Regularity theory and singularity analysis for certain geometric partial differential equations

Florian Georg Litzinger

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School of Mathematical Sciences
Queen Mary University of London
United Kingdom

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Abstract

This work considers problems pertaining to the regularity theory and the analysis of singularities of geometric partial differential equations that stem from the theory of isometric immersions and geometric flows.

In the first of two largely independent parts, we employ the Uhlenbeck–Rivière theory of Coulomb gauges to prove that a Pfaffian system with coefficients in the critical space L^2_{loc} on a simply connected open subset of $\mathbb{R}^2$ has a non-trivial solution in the Sobolev space $W^{1,2}_{\mathrm{loc}}$ if the coefficients are antisymmetric and satisfy a compatibility condition. As an application of this result, we show that the fundamental theorem of surface theory holds for prescribed first and second fundamental forms of optimal regularity in the classes $W^{1,2}_{\mathrm{loc}}$ and L^2_{loc}, respectively, that satisfy a compatibility condition equivalent to the Gauss–Codazzi–Mainardi equations. Finally, we give a weak compactness theorem for surface immersions in the class $W^{2,2}_{\mathrm{loc}}$.

The second part of this work is concerned with the analysis of singularities of the curve shortening and mean curvature flows. In particular, we show a cylindrical estimate for the mean curvature flow of k-convex hypersurfaces, extending estimates that had previously been introduced in the context of Huisken–Sinestrari's surgery procedure for 2-convex flows. Furthermore, we consider curve shortening flow of arbitrary codimension in an Euclidean background. For type-II singularities, we prove the existence of a sequence of space-time points along which the curvature tends to infinity such that a rescaling of the solution along it converges to the Grim Reaper solution, paralleling Altschuler's work in the case of space curves. Finally, we demonstrate that the curve shortening flow of initial curves with an entropy bound converges to a round point in finite time.

To my parents

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Chapter 1

Introduction

Ever since the development of modern differential geometry, the worlds of analysis and geometry have been firmly intertwined. Indeed, the possibilities of either treating a geometric problem with the machinery of contemporary analysis or gaining valuable insight into the analysis of partial differential equations by geometric analogies have proven incredibly fruitful. It is thus no surprise that despite being one of the most active areas of the mathematics of today, new and interesting problems continue to surface in geometric analysis.

A common challenge in this interdisciplinary field is to deal with non-smooth objects. While many applications, in mathematics itself and further afield, require the study of geometric, often non-linear, partial differential equations allowing functions outside of the realm of classical differentiability, geometric flows are likely to produce objects for which some geometric quantities tend to infinity, thus breaking down their description in terms of differential geometry. Nevertheless, in order to obtain meaningful statements, one must endeavour to salvage as much as possible of what can still be said. Therefore, the regularity theory for solutions of partial differential equations and the analysis of singularities of evolving geometric objects remain at the forefront of research in geometric analysis.

The isometric immersion problem lies firmly at the core of Riemannian geometry; as the availability of an isometric immersion of any Riemannian manifold into an Euclidean space of some dimension instantly makes a breadth of knowledge about the arguably simplest manifold, Euclidean space, available for the study of more complicated objects. For example, in the case of immersions of two-dimensional surfaces into three-space, the fundamental theorem of surface theory, that is, the existence of an immersion of a surface with prescribed metric and second fundamental form, is of great importance in the theory of non-linear elasticity. In this context, the functions under consideration are naturally merely weakly differentiable.

Meanwhile, the general idea of using geometric heat flow to deform an object into a canonical one can be used to obtain topological consequences to the greatest effect, as is nowadays well-known. Even though such a flow often exhibits a certain smoothing behaviour common to heat-type flows, geometric singularities are likely to occur due to its non-linear nature. However, thorough analysis and classification of the kinds of singularities that can possibly develop might nevertheless still allow making *a posteriori* statements about the initial object. A successful approach are geometric surgery procedures, which enable the continuation of the flow past a singularity while keeping track of the changes in its topology.

1.1 Historical overview

In this section, we give a short historical introduction to both sets of problems that are central to this work.

Isometric immersions

The *isometric immersion problem* is a classical problem of differential geometry [And02; HH06]. Given a smooth manifold (M^n, g), it is a natural question whether M can be isometrically immersed (or even embedded) into Euclidean space $\mathbb{R}^N$ for some dimension N. A related problem is to determine the minimal dimension N for which this is possible.

Naturally, these problems are at the heart of geometric analysis in the sense that they are geometric problems that lend themselves to a treatment by means of analysis.

In this framework, the problem is the following: Find a smooth immersion $u : M^n \to \mathbb{R}^N$ such that

$$\mathrm{d}u^2 = g,$$

that is, $u : (M^n, g) \to u(M)$ is an isometry, where $u(M)$ is equipped with the induced (Euclidean) metric. Once a system of local coordinates in some neighbourhood U is given such that

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

we need to find a local immersion $u : U \to \mathbb{R}^N$ such that

$$\sum_{k=1}^{N} \partial_i u_k \partial_j u_k = g_{ij}$$

in U for $1 \leq i, j \leq n$. Since the number of first-order partial differential equations in this non-linear system is $s_n = \frac{n(n+1)}{2}$, we must have $N \geq s_n$.

Indeed, Schläfli [Sch71] conjectured that for every smooth manifold (M^n, g) of dimension n, there exists a smooth local isometric immersion into Euclidean space $\mathbb{R}^N$ of dimension $N = s_n$, which was proved for analytic manifolds by Janet [Jan26; see also Bur31] for $n = 2$, using the Cauchy–Kovalevskaya theorem, and by Cartan [Car27] for general n using his work on exterior differential systems. (The number s_n is called *Janet dimension*.) In the smooth domain, the main difficulty lies in substituting the Cauchy–Kovalevskaya theorem [Gro86].

The global embedding problem, without requiring isometry, was first considered by Whitney [Whi44a; Whi44b], who showed that any compact manifold M^n of dimension n can be embedded into $\mathbb{R}^{2n}$ and immersed into $\mathbb{R}^{2n-1}$. The most celebrated result on isometric immersions as a whole is certainly the Nash embedding theorem: Any smooth, compact, n-dimensional Riemannian manifold can be smoothly isometrically embedded into $\mathbb{R}^N$ for some dimension N [Nas54; cf. Kui55; Nas56, for the C^1-problem]. The precise number N has been subject to improvements due to Gromov [Gro86] and then Günther [Gün91], who achieved $N = \max\{s_n + 2n, s_n + n + 5\}$. The importance of Nash's work goes beyond its result in that he pioneered a novel technique that has come into widespread use and is now known as Nash–Moser iteration. Interestingly, Günther was able to circumvent the problem that required its use altogether in his improvement of the embedding dimension N.

In the case of surfaces, i. e., in dimension $n = 2$, it is well-known that a necessary condition for the existence of an isometric immersion of $(M, g) \to \mathbb{R}^3$ is that the second fundamental form $A = (h_{ij})$ satisfies the *Gauss–Codazzi–Mainardi equations*,

$$\begin{aligned} &R_{ijk\ell} = h_{ik}h_{j\ell} - h_{i\ell}h_{jk} && \text{(Gauss)}, \\ &\nabla_i h_{jk} = \nabla_j h_{ik} && \text{(Codazzi–Mainardi)}, \end{aligned}$$

where R denotes the Riemannian curvature tensor and ∇ is the covariant derivative induced by the Levi–Civita connection. Assuming that M is simply connected, the Gauss–Codazzi–Mainardi equations are also a sufficient condition for an isometric immersion to exist. In that sense, the equations can be seen as compatibility equations for the isometric immersion problem in this case. The first proof of this fact was given by Bonnet [Bon67] (see Tenenblat [Ten71] for a similar result in higher dimension and codimension, which introduces an additional condition related to the induced connection on the normal bundle). The statement that an immersion of a surface with prescribed first and second fundamental forms exists and is uniquely determined if the prescribed forms satisfy a set of compatibility conditions is now known as Bonnet's theorem or *fundamental theorem of surface theory*.

The proof of the fundamental theorem of surface theory relies on the construction of a suitable frame on the surface that is orthonormal with respect to the prescribed metric. Analytically, this amounts to the solution of a *Pfaffian system* of partial differential equations of the form

$$\nabla P + \Omega P = 0$$

for the unknown matrix-valued function P given a matrix-valued 1-form Ω of coefficients satisfying a compatibility condition of the form

$$\mathrm{d}\Omega + \Omega \wedge \Omega = 0.$$

In the smooth case, supposing for the moment that P be invertible, the fact that second derivatives of P commute implies that the compatibility conditions are necessary.

In the literature, one then finds several iterations of the following *proto-theorem* for Pfaffian systems of any dimension:

Theorem (Existence and uniqueness for Pfaffian systems)**.** *Let $U \subset \mathbb{R}^n$ be a simply connected domain. Suppose that $\Omega \in Y(U, L(m) \otimes \wedge^1\mathbb{R}^n)$ satisfies the compatibility condition*

$$\mathrm{d}\Omega + \Omega \wedge \Omega = 0$$

in an appropriate sense. Then there exists $P \in X(U, K(m))$ such that

$$\nabla P + \Omega P = 0.$$

Moreover, P is unique up to a constant factor in $K(m)$.

In order to assess the validity of the above, the function spaces X and Y as well as the matrix subgroups $K(m), L(m) \subset \mathrm{gl}(m)$ need to be specified. Classically, going back to the work of Cartan [Car27; Car83] and Thomas [Tho34], the theorem is known to hold for $X = C^2$, $Y = C^1$, and $K(m) = L(m) = \mathrm{gl}(m)$. In the two-dimensional case, $n = 2$, Hartman and Wintner [HW50b] proved it for $X = C^1$, $Y = C^0$, considering the compatibility equation in an integrated sense, while S. Mardare, in a series of papers [Mar03b; Mar05; Mar07], improved it to $X = W^{1,\infty}$, $Y = L^\infty$ and finally $X = W^{1,p}$,

$Y = L^p$, $p > n = 2$. (In higher dimensions, see C. Mardare [Mar03a], S. Mardare [Mar04], and Szopos [Szo08].) Indeed, without any further assumptions on the coefficients Ω, this is optimal [Mar05].

Further recent developments on the isometric immersion problem for surfaces and applications, particularly in non-linear elasticity, include the work by Ciarlet and coworkers [CL02; Cia03; CM05; CGM08; Cia13; CM16; CM19a], Chen–Li [CL18], and Li [Li19; Li20]. We should also mention the related subject of immersions with L^2-bounded second fundamental form, which is particularly relevant in the context of Willmore surfaces [Lan85; Tor94; Riv14; Bre15; Riv16; LR18].

Curve shortening and mean curvature flow

The *mean curvature flow* and its one-dimensional analogue, the *curve shortening flow*, are the most studied examples of extrinsic geometric flows [ACGL20]. Brakke's influential work [Bra78] was the first to mathematically consider and thoroughly analyse mean curvature flow from the viewpoint of geometric measure theory. Since then, mean curvature flow has enjoyed continued attention from many geometers and analysts alike in the hope of obtaining profound topological results through it. Both flows evolve a geometric object (a curve or a hypersurface) in normal direction with speed proportional to the (mean) curvature.

For curve shortening flow of embedded curves in the plane, Gage and Hamilton [Gag84; GH86] showed that convex curves eventually become circular and shrink to a point. In particular, the flow stays smooth throughout the evolution until the curvature tends to infinity at the final time. Grayson [Gra87; Gra89b; see also Hui98] showed that any embedded curve continues to be embedded and eventually becomes convex without developing singularities in the process, thus completing the analysis of the flow of embedded curves in the plane. Immersed curves, on the other hand, can exhibit much more complex behaviour [Gra89b]. A particular type of solutions, namely those that move self-similarly under the flow, has been classified [AL86; Hal12].

One of the most prominent applications of curve shortening flow, following an idea of Uhlenbeck, is a proof of the theorem of the three geodesics [Gra89a] which states that any 2-sphere with a smooth metric admits at least three simple closed geodesics. Other applications even include Perelman's implementation of a surgery procedure for Ricci flow [Per03].

In higher dimension, Huisken [Hui84; see also Hui86] proved a result analogous to the Gage–Hamilton theorem, namely, that convex hypersurfaces converge to a *round point* under mean curvature flow, that is, a rescaling of the solution eventually produces the round sphere. Moreover, he introduced his influential monotonicity formula [Hui90]. In the following decades, the body of literature on various aspects of mean curvature flow, such as the analysis of singularities, special solutions, topological applications, and related flows, has grown immensely. For an extensive overview of both mean curvature and curve shortening flow, we refer to Ecker's book [Eck04], lectures by Mantegazza [Man11], White [Whi15], Haslhofer [Has16], and Schulze [Sch17] as well as the comprehensive book by Andrews et al. [ACGL20].

Since the formation of singularities of the flow, that is, points where the differential geometric description breaks down, is an inevitable phenomenon, several approaches have been developed to define an extension of mean curvature flow past the first singular time, including Brakke [Bra78] in the GMT sense; Chen–Giga–Goto [CGG91], Evans–Spruck [ES91], and Ilmanen [Ilm92] by considering a flow of level sets; and Huisken–Sinestrari [HS09], Brendle–Huisken [BH16], Haslhofer–Kleiner [HK17], and Mramor–Wang [MW21] by defining a mean curvature flow with surgery.

Recently, both curve shortening and mean curvature flow have been increasingly considered in codimensions greater than one. While Altschuler had studied singularities of space curves [Alt91] and Altschuler–Grayson defined a flow through singularities for planar curves by means of a special class of space curves [AG92] in the early 1990s, more results concerning also the flow of curves immersed in arbitrarily dimensional spaces have begun to appear [YJ05; MC07; He12; AAAW13; Hät15; Kha15; Cor16; MB20]. In mean curvature flow, Ambrosio and Soner [AS97] developed an approach in arbitrary codimension using a varifold ansatz, while the more traditional submanifold approach has been followed as well [Smo05; Bak10; Coo11; see also Wan08; Smo12]. Most notably, a notion of mean curvature flow with surgery in any codimension has been introduced by Nguyen [Ngu20].

More and more attention has also been paid to the study of mean curvature flow with entropy bounds. The entropy is a quantity that is monotone non-increasing under the flow and is particularly attractive in higher codimension in order to cope with new technical difficulties that arise when the codimension is greater than one, such as the lack of a maximum principle and the resulting fact that an initially embedded surface need not stay embedded. Due to the monotonicity, upper bounds that are imposed on the entropy of the initial surface will continue to hold throughout the flow.

In particular, entropy has been employed in the study of generic singularities of mean curvature flow by Colding–Minicozzi [CM12] and Chodosh et al. [CCMS20; CCMS21] and in Bernstein–Wang's low-entropy Schoenflies theorem [BW20]. Colding et al. showed that the round sphere minimises entropy among closed self-shrinkers [CIMW13], while Bernstein–Wang [BW16; BW17; see also BW18b] and Ketover–Zhou [KZ18] proved the same statement for closed embedded surfaces in $\mathbb{R}^3$. An extension to higher dimensions is due to Zhu [Zhu20]. Hershkovits–White proved sharp entropy bounds for self-shrinkers of any dimension [HW19]. Bernstein–Wang [BW18a] and S. Wang [Wan20] proved the Hausdorff stability of round spheres under small perturbations of the entropy. Moreover, in any codimension, Colding and Minicozzi [CM19b] gave uniform bounds on the entropy and codimension of generic singularities.

1.2 Main results

We now give a short summary of the main results presented herein.

Pfaffian systems and the fundamental theorem of surface theory

The original results in the first part of the thesis previously appeared in JGA [Lit21].

In an open set $U \subset \mathbb{R}^2$, we consider a Pfaffian system of the form

$$\nabla P = P\Omega, \tag{1.1}$$

where P is a matrix-valued function and Ω is a given matrix-valued 1-form. Local existence of a non-trivial solution P to this partial differential equation, and its regularity, manifestly depend on the regularity properties of the coefficients. It is a classical result that a twice continuously differentiable solution exists if every component Ω_i is continuously differentiable and they satisfy the compatibility condition

$$\partial_i \Omega_j - \partial_j \Omega_i = \Omega_j \Omega_i - \Omega_i \Omega_j. \tag{1.2}$$

In this work, we show the corresponding result for solutions $P \in W^{1,2}_{\text{loc}}$ and coefficients $\Omega \in L^2$ satisfying an additional structural assumption. This is the case of least possible regularity for an equation such as (1.2) to make sense in an integrated form. We then have the following

Theorem. *Let $U \subset \mathbb{R}^2$ be a connected and simply connected open set and let $\Omega \in L^2(U, \mathrm{so}(m) \otimes \wedge^1 \mathbb{R}^2)$ satisfy the compatibility condition (1.2) in the distributional sense. Then there exists $P \in W^{1,2}_{\text{loc}}(U, \mathrm{SO}(m))$ such that $\nabla P = P\Omega$ in U. Moreover, any two such solutions P_0, P_1 are related by $P_0 = CP_1$ with a constant $C \in \mathrm{SO}(m)$.*

Over the years, there have been several incremental improvements to the classical theory. In particular, Hartman and Wintner [HW50b] showed that the above existence result holds if the given form Ω is continuous, with a continuously differentiable solution P. Following this, Mardare [Mar03b; Mar05] first showed the existence of a solution P to (1.1) in the Sobolev class $W^{1,\infty}_{\text{loc}}$ for locally essentially bounded coefficients and later improved the theorem to hold in the class $W^{1,p}_{\text{loc}}$ for $\Omega \in L^p_{\text{loc}}$, where $p > 2$. It is important to note that without any further structural assumptions on the coefficients Ω, this result has been demonstrated to be optimal [Mar05]. However, once one supposes that the components of the matrix-valued 1-form Ω be *antisymmetric*, it is possible to improve the regularity to the critical case above.

Meanwhile, there have been developments in the theory of non-linear PDE that attempt to exploit a particular structure of the equation in order to gain additional regularity of the solution beyond what would usually be expected; and these compensated compactness methods [CLMS93; Riv07; Wen69] have been markedly successful in that regard. In particular, in his 2007 paper, Rivière [Riv07] provided a proof of the regularity of two-dimensional weakly harmonic maps, from which we recall an important intermediate result:

Lemma (Uhlenbeck–Rivière decomposition [Riv07, Lemma A.3; Sch10])**.** *Let $U \subset \mathbb{R}^2$ be a contractible bounded regular domain and let $\Omega \in L^2(U, \mathrm{so}(m) \otimes \wedge^1\mathbb{R}^2)$. Then there exist $\xi \in W_0^{1,2}(U, \mathrm{so}(m))$ and $P \in W^{1,2}(U, \mathrm{SO}(m))$ such that*

$$P^{-1}\nabla P + P^{-1}\Omega P = \nabla^{\perp}\xi,$$

$$\|\nabla\xi\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \leq 5\,\|\Omega\|_{L^2}^2 .$$

Thanks to the Riemann mapping theorem, this also holds true if $U \subset \mathbb{R}^2$ is an open, connected, and simply connected bounded set with sufficiently smooth boundary. While the techniques employed in the original proof [Riv07] are quite involved, Schikorra [Sch10] gave an alternative proof using variational methods, which in addition removes the need for a smallness condition on Ω.

The above result is of particular interest to us because the given form Ω is only assumed to be square-integrable. In order to achieve existence and regularity of the solution $P \in W^{1,2}$, the additional structure assumed, that is, the antisymmetry of each Ω_i, is utilised in a crucial way. In the same vein, it is this additional structural assumption that enables us to employ Rivière's lemma to extend the previous results on the solvability of the above Pfaffian system in (1.1) to the critical $p = 2$ case.

The possibility of finding a solution to this Pfaffian system, in turn, has been an essential ingredient in the proof of weak versions of the fundamental theorem of surface theory. As for Pfaffian systems, there have been incremental improvements to this classical geometric result. The theorem answers the question of whether it is possible to find an immersion of a surface in three-dimensional space with prescribed first and second fundamental forms—this turns out to be true if, and only if, the fundamental forms satisfy the Gauss–Codazzi–Mainardi equations. We obtain the following

Theorem. *Let U be a connected and simply connected open subset of $\mathbb{R}^2$ and let $(a_{ij}) \in W^{1,2}_{\text{loc}}(U, \text{Sym}^+(2)) \cap L^\infty_{\text{loc}}(U, \text{Sym}^+(2))$ and $(b_{ij}) \in L^2_{\text{loc}}(U, \text{Sym}(2))$ be given. Suppose that the eigenvalues of (a_{ij}) are locally uniformly bounded from below and that the matrix fields (a_{ij}), (b_{ij}) are such that*

$$\partial_1\Omega_2 - \partial_2\Omega_1 = \Omega_2\Omega_1 - \Omega_1\Omega_2,$$

where $\Omega \in L^2_{\text{loc}}(U, \text{so}(3) \otimes \wedge^1\mathbb{R}^2)$ is given by the following sequence of definitions, see also Section 4.4.2:

$$\begin{aligned}
(a^{ij}) &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \\
b_i^j &= a^{jk} b_{ik}, \\
\Gamma_{ij}^k &= \frac{1}{2} a^{k\ell}(\partial_j a_{i\ell} + \partial_i a_{j\ell} - \partial_\ell a_{ij}), \\
G &= \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\frac{1}{2}},
\end{aligned}$$

$$\Gamma_i = \begin{pmatrix} \Gamma_{i1}^1 & \Gamma_{i2}^1 & -b_i^1 \\ \Gamma_{i1}^2 & \Gamma_{i2}^2 & -b_i^2 \\ b_{i1} & b_{i2} & 0 \end{pmatrix},$$

$$\Omega_i = (G\Gamma_i - \partial_i G)G^{-1}.$$

Then there exists an immersion $\theta \in W^{2,2}_{\mathrm{loc}}(U, \mathbb{R}^3)$ such that

$$\begin{aligned} a_{ij} &= \partial_i\theta \cdot \partial_j\theta && \text{in } W^{1,2}_{\mathrm{loc}}(U), \\ b_{ij} &= \partial_{ij}\theta \cdot \frac{\partial_1\theta \times \partial_2\theta}{|\partial_1\theta \times \partial_2\theta|} && \text{in } L^2_{\mathrm{loc}}(U). \end{aligned}$$

Moreover, the map θ is unique in $W^{2,2}_{\mathrm{loc}}(U, \mathbb{R}^3)$ up to proper isometries of $\mathbb{R}^3$.

We remark that the compatibility condition assumed in the theorem is in fact equivalent to the Gauss–Codazzi–Mainardi equations, see Proposition 4.27. As for the Pfaffian system mentioned above, one needs to consider the compatibility equations in the distributional sense.

In the works mentioned above [HW50b; Mar03b; Mar05], the fundamental theorem of surface theory has been extended to hold true for, finally, first and second fundamental forms in the classes $W^{1,p}_{\mathrm{loc}}$ and L^p_{loc}, respectively, where $p > 2$. The method of proof, whose lines we also follow in this work, is the following: First, a Pfaffian system as in (1.1) is solved for a proper orthogonal matrix field P, and then the sought-after surface immersion is found by means of a weak version of the Poincaré lemma, solving the equation $\nabla\theta = PG$, where G is the matrix square root of the three-dimensional extension of the given metric. Since the Poincaré lemma is known to hold for all $p \geq 1$ (see Lemma 4.22), the premier challenge in extending the fundamental theorem of surface theory to the critical exponent $p = 2$ lies in the extension of the corresponding existence theorem on Pfaffian systems.

Therefore, in order to be able to apply our optimal regularity theorem, an appropriate *antisymmetric* matrix-valued 1-form Ω of coefficients of the Pfaffian system has to be constructed as above. While the connection form Γ does not possess this property in an arbitrary frame, it is known to be antisymmetric in an orthonormal frame. This approach to the fundamental theorem of surface theory, via an antisymmetric field of coefficients, has previously been introduced by Ciarlet, Gratie, and C. Mardare [CGM08], who identified the solution P of the Pfaffian system as the rotation field appearing in the polar factorisation of the gradient of the three-dimensional extension of the immersion θ.

As a consequence of our approach, we finally obtain a weak rigidity of the compatibility equation and a weak compactness theorem for surface immersions in the class $W^{2,2}_{\mathrm{loc}}$.

Theorem. *Let $\{\theta^k\} \subset W^{2,2}_{\mathrm{loc}}(U,\mathbb{R}^3)$ be a uniformly bounded sequence of immersions with corresponding sequences of first and second fundamental forms denoted by $\{(a_{ij})^k\}$ and $\{(b_{ij})^k\}$, respectively. Suppose that $\partial_i\theta^k \in W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ and that the first fundamental forms $(a_{ij})^k$, $a^k_{ij} = \partial_i\theta^k \cdot \partial_j\theta^k$, have eigenvalues bounded from below by a positive constant uniformly in the domain U and in k. Then, after passing to subsequences, $\{\theta^k\}$ converges weakly in $W^{2,2}_{\mathrm{loc}}$ to an immersion $\theta \in W^{2,2}_{\mathrm{loc}}(U,\mathbb{R}^3)$, whose first and second fundamental forms (a_{ij}), (b_{ij}) are limit points of the sequences $\{(a_{ij})^k\}$, $\{(b_{ij})^k\}$ in the weak $W^{1,2}_{\mathrm{loc}}$- and L^2_{loc}-topologies, respectively.*

In the context of immersions of Riemannian manifolds, results in this spirit already appeared in a recent work by Chen and Li [CL18]. Moreover, sequences of weak immersions have previously been investigated without any assumptions about the first fundamental form, supposing instead a uniform bound on the L^2-norm of the second fundamental form—see the paper of Laurain and Rivière [LR18] and the references therein.

Curve shortening and mean curvature flow

In the second part of the thesis, we first consider k-convex solutions of mean curvature flow. A k-convex surface is one whose smallest k principal curvatures $\lambda_1, \ldots, \lambda_k$ satisfy $\lambda_1 + \cdots + \lambda_k \geq 0$. The k-convexity property is preserved by mean curvature flow, hence it is a natural curvature condition to consider. In the past, the convex ($k = 1$) and mean convex cases ($k = n$) have been of particular interest. However, in their 2009 work, Huisken and Sinestrari introduced a surgery procedure for 2-convex mean curvature flow [HS09]. Central to their analysis are the convexity and cylindrical estimates. They proved the former using an induction argument on symmetric polynomials and then obtained the latter from it. Denoting the mean curvature by H, their results read as follows:

Theorem (Convexity estimate [HS99a; see also HS09, Thm. 1.4])**.** *Let $\{M_t\}$ be a closed mean convex mean curvature flow. Then for any $\eta > 0$ there exists $C_\eta = C(\eta, M_0) > 0$ such that*

$$\lambda_1 \geq -\eta H - C_\eta \tag{1.3}$$

on M_t for any $t \in [0, T)$.

Theorem (Cylindrical estimate [HS09, Thm. 1.5])**.** *Let $\{M_t\}$ be a closed 2-convex mean curvature flow. Then for any $\eta > 0$ there exist constants $C_\eta = C(\eta, M_0) > 0$ and $c = c(n)$ such that*

$$|\lambda_1| \leq \eta H \quad \Longrightarrow \quad |\lambda_i - \lambda_j| \leq c\eta H + C_\eta \tag{1.4}$$

for any $1 < i, j \leq n$ on M_t for any $t \in [0, T)$.

In essence, these estimates are used to find regions, so-called necks, on the hypersurface that are suitable for surgery. Moreover, it has to be ensured that the estimates continue to hold after the surgery while maintaining control on the relevant constants in them. We describe the procedure in a bit more detail in Chapter 7.

Recently, a new proof strategy for these estimates has been devised by Nguyen [Sch17] according to which one first proves the cylindrical estimate directly from the 2-convexity assumption. Then the convexity estimate can be shown to be a consequence of the cylindrical one. We carry out this strategy in the general k-convex case by means of a careful analysis of the terms in Simons' identity, combined with a Poincaré-type inequality, which then leads to the derivation of L^p-bounds for the function

$$G_{\sigma,\eta} = \frac{|A| - \left(\frac{1}{\sqrt{n-(k-1)}} + \eta\right) H}{H^{1-\sigma}},$$

where $\eta > 0$ and $\sigma \in [0,1]$ and A denotes the second fundamental form. Using the well-established method via the Michael–Simon Sobolev inequality and Stampacchia iteration, used by Huisken [Hui84] to great effect, we can obtain a L^∞-bound on this function, which gives our main theorem of this section:

Theorem. *Let $\{M_t\}$ be a mean curvature flow of closed n-dimensional k-convex hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 3$. Then for any $\eta > 0$ there exists $C_\eta = C(\eta, M_0) > 0$ such that*

$$|A|^2 - \frac{1}{n-(k-1)} H^2 \leq \eta H^2 + C_\eta.$$

We note that a similar result has been obtained by Andrews and Langford [AL14] for a more general class of flows with different methods. In the 2-convex case, we then indicate how to recover Huisken–Sinestrari's estimates (1.3) and (1.4).

Finally, we study singularities of curve shortening flow in arbitrary codimension. Throughout, let a one-parameter family of immersions $\gamma : S^1 \times [0,T) \to \mathbb{R}^n$ satisfy

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(p,t) &= (\kappa N)(p,t), \\ \gamma(p,0) &= \gamma_0(p), \end{aligned} \tag{CSF}$$

where κ denotes the curvature and N a choice of normal vector of the time-dependent curve and γ_0 is a smooth initial curve. While the planar and, to a lesser extent, the space curve case have historically garnered most of the attention, recently, as in mean curvature flow, the higher codimension case has been subject to more detailed investigation. Since common tools such as the maximum principle are not as applicable as in the codimension one case, one commonly resorts to Huisken's monotonicity formula. In particular, it implies that the entropy $\lambda(\gamma)$ of a curve γ, which is a functional that can be seen as a measure for geometric complexity and is defined by

$$\lambda(\gamma) = \sup_{x_0 \in \mathbb{R}^n,\, t_0 > 0} (4\pi t_0)^{-\frac{1}{2}} \int_\gamma \mathrm{e}^{-\frac{|x-x_0|^2}{4t_0}},$$

is monotone under curve shortening flow. This property makes the entropy particularly interesting for the study of singularities, since a bound on this quantity for the initial curve propagates with the flow.

In this work, we first follow the strategy of Altschuler's work on space curves to show that singularity formation is an essentially planar phenomenon. That is, we show that blow-up limits of the flow are confined to two-dimensional subspaces of $\mathbb{R}^n$. Moreover, we argue that, as in the $n = 3$ case, for any blow-up sequence of a type-I singularity (that is, the curvature does not grow faster than $(T-t)^{\frac{1}{2}}$) there exists a subsequence such that a rescaling of the curve along it converges to a planar self-similarly shrinking solution, while for a type-II singularity (that is, it is not of type-I), there exists an essential blow-up sequence such that a sequence of rescalings along it converges to the translating Grim Reaper solution. Since the entropy of the Grim Reaper is known [Gua19], we are thus able to rule out the occurrence of type-II singularities altogether, and combined with the classification of self-similarly shrinking curves in the plane [AL86; see also Hal12], we can show that for curve shortening flows with initially 'small' entropy, the only possible singularity is the round circle:

Theorem. *Suppose that $\gamma : S^1 \times [0,T) \to \mathbb{R}^n$ is a smooth solution of the curve shortening flow with initial data $\gamma(\cdot,0) = \gamma_0$ and assume that the entropy of γ_0 satisfies*

$$\lambda(\gamma_0) \leq 2.$$

Then T is finite, and the rescaled flow converges to the round circle.

1.3 Outline of the thesis

This thesis is divided into two largely independent parts.

In the first part (Chapters 2 to 4), we are concerned with regularity theory for some exterior differential systems, specifically Pfaffian systems and isometric immersions. In Chapter 2, we introduce Cartan's moving frame method for submanifolds of Euclidean space, which is a convenient formalism to express the isometric immersion problem in. Moreover, we avail ourselves of ideas from the theory of compensated compactness, particularly the Uhlenbeck–Rivière decomposition, a very short summary of which is provided in Chapter 3.

These methods are key for the regularity theory for Pfaffian systems in two dimensions carried out in Chapter 4. After collecting some facts about the moving frame method in the case of surfaces in $\mathbb{R}^3$, we summarise the smooth theory of Pfaffian systems and the fundamental theorem of surface theory and previous efforts in the regularity theory thereof. In what follows, we show an optimal regularity theorem for Pfaffian systems with antisymmetric coefficients that satisfy a natural compatibility condition. Then we apply this result to the fundamental theorem of surface theory, for which we can easily show that the connection form satisfies the requirements of the regularity theorem in a suitable frame. We also show that the compatibility conditions are equivalent to the Gauss–Codazzi–Mainardi equations in the distributional sense. In the final section of the chapter, we use our previous result to obtain a weak compactness theorem for sequences of $W^{2,2}$-immersions.

The second part of the thesis (Chapters 5 to 8) deals with singularities of extrinsic curvature flows, i. e., mean curvature and curve shortening flow. In Chapter 5, we summarise well-known results about submanifolds of Euclidean space, in particular n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ and curves in arbitrary codimension. We then give a short introduction to the theory of curve shortening and mean curvature flow in Chapter 6 with particular attention to the classification of singularities of the flow, Huisken's monotonicity formula and the entropy functional. We list basic properties of the entropy which will be used in the sequel.

Chapter 7 deals with two particular estimates that are relevant in Huisken–Sinestrari's surgery procedure for 2-convex mean curvature flow, one of the known techniques to continue the flow past the first singular time. We first give a brief introduction to Huisken–Sinestrari's work, particularly where the cylindrical and convexity estimates are relevant. We then prove the cylindrical estimate in the general k-convex case by means of a generalisation of a recently introduced technique to directly prove this estimate from the assumptions, considerably simplifying the original method. We then indicate how the estimates in the 2-convex case can be recovered from our result.

Finally, in Chapter 8, we are concerned with singularities of the curve shortening flow in arbitrary codimension. The first two sections parallel Altschuler's work on singularities of space curves, showing estimates for derivatives of the curvature as well as results on blow-up limits of solutions of the flow. In the final section of the chapter, we then prove our main theorem, which combines the previous results on singularities of the curve shortening flow with an entropy bound on the initial curve to show that the flow converges to a round circle in finite time.

1.4 General notation

While most of our notation is standard, we give a few definitions for clarity.

Throughout the thesis, the summation convention will be employed, such that summation over repeated 'upper' and 'lower' indices is implied. If not stated otherwise, we will be working with differentiable, i. e., smooth, objects.

We denote the set of real matrices of size $n \times n$ by $\mathrm{gl}(n)$, the set of invertible matrices by $\mathrm{GL}(n)$, the set of symmetric matrices by $\mathrm{Sym}(n)$, the set of symmetric positive definite matrices by $\mathrm{Sym}^+(n)$, the set of antisymmetric matrices by $\mathrm{so}(n)$, the set of orthogonal matrices by $\mathrm{O}(n)$, and the set of proper orthogonal matrices by $\mathrm{SO}(n)$.

Moreover, we denote the elements of a matrix $A \in \mathrm{gl}(n)$ by a_{ij}, $i, j = 1, \ldots, n$, such that $A = (a_{ij})$, and the j-th column of A is denoted by $A_{(j)} = a_j$. The inverse A^{-1} of A is denoted by (a^{ij}) and the transpose of A by $A^T = (a_{ji})$. We enumerate the real eigenvalues of $A \in \mathrm{Sym}(n)$ as $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ and with any $A \in \mathrm{Sym}^+(n)$ we associate the unique matrix square root $A^{\frac{1}{2}}$.

Partial derivatives $\frac{\partial u}{\partial x_i}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ are sometimes denoted by $\partial_i u$. For any function u of two variables, we write $\nabla^{\perp} u = (-\partial_2 u, \partial_1 u)$ for the curl operator (or orthogonal gradient) of u.

Part I

Exterior differential systems

Chapter 2

Cartan geometry

The aim of this part is to prove optimal regularity results for a specific form of Pfaffian system and to treat the optimal regularity case of a corresponding problem in geometry, the isometric immersion problem for surfaces. In general, a Pfaffian system is an exterior differential system that is specified solely by 1-forms on a smooth manifold. In particular, the Pfaffian system we consider arises from the isometric immersion problem, which is, in the simplest terms, to find an immersion of a surface with prescribed first and second fundamental forms.

It is convenient to consider the isometric immersion problem in terms of Cartan's moving frame formalism. To that end, we summarise the relevant theory of moving frames for surfaces in Euclidean space below, largely following the excellent exposition in Clelland's book [Cle17]. We first define the relevant objects in arbitrary dimension and later move to the case $n = 2$ which is the one considered in Chapter 4.

Cartan's simple, yet brilliant idea is the following: We consider a map on a manifold (in our case, $\mathbb{R}^n$) that assigns a basis of the tangent space, which is not necessarily a coordinate basis, to every point of the manifold (this is called a 'moving frame'). Then we express the derivatives of the components of the moving frame in terms of the components themselves. It turns out that the geometric properties of submanifolds of $\mathbb{R}^n$, or indeed of any manifold, can be expressed rather concisely in this framework. In a sense, this is a generalisation of the concept of the Frenet–Serret frame for curves. In particular, we will be considering orthonormal moving frames, which we can always obtain from an arbitrary frame by means of the Gram–Schmidt orthonormalisation process.

2.1 Orthonormal frames

We denote Euclidean space $(\mathbb{R}^n, \langle \cdot\,, \cdot \rangle)$ by $\mathbb{E}^n$ and the Euclidean group by $E(n)$. The latter is a Lie group and can be represented as the matrix group

$$E(n) = \left\{ \begin{pmatrix} 1 & 0 \\ b & A \end{pmatrix} : A \in \mathrm{SO}(n), b \in \mathbb{E}^n \right\}$$

with the associated Lie algebra

$$T_I E(n) = \mathfrak{e}(n) = \left\{ \begin{pmatrix} 0 & 0 \\ b & B \end{pmatrix} : B \in \mathrm{so}(n), b \in \mathbb{E}^n \right\},$$

where $I \in E(n)$ is the identity matrix. The isotropy groups of $E(n)$ are all isomorphic to $\mathrm{SO}(n)$, and there is the correspondence

$$\mathbb{E}^n \cong E(n)/\mathrm{SO}(n).$$

Definition 2.1. Let $x \in \mathbb{E}^n$ and $(e_1, \ldots, e_n)$ be an oriented orthonormal basis for $T_x\mathbb{E}^n$. Then we say that $\mathrm{f} = (x; e_1, \ldots, e_n)$ is an *orthonormal moving frame on* $\mathbb{E}^n$, or, equivalently, that $(e_1, \ldots, e_n)$ is an *orthonormal frame based at* x.

Remark. We will usually work with orthonormal (moving) frames only and thus just refer to them as (moving) frames, omitting the adjective.

Note that, as mentioned above, we do not require that a moving frame should arise naturally from some coordinate system. Indeed, this is one of the strengths of the method of moving frames, as it can be applied to regions that cannot be included in a coordinate system [Spi99a, p. 260].

Considering the vectors $(e_1, \dots, e_n)$ as columns of a matrix $A \in \mathrm{SO}(n)$ shows that there is a bijection between the set of orthonormal frames on $\mathbb{E}^n$ and the Euclidean group $E(n)$. Thus $E(n)$ is also called the *orthonormal frame bundle* of $\mathbb{E}^n$ and denoted $\mathcal{F}(\mathbb{E}^n)$. The projection map $\pi : E(n) \to \mathbb{E}^n$ defined by

$$\pi(x; e_1, \dots, e_n) = x$$

furnishes a description of $E(n)$ as a principal bundle over $\mathbb{E}^n$ with fibre group $\mathrm{SO}(n)$.

Using the isomorphisms $T_x\mathbb{E}^n \cong \mathbb{E}^n$, we can think of the components $x, e_1, \dots, e_n$ of a moving frame as $\mathbb{E}^n$-valued functions on $\mathcal{F}(\mathbb{E}^n)$. Their exterior derivatives are then given by the differentials as such functions; and they map $T_{\mathrm{f}}\mathcal{F}(\mathbb{E}^n) \to T_x\mathbb{E}^n$ for any $\mathrm{f} = (x; e_1, \dots, e_n) \in \mathcal{F}(\mathbb{E}^n)$ (see [Cle17, p. 76] for details).

In particular, as the set $(e_1, \dots, e_n)$ forms a basis of $T_x\mathbb{E}^n$ for any frame $\mathrm{f} = (x; e_1, \dots, e_n)$, the 1-forms $\mathrm{d}x, \mathrm{d}e_1, \dots, \mathrm{d}e_n$ can be written as linear combinations of $(e_1, \dots, e_n)$, defining scalar-valued differentiable 1-forms (ω^i, ω^i_j) on $\mathcal{F}(\mathbb{E}^n)$:

$$\mathrm{d}x = e_i\omega^i, \tag{2.1}$$

$$\mathrm{d}e_i = e_j\omega^j_i, \qquad 1 \leq i, j \leq n. \tag{2.2}$$

Definition 2.2. The 1-forms $(\omega^1, \dots, \omega^n)$ are called *dual forms to* $(e_1, \dots, e_n)$. The set (ω^i) of dual forms is called the *coframe associated to the frame* f. The 1-forms (ω^i_j), $1 \leq i, j \leq n$, are called *connection forms* in the moving frame f.

Remark. Since the 1-forms $(\omega^1, \dots, \omega^n)$ satisfy

$$\omega^i(e_j) = \delta^i_j, \qquad 1 \leq i, j \leq n$$

for any f $\in \mathcal{F}(\mathbb{E}^n)$, we see that, at every point x, the basis (ω^i) of $T_x^*\mathbb{E}^n$ is indeed dual to the basis (e_i) of $T_x\mathbb{E}^n$, hence the name.

Moreover, from (2.2) we see that since $\mathrm{d}e_i(v_p)$ is the directional derivative of e_i in the direction v_p, the scalar $\omega_i^j(v_p)$ can be interpreted as the rate at which e_i rotates toward $e_j(p)$ as we move along a curve with tangent vector v_p [Spi99a, p. 260], eventually giving rise to a connection ∇ and thus the term 'connection forms'.

The connection forms (ω_j^i) are not fully independent from each other. Indeed, the vectors $e_1, \dots, e_n \in \mathbb{R}^n$ are orthonormal with respect to the inner product $\langle \cdot\,, \cdot \rangle$, that is,

$$\langle e_i, e_j \rangle = \delta_{ij}, \qquad 1 \leq i, j \leq n.$$

We thus obtain

$$\langle \mathrm{d}e_i, e_j \rangle + \langle e_i, \mathrm{d}e_j \rangle = 0,$$

which implies that the connection forms (ω_j^i) are *antisymmetric* in their indices i and j, i. e.,

$$\omega_j^i = -\omega_i^j, \qquad 1 \leq i, j \leq n.$$

However, this is not the only constraint on the dual and connection forms, for they also necessarily satisfy certain integrability conditions, which are the topic of the following section.

Meanwhile, it is important to note that the dual and connection forms (ω^i, ω_j^i) are defined relative to the choice of frame $(e_1, \dots, e_n)$ for $T_x\mathbb{E}^n$. Still, the dual forms $(\omega^1, \dots, \omega^n)$ form a basis for the 1-forms on $\mathbb{E}^n$. We can rewrite (2.1) and (2.2) as

$$\begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix} = A^{-1}\,\mathrm{d}x,$$

$$\begin{pmatrix} \omega_1^1 & \cdots & \omega_n^1 \\ \vdots & & \vdots \\ \omega_1^n & \cdots & \omega_n^n \end{pmatrix} = A^{-1}\,\mathrm{d}A,$$

where the columns of $A \in \mathrm{SO}(n)$ are given by $e_1, \ldots, e_n$ [Cle17, p. 77].

2.2 The Cartan structure equations

The key property of the method of moving frames is that the derivatives of the dual and connection forms can be expressed in terms of the dual and connection forms themselves. This fact is reflected in the *Cartan structure equations* (or *structure equations of* $\mathbb{R}^n$).

Proposition 2.3. *Let* $\mathrm{f} = (x; e_1, \ldots, e_n)$ *be a moving frame on* $\mathbb{E}^n$*,* (ω^i) *its coframe and* (ω_j^i) *the connection forms in* f*. Then*

$$\mathrm{d}\omega^i = -\omega_j^i \wedge \omega^j, \tag{2.3}$$

$$\mathrm{d}\omega_j^i = -\omega_k^i \wedge \omega_j^k, \qquad 1 \leq i, j \leq n. \tag{2.4}$$

The structure equations can be derived by differentiating (2.1) and (2.2) and applying the Leibniz rule [Cle17, p. 80]. Intuitively, the defining equations for the dual and connection forms express how the moving frame f varies along a curve $x(t)$. Then, indeed, the structure equations are simply a consequence of the fact that $\mathrm{d}^2 = 0$ [dCar94, p. 79].

Proof. We interpret e_i as an $\mathbb{E}^n$-valued 0-form. From (2.1) we obtain

$$0 = \mathrm{d}^2 x = \mathrm{d}e_i \wedge \omega^i + e_i \wedge \mathrm{d}\omega^i = e_i \omega_j^i \wedge \omega^j + e_i\,\mathrm{d}\omega^i,$$

which yields (2.3). Similarly, (2.2) implies that

$$0 = \mathrm{d}^2 e_i = \mathrm{d}e_j \wedge \omega_i^j + e_j \wedge \mathrm{d}\omega_i^j = e_k \omega_j^k \wedge \omega_i^j + e_k\,\mathrm{d}\omega_i^k,$$

which gives (2.4). □

It is interesting to ask about the extent to which the structure equations are sufficient conditions. This is the content of the following proposition (see also Section 4.2.2).

Proposition 2.4 ([Spi99a, Prop. 7.2])**.** *Let* (ω^i_j) *be a matrix of* 1*-forms satisfying the second structure equation* (2.4)*. Then the following statements are true:*

1. *In a neighbourhood of* 0*, for any* $A_0 \in \mathbb{R}^{n\times n}$ *there exists a matrix-valued function* $A = (A^i_j)$ *such that*

$$\begin{aligned} \mathrm{d}A^i_j &= -\omega^i_k \wedge A^k_j, \\ A(0) &= A_0. \end{aligned}$$

2. *In a neighbourhood of* 0*, for any basis* $(e_{1,0}, \ldots, e_{n,0})$ *of* $\mathbb{E}^n$ *there exists a moving frame* $(e_1, \ldots, e_n)$ *such that it and its coframe* $(\omega^1, \ldots, \omega^n)$ *satisfy*

$$\begin{aligned} \mathrm{d}\omega^i &= -\omega^i_k \wedge \omega^k_j, \\ e_i(0) &= e_{i,0}. \end{aligned}$$

Remark. The second structure equation (2.4) also expresses the fact that $\mathbb{E}^n$ is flat, for on a general Riemannian manifold M^n the second structure equation reads

$$\mathrm{d}\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j.$$

The 2-forms Ω^i_j, called *curvature forms*, vanish if and only if the Riemannian curvature tensor R does, i. e., M^n is locally isometric to $\mathbb{E}^n$ [Spi99a, Thm. 7.6].

2.3 The Maurer–Cartan form

We can express the Cartan structure equations in an even more compact form. To that end, we consider $\mathcal{F}(\mathbb{E}^n)$ again as the Lie group $E(n)$ and define the *left translation map* $L_h : E(n) \to E(n)$ for any fixed element $h \in E(n)$ by

$$L_h(g) = hg.$$

Its differential is a map $(\mathrm{d}L_h)_g = (L_h)_* : T_gE(n) \to T_{hg}E(n)$ for any $g \in E(n)$.

Definition 2.5. The *Maurer–Cartan form* is the $\mathfrak{e}(n)$-valued 1-form ω on $E(n)$ defined by

$$\omega(v) = (L_{g^{-1}})_*(v)$$

for any $g \in E(n)$, $v \in T_gE(n)$.

Remark. Recall that $T_IE(n) = \mathfrak{e}(n)$, so that $(L_{g^{-1}})_*$ maps $T_gE(n) \to \mathfrak{e}(n)$. The Maurer–Cartan form is left-invariant, i. e., $L_h^*\omega = \omega$ for any $h \in E(n)$ [Cle17, p. 81].

Equivalently, we can define ω extrinsically via the identity map $g : E(n) \to E(n)$, which represents elements of the Lie group $E(n)$ as matrices. For any $\mathrm{f} = (x; e_1, \dots, e_n)$ we have

$$g(\mathrm{f}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ x & e_1 & \cdots & e_n \end{pmatrix}.$$

Then we can write

$$\omega = g^{-1}\,\mathrm{d}g\,. \tag{2.5}$$

Remark. Here, it is crucial to think of $E(n)$ being realised as a matrix group for the product in (2.5) to make sense [Cle17, p. 82].

While the previous definitions can be made for other Lie groups than just $E(n)$, we have the following explicit expression for the Maurer–Cartan form on $E(n)$ in terms of the dual and connection forms on $\mathcal{F}(\mathbb{E}^n)$:

$$\omega = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \omega^1 & \omega^1_1 & \cdots & \omega^1_n \\ \vdots & \vdots & & \vdots \\ \omega^n & \omega^n_1 & \cdots & \omega^n_n \end{pmatrix}.$$

As a result, the 1-forms (ω^i, ω^i_j) are collectively referred to as Maurer–Cartan forms as well. Moreover, the expression shows that, indeed, the antisymmetry of the connection forms implies that for any $v \in TE(n)$ we have $\omega(v) \in \mathfrak{e}(n)$.

Proposition 2.6. *The Cartan structure equations* (2.3) *and* (2.4) *are equivalent to the* Maurer–Cartan equation*,*

$$\mathrm{d}\omega = -\omega \wedge \omega. \tag{2.6}$$

Remark. The term $\omega \wedge \omega$ in (2.6) does not vanish, as the exterior product of matrix-valued 1-forms is not antisymmetric. Instead, for a matrix-valued k-form $\alpha = (\alpha^i_j)$ and a matrix-valued ℓ-form $\beta = (\beta^i_j)$, their exterior product $\gamma = \alpha \wedge \beta$ as a matrix-valued $(k + \ell)$-form is given by

$$\gamma^i_j = \alpha^i_k \wedge \beta^k_j.$$

In other words, the product $\alpha \wedge \beta$ has the structure of the usual matrix product, but the multiplication of individual terms is done using the exterior product $\wedge$.

If we choose a particular orthonormal frame field on $\mathbb{E}^n$, that is, a section $\sigma : \mathbb{E}^n \to \mathcal{F}(\mathbb{E}^n)$ of the orthonormal frame bundle $\pi : \mathcal{F}(\mathbb{E}^n) \to \mathbb{E}^n$, all the pullbacks $\bar{\omega}^i := \sigma^*\omega^i$, $\bar{\omega}^i_j := \sigma^*\omega^i_j$, $1 \leq i, j \leq n$, are 1-forms on $\mathbb{E}^n$. Arranging $e_1(x), \dots, e_n(x)$ column-wise into a matrix field $A(x)$ for $x \in \mathbb{E}^n$, we obtain

$$\begin{pmatrix} \bar{\omega}^1 \\ \vdots \\ \bar{\omega}^n \end{pmatrix} = A(x)^{-1}\, \mathrm{d}x,$$

$$\begin{pmatrix} \bar{\omega}^1_1 & \cdots & \bar{\omega}^1_n \\ \vdots & & \vdots \\ \bar{\omega}^n_1 & \cdots & \bar{\omega}^n_n \end{pmatrix} = A(x)^{-1}\,\mathrm{d}A(x)\,.$$

Writing

$$g(x) = \begin{pmatrix} 1 & 0 \\ x & A(x) \end{pmatrix},$$

we can define a 'pulled-back' Maurer–Cartan form $\bar{\omega}$,

$$\bar{\omega} := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \bar{\omega}^1 & \bar{\omega}^1_1 & \cdots & \bar{\omega}^1_n \\ \vdots & \vdots & & \vdots \\ \bar{\omega}^n & \bar{\omega}^n_1 & \cdots & \bar{\omega}^n_n \end{pmatrix},$$

which satisfies

$$\bar{\omega} = g(x)^{-1}\,\mathrm{d}g(x)\,.$$

However, note that while the pullbacks of the dual forms are linearly independent on $\mathbb{E}^n$, just like both the dual and connection forms on the frame bundle $\mathcal{F}(\mathbb{E}^n)$, the pullbacks of the connection forms can be expressed as linear combinations of the pulled-back dual forms [Cle17, p. 83].

Chapter 3

Compensated compactness theory

The basic idea behind the concept of *compensation* is that for some partial differential equations, their regularity theory can be improved beyond what one would normally expect if some additional structure is present. Often, the PDE in question is non-linear and stems from geometry. The first result in this direction was given by Wente [Wen69] in the context of parametrised constant mean curvature surfaces, who realised that if the right hand side of Poisson's equation has the structure of a Jacobian determinant,

$$\nabla^{\perp} a \cdot \nabla b = (\partial_1 a)(\partial_2 b) - (\partial_2 a)(\partial_1 b),$$

where $a, b \in W^{1,2}(D^2)$ and $D^2 \subset \mathbb{R}^2$ is the open unit disk, then $\nabla^{\perp} a \cdot \nabla b$ does not only belong to $L^1(D^2)$, but the solution u of Poisson's equation is bounded and its gradient is square-integrable. In fact, it is also true that $D^2 u \in L^1(D^2)$ [CLMS93]. More results in this direction followed, and they were systemised by the introduction of the Hardy space $\mathcal{H}^1$, a subspace of L^1 [Mül90; CLMS93; Sem94]. A very successful application of these insights is the regularity theory of weakly harmonic maps [Hél02; Riv07].

Compensated compactness, meanwhile, refers to a concept which sometimes allows to prove weak continuity of a certain non-linear functional using additional control on the sequence of arguments. In that sense the latter compensates for the lack of compactness, allowing conclusions analogous to those of a compactness argument. For example, the inner product of vector fields turns out to be weakly continuous whenever their divergence and rotational curl, respectively, are controlled, thus enabling one to pass a sequence of products to a weak limit which is the product of the weak limits of the individual sequences.

This technique was pioneered by Murat and Tartar's div-curl lemma [Mur78; Mur79; Tar79; Mur81; Tar83; Tar85] and has enjoyed considerable popularity [Eva90; CLMS93; BCM09; CDM11]. Obviously, this is particularly useful when considering a sequence of approximating solutions to a partial differential equation to obtain a limit solution to the equation. In particular, it does not require the system in question to be of a specific type (such as elliptic or hyperbolic).

Indeed, the Gauss–Codazzi–Ricci equations, which have no type, have recently been cast in a compensated compactness framework [CSW10a; CSW10b; Chr12; CHW15; Che15; CS15] and an intrinsic div-curl lemma on Riemannian manifolds of any dimension n has been used to prove weak rigidity for the Gauss–Codazzi–Ricci equations and $W^{2,p}$-isometric immersions [CL18]. In particular, the critical case $n = p = 2$ has been treated in the latter, using a corresponding 'critical' div-curl lemma [CDM11] based on a Lipschitz truncation argument. While the rigidity result is equivalent to the existence of local isometric immersions in the smooth category, in the lower regularity case this is not automatic.

In contrast, our approach in Chapter 4 is not based on the div-curl lemma, but instead on Rivière's Coulomb gauge construction [Riv07] following Uhlenbeck's work [Uhl82]. Essentially, this can be understood as a non-linear decomposition of $\Omega \in L^2(D^2, \mathrm{so}(m) \otimes \mathbb{R}^2)$ into $\xi \in W^{1,2}(D^2, \mathrm{so}(m))$ and $P \in W^{1,2}(D^2, \mathrm{SO}(m))$ via

$$\nabla^{\perp}\xi = P^{-1}\nabla P + P^{-1}\Omega P.$$

It turns out that, in fact, for any Ω satisfying an additional compatibility condition we must have that $\xi = 0$, so that P solves the corresponding Pfaffian system $\nabla P + \Omega P = 0$. While Rivière's original proof is based on Wente's inequality, Theorem 3.1, and the Poincaré lemma, Schikorra later gave a proof using variational methods [Sch10].

3.1 Wente's inequality

Wente's result originally appeared in the context of parametrised constant mean curvature surfaces [Wen80; BC84]. Since then, it has found applications and inspired similar results in many other areas of PDE, such as the original proof of Lemma 3.5.

Theorem 3.1 ([Wen69; see also BC84; Hél02; GM12])**.** *Let $a, b \in W^{1,2}(D^2)$ and suppose that $u \in W_0^{1,2}(D^2)$ is a weak solution of*

$$\begin{aligned} -\Delta u &= \nabla^\perp a \cdot \nabla b && \text{in } D^2, \\ u &= 0 && \text{on } \partial D^2. \end{aligned}$$

Then $u \in C^0(\overline{D^2}) \cap W^{1,2}(D^2)$ and there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty} + \|\nabla u\|_{L^2} \leq C \|\nabla a\|_{L^2} \|\nabla b\|_{L^2}.$$

Proof. We sketch the proof from Giaquinta–Martinazzi's book [GM12, Thm. 7.8].

Assume that $a, b \in C^\infty(\overline{D^2})$. Note that by an integration by parts, followed by Hölder's and Young's inequalities, it suffices to prove the estimate

$$\|u\|_{L^\infty} \leq C \|\nabla a\|_{L^2} \|\nabla b\|_{L^2}.$$

Extend a, b to functions $\tilde{a}, \tilde{b} \in C_0^\infty(\mathbb{R}^2)$ with compact support such that

$$\|\nabla \tilde{a}\|_{L^2(\mathbb{R}^2)} \leq C \|\nabla a\|_{L^2(D^2)},$$
$$\|\nabla \tilde{b}\|_{L^2(\mathbb{R}^2)} \leq C \|\nabla b\|_{L^2(D^2)}$$

for some constant C and define

$$\tilde{u} = \psi * (\nabla^\perp \tilde{a} \cdot \nabla \tilde{b}),$$

where $\psi(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$ is a fundamental solution of Laplace's equation. Then $\Delta \tilde{u} = \Delta u$ in D^2.

We introduce polar coordinates (r, θ). Using

$$\partial_r = \cos\theta\, \partial_1 + \sin\theta\, \partial_2,$$
$$\partial_\theta = -r \sin\theta\, \partial_1 + r\cos\theta\, \partial_2,$$

we find

$$\begin{aligned} \nabla^\perp \tilde{a} \cdot \nabla \tilde{b} &= \frac{1}{r}(\partial_r \tilde{a})(\partial_\theta \tilde{b}) - \frac{1}{r}(\partial_\theta \tilde{a})(\partial_r \tilde{b}) \\ &= \frac{1}{r}\partial_r(\tilde{a}\, \partial_\theta \tilde{b}) - \frac{1}{r}\partial_\theta(\tilde{a}\, \partial_r \tilde{b}). \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{u}(0) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x|}(\nabla^\perp \tilde{a} \cdot \nabla \tilde{b})(x)\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \log\frac{1}{r}\left(\partial_r(\tilde{a}\,\partial_\theta \tilde{b}) - \partial_\theta(\tilde{a}\,\partial_r \tilde{b})\right)\mathrm{d}\theta\,\mathrm{d}r \\ &= \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \log\frac{1}{r}\,\partial_r(\tilde{a}\,\partial_\theta \tilde{b})\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \frac{1}{2\pi}\int_0^\infty \frac{1}{r}\int_0^{2\pi} \tilde{a}\,\partial_\theta \tilde{b}\,\mathrm{d}\theta\,\mathrm{d}r\,. \end{aligned}$$

Define

$$\bar{\tilde{a}}(r) = \int_0^{2\pi} \tilde{a}(r,\sigma)\,\mathrm{d}\sigma\,.$$

Then we have

$$\int_0^{2\pi} \tilde{a}\,\partial_\theta \tilde{b}\,\mathrm{d}\theta = \int_0^{2\pi} (\tilde{a} - \bar{\tilde{a}})\,\partial_\theta \tilde{b}\,\mathrm{d}\theta$$

for any $r > 0$, which implies that

$$\left| \int_0^{2\pi} \tilde{a}\,\partial_\theta \tilde{b}\,\mathrm{d}\theta \right| \le \|\tilde{a} - \bar{\tilde{a}}\|_{L^2(0,2\pi)}\,\|\partial_\theta \tilde{b}\|_{L^2(0,2\pi)} \le \|\partial_\theta \tilde{a}\|_{L^2(0,2\pi)}\,\|\partial_\theta \tilde{b}\|_{L^2(0,2\pi)},$$

using the Poincaré inequality, whereby

$$\begin{aligned} |\tilde{u}(0)| &\le \frac{1}{2\pi} \int_0^\infty \|\partial_\theta \tilde{a}\|_{L^2(0,2\pi)}\,\|\partial_\theta \tilde{b}\|_{L^2(0,2\pi)} \frac{1}{r}\,\mathrm{d}r \\ &\le \frac{1}{2\pi} \|\nabla \tilde{a}\|_{L^2(\mathbb{R}^2)}\,\|\nabla \tilde{b}\|_{L^2(\mathbb{R}^2)} \\ &\le C_0\,\|\nabla a\|_{L^2(D^2)}\,\|\nabla b\|_{L^2(D^2)} \end{aligned}$$

with $C_0 = \frac{C^2}{2\pi}$. By translation invariance, we thus obtain

$$\|\tilde{u}\|_{L^\infty} \le C_0\,\|\nabla a\|_{L^2}\,\|\nabla b\|_{L^2}\,.$$

Since $v := \tilde{u} - u$ is harmonic, the maximum principle implies

$$\sup_{D^2} |\tilde{u} - u| \le \sup_{\partial D^2} |\tilde{u}| \le \|\tilde{u}\|_{L^\infty}\,,$$

so that

$$\sup_{D^2} |u| \le 2\,\|\tilde{u}\|_{L^\infty} \le 2C_0\,\|\nabla a\|_{L^2}\,\|\nabla b\|_{L^2}\,.$$

The general case follows by a standard approximation argument and L^p-estimates. $\square$

3.2 Uhlenbeck–Rivière decomposition

This non-linear decomposition result is key to our treatment of Pfaffian systems with L^2-coefficients. Instead of the original proof, we sketch Schikorra's variational construction [Sch10] of the Uhlenbeck–Rivière Coulomb gauge [Uhl82; Riv07], which is inspired by a similar technique in Hélein's moving frame method [Hél02], without requiring a smallness condition on Ω.

Rivière's original proof is also summarised in Müller–Schikorra [MS09], and a more accessible exposition of Uhlenbeck's work can be found in Wehrheim's book [Weh04]. Moreover, Goldstein and Zatorska-Goldstein give an overview of the developments and generalisations that followed Rivière's work [GZ18].

In the following, let $E = E_\Omega$ denote the functional on $W^{1,2}(U, \mathrm{SO}(m))$ defined by

$$E(Q) = \int_U \left| Q^{-1}\nabla Q - Q^{-1}\Omega Q \right|^2 \mathrm{d}x, \qquad Q \in W^{1,2}(U, \mathrm{SO}(m)).$$

Lemma 3.2 ([Sch10, Lemma 2.2; GM12, Lemma 10.49])**.** *Let $U \subset \mathbb{R}^n$ be a bounded regular domain and let $\Omega \in L^2(U, \mathrm{so}(m) \otimes \wedge^1\mathbb{R}^n)$. Then there exists $P \in W^{1,2}(U, \mathrm{SO}(m))$ minimising the functional E on $W^{1,2}(U, \mathrm{SO}(m))$. Moreover,*

$$\begin{aligned} \|\nabla P\|_{L^2} &\leq 2\,\|\Omega\|_{L^2}\,, \\ \left\| P^{-1}\nabla P - P^{-1}\Omega P \right\|_{L^2} &\leq \|\Omega\|_{L^2}\,. \end{aligned}$$

Proof. The constant matrix field $I = (\delta_{ij})$ belongs to the admissible set. Therefore, there exists a minimising sequence $\{Q_k\} \subset W^{1,2}(U, \mathrm{SO}(m))$ such that

$$E(Q_k) \leq E(I) = \|\Omega\|_{L^2}^2$$

for all $k \in \mathbb{N}$. Since, for any k, Q_k is orthogonal almost everywhere, Q_k is bounded a. e.

and

$$|\nabla Q_k| = \left|Q_k^T \nabla Q_k\right| \leq \left|Q_k^T \nabla Q_k - Q_k^T \Omega Q_k\right| + |\Omega|$$

a. e., so that

$$\|\nabla Q_k\|_{L^2}^2 \leq 2\left(E(Q_k) + \|\Omega\|_{L^2}^2\right) \leq 4\,\|\Omega\|_{L^2}^2 .$$

Up to a subsequence, we can thus assume that $\{Q_k\}$ converges weakly in $W^{1,2}$ to $P \in W^{1,2}(U, \mathrm{gl}(m))$, strongly in L^2, and pointwise almost everywhere. Thus $P^T P = 1$ and $\det P = 1$, so that $P(x) \in \mathrm{SO}(m)$ for a. e. $x \in U$.

Let $\Omega^P = P^T \nabla P - P^T \Omega P$ and denote the Hilbert–Schmidt inner product for tensors by $\langle \cdot, \cdot \rangle$. A computation then shows that

$$\begin{aligned} E(Q_k) &= \int_U \left|\nabla(P^T Q_k)\right|^2 + 2\int_U \langle \nabla(P^T Q_k), \Omega^P P^T Q_k\rangle + E(P) \\ &\geq \int_U \left|\nabla(P^T Q_k)\right|^2 + 2\int_U \langle \nabla(P^T Q_k), \Omega^P P^T Q_k\rangle + \inf_Q E(Q). \end{aligned}$$

One can then show that the second integral above converges to zero as $k \to \infty$. Therefore, since $\{Q_k\}$ is a minimising sequence, $P^T Q_k$ must converge to I in $W^{1,2}$, so that Q_k converges to P, which implies that P is minimal. □

Lemma 3.3 ([Sch10, Lemma 2.4; GM12, Lemma 10.50])**.** *Let $U \subset \mathbb{R}^n$ be a bounded regular domain. Critical points $P \in W^{1,2}(U, \mathrm{SO}(m))$ of the functional E on $W^{1,2}(U, \mathrm{SO}(m))$ with $\Omega \in L^2(U, \mathrm{so}(m) \otimes \wedge^1 \mathbb{R}^n)$ satisfy*

$$\operatorname{div}(P^{-1}\nabla P - P^{-1}\Omega P) = 0 \qquad \text{in } U,$$

and, denoting the unit normal to ∂U by ν,

$$\nu \cdot (P^{-1}\nabla P - P^{-1}\Omega P) = 0 \qquad \text{on } \partial U.$$

Depending on the regularity of $P^{-1}\nabla P - P^{-1}\Omega P$, these equations have to be understood in the distributional sense.

Proof. Let P be a critical point of $E(Q)$ and consider the perturbation

$$P_\varepsilon = P\mathrm{e}^{\varepsilon\varphi\alpha} = P + \varepsilon\varphi P\alpha + o(\varepsilon) \in W^{1,2}(U, \mathrm{SO}(m)),$$

where $\varphi \in C^\infty(\bar{U})$, $\alpha \in \mathrm{so}(m)$ and ε small. Geometrically speaking, this uses the fact that $\mathrm{so}(m) = T_I\mathrm{SO}(m)$. Then we have that

$$\begin{aligned} P_\varepsilon^T &= P^T - \varepsilon\varphi\alpha P^T + o(\varepsilon), \\ \nabla P_\varepsilon &= \nabla P + \varepsilon\varphi\nabla P\alpha + \varepsilon\nabla\varphi P\alpha + o(\varepsilon), \end{aligned}$$

so that, with $\Omega^P = P^T\nabla P - P^T\Omega P$, we obtain

$$\Omega^{P_\varepsilon} = \Omega^P + \varepsilon\varphi(\Omega^P\alpha - \alpha\Omega^P) + \varepsilon\nabla\varphi\alpha + o(\varepsilon).$$

Since Ω^P is antisymmetric, we have that

$$\sum_{i,j=1}^m (\Omega^P)^i_{\ell j}(\Omega^P\alpha - \alpha\Omega^P)^i_{\ell j} = 0$$

almost everywhere, $1 \leq \ell \leq n$. Therefore,

$$\left|\Omega^{P_\varepsilon}\right|^2 = \left|\Omega^P\right|^2 + 2\varepsilon(\Omega^P)^i_j\alpha^i_j\nabla\varphi + o(\varepsilon),$$

which gives

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(P_\varepsilon) = \int_U (\Omega^P)^i_j\alpha^i_j\nabla\varphi$$

for any $\varphi \in C^\infty(\bar{U})$ and $\alpha \in \mathrm{so}(m)$. Once we define $\alpha^i_j = \delta^i_s\delta^t_j - \delta^s_j\delta^i_t$ for arbitrary $1 \leq s, t \leq m$, we obtain

$$\int_U (P^{-1}\nabla P - P^{-1}\Omega P) \cdot \nabla\varphi = 0, \qquad \varphi \in C^\infty(\bar{U}, \mathrm{gl}(m)),$$

proving the claim. $\square$

From the Hodge–Morrey decomposition of k-forms in L^2 [GM12, Thm. 10.66], we have the following decomposition of L^2-vector fields:

Proposition 3.4 ([GM12, Cor. 10.70])**.** *Let $U \subset \mathbb{R}^2$ be a contractible bounded regular domain. Then any vector field $X \in L^2(U)$ can be decomposed as*

$$X = \nabla p + \nabla^\perp \xi,$$

where $p \in W_0^{1,2}(U)$ and $\xi \in W^{1,2}(U)$. If in addition it holds that $\operatorname{div} X = 0$ then $p \equiv 0$.

Together, Lemmas 3.2 and 3.3 and Proposition 3.4 immediately imply Rivière's lemma and thus the existence of an Uhlenbeck–Rivière decomposition.

Lemma 3.5 ([Riv07, Lemma A.3; see also Sch10, Thm. 2.1; GM12, Thm. 10.48])**.** *Let $U \subset \mathbb{R}^2$ be a contractible bounded regular domain and let $\Omega \in L^2(U, \mathrm{so}(m) \otimes \wedge^1 \mathbb{R}^2)$. Then there exist $\xi \in W_0^{1,2}(U, \mathrm{so}(m))$ and $P \in W^{1,2}(U, \mathrm{SO}(m))$ such that*

$$\begin{aligned} P^{-1}\nabla P + P^{-1}\Omega P &= \nabla^\perp \xi, \\ \|\nabla \xi\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 &\leq 5 \|\Omega\|_{L^2}^2 . \end{aligned}$$

Chapter 4

Regularity theory for two-dimensional Pfaffian systems

This chapter is devoted to proving optimal regularity results for the fundamental theorem of surface theory and the related two-dimensional Pfaffian system. In particular, in Theorem 4.25 we prove that a Pfaffian system over a simply connected domain with antisymmetric coefficients in the critical space L^2 has a unique solution in $W^{1,2}_{\mathrm{loc}}$ as long as the coefficients satisfy a compatibility condition. To this end, in a novel approach we utilise the additional structural assumption on the coefficients, that is, their antisymmetry, to cast the problem in the Coulomb gauge *à la* Uhlenbeck–Rivière and thus extend the previously known results due to Hartman–Wintner [HW50a] and Mardare [Mar05] on weak solutions of Pfaffian systems. We then apply the optimal regularity theorem for Pfaffian systems to improve the fundamental theorem of surface theory to its optimal regularity case in Theorem 4.26. The latter is then used to prove Theorem 4.29, which states that the space of $W^{2,2}_{\mathrm{loc}}$-immersions is weakly compact.

In addition to the general notation laid out in Chapter 1, we make some further arrangements.

Throughout the chapter, let U be an open, connected and simply connected subset of $\mathbb{R}^2$. A continuously differentiable mapping $\theta : U \to \mathbb{R}^3$ is called an immersion if the vectors $\partial_i \theta(y)$, $i = 1, 2$, are linearly independent for all $y \in U$.

We write the space of so(n)-valued 1-forms on $\mathbb{R}^2$ as so$(n) \otimes \wedge^1 \mathbb{R}^2$. The components of $\Omega \in \mathrm{so}(n) \otimes \wedge^1 \mathbb{R}^2$ are denoted by Ω_i, $i = 1, 2$, such that $\Omega_i \in \mathrm{so}(n)$.

Therefore, we remark that a Pfaffian system of the form $\nabla P = P\Omega$ as studied in this chapter can be understood in the following way: We interpret $\Omega \in \mathrm{so}(m) \otimes \wedge^1 \mathbb{R}^2$ as a tensor $\Omega^i_{j\ell}$ that is antisymmetric in i and j. The above equation then reads, for $\ell = 1, 2$,

$$\partial_\ell P = P\Omega_\ell,$$

that is, assuming the summation convention,

$$\partial_\ell P^i_j = P^i_k \Omega^k_{j\ell}.$$

We write $\mathcal{D}(U)$ for the space of smooth functions with compact support contained in U and $\mathcal{D}'(U)$ for the space of distributions over U. As usual, we denote the Lebesgue spaces by $L^p(U)$, $1 \le p \le \infty$, and the Sobolev spaces of (equivalence classes of) weakly differentiable functions by $W^{k,p}(U)$, $k = 0, 1, \dots,$ $1 \le p \le \infty$. The closure of $\mathcal{D}(U)$ in $W^{1,2}(U)$ is denoted by $W^{1,2}_0(U)$. Furthermore, we write

$$W^{k,p}_{\mathrm{loc}}(U) = \{T \in \mathcal{D}'(U) : T \in W^{k,p}(V) \text{ for all open sets } V \subset\subset U\}.$$

Whenever X is a finite-dimensional space, let $\mathcal{D}(U, X)$, $L^p(U, X)$, and $W^{k,p}(U, X)$ designate the spaces of X-valued objects whose components belong to $\mathcal{D}(U)$, $L^p(U)$, and $W^{k,p}(U)$, respectively. We shall omit the additional symbol if it is implied by the context.

In passing, we note that the space $W^{1,2}(B) \cap L^\infty(B)$ is an algebra for all open balls $B \subset\subset U$, so that $fg \in W^{1,2}_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U)$ whenever $f, g \in W^{1,2}_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U)$.

4.1 Moving frames for hypersurfaces

In this section we consider moving frames in the special case of two-dimensional submanifolds of three-space, that is, hypersurfaces in $\mathbb{E}^3$.

Let $U \subset \mathbb{R}^2$ be open, connected, and simply connected and let $\theta : U \to \mathbb{E}^3$ be a smooth immersion whose image $\Sigma = \theta(U)$ is a regular surface. In particular, the rank of the differential $D\theta$ is two at every point. We choose a frame on $\mathbb{E}^3$ along Σ by defining a *lifting* $\tilde{\theta} : U \to E(3)$,

$$\tilde{\theta}(u) = (\theta(u); e_1(u), e_2(u), e_3(u)),$$

such that for each $u \in U$, $(e_1(u), e_2(u), e_3(u))$ is an oriented orthonormal basis of $T_{\theta(u)}\mathbb{E}^3$. We have, for any $u \in U$,

$$(\pi \circ \tilde{\theta})(u) = \theta(u) \in E(3)/\mathrm{SO}(3) \cong \mathbb{E}^3,$$

where $\pi : E(3) \to \mathbb{E}^3$ is the projection map (cf. Section 2.1). We then let $\tilde{\theta}$ be *adapted*, that is, $e_3(u)$ is orthogonal to $T_{\theta(u)}\Sigma$.

As before, associated to the frame (e_1, e_2, e_3) on $\mathbb{E}^3$ we have dual and connection 1-forms (ω^i, ω^i_j), $1 \leq i, j \leq 3$ on $\mathcal{F}(\mathbb{E}^3)$. Recall that they have the properties that

$$\begin{aligned} \omega^i(e_j) &= \delta^i_j, \\ \omega^j_i &= -\omega^i_j, \qquad 1 \leq i, j \leq 3, \end{aligned}$$

and they satisfy the Cartan structure equations,

$$\mathrm{d}\omega^1 = -\omega^1_2 \wedge \omega^2 - \omega^1_3 \wedge \omega^3, \tag{4.1}$$

$$\mathrm{d}\omega^2 = -\omega^2_1 \wedge \omega^1 - \omega^2_3 \wedge \omega^3, \tag{4.2}$$

$$\mathrm{d}\omega^3 = -\omega^3_1 \wedge \omega^1 - \omega^3_2 \wedge \omega^2, \tag{4.3}$$

$$\mathrm{d}\omega^1_2 = -\omega^1_3 \wedge \omega^3_2, \tag{4.4}$$

$$\mathrm{d}\omega^1_3 = -\omega^1_2 \wedge \omega^2_3, \tag{4.5}$$

$$\mathrm{d}\omega^2_3 = -\omega^2_1 \wedge \omega^1_3. \tag{4.6}$$

Furthermore, the Maurer–Cartan form on $\mathcal{F}(\mathbb{E}^3)$ is given by

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & \omega^1_2 & \omega^1_3 \\ \omega^2 & \omega^2_1 & 0 & \omega^2_3 \\ \omega^3 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix}$$

and the Cartan structure equations are equivalent to the Maurer–Cartan equation

$$\mathrm{d}\omega = -\omega \wedge \omega. \tag{4.7}$$

For later use, we state the following general technical lemma due to Cartan.

Lemma 4.1 (Cartan's lemma [dCar94, Lemma 5.1; see also Cle17, Lemma 2.49])**.** *Let $\alpha^1, \dots, \alpha^r$, $r \leq n$ be linearly independent differentiable 1-forms on a manifold M^n. Suppose that $\beta^1, \dots, \beta^r$ are differentiable 1-forms on M such that*

$$\sum_{i=1}^{r} \alpha^i \wedge \beta^i = 0.$$

Then there exist differentiable functions $c^i_j : M \to \mathbb{R}$, $1 \leq i, j \leq r$ such that $c^i_j = c^j_i$ and

$$\beta^i = c^i_j \, \alpha^j.$$

A direct consequence of Cartan's lemma is that a set of antisymmetric 1-forms (ω^i_j) satisfying the first structure equation (2.3) is unique.

Lemma 4.2 ([dCar94, Lemma 5.2])**.** *Let $U \subset \mathbb{R}^n$ and let $\omega^1, \dots, \omega^n$ be linearly independent differential 1-forms in U. Suppose that there exists a set of differential 1-forms*

(ω^i_j), $1 \leq i, j \leq n$ such that

$$\omega^i_j = -\omega^j_i,$$
$$\mathrm{d}\omega^i = -\omega^i_j \wedge \omega^j.$$

Then the set (ω^i_j) is unique.

Let us return to the hypersurface case. Using the lifting $\tilde{\theta}$, we can pull the dual and connection forms (ω^i, ω^i_j) back to U. We will denote them by $(\bar{\omega}^i, \bar{\omega}^i_j) = (\tilde{\theta}^*\omega^i, \tilde{\theta}^*\omega^i_j)$. Immediately, we have

Proposition 4.3 ([Cle17, Prop. 4.18; see also dCar94, p. 82])**.** *Let $U \subset \mathbb{R}^2$ be open, connected, and simply connected and let $\theta : U \to \mathbb{E}^3$ be an immersion. Moreover, suppose that (e_1, e_2, e_3) is an adapted moving frame along $\Sigma = \theta(U)$ with dual and connection forms $(\bar{\omega}^i, \bar{\omega}^i_j)$, $1 \leq i, j \leq 3$ on U. Then $(\bar{\omega}^i, \bar{\omega}^i_j)$ satisfy the Cartan structure equations and furthermore, we have that $(\bar{\omega}^1, \bar{\omega}^2)$ form a basis for the 1-forms in U and*

$$\bar{\omega}^3 = 0.$$

From $\bar{\omega}^3 = 0$ and the structure equation (4.3) we immediately obtain

$$0 = \mathrm{d}\bar{\omega}^3 = -\bar{\omega}^3_1 \wedge \bar{\omega}^1 - \bar{\omega}^3_2 \wedge \bar{\omega}^2.$$

By Cartan's lemma, there exist differentiable functions $h_{ij} = h_{ji}$, $1 \leq i, j \leq 2$ on U such that

$$\begin{pmatrix} \bar{\omega}^3_1 \\ \bar{\omega}^3_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{pmatrix}. \tag{4.8}$$

Geometrically, from our interpretation of the connection forms (ω^i_j) (see page 32) we can infer that h_{ij} can be interpreted as the rate at which e_3 rotates towards e_i if we move in the direction e_j.

Moreover, the first two structure equations (4.1) and (4.2) imply, together with $\bar{\omega}^3 = 0$, that

$$\begin{aligned} \mathrm{d}\bar{\omega}^1 &= -\bar{\omega}^1_2 \wedge \bar{\omega}^2, \\ \mathrm{d}\bar{\omega}^2 &= \bar{\omega}^1_2 \wedge \bar{\omega}^1. \end{aligned}$$

Writing $\mathrm{d}\bar{\omega}^1 = a\bar{\omega}^1 \wedge \bar{\omega}^2$, $\mathrm{d}\bar{\omega}^2 = b\bar{\omega}^1 \wedge \bar{\omega}^2$ for some real-valued functions a, b we obtain

$$\begin{aligned} a\bar{\omega}^1 \wedge \bar{\omega}^2 &= -\bar{\omega}^1_2 \wedge \bar{\omega}^2, \\ b\bar{\omega}^1 \wedge \bar{\omega}^2 &= -\bar{\omega}^1 \wedge \bar{\omega}^1_2. \end{aligned}$$

Therefore, $\bar{\omega}^1_2$ is completely determined by $\bar{\omega}^1$, $\bar{\omega}^2$:

$$\bar{\omega}^1_2 = -a\bar{\omega}^1 - b\bar{\omega}^2.$$

Definition 4.4. The 1-form $\bar{\omega}^1_2$ is called *Levi–Civita connection form.*

Remark. The reason for the name ‘Levi–Civita form’ will become apparent later.

It turns out that the Gauss curvature is determined by the Levi–Civita connection form $\bar{\omega}^1_2$. Since $\bar{\omega}^1_2$ is itself determined by $\bar{\omega}^1$, $\bar{\omega}^2$, this implies that the Gauss curvature belongs to the inner geometry of the surface, confirming Gauss’ *remarkable theorem* (see do Carmo [dCar94, p. 84] and Agricola–Friedrich [AF02, Section 5.4] for details).

So far, we have used the first three structure equations. The remaining equations (4.4) to (4.6) read

$$\mathrm{d}\bar{\omega}^1_2 = -\bar{\omega}^1_3 \wedge \bar{\omega}^3_2, \tag{4.9}$$

$$\mathrm{d}\bar{\omega}^1_3 = -\bar{\omega}^1_2 \wedge \bar{\omega}^2_3, \tag{4.10}$$

$$\mathrm{d}\bar{\omega}^2_3 = -\bar{\omega}^2_1 \wedge \bar{\omega}^1_3. \tag{4.11}$$

Definition 4.5. We call (4.9) the *Gauss equation* and (4.10) and (4.11) the *Codazzi–Mainardi equations*.

Continuing our quest to formulate the basic geometric quantities of the hypersurface Σ in the language of moving frames and differential forms, we define another product operation on forms.

Definition 4.6. Let α, β be 1-forms on a manifold M. Then their *symmetric product* is a symmetric bilinear form $\alpha\beta : TM \times TM \to \mathbb{R}$ defined by

$$\alpha\beta(v, w) = \frac{1}{2}(\alpha(v)\beta(w) + \alpha(w)\beta(v)), \qquad v, w \in TM.$$

The square α^2 of a 1-form α is meant to be the symmetric product of α with itself.

We then have concise descriptions of the first and second fundamental forms of the surface Σ.

Definition 4.7. The *first fundamental form* of Σ is the quadratic form $\mathrm{I} : TU \to \mathbb{R}$ defined by

$$\mathrm{I}(v) = \langle \mathrm{d}\theta(v), \mathrm{d}\theta(v) \rangle, \qquad v \in T_u U.$$

From the definition of the dual forms (2.2), we find that the first fundamental form can be written as

$$\mathrm{I} = (\bar{\omega}^1)^2 + (\bar{\omega}^2)^2.$$

Remark. By polarisation, we can think of I as a symmetric positive definite bilinear form on $TU \times TU$ [Cle17, p. 119].

Definition 4.8. The *Gauss map* $N : \Sigma \to S^2$ of Σ is defined by

$$N(\theta(u)) = e_3(u), \qquad u \in U.$$

The differential $\mathrm{d}N = e_1\bar{\omega}_3^1 + e_2\bar{\omega}_3^2$ of the Gauss map is called *shape operator* of Σ.

Remark. As such, the Gauss map is only well-defined up to a sign. Once we have fixed orientations on U and $\mathbb{R}^3$, however, we can choose a moving frame (e_1, e_2, e_3) so that (e_1, e_2) is in the orientation of U and (e_1, e_2, e_3) is in the orientation of $\mathbb{R}^3$. Then the Gauss map is well-defined and independent of our choice of moving frame [dCar94, p. 83].

Definition 4.9. The *second fundamental form* of Σ is the quadratic form $\mathrm{II} : TU \to \mathbb{R}$ defined by

$$\mathrm{II}(v) = -\langle \mathrm{d}e_3(v), \mathrm{d}\theta(v)\rangle, \qquad v \in T_uU.$$

The definition of the dual and connection forms then implies that, using (4.8),

$$\begin{aligned}\mathrm{II} &= \bar{\omega}_1^3\bar{\omega}^1 + \bar{\omega}_2^3\bar{\omega}^2 \\ &= h_{11}(\bar{\omega}^1)^2 + 2h_{12}\bar{\omega}^1\bar{\omega}^2 + h_{22}(\bar{\omega}^2)^2.\end{aligned}$$

The next geometric object we will write in terms of moving frames is the covariant derivative.

Definition 4.10. Given two vector fields v, w on Σ, we define the *covariant derivative* of w with respect to v as the tangential part of the Euclidean directional derivative $\mathrm{d}v(w)$,

$$\nabla_v w = \mathrm{d}w(v) - \langle \mathrm{d}w(v), e_3\rangle e_3.$$

Then the induced *Levi–Civita connection* ∇ on Σ is given once we define the TU-valued 1-forms ∇w via

$$\nabla w(v) = \nabla_v w.$$

We have that

$$\begin{aligned}\nabla_v e_1 &= \mathrm{d}e_1(v) + \langle \mathrm{d}e_1(v), e_3\rangle e_3 = e_2\bar{\omega}_1^2(v), \\ \nabla_v e_2 &= \mathrm{d}e_2(v) + \langle \mathrm{d}e_2(v), e_3\rangle e_3 = e_1\bar{\omega}_2^1(v),\end{aligned}$$

so that, writing the vector field w as $w = w^1 e_1 + w^2 e_2$ for some functions $w^1, w^2 : U \to \mathbb{R}$, we obtain

$$\nabla_v w = (\mathrm{d}w^1(v) + \bar{\omega}_1^2(v))\, e_1 + (\mathrm{d}w^2(v) + \bar{\omega}_2^1(v))\, e_2.$$

Thus the covariant derivative is determined by the Levi–Civita connection form $\bar{\omega}_2^1$, which also explains its name. Recall that $\bar{\omega}_2^1$ only depends on the dual forms $\bar{\omega}^1$, $\bar{\omega}^2$ and therefore the covariant derivative is a quantity belonging to the inner geometry of the surface.

Remark. The covariant derivative ∇ thus defined has the usual properties of a covariant derivative [AF02, Thm. 5.13].

Finally, we define the curvature tensor of the surface.

Definition 4.11. The *curvature tensor* of Σ is a map $R : TU \times TU \times TU \to TU$ defined by

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w,$$

where u, v, w are vector fields and $[u, v]$ denotes the commutator of u and v.

As for the covariant derivative, we can write the curvature tensor in terms of a moving frame, its coframe and the Levi–Civita connection form [AF02, Thm. 5.16]:

$$R(u, v)w = \mathrm{d}\bar{\omega}_1^2(u, v)(e_2 \omega^1(w) - e_1 \omega^2(w)).$$

Since the first fundamental form is positive definite, we can represent the second fundamental form as a symmetric bilinear form II on $TU \times TU$ by the self-adjoint shape operator $S = \mathrm{d}e_3$, that is,

$$\mathrm{II}(v, w) = \mathrm{I}(v, S(w)).$$

The covariant exterior derivative of the shape operator $S : TU \to TU$ (or any endomorphism of TU, for that matter) is the 2-form $\nabla S : TU \times TU \to TU$ given by

$$\nabla S(v, w) = \nabla_v(S(w)) - \nabla_w(S(v)) - S([v, w]),$$

where v, w are vector fields.

We then have

Proposition 4.12 ([AF02, Thm. 5.17])**.** *The shape operator S, the curvature tensor R and the second fundamental form* II *satisfy the Gauss and Codazzi–Mainardi equations,*

$$\begin{aligned} R(u, v)w &= \mathrm{II}(v, w)S(u) - \mathrm{II}(u, w)S(v), \\ \nabla S &= 0, \end{aligned}$$

for any vector fields u, v, w.

4.2 Smooth theory and previous regularity results

As an introduction to Pfaffian systems and the fundamental theorem of surface theory, in this section we review the classical smooth case. We also summarise the previous efforts of Hartman and Wintner [HW50a; HW50b] and Mardare [Mar03b; Mar05; Mar07] in the regularity theory of these equations.

4.2.1 Pfaffian systems

In order not to have to deal with too much abstract theory of exterior differential systems, in this and the following subsections on the smooth theory we follow closely the nice exposition by Agricola and Friedrich [AF02].

Suppose that we are given $m - k$ smooth functions $f_1, \dots, f_{m-k}$ on $\mathbb{R}^m$ such that their differentials $\mathrm{d}f_1, \dots, \mathrm{d}f_{m-k}$ are linearly independent. Then the level set

$$\{x \in \mathbb{R}^m : f_1(x) = c_1, \dots, f_{m-k}(x) = c_{m-k}\}$$

is a k-dimensional manifold with tangent bundle

$$\{\nu \in T\mathbb{R}^m : \mathrm{d}f_1(\nu) = 0, \ldots, \mathrm{d}f_m(\nu) = 0\}.$$

Equivalently, we can describe these k-dimensional subspaces of $T\mathbb{R}^m$ as the zero level sets of any set of 1-forms $\{\omega^1, \ldots, \omega^{m-k}\}$ defined by $\omega^i = h_i^j \, \mathrm{d}f_j$ for any $\mathrm{gl}(m-k)$-valued function $h = (h_i^j)$.

We are thus interested in the compatibility conditions that allow the recovery of a family of k-dimensional manifolds from a set of $m - k$ linearly independent 1-forms $\{\omega^1, \ldots, \omega^{m-k}\}$ via the exterior differential system

$$\omega^1 = \cdots = \omega^{m-k} = 0.$$

Definition 4.13. A k-dimensional *Pfaffian system* (or *geometric distribution*) on a manifold M^m is a family

$$\mathcal{E}^k = \{E^k(x) \subset T_x M : x \in M\}$$

such that the subspaces $E^k(x)$ depend smoothly on x in the sense that for each point $x_0 \in M$, there exist a neighbourhood $U \subset M$ containing x_0 and vector fields $v_1, \ldots, v_k$ on U such that

$$E^k(x) = \mathrm{span}\{v_1(x), \ldots, v_k(x)\}, \qquad x \in U.$$

In the above setting, a Pfaffian system is determined by linearly independent 1-forms $\{\omega^1, \ldots, \omega^{m-k}\}$ on a manifold M^m via

$$E^k(x) = \{\nu \in T_x M : \omega^1(\nu) = \cdots = \omega^{m-k}(\nu) = 0\}.$$

Definition 4.14. Let $\mathcal{E}^k$ be a k-dimensional Pfaffian system on M^m. A k-dimensional submanifold $N^k \subset M^m$ is called an *integral manifold* of $\mathcal{E}^k$ if

$$T_x N = E^k(x), \qquad x \in N.$$

Definition 4.15. A k-dimensional Pfaffian system $\mathcal{E}^k$ on M^m is called *integrable* if for every point $x \in M$, there exists an integral manifold through x.

If $\mathcal{E}^k$ is defined by $\{\omega^1, \dots, \omega^{m-k}\}$ as above, an immersed submanifold $\theta : N^k \to M^m$ is an integral manifold of $\mathcal{E}^k$ precisely when the pullbacks $\theta^*\omega^1, \dots, \theta^*\omega^{m-k}$ vanish.

4.2.2 Frobenius' theorem

The main ingredient of the proof of the smooth fundamental theorem of surface theory, besides the Poincaré lemma for differential forms, is Frobenius' theorem of involutive distributions.

Definition 4.16. A k-dimensional Pfaffian system $\mathcal{E}^k = \{E^k(x)\}$ on M^m is called *involutive* if for every two vector fields v, w on M such that $v(x), w(x) \in E^k(x)$, their commutator satisfies $[v, w](x) \in E^k(x)$.

Theorem 4.17 ([Fro77; AF02, Thm. 4.1])**.** *Let $\mathcal{E}^k$ be a k-dimensional distribution on the manifold M^m defined by $m-k$ linearly independent 1-forms $\omega^1, \dots, \omega^{m-k}$, that is,*

$$\mathcal{E}^k = \{\nu \in TM : \omega^1(\nu) = \dots = \omega^{m-k}(\nu) = 0\}.$$

Then the following statements are equivalent:

1. *$\mathcal{E}^k$ is integrable,*

2. *$\mathcal{E}^k$ is involutive,*

3. *for every $x_0 \in M$, there exist a neighbourhood $U \subset M$ containing x_0 and 1-forms*

θ^i_j, $1 \leq i, j \leq m-k$ *on* U *such that*

$$\mathrm{d}\omega^i = \theta^i_j \wedge \omega^j, \qquad 1 \leq i \leq m-k,$$

4. *for all* $1 \leq i \leq m-k$,

$$\mathrm{d}\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{m-k} = 0.$$

Specifically, we will use a consequence of the implication '3. $\Rightarrow$ 1.' of Frobenius' theorem. Indeed, we will later be concerned with generalisations of this type of theorem, referred to as an existence and uniqueness theorem for Pfaffian systems, to the case of merely integrable coefficients.

Theorem 4.18 ([AF02, Thm. 4.6])**.** *Let* $\Omega = (\omega_{ij}) \in \mathrm{gl}(k) \otimes \wedge^1\mathbb{R}^m$ *be a matrix-valued* 1*-form defined on a neighbourhood of* $0 \in \mathbb{R}^m$ *and let* $A_0 \in \mathrm{GL}(k)$ *be an invertible matrix. Then the following statements are equivalent:*

1. *In a connected neighbourhood* $V \subset \mathbb{R}^m$ *of* 0 *there exists a matrix-valued function* $A = (a_{ij})$ *of size* $k \times k$ *such that*

$$\Omega = \mathrm{d}A \cdot A^{-1},$$
$$A(0) = A_0.$$

2. *The matrix-valued* 1*-form* Ω *satisfies the compatibility condition,*

$$\mathrm{d}\Omega = \Omega \wedge \Omega.$$

If such a function A *exists, it is uniquely determined. If in addition* $\Omega \in \mathrm{so}(k) \otimes \wedge^1\mathbb{R}^m$ *and* $A_0 \in \mathrm{SO}(k)$*, then* $A(x) \in \mathrm{SO}(k)$ *for any* $x \in V$*.*

4.2.3 The fundamental theorem of surface theory

We give two formulations of the fundamental theorem. The first is in terms of differential forms.

Theorem 4.19 (Fundamental theorem of surface theory I [AF02, Thm. 5.11; Cle17, Thm. 4.39])**.** *Let $U \subset \mathbb{R}^2$ be open, connected and simply connected. Suppose that four differential 1-forms $\tilde{\omega}^1$, $\tilde{\omega}^2$, $\tilde{\omega}_1^3$, $\tilde{\omega}_2^3$ on U are given such that the forms $\tilde{\omega}^1$ and $\tilde{\omega}^2$ are linearly independent and that they define a 1-form $\tilde{\omega}_1^2$ via*

$$
\begin{aligned}
\mathrm{d}\tilde{\omega}^1 &= \tilde{\omega}_1^2 \wedge \tilde{\omega}^2, \\
\mathrm{d}\tilde{\omega}^2 &= -\tilde{\omega}_1^2 \wedge \tilde{\omega}^1.
\end{aligned}
$$

Moreover, define $\tilde{\omega}_j^i = -\tilde{\omega}_i^j$. Assume that the system $(\tilde{\omega}^i, \tilde{\omega}_i^j)$ of 1-forms satisfies the structure equations,

$$
\begin{aligned}
0 &= \tilde{\omega}^1 \wedge \tilde{\omega}_1^3 + \tilde{\omega}^2 \wedge \tilde{\omega}_2^3, \\
\mathrm{d}\tilde{\omega}_2^1 &= -\tilde{\omega}_3^1 \wedge \tilde{\omega}_2^3, \\
\mathrm{d}\tilde{\omega}_3^1 &= -\tilde{\omega}_2^1 \wedge \tilde{\omega}_3^2, \\
\mathrm{d}\tilde{\omega}_3^2 &= -\tilde{\omega}_1^2 \wedge \tilde{\omega}_3^1.
\end{aligned}
$$

Then there exists an immersion $\theta : U \to \mathbb{R}^3$ of a surface Σ and an orthonormal frame of vector fields tangent to Σ such that the induced dual and connection forms $(\bar{\omega}^i, \bar{\omega}_i^j)$ coincide with $(\tilde{\omega}^i, \tilde{\omega}_i^j)$. The surface Σ and its orthonormal frame are uniquely determined up to proper isometries of $\mathbb{R}^3$.

The proof proceeds as follows: First, we construct an orthonormal frame on U such that the given forms are its dual and connection forms, respectively, using Theorem 4.18. Then, using the Poincaré lemma, we construct an immersion of U into $\mathbb{R}^3$ from this orthonormal frame.

Proof. By assumption, the matrix-valued function $\tilde{\Omega} = (\tilde{\omega}_i^j)$ satisfies $\tilde{\Omega} \in \mathrm{so}(3) \wedge^1 \mathbb{R}^2$ as well as $\mathrm{d}\tilde{\Omega} = \tilde{\Omega} \wedge \tilde{\Omega}$. By Theorem 4.18, therefore, for any $A_0 \in \mathrm{SO}(3)$ we have a matrix-valued function $\tilde{A}$ such that $\mathrm{d}\tilde{A} = \tilde{\Omega}\tilde{A}$ and $\tilde{A} \in \mathrm{SO}(3)$ at every point. Thus the rows of $\tilde{A}$ define an orthonormal frame $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3 : U \to \mathbb{R}^3$ such that

$$\mathrm{d}\tilde{e}_i = \tilde{\omega}_i^j \tilde{e}_j.$$

Define the 1-form ϕ by

$$\phi = \tilde{\omega}^1 \tilde{e}_1 + \tilde{\omega}^2 \tilde{e}_2.$$

Then $\mathrm{d}\phi = 0$ and by the Poincaré lemma the 1-form ϕ is exact, so there exists $\theta : U \to \mathbb{R}^3$ such that

$$\mathrm{d}\theta = \phi.$$

We note that θ is an immersion by construction. Observe that the solution $\tilde{A}$ of $\mathrm{d}\tilde{A} = \tilde{\Omega}\tilde{A}$ is unique up to the prescribed initial condition $A_0 \in \mathrm{SO}(3)$, and the solution θ of $\mathrm{d}\theta = \phi$ is unique up to a constant $a \in \mathbb{R}^3$. As a result, the surface thus obtained is uniquely determined up to an Euclidean motion in $\mathbb{R}^3$. $\square$

The second formulation of the fundamental theorem is in terms of the first and second fundamental forms. Its proof can be reduced to follow from the first formulation of the theorem.

Theorem 4.20 (Fundamental theorem of surface theory II [AF02, Thm. 5.18])**.** *Let $U \subset \mathbb{R}^2$ be open, connected and simply connected. Suppose that two symmetric bilinear forms $\tilde{\mathrm{I}}, \tilde{\mathrm{II}} : TU \times TU \to \mathbb{R}$ are given such that $\tilde{\mathrm{I}}$ is positive definite at each point. Define a covariant derivative $\tilde{\nabla}$ on vector fields via*

$$\begin{aligned} 2\tilde{\mathrm{I}}(\tilde{\nabla}_u v, w) = {} & u(\tilde{\mathrm{I}}(v,w)) + v(\tilde{\mathrm{I}}(u,w)) - w(\tilde{\mathrm{I}}(u,v)) \\ & + \tilde{\mathrm{I}}([u,v],w) + \tilde{\mathrm{I}}(v,[w,u]) - \tilde{\mathrm{I}}(u,[v,w]), \end{aligned} \tag{4.12}$$

and a curvature tensor $\tilde{R}$ *via*

$$\tilde{R}(u,v)w = \tilde{\nabla}_u \tilde{\nabla}_v w - \tilde{\nabla}_v \tilde{\nabla}_u w - \tilde{\nabla}_{[u,v]} w. \tag{4.13}$$

Assume that the symmetric endomorphism $\tilde{S} : TU \to TU$ *induced by* $\tilde{\mathrm{II}}$ *via* $\tilde{\mathrm{II}}(u,v) = \tilde{\mathrm{I}}(u, \tilde{S}(v))$ *satisfies*

$$\tilde{\nabla}\tilde{S} = 0, \tag{4.14}$$

$$\tilde{R}(u,v)w = \tilde{\mathrm{II}}(v,w)\tilde{S}(u) - \tilde{\mathrm{II}}(u,w)\tilde{S}(v). \tag{4.15}$$

Then there exists an immersion $\theta : U \to \mathbb{R}^3$ *of a surface* Σ *such that the induced first and second fundamental forms* I, II *coincide with* $\tilde{\mathrm{I}}$, $\tilde{\mathrm{II}}$, *respectively, that is,*

$$\tilde{\mathrm{I}} = \theta^*(\mathrm{I}),$$

$$\tilde{\mathrm{II}} = \theta^*(\mathrm{II}).$$

The surface Σ *is uniquely determined up to proper isometries of* $\mathbb{R}^3$.

Proof. Since $\tilde{\mathrm{I}}$ is a positive definite symmetric bilinear form, we can choose a frame $\{\tilde{e}_1, \tilde{e}_2\}$ on U that is orthonormal with respect to $\tilde{\mathrm{I}}$. Denote the corresponding dual frame of 1-forms by $\{\tilde{\omega}^1, \tilde{\omega}^2\}$, and define the 1-forms $\tilde{\omega}_1^2, \tilde{\omega}_1^3, \tilde{\omega}_2^3$ by

$$\tilde{\omega}_1^2(v) = \tilde{\mathrm{I}}(\tilde{\nabla}_v \tilde{e}_1, \tilde{e}_2),$$

$$\tilde{\omega}_1^3(v) = \tilde{\mathrm{II}}(v, \tilde{e}_1),$$

$$\tilde{\omega}_2^3(v) = \tilde{\mathrm{II}}(v, \tilde{e}_2).$$

We extend these 1-forms $(\tilde{\omega}_i^j)$ antisymmetrically by requiring $\tilde{\omega}_i^j = -\tilde{\omega}_j^i$. After some computation, our assumptions (4.12) to (4.15) then imply that $(\tilde{\omega}_i^j)$ satisfy the requirements of Theorem 4.19. $\square$

4.2.4 Regularity theory

Recall that for a given surface immersion $\theta : U \to \mathbb{R}^3$, if we introduce coordinates (x_1, x_2, x_3), its first and second fundamental forms (g_{ij}), (h_{ij}), $1 \leq i, j \leq 3$ are given by

$$g_{ij} = \partial_i\theta \cdot \partial_j\theta, \tag{4.16}$$

$$h_{ij} = \partial_{ij}\theta \cdot \frac{\partial_1\theta \times \partial_2\theta}{|\partial_1\theta \times \partial_2\theta|}. \tag{4.17}$$

The Levi–Civita connection on $\Sigma = \theta(U)$ is represented by the Christoffel symbols Γ^k_{ij},

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\partial_j g_{i\ell} + \partial_i g_{j\ell} - \partial_\ell g_{ij}\right). \tag{4.18}$$

Clearly, whenever $\theta \in C^2(U, \mathbb{R}^3)$, it follows that $g_{ij} \in C^1$, $h_{ij} \in C^0$, and $\Gamma^k_{ij} \in C^0$. However, conversely, once we are given symmetric matrices of functions g_{ij}, h_{ij} and define Γ^k_{ij} by (4.18), writing the Gauss–Codazzi–Mainardi equations in coordinates,

$$\partial_\ell\Gamma^k_{ij} - \partial_j\Gamma^k_{i\ell} + \Gamma^m_{ij}\Gamma^k_{m\ell} - \Gamma^m_{i\ell}\Gamma^k_{mj} = h_{ij}h^k_\ell - h_{i\ell}h^k_j, \tag{4.19}$$

$$\partial_\ell h_{ij} - \partial_j h_{i\ell} + \Gamma^k_{ij}h_{k\ell} - \Gamma^k_{i\ell}h_{kj} = 0, \tag{4.20}$$

shows that in order for them to hold, we need to require that $g_{ij} \in C^2$, $h_{ij} \in C^1$. Indeed, the classical proof of the fundamental theorem of surface theory succeeds in this case, yielding an immersion $\theta \in C^3(U, \mathbb{R}^3)$.

This *regularity paradox* had been identified by Hartman and Wintner [HW50a], who then proved the fundamental theorem of surface theory [HW50b] for first and second fundamental forms in C^1 and C^0, respectively, by considering the Gauss–Codazzi–Mainardi equations in an integrated form, that is,

$$\int_J (\Gamma^k_{i1}\,\mathrm{d}x_1 + \Gamma^k_{i2}\,\mathrm{d}x_2) = \int_{\mathrm{dom}\,J} (\Gamma^m_{i1}\Gamma^k_{m2} - \Gamma^m_{i2}\Gamma^k_{m1} - h_{i1}h^k_2 - h_{i2}h^k_1)\,\mathrm{d}x,$$

$$\int_J (h_{i1}\,\mathrm{d}x_1 + h_{i2}\,\mathrm{d}x_2) = \int_{\mathrm{dom}\,J} (\Gamma^m_{i1}h_{m2} - \Gamma^m_{i2}h_{m1})\,\mathrm{d}x$$

for all C^1-Jordan curves J in U, where $\operatorname{dom} J$ is the bounded open set with boundary J. Their method of proof paralleled the smooth case in the sense that they proved and used a corresponding generalised version of the existence theorem for systems of total differential equations.

The results of Hartman and Wintner have been generalised further to the realm of weakly differentiable functions in a series of papers by Mardare [Mar03b; Mar05] (see also [Mar07] for the theory of weak solutions of more general systems). Specifically, he proved the following

Theorem 4.21 ([Mar03b; Mar05]). *Let $U \subset \mathbb{R}^2$ be a connected and simply connected open set and let $2 < p \leq \infty$. Suppose that $(a_{ij}) \in W^{1,p}_{\mathrm{loc}}(U, \mathrm{Sym}^+(2))$, $(b_{ij}) \in L^p_{\mathrm{loc}}(U, \mathrm{Sym}(2))$ are given such that the Gauss–Codazzi–Mainardi equations (4.19) and (4.20) are satisfied in the distributional sense, i. e.,*

$$\int_U (\Gamma^k_{i\ell}\partial_j\phi - \Gamma^k_{ij}\partial_\ell\phi + \Gamma^m_{ij}\Gamma^k_{m\ell}\phi - \Gamma^m_{i\ell}\Gamma^k_{m\ell}\phi)\,\mathrm{d}x = \int_U (b_{ij}b^k_\ell - b_{i\ell}b^k_j)\phi\,\mathrm{d}x,$$
$$\int_U (b_{i\ell}\partial_j\phi - b_{ij}\partial_\ell\phi + \Gamma^m_{ij}b_{m\ell}\phi - \Gamma^m_{i\ell}b_{mj}\phi)\,\mathrm{d}x = 0$$

for any $\phi \in \mathcal{D}(U)$. Then there exists an immersion $\theta \in W^{2,p}_{\mathrm{loc}}(U, \mathbb{R}^3)$, unique up to proper isometries of $\mathbb{R}^3$, such that the first and second fundamental forms of the surface $\Sigma = \theta(U)$ are given by (a_{ij}) and (b_{ij}), respectively, as in (4.16) and (4.17).

In order to prove this result, first for $p = \infty$ and then for any $p > 2$, Mardare utilised the fact that the Gauss–Codazzi–Mainardi equations are equivalent to the equation

$$\partial_i\Gamma_j + \Gamma_i\Gamma_j = \partial_j\Gamma_i + \Gamma_j\Gamma_i, \qquad 1 \leq i, j \leq 2$$

being satisfied in the distributional sense. Here, $\Gamma_i : U \to \mathrm{gl}(3)$ is given by

$$\Gamma_i = \begin{pmatrix} \Gamma^1_{i1} & \Gamma^1_{i2} & -b^1_i \\ \Gamma^2_{i1} & \Gamma^2_{i2} & -b^2_i \\ b_{i1} & b_{i2} & 0 \end{pmatrix},$$

where Γ^k_{ij} is computed from (a_{ij}) via (4.18). Then, Theorem 4.21 could be proved much in the same way as in the smooth case, that is, by solving the Pfaffian system

$$\partial_i P = P\Gamma_i, \qquad 1 \leq i \leq 2,$$

and applying the Poincaré lemma.

Generalisations of the Poincaré lemma to right hand sides in L^p are well-known in the literature [e. g., BBM00; GM12], even for $p \geq 1$.

Lemma 4.22 (Weak Poincaré lemma [Mar07, Thm. 6.5])**.** *Let $U \subset \mathbb{R}^2$ be a connected and simply connected open set and let $p \geq 1$. Let $f_i \in L^p_{\mathrm{loc}}(U)$, $i = 1, 2$, be functions that satisfy*

$$\partial_1 f_2 = \partial_2 f_1 \qquad \text{in } \mathcal{D}'(U).$$

Then there exists a function $\theta \in W^{1,p}_{\mathrm{loc}}(U)$, unique up to an additive constant, such that

$$\partial_i \theta = f_i \qquad \text{in } L^p_{\mathrm{loc}}(U).$$

Hence, to prove Theorem 4.21, Mardare proved a corresponding existence and uniqueness theorem for two-dimensional Pfaffian systems with coefficients in L^p, $2 < p \leq \infty$:

Theorem 4.23 ([Mar03b; Mar05])**.** *Let $U \subset \mathbb{R}^2$ be a connected and simply connected open set, $x_0 \in U$, $P_0 \in \mathrm{gl}(k)$, and let $2 < p \leq \infty$. Suppose that $\Gamma_i \in L^p_{\mathrm{loc}}(U, \mathrm{gl}(k))$, $i = 1, 2$, satisfy*

$$\partial_1 \Gamma_2 + \Gamma_1 \Gamma_2 = \partial_2 \Gamma_1 + \Gamma_2 \Gamma_1 \tag{4.21}$$

in $\mathcal{D}'(U, \mathrm{gl}(k))$. *Then the Pfaffian system*

$$\begin{aligned} \partial_i P &= P\Gamma_i \qquad \text{in } \mathcal{D}'(U, \mathrm{gl}(k)), \\ P(x_0) &= P_0 \end{aligned}$$

has a unique solution $P \in W^{1,p}_{\mathrm{loc}}(U, \mathrm{gl}(k))$.

The method of proof of this theorem in the $p = \infty$ case [Mar03b] is to solve the system of differential equations locally by integrating the equations along a set of 'admissible' straight lines, using an approximation argument. The existence of such 'admissible' lines is ensured by a Lebesgue–Besicovitch-type theorem for locally integrable functions. The local solutions are then extended to a global solution on U using a gluing procedure, where the uniqueness of the global solution is implied by the fact that the domain is simply connected. Meanwhile, for $2 < p < \infty$ [Mar05] an intricate smoothing argument based on a stability result for Pfaffian systems with L^p-coefficients is carried out.

4.3 Optimal regularity theorem

In view of Mardare's results, an immediate question is whether they are optimal as far as the requirements on the regularity of the prescribed first and second fundamental forms (in Theorem 4.21), or, respectively, the coefficients of the Pfaffian system (in Theorem 4.23) are concerned. Starting with the latter [Mar05], we see immediately that in order for the compatibility conditions as in (4.21) to make sense distributionally we need to require that $\Gamma_i \in L^2_{\mathrm{loc}}$ so that the product terms are integrable.

Moreover, in the scalar case, in which we seek a solution $q : U \to \mathbb{R}$ of

$$\partial_i q = \gamma_i q, \qquad i = 1, 2,$$

the compatibility condition for $\gamma_1, \gamma_2 : U \to \mathbb{R}$ reads

$$\partial_1 \gamma_2 - \partial_2 \gamma_1 = 0.$$

Therefore, whenever $\gamma_i \in L^p_{\mathrm{loc}}(U)$, $p \geq 1$, the weak Poincaré lemma 4.22 ensures the existence of a solution $\theta \in W^{1,p}_{\mathrm{loc}}(U)$ of

$$\partial_i \theta = \gamma_i,$$

whereby a solution to the Pfaffian system is given by

$$q(x) = C\mathrm{e}^{\theta(x)}$$

for almost every $x \in U$ and any constant $C \in \mathbb{R}$. Now, in order for q to be integrable, we need to assume that $p > 2$ so that $\theta \in L^\infty_{\mathrm{loc}}(U)$ by virtue of the Sobolev embedding $W^{1,p}_{\mathrm{loc}}(U) \subset L^\infty_{\mathrm{loc}}(U)$ [Mar05]. Finally, in order to consider the Cauchy problem as in Theorem 4.23, the embedding $W^{1,p}_{\mathrm{loc}}(U) \subset C^0(U)$ is necessary. Therefore, *under the given assumptions*, Theorem 4.23 is indeed optimal.

In order to bridge the gap between the result for $p > 2$ and the endpoint case $p = 2$, in the spirit of compensated compactness theory, we are thus led to search for additional algebraic structure to improve the regularity properties of the equation. As far as the application to the fundamental theorem of surface theory that we have in mind is concerned, the key insight is that in an orthonormal frame, the connection forms ω^i_j are antisymmetric. Consequently, we consider Pfaffian systems with antisymmetric coefficients $\Omega_i \in L^2(U, \mathrm{so}(m))$. It then turns out that Rivière's lemma 3.5 on Coulomb gauges is precisely the ingredient we need to prove the existence and uniqueness theorem for Pfaffian systems in the optimal regularity case. First, we prove the local result.

Proposition 4.24. *Let $U \subset \mathbb{R}^2$ be a contractible bounded regular domain and let $\Omega \in L^2(U, \mathrm{so}(m) \otimes \wedge^1\mathbb{R}^2)$. Suppose that Ω satisfies the compatibility equation*

$$\partial_i \Omega_j - \partial_j \Omega_i = \Omega_j \Omega_i - \Omega_i \Omega_j \tag{4.22}$$

in the distributional sense. Then there exists $P \in W^{1,2}(U, \mathrm{SO}(m))$ such that

$$\nabla P + \Omega P = 0. \tag{4.23}$$

Moreover, if P_0 and P_1 are two such solutions then there exists a constant $C \in \mathrm{SO}(m)$ such that

$$P_0 = P_1 C.$$

Proof. By Lemma 3.5, there exist $\xi \in W_0^{1,2}(U, \mathrm{so}(m))$ and $P \in W^{1,2}(U, \mathrm{SO}(m))$ such that

$$P^{-1}\nabla P + P^{-1}\Omega P = \nabla^\perp \xi.$$

Recall that we write Ω_1, Ω_2 for the components of the $\mathrm{so}(m)$-valued 1-form Ω. We rewrite the above equation as

$$\begin{aligned}
\partial_1 P + \Omega_1 P &= -P\partial_2 \xi, \\
\partial_2 P + \Omega_2 P &= P\partial_1 \xi,
\end{aligned}$$

so that

$$\begin{aligned}
-\partial_2\partial_1 P - (\partial_2\Omega_1)P - \Omega_1(\partial_2 P) &= (\partial_2 P)(\partial_2\xi) + P\partial_2\partial_2\xi, \\
\partial_1\partial_2 P + (\partial_1\Omega_2)P + \Omega_2(\partial_1 P) &= (\partial_1 P)(\partial_1\xi) + P\partial_1\partial_1\xi.
\end{aligned}$$

We obtain, using the compatibility equation (4.22),

$$\begin{aligned}(\partial_1 P)(\partial_1\xi)+(\partial_2 P)(\partial_2\xi)+P\Delta\xi &= (\Omega_2\Omega_1-\Omega_1\Omega_2)P+\Omega_2(\partial_1 P)-\Omega_1(\partial_2 P)\\ &=\Omega_2(-\partial_1 P-P\partial_2\xi)-\Omega_1(-\partial_2 P+P\partial_1\xi)\\ &\quad+\Omega_2(\partial_1 P)-\Omega_1(\partial_2 P)\\ &=\ -\Omega_2 P(\partial_2\xi)-\Omega_1 P(\partial_1\xi).\end{aligned}$$

Note that we can rewrite this equation more succinctly as

$$P\Delta\xi=-\nabla^\perp P\cdot\nabla^\perp\xi-(\Omega P)\cdot\nabla\xi.$$

On the right hand side, we get

$$\begin{aligned}-\nabla^\perp P\cdot\nabla^\perp\xi-(\Omega P)\cdot\nabla\xi = &\ -(\partial_1 P)(\partial_1\xi)-(\partial_2 P)(\partial_2\xi)\\ &-(-P\partial_2\xi-\partial_1 P)(\partial_1\xi)-(P\partial_1\xi-\partial_2 P)(\partial_2\xi)\\ =&\ P(\partial_2\xi)(\partial_1\xi)-P(\partial_1\xi)(\partial_2\xi),\end{aligned}$$

and thus

$$P\Delta\xi=-P\nabla^\perp\xi\cdot\nabla\xi,$$

or equivalently

$$\Delta\xi=(\partial_2\xi)(\partial_1\xi)-(\partial_1\xi)(\partial_2\xi). \tag{4.24}$$

While the right hand side of this equation is not necessarily equal to zero, we claim that (4.24) does imply that $\xi\equiv 0$, using that $\xi|_{\partial U}=0$.

We follow an argument by Wente [Wen75]. We may assume that $U=B_1(0)\subset\mathbb{R}^2$. By Theorem 3.1, $\xi\in W_0^{1,2}$ is continuous in $\bar U$ and, indeed, $\xi\in C^\infty$ in the interior. Extend ξ to $\mathbb{R}^2\cong\mathbb{C}$ by inversion in the unit circle, which is a conformal map, and let the same letter now refer to the extension. It satisfies $\xi \in C^\infty(\mathbb{R}^2, \mathrm{so}(m))$, (4.24), $\xi|_{S^1} = 0$, and

$$\|\nabla\xi\|_{L^2}^2 \le C\,\|\Omega\|_{L^2(U)}^2 \,.$$

Now, denoting by $\langle\cdot\,,\cdot\rangle$ the standard inner product on real matrices, extended complex linearly to complex matrices, and writing $\xi_z := \partial_z \xi = \frac{1}{2}(\xi_x - i\xi_y)$ and $\xi_{\bar z} := \frac{1}{2}(\xi_x + i\xi_y)$, define

$$\Phi = \langle \xi_z, \xi_z\rangle = \frac{1}{4}\left(|\xi_x|^2 - 2i\langle \xi_x, \xi_y\rangle - |\xi_y|^2\right).$$

In order to show that Φ is holomorphic, we compute

$$\begin{aligned}
\Phi_{\bar z} &= 2\langle \xi_{z\bar z}, \xi_z\rangle \\
&= \frac{1}{2}\,\mathrm{tr}\left((\Delta\xi)(\xi_z)^T\right) \\
&= \frac{1}{4}\,\mathrm{tr}\left((\xi_y\xi_x - \xi_x\xi_y)\xi_x - i(\xi_y\xi_x - \xi_x\xi_y)\xi_y\right) \\
&= \frac{1}{4}\,\mathrm{tr}\left((\xi_x\xi_y\xi_x - \xi_x\xi_y\xi_x) - i(\xi_y\xi_x\xi_y - \xi_y\xi_x\xi_y)\right) \\
&= 0,
\end{aligned}$$

where we have used $\Delta\xi = 4\xi_{z\bar z}$, (4.24), and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. But we also have that $\Phi \in L^1(\mathbb{C})$ and hence $\Phi \equiv 0$.

As a result, in addition to (4.24), ξ satisfies

$$|\xi_x|^2 - |\xi_y|^2 = \langle \xi_x, \xi_y\rangle = 0.$$

Let us view ξ as a map into $\mathbb{R}^{m^2}$, which is smooth and conformal. By the Hartman–Wintner lemma [Jos11, Lemma 9.1.7] and (4.24), we deduce that the points where $|\xi_x| = |\xi_y| = 0$ are isolated whenever ξ is non-constant. However, we know that $\xi = \nabla\xi = 0$ on the unit circle. Therefore, ξ must be constant, and as $\xi|_{S^1} = 0$, we conclude that $\xi \equiv 0$. This yields (4.23).

Now suppose that $P_0, P_1 \in W^{1,2}(U, \mathrm{SO}(m))$ solve

$$
\begin{aligned}
\nabla P_0 + \Omega P_0 &= 0, \\
\nabla P_1 + \Omega P_1 &= 0
\end{aligned}
$$

in U, respectively. Since $\nabla(P_1^{-1}) = -P_1^{-1}(\nabla P_1)P_1^{-1}$, we have

$$
\begin{aligned}
\nabla(P_1^{-1}P_0) &= \nabla(P_1^{1})P_0 + P_1^{-1}\nabla P_0 \\
&= -P_1^{-1}(\nabla P_1)P_1^{-1}P_0 + P_1^{-1}\nabla P_0 \\
&= P_1^{-1}\Omega P_1 P_1^{-1}P_0 - P_1^{-1}\Omega P_0 \\
&= 0.
\end{aligned}
$$

Thus $P_1^{-1}P_0 = C$, a constant invertible matrix, so that $P_0 = P_1 C$. We also have $C^T C = P_0^{-1}P_1P_1^{-1}P_0 = I$ and $\det C = (\det P_1)^{-1}\det P_0 = 1$, whereby $C \in \mathrm{SO}(m)$. $\square$

Using the local existence and uniqueness, we can then prove the corresponding global result on simply connected domains by means of a gluing procedure.

Theorem 4.25. *Let $U \subset \mathbb{R}^2$ be a connected and simply connected open set and let $\Omega \in L^2(U, \mathrm{so}(m) \otimes \wedge^1\mathbb{R}^2)$ satisfy the compatibility condition (4.22) in the distributional sense. Then there exists $P \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{SO}(m))$ such that $\nabla P = P\Omega$ in U. Moreover, any two such solutions P_0, P_1 are related by $P_0 = CP_1$ with a constant $C \in \mathrm{SO}(m)$.*

Proof. Equipped with the local existence result in Proposition 4.24, we intend to leverage the simple-connectedness of U to construct $P \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{SO}(m))$. While this type of construction can be found in various places in the literature, we reproduce such a proof [Mar08, Thm. 2.1] here almost verbatim, adapting where necessary, for the sake of completeness. In particular, we do not claim any originality.

Note that instead of solving the equation

$$\nabla P = P\Omega,$$

we solve

$$\nabla P + \Omega P = 0.$$

Once a solution P to the latter equation is given, a solution to the former is readily found by transposition and using that $\Omega_i \in \mathrm{so}(m)$.

Let $x_0 \in U$, $B_{r_0}(x_0) \subset U$, $r_0 < \frac{1}{2}\operatorname{dist}(x_0, \mathbb{R}^2 \setminus U)$. Then there exists $P_0 \in W^{1,2}(B_{r_0}(x_0), \mathrm{SO}(m))$ such that

$$\nabla P_0 + \Omega P_0 = 0 \qquad \text{in } B_{r_0}(x_0).$$

Let $x \in U$ be arbitrary and $(\gamma, \Delta, (B_j))$ a triple such that $\gamma : [0,1] \to U$ is continuous, $\gamma(0) = x_0$, $\gamma(1) = x$, $\Delta = (t_0, t_1, \ldots, t_n, t_{n+1})$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, and $(B_j)_{j=0}^n$ a sequence of open balls contained in U such that $B_0 = B_{r_0}(x_0)$ and $\gamma([t_j, t_{j+1}]) \subset B_j$, $j = 1, \ldots, n$.

An example of such a triple can be constructed in the following way: Given any continuous path γ as above, choose $r < \frac{1}{2}\operatorname{dist}(\gamma([0,1]), \mathbb{R}^2 \setminus U)$ and Δ such that $|t_{j+1} - t_j| \leq \delta_r$, where δ_r is such that $|\gamma(t) - \gamma(s)| < \min\{r, r_0\}$ whenever $|t - s| \leq \delta_r$. Then set $B_0 = B_{r_0}(x_0)$, $B_j = B_r(\gamma(t_j))$, $j = 1, \ldots, n$.

Define recursively $P_j \in W^{1,2}(B_j, \mathrm{SO}(m))$, $j = 1, \ldots, n$, such that

$$\begin{aligned} \nabla P_j + \Omega P_j = 0 \qquad & \text{in } B_j, \\ P_j = P_{j-1} \qquad & \text{in } B_j \cap B_{j-1}. \end{aligned}$$

By Proposition 4.24, we know that $P_j = P_{j-1}C$ in $B_j \cap B_{j-1}$, so we can, if necessary, replace P_j by $P_j C^{-1}$ to ensure the validity of the second equation. We retain $P_n \in$

$W^{1,2}(B_n, \mathrm{SO}(m))$ and claim that it is independent of our choice of $(\gamma, \Delta, (B_j))$.

Fix γ, Δ and consider two sequences of balls (B_j), $(\tilde{B}_j)$. We know that $P_0 = \tilde{P}_0$ in $B_0 = \tilde{B}_0$ by definition. Now assume that $P_j = \tilde{P}_j$ in $B_j \cap \tilde{B}_j$. Then we have that

$$P_{j+1} = P_j = \tilde{P}_j = \tilde{P}_{j+1}$$

in the open set $B_{j+1} \cap B_j \cap \tilde{B}_j \cap \tilde{B}_{j+1}$. Note that this set is non-empty since $\gamma(t_{j+1})$ is contained in it. Therefore, we must have $P_{j+1} = \tilde{P}_{j+1}$ in $B_{j+1} \cap \tilde{B}_{j+1}$.

In order to prove that P_n is independent of the division of the unit interval Δ, first consider two divisions Δ, $\tilde{\Delta}$ that differ only by an additional point t^* inserted between t_k and t_{k+1} in $\tilde{\Delta}$. Let $(B_j)_{j=0}^n$ be an admissible family of open balls for Δ and define the family $(\tilde{B}_j)_{j=0}^{n+1}$ for $\tilde{\Delta}$ by $\tilde{B}_j = B_j$, $j = 0, \ldots, k$ and $\tilde{B}_{j+1} = B_j$, $j = k, \ldots, n$. By definition, this family is admissible and $\tilde{P}_j = P_j$ in B_j for $j = 0, \ldots, k$. According to the recursive definition of $\tilde{P}_j$, we then have $\tilde{P}_{k+1} = \tilde{P}_k = P_k$ in $\tilde{B}_{k+1} \cap \tilde{B}_k = B_k$. Therefore, $\tilde{P}_j = P_j$ in B_j for $j = k, \ldots, n$.

Now let Δ and $\tilde{\Delta}$ be two arbitrary divisions of the unit interval with associated admissible families of open balls $(B_j)_{j=0}^n$ and $(\tilde{B}_j)_{j=0}^{\tilde{n}}$, respectively. In addition, consider the joint division $\bar{\Delta} = (s_0, \ldots, s_{m+1})$ with

$$\{s_0, \ldots, s_{m+1}\} = \{t_0, \ldots, t_{n+1}\} \cup \{\tilde{t}_0, \ldots, \tilde{t}_{\tilde{n}+1}\}, \qquad m \leq n + \tilde{n}.$$

Then, applying the above argument $(m - n)$ times, one obtains $\bar{P}_m = P_n$ in B_n, and similarly, after $(m - \tilde{n})$ steps, $\bar{P}_m = \tilde{P}_{\tilde{n}}$ in $\tilde{B}_{\tilde{n}}$. As we have already shown that the solution $\bar{P}_m$ does not depend on the family of admissible open balls, we get

$$P_n = \tilde{P}_{\tilde{n}} \qquad \text{in } B_n \cap \tilde{B}_{\tilde{n}}.$$

Lastly, in proving that the final solution is independent of the path γ, we make use of the fact that U is simply connected. Let $(\gamma, \Delta, (B_j)_{j=0}^n)$ and $(\tilde{\gamma}, \tilde{\Delta}, (\tilde{B}_j)_{j=0}^{\tilde{n}})$ be two admissible triples associated with $x \in U$. Then there exists $H \in C^0([0,1]^2, U)$ such that

$$H(0,\cdot) = \gamma, \qquad H(1,\cdot) = \tilde{\gamma},$$
$$H(\cdot,0) = x_0, \qquad H(\cdot,1) = x.$$

For each $s \in [0,1]$, choose $(\gamma_s, \Delta_s, (B_j^s)_{j=0}^{n_s})$ to be an admissible triple associated with x such that

$$\gamma_s = H(s,\cdot)$$

and that they agree with those already chosen at $s = 0$ and $s = 1$, respectively. Denoting

$$s^* := \sup\{s \in [0,1] : P_{n_s}^s = P_{n_0}^0 \text{ in } B_{n_s}^s \cap B_{n_0}^0\},$$

we thus need to prove that $s^* = 1$ and the supremum is attained.

First, we show that for small enough $s > 0$, we have that

$$P_{n_s}^s = P_{n_0}^0 \qquad \text{in } B_{n_s}^s \cap B_{n_0}^0.$$

This implies that $s^* > 0$. We note that the triple $(\gamma_s, \Delta, (B_j))$ associated with x is admissible if s is small enough. This results from the fact that for such s, $\gamma_s([t_j, t_{j+1}]) \subset B_j$, $j = 0, \ldots, n$. Suppose that this is not the case and there exist sequences (s_m), (t_m) with $s_m \to 0$ as $m \to \infty$ and $t_m \in [t_j, t_{j+1}]$ such that $\gamma_{s_m}(t_m) = H(s_m, t_m) \notin B_j$. Then, up to the choice of a subsequence, we have that $t_m \to t \in [t_j, t_{j+1}]$. By the continuity of H, we conclude that $H(s_m, t_m)$ converges to $\gamma(t) \in B_j$. However, this contradicts our assumption that $H(s_m, t_m) \notin B_j$ for all m. Therefore, since we have already shown that solutions are independent of the choice of a division and a family of open balls, the claim follows.

Second, in the same vein we also have that for $\alpha > 0$ sufficiently small and $\varepsilon \in [0, \alpha)$, the triple $(\gamma_{s^*-\varepsilon}, \Delta_{s^*}, (B_j^{s^*}))$ is admissible for x and thus

$$P_{n_{s^*}}^{s^*} = P_{n_{s^*-\varepsilon}}^{s^*-\varepsilon} \qquad \text{in } B_{n_{s^*}}^{s^*} \cap B_{n_{s^*-\varepsilon}}^{s^*-\varepsilon}.$$

On the other hand, since s^* is a supremum, it holds for some $\delta \in [0, \alpha)$ that

$$P_{n_{s^*-\delta}}^{s^*-\delta} = P_{n_0}^0 \qquad \text{in } B_{n_{s^*-\delta}}^{s^*-\delta} \cap B_{n_0}^0,$$

whereby

$$P_{n_{s^*}}^{s^*} = P_{n_0}^0 \qquad \text{in } B_{n_{s^*}}^{s^*} \cap B_{n_0}^0$$

and s^* is a maximum.

Finally, if $s^* < 1$ then in a similar fashion one concludes that for $\varepsilon > 0$ sufficiently small the triple $(\gamma_{s^*+\varepsilon}, \Delta_{s^*}, (B_j^{s^*}))$ is admissible, contradicting the definition of s^*. Thus $P_{n_0}^0 = P_{n_1}^1$.

It is then possible to define a global solution by means of a gluing procedure. For any $x \in U$, let $(\gamma, \Delta, (B_j)_{j=0}^n)$ be an admissible triple and let $B_x = B_n$ and $P_x = P_n \in W^{1,2}(B_x, \mathrm{SO}(m))$, constructed as above. We claim that for any $x, y \in U$ such that $B_x \cap B_y$ is non-empty it holds that

$$P_x = P_y \qquad \text{in } B_x \cap B_y.$$

Let $z \in B_x \cap B_y$ and let $(\gamma, \Delta, (B_j)_{j=0}^n)$ and $(\tilde{\gamma}, \tilde{\Delta}, (\tilde{B}_j)_{j=0}^{\tilde{n}})$ be two admissible triples for x and y, respectively. If we define γ^* to be the path obtained by joining γ and the segment $[x.z]$, which lies entirely within B_x, parametrised such that $\gamma^*(\frac{1}{2}) = x$ and $\gamma^*(1) = z$, then we obtain an admissible triple for z by letting, for $j \leq n$,

$$\begin{aligned} \Delta^* &:= \left(\frac{t_0}{2}, \dots, \frac{t_{n+1}}{2}, t_{n+1}\right), \\ B_j^* &:= B_j, \\ B_{n+1}^* &:= B_n = B_x. \end{aligned}$$

The same construction applied to $\tilde{\gamma}$ yields another admissible triple for z and we thus obtain

$$\begin{aligned} P_{n+1}^* &= \tilde{P}_{\tilde{n}+1}^* && \text{in } B_x \cap B_y, \\ P_{n+1}^* &= P_n = P_x && \text{in } B_x, \\ \tilde{P}_{\tilde{n}+1}^* &= \tilde{P}_{\tilde{n}} = P_y && \text{in } B_y, \end{aligned}$$

proving the claim.

Therefore, we define a distribution P on the set $U = \bigcup_{x \in U} B_x$ as follows: Let $\phi \in \mathcal{D}(U)$. Since ϕ has compact support, there is a finite number of points $x_i \in U$ such that $\operatorname{spt} \phi \subset \bigcup_{i=1}^m B_{x_i}$. Moreover, let $(\theta_i)_{i=1}^m$ be a partition of unity subordinate to the covering $(B_{x_i})_{i=1}^m$ of $\operatorname{spt} \phi$. We then define

$$\langle P, \phi \rangle := \sum_{i=1}^m \langle P_{x_i}, \theta_i \phi \rangle.$$

By the gluing principle of Schwartz [Mar08, Thm. 1.2], the result is a distribution P on U that satisfies $P = P_x$ in B_x for all $x \in U$.

In order to show that P solves $\nabla P + \Omega P = 0$ in the distributional sense, we let $\phi \in \mathcal{D}(U)$ and $K \subset U$ be a compact neighbourhood of $\operatorname{spt} \phi$ such that $K \subset \bigcup_{i=1}^m B_{x_i}$ for some family of open balls $(B_{x_i})_{i=1}^m$. As above, let $(\theta_i)_{i=1}^m$ be a partition of unity subordinate to the covering (B_{x_i}). We compute

$$\begin{aligned}
\langle P, \nabla\phi\rangle &= \sum_{i=1}^{m} \langle P_{x_i}, \theta_i \nabla\phi\rangle \\
&= \sum_{i=1}^{m} \langle P_{x_i}, \nabla(\theta_i\phi) - (\nabla\theta_i)\phi\rangle \\
&= \sum_{i=1}^{m} \langle -\Omega P_{x_i}, \theta_i\phi\rangle - \langle P_{x_i}, (\nabla\theta_i)\phi\rangle \\
&= \langle -\Omega P, \phi\rangle.
\end{aligned}$$

Here, we have used that $\sum_{i=1}^{m} \theta_i = 1$ in K and hence $\nabla(\sum_{i=1}^{m} \theta_i)\phi = 0$ in U.

Since, by construction, $P = P_{x_i}$ in B_{x_i} and $P_{x_i} \in W^{1,2}(B_{x_i}, \mathrm{SO}(m))$, it follows that $P \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{SO}(m))$. Furthermore, we may repeat the same calculation as in the proof of Proposition 4.24 to infer that any two such solutions differ by a multiplicative constant in $\mathrm{SO}(m)$. □

4.4 Application to surfaces

In this section, we shall apply Theorem 4.25 in order to prove the existence of a $W^{2,2}_{\mathrm{loc}}$-immersion of a surface with prescribed first and second fundamental forms in the classes $W^{1,2}_{\mathrm{loc}}$ and L^2_{loc}, respectively. First, we motivate the definition of appropriate antisymmetric matrix fields Ω_i that serve as the coefficients of a Pfaffian system. After that, we show that the quantities derived from the given matrix fields that are to be realised as fundamental forms of a surface possess the required regularity. We then prove Theorem 4.26. Lastly, we demonstrate that the compatibility equation satisfied by the matrix fields Ω_i is equivalent to the Gauss–Codazzi–Mainardi equations in the present setting.

4.4.1 Derivation of antisymmetric coefficients

There is no reason to believe that the connection form should be antisymmetric in an arbitrarily given frame. However, we can always arrange for an antisymmetric matrix of connection forms in a frame that is orthonormal with respect to a given Riemannian metric on Σ, as we now show. While these antisymmetric matrix fields have previously been introduced [CGM08] in the context of the fundamental theorem, the viewpoint of Cartan geometry is advantageous in that the antisymmetric connection form arises naturally from the change of frame.

As in Section 4.1, let $\theta : U \to \mathbb{E}^3$ be a smooth immersion whose image $\Sigma = \theta(U)$ is a regular surface for some open, connected, and simply connected subset U of $\mathbb{R}^2$, and let (e_1, e_2, e_3) be an adapted frame. Again, we denote the pullbacks of the dual and connection forms by $(\bar{\omega}^i, \bar{\omega}^i_j)$. We then write the connection form as

$$\underline{\omega} = \begin{pmatrix} 0 & \bar{\omega}^1_2 & \bar{\omega}^1_3 \\ \bar{\omega}^2_1 & 0 & \bar{\omega}^2_3 \\ \bar{\omega}^3_1 & \bar{\omega}^3_2 & 0 \end{pmatrix},$$

and define

$$\Gamma_i := \underline{\omega}(e_i) = \begin{pmatrix} 0 & \Gamma^1_{i2} & -h_{1i} \\ \Gamma^2_{i1} & 0 & -h_{2i} \\ h_{1i} & h_{2i} & 0 \end{pmatrix}.$$

Now, given a metric $\bar{g}$ on Σ and an orthonormal frame $\mathrm{f} = (e_1, e_2, e_3)$, we set

$$g = \begin{pmatrix} \bar{g}_{11} & \bar{g}_{12} & 0 \\ \bar{g}_{21} & \bar{g}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\frac{1}{2}}.$$

Defining the frame $\mathrm{f}' = \mathrm{f}g^{-1}$, which is orthonormal with respect to g^2, the Maurer–Cartan form in this frame is given by means of the gauge transformation

$$\underline{\omega}' = (g\underline{\omega} + \mathrm{d}g)g^{-1},$$

which implies in components that

$$\Gamma_i' = (g\Gamma_i - \partial_i g)g^{-1}.$$

4.4.2 Regularity of coefficients

Let $(a_{ij}) \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{Sym}^+(2)) \cap L^\infty_{\mathrm{loc}}(U, \mathrm{Sym}^+(2))$ and $(b_{ij}) \in L^2_{\mathrm{loc}}(U, \mathrm{Sym}(2))$ and define

$$\begin{aligned}
(a^{ij}) &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \\
b_i^j &= a^{jk} b_{ik}, \\
\Gamma_{ij}^k &= \frac{1}{2} a^{k\ell}(\partial_j a_{i\ell} + \partial_i a_{j\ell} - \partial_\ell a_{ij}), \\
G &= \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\frac{1}{2}}, \\
\Gamma_i &= \begin{pmatrix} \Gamma_{i1}^1 & \Gamma_{i2}^1 & -b_i^1 \\ \Gamma_{i1}^2 & \Gamma_{i2}^2 & -b_i^2 \\ b_{i1} & b_{i2} & 0 \end{pmatrix}, \\
\Omega_i &= (G\Gamma_i - \partial_i G)G^{-1}.
\end{aligned}$$

Since $W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ is an algebra, we see that

$$\det(a_{ij}) = a_{11}a_{22} - a_{12}a_{21} \in W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}.$$

Now assume in addition that the (positive) eigenvalues of (a_{ij}) are locally uniformly bounded away from zero, i. e., there exists $C > 0$ such that $0 < C < \lambda_1 < \lambda_2$ almost everywhere in $K \subset\subset U$. Then $\det(a_{ij})^{-1} \in L^\infty_{\mathrm{loc}}$. Therefore, we have that $(a^{ij}) \in L^\infty_{\mathrm{loc}}$. Moreover, the fact that

$$D(\det(a_{ij})^{-1}) = -\frac{D(\det(a_{ij}))}{\det(a_{ij})^2}$$

implies that

$$\det(a_{ij})^{-1} \in W^{1,2}_{\mathrm{loc}} \cap L^{\infty}_{\mathrm{loc}}.$$

Hence

$$(a^{ij}) \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{Sym}^+(2)) \cap L^{\infty}_{\mathrm{loc}}(U, \mathrm{Sym}^+(2)).$$

Furthermore, by the boundedness of (a^{ij}) and as $(a_{ij}) \in W^{1,2}_{\mathrm{loc}}$, we obtain that

$$\Gamma^k_{ij} \in L^2_{\mathrm{loc}}(U).$$

From the formula

$$A^{\frac{1}{2}} = \frac{1}{\sqrt{\operatorname{tr} A + 2\sqrt{\det A}}}(A + \sqrt{\det A} I),$$

valid for any $A \in \mathrm{Sym}^+(2)$, we infer, using again $(a_{ij}) \in W^{1,2}_{\mathrm{loc}} \cap L^{\infty}_{\mathrm{loc}}$ and the boundedness of the eigenvalues away from zero, that

$$(a_{ij})^{\frac{1}{2}}, (a_{ij})^{-\frac{1}{2}} \in W^{1,2}_{\mathrm{loc}} \cap L^{\infty}_{\mathrm{loc}}.$$

Finally, as $\Gamma_i \in L^2_{\mathrm{loc}}$ we conclude that

$$\Omega_i \in L^2_{\mathrm{loc}}(U, \mathrm{gl}(3)).$$

It remains to show that each matrix Ω_i is antisymmetric. (The following argument is taken from the proof of Theorem 7 in Ciarlet, Gratie, and C. Mardare [CGM08].) Equivalently, we may show that

$$G\Omega_i G = G^2\Gamma_i - G\partial_i G$$

is antisymmetric. By a direct computation, using the symmetry of (a_{ij}), we find that

$$G^2\Gamma_i + \Gamma_i^T G^2 = \begin{pmatrix} 2\Gamma_{i11} & \Gamma_{i12} + \Gamma_{i21} & 0 \\ \Gamma_{i21} + \Gamma_{i12} & 2\Gamma_{i22} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \partial_i G^2.$$

Here, as usual, $\Gamma_{ijk} = a_{k\ell}\Gamma_{ij}^{\ell}$. We thus compute

$$\begin{aligned} G\Omega_i G &= G^2\Gamma_i - G\partial_i G \\ &= \frac{1}{2}G^2\Gamma_i + \frac{1}{2}(\partial_i G^2 - \Gamma_i^T G^2) - G\partial_i G \\ &= \frac{1}{2}(G^2\Gamma_i - \Gamma_i^T G^2) + \frac{1}{2}((\partial_i G)G + G\partial_i G) - G\partial_i G \\ &= \frac{1}{2}(G^2\Gamma_i - \Gamma_i^T G^2) + \frac{1}{2}((\partial_i G)G - G\partial_i G), \end{aligned}$$

whereby, indeed, $\Omega_i \in \mathrm{so}(3)$.

Therefore, we have shown that if $(a_{ij}) \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{Sym}^+(2)) \cap L^\infty_{\mathrm{loc}}(U, \mathrm{Sym}^+(2))$ and $(b_{ij}) \in L^2_{\mathrm{loc}}(U, \mathrm{Sym}(2))$ are given and the eigenvalues of (a_{ij}) are locally uniformly bounded from below then $\Omega \in L^2_{\mathrm{loc}}(U, \mathrm{so}(3) \otimes \wedge^1\mathbb{R}^2)$.

4.4.3 Optimal regularity for the fundamental theorem

We are now in a position to state and prove the optimal regularity case of the fundamental theorem of surface theory. By and large, we follow the structure of the proof of the corresponding Theorem 7 in Ciarlet, Gratie, and C. Mardare [CGM08].

Theorem 4.26. *Let U be a connected and simply connected open subset of $\mathbb{R}^2$ and let $(a_{ij}) \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{Sym}^+(2)) \cap L^\infty_{\mathrm{loc}}(U, \mathrm{Sym}^+(2))$ and $(b_{ij}) \in L^2_{\mathrm{loc}}(U, \mathrm{Sym}(2))$ be given. Suppose that the eigenvalues of (a_{ij}) are locally uniformly bounded from below and that the matrix fields (a_{ij}), (b_{ij}) are such that*

$$\partial_1\Omega_2 - \partial_2\Omega_1 = \Omega_2\Omega_1 - \Omega_1\Omega_2, \tag{4.25}$$

where $\Omega \in L^2_{\mathrm{loc}}(U, \mathrm{so}(3) \otimes \wedge^1\mathbb{R}^2)$ *is given by the following sequence of definitions, see also Section 4.4.2:*

$$\begin{aligned}
(a^{ij}) &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \\
b_i^j &= a^{jk} b_{ik}, \\
\Gamma_{ij}^k &= \frac{1}{2} a^{k\ell}(\partial_j a_{i\ell} + \partial_i a_{j\ell} - \partial_\ell a_{ij}), \\
G &= \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\frac{1}{2}}, \\
\Gamma_i &= \begin{pmatrix} \Gamma_{i1}^1 & \Gamma_{i2}^1 & -b_i^1 \\ \Gamma_{i1}^2 & \Gamma_{i2}^2 & -b_i^2 \\ b_{i1} & b_{i2} & 0 \end{pmatrix}, \\
\Omega_i &= (G\Gamma_i - \partial_i G)G^{-1}.
\end{aligned}$$

Then there exists an immersion $\theta \in W^{2,2}_{\mathrm{loc}}(U, \mathbb{R}^3)$ *such that*

$$\begin{aligned}
a_{ij} &= \partial_i\theta \cdot \partial_j\theta && \textit{in } W^{1,2}_{\mathrm{loc}}(U), \\
b_{ij} &= \partial_{ij}\theta \cdot \frac{\partial_1\theta \times \partial_2\theta}{|\partial_1\theta \times \partial_2\theta|} && \textit{in } L^2_{\mathrm{loc}}(U).
\end{aligned}$$

Moreover, the map θ *is unique in* $W^{2,2}_{\mathrm{loc}}(U, \mathbb{R}^3)$ *up to proper isometries of* $\mathbb{R}^3$.

Proof. We have shown in the previous section that $\Omega \in L^2(U, \mathrm{so}(3) \otimes \wedge^1\mathbb{R}^2)$ and by assumption the compatibility equation is satisfied. Therefore, by Theorem 4.25, there exists $P \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{SO}(3))$ such that

$$\partial_i P = P\Omega_i.$$

Let $G_{(i)} = g_i$ denote the i-th column of G. We know that $P \in W^{1,2}_{\text{loc}}$ and $G \in W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}$. Furthermore, as $P \in \mathrm{SO}(3)$, P is essentially bounded. Thus we conclude that $Pg_i \in W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}$.

In order to apply Lemma 4.22, we require that

$$\partial_j(Pg_i) = \partial_i(Pg_j).$$

As $\partial_i P = P\Omega_i$ and $P \in \mathrm{SO}(3)$, we obtain

$$\begin{aligned}\partial_j(Pg_i) - \partial_i(Pg_j) &= (\partial_j P)g_i + P\partial_j g_i - (\partial_i P)g_j - P\partial_i g_j \\ &= P\Omega_j g_i + P\partial_j g_i - P\Omega_i g_j - P\partial_i g_j,\end{aligned}$$

which is equal to zero if and only if

$$\begin{aligned}0 &= \Omega_j g_i + \partial_j g_i - \Omega_i g_j - \partial_i g_j \\ &= (G\Gamma_j - \partial_j G)G^{-1} g_i + \partial_j g_i - (G\Gamma_i - \partial_i G)G^{-1} g_j + \partial_i g_j \\ &= (G\Gamma_j - \partial_j G)e_i + \partial_j g_i - (G\Gamma_i - \partial_i G)e_j + \partial_i g_j \\ &= (G\Gamma_j)_{(i)} - (G\Gamma_i)_{(j)} \\ &= G\begin{pmatrix}\Gamma^1_{ji} \\ \Gamma^2_{ji} \\ b_{ji}\end{pmatrix} - G\begin{pmatrix}\Gamma^1_{ij} \\ \Gamma^2_{ij} \\ b_{ij}\end{pmatrix},\end{aligned}$$

where e_i denotes the i-th unit vector in $\mathbb{R}^3$. Since $\Gamma^k_{ij} = \Gamma^k_{ji}$ and $b_{ij} = b_{ji}$, it follows that

$$\partial_j(Pg_i) - \partial_i(Pg_j) = 0.$$

As a result, by Lemma 4.22, there exists $\theta \in W^{1,2}_{\text{loc}}(U, \mathbb{R}^3)$ such that

$$\partial_i \theta = Pg_i$$

in L^2_{loc}. Since $Pg_i \in W^{1,2}_{\text{loc}}$, we conclude that in fact $\theta \in W^{2,2}_{\text{loc}}(U, \mathbb{R}^3)$. Moreover, as the vectors Pg_i are linearly independent, the map θ is an immersion.

Defining $F := PG \in W^{1,2}_{\text{loc}} \cap L^\infty_{\text{loc}}$ and $f_i = F_{(i)}$ (here, $i = 1, 2, 3$), we have that

$$\partial_i \theta = f_i,$$
$$F^T F = G^2 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$f_i^T f_j = a_{ij},$$

whence

$$\partial_i \theta \cdot \partial_j \theta = a_{ij},$$

and the matrix field (a_{ij}) is indeed the first fundamental form of the surface $\theta(U)$.

Furthermore, it is clear that $f_i^T f_3 = \delta_{i3}$, $i = 1, 2, 3$. Therefore, taking into account that F is positive definite almost everywhere, it follows that

$$f_3 = \frac{f_1 \times f_2}{|f_1 \times f_2|}.$$

Meanwhile, we compute

$$\begin{aligned} \partial_{ij} \theta &= \partial_j (Pg_i) \\ &= (\partial_j P) g_i + P \partial_j g_i \\ &= P(\Omega_j g_i + \partial_j g_i) \\ &= P(\Omega_j G + \partial_j G)_{(i)} \\ &= P(G \Gamma_j)_{(i)} \\ &= F(\Gamma_j)_{(i)}. \end{aligned}$$

As a result, we obtain that

$$\begin{aligned}\partial_{ij}\theta \cdot f_3 &= (\partial_{ij}\theta)^T f_3 \\ &= \left((\Gamma_j)_{(i)}\right)^T F^T f_3 \\ &= \begin{pmatrix}\Gamma^1_{ji} & \Gamma^2_{ji} & b_{ji}\end{pmatrix} \cdot e_3 \\ &= b_{ji},\end{aligned}$$

whereby the matrix field (b_{ij}) is the second fundamental form of $\theta(U)$.

Regarding the question of uniqueness of the immersion thus obtained, we note that by Theorem 4.25, the matrix field P is unique up to a multiplicative constant $C \in \mathrm{SO}(3)$, while the function θ that results from the application of Lemma 4.22 is unique up to an additive constant $b \in \mathbb{R}^3$. Therefore, any two immersions θ, $\tilde{\theta}$ constructed by means of the above procedure are related by

$$\theta = C\tilde{\theta} + b,$$

and the proof is complete. □

4.4.4 Equivalence of compatibility conditions

By means of a direct computation, we argue that the compatibility condition (4.25) is equivalent to the Gauss–Codazzi–Mainardi equations.

Proposition 4.27. *In the $W^{2,2}_{\mathrm{loc}}$-setting of Theorem 4.26, the compatibility condition* (4.25) *is necessary and sufficient for the Gauss–Codazzi–Mainardi equations to hold.*

Proof. Assuming the compatibility condition, we have shown the existence of a $W^{2,2}_{\mathrm{loc}}$-immersion with associated first and second fundamental forms (a_{ij}), (b_{ij}) which necessarily satisfy the Gauss–Codazzi–Mainardi equations in the distributional sense.

Moreover, we have

$$\begin{aligned}
0 &= \partial_1\Omega_2 - \partial_2\Omega_1 - \Omega_2\Omega_1 + \Omega_1\Omega_2 \\
&= \partial_1\big((G\Gamma_2 - \partial_2 G)G^{-1}\big) \\
&\quad - \partial_2\big((G\Gamma_1 - \partial_1 G)G^{-1}\big) \\
&\quad - (G\Gamma_2 - \partial_2 G)G^{-1}(G\Gamma_1 - \partial_1 G)G^{-1} \\
&\quad + (G\Gamma_1 - \partial_1 G)G^{-1}(G\Gamma_2 - \partial_2 G)G^{-1} \\
&= (\partial_1(G\Gamma_2) - \partial_1\partial_2 G)G^{-1} - (G\Gamma_2 - \partial_2 G)G^{-1}(\partial_1 G)G^{-1} \\
&\quad - (\partial_2(G\Gamma_1) - \partial_2\partial_1 G)G^{-1} + (G\Gamma_1 - \partial_1 G)G^{-1}(\partial_2 G)G^{-1} \\
&\quad - (G\Gamma_2 - \partial_2 G)G^{-1}(G\Gamma_1 - \partial_1 G)G^{-1} \\
&\quad + (G\Gamma_1 - \partial_1 G)G^{-1}(G\Gamma_2 - \partial_2 G)G^{-1}
\end{aligned}$$

if and only if

$$\begin{aligned}
0 &= \partial_1(G\Gamma_2) - \partial_1\partial_2 G - (G\Gamma_2 - \partial_2 G)G^{-1}(\partial_1 G) \\
&\quad - \partial_2(G\Gamma_1) + \partial_2\partial_1 G + (G\Gamma_1 - \partial_1 G)G^{-1}(\partial_2 G) \\
&\quad - (G\Gamma_2 - \partial_2 G)(\Gamma_1 - G^{-1}\partial_1 G) \\
&\quad + (G\Gamma_1 - \partial_1 G)(\Gamma_2 - G^{-1}\partial_2 G) \\
&= (\partial_1 G)\Gamma_2 + G\partial_1\Gamma_2 - G\Gamma_2 G^{-1}(\partial_1 G) + (\partial_2 G)G^{-1}(\partial_1 G) \\
&\quad - (\partial_2 G)\Gamma_1 - G\partial_2\Gamma_1 + G\Gamma_1 G^{-1}(\partial_2 G) - (\partial_1 G)G^{-1}(\partial_2 G) \\
&\quad - G\Gamma_2\Gamma_1 + G\Gamma_2 G^{-1}(\partial_1 G) + (\partial_2 G)\Gamma_1 - (\partial_2 G)G^{-1}(\partial_1 G) \\
&\quad + G\Gamma_1\Gamma_2 - G\Gamma_1 G^{-1}(\partial_2 G) - (\partial_1 G)\Gamma_2 + (\partial_1 G)G^{-1}(\partial_2 G) \\
&= G(\partial_1\Gamma_2 - \partial_2\Gamma_1 - \Gamma_2\Gamma_1 + \Gamma_1\Gamma_2).
\end{aligned}$$

Therefore, the compatibility condition is equivalent to

$$\partial_i\Gamma_j + \Gamma_i\Gamma_j = \partial_j\Gamma_i + \Gamma_j\Gamma_i.$$

On the other hand, in Mardare [Mar05], it has been shown that these equations are indeed equivalent to the Gauss–Codazzi–Mainardi equations, understood in the sense of distributions. We note that their argument readily carries over to the present $p = 2$ case. □

4.5 Weak compactness theorem

In order to prove the weak compactness theorem, we first show a corresponding statement for the Pfaffian system $\nabla P = P\Omega$.

Lemma 4.28. *Let $\{\Omega^k\} \subset L^2(U, \mathrm{so}(3) \otimes \wedge^1\mathbb{R}^2)$ be a sequence such that $\Omega^k \rightharpoonup \Omega$ in L^2 as $k \to \infty$ and suppose that Ω^k satisfies the compatibility condition (4.22) for every k. Then, up to the choice of a subsequence, there exists a sequence $\{P^k\} \subset W^{1,2}_{\mathrm{loc}}(U, \mathrm{SO}(3))$ of solutions to the equation $\nabla P^k = P^k\Omega^k$ such that $P^k \rightharpoonup P$ in $W^{1,2}_{\mathrm{loc}}$ as $k \to \infty$ and $\nabla P = P\Omega$.*

Proof. By Theorem 4.25, there exists a sequence $\{P^k\} \subset W^{1,2}_{\mathrm{loc}}(U, \mathrm{SO}(3))$ such that, for each k, $\partial_i P^k = P^k\Omega^k_i$ and

$$\left\|\nabla P^k\right\|_{L^2_{\mathrm{loc}}} \leq C \left\|\Omega^k\right\|_{L^2_{\mathrm{loc}}}.$$

Then, as $P^k \in \mathrm{SO}(3)$ and $\{\Omega^k\}$ is uniformly bounded in L^2_{loc}, so are $\{P^k\}$ and $\{\nabla P^k\}$. As a result, there exists a subsequence, still denoted $\{P^k\}$, that converges weakly to some P in $W^{1,2}_{\mathrm{loc}}$, and strongly in L^2_{loc}. It remains to show that $\nabla P = P\Omega$. We know that $\nabla P^k \rightharpoonup \nabla P$ in L^2_{loc}. Moreover, since $P^k \to P$ and $\Omega^k \rightharpoonup \Omega$ in L^2_{loc} we infer that the product sequence $P^k\Omega^k$ is weakly convergent to some v in L^1_{loc}. On the other hand, since $P^k\Omega^k = \nabla P^k$ for every k, we must have for every $\varphi \in L^\infty_{\mathrm{loc}} \subset L^2_{\mathrm{loc}}$ that

$$\int P^k\Omega^k\varphi = \int \nabla P^k\varphi \to \int \nabla P\varphi = \int P\Omega\varphi,$$

whereby $v = P\Omega$, by the uniqueness of weak limits, and thus $\nabla P = P\Omega$. □

Finally, we have the weak compactness theorem.

Theorem 4.29. *Let $\{\theta^k\} \subset W^{2,2}_{\mathrm{loc}}(U, \mathbb{R}^3)$ be a uniformly bounded sequence of immersions with corresponding sequences of first and second fundamental forms denoted by $\{(a_{ij})^k\}$ and $\{(b_{ij})^k\}$, respectively. Suppose that $\partial_i \theta^k \in W^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ and that the first fundamental forms $(a_{ij})^k$, $a^k_{ij} = \partial_i \theta^k \cdot \partial_j \theta^k$, have eigenvalues bounded from below by a positive constant uniformly in the domain U and in k. Then, after passing to subsequences, $\{\theta^k\}$ converges weakly in $W^{2,2}_{\mathrm{loc}}$ to an immersion $\theta \in W^{2,2}_{\mathrm{loc}}(U, \mathbb{R}^3)$, whose first and second fundamental forms (a_{ij}), (b_{ij}) are limit points of the sequences $\{(a_{ij})^k\}$, $\{(b_{ij})^k\}$ in the weak $W^{1,2}_{\mathrm{loc}}$- and L^2_{loc}-topologies, respectively.*

Proof. Let such a sequence $\{\theta^k\}$ of immersions be given. Then we denote the corresponding sequences of first and second fundamental forms by $\{(a_{ij})^k\}$, $\{(b_{ij})^k\}$, respectively. By assumption, we have that $(a_{ij})^k \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{Sym}^+(2)) \cap L^\infty_{\mathrm{loc}}(U, \mathrm{Sym}^+(2))$ and $(b_{ij})^k \in L^2_{\mathrm{loc}}(U, \mathrm{Sym}(2))$. Moreover, for each k, we may define $\Omega^k_i \in L^2_{\mathrm{loc}}(U, \mathrm{so}(3))$ as in Section 4.4.2.

For each k, the Ω^k_i necessarily satisfy the compatibility equation (4.25) (the proof of Theorem 1 of Ciarlet, Gratie, and C. Mardare [CGM08] carries over to the present $p = 2$ case). Furthermore, it is straightforward to see from the estimates in Section 4.4.2 that the sequence $\{\Omega^k_i\}$ is uniformly bounded in L^2_{loc} and thus subsequentially weakly convergent to some limit $\Omega_i \in L^2_{\mathrm{loc}}(U, \mathrm{so}(3))$. By Lemma 4.28, therefore, up to the choice of a subsequence, there exists a sequence $\{P^k\} \subset W^{1,2}_{\mathrm{loc}}(U, \mathrm{SO}(3))$ of solutions to the equation $\nabla P^k = P^k \Omega^k$ such that $P^k \rightharpoonup P$ in $W^{1,2}_{\mathrm{loc}}$ as $k \to \infty$ and $\nabla P = P\Omega$. Since $\partial_j \partial_i P = \partial_i \partial_j P$ we thus have that $\partial_j(P\Omega_i) = \partial_i(P\Omega_j)$, which shows after a short computation that the compatibility equation is satisfied by the weak limit Ω_i.

At the same time, the uniformly bounded sequences $\{(a_{ij})^k\}$, $\{(b_{ij})^k\}$ possess subsequences that are weakly convergent to some (a_{ij}), (b_{ij}) in $W^{1,2}_{\mathrm{loc}}$ and L^2_{loc}, respectively. They satisfy $(a_{ij}) \in W^{1,2}_{\mathrm{loc}}(U, \mathrm{Sym}^+(2)) \cap L^\infty_{\mathrm{loc}}(U, \mathrm{Sym}^+(2))$ and $(b_{ij}) \in L^2_{\mathrm{loc}}(U, \mathrm{Sym}(2))$ and the eigenvalues of (a_{ij}) are uniformly bounded from below in U. As a result, we have that Ω_i and the components of the connection form induced by (a_{ij}) and (b_{ij}) coincide. Hence we obtain from Theorem 4.26 an immersion $\theta \in W^{2,2}_{\mathrm{loc}}(U, \mathbb{R}^3)$ with first and second fundamental forms (a_{ij}) and (b_{ij}), respectively. On the other hand, the given sequence $\{\theta^k\}$ must have a weakly convergent subsequence in $W^{2,2}_{\mathrm{loc}}$ with a weak limit $\bar{\theta}$, which coincides with the immersion θ modulo an ambient isometry due to the uniqueness of distributional limits. $\square$

Part II

Extrinsic curvature flows

Chapter 5

Geometry of submanifolds

In this chapter, we briefly summarise the geometric objects associated to submanifolds of Euclidean space that are relevant to their flow by curvature.

5.1 Submanifolds of arbitrary codimension

We follow and use the notation of Smoczyk's survey paper [Smo12; see also Smo05] in this and the following section.

Let $F : M^m \to \mathbb{R}^n$ be a smooth immersion of an m-dimensional manifold M into $\mathbb{R}^n$. We call $k = n - m$ the *codimension* of M in $\mathbb{R}^n$. Local coordinates on M are denoted by $(x^i)_{i=1,\ldots,m}$ and Cartesian coordinates on $\mathbb{R}^n$ by $(y^\alpha)_{\alpha=1,\ldots,n}$, with summation over doubled indices implied.

In local coordinates, we write $F^\alpha = y^\alpha(F)$ and $F_i^\alpha = \frac{\partial F^\alpha}{\partial x^i}$. Then the differential DF of F can be written as

$$DF = F_i^\alpha \frac{\partial}{\partial y^\alpha} \otimes \mathrm{d}x^i \,.$$

The Euclidean metric δ on $\mathbb{R}^n$ induces a Riemannian metric $F^*\delta = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ on M, the *first fundamental form*, with coefficients

$$g_{ij} = \langle F_i, F_j \rangle = \delta_{\alpha\beta} F_i^\alpha F_j^\beta.$$

Therefore, TM, T^*M, $F^*T\mathbb{R}^n$ and their product bundles are equipped with Riemannian metrics. A *connection* ∇ on TM is given as the Levi–Civita connection of the induced metric g_{ij} with the usual Christoffel symbols,

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right).$$

The induced connections on bundles over M are also denoted by ∇.

Given $p \in M$, we denote the *normal space* of M at p by

$$T_p^\perp M := \{\nu \in T_{F(p)}\mathbb{R}^n \cong \mathbb{R}^n : g(\nu, DF|_p(W)) = 0 \ \forall W \in T_pM\}$$

and the *normal bundle* of M by

$$T^\perp M = \cup_{p \in M} T_p^\perp M.$$

The normal bundle is a rank-k subbundle of $F^*T\mathbb{R}^n$. The connection on $T^\perp M$ is denoted by $\nabla^\perp$.

We define the *second fundamental form* $A \in \Gamma(F^*T\mathbb{R}^n \otimes T^*M \otimes T^*M)$ by

$$A := \nabla DF = A_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j = A_{ij}^\alpha \frac{\partial}{\partial y^\alpha} \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

or in local coordinates

$$A_{ij}^\alpha = F_{ij}^\alpha - \Gamma_{ij}^k F_k^\alpha,$$

where $F_{ij}^\alpha = \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j}$, $F_k^\alpha = \frac{\partial F^\alpha}{\partial x^k}$. The second fundamental form is *symmetric*,

$$A_{ij}^\alpha = A_{ji}^\alpha,$$

and *normal*, that is,

$$\langle F_k, A_{ij} \rangle := \delta_{\alpha\beta} F_k^\alpha A_{ij}^\beta = 0 \qquad \forall i, j, k = 1, \dots, m,$$

so that $A \in \Gamma(T^\perp M \otimes T^*M \otimes T^*M)$. Finally, the *mean curvature vector field* $\vec{H} = H^\alpha \frac{\partial}{\partial y^\alpha}$ is defined as the trace of the second fundamental form, that is,

$$\vec{H} = g^{ij} A_{ij} = g^{ij} A_{ij}^\alpha \frac{\partial}{\partial y^\alpha}.$$

Since A is normal, this gives a section $\vec{H} \in \Gamma(T^\perp M)$ of the normal bundle of M.

The *Riemannian curvature tensor* is denoted by $R_{ijk\ell} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right)$, where

$$R(T, U, V, W) = \langle T, (\nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]})U \rangle$$

for any $T, U, V, W \in TM$. The curvature tensor $R_{ij}^\perp$ on the normal bundle is defined analogously. The covariant derivative ∇A of the second fundamental form $A \in \Gamma(F^*T\mathbb{R}^n \otimes T^*M \otimes T^*M)$ is given by

$$(\nabla_U A)(V, W) = \nabla_U (A(V, W)) - A(\nabla_U V, W) - A(V, \nabla_U W).$$

Using the connection $\nabla^\perp$ on the normal bundle $T^\perp M$, we can write

$$(\nabla_U^\perp A)(V, W) = ((\nabla_U A)(V, W))^\perp.$$

In local coordinates, the components of the tensor ∇A are given by $\nabla_i A_{jk}^\alpha$, where

$$\left(\nabla_{\frac{\partial}{\partial x^i}} A\right)\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \nabla_i A_{jk}^\alpha \frac{\partial}{\partial y^\alpha}.$$

The structure equations can then be written as follows:

$$\begin{aligned} R_{ijk\ell} &= \langle A_{ik}, A_{j\ell}\rangle - \langle A_{i\ell}, A_{jk}\rangle && \text{(Gauss)}, \\ \nabla^\perp_i A_{jk} &= \nabla^\perp_j A_{ik} && \text{(Codazzi)}, \\ R^\perp_{ij} &= A_{ik} \wedge A^k_j && \text{(Ricci)}. \end{aligned}$$

Moreover, we have *Simons' identity*,

$$\nabla^\perp_k \nabla^\perp_\ell \vec{H} = \Delta^\perp A_{k\ell} + R_{ki\ell j} A^{ij} - R^i_k A_{i\ell} + Q^i_\ell A_{ik} - S_{ki\ell j} A^{ij},$$

where the *Ricci curvature* is given by $R_{ij} = g^{k\ell} R_{ikj\ell}$ and we define $Q_{ij} = \langle A^k_i, A_{kj}\rangle$ and $S_{ijk\ell} = \langle A_{ij}, A_{kl}\rangle$ [Smo05].

5.2 The hypersurface case

In the special case of an immersion of an orientable hypersurface $F : M^m \to \mathbb{R}^{m+1}$, some of the above quantities simplify [Smo12]. In particular, there exists a unique unit normal vector field $\nu \in \Gamma(T^\perp M)$, the principal normal, such that at any $p \in M$, $DF(e_1), \ldots, DF(e_m), \nu|_p$ form a basis of $T_{F(p)}\mathbb{R}^m$ of positive orientation for any basis $e_1, \ldots, e_m$ of T_pM of positive orientation.

The scalar second fundamental form $h \in \Gamma(T^*M \otimes T^*M)$ is defined by

$$h(U, V) = \langle A(U, V), \nu\rangle, \qquad U, V \in TM.$$

The components of h are denoted by h_{ij}. The scalar mean curvature is its trace, $H = \operatorname{tr} h$, so that $\vec{H} = H\nu$. Using the bundle isomorphism $\sharp : T^*M \to TM$, the shape operator $S : TM \to TM$ is given by

$$S(U) = (h(U, \cdot))^\sharp, \qquad U \in TM.$$

The shape operator is self-adjoint, so its eigenvalues are real numbers $\lambda_1, \ldots, \lambda_m$, called principal curvatures. Then $H = \lambda_1 + \cdots + \lambda_m$.

Since $A^\alpha_{ij} = \nu^\alpha h_{ij}$, the Gauss–Codazzi equations simplify [Smo12], while the Ricci equation is vacuous (as the codimension is equal to one):

$$\begin{aligned} R_{ijk\ell} &= h_{ik}h_{j\ell} - h_{i\ell}h_{jk} && \text{(Gauss)}, \\ \nabla_i h_{jk} &= \nabla_j h_{ik} && \text{(Codazzi)}. \end{aligned}$$

Moreover, Simons' identity simplifies to

$$\nabla_k \nabla_\ell h_{ij} - \nabla_i \nabla_j h_{k\ell} = h_{k\ell} h_i^p h_{jp} - h_{i\ell} h_j^p h_{kp} + h_{jk} h_i^p h_{\ell p} - h_{ij} h_k^p h_{\ell p}.$$

5.3 Curves in $\mathbb{R}^n$

In Chapters 6 and 8, we consider curve shortening flow of curves in $\mathbb{R}^n$. It is convenient to work in the Frenet–Serret frame, and we will largely adopt the notation from Gage–Hamilton's work on planar curves [GH86].

Let $\gamma : S^1 \to \mathbb{R}^n$, $p \mapsto \gamma(p)$ be an immersion of the unit circle into $\mathbb{R}^n$ with the parameter p taken to be modulo 2π. We always assume that γ is smooth and rectifiable. We define the *velocity* of the parametrisation by

$$v = \left| \frac{\partial \gamma}{\partial p} \right|.$$

Since γ is rectifiable, we can parametrise it by arclength. The length s of γ is given by

$$s(p) = \int_0^p v(q) \, \mathrm{d}q,$$

and by the fundamental theorem of calculus we have that

$$\frac{ds}{dp}(p) = v(p).$$

Then there exists an inverse function p such that

$$\frac{dp}{ds}(s) = \frac{1}{v(p)}.$$

Now if $\gamma(p(s))$ is parametrised by arclength, it follows that

$$v(s) = \frac{d\gamma(p(s))}{dp}\frac{dp}{ds} = \frac{v(p)}{v(p)} = 1.$$

Moreover, the operator $\frac{\partial}{\partial s}$ is given by

$$\frac{\partial}{\partial s} = \frac{1}{v(p)}\frac{\partial}{\partial p},$$

and for any $U \subset S^1$, the induced measure $\mathrm{d}s$ is given by

$$\int_U \mathrm{d}s = \int_U v(p)\,\mathrm{d}p\,.$$

In the following, we will usually assume that γ is parametrised by arclength.

A *Frenet–Serret frame* is a moving frame of orthonormal vectors $(T, N, B_1, \ldots, B_{n-2})$, where T is called the *tangent vector*, N the *normal vector* and $B_1, \ldots, B_{n-2}$ the *binormal vectors*. Define the tangent vector by

$$T = \frac{\partial \gamma}{\partial s}$$

and note that T has unit length since γ is parametrised by arclength. By differentiating the equation $\langle T, T\rangle = 1$, we find that the curvature vector $\frac{\partial T}{\partial s}$ is orthogonal to T. The *curvature* κ is given by

$$\kappa = \left|\frac{\partial T}{\partial s}\right| = \left|\frac{\partial^2 \gamma}{\partial s^2}\right|.$$

Assuming that the curvature does not vanish, let

$$N = \kappa^{-1}\frac{\partial T}{\partial s}$$

be the unit normal vector, which is orthogonal to T. Once again, the fact that N has unit length implies that $\frac{\partial N}{\partial s}$ is orthogonal to N, and $\langle T, N\rangle = 0$ gives

$$\kappa + \left\langle T, \frac{\partial N}{\partial s}\right\rangle = 0.$$

Hence $\frac{\partial N}{\partial s} + \kappa T$ must be a vector that is orthogonal to both T and N. We let

$$\tau_1 = \left|\frac{\partial N}{\partial s} + \kappa T\right|$$

denote the first *torsion* and, assuming that $\tau_1 \neq 0$, define the first binormal vector by

$$B_1 = \tau_1^{-1}\left(\frac{\partial N}{\partial s} + \kappa T\right).$$

Continuing this process inductively, we obtain the binormal vectors $B_2, \dots, B_{n-2}$ and the torsions $\tau_2, \dots, \tau_{n-2}$ [Spi99b]. If none of $\kappa, \tau_1, \dots, \tau_{n-2}$ vanish, the frame is uniquely defined. In particular, we have

Theorem 5.1. *The generalised Frenet–Serret equations hold:*

$$\frac{\partial}{\partial s}\begin{pmatrix} T \\ N \\ B_1 \\ B_2 \\ \vdots \\ B_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 & 0 & \dots & 0 \\ -\kappa & 0 & \tau_1 & 0 & \dots & 0 \\ 0 & -\tau_1 & 0 & \tau_2 & \dots & 0 \\ 0 & 0 & -\tau_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \tau_{n-2} \\ 0 & 0 & 0 & 0 & -\tau_{n-2} & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B_1 \\ B_2 \\ \vdots \\ B_{n-2} \end{pmatrix}. \tag{5.1}$$

So, for example,

$$\frac{\partial T}{\partial s} = \kappa N.$$

In the following chapter, we will consider time-dependent immersions $\gamma_t = \gamma(\cdot, t)$ moving in the direction N with speed κ and compute the evolution equations for the Frenet–Serret frame.

Finally, in order to show that a blow-up limit of a solution of curve shortening flow is planar, we require the following theorem. It generalises the well-known fact that a space curve with vanishing torsion is planar to the case of curves in $\mathbb{R}^n$.

Theorem 5.2 ([Spi99b, Thm. 7.B.5])**.** *Let $\gamma : S^1 \to \mathbb{R}^n$ be a curve parametrised by arclength such that $\kappa, \tau_1, \ldots, \tau_{j-2}$ do not vanish at any point and τ_{j-1} vanishes everywhere. Then γ lies in some $(j-1)$-dimensional plane in $\mathbb{R}^n$.*

Chapter 6

Curve shortening and mean curvature flow

In this chapter, we consider both mean curvature flow (MCF) and its one-dimensional variant, the curve shortening flow (CSF). We state some classical results, in particular Huisken's monotonicity formula. Due to the extensive body of literature spanning more than three decades, we merely present a small subset of known results.

Let M^m be an m-dimensional smooth manifold, $T > 0$, and let $F : M \times [0, T) \to \mathbb{R}^n$ be a smooth 1-parameter family of immersions with codimension $k = n - m$. This means that every $F_t := F(\cdot, t)$, $t \in [0, T)$ is an immersion of M into $\mathbb{R}^n$. Then F evolves by mean curvature flow with initial data $F_0 : M \to \mathbb{R}^n$ if

$$\begin{aligned} \frac{\partial F}{\partial t}(p,t) &= \vec{H}(p,t), \\ F(p,0) &= F_0(p), \end{aligned} \tag{MCF}$$

where $\vec{H}(\cdot, t)$ is the mean curvature vector field of F_t. In codimension one, it is common to choose the normal vector so that $\vec{H} = -H\nu$. Writing $M_t := F_t(M)$, we call the set $\{M_t : t \in [0, T)\}$ a *mean curvature flow.*

Meanwhile, in the one-dimensional case $m = 1$ we consider a family of smooth immersions $\gamma : S^1 \times [0,T) \to \mathbb{R}^n$ of the unit circle satisfying curve shortening flow with initial data $\gamma_0 : S^1 \to \mathbb{R}^n$:

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(p,t) &= (\kappa N)(p,t), \\ \gamma(p,0) &= \gamma_0(p). \end{aligned} \tag{CSF}$$

Here, $\kappa(\cdot,t)$ is the curvature of $\gamma_t := \gamma(\cdot,t)$ and $N(\cdot,t)$ our choice of unit normal vector field. While in the planar case, $n = 2$, there is a notion of the normal vector pointing inwards or outwards, in arbitrary codimension every curve γ_t still has two well-defined orientations. We can thus choose the sign of the orientation to make (CSF) (weakly) parabolic forward in time. In particular, the product κN makes sense even at the points where N is not defined. We also assume that every curve γ_t is smooth and rectifiable.

It is well-known that the mean curvature flow equation is a quasilinear weakly parabolic evolution equation [Smo12]. The existence of null directions stems from the diffeomorphism invariance of the flow, i. e., if $\phi : M \to M$ is a diffeomorphism and $F : M \times [0,T) \to \mathbb{R}^n$ a solution of (MCF) then $\tilde{F} : M \times [0,T) \to \mathbb{R}^n$, $\tilde{F}(p,t) = F(\phi(p),t)$ is also a solution, and the submanifolds $\tilde{M}_t$ and M_t coincide [Smo12, Prop. 3.1].

6.1 Evolution equations

Mean curvature flow

Given a smooth solution $F : M^m \times [0,T) \to \mathbb{R}^n$ of (MCF), we note the evolution equations of the basic geometric quantities. In codimension one, these are well-known, and in full generality they are derived in detail in Smoczyk's survey paper [Smo12].

Proposition 6.1 ([Smo12]). *The induced Riemannian metric $g = g(t)$ satisfies*

$$\frac{d}{dt} g_{ij} = -2\langle \vec{H}, A_{ij} \rangle.$$

Thus, the induced volume form $\mathrm{d}\mu_t = \sqrt{\det g}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ *evolves by*

$$\frac{d}{dt}\,\mathrm{d}\mu_t = -\left|\vec{H}\right|^2 \mathrm{d}\mu_t\,.$$

Moreover, the Christoffel symbols of the Levi–Civita connection on M *satisfy*

$$C_{ij}^k := \frac{d}{dt}\Gamma_{ij}^k = -g^{k\ell}\left(\nabla_i\langle\vec{H}, A_{j\ell}\rangle + \nabla_j\langle\vec{H}, A_{i\ell}\rangle - \nabla_\ell\langle\vec{H}, A_{ij}\rangle\right).$$

The second fundamental form A *evolves according to*

$$\nabla_{\frac{d}{dt}} A_{ij}^\alpha = \nabla_i\nabla_j H^\alpha - C_{ij}^k F_k^\alpha.$$

Finally, the mean curvature vector $\vec{H}$ *satisfies*

$$\nabla_{\frac{d}{dt}} H^\alpha = \Delta H^\alpha - g^{ij}C_{ij}^k F_k^\alpha + 2\langle A_{k\ell}, \vec{H}\rangle A^{\alpha k\ell}.$$

For a mean curvature flow of hypersurfaces in codimension one, the evolution equations simplify [Hui84].

Proposition 6.2. *Let* $F : M^m \times [0,T) \to \mathbb{R}^{m+1}$ *be a smooth solution of* (MCF). *Then the following equations hold:*

$$\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}, \tag{6.1}$$

$$\frac{\partial}{\partial t}\,\mathrm{d}\mu_t = -H^2\,\mathrm{d}\mu_t, \tag{6.2}$$

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2Hh_{ik}g^{k\ell}h_{j\ell} + |A|^2\,h_{ij}, \tag{6.3}$$

$$\frac{\partial}{\partial t} H = \Delta H + |A|^2\,H, \tag{6.4}$$

$$\frac{\partial}{\partial t}\,|A|^2 = \Delta\,|A|^2 - 2\,|\nabla A|^2 + 2\,|A|^4\,. \tag{6.5}$$

Therefore, by the evolution equation for H and the maximum principle, we conclude that the condition $H > 0$ is preserved under the flow. The evolution equation for the measure implies that the total area of M_t is monotonically decreasing.

Curve shortening flow

In the case of curve shortening flow, we have corresponding evolution equations in terms of the Frenet–Serret frame. They have previously been derived for planar curves by Gage and Hamilton [GH86] and for space curves by Altschuler [Alt91]; in the general case some of them appeared in various places in the literature [YJ05; MC07; Hät15].

Let $\gamma : S^1 \times [0,\omega) \to \mathbb{R}^n$ be a solution of (CSF). We employ the notation of Section 5.3, in particular, s is the arclength parameter and $v = \left|\frac{\partial \gamma}{\partial p}\right|$. In order to avoid notational ambiguities, the final time of existence of γ is denoted by either T or ω, depending on the context. The following two statements can be proved exactly as in the planar case [GH86].

Proposition 6.3. *The evolution of v is given by*

$$\frac{\partial}{\partial t} v = -\kappa^2 v.$$

Proof. The operators $\frac{\partial}{\partial p}$ and $\frac{\partial}{\partial t}$ commute, hence

$$\begin{aligned} 2v\frac{\partial}{\partial t} v = \frac{\partial}{\partial t}(v^2) &= 2\left\langle \frac{\partial \gamma}{\partial p}, \frac{\partial^2 \gamma}{\partial p \partial t} \right\rangle \\ &= 2\left\langle vT, \frac{\partial}{\partial p}(\kappa N) \right\rangle \\ &= 2\left\langle vT, \frac{\partial \kappa}{\partial p} N - v\kappa^2 T + v\kappa\tau_1 B_1 \right\rangle \\ &= -2v^2\kappa^2, \end{aligned}$$

where we have used the Frenet–Serret equations (5.1), (CSF), and that the vectors (T, N, B_1) are orthonormal. □

Thus, the length of the curve is monotonically decreasing under curve shortening flow, hence the name.

Since the arclength parameter s depends on t, we cannot expect that the operators $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ commute. Instead, we have

Proposition 6.4. *Differentiation with respect to s and t is related by the commutation formula*

$$\frac{\partial^2}{\partial t \partial s} = \frac{\partial^2}{\partial s \partial t} + \kappa^2 \frac{\partial}{\partial s}. \tag{6.6}$$

Proof. Using $\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial p}$, we obtain

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \kappa^2 \frac{1}{v}\frac{\partial}{\partial p} + \frac{1}{v}\frac{\partial}{\partial p}\frac{\partial}{\partial t} = \kappa^2 \frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t},$$

proving the claim. □

These two propositions and the Frenet–Serret equations (5.1) enable us to derive the evolution equations for the moving frame and the curvature and torsions.

Proposition 6.5. *We have*

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial s^2} + \kappa^2 T = \frac{\partial \kappa}{\partial s} N + \kappa \tau_1 B_1. \tag{6.7}$$

Proof. Using the commutation formula first, followed by the curve shortening flow equation (CSF), we compute

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \frac{\partial^2 \gamma}{\partial t \partial s} \\
&= \frac{\partial^2 \gamma}{\partial s \partial t} + \kappa^2 \frac{\partial \gamma}{\partial s} \\
&= \frac{\partial^3 \gamma}{\partial s^3} + \kappa^2 T \\
&= \frac{\partial^2 T}{\partial s^2} + \kappa^2 T.
\end{aligned}$$

In order to obtain the second equality, observe that

$$\begin{aligned}\frac{\partial^2 T}{\partial s^2} &= \left(\frac{\partial \kappa}{\partial s}\right) N + \kappa \frac{\partial N}{\partial s} \\ &= \frac{\partial \kappa}{\partial s} N + \kappa\left(-\kappa T + \tau_1 B_1\right) \end{aligned} \tag{6.8}$$

by (5.1), so that

$$\frac{\partial T}{\partial t} = \frac{\partial \kappa}{\partial s} N + \kappa \tau_1 B_1,$$

as required. □

Proposition 6.6. *The evolution of the curvature κ is given by*

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} + \kappa^3 - \kappa \tau_1^2 \tag{6.9}$$

whenever $\kappa > 0$.

Proof. We use the fact that $\kappa^2 = \langle \frac{\partial T}{\partial s}, \frac{\partial T}{\partial s} \rangle$ and get, using the Frenet–Serret equations (5.1) and (6.7), that

$$\begin{aligned}\kappa \frac{\partial \kappa}{\partial t} &= \left\langle \frac{\partial^2 T}{\partial t \partial s}, \frac{\partial T}{\partial s} \right\rangle \\ &= \left\langle \frac{\partial^2 T}{\partial s \partial t} + \kappa^3 N, \kappa N \right\rangle \\ &= \left\langle \frac{\partial^2 \kappa}{\partial s^2} N + \frac{\partial \kappa}{\partial s}\frac{\partial N}{\partial s} + \frac{\partial \kappa}{\partial s}\tau_1 B_1 + \kappa \frac{\partial \tau_1}{\partial s} B_1 + \kappa \tau_1 \frac{\partial B_1}{\partial s} + \kappa^3 N, \kappa N \right\rangle \\ &= \kappa \frac{\partial^2 \kappa}{\partial s^2} + \kappa^4 + \langle \kappa^2 \tau_1(-\tau_1 N + \tau_2 B_2), N \rangle \\ &= \kappa \frac{\partial^2 \kappa}{\partial s^2} + \kappa^4 - \kappa^2 \tau_1^2, \end{aligned}$$

which implies the claim. □

Corollary 6.7. *We have*

$$\frac{\partial \kappa^2}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} - 2\left(\frac{\partial \kappa}{\partial s}\right)^2 + 2\kappa^4 - 2\kappa^2\tau_1^2.$$

Proposition 6.8. *The normal vector field N satisfies*

$$\frac{\partial N}{\partial t} = -\frac{\partial \kappa}{\partial s}T + \left(\frac{\partial \tau_1}{\partial s} + \frac{2\tau_1}{\kappa}\frac{\partial \kappa}{\partial s}\right)B_1 + \tau_1\tau_2 B_2. \tag{6.10}$$

Proof. From the Frenet–Serret equations (5.1) we have $\frac{\partial T}{\partial s} = \kappa N$, so

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{\kappa}\frac{\partial T}{\partial s}\right).$$

The result then follows by a computation similar to the ones above using the Frenet–Serret equations (5.1), the commutation formula (6.6), and the evolution equations for T and κ, (6.7) and (6.9). □

Proposition 6.9. *The evolution of the first torsion τ_1 is given by*

$$\frac{\partial \tau_1}{\partial t} = \frac{\partial^2 \tau_1}{\partial s^2} + \frac{2}{\kappa}\frac{\partial \kappa}{\partial s}\frac{\partial \tau_1}{\partial s} + \frac{2\tau_1}{\kappa}\left(\frac{\partial^2 \kappa}{\partial s^2} - \frac{1}{\kappa}\left(\frac{\partial \kappa}{\partial s}\right)^2 + \kappa^3\right) - \tau_1\tau_2^2.$$

Proof. Note that, by the Frenet–Serret equations (5.1),

$$\left\langle \frac{\partial N}{\partial s}, \frac{\partial N}{\partial s}\right\rangle = \kappa^2 + \tau_1^2.$$

Differentiating this equation with respect to t yields [YJ05]

$$\tau_1\frac{\partial \tau_1}{\partial t} + \kappa\frac{\partial \kappa}{\partial t} = \left\langle \frac{\partial^2 N}{\partial t \partial s}, \frac{\partial N}{\partial s}\right\rangle.$$

Using (5.1), the commutation formula (6.6) and the evolution equations for N and κ, (6.9) and (6.10), we obtain the result. □

Proposition 6.10. *We have*

$$\frac{\partial |\gamma|^2}{\partial t} = \frac{\partial^2 |\gamma|^2}{\partial s^2} - 2. \tag{6.11}$$

Proof. Since $\frac{\partial T}{\partial s} = \kappa N$, the curve shortening flow equation can also be written as

$$\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial s^2}.$$

While this resembles a heat equation, note that the arclength parameter s depends on the time t. This implies that

$$\begin{aligned} \frac{\partial |\gamma|^2}{\partial t} &= 2 \left\langle \frac{\partial \gamma}{\partial t}, \gamma \right\rangle \\ &= 2 \left\langle \frac{\partial^2 \gamma}{\partial s^2}, \gamma \right\rangle. \end{aligned}$$

Finally, note that

$$\begin{aligned} \frac{\partial^2 |\gamma|^2}{\partial s^2} &= \frac{\partial^2}{\partial s^2} \langle \gamma, \gamma \rangle \\ &= 2 \left\langle \frac{\partial^2 \gamma}{\partial s^2}, \gamma \right\rangle + 2 \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial s} \right\rangle \\ &= 2 \left\langle \frac{\partial^2 \gamma}{\partial s^2}, \gamma \right\rangle + 2, \end{aligned}$$

which proves the claim. □

6.2 Existence and uniqueness

The short-time existence and uniqueness for solutions of the mean curvature flow is by now classical. The difficulty in proving it lies in the diffeomorphism invariance, as remarked above. Since the evolution equation is merely weakly parabolic, standard PDE theory does not apply. Following Hamilton's approach for the Ricci flow, the theorem can be proved using the Nash–Moser inverse function theorem [Ham82; GH86]. Alternatively, the DeTurck trick can be employed to remove the diffeomorphism invariance, transforming the system into a strongly parabolic one [Zhu02]. Finally, the surfaces $M_t = F_t(M)$ can locally be written as normal graphs over M_0, which results in a quasilinear parabolic equation for the height function to which standard theory can be applied [HP99].

Theorem 6.11. *Let M^m be a closed smooth manifold and $F_0 : M \to \mathbb{R}^n$ a smooth immersion. Then there exists a unique smooth solution of* (MCF) *on* $[0, T)$ *for some* $0 < T \leq \infty$.

While the above theorem deals with a closed initial manifold, there are also results in the non-compact case, e. g., for complete hypersurfaces, Ecker and Huisken showed short-time existence assuming a uniform local Lipschitz condition [EH91].

Examples of simple yet instructive and powerful explicit solutions include the sphere in any dimension and the *Grim Reaper* and the *Abresch–Langer curves* [AL86] in the plane. If F_0 is the standard embedding of the sphere $S^n_{r_0} \subset \mathbb{R}^{n+1}$ of radius r_0, the mean curvature flow equation reduces to the ordinary differential equation

$$\frac{dr}{dt} = -\frac{n}{r}$$

for the radius $r(t)$ with $r(0) = r_0$, whereby $r(t) = \sqrt{r_0^2 - 2nt}$. The maximal time of existence $T = \frac{r_0^2}{2n}$ is thus indeed finite. Under mean curvature flow, the sphere moves solely by scaling and is an example of a *self-similarly shrinking solution.*

More generally, self-shrinking solutions of the mean curvature flow are *ancient solutions* $\{M_t\}$, $t \in (-\infty, 0)$ that are completely determined by their time-slice at $t = -1$, so that they satisfy

$$M_t = \sqrt{-t} M_{-1}, \qquad t < 0.$$

The surface M_{-1} is then called a *self-shrinker.* Examples of self-shrinking solutions other than the sphere include the flat plane (which is of course wholly unaffected by mean curvature flow), generalised cylinders $S^k \times \mathbb{R}^{m-k}$, and the *Angenent torus* [Ang92].

Equivalently, self-shrinkers $M^m \subset \mathbb{R}^{m+1}$ are characterised by the equation

$$H = \frac{x^\perp}{2},$$

where x denotes the position vector [CM12], that is, an ancient solution $\{M_t\}$ is a self-shrinking solution if and only if $H = \frac{x^\perp}{2t}$ for all $t < 0$.

The Grim Reaper is a non-compact solution of curve shortening flow in the plane given by $\Gamma(x,t) = (x, -\log(\cos x) + t)$, $|x| \leq \frac{\pi}{2}$ and moves solely by translation. Abresch–Langer curves, on the other hand, are a class of planar self-shrinking convex closed curves $\gamma_{m,n} \subset \mathbb{R}^2$ with turning number m and n 'petals', that is, $2n$ critical points of the curvature, where the coprime integers $m, n \in \mathbb{N}$ must satisfy

$$\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}.$$

In fact, any closed self-shrinker in the plane except the circle is of this form [AL86; see also Hal12].

These examples suggest that while long-time existence of the flow can occur, it would be wrong to expect it in general. Indeed, the parabolic maximum principle implies that the flow of a compact submanifold must cease to exist in finite time.

Theorem 6.12. *Let M^m be a closed smooth manifold and $F_0 : M \to \mathbb{R}^n$ a smooth immersion. Then the maximal time T of existence of the solution $F : M \times [0, T) \to \mathbb{R}^n$ of* (MCF) *starting from F_0 is finite.*

This follows directly from the evolution equation for $|F|$. In the one-dimensional case, (6.11) implies that

$$\max |\gamma_t|^2 \leq \max |\gamma_0|^2 - 2t,$$

whereby $T \leq \frac{1}{2} \max |\gamma_0|^2$. Geometrically, in codimension one this argument can be rephrased in terms of the avoidance principle: Any two initially disjoint mean curvature flows $\{M_t\}$, $\{M'_t\}$ must stay disjoint throughout the evolution. Therefore, enclosing a compact initial hypersurface M_0 in a sphere of sufficiently large radius, we see that M_0 must cease to exist before the sphere does.

However, non-compact solutions may exist for arbitrarily long times, for example entire n-dimensional graphs in $\mathbb{R}^{n+1}$ exhibit long-term existence [EH89]. In any codimension there is the result of Wang [Wan02].

In addition, we have that for a closed initial manifold, the curvature must tend to infinity as we approach the maximal time of existence [Smo12].

Theorem 6.13. *Let M^m be a closed smooth manifold and $F : M \times [0, T) \to \mathbb{R}^n$ a smooth solution of* (MCF) *for which the maximal time of existence T is finite. Then we have that*

$$\limsup_{t\to T} \max_{M_t} |A|^2 = \infty.$$

In the hypersurface case, this result goes back to Huisken [Hui84; Hui90], whose proof used the maximum principle and a uniform bound on the derivatives $\nabla^k A$ of A, given a uniform bound on A itself.

6.3 Classification of singularities

Since singularities are inevitable in general, it is a worthwhile endeavour to analyse their geometric structure. To that end, they are commonly classified according to the growth rate of the curvature. The likely origin of this concept is the fact that under the assumptions of Theorem 6.13, the growth rate is bounded from below, i. e.,

$$\limsup_{t\to T} \max_{M_t} |A|^2 \geq \frac{c}{T-t},$$

for some constant $c > 0$ depending on M.

We thus distinguish singularities by the presence of a bound on the growth rate from above [Hui90]. Note that the curvature of the sphere realises this growth rate exactly.

Definition 6.14. Let $F : M \times [0, T) \to \mathbb{R}^n$, $T < \infty$ be a smooth solution of (MCF) such that

$$\limsup_{t \to T} \max_{M_t} |A|^2 = \infty.$$

If the growth rate is optimal, that is, there exists a constant $c > 0$ such that

$$\max_{M_t} |A|^2 \leq \frac{c}{T - t}$$

for all $t \in [0, T)$, then M is said to develop a *type-I singularity* at time T. Otherwise, that is, if

$$\limsup_{t \to T} \max_{M_t} |A|^2 (T - t) = \infty$$

then the singularity is said to be of *type-II*.

The sphere exhibits a type-I singularity at its final time of existence. In some cases, it is the only one possible, according to Huisken's seminal result:

Theorem 6.15 ([Hui84])**.** *Under mean curvature flow, any closed convex hypersurface immediately becomes strictly convex and converges to a round point in finite time.*

The one-dimensional case was treated by Grayson and Gage–Hamilton.

Theorem 6.16 ([GH86; Gra87; cf. Hui98; AB11])**.** *Under curve shortening flow, any simple closed curve becomes convex and converges to a round point in finite time.*

A common technique for the analysis of singularities of curvature flows is to study sequences of rescalings of the solution along sequences of space-time points with the property that the curvature tends to infinity. The objective is then to extract, in a suitable sense, a limit solution, whose classification is still subject to current research [see, e. g., CM12].

Definition 6.17. Let $F : M \times [0,T) \to \mathbb{R}^n$, $T < \infty$ be a smooth solution of (MCF). If a sequence (p_k, t_k) of points $p_k \in M$ and times $t_k \in [0,T)$ is such that $t_k \to T$ and

$$\limsup_{k\to\infty} |A|(p_k, t_k) = \infty,$$

it is called a *blow-up sequence*. A blow-up sequence (p_k, t_k) is called *essential* if there exists a constant $\delta > 0$ such that

$$|A|^2 (p_k, t_k) \leq \frac{\delta}{T - t_k}.$$

If there exists a sequence of points $p_k \to p$ in M and a sequence of times $t_k \to T$ such that

$$\limsup_{k\to\infty} |A|(p_k, t_k) = \infty$$

then $p \in M$ is called a *singular point* of the flow. If there exists a sequence of times $t_k \to T$ such that

$$\limsup_{k\to\infty} |A|(p, t_k) = \infty$$

for some $p \in M$ then p is called a *special singular point*.

Given a mean curvature flow $\{M_t\}$, $\lambda > 0$ and a space-time point (x_0, t_0), we consider the *parabolic rescalings*

$$M_{t'}^{\lambda} = \lambda \left(M_{\lambda^{-2}t'+t_0} - x_0\right).$$

Then $\{M_t^\lambda\}$ is again a mean curvature flow with variables $x' = \lambda(x - x_0)$, $t' = \lambda^2(x - x_0)$. If (x_0, T) is a type-I singularity and $\{\lambda_j\}$ a sequence of positive numbers with $\lambda_j \to \infty$, the bound on $|A|$ can then be used to show that the sequence $\left\{M_{t'}^{\lambda_j}\right\}$ subconverges to an ancient smooth limit flow, which is called *type-I blow-up* or *tangent flow*. Note that even if (x_0, T) is of type-II, one can still carry out the rescaling procedure and obtain a weak limit as a Brakke flow of rectifiable varifolds [Bra78], which is a weak, non-smooth notion of mean curvature flow.

It turns out that, in fact, a type-I blow-up produces a self-similarly shrinking solution [Hui90]. Therefore, self-shinkers are used as models for type-I singularities. This is a consequence of Huisken's monotonicity formula, which is the topic of the following section.

6.4 The monotonicity formula

Introduced by Huisken, the monotonicity formula is one of the central tools in the study of type-I singularities of mean curvature flow.

Let M^m be a closed manifold and $F : M \times [0, T) \to \mathbb{R}^n$ a smooth family of immersions satisfying (MCF). For $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ the scaled backward heat kernel centered at (x_0, t_0) is given by

$$k_{x_0,t_0}(x, t) = (4\pi(t_0 - t))^{-\frac{m}{2}} \mathrm{e}^{-\frac{|x-x_0|^2}{4(t_0-t)}}.$$

Note that k_{x_0,t_0} is well defined on $\mathbb{R}^n \times (-\infty, t_0)$.

We then have

Theorem 6.18 ([Hui90, Thm. 3.1; see also EH89; Ham93; Eck04])**.** *Let $\{M_t\}$ be a mean curvature flow as above and $(x_0, t_0) \in \mathbb{R}^n \times [0, T)$. It holds that*

$$\frac{d}{dt} \int_M k_{x_0,t_0} \, \mathrm{d}\mu_t = -\int_M \left| \vec{H} + \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 k_{x_0,t_0} \, \mathrm{d}\mu_t, \qquad t < t_0,$$

where $(\cdot)^\perp$ denotes the part of a vector normal to M.

Proof. We can assume $(x_0, t_0) = (0, 0)$ and set $k = k_{0,0}$. Recall the evolution equation for the measure

$$\frac{d}{dt} \, \mathrm{d}\mu_t = -|\vec{H}|^2 \, \mathrm{d}\mu_t,$$

which implies that, by (MCF),

$$\begin{aligned}\frac{d}{dt}\int_M k(F(p,t),t)\,\mathrm{d}\mu_t(p) &= \int_M \frac{\partial k}{\partial t}(F(p,t),t) + \left\langle \frac{\partial F}{\partial t}(p,t), Dk(F(p,t),t)\right\rangle \\ &\qquad - \left|\vec{H}(p,t)\right|^2 k(F(p,t),t)\,\mathrm{d}\mu_t(p) \\ &= \int_M \frac{\partial k}{\partial t}(F(p,t),t) + \langle \vec{H}(p,t), Dk(F(p,t),t)\rangle \\ &\qquad - \left|\vec{H}(p,t)\right|^2 k(F(p,t),t)\,\mathrm{d}\mu_t(p)\,.\end{aligned}$$

By the divergence theorem, we have that

$$\int_M \mathrm{div}_M\, Dk = -\int_M \langle \vec{H}, Dk\rangle.$$

We compute, denoting by x^T the part of x tangential to M,

$$\begin{aligned}\frac{\partial k}{\partial t} &= \left(-\frac{m}{2t} - \frac{|x|^2}{4t^2}\right)k, \\ Dk &= \frac{k}{2t}x, \\ \mathrm{div}_M\, Dk &= \left(\frac{m}{2t} + \frac{\left|x^T\right|^2}{4t^2}\right)k,\end{aligned}$$

which gives

$$\begin{aligned}\frac{d}{dt}\int_M k\,\mathrm{d}\mu_t &= \int_M \frac{\partial k}{\partial t} + \langle \vec{H}, Dk\rangle - \left|\vec{H}\right|^2 k\,\mathrm{d}\mu_t \\ &= \int_M \frac{\partial k}{\partial t} + \mathrm{div}_M\, Dk + 2\langle \vec{H}, Dk\rangle - \left|\vec{H}\right|^2 k\,\mathrm{d}\mu_t \\ &= -\int_M \left(\frac{|x|^2}{4t^2} - \frac{\left|x^T\right|^2}{4t^2} - 2\left\langle \vec{H}, \frac{x}{2t}\right\rangle + \left|\vec{H}\right|^2\right) k\,\mathrm{d}\mu_t\,.\end{aligned}$$

Therefore, we obtain

$$\frac{d}{dt}\int_M k\,\mathrm{d}\mu_t = -\int_M \left|\vec{H} - \frac{x^\perp}{2t}\right|^2 k\,\mathrm{d}\mu_t, \qquad t<0,$$

completing the proof. $\square$

As mentioned above, the monotonicity formula implies that a type-I blow-up is a self-shrinker.

Theorem 6.19 ([Hui90, Thm. 3.5])**.** *Let $\{M_t\}$ be a mean curvature flow of closed hypersurfaces. Then any tangent flow at a type-I singularity is self-similarly shrinking.*

6.5 The entropy functional

Entropy is a functional on the space of surfaces that can be seen as a measure of geometric complexity. Since it is monotone under mean curvature flow, a bound on the entropy of the initial surface implies a bound on the entropy for all later singularities. Most prominently, it has been employed in Colding–Minicozzi's analysis of generic singularities of mean curvature flow [CM12].

Let $M^m \subset \mathbb{R}^n$ be an immersed surface. We define the functional F_{x_0,t_0}, $x_0 \in \mathbb{R}^n$, $t_0 > 0$, by [CM12; Gua19]

$$F_{x_0,t_0}(M) = (4\pi t_0)^{-\frac{m}{2}} \int_M \mathrm{e}^{-\frac{|x-x_0|^2}{4t_0}} \,\mathrm{d}\mu = \int_M k_{x_0,t_0}(x,0)\,\mathrm{d}\mu\,.$$

Definition 6.20. The *entropy* of M is defined by

$$\lambda(M) = \sup_{x_0\in\mathbb{R}^n,\, t_0>0} F_{x_0,t_0}(M).$$

Suppose that $\{M_t\}$ is a mean curvature flow of closed surfaces. For any $s < t < t_0$, the monotonicity formula yields

$$\frac{d}{dt}\int_{M_t} k_{x_0,t_0}\,\mathrm{d}\mu \le 0,$$

whereby

$$F_{x_0,t_0}(M_t) \le F_{x_0,t_0+(t-s)}(M_s),$$

which implies that $\lambda(M_t) \leq \lambda(M_s)$ for any $s < t$.

From these considerations and the definition of the entropy, we thus immediately obtain [CM12]

Proposition 6.21. *The entropy functional λ is non-negative and invariant under dilations, rotations and translations of M. Moreover, if $\{M_t\}$ is a mean curvature flow, $\lambda(M_t)$ is non-increasing in t. Finally, the critical points of λ are self-similarly shrinking solutions of the mean curvature flow.*

In addition, among all closed planar curves, the entropy λ is minimised by the circle. By the Gage–Hamilton–Grayson theorem 6.16, any closed curve γ in the plane becomes convex and eventually shrinks to a round point. But the entropy is non-increasing under curve shortening flow, so we must have $\lambda(\gamma) \geq \lambda(S^1)$. In fact, in any dimension, the round sphere minimises entropy among all closed self-shrinking solutions [CIMW13].

Note that the entropy of a self-shrinker is equal to the functional $F_{0,1}$, the Gaussian area [CM12], since by the monotonicity formula, the critical points of $F_{0,1}$ are precisely the self-shrinkers. In other words, self-shrinkers are the minimal hypersurfaces of $(\mathbb{R}^{n+1}, g_{ij})$ with the conformal metric $g_{ij} = \mathrm{e}^{-\frac{|x|^2}{2n}}\delta_{ij}$. Equivalently, self-shrinkers are the critical points of the functional $F_{x_0,t_0}(\cdot)$ with respect to variations in all three parameters [CM12].

For some examples, it is possible to compute the entropy explicitly. A related quantity is Huisken's density, which is defined as the limit

$$\Theta_{x_0,t_0} = \lim_{t \to t_0} \int_{M_t} k_{x_0,t_0} \, \mathrm{d}\mu,$$

which exists thanks to the monotonicity formula. The density of the sphere, and thus its entropy (as the sphere is a self-shrinker) has been computed by Stone [Sto94]:

$$2 > \lambda(S^1) = \sqrt{\frac{2\pi}{\mathrm{e}}} \approx 1.52 > \frac{3}{2} > \lambda(S^2) = \frac{4}{\mathrm{e}} \approx 1.47 > \cdots > \lambda(S^n) > \sqrt{2}.$$

For a cylinder $S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}$, it holds that

$$\lambda\left(S^k_{\sqrt{2k}} \times \mathbb{R}^{n-k}\right) = \lambda(S^k).$$

In fact, for any hypersurface M, we have $\lambda(M \times \mathbb{R}) = \lambda(M)$. Furthermore, the entropy of any plane is equal to one. To see this, note that we may rewrite the entropy as the supremum of the Gaussian area,

$$\lambda(M) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} F(t_0^{-1}M + x_0),$$

where $F = F_{0,1}$. Therefore, for an m-dimensional plane $P \subset \mathbb{R}^n$, which we may assume to be $\mathbb{R}^m$,

$$\begin{aligned} F(\mathbb{R}^m) &= (4\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} \mathrm{e}^{-\frac{|x|^2}{4}} \, \mathrm{d}x \\ &= (4\pi)^{-\frac{m}{2}} \prod_{i=1}^{m} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x_i^2}{4}} \, \mathrm{d}x_i \\ &= 1, \end{aligned}$$

by Fubini's theorem. As a result, we obtain $\lambda(P) = 1$ for any m-dimensional plane $P \subset \mathbb{R}^n$.

In Chapter 8, we will need the explicit value of the entropy of some special solutions of the curve shortening flow. First, for the translating Grim Reaper solution the entropy has been computed by Guang:

Proposition 6.22 ([Gua19, Thm. 1.3])**.** *Let $\Gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, \infty) \to \mathbb{R}^2$ denote the Grim Reaper. For any point $(x_0, y_0) \in \mathbb{R}^2$, $t_0 \in (0, \infty)$, we have that*

$$F_{(x_0,y_0),t_0}(\Gamma) \leq 2.$$

In fact,

$$\lim_{N \to \infty} F_{(0,N),N}(\Gamma) = 2.$$

Therefore, the entropy of the Grim Reaper satisfies

$$\lambda(\Gamma) = 2.$$

Moreover, for the self-shrinking Abresch–Langer solutions [AL86] there is a lower bound on the entropy due to Baldauf and Sun.

Proposition 6.23 ([BS20])**.** *Let $\gamma_{m,n}$ denote an Abresch–Langer curve, that is, a closed convex self-shrinking solution to the curve shortening flow with turning number m and $2n$ critical points of the curvature, where $m, n \in \mathbb{N}$ are coprime integers such that*

$$\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}.$$

Then the entropy of $\gamma_{m,n}$ satisfies

$$\lambda(\gamma_{m,n}) \geq m\lambda(S^1) = m\sqrt{\frac{2\pi}{\mathrm{e}}}.$$

As a function of $(x_0, t_0) \in \mathbb{R}^{n+1} \times (0, \infty)$, $F_{x_0,t_0}(M)$ is a smooth function for any smooth closed embedded hypersurface $M^n \subset \mathbb{R}^{n+1}$ with polynomial volume growth [CM12]. However, $\lambda(M)$ does not depend smoothly on M: In fact, the entropy functional is not continuous on the space of hypersurfaces, for a sequence of rescalings of the sphere converges to a hyperplane at any point, but the sphere has entropy greater than $\sqrt{2}$, while the plane has entropy 1. Still, λ is lower semicontinuous, since it is defined as the supremum of the collection $\{F_{x_0,t_0} : (x_0, t_0) \in \mathbb{R}^n \times (0, \infty)\}$ of lower semicontinuous (in fact continuous) functions on submanifolds.

Finally, we have an estimate for the entropy in terms of the Euclidean density at each point.

Proposition 6.24 ([Whi15]). *Let $M^m \subset \mathbb{R}^n$ be an immersed surface. Define the Euclidean density of M at x by*

$$\Theta^m(M, x) = \lim_{r \to 0} \frac{\mathcal{H}^m(B_r(x) \cap M)}{\omega_m r^m},$$

where ω_m is the volume of the m-dimensional unit ball. Then, for any $x \in M$, we have that

$$\lambda(M) \geq \Theta^m(M, x)$$

whenever the limit exists.

Chapter 7

Convexity and cylindrical estimates for k-convex mean curvature flow

In this chapter, we consider two estimates that are central to Huisken and Sinestrari's surgery procedure for 2-convex mean curvature flow in the more general case of k-convex mean curvature flow. The asymptotic convexity estimate was originally proved by Huisken–Sinestrari [HS99a; HS99b] using an intricate argument based on induction on elementary symmetric polynomials, which was then used to show the cylindrical estimate. We follow instead an approach introduced by Nguyen (which appeared in Schulze's lecture notes [Sch17]) that shares many similarities with the strategy of the proof of Langford's very general pinching principle [Lan17]. Our main result, Theorem 7.11, which gives an estimate akin to the cylindrical estimate of Huisken–Sinestrari, but in the case of k-convex mean curvature flow, is proved directly from the assumption using the Stampacchia iteration technique [Hui84]. In the 2-convex case, the convexity estimate then follows from the cylindrical one, shortening the original proof considerably. Estimates that are similar in spirit have been employed in higher codimension [Ngu18; LN20b], which eventually led to a surgery procedure for mean curvature flow of high codimension with a suitably pinched second fundamental form [Ngu20].

In the following, we denote the principal curvatures of an immersed hypersurface $M^n \subset \mathbb{R}^{n+1}$ by

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Definition 7.1. Let $M^n \subset \mathbb{R}^{n+1}$ be an immersed hypersurface. Suppose that $1 \leq k \leq n$. We say that M is *k-convex* if

$$\lambda_1 + \cdots + \lambda_k \geq 0$$

everywhere in M. In particular, if M is 1-convex it is called *convex*, and if M is n-convex, that is, $H \geq 0$, it is called *mean convex*.

By Hamilton's maximum principle for tensors [Ham86], k-convexity is a property that is preserved by the flow [Sch17, Prop. 6.0.4].

Proposition 7.2. *Suppose that $\{M_t\}$ is a mean curvature flow of immersed hypersurfaces for which M_0 satisfies*

$$\lambda_1 + \cdots + \lambda_k \geq \alpha H$$

for some $\alpha \geq 0$ and $1 \leq k \leq n$. Then this inequality is preserved under mean curvature flow. In particular, if M_0 is k-convex then so is M_t.

7.1 Huisken–Sinestrari's surgery procedure

One of the main motivations of the study of curvature flows in general is the possibility of obtaining topological statements from them, as most prominently evidenced by the Ricci flow of three-manifolds. Obviously, the presence of finite-time singularities in mean curvature flow prevents a further description of the flow in terms of differential geometry. The basic idea of a surgery procedure is to be able to continue a smooth flow through singularities up to errors that are introduced in a controlled manner, so that statements about the changes in its topology can be made.

More precisely, assuming that the flow does not completely vanish at a singular time, at a time shortly before the singular time the flow is stopped and a neck, that is, a part of the surface close to a cylinder, is removed and replaced by two regions diffeomorphic to disks such that the resulting surface, possibly disconnected, is again smooth and the flow can be continued until the next singular time, when the process is repeated. This enables one to keep track of the topological changes. Since area is non-increasing along mean curvature flow and every surgery decreases area by a certain amount, the procedure must terminate eventually. Moreover, one must ensure that the relevant estimates, such as the ones below, continue to hold through the surgeries with the same constants. Huisken–Sinestrari showed, in a complex technical work, that this procedure can be carried out for 2-convex mean curvature flow and stated its topological implication.

Theorem 7.3 ([HS09, Thm. 1.1])**.** *Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed hypersurface with $n \geq 3$. Suppose that $M_0 = F_0(M)$ is 2–convex. Then there exists a mean curvature flow with surgeries starting from M_0 which terminates after a finite number of steps.*

Corollary 7.4 ([HS09, Cor. 1.2])**.** *Any smooth closed 2-convex immersed hypersurface $M^n \subset \mathbb{R}^{n+1}$ with $n \geq 3$ is diffeomorphic either to S^n or to a finite connected sum of $S^{n-1} \times S^1$.*

Note that these results have been extended to the $n = 3$ case by Brendle–Huisken [BH16].

Within the surgery procedure, the convexity and cylindrical estimates are used to establish the presence of a suitable neck region. In particular, combined with a gradient estimate for the curvature they quantify the closeness of a high curvature region to a cylinder or a sphere. In Huisken's work on mean curvature flow of convex hypersurfaces [Hui84], he established an *umbilic estimate*: For any $\eta > 0$ there exists $C_\eta = C(\eta, M_0)$

such that

$$|\mathring{A}|^2 \leq \eta H^2 + C_\eta.$$

Here $\mathring{A}$ denotes the trace-free second fundamental form. Therefore, the ratio $\frac{|\mathring{A}|^2}{H^2}$ is close to zero wherever H tends to infinity, so that at a singular point the hypersurface is essentially umbilic. The convexity estimate, meanwhile, shows that an almost singular region becomes asymptotically convex as one approaches the singular time [HS09].

Theorem 7.5 ([HS99a; see also HS09, Thm. 1.4])**.** *Let $\{M_t\}$ be a closed mean convex mean curvature flow. Then for any $\eta > 0$ there exists $C_\eta = C(\eta, M_0) > 0$ such that*

$$\lambda_1 \geq -\eta H - C_\eta \tag{7.1}$$

on M_t for any $t \in [0, T)$.

Clearly, for a cylinder $S^{n-1} \times \mathbb{R}$ we have that $|A|^2 = \frac{1}{n-1}H^2$. Conversely, if $|A|^2 = \frac{1}{n-1}H^2$ and $\lambda_1 = 0$ at a point, then $\lambda_2 = \cdots = \lambda_n$. The cylindrical estimate can be seen as a quantitative version of this statement in the sense that it implies that the curvature at points with small first principal curvature is close to that of a cylinder.

Theorem 7.6 ([HS09, Thm. 1.5])**.** *Let $\{M_t\}$ be a closed 2-convex mean curvature flow. Then for any $\eta > 0$ there exist constants $C_\eta = C(\eta, M_0) > 0$ and $c = c(n)$ such that*

$$|\lambda_1| \leq \eta H \quad \Longrightarrow \quad |\lambda_i - \lambda_j| \leq c\eta H + C_\eta \tag{7.2}$$

for any $1 < i, j \leq n$ on M_t for any $t \in [0, T)$.

In particular, we know that the limit of a sequence of rescalings at a type-I singularity as in Section 6.3 of a mean convex mean curvature flow can only be a shrinking sphere or a generalised cylinder $S^{n-m}_{\sqrt{2(n-m)}} \times \mathbb{R}^m$, by Huisken's classification of self-similarly shrinking solutions [Hui90]. The convexity and cylindrical estimates then further constrain these possibilities, as the former implies that in the mean convex case the limit must be convex, while the latter implies that in the 2-convex case the tangent flow is either a shrinking sphere or a cylinder $S^{n-1} \times \mathbb{R}$.

7.2 Poincaré-type inequality

The key to the proof of the cylindrical estimate presented here is a careful analysis of the curvature terms in Simons' identity [LN20b; cf. BH17]. In the following, we assume that $\{M_t\}$ is a mean curvature flow of n-dimensional k-convex immersed hypersurfaces in $\mathbb{R}^{n+1}$. Recall

Proposition 7.7 (Simons' identity)**.** *Let $M^n \subset \mathbb{R}^{n+1}$ be a hypersurface. Then it holds that*

$$\nabla_k \nabla_\ell h_{ij} - \nabla_i \nabla_j h_{k\ell} = h_{k\ell} h_i^p h_{jp} - h_{i\ell} h_j^p h_{kp} + h_{jk} h_i^p h_{\ell p} - h_{ij} h_k^p h_{\ell p}. \tag{7.3}$$

We symmetrise (7.3) and obtain

$$\begin{aligned}\nabla_k \nabla_\ell h_{ij} + \nabla_\ell \nabla_k h_{ji} - \nabla_i \nabla_j h_{k\ell} - \nabla_j \nabla_i h_{k\ell} &= h_{k\ell} h_i^p h_{jp} - h_{i\ell} h_j^p h_{kp} \\ &\quad + h_{jk} h_i^p h_{\ell p} - h_{ij} h_k^p h_{\ell p} \\ &\quad + h_{\ell k} h_j^p h_{ip} - h_{jk} h_i^p h_{\ell p} \\ &\quad + h_{i\ell} h_j^p h_{kp} - h_{ji} h_\ell^p h_{kp} \\ &= 2 h_{k\ell} h_i^p h_{jp} - 2 h_{ij} h_k^p h_{\ell p}.\end{aligned}$$

Defining $C_{ijk\ell} = h_{k\ell} h_i^p h_{jp} - h_{ij} h_k^p h_{\ell p}$, we can write this more compactly as

$$\nabla_{(k} \nabla_{\ell)} h_{ij} - \nabla_{(i} \nabla_{j)} h_{k\ell} = 2 C_{ijk\ell}.$$

Take the trace on both sides with respect to $C_{ijk\ell}$ to get

$$\left(\nabla_{(k} \nabla_{\ell)} h_{ij} - \nabla_{(i} \nabla_{j)} h_{k\ell} \right) C^{ijk\ell} = 2 \left| C \right|^2 .$$

However, note that the tensor $C_{ijk\ell}$ is symmetric in the pairs (i,j) and (k,ℓ), so that

$$2\left(\nabla_k\nabla_\ell h_{ij}-\nabla_i\nabla_j h_{k\ell}\right)C^{ijk\ell}=2\left|C\right|^2.$$

The norm of C can be computed as

$$\begin{aligned}|C|^2&=(h_{k\ell}h_{ij}^2-h_{ij}h_{k\ell}^2)(h^{k\ell}(h^{ij})^2-h^{ij}(h^{k\ell})^2)\\&=2\left|A\right|^2\operatorname{tr}(A^4)-2\operatorname{tr}(A^3)^2.\end{aligned}$$

Recall that the second fundamental form A can be diagonalised by its eigenvalues $\lambda_1,\dots,\lambda_n$. We thus have that

$$\begin{aligned}\sum_{i,j=1}^n(\lambda_i-\lambda_j)^2\lambda_i^2\lambda_j^2&=\sum_{i,j=1}^n(\lambda_i^2+\lambda_j^2-2\lambda_i\lambda_j)\lambda_i^2\lambda_j^2\\&=\sum_{i,j=1}^n(\lambda_i^4\lambda_j^2+\lambda_j^4\lambda_i^2-2\lambda_i^3\lambda_j^3)\\&=2\left|A\right|^2\operatorname{tr}(A^4)-2\operatorname{tr}(A^3)^2.\end{aligned}$$

Therefore,

$$2\left(\nabla_k\nabla_\ell h_{ij}-\nabla_i\nabla_j h_{k\ell}\right)C^{ijk\ell}=2\sum_{i,j=1}^n(\lambda_i-\lambda_j)^2\lambda_i^2\lambda_j^2.$$

We claim that at any point where $H>0$ and

$$|A|-\frac{1}{\sqrt{n-(k-1)}}H>0,$$

the term $|C|^2$ is strictly positive. To the contrary, assume that $|C|^2=0$, that is,

$$\sum_{j=1}^n\sum_{i=1}^n(\lambda_i-\lambda_j)^2\lambda_i^2\lambda_j^2=0,$$

which yields in particular that

$$\sum_{i=1}^{n}(\lambda_i - \lambda_n)^2\lambda_i^2\lambda_n^2 = 0.$$

Since $H > 0$ implies that $\lambda_n > 0$, we must have either $\lambda_i = 0$ or $\lambda_i = \lambda_n$ for any $i = 1, \dots, n-1$. Therefore, there exists $0 \leq i_0 \leq n-1$ such that $\lambda_{i_0} = 0$, $\lambda_{i_0+1} = \lambda_n$. By k-convexity, we must have $\lambda_k > 0$, whence $i_0 \leq k-1$. We obtain

$$\begin{aligned}|A|^2 &= (n-i_0)\lambda_n^2,\\ H &= (n-i_0)\lambda_n,\end{aligned}$$

and thus

$$|A| - \frac{1}{\sqrt{n-(k-1)}}H = \left(\sqrt{n-i_0} - \frac{n-i_0}{\sqrt{n-(k-1)}}\right)\lambda_n.$$

But as $i_0 \leq k-1$, we conclude that $|A| - \frac{1}{\sqrt{n-(k-1)}}H \leq 0$, which is a contradiction.

Using the claim, we can prove the following Poincaré-type inequality.

Lemma 7.8 ([cf. Lan17, Prop. 2.7; LN20a, Prop. 2.2])**.** *Given $n \geq 3$, $\varepsilon \in (0,1)$, and $\eta > 0$, there exists $\gamma = \gamma(n, \varepsilon, \eta) > 0$ with the following property: Let $F : M^n \to \mathbb{R}^{n+1}$ be a smoothly immersed k-convex hypersurface and $u \in W^{2,2}(M)$ a function satisfying* $\operatorname{spt} u \subset U_{\varepsilon,\eta,M}$*, where, introducing the functions*

$$f_{1,\eta} := |A| - \frac{1}{\sqrt{n-(k-1)}}H - \eta H$$

and

$$f_{2,\varepsilon} := \sum_{i=1}^{k}\lambda_i - \varepsilon H,$$

the set $U_{\varepsilon,\eta,M} \subset M$ is defined by

$$U_{\varepsilon,\eta,M} := \{x \in M : f_{1,\eta} \geq 0,\ f_{2,\varepsilon} \geq 0\},$$

that is,

$$U_{\varepsilon,\eta,M} = \left\{ x \in M : |A| - \frac{1}{\sqrt{n-(k-1)}} H - \eta H \geq 0,\ \sum_{i=1}^{k} \lambda_i - \varepsilon H \geq 0 \right\} \subset M.$$

Then, for any $r \geq 1$,

$$\gamma \int u^2 \left|A\right|^2 \mathrm{d}\mu \leq r^{-1} \int \left|\nabla u\right|^2 \mathrm{d}\mu + (1+r) \int u^2 \frac{\left|\nabla A\right|^2}{H^2} \,\mathrm{d}\mu\,.$$

Remark. Note that for strictly k-convex hypersurfaces evolving by mean curvature flow, $f_{2,\varepsilon} \geq 0$ follows immediately, so that the condition $f_{2,\varepsilon} \geq 0$ is automatic in this case.

Proof. We claim that

$$\gamma(n,\varepsilon,\eta) \left|A\right|^2 H^4 \leq \left|C\right|^2 \qquad \text{in } U_{\varepsilon,\eta,M} \tag{7.4}$$

for any immersed hypersurface $F : M^n \to \mathbb{R}^{n+1}$. This follows by a rescaling and compactness argument, as we now show.

Indeed, if this is not the case, then there exists a sequence of points $\{\lambda^m\} \subset \mathbb{R}^n$ satisfying

$$f_{1,\eta}(\lambda^m) := |\lambda^m| - \frac{1}{\sqrt{n-(k-1)}} \operatorname{tr}(\lambda^m) - \eta \operatorname{tr}(\lambda^m) \geq 0$$

and

$$f_{2,\varepsilon}(\lambda^m) := \sum_{i=1}^{k} \lambda_i^m - \varepsilon \operatorname{tr}(\lambda^m) \geq 0,$$

where $\operatorname{tr}(\lambda) := \sum_{i=1}^n \lambda_i$, but

$$\frac{\left|C(\lambda^m)\right|^2}{\left|\lambda^m\right|^2 \operatorname{tr}(\lambda^m)^4} = \frac{\left|C(\lambda^m)\right|^2}{W(\lambda^m)} \to 0 \tag{7.5}$$

as $m \to \infty$, with $W(\lambda^m) := \left|\lambda^m\right|^2 \operatorname{tr}(\lambda^m)^4$ and

$$\left|C(\lambda)\right|^2 := \sum_{i,j=1}^{n} (\lambda_j - \lambda_i)^2 \lambda_i^2 \lambda_j^2.$$

Now define $r_m := W(\lambda^m)^{-\frac{1}{6}}$ and $\hat{\lambda}^m := r_m \lambda^m$. Using the inequality [HS99b, (2.5)]

$$|A|^2 \le c_0 H^2,$$

where c_0 is a constant, we observe that

$$c_0^{-\frac{1}{6}} \operatorname{tr}(\lambda^m)^{-1} \le W(\lambda^m)^{-\frac{1}{6}} \le c_0^{\frac{1}{3}} |\lambda^m|^{-1},$$

which implies

$$\begin{aligned} \left|\hat{\lambda}^m\right| &\le c_0^{\frac{1}{3}} < \infty, \\ \operatorname{tr}(\hat{\lambda}^m) &\ge c_0^{-\frac{1}{6}} > 0, \end{aligned}$$

and hence, up to a subsequence, $\hat{\lambda}^m \to \hat{\lambda} \in \mathbb{R}^n$ as $m \to \infty$. Since

$$\begin{aligned} r_m f_{1,\eta}(\lambda^m) &\ge 0, \\ r_m f_{2,\varepsilon}(\lambda^m) &\ge 0, \end{aligned}$$

we find

$$\left|\hat{\lambda}\right| - \frac{1}{\sqrt{n-(k-1)}} \operatorname{tr}(\hat{\lambda}) \ge \eta \operatorname{tr}(\hat{\lambda}) > 0 \tag{7.6}$$

and

$$\sum_{i=1}^{k} \hat{\lambda}_i \ge \varepsilon \operatorname{tr}(\hat{\lambda}) > 0. \tag{7.7}$$

On the other hand,

$$\sum_{i,j=1}^{n} \left(\hat{\lambda}_i^m \hat{\lambda}_j^m (\hat{\lambda}_j^m - \hat{\lambda}_i^m)\right)^2 = r_m^6 \, |C(\lambda^m)|^2$$

so that, by (7.5),

$$\sum_{i,j=1}^{n} \hat{\lambda}_i^2 \hat{\lambda}_j^2 (\hat{\lambda}_j - \hat{\lambda}_i)^2 = 0. \tag{7.8}$$

Together, (7.6) to (7.8) are in contradiction by the claim we proved before. This proves (7.4).

Using (7.4), we can estimate [Lan17]

$$\begin{aligned}
\gamma \int u^2 |A|^2 \, \mathrm{d}\mu \le & \int u^2 H^{-4} |C|^2 \, \mathrm{d}\mu \\
= & \int u^2 H^{-4} C^{ijk\ell} \left(\nabla_k \nabla_\ell h_{ij} - \nabla_i \nabla_j h_{k\ell} \right) \mathrm{d}\mu \\
= & \int u^2 \left(2H^{-4} C^{ijk\ell} \frac{\nabla_k u}{u} - 4C^{ijk\ell} \frac{\nabla_k H}{H^5} + H^{-4} \nabla_k C^{ijk\ell} \right) \nabla_\ell h_{ij} \, \mathrm{d}\mu \\
& - \int u^2 \left(2H^{-4} C^{ijk\ell} \frac{\nabla_i u}{u} - 4C^{ijk\ell} \frac{\nabla_i H}{H^5} + H^{-4} \nabla_i C^{ijk\ell} \right) \nabla_j h_{k\ell} \, \mathrm{d}\mu \\
\le & \, c \int u^2 \left(\frac{|\nabla u|}{u} + \frac{|\nabla A|}{H} \right) \frac{|\nabla A|}{H} \, \mathrm{d}\mu,
\end{aligned}$$

where $c = c(n, \varepsilon, \eta)$ is a constant. The claim now follows from Young's inequality. $\square$

7.3 Cylindrical estimate

For a uniformly k-convex mean curvature flow, the inequality $|A|^2 \ge \frac{1}{n} H^2$ is preserved, so that $|A|$ is a smooth function along the flow. From the evolution equation (6.5) we can thus compute

$$\frac{\partial}{\partial t} |A| = \Delta |A| - \frac{1}{2|A|^3} |A \otimes \nabla A - \nabla A \otimes A|^2 + |A|^3 .$$

The gradient term in this equation has an interesting structure, as it only vanishes if A is a multiple of the second fundamental form of $\mathbb{R}^{n-1} \times S^1$. However, in the k-convex case, we bound this term from below.

Lemma 7.9 ([Lan17, Lemma 2.1; cf. Hui84, Lemma 2.3])**.** *Let $F : M^n \to \mathbb{R}^{n+1}$ be a strictly k-convex hypersurface, i. e., $\sum_{i=1}^k \lambda_i \ge \varepsilon H > 0$ for some $\varepsilon \in (0, 1]$. In the mean convex case $k = n$, suppose that $|A| - H \ge \eta H$ for $\eta > 0$. Then there exists a constant*

$\gamma = \gamma(n, \varepsilon, \eta) > 0$ *such that*

$$|A \otimes \nabla A - \nabla A \otimes A|^2 \geq \gamma |A|^2 |\nabla A|^2 .$$

Proof. Fix $x \in M$ such that $|A| |\nabla A| (x) \neq 0$. (If there is no such x, then the claim is trivially true.) By rescaling, we can assume that $|A| |\nabla A| = 1$ at x. Since the set

$$\{(W,T) \in \mathrm{Sym}^2 \times \mathrm{Sym}^3 : \sum_{i=1}^{k} \lambda_i(W) \geq \varepsilon \operatorname{tr}(W) > 0, |W| = |T| = 1\},$$

where Sym^m denotes the set of totally symmetric $(0,m)$-tensors is compact, and $\operatorname{tr}(W)$ is uniformly bounded from below for such W, we only need to prove that

$$|A \otimes \nabla A - \nabla A \otimes A|^2 > 0.$$

Therefore, assume that we have

$$A \otimes \nabla A = \nabla A \otimes A.$$

We choose a diagonalising frame and after applying the Codazzi identity, we get

$$\begin{aligned} \lambda_i \delta_{ij} \nabla_p h_{\ell m} &= \nabla_i h_{jp} \lambda_\ell \delta_{\ell m} \\ &= \lambda_\ell \delta_{\ell m} \nabla_p h_{ij} \end{aligned} \tag{7.9}$$

for each i, j, ℓ, m, p. Since $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, $H > 0$ implies that $\lambda_n > 0$. Let i, j, p be such that $\nabla_p h_{ij} \neq 0$. (If $\nabla_p h_{ij} = 0$ for all i, j, p, then the claim is trivially true.) Then, in particular,

$$\lambda_i \delta_{ij} \nabla_p h_{nn} = \lambda_n \nabla_p h_{ij} \neq 0$$

and hence $i = j$. By the same reasoning and the Codazzi identity, we obtain that $p = j$.

Thus $\nabla_p h_{ij} \neq 0$ only if $i = j = p$. Therefore,

$$0 \neq \lambda_n \nabla_p h_{pp} = \lambda_p \nabla_p h_{nn},$$

so that $p = n$ by the above. That is, $\lambda_n \nabla_p h_{ij} \neq 0$ only if $i = j = p = n$. On the other hand, if we set $p = \ell = m = n$ and $i = j \neq n$ in (7.9), we get

$$\lambda_i \nabla_n h_{nn} = \nabla_n h_{ii} \lambda_n = 0,$$

from which we conclude that $\lambda_i = 0$ for all $i = 1, \ldots, n-1$. Hence

$$\begin{aligned} |A|^2 &= \lambda_n^2, \\ H &= \lambda_n. \end{aligned}$$

Consequently, in the mean convex case our assumption that $|A| - H \geq \eta H$ implies

$$0 \geq \eta \lambda_n,$$

a contradiction. On the other hand, if $k < n$ then k-convexity directly yields $\lambda_k > 0$. $\square$

In order to derive the cylindrical estimate, we first define a suitable function and derive L^p-bounds [Hui84; Lan17; cf. LN20a]. Consider for $\eta > 0$ and $\sigma \in [0,1]$ the following function

$$G_{\sigma,\eta} = \frac{|A| - \left(\frac{1}{\sqrt{n-(k-1)}} + \eta \right) H}{H^{1-\sigma}}.$$

The evolution equation of $G_{\sigma,\eta}$ is given by

$$\begin{aligned} \frac{\partial}{\partial t} G_{\sigma,\eta} = \Delta G_{\sigma,\eta} &+ \frac{2(1-\sigma)}{H} \langle \nabla G_{\sigma,\eta}, \nabla H \rangle \\ &- \frac{1}{2H^{1-\sigma} |A|^3} |A \otimes \nabla A - \nabla A \otimes A|^2 \\ &- \frac{\sigma(1-\sigma) G_{\sigma,\eta}}{H^2} |\nabla H|^2 + \sigma |A|^2 G_{\sigma,\eta}. \end{aligned}$$

Therefore, at all points where $G_{\sigma,\eta} \geq 0$, we may estimate, using Lemma 7.9, that

$$\frac{\partial}{\partial t} G_{\sigma,\eta} \leq \Delta G_{\sigma,\eta} + 2\left|\nabla G_{\sigma,\eta}\right| \frac{|\nabla H|}{H} - \frac{\gamma_1 G_{\sigma,\eta}}{H^2} |\nabla A|^2 + \sigma |A|^2 G_{\sigma,\eta}.$$

We let $G_{\sigma,\eta,+} = \max\{G_{\sigma,\eta}, 0\}$ and obtain the evolution equation for its L^p-norm,

$$\frac{d}{dt} \int G^p_{\sigma,\eta,+} \, \mathrm{d}\mu = p \int G^{p-1}_{\sigma,\eta,+} \frac{\partial}{\partial t} G_{\sigma,\eta} \, \mathrm{d}\mu - \int G^p_{\sigma,\eta,+} H^2 \, \mathrm{d}\mu \, .$$

Discarding the second term, we get

$$\begin{aligned} \frac{d}{dt} \int G^p_{\sigma,\eta,+} \, \mathrm{d}\mu \leq & - p(p-1) \int G^{p-2}_{\sigma,\eta,+} \left|\nabla G_{\sigma,\eta}\right|^2 \mathrm{d}\mu - \gamma_1 p \int G^p_{\sigma,\eta,+} \frac{|\nabla A|^2}{H^2} \, \mathrm{d}\mu \\ & + 2p \int G^{p-1}_{\sigma,\eta,+} \left|\nabla G_{\sigma,\eta}\right| \frac{|\nabla H|}{H} \, \mathrm{d}\mu + \sigma p \int G^p_{\sigma,\eta,+} |A|^2 \, \mathrm{d}\mu \, . \end{aligned}$$

We use Young's inequality and the inequality $\frac{3}{n+2} |\nabla H|^2 \leq |\nabla A|^2$ [Hui84, Lemma 2.2] to divide one of the terms as follows:

$$\begin{aligned} 2p \int G^{p-1}_{\sigma,\eta,+} \left|\nabla G_{\sigma,\eta}\right| \frac{|\nabla H|}{H} \, \mathrm{d}\mu \leq & \; p^{\frac{3}{2}} \int G^{p-2}_{\sigma,\eta,+} \left|\nabla G_{\sigma,\eta}\right|^2 \mathrm{d}\mu \\ & + \frac{n+2}{3} p^{\frac{1}{2}} \int G^p_{\sigma,\eta,+} \frac{|\nabla A|^2}{H^2} \, \mathrm{d}\mu \, . \end{aligned}$$

This implies that

$$\begin{aligned} \frac{d}{dt} \int G^p_{\sigma,\eta,+} \, \mathrm{d}\mu \leq & - (p^2 - p^{\frac{3}{2}} - p) \int G^{p-2}_{\sigma,\eta,+} \left|\nabla G_{\sigma,\eta}\right|^2 \mathrm{d}\mu \\ & - \left(\gamma_1 p - \frac{n+2}{3} p^{\frac{1}{2}}\right) \int G^p_{\sigma,\eta,+} \frac{|\nabla A|^2}{H^2} \, \mathrm{d}\mu \\ & + \sigma p \int G^p_{\sigma,\eta,+} |A|^2 \, \mathrm{d}\mu \, . \end{aligned}$$

We use the Poincaré inequality, Lemma 7.8, with $u^2 = G^p_{\sigma,\eta,+}$, $r = p^{\frac{1}{2}}$, so that

$$|\nabla u|^2 = \frac{p^2}{4} G^{p-2}_{\sigma,\eta,+} \left|\nabla G_{\sigma,\eta}\right|^2$$

to get

$$\gamma_2 \int G_{\sigma,\eta,+}^p |A|^2 \, d\mu \le \frac{p^{\frac{3}{2}}}{4} \int G_{\sigma,\eta,+}^{p-2} |\nabla G_{\sigma,\eta}|^2 \, d\mu + (p^{\frac{1}{2}} + 1) \int G_{\sigma,\eta,+}^p \frac{|\nabla A|^2}{H^2} \, d\mu \, .$$

We thus have

$$\begin{aligned} \frac{d}{dt} \int G_{\sigma,\eta,+}^p \, d\mu \le & -\left(p^2 - p^{\frac{3}{2}} - p - \frac{1}{\gamma_2}\sigma p^{\frac{5}{2}}\right) \int G_{\sigma,\eta,+}^{p-2} |\nabla G_{\sigma,\eta}|^2 \, d\mu \\ & - \left(\gamma_1 p - \frac{n+2}{3} p^{\frac{1}{2}} - \frac{1}{\gamma_2}\sigma(p^{\frac{3}{2}} + p)\right) \int G_{\sigma,\eta,+}^p \frac{|\nabla A|^2}{H^2} \, d\mu \, . \end{aligned}$$

Therefore, if we choose p large enough and $\sigma \sim p^{-\frac{1}{2}}$, we can show that the right hand side is non-positive.

Proposition 7.10 ([Lan17, Prop. 3.1])**.** *There exists $l = l(n, \eta) > 0$ such that*

$$\frac{d}{dt} \int G_{\sigma,\eta,+}^p \, d\mu \le 0$$

if $p \ge \frac{1}{l}$, $\sigma \le \frac{l}{\sqrt{p}}$.

We can then apply the Michael–Simon Sobolev inequality and Stampacchia iteration to obtain an L^∞-bound for $G_{\sigma,\eta,+}$ from the L^p-bounds [Hui84; Lan17]. That is, we can show that for all $\eta > 0$ there exist $\sigma \in (0, 1)$ and $C(\eta)$ such that

$$|A| \le \left(\frac{1}{\sqrt{n - (k-1)}} + \eta\right) H + C(\eta) H^{1-\sigma}.$$

Taking the square and applying Young's inequality gives, with a different constant $C(\eta)$,

$$|A|^2 \le \left(\frac{1}{n - (k-1)} + \eta\right) H^2 + C(\eta) H^{2-2\sigma}.$$

Theorem 7.11 ([cf. HS09, Thm. 5.3; AL14, Thm. 1.3])**.** *Let $\{M_t\}$ be a mean curvature flow of closed n-dimensional k-convex hypersurfaces in $\mathbb{R}^{n+1}$, $n \ge 3$. Then for any $\eta > 0$*

there exists $C_\eta = C(\eta, M_0) > 0$ *such that*

$$|A|^2 - \frac{1}{n-(k-1)}H^2 \leq \eta H^2 + C_\eta.$$

In the 2-convex case, $k = 2$, we thus have

$$|A|^2 - \frac{1}{n-1}H^2 \leq \eta H^2 + C_\eta. \tag{7.10}$$

Recall the general identity

$$|A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}(\lambda_i - \lambda_j)^2,$$

which implies

$$|A|^2 - \frac{1}{n-1}H^2 = \frac{1}{n-1}\left(\sum_{1<i<j\leq n}(\lambda_i - \lambda_j)^2 + \lambda_1(n\lambda_1 - 2H)\right). \tag{7.11}$$

As a result, we obtain the cylindrical estimate (7.2).

7.4 Convexity estimate in the 2-convex case

In order to recover the convexity estimate (7.1), we may assume that $\lambda_1 \leq 0$, since otherwise the estimate is trivial. We estimate (7.11) to obtain

$$|A|^2 - \frac{1}{n-1}H^2 \geq |\lambda_1|\,(n\,|\lambda_1| + 2H).$$

Therefore, the estimate (7.10) gives

$$|\lambda_1|\,(n\,|\lambda_1| + 2H) \leq \eta H^2 + C_\eta.$$

This yields the convexity estimate (7.1). It should be noted, however, that Huisken–Sinestrari's proof of the convexity estimate only assumed mean convexity, i. e., $H > 0$.

Chapter 8

Singularity analysis for high codimension curve shortening flow

In this final chapter, we consider curve shortening flow of curves in $\mathbb{R}^n$. Using the same methods as in Altschuler's work in codimension two [Alt91], we extend his results to the case of arbitrary codimension to show in Theorems 8.6 and 8.7 that close to a singularity the solution is essentially planar, that is, a subsequential limit of a sequence of rescalings is a family of convex planar curves. Moreover, close to a type-I singularity, Theorem 8.8 implies that a sequence of rescalings along a blow-up sequence converges to a planar self-similarly shrinking solution, while for a type-II singularity, we show the existence of an essential blow-up sequence converging to the Grim Reaper in Theorem 8.9. Finally, we prove our main result, Theorem 8.10, which analyses the long-time behaviour of solutions of curve shortening flow with an entropy bound. More precisely, we show that for an initial curve with entropy less than that of the Grim Reaper, the curve shortening flow converges to a round point in finite time. This represents the first such convergence result for curve shortening flow in arbitrary codimension.

8.1 Curve shortening flow in any codimension

In the following, let $\gamma : S^1 \times [0, T) \to \mathbb{R}^n$ be a one-parameter family of smooth immersions of curves evolving by curve shortening flow,

$$\begin{aligned} \frac{\partial \gamma}{\partial t}(p,t) &= (\kappa N)(p,t), \\ \gamma(p,0) &= \gamma_0(p). \end{aligned} \tag{CSF}$$

Throughout, we employ the notation from Chapter 5, with the final time of existence of the flow denoted by either T or ω. Recall that by Theorem 6.11, for any smooth initial curve $\gamma_0 : S^1 \to \mathbb{R}^n$, there exists a unique smooth solution on some time interval $[0, T)$, $0 < T \leq \infty$. In fact, by Theorem 6.12, we have $T < \infty$.

As in the space curve case, we have scaling-invariant estimates on the derivatives of the tangent vector, and thus the derivatives of the curvature, depending only on the maximal curvature at the initial time. The proof proceeds just like Altschuler's. For brevity, we write, e. g., $\frac{\partial T}{\partial s} \equiv T_s$ and $T^{(m)} \equiv \frac{\partial^m T}{\partial s^m}$ for the derivatives of the tangent vector T [cf. ACGL20, Ch. 2].

Theorem 8.1 ([cf. Alt91, Thm. 3.1; YJ05, Thm. 3.1; Hät15, Thm. 3.6])**.** *For any $m \geq 1$ there exists $C_m < \infty$ such that for $t \in (0, \frac{1}{8K_0}]$, where $K_t := \sup \kappa^2(\cdot, t)$, it holds that*

$$\left| \frac{\partial^m T}{\partial s^m} \right|^2 \leq \frac{C_m K_0}{t^{m-1}}.$$

Proof. The commutation formula (6.6) implies

$$T_t = T_{ss} + |T_s|^2 \, T,$$

because γ evolves by curve shortening flow. Therefore,

$$\begin{aligned} |T_s|_t^2 &= 2\langle (T_s)_t, T_s\rangle \\ &= 2\langle (T_t)_s + |T_s|^2 T_s, T_s\rangle \\ &= 2\langle (T_{ss} + |T_s|^2 T)_s, T_s\rangle + 2|T_s|^4 \\ &= 2\langle T_{sss}, T_s\rangle + 2|T_s|_s^2 \langle T, T_s\rangle + 4|T_s|^4 \\ &= |T_s|_{ss}^2 - 2|T_{ss}|^2 + 4|T_s|^4, \end{aligned}$$

as $|T_s|_{ss}^2 = 2\langle T_{sss}, T_s\rangle + 2|T_{ss}|^2$. We thus have the differential inequality

$$|T_s|_t^2 - |T_s|_{ss}^2 = -2|T_{ss}|^2 + 4|T_s|^4 \leq 4|T_s|^4.$$

By the ODE comparison principle and $|T_s|^2 = \kappa^2 \leq K_0$ at $t = 0$, we have

$$|T_s|^2 \leq \frac{K_0}{1 - 4K_0 t} \leq 2K_0,$$

as $t \leq \frac{1}{8K_0}$ by assumption. We thus choose $C_1 = 2$.

We define

$$Z_m := T_t^{(m)} - T_{ss}^{(m)},$$

using the notation $T^{(m)} = \frac{\partial^m T}{\partial s^m}$. Then

$$Z_{m+1} = (Z_m)_s + |T_s|^2 T^{(m+1)}$$

and

$$\begin{aligned} \left|T^{(m)}\right|_t^2 - \left|T^{(m)}\right|_{ss}^2 &= 2\langle T_t^{(m)} - T_{ss}^{(m)}, T^{(m)}\rangle - 2\left|T^{(m+1)}\right|^2 \\ &= 2\langle Z_m, T^{(m)}\rangle - 2\left|T^{(m+1)}\right|^2. \end{aligned}$$

We already know

$$
\begin{aligned}
Z_0 &= |T_s|^2 T, \\
Z_1 &= 2\langle T_{ss}, T_s\rangle T + 2\,|T_s|^2 T_s,
\end{aligned}
$$

and moreover

$$
Z_2 = 2\langle T_{sss}, T_s\rangle T + 2\,|T_{ss}|^2 T + 6\langle T_{ss}, T_s\rangle T_s + 3\,|T_s|^2 T_{ss}.
$$

For $m = 2$, define $\Phi := t\,|T_{ss}|^2 + 4\,|T_s|^2$. Then

$$
\begin{aligned}
\Phi_t - \Phi_{ss} = {} & |T_{ss}|^2 + 2t\langle (T_{ss})_t, T_{ss}\rangle + 4(|T_s|^2_{ss} - 2\,|T_{ss}|^2 + 4\,|T_s|^4) \\
& - (2t\langle (T_{ss})_{ss}, T_{ss}\rangle + 2t\,|T_{sss}|^2 + 4\,|T_s|^2_{ss}) \\
= {} & -7\,|T_{ss}|^2 + 2t(\langle (T_{ss})_t - (T_{ss})_{ss}, T_{ss}\rangle - |T_{sss}|^2) + 16\,|T_s|^4 \\
= {} & -7\,|T_{ss}|^2 + 2t(\langle Z_2, T_{ss}\rangle - |T_{sss}|^2) + 16\,|T_s|^4
\end{aligned}
$$

Since

$$
\begin{aligned}
\langle Z_2, T_{ss}\rangle &= 2\langle T_{sss}, T_s\rangle\langle T, T_{ss}\rangle + 2\,|T_{ss}|^2 \langle T, T_{ss}\rangle \\
&\quad + 6\langle T_{ss}, T_s\rangle^2 + 3\,|T_s|^2 |T_{ss}|^2 \\
&= 2\langle T_{sss}, T_s\rangle\langle T, T_{ss}\rangle - 2\,|T_{ss}|^2 |T_s|^2 \\
&\quad + 6\langle T_{ss}, T_s\rangle^2 + 3\,|T_s|^2 |T_{ss}|^2,
\end{aligned}
$$

where we have used that $T_{ss} = -\kappa^2 T + \kappa_s N + \kappa\tau_1 B_1$ (see (6.8)) and $\kappa = |T_s|^2$ imply that

$\langle T, T_{ss}\rangle = -|T_s|^2$, we obtain

$$\begin{aligned}
\Phi_t - \Phi_{ss} = & -7|T_{ss}|^2 + 4t\langle T_{sss}, T_s\rangle\langle T, T_{ss}\rangle - 4t|T_{ss}|^2|T_s|^2 \\
& + 12t\langle T_{ss}, T_s\rangle^2 + 6t|T_s|^2|T_{ss}|^2 - 2t|T_{sss}|^2 + 16|T_s|^4 \\
\le & -7|T_{ss}|^2 + 4t|T_{sss}||T_{ss}||T_s| + 14t|T_{ss}|^2|T_s|^2 - 2t|T_{sss}|^2 + 16|T_s|^4 \\
= & -7|T_{ss}|^2 - 2t(|T_{sss}| - |T_{ss}||T_s|)^2 + 16t|T_{ss}|^2|T_s|^2 + 16|T_s|^4 \\
\le & \, 64K_0^2 + (32K_0 t - 7)|T_{ss}|^2 \\
\le & \, 64K_0^2,
\end{aligned}$$

using $|T_s|^2 \le 2K_0$ and $t \le \frac{1}{8K_0}$. At $t = 0$ we have $\Phi \le 4K_0$, so the ODE comparison principle implies

$$\Phi \le 64K_0^2 t + 4K_0 \le 12K_0$$

for any $t \le \frac{1}{8K_0}$. Therefore,

$$|T_{ss}|^2 \le \frac{12K_0}{t},$$

and we may choose $C_2 = 12$.

In general, for $m \ge 3$ we have

$$\begin{aligned}
Z_m = & \, 2\langle T^{(m+1)}, T^{(1)}\rangle T \\
& + 2m\langle T^{(m)}, T^{(2)}\rangle T \\
& + 2(m+1)\langle T^{(m)}, T^{(1)}\rangle T^{(1)} \\
& + (m+1)\left|T^{(1)}\right|^2 T^{(m)} \\
& + \sum_{\substack{0\le i,j,k<m \\ i+j+k=m+2}} N_{ijk}\langle T^{(i)}, T^{(j)}\rangle T^{(k)},
\end{aligned} \tag{8.1}$$

where $N_{ijk} = N_{ijk}(m)$ are non-negative integers. Indeed,

$$
\begin{aligned}
Z_3 &= (Z_2)_s + \left|T^{(1)}\right|^2 T^{(3)} \\
&= 2\langle T^{(4)}, T^{(1)}\rangle T \\
&\quad + 6\langle T^{(3)}, T^{(2)}\rangle T \\
&\quad + 8\langle T^{(3)}, T^{(1)}\rangle T^{(1)} \\
&\quad + 4\left|T^{(1)}\right|^2 T^{(3)} \\
&\quad + 8\left|T^{(2)}\right|^2 T^{(1)} \\
&\quad + 12\langle T^{(2)}, T^{(1)}\rangle T^{(2)}.
\end{aligned}
$$

Now assume that (8.1) holds for some m. Then we have that

$$
\begin{aligned}
Z_{m+1} &= (Z_m)_s + |T_s|^2 T^{(m+1)} \\
&= 2\langle T^{(m+2)}, T^{(1)}\rangle T + 2\langle T^{(m+1)}, T^{(2)}\rangle T + 2\langle T^{(m+1)}, T^{(1)}\rangle T^{(1)} \\
&\quad + 2m\langle T^{(m+1)}, T^{(2)}\rangle T + 2m\langle T^{(m)}, T^{(3)}\rangle T + 2m\langle T^{(m)}, T^{(2)}\rangle T^{(1)} \\
&\quad + 2(m+1)\langle T^{(m+1)}, T^{(1)}\rangle T^{(1)} + 2(m+1)\langle T^{(m)}, T^{(2)}\rangle T^{(1)} \\
&\quad + 2(m+1)\langle T^{(m)}, T^{(1)}\rangle T^{(2)} + 2(m+1)\langle T^{(2)}, T^{(1)}\rangle T^{(m)} \\
&\quad + (m+1)\left|T^{(1)}\right|^2 T^{(m+1)} \\
&\quad + \sum_{\substack{0\le i,j,k<m \\ i+j+k=m+2}} N_{ijk}\langle T^{(i+1)}, T^{(j)}\rangle T^{(k)} \\
&\quad + \sum_{\substack{0\le i,j,k<m \\ i+j+k=m+2}} N_{ijk}\langle T^{(i)}, T^{(j+1)}\rangle T^{(k)} \\
&\quad + \sum_{\substack{0\le i,j,k<m \\ i+j+k=m+2}} N_{ijk}\langle T^{(i)}, T^{(j)}\rangle T^{(k+1)} + |T_s|^2 T^{(m+1)} \\
&= 2\langle T^{(m+2)}, T^{(1)}\rangle T + 2(m+1)\langle T^{(m+1)}, T^{(2)}\rangle T \\
&\quad + 2(m+2)\langle T^{(m+1)}, T^{(1)}\rangle T^{(1)} + (m+2)\left|T^{(1)}\right|^2 T^{(m+1)} \\
&\quad + \sum_{\substack{0\le i,j,k<m+1 \\ i+j+k=m+3}} \tilde{N}_{ijk}\langle T^{(i)}, T^{(j)}\rangle T^{(k)},
\end{aligned}
$$

proving the claim.

We obtain

$$
\begin{aligned}
\left|T^{(m)}\right|_t^2 - \left|T^{(m)}\right|_{ss}^2 &= 2\langle Z_m, T^{(m)}\rangle - 2\left|T^{(m+1)}\right|^2 \\
&= 4\langle T^{(m+1)}, T^{(1)}\rangle\langle T, T^{(m)}\rangle + 4m\langle T^{(m)}, T^{(2)}\rangle\langle T, T^{(m)}\rangle \\
&\quad + 4(m+1)\langle T^{(m)}, T^{(1)}\rangle^2 + 2(m+1)\left|T^{(1)}\right|^2\left|T^{(m)}\right|^2 \\
&\quad + 2\sum_{\substack{0\le i,j,k<m\\ i+j+k=m+2}} N_{ijk}\langle T^{(i)}, T^{(j)}\rangle\langle T^{(k)}, T^{(m)}\rangle \\
&\quad - 2\left|T^{(m+1)}\right|^2 \\
&\le -2\left|T^{(m+1)}\right|^2 + 4\left|T^{(m+1)}\right|\left|T^{(1)}\right|\left|T^{(m)}\right| \\
&\quad - 2\left|T^{(1)}\right|^2\left|T^{(m)}\right|^2 + 2(m+2)\left|T^{(1)}\right|^2\left|T^{(m)}\right|^2 \\
&\quad + 4m\langle T^{(m)}, T^{(2)}\rangle\langle T, T^{(m)}\rangle + 4(m+1)\langle T^{(m)}, T^{(1)}\rangle^2 \\
&\quad + 2\sum_{\substack{0\le i,j,k<m\\ i+j+k=m+2}} N_{ijk}\langle T^{(i)}, T^{(j)}\rangle\langle T^{(k)}, T^{(m)}\rangle \\
&= -2\left(\left|T^{(m+1)}\right| - \left|T^{(1)}\right|\left|T^{(m)}\right|\right)^2 \\
&\quad + 2(m+2)\left|T^{(1)}\right|^2\left|T^{(m)}\right|^2 \\
&\quad + 4m\langle T^{(m)}, T^{(2)}\rangle\langle T, T^{(m)}\rangle + 4(m+1)\langle T^{(m)}, T^{(1)}\rangle^2 \\
&\quad + 2\sum_{\substack{0\le i,j,k<m\\ i+j+k=m+2}} N_{ijk}\langle T^{(i)}, T^{(j)}\rangle\langle T^{(k)}, T^{(m)}\rangle.
\end{aligned}
$$

By the induction hypothesis, we have

$$\left|T^{(i)}\right|^2 \le \frac{C_i K_0}{t^{i-1}}$$

for any $i = 1, \ldots, m-1$. Thus

$$\begin{aligned}\left|T^{(m)}\right|_t^2 - \left|T^{(m)}\right|_{ss}^2 \leq A_1 K_0 \left|T^{(m)}\right|^2 + A_2 \sqrt{\frac{K_0}{t}} \left|T^{(m)}\right|^2 \\ + 2 \sum_{\substack{0 \leq i,j,k < m \\ i+j+k = m+2}} N_{ijk} \left|T^{(i)}\right| \left|T^{(j)}\right| \left|T^{(k)}\right| \left|T^{(m)}\right| \\ \leq A_3 K_0 \left|T^{(m)}\right|^2 + A_4 K_0^3 t^{-(m-2)} + \frac{A_5}{t} \left|T^{(m)}\right|^2,\end{aligned}$$

using the Peter–Paul inequality with $\varepsilon = t$, where the constants A_i depend on m and $C_1, \ldots, C_{m-1}$. Therefore,

$$\begin{aligned}\left(t^{m-1}\left|T^{(m)}\right|^2\right)_t - \left(t^{m-1}\left|T^{(m)}\right|^2\right)_{ss} \leq (m-1)t^{m-2}\left|T^{(m)}\right|^2 \\ + A_3 K_0 t^{m-1}\left|T^{(m)}\right|^2 \\ + A_4 K_0^3 t + A_5 t^{m-2}\left|T^{(m)}\right|^2.\end{aligned}$$

For a large enough constant $C > 0$, we set $\Phi_m = t^{m-1}\left|T^{(m)}\right|^2 + Ct^{m-2}\left|T^{(m-1)}\right|^2$ and obtain

$$\begin{aligned}(\Phi_m)_t - (\Phi_m)_{ss} &\leq t^{m-2}(A_3 K_0 t + A_6 - 2C)\left|T^{(m)}\right|^2 + A_7 K_0^2 \\ &\leq A_7 K_0^2.\end{aligned}$$

We then proceed as in the $m = 2$ case to obtain C_m. □

Using the estimates and the short-time existence, we have long-time existence in the sense that as long as the curvature stays bounded, the flow can be continued for some time. In particular, the torsions do not play a role.

Theorem 8.2 ([AG92, Thm. 1.13]). *Assume that the curvature κ is bounded on the time interval $[0, t_0)$. Then there exists $\varepsilon > 0$ such that the curve shortening flow $\{\gamma_t\}$ exists and is smooth on the interval $[0, t_0 + \varepsilon)$.*

Equivalently, we may say that if T is the maximal time of existence of the flow, the curvature must tend to infinity as t approaches T.

Corollary 8.3 (cf. Theorem 6.13)**.** *Suppose that $\gamma : S^1 \times [0, T) \to \mathbb{R}^n$ is a solution of* (CSF) *with initial data $\gamma(\cdot, 0) = \gamma_0$. Then T is finite, and furthermore, $\max_{\gamma_t} \kappa^2 \to \infty$ as $t \to T$.*

8.2 Blow-up limits

Recall that we have set $K_t = \sup \kappa^2(\cdot, t)$. Assume that $\{\gamma_t\}$ is a curve shortening flow with a singularity forming at time T. If there exists a constant $c > 0$ such that

$$K_t \leq \frac{c}{T - t}, \qquad t < T,$$

we say that the singularity at T is of *type-I* (cf. Definition 6.14). Otherwise, that is, if

$$\limsup_{t \to T} K_t (T - t) = \infty,$$

we say that it is of *type-II*.

Analogous to Definition 6.17 [cf. Alt91], we say that $\{(p_j, t_j)\} \subset S^1 \times [0, T)$ is a *blow-up sequence* if $t_j \to T$ as $j \to \infty$ and

$$\lim_{j \to \infty} \kappa^2(p_j, t_j) = \infty.$$

In particular, a blow-up sequence is called *essential* if there exists a constant $\rho > 0$ such that

$$\rho K_t \leq \kappa^2(p_j, t_j), \qquad t < t_j.$$

For any immersed curve $\gamma : S^1 \to \mathbb{R}^n$, the total absolute curvature $\int_\gamma |\kappa| \, \mathrm{d}s$ is a scaling-invariant quantity. In order to show that blow-up limits of the curve shortening flow are planar, we recall Altschuler's estimate on the derivative of the total absolute curvature, which also holds for evolving curves in $\mathbb{R}^n$.

Theorem 8.4 ([Alt91, Thm. 5.1; see also YJ05]). *Let $\gamma : S^1 \times [0,T) \to \mathbb{R}^n$ be a solution of (CSF). Then the integral estimate*

$$\frac{d}{dt}\int_\gamma |\kappa| \, \mathrm{d}s \le -\int_\gamma |\kappa| \, \tau_1^2 \, \mathrm{d}s$$

holds for $t \in [0,T)$.

Proof. From Corollary 6.7, we have that

$$\frac{\partial \kappa^2}{\partial t} = \frac{\partial^2 \kappa}{\partial s^2} - 2\left(\frac{\partial \kappa}{\partial s}\right)^2 + 2\kappa^4 - 2\kappa^2\tau_1^2.$$

As in Altschuler's proof, we then define $\kappa_\varepsilon = \sqrt{\kappa^2 + \varepsilon}$, where $\varepsilon > 0$ is arbitrary, and obtain

$$\frac{d}{dt}\int_\gamma \kappa_\varepsilon \, \mathrm{d}s \le -\int_\gamma \frac{1}{\kappa_\varepsilon}\kappa^2\tau_1^2 \, \mathrm{d}s,$$

which implies the claim. ☐

For a planar curve we obtain the more precise formula

Theorem 8.5 ([Alt91, Thm. 5.14]). *For a planar solution γ to the curve shortening flow, we have*

$$\frac{d}{dt}\int_\gamma |\kappa| \, \mathrm{d}s = -2 \sum_{\{p \,:\, \kappa(p,\cdot)=0\}} \left|\frac{\partial \kappa}{\partial s}\right|.$$

Given a blow-up sequence $\{(p_j, t_j)\}$, we define a *blow-up procedure* as follows: Define $\gamma_j : S^1 \times [\alpha_j, \omega_j) \to \mathbb{R}^n$, where $\alpha_j = -\lambda_j^2 t_j$, $\omega_j = \lambda_j^2(\omega - t_j)$, by

$$\gamma_j(\cdot, \bar{t}) = \lambda_j(A_j\gamma(\cdot, t) + b_j), \qquad \bar{t} = \lambda_j^2(t - t_j).$$

Here, ω denotes the final time of existence of γ. Moreover, $\lambda_j > 0$, $A_j \in \mathrm{SO}(n)$, $b_j \in \mathbb{R}^n$

are such that

$$\begin{aligned}
\gamma_j(p_j,0) &= 0 \in \mathbb{R}^n,\\
T_j(p_j,0) &= (1,0,\dots,0) = e_1,\\
N_j(p_j,0) &= (0,1,0,\dots,0) = e_2,\\
(B_i)_j(p_j,0) &= (0,\dots,0,1,0,\dots,0) = e_{i+2}, \qquad i = 1,\dots,n-2.
\end{aligned}$$

In order to parametrise γ_j by arclength, note that

$$\left|\frac{\partial \gamma_j}{\partial s}\right| = \lambda_j \left|A_j \frac{\partial \gamma}{\partial s}\right| = \lambda_j.$$

Therefore, if we let $\bar{s} = \lambda_j s$ then γ_j is parametrised by arclength $\bar{s}$ once we define

$$\gamma_j(\bar{s},\bar{t}) = \lambda_j(A_j\gamma(s,t) + b_j).$$

This then gives

$$\frac{\partial \gamma_j}{\partial \bar{s}} = \frac{\partial \gamma_j}{\partial s}\frac{\partial s}{\partial \bar{s}} = \lambda_j A_j \frac{\partial \gamma}{\partial s}\frac{1}{\lambda_j} = A_j \frac{\partial \gamma}{\partial s},$$

that is,

$$T_j(\bar{s},\bar{t}) = A_j T(s,t), \tag{8.2}$$

and in the same way we obtain

$$\begin{aligned}
N_j(\bar{s},\bar{t}) &= A_j N(s,t),\\
(B_i)_j(\bar{s},\bar{t}) &= A_j B_i(s,t),\\
\kappa_j(\bar{s},\bar{t}) &= \frac{1}{\lambda_j}\kappa(s,t),\\
(\tau_i)_j(\bar{s},\bar{t}) &= \frac{1}{\lambda_j}\tau_i(s,t).
\end{aligned}$$

Then γ_j is indeed parametrised by arclength $\bar{s}$,

$$\left|\frac{\partial \gamma_j}{\partial \bar{s}}\right| = \left|A_j \frac{\partial \gamma}{\partial s}\right| = 1,$$

since γ is parametrised by arclength s and $|A_j| = 1$. As a result, each γ_j is a solution of curve shortening flow,

$$\frac{\partial \gamma_j}{\partial \bar{t}}(\bar{s}, \bar{t}) = (\kappa_j N_j)(\bar{s}, \bar{t}).$$

Indeed,

$$\begin{aligned}\frac{\partial \gamma_j}{\partial \bar{t}}(\bar{s}, \bar{t}) &= \lambda_j A_j \frac{\partial \gamma}{\partial t}(s,t)\frac{\partial t}{\partial \bar{t}} \\ &= \frac{1}{\lambda_j} A_j (\kappa N)(s,t) \\ &= (\kappa_j N_j)(\bar{s}, \bar{t}).\end{aligned}$$

Using the same technique as Altschuler, we obtain

Theorem 8.6 ([Alt91, Thm. 7.3])**.** *Let $\gamma : S^1 \times [0,\omega) \to \mathbb{R}^n$ be a solution of* (CSF)*. Assume that $\{(p_j, t_j)\}$ is an essential blow-up sequence. Then there exists a subsequence of $\{(p_j, t_j)\}$ along which the rescaled solutions γ_j converge to a smooth nontrivial limit γ_∞ which exists at least on the time interval $[-\infty, 0]$.*

Proof. Set $\lambda_j := \kappa(p_j, t_j)$, so that $\kappa_j^2(p_j, 0) = 1$ (note that $\bar{t} = 0 \Leftrightarrow t = t_j$).

Since $\{(p_j, t_j)\}$ is a blow-up sequence, we have that $\lim_{j\to\infty} \alpha_j = -\infty$. If a type-I singularity occurs (that is, $\lim_{t\to\omega} K_t(\omega - t) < \infty$), then $\lim_{j\to\infty} \omega_j < \infty$, for a type-II singularity ($\lim_{t\to\omega} K_t(\omega - t) = \infty$), we can choose an essential blow-up sequence such that $\lim_{j\to\infty} \omega_j = \infty$, since from the definition of λ_j we have that $\lambda_j^2 \le K_{t_j}$.

As a limit solution might be a family of noncompact curves, we consider the solutions γ_j instead as a family of curves $\tilde{\gamma}_j : \mathbb{R} \times [\alpha_j, \omega_j) \to \mathbb{R}^n$, periodic in space, such that $\tilde{\gamma}_j(0, \cdot) = \gamma_j(p_j, \cdot)$. Denote the arclength parameter for $\tilde{\gamma}_j(\cdot, \bar{t})$ from the origin $0 \in \mathbb{R}$ by $\bar{s}$

and recall that, in general, $\bar{s}$ depends on $\bar{t}$ in the sense that $\frac{\partial \bar{s}}{\partial \bar{t}} \neq 0$.

Define the differential operator

$$\frac{\delta}{\delta \bar{t}} = \frac{\partial}{\partial \bar{t}} + \phi_j(\bar{s})\frac{\partial}{\partial \bar{s}},$$

where $\phi_j(\bar{s}) = \int_0^{\bar{s}} \kappa_j^2(\sigma, \bar{t})\,\mathrm{d}\sigma$ and thus $\frac{\partial \phi_j}{\partial \bar{s}} = \kappa_j^2$. Then,

$$\begin{aligned}\left[\frac{\delta}{\delta \bar{t}}, \frac{\partial}{\partial \bar{s}}\right] &= \frac{\partial}{\partial \bar{t}}\frac{\partial}{\partial \bar{s}} + \phi_j \frac{\partial}{\partial \bar{s}}\frac{\partial}{\partial \bar{s}} - \frac{\partial}{\partial \bar{s}}\frac{\partial}{\partial \bar{t}} - \frac{\partial \phi_j}{\partial \bar{s}}\frac{\partial}{\partial \bar{s}} - \phi_j \frac{\partial}{\partial \bar{s}}\frac{\partial}{\partial \bar{s}} \\ &= \frac{\partial}{\partial \bar{t}}\frac{\partial}{\partial \bar{s}} - \frac{\partial}{\partial \bar{s}}\frac{\partial}{\partial \bar{t}} - \kappa_j^2 \frac{\partial}{\partial \bar{s}} \\ &= 0.\end{aligned}$$

Therefore, denoting $v = \left|\frac{\partial \gamma}{\partial p}\right|$ and $\bar{s} = \int_{p_0}^{p} v\,\mathrm{d}q$,

$$\begin{aligned}\frac{\delta \bar{s}}{\delta \bar{t}} &= \frac{\partial \bar{s}}{\partial \bar{t}} + \phi_j \\ &= -\int_{p_0}^{p} \kappa_j^2 v\,\mathrm{d}q + \int_0^{\bar{s}} \kappa_j^2\,\mathrm{d}\sigma \\ &= 0,\end{aligned}$$

where we have used that $\frac{\partial v}{\partial \bar{t}} = -\kappa_j^2 v$ and $\mathrm{d}\bar{s} = v\,\mathrm{d}p$.

Since $\{(p_j, t_j)\}$ is an essential blow-up sequence, there exists $\rho > 0$ independent of j such that $\rho K_t \leq \kappa^2(p_j, t_j)$ whenever $t \leq t_j$. In particular, for the curves $\tilde{\gamma}_j$, $\rho \sup \kappa_j^2(\cdot, \bar{t}) \leq \kappa_j^2(p_j, 0) = 1$ for $\bar{t} \leq 0$.

Then differentiating (8.2) with respect to $\bar{s}$ we get

$$\frac{\partial T_j}{\partial \bar{s}} = A_j \frac{\partial T}{\partial s}\frac{\partial s}{\partial \bar{s}} = \frac{1}{\lambda_j} A_j \frac{\partial T}{\partial s}.$$

Hence, by Theorem 8.1, we have that

$$\left|\frac{\partial T_j}{\partial \bar{s}}\right|^2 = \frac{1}{\lambda_j^2}\left|\frac{\partial T}{\partial s}\right|^2 \leq \frac{\tilde{c}_1 K_{t_j}}{\lambda_j^2} \leq c_1,$$

since $\{(p_j, t_j)\}$ is an essential blow-up sequence and we have $\frac{K_{t_j}}{\lambda_j^2} \leq \frac{1}{\rho} < \infty$.

Taking further derivatives with respect to $\bar{s}$, we obtain

$$\frac{\partial^\ell T_j}{\partial \bar{s}^\ell} = \frac{1}{\lambda_j^\ell} A_j \frac{\partial^\ell T}{\partial s^\ell},$$

whereby

$$\left|\frac{\partial^\ell T_j}{\partial \bar{s}^\ell}\right|^2 = \frac{1}{\lambda_j^{2\ell}} \left|\frac{\partial^\ell T}{\partial s^\ell}\right|^2 \leq c_\ell.$$

Using the commutation formula for $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \bar{s}}$, we get

$$\left|\frac{\partial T_j}{\partial \bar{t}}\right|^2 \leq \left|\frac{\partial^2 T_j}{\partial \bar{s}^2}\right|^2 + \left|\frac{\partial T_j}{\partial \bar{s}}\right|^4 \leq c_2 + c_1^2.$$

Then, taking derivatives with respect to $\bar{t}$ and using the commutation formula, we see that $\left|\frac{\partial^\ell T_j}{\partial \bar{t}^\ell}\right|^2$ is bounded by a sum of products of $\left|\frac{\partial^k T_j}{\partial \bar{t}^k}\right|^2$ for $k < \ell$ and $\left|\frac{\partial^m T_j}{\partial \bar{s}^m}\right|^2$ for $m \leq 2\ell$ and is thus itself bounded.

By the definition of $\frac{\delta}{\delta \bar{t}}$, therefore, the fact that $\frac{\delta}{\delta \bar{t}}$ and $\frac{\partial}{\partial \bar{s}}$ commute, and the estimate $\phi_j(\bar{s}) \leq \rho^{-1}\bar{s}$ we conclude that $\left|\frac{\delta^\ell T_j}{\delta \bar{t}^\ell}\right|^2$ is bounded for any ℓ independently of j on compact subsets of $\mathbb{R} \times [-\infty, \omega_\infty)$, and, again, since $\frac{\delta}{\delta \bar{t}}$ and $\frac{\partial}{\partial \bar{s}}$ commute, the same is true for all mixed derivatives $\left|\frac{\delta^j \partial^k T_j}{\delta^j \bar{t}\, \partial^k \bar{s}}\right|^2$.

By the Arzelà–Ascoli theorem, there exists a subsequence of $\{(p_j, t_j)\}$, denoted the same, along which the tangent vectors $T_j(\bar{s}, \bar{t})$ converge uniformly on compact sets of $\mathbb{R} \times [-\infty, \tau_\infty)$ to a smooth limit $T_\infty(\bar{s}, \bar{t})$ as $j \to \infty$. We may thus define a smooth limit solution $\tilde{\gamma}_\infty$ by integrating T_∞. If $\tilde{\gamma}_\infty$ is periodic, we denote by γ_∞ one period of $\tilde{\gamma}_\infty$, if not, we set $\gamma_\infty = \tilde{\gamma}_\infty$.

Finally, γ_∞ cannot be trivial, i. e., a straight line, for

$$\kappa_\infty^2(0,0) = \lim_{j\to\infty} \tilde{\kappa}_j^2(0,0) = \lim_{j\to\infty} \kappa_j^2(p_j, 0) = 1,$$

finishing the proof. □

The fact that blow-up limits of evolving space curves are planar goes back to Altschuler's work. Using Theorem 8.4 and Huisken's monotonicity formula, we can give a simpler proof also in the general case.

Theorem 8.7 ([Alt91, Thm. 7.7])**.** *Let $\gamma : S^1 \times [0,T) \to \mathbb{R}^n$ be a solution of (CSF). Then any nontrivial blow-up limit of γ is planar and convex.*

Proof. By Huisken's monotonicity formula, any blow-up limit γ_∞ of curve shortening flow is self-similar. Moreover, the total absolute curvature is scaling-invariant. Therefore, Theorem 8.4 implies that

$$0 \leq -\int_{\gamma_\infty} |\kappa|\, \tau_1^2 \,\mathrm{d}s\,.$$

This implies that, at almost every point of the smooth limit curve γ_∞, we must have either $\kappa = 0$ or $\tau_1 = 0$. Then Theorem 5.2 implies that γ_∞ must be contained in a 2-dimensional subspace of $\mathbb{R}^n$. Note that γ_∞ cannot have any inflection points, since by Theorem 8.5, any inflection point must be degenerate, that is, $\kappa = \frac{\partial \kappa}{\partial s} = 0$, but a result of Angenent [Ang91] implies that any solution with degenerate inflection points must be a line. □

8.2.1 Type-I singularities

In order to analyse the behaviour of type-I singularities, we can employ Huisken's argument for singularities of mean curvature flow [Hui90], which was also used by Altschuler to prove the corresponding result for curve shortening flow of space curves.

To that end, let $\gamma : S^1 \times [0,T) \to \mathbb{R}^n$ be a solution of (CSF) and assume that $(0,T) \in \mathbb{R}^n \times \mathbb{R}$ is a special singular point of type-I reached by the flow. We then define a continuous rescaling of the flow via

$$\tilde{\gamma}(s,\tilde{t}) = \frac{1}{\sqrt{2(T-t)}}\gamma(s,t),$$

where $\tilde{t} = -\frac{1}{2}\log(T-t)$. The rescaled flow $\{\tilde{\gamma}_t\}$ is thus defined for $-\frac{1}{2}\log T \leq \tilde{t} < \infty$, and in terms of the differential operators

$$\frac{\partial}{\partial \tilde{t}} = 2(T-t)\frac{\partial}{\partial t},$$
$$\frac{\partial}{\partial \tilde{s}} = \sqrt{2(T-t)}\frac{\partial}{\partial s},$$

it satisfies

$$\frac{\partial}{\partial \tilde{t}}\tilde{\gamma} = \frac{\partial^2}{\partial \tilde{s}^2}\tilde{\gamma} + \tilde{\gamma}.$$

The reason to choose this particular rescaling is that the type-I assumption then implies that the curvature of the rescaled flow is uniformly bounded for all time, since

$$\tilde{\kappa}(s,\tilde{t}) = \sqrt{2(T-t)}\kappa(s,t).$$

In the rescaled setting, we then have the monotonicity formula, cf. Theorem 6.18,

$$\frac{d}{d\tilde{t}}\int_{\tilde{\gamma}} \tilde{k}\,\mathrm{d}\tilde{s} = -\int_{\tilde{\gamma}} \left|\frac{\partial \tilde{\gamma}}{\partial \tilde{s}} + \tilde{\gamma}^{\perp}\right|^2 \tilde{k}\,\mathrm{d}\tilde{s}, \tag{8.3}$$

where the rescaled backwards heat kernel on $\mathbb{R}^n$ is given by

$$\tilde{k}(x,\tilde{t}) = \mathrm{e}^{-|x|^2}.$$

We can then perform the blow-up procedure as in the proof of Theorem 8.6, noting that the type-I assumption also implies that any blow-up sequence is necessarily essential, to obtain a subsequential limit. By the rescaled monotonicity formula (8.3), we conclude that the limit is self-similarly shrinking [Hui90], and by Theorem 8.7 the limit is planar. Moreover, by continuity of the total absolute curvature, the winding number cannot change. We thus have

Theorem 8.8 ([Alt91, Thm. 8.15]). *Suppose that a type-I singularity is forming at time T. Let $\{(p_j, t_j)\}$ be a blow-up sequence. Then there exists a subsequence of $\{(p_j, t_j)\}$ such that a rescaling of γ along it converges to a planar self-similarly shrinking solution γ_∞ with the same winding number.*

8.2.2 Type-II singularities

We now assume that the special singular point $(0, T) \in \mathbb{R}^n \times \mathbb{R}$ reached by the flow is of type-II. Since the argument is exactly the same as for space curves, we do not repeat the details and instead refer to Altschuler's work [Alt91].

We already know that by Theorem 8.6, a limit of rescalings γ_∞ must exist on the interval $[-\infty, 0]$. Moreover, it is planar and convex. It is then possible to show that, since the singularity is of type-II, there exists an essential blow-up sequence such that a limit of rescalings along it is in fact eternal, that is, it exists on the time interval $[-\infty, \infty]$. In addition, the limit solution is embedded and its total curvature is equal to π. Furthermore, by showing that the curvature and all its derivatives tend to zero at the ends, one then proves that this limit must be the Grim Reaper. Finally, one has

Theorem 8.9 ([Alt91, Thm. 8.16]). *Suppose that a type-II singularity is forming at time T. Then there exists an essential blow-up sequence $\{(p_j, t_j)\}$ such that a sequence of rescalings along it converges to the Grim Reaper.*

8.3 Convergence analysis

We now come to the proof of our main theorem. Simply put, we show that initial curves with entropy less than that of the Grim Reaper converge to a round point in finite time.

Theorem 8.10. *Suppose that $\gamma : S^1 \times [0, T) \to \mathbb{R}^n$ is a smooth solution of (CSF) with initial data $\gamma(\cdot, 0) = \gamma_0$ and assume that the entropy of γ_0 satisfies*

$$\lambda(\gamma_0) \leq 2.$$

Then T is finite, and the rescaled flow converges to the round circle.

Proof. Since the entropy is non-increasing under curve shortening flow, we may assume that $\lambda(\gamma) < 2$. For if $\lambda(\gamma_0)$ was equal to 2, the initial curve γ_0 would have to be a self-shrinker, but $\lambda(S^1) < 2$.

Assume that a type-II singularity forms at time T. Then by Theorem 8.9, there exists an essential blow-up sequence such that a limit γ_∞ of rescalings along it is the Grim Reaper. Since the entropy is lower semicontinuous with regard to the locally smooth convergence, we must have $\lambda(\gamma_\infty) < 2$, however, we know from Proposition 6.22 that the entropy of the Grim Reaper equals 2. Thus the singularity cannot be of type-II.

Therefore, assume that a type-I singularity forms, so that by Theorem 8.8, we have a limit γ_∞ of a sequence of rescalings $\{\gamma_j\}$ that is self-similarly shrinking in the plane. By the classification of Abresch–Langer [AL86], γ_∞ could be one or more lines through the origin, a singly or multiply-covered circle, or one of the Abresch–Langer curves $\gamma_{m,n}$, $m \geq 2$. Proposition 6.23 implies that $\lambda(\gamma_{m,n}) \geq m\sqrt{\frac{2\pi}{\mathrm{e}}}$, so that the latter possibility cannot occur. Moreover, from Proposition 6.24 we have that $\lambda(\gamma_\infty) \geq \Theta^2(\gamma_\infty, x)$ for any point x, which implies that γ_∞ is embedded. Hence γ_∞ cannot be a multiply-covered circle or a family of intersecting lines.

By standard theory [Bra78; Whi05], should a single line appear as a limit of rescalings of the flow, the fact that its Gaussian density is 1 in a suitable space-time region implies that the curvature is bounded there after all, so that the blow-up point is not a singularity, which is a contradiction. Therefore, the blow-up limit γ_∞ must be the standard circle around the origin, that is, the tangent flow at the singularity is a smooth, closed, embedded self-shrinker. Then Schulze's uniqueness result for compact tangent flows [Sch14] implies that this is the only possible tangent flow, whereby the rescaled flow converges to the round circle. □

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