \leq M-i+1$, and from $1-\bar{F}_j = \bar{F}_{M-j+1}$, we get

$$\frac{\bar{F}_{M-j+1}}{\bar{F'}_{M-j+1}} \geq \frac{\bar{F}_{M-i+1}}{\bar{F'}_{M-i+1}},$$

which means $\{\alpha'_{M-k+1}\} \succ_{HR} \{\alpha_{M-k+1}\}$, and HR order holds the mirror property.
It can be shown that $\{\alpha_k\} \succ_{HR} \{\alpha'_k\} \Rightarrow \{\alpha_k\} \succ_{FSD} \{\alpha'_k\}$ and then HR order holds Theorem 4 and Eq. 14 holds for any mean $\mathcal{M}$.

7. Example: Linear combination of Monte Carlo techniques

Let us estimate an (intractable) integral $\mu = \int h(x)dx$ using importance sampling [12]. There are n techniques (also called proposals) where sampling from them is possible. Each technique has an associated pdf $\{p_k(x)\}$, $1 \leq k \leq n$, and a related primary estimator $\{I_{k,1} = \frac{h(x)}{p_k(x)}\}$. If for every x where $h(x) > 0$ we have that $p_k(x) > 0$, then the technique is unbiased. We are interested in optimal ways of combining the techniques. One option is the linear combination of the different estimators $\{I_{i,n_i}\}$, $\sum_{i=1}^{M} w_i I_{i,n_i}$, with weights $\{w_k\}$ and sampling proportions $\{n_i = \alpha_i N\}$, with $N = \sum_{i=1}^{M} n_i$. If all techniques are unbiased the resulting combination is also unbiased. The variance is given by

$$V_N = \sum_{i=1}^{M} w_i^2 \frac{v_i}{n_i} = \frac{1}{N} \sum_{i=1}^{M} w_i^2 \frac{v_i}{\alpha_i} = \frac{1}{N} V_1, \tag{31}$$

where v_i are the variances for the primary estimators of each technique.

The optimal combination of weights, i.e. the ones that lead to minimum variance V_1, has been studied in [13], [14], [15]. The variance of the combination depends on two sets of weights, $\{w_i\}$ and $\{\alpha_i\}$, but there are cases where it can be reduced to a single set. Several examples can be found:

- when $\alpha_i = w_i$, then $V_1 = \sum_{i=1}^{M} \alpha_i v_i$, the weighted arithmetic mean of $\{v_i\}$.
- when the sampling proportions are fixed, the optimal variance is given by $\mathcal{H}(\{\frac{v_k}{\alpha_k}\})$, the weighted harmonic mean of $\{v_i\}$.

- when weights $\{w_i\}$ are fixed, the optimal variance is given by $(\sum_{k=1}^{M} w_k \sqrt{v_k})^2$, which is the weighted power mean with exponent $r = 1/2$.

Observe that the three cases correspond to quasi-arithmetic means. Considering the sequences $\alpha'_k = q(v_k)$, $q(x)$ decreasing, and $\alpha_k = 1/M$, we have that $\{\alpha_k\} \succ_{LR} \{\alpha'_k\}$ and thus in all cases the variance is less when taking sampling proportions or coefficients decreasing in v_k than for equal sampling or equal weighting. See [15] for more details.

7.1. Liabilities vs utilities

In many problems in economics, it can be argued that the goal is the maximization of the utilities instead of the minimization of the liabilities. However, a liability can be always considered as the inverse of an utility; Theorem 1 shows that establishing invariance properties on the order between means of utilities, i.e., increasing $f(x)$, is equivalent to doing the same with the liabilities with inverted order, i.e., decreasing $f(x)$. As an example, if we consider the variance V_1 in the Monte Carlo problem in Section 7 as a liability, we can define the efficiency $E_1 = 1/V_1$ as a utility, and obtain it as a weighted mean of individual techniques efficiencies, $e_i = 1/v_i$. Consider the first case above in Section 7, where $V_1 = \sum_{i=1}^{M} \alpha_i v_i$. Then,

$$E_1 = \frac{1}{V_1} = \frac{1}{\sum_{i=1}^{M} \alpha_i v_i} = \frac{1}{\sum_{i=1}^{M} \alpha_i e_i^{-1}} = \mathcal{H}(\{\frac{e_i}{\alpha_i}\}),$$

where $\mathcal{H}$ denotes weighted harmonic.
Considering now the second case, we obtain

$$E_1 = \frac{1}{\mathcal{H}(\{\frac{v_k}{\alpha_k}\})} = \sum_{i=1}^{M} \alpha_i v_i^{-1} = \sum_{i=1}^{M} \alpha_i e_i.$$

Finally, the third case yields

$$E_1 = \frac{1}{(\sum_{k=1}^{M} w_k \sqrt{v_k})^2} = (\sum_{k=1}^{M} w_k e_k^{-1/2})^{-2},$$

which is the weighted power mean with $r = -1/2$. Symmetrically to the variances, taking now the sequences $\alpha'_k = p(e_k)$, $p(x)$ increasing, and $\alpha_k = 1/M$, we have that $\{\alpha'_k\} \succ_{LR} \{\alpha_k\}$. Thus, in all cases, the resulting efficiency is higher when taking sampling proportions or coefficients increasing in e_k instead for equal sampling or equal weighting.

8. Related work

Yuji Yoshida has pioneered the study of stochastic orders related to the invariance of quasi-arithmetic mean. We thus discuss here the relationships between this paper and his work in [4], [5]. Yuji Yoshida has considered invariance of continuous generalized or quasi-arithmetic weighted means of utility functions. The quasi-arithmetic mean has been defined using as its defining function (i.e., the $g(x)$ function in definition 2) the same utility function. Several results are obtained, for instance in Theorem 3.1 in [4] it is shown that a necessary condition for a weighted mean on continuous and derivable weights v to be greater than the weighted mean on continuous and derivable weights w is that $w'/w \leq v'/v$, where w', v' are the derivatives of w, v, respectively. A necessary and sufficient condition for $w'/w \leq v'/v$ is derived in Theorem 3.2 in [4] or Lemma 2 in [5]. Let us now present the main differences to our work (apart from the fact that we work with discrete distributions and in [4], [5] with continuous ones) are:

- In [4], [5], the function that defines the quasi-arithmetic mean, i.e. the $g(x)$ function in definition 2, is the same as the utility function used. Observe that, on the one hand, this restricts the utility functions to being *strictly* increasing, assumption not necessary in our framework. On the other hand, as the utility function is always an increasing function, this implies that means such as harmonic mean, which correspond to decreasing function $1/x$, are not considered for invariance. As a consequence, the symmetry between utility and cost, discussed in section 7.1, can not be brought forward.

- Only likelihood-ratio order is considered (implicitly) in [4], [5]. This is, the condition for mean invariance is $w'/w \leq v'/v$, where w and v are weighting functions. This can be easily shown to be equivalent to likelihood-ratio dominance, section 6.3. Likelihood-ratio dominance implies first order stochastic dominance, but not the reverse (see for instance Proposition 4.2 in [4] that states that $w'/w \leq v'/v$ implies first order stochastic dominance). Thus Yoshida considers in Theorem 3.2 in [4] or Lemma 2 in [5] a necessary and sufficient condition for likelihood ratio, while we consider in the present paper the, more general, necessary and sufficient condition for first order stochastic dominance.

9. Conclusions and future work

We have presented in this paper the relationship between stochastic orders and quasi-arithmetic means. We have proved several ordering invariance theorems, that show that given two distributions under a certain stochastic order, the ordering of the means is preserved for any quasi-arithmetic mean we might consider, this is, not only for arithmetic mean (or expected value). We have showed how the results apply to first order, second order, likelihood ratio, hazard-rate, and increasing convex stochastic orders. We have presented an application example based on the linear combination of Monte Carlo estimators, and shown that the invariance allows to consider costs or liabilities as the symmetric case of utilities. We have established too the relationship of our results with previous work.

In the future we want to generalize our results to spatial weight matrices [16]. The rows in a spatial weight matrix are weights, that give the influence of n entities over each other. Different weighted means as arithmetic, harmonic, or geometric [17] can be used to compute this influence. We can thus apply to each row our invariance results.

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Appendix. Proof of Lemma 1

We present here the proof of Lemma1.

Proof. Subtracting 1 from both sides of each inequality proves $c) \Leftrightarrow d)$. To prove $a) \Rightarrow c)$ we proceed in the following way. Consider the increasing sequence $\{g(f(1)), \ldots, g(f(1)), g(f(M)), \ldots, g(f(M))\}$, $f(1) < f(M)$ (and thus $g(f(1)) < g(f(M))$ by the strict monotonicity of $g(x)$), and where $g(f(1))$ is written l times, denote $\mathbf{L} = a_1 + \ldots + a_l$, $\mathbf{L'} = a'_1 + \ldots + a'_l$. Since $a_{l+1} + \ldots + a_M = 1 - \mathbf{L}$, $a'_{l+1} + \ldots + a'_M = 1 - \mathbf{L'}$ then a) gives

$$\mathbf{L'}g(f(1)) + (1 - \mathbf{L'})g(f(M)) - \mathbf{L}g(f(1)) - (1 - \mathbf{L})g(f(M)) \leq 0,$$

i.e.,

$$(\mathbf{L'} - \mathbf{L})(g(f(1)) - g(f(M))) \leq 0 \Rightarrow \mathbf{L'} \geq \mathbf{L}.$$

This proves the first $M-1$ inequalities. Observe now that

$$\mathbf{L'} \geq \mathbf{L} \Rightarrow \mathbf{1\text{-}L} \geq \mathbf{1\text{-}L'}$$

and this accounts for the last $M-1$ inequalities, thus $a) \Rightarrow d)$.
To prove that $b)$ implies $c)$ and $d)$, consider the sequence,

$$\{g(f(1)), \ldots, g(f(1)), g(f(M)), \ldots, g(f(M))\},$$

$f(1) < f(M)$, $g(f(M))$ is written l times, and the same definitions as before for $\mathbf{L}, \mathbf{L'}$, then $b)$ gives

$$(1-\mathbf{L})g(f(1)) + \mathbf{L}g(f(M)) - (1-\mathbf{L'})g(f(1)) - \mathbf{L'}g(f(M)) \leq 0,$$

and we proceed as above. Thus $a) \Rightarrow c)$. Let us see now that $c)$ implies $a)$.

Define for $1 \leq M$, $A_k = \sum_{j=1}^{k} \alpha_k$, $A'_k = \sum_{j=1}^{k} \alpha'_k$, and $A_0 = A'_0 = 0$. Then

$$\sum_{k=1}^{M} \alpha_k g(f(k)) - \sum_{k=1}^{M} \alpha'_k g(f(k)) = \sum_{k=1}^{M} (\alpha_k - \alpha'_k) g(f(k)) \tag{32}$$

$$= \sum_{k=1}^{M} (A_k - A_{k-1} - A'_k + A'_{k-1}) g(f(k))$$

$$= \sum_{k=1}^{M} (A_k - A'_k) g(f(k)) - \sum_{k=1}^{M} (A_{k-1} - A'_{k-1}) g(f(k))$$

$$= \sum_{k=1}^{M-1} (A_k - A'_k) g(f(k)) - \sum_{k=0}^{M-1} (A_k - A'_k) g(f(k+1))$$

$$= \sum_{k=1}^{M-1} (A_k - A'_k) g(f(k)) - \sum_{k=1}^{M-1} (A_k - A'_k) g(f(k+1))$$

$$= \sum_{k=1}^{M-1} (A_k - A'_k)(g(f(k)) - g(f(k+1))) \geq 0,$$

as $c)$ implies that for all k, $A'_k - A_k \geq 0$, and $\{g(f(k))\}$ is an increasing sequence. Thus $c) \Rightarrow a)$.
Repeating the proof for the sequences $\{\alpha_{M-k+1}\}$ and $\{\alpha'_{M-k+1}\}$ we obtain that $d) \Rightarrow b)$.

Let now be $g(x)$ decreasing. Observe that

$$\sum_{k=1}^{M} \alpha_{M-k+1} g(f(k)) = \sum_{k=1}^{M} \alpha_{M-k+1} g(f(m(M-k+1))) = \sum_{k=1}^{M} \alpha_k g(f(m(k))),$$

where $m(k) = M - k + 1$, and thus $a) \Rightarrow b')$ because $\{g(f(m(k)))\}$ is an increasing sequence. Now suppose $g(x)$ is increasing, from

$$\sum_{k=1}^{M} \alpha_k g(f(k)) = \sum_{k=1}^{M} \alpha_{M-k+1} g(f(M-k+1)) = \sum_{k=1}^{M} \alpha_{M-k+1} g(f(m(k)))$$

we have that $b') \Rightarrow a)$ because $\{g(f(m(k)))\}$ is a decreasing sequence. We can show similarly that $b) \Leftrightarrow a')$.
Consider now $f(x)$ decreasing. Reversed $a)$ is

$$\sum_{k=1}^{M} \alpha_k g(f(k)) \leq \sum_{k=1}^{M} \alpha'_k g(f(k)) \tag{33}$$

but it can be written as

$$\sum_{k=1}^{M} \alpha_{M-k+1} g(f(M-k+1)) \leq \sum_{k=1}^{M} \alpha'_{M-k+1} g(f(M-k+1)) \tag{34}$$

$$= \sum_{k=1}^{M} \alpha_{M-k+1} g(f(m(k))) \leq \sum_{k=1}^{M} \alpha'_{M-k+1} g(f(m(k))),$$

where $\{g(f(m(k)))\}$ is an increasing sequence, and thus $c) \Rightarrow a)$ when $f(x)$ is decreasing and for order of inequality in $a)$ reversed. The other cases for $f(x)$ decreasing can be analogously proven. □

Appendix. Proof of Lemma 2

Proof. Be $g(x)$ the strictly monotonic function associated with mean $\mathcal{M}$. Consider first $g(x)$ convex and increasing, $g^{-1}(x)$ is concave and increasing, $f(x)$ concave and increasing. Supposing Eq. 19 holds,

$$\sum_{k=1}^{M} \alpha'_k g(f(k)) \leq \sum_{k=1}^{M} \alpha_k g(f(k)), \tag{35}$$

Suppose we have another mean with concave function $g^{\star}(x)$. Suppose first $g^{\star}(x)$ is increasing. We have to show that

$$\sum_{k=1}^{M} \alpha'_k g^{\star}(f(k)) \leq \sum_{k=1}^{M} \alpha_k g^{\star}(f(k)), \tag{36}$$

as $g^{\star-1}(x)$ is also increasing. But Eq. 36 can be written as

$$\sum_{k=1}^{M} \alpha'_k g(g^{-1}(g^{\star}(f(k)))) \leq \sum_{k=1}^{M} \alpha_k g(g^{-1}(g^{\star}(f(k)))), \tag{37}$$

which holds because $g^{-1}(g^{\star}(f(x)))$ is concave and increasing.

Consider now $g^{\star}(x)$ is decreasing. As $g^{\star-1}(x)$ is also decreasing, we have to show that

$$\sum_{k=1}^{M} \alpha'_k g^{\star}(f(k)) \geq \sum_{k=1}^{M} \alpha_k g^{\star}(f(k)). \tag{38}$$

Eq. 38 can be written as

$$\sum_{k=1}^{M} \alpha'_{M-k+1} g^{\star}(f(m(k))) \geq \sum_{k=1}^{M} \alpha_{M-k+1} g^{\star}(f(m(k))), \tag{39}$$

where $m(k) = M - k + 1$, or

$$\sum_{k=1}^{M} \alpha'_{M-k+1} g(g^{-1}(g^{\star}(f(m(k))))) \geq \sum_{k=1}^{M} \alpha_{M-k+1} g(g^{-1}(g^{\star}(f(m(k))))), \tag{40}$$

which holds because $g^{-1}(g^{\star}(f(m(x))))$ is concave and increasing function, as $m(x)$ is both convex and concave.

The case when $g(x)$ is decreasing is dealt with analogously. For $g(x)$ concave and $f(x)$ convex and increasing we can proceed as above observing that for $g^{\star}(x)$ convex, $g^{-1}(g^{\star}(f(x)))$ and $g^{-1}(g^{\star}(f(m(x))))$ are convex and increasing. □

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