K-STABILITY OF COMPLEXITY ONE G-VARIETIES

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Jack A. Rogers
School of Natural Sciences
Department of Mathematics

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Jack A. Rogers
Doctor of Philosophy
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This thesis concerns the problem of existence of Kähler-Einstein metrics on Fano manifolds equipped with actions by reductive algebraic groups. We describe the basic ideas of Kähler geometry and the significance of Kähler-Einstein metrics before explaining how the solution of the Yau-Tian-Donaldson conjecture by Chen-Donaldson-Sun allows these metrics to be studied via the algebro-geometric concept of K-stability. The *equivariant* K-stability of Datar-Székelyhidi, which allows concrete criteria for K-stability to be found for varieties with group actions, is then described. After discussing various aspects of the theory of algebraic group actions on varieties, and the combinatorial description of varieties of complexity one due to Timashev in particular, we apply this theory to the specific case of smooth Fano threefolds admitting SL_2-actions. We give a detailed combinatorial description of these varieties, which we then use to prove the K-stability of several examples, via the β-invariant of Fujita and Li.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

1.1 Overview

This thesis outlines research arising from a long-standing open problem in complex geometry which, due to the intimate connection between the two fields, has recently been reformulated in purely algebro-geometric terms. There is a particularly interesting class of manifolds, Kähler manifolds[1], which have mutually compatible complex, Riemannian and symplectic structures. One definition is that a complex manifold X is *Kähler* if it admits a Hermitian metric h such that the 2-form $\omega\colon (u,v) \mapsto \operatorname{Re} h(iu,v)$, called the *Kähler form* or *Kähler metric*, is closed. In particular, this class includes all complex projective varieties.

A given compact complex manifold X may admit many different Kähler metrics, and we want to find those which are well-behaved or canonical in some way. Of particular interest are the following: a Kähler metric ω on X is *Kähler-Einstein*[2] if the Ricci form on X is proportional to ω, i.e. if $\operatorname{Ric}\omega = \lambda\omega$ for some $\lambda \in \mathbb{R}$. One reason that these metrics are important is that the Einstein condition implies that X has constant scalar curvature $\lambda \dim X$.

The natural question to ask is: given a compact Kähler manifold (X, ω), does there exist a Kähler-Einstein metric ω' in the cohomology class $[\omega]$, and if so, is such a metric unique?

[1] It is regrettable that such fundamental objects are named after a man who remained an apologist for national socialism throughout his life (see [Seg14]), but the terminology seems too firmly established to be avoided.

[2] Named so because such a metric gives a vacuum solution to the Einstein field equations of general relativity.

As a first step in answering the question, note that the de Rham cohomology class of $\operatorname{Ric}\omega$ defines a characteristic class

$$c_1(X) = \frac{1}{2\pi}[\operatorname{Ric}\omega] \in H^2_{dR}(X, \mathbb{R}),$$

the *first Chern class* of X. This is an integral cohomology class which turns out to be independent of ω [Szé14, Lemma 1.22]. Now if the Einstein condition holds, then $c_1(X)$ must be zero or a *definite class*, in the sense that the bilinear form corresponding to $\operatorname{Ric}\omega$ must be positive or negative definite (being proportional to ω, which is positive definite). So we can split the question of the existence of Kähler-Einstein metrics into cases in which the first Chern class is negative, zero or positive.

Aubin proved that when $c_1(X) < 0$, there exists a unique Kähler-Einstein metric in each Kähler class $[\omega]$ [Aub76], and Yau independently proved the same result for $c_1(X) \leq 0$ as a consequence of his proof of the Calabi conjecture [Yau78]. These cases are thus completely solved.

When $c_1(X) > 0$, the question turns out to be far more difficult and interesting. Matsushima proved [Mat57] that in this case, for X to admit a KE metric, it is necessary that the Lie algebra $\eta(X)$ of holomorphic vector fields on X is reductive. But there are certain Kähler manifolds with positive first Chern class, such as the projective plane blown up at one or two points, for which $\eta(X)$ is not reductive, and hence these manifolds do not admit a KE metric [Tia97]. The question of exactly which manifolds do admit such a metric is still open. In 2012, Chen, Donaldson and Sun proved [CDS15] that when $c_1(X) > 0$, the existence of a Kähler-Einstein metric on X is equivalent to another condition called K-stability, which is defined purely in terms of algebraic geometry.

Unfortunately, K-stability is in general a difficult condition to check, for reasons which will be clarified in the following chapter, where the notion is defined properly. Because of this, certain specialised variants of K-stability have been developed. Of most interest for this thesis is the case in which X comes equipped with the action of a reductive algebraic group G, where, due to results of Datar and Székelyhidi [DS15], an equivariant version of K-stability makes the situation far more amenable to analysis.

The *complexity* of the G-action on X is the minimal codimension of the orbits of a Borel subgroup B of G. Thus if $G = B = T$ is an algebraic torus, the complexity zero T-varieties are exactly the toric varieties. A simple criterion for the existence of Kähler-Einstein metrics in this case was proved by Wang and Zhu [WZ04]. Conditions for K-stability of complexity one T-varieties have been found by Ilten and Süß [IS17], and the complexity zero case for general G (the *spherical varieties*) has been solved by Delcroix [Del16]. In this thesis similar techniques are used to find conditions for equivariant K-stability of complexity one G-varieties, for general reductive G.

1.2 Structure of the Thesis

Here we summarise the content of the remaining chapters of this thesis.

Chapter 2 - Background

In this chapter we collect some background theory which will be required later. We begin with the basics of Kähler geometry, then describe the translation of the problem of Kähler-Einstein metrics to algebraic geometry, which brings us to the notion of K-stability. Next we discuss the representation theory of reductive algebraic groups and the structure of varieties and line bundles on which they act. Following that, we describe how these actions and representations can be exploited using geometric valuations to obtain combinatorial descriptions of varieties.

Chapter 3 - Combinatorial Description of Smooth Fano SL_2-Threefolds

Here we explain the combinatorial classification due to Timashev of complexity one G-varieties in terms of coloured hyperfans, and describe all homogeneous complexity one SL_2-spaces. Finally, the author's work in applying this classification to smooth Fano SL_2-threefolds is presented. Specifically, we classify the smooth Fano SL_2-threefolds with reductive automorphism group and which do not admit a faithful action of a 2- or 3-torus.

Chapter 4 - β-Invariant and K-Stability

This chapter begins with some results, also due to Timashev, describing the combinatorial properties of divisors on complexity one varieties, then discusses the β-invariant of Fujita-Li, which they have shown to be an indicator of K-stability. We finally apply the preceding results to prove the K-stability of all smooth Fano SL_2-threefolds classified in Chapter 3.

Chapter 2

Background

2.1 Kähler Geometry

2.1.1 Basic Definitions

Here we give a brief overview of Kähler geometry. The material and proofs mostly follow [Huy05, Szé14]. We first discuss almost-complex structures on real manifolds and how their behaviour with respect to complexified differential forms can induce a complex structure.

Definition 2.1. Let X be a real manifold of even dimension. An *almost-complex structure* on X is an endomorphism $J\colon TX \to TX$ of the tangent bundle such that $J^2 = -Id$.

Example. Let X be a complex manifold and identify T_pX with $\mathbb{C}^n$ for each $p \in X$. Then multiplication by $\sqrt{-1}$ is an almost-complex structure on X.

Let (X, J) be an almost-complex manifold. The endomorphism J on TX extends to a complex-linear endomorphism of the complexified tangent bundle $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$ and induces a direct sum decomposition $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ into $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces. In turn this induces a decomposition of the complexified cotangent bundle $T^*_{\mathbb{C}}X = T^*_{1,0}X \oplus T^*_{0,1}X$. Finally, this decomposition extends to higher-degree forms, giving

$$\Omega^k_{\mathbb{C}}X = \bigoplus_{p+q=k} \Omega^{p,q}X.$$

We call an element of $\Omega^{p,q}X$ a form of *type* (p, q), or simply a (p, q)-form.

Let $d\colon \Omega^k_{\mathbb{C}}X \to \Omega^{k+1}_{\mathbb{C}}X$ be the complex-linear extension of the exterior derivative on X. If $p+q=k$, then composing d with the projections from $\Omega^{k+1}_{\mathbb{C}}X$ to forms of type $(p+1,q)$ and $(p,q+1)$ induces maps $\partial\colon \Omega^{p,q}X \to \Omega^{p+1,q}X$ and $\overline{\partial}\colon \Omega^{p,q}X \to \Omega^{p,q+1}X$.

Definition 2.2. An almost-complex structure J on a manifold X is called *integrable* if $d\alpha = \partial\alpha + \overline{\partial}\alpha$ for all forms α on X.

For an integrable almost-complex structure, we have $\overline{\partial}^2 = \partial^2 = 0$ and $\overline{\partial}\partial = -\partial\overline{\partial}$.

Newlander-Nirenberg theorem. *[NN57] The almost-complex structure induced by the holomorphic atlas on a complex manifold X is integrable. Any integrable almost-complex structure on a real manifold X of even dimension is induced by a holomorphic atlas on X.*

We will call an integrable almost-complex structure a *complex structure*, and the theorem says that a complex structure is essentially the same as a holomorphic atlas.

Now we discuss Hermitian metrics on complex manifolds, which are the complex version of Riemannian metrics. Recall that a *Riemannian metric* on a real manifold X is a smooth section of the bundle $T^*X \otimes T^*X$ inducing an inner product on each tangent space, or equivalently a smooth map $: TX \times_X TX \to \mathbb{R}$ where the restriction to each fibre is an inner product.

Definition 2.3. Let (X, J) be a complex manifold with its complex structure. A Riemannian metric g on the underlying real manifold of X is called *Hermitian* if $g(U,V) = g(JU, JV)$ for all vector fields U, V on X.

Given a Hermitian metric g on a complex manifold (X, J), we define an antisymmetric real (1,1)-form ω on X, called the *fundamental form* of g, by $\omega(U,V) = g(JU,V)$.

Definition 2.4. A Hermitian metric g on X is *Kähler* if the fundamental form ω of g is closed, i.e. $d\omega = 0$. A complex manifold admitting a Kähler metric is called a *Kähler manifold.*

An almost universal abuse of terminology in the field of Kähler geometry, which we heartily adopt, is to ignore g as often as possible and refer to ω as the Kähler metric. This is harmless as g is recovered from ω by $g(U,V) = \omega(U, JV)$.

Note that ω is a closed, non-degenerate 2-form on X, so induces a symplectic structure on X along with the Riemannian and complex structures, and all three of these structures are mutually compatible.

Example. [Szé14, Ex 1.12] Let $X = \mathbb{P}^n$ be complex projective space. Let π be the projection map $\mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ and let s be some holomorphic section of π over an open subset $U \subseteq \mathbb{P}^n$. Define ω_{FS}, the *Fubini-Study metric* on $\mathbb{P}^n$, by $\omega_{FS} = \sqrt{-1}\partial\overline{\partial}\log\|s\|^2$. This is well-defined: first, such a section always exists for a given open subset U, and second, given another holomorphic section s' on an open subset V, we have $s' = fs$ on $U \cap V$ for some holomorphic function $f\colon U \cap V \to \mathbb{C}^\times$, and

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\log\|fs\|^2 &= \sqrt{-1}\partial\overline{\partial}\log\|s\|^2 + \sqrt{-1}\partial\overline{\partial}\log f + \sqrt{-1}\partial\overline{\partial}\log\bar{f} \\ &= \sqrt{-1}\partial\overline{\partial}\log\|s\|^2\end{aligned}$$

since holomorphicity of $\log f$ gives $\overline{\partial}\log f = \partial\log\bar{f} = 0$.

Thus ω_{FS} is a well-defined, closed (1,1)-form and it remains to check positive-definiteness. To do so, note that the unitary group U_{n+1} acts transitively on $\mathbb{P}^n$ and since $\|As\|^2 = \|s\|^2$ for any $A \in U_{n+1}$, this action preserves ω_{FS}. We can therefore check positive-definiteness at a single point. Let $P = [1 : 0 : \ldots : 0] \in \mathbb{P}^n$ and introduce local holomorphic co-ordinates z^i on the chart $U_0 \ni P$ defined by non-vanishing of the first co-ordinate. We can choose a section $s\colon (z^1, \ldots, z^n) \mapsto (1, z^1, \ldots, z^n)$ of π over U_0, so that $\omega_{FS} = \sqrt{-1}\partial\overline{\partial}\log(1 + z^1\bar{z}^1 + \ldots z^n\bar{z}^n)$. At P this gives $\omega_{FS}(P) = \sqrt{-1}\sum_i \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i$, which is positive definite. Hence $\mathbb{P}^n$ is Kähler.

Example. Let $X \subseteq \mathbb{P}^n$ be a closed submanifold of complex projective space. Let $i\colon X \to \mathbb{P}^n$ be the inclusion, so that $\omega_{FS}|_X = i^*(\omega_{FS})$ is a positive-definite real $(1,1)$-form on X. Then $\omega_{FS}|_X$ is closed, since d commutes with pullbacks, so gives a Kähler metric on X. Hence all complex projective manifolds are Kähler.

2.1.2 Ricci Form and Einstein Condition

We can now approach the *Einstein condition.* To do so we must discuss the curvature of Kähler manifolds and vector bundles over them. Recall that in Riemannian geometry, we define connections on the manifold, which essentially allow us to move tangent vectors from point to point, and curvature measures the failure of a tangent vector moving along a closed path to return to its original starting point. We can do the same on an arbitrary complex vector bundle.

Let $E \to X$ be a complex vector bundle. We denote the sheaf of differentiable E-valued k-forms on X by $\Omega^k(E)$. Hence $\Omega^0(E)$ in particular is the space of holomorphic sections of E.

Definition 2.5. Let $E \to X$ be a complex vector bundle over a complex manifold (X, J). A *connection* on E is a $\mathbb{C}$-linear map $\nabla\colon \Omega^0(E) \to \Omega^1(E)$ satisfying the Leibniz rule $\nabla(fs) = f\nabla s + \mathrm{d}f \otimes s$ for all holomorphic functions f and sections s of E.

We can also introduce metrics on vector bundles:

Definition 2.6. A *Hermitian structure* on $E \to X$ is a smooth section h of $E^* \otimes \overline{E}^*$ inducing a Hermitian inner product on each fibre.

A Riemannian manifold has a canonical connection (the *Levi-Civita connection*) on its tangent bundle which is compatible with the metric, and this situation is replicated in the complex case.

Theorem 2.1. *Let (E, h) be a holomorphic Hermitian vector bundle over a complex manifold. There exists a unique connection on E, called the* Chern connection*, such that:*

(i) *for any section s of E, the $(0, 1)$-component of the E-valued 1-form ∇s is $\overline{\partial} s$;*

(ii) *for any sections s, t of E we have*

$$\mathrm{d}(h(s, t)) = h(\nabla s, t) + h(s, \nabla t).$$

Another interesting characterisation of Kähler manifolds is that they are exactly the Hermitian manifolds on which the Chern connection on the holomorphic tangent bundle coincides with the complexification of the Levi-Civita connection on the underlying real manifold [Szé14, Ex 1.33].

To study the curvature of a connection, we extend an arbitrary connection ∇ to higher-degree forms as follows: if $\alpha \in \Omega^k(E)$ is an E-valued k-form, and s is a smooth section of E, set $\nabla(\alpha \otimes s) = \mathrm{d}\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$.

Definition 2.7. Let ∇ be a connection on a holomorphic vector bundle $E \to X$. The *curvature* of ∇ is the map $F_\nabla = \nabla \circ \nabla \colon \Omega^0(E) \to \Omega^2(E)$. It is an $\operatorname{End} E$-valued 2-form.

We can now introduce one of the most important invariants of a Kähler manifold, the Ricci form:

Definition 2.8. Let (X, g, ω) be a Kähler manifold. Then g induces a Hermitian metric on the *anticanonical line bundle* K_X^{-1}, which is the top exterior power of the tangent bundle TX. Let F_∇ be the curvature of the Chern connection on K_X^{-1}. The form given by $(U, V) \mapsto F_\nabla(JU, V)$ is called the *Ricci form* of X and denoted $\operatorname{Ric}\omega$.

Proposition 2.1. *The Ricci form* $\operatorname{Ric}\omega$ *on a Kähler manifold* (X, g, ω) *is a real closed* $(1,1)$*-form given by* $\operatorname{Ric}\omega = -\sqrt{-1}\partial\overline{\partial}\log(\det g)$*, and hence the cohomology class* $[\operatorname{Ric}\omega] \in H^2(X, \mathbb{R})$ *is independent of the choice of* ω*.*

Proof. [Szé14, Lemma 1.22] We can split ∇ into $\nabla^{1,0} + \nabla^{0,1}$, and since ∇ is a Chern connection, we have $\nabla^{0,1} = \overline{\partial}$. Now if s is any holomorphic section, we have $\overline{\partial}s = 0$, so $F_\nabla(s) = \nabla(\nabla^{1,0}s) = \nabla^{1,0}(\nabla^{1,0}s) + \overline{\partial}(\nabla^{1,0}s)$.

Consider the norm $h(s, s)$ of a holomorphic section s of K_X^{-1} under its Hermitian metric. By property (ii) of Chern connections, and by checking degrees, we have

$$\overline{\partial}h(s, s) = h(\nabla^{1,0}s, s) + h(s, \nabla^{0,1}s) = h(\nabla^{1,0}s, s) = h(s, s)\nabla^{1,0}(s).$$

So $\nabla^{1,0} = h^{-1}\overline{\partial}h$. Now since K_X^{-1} is the top exterior power of the tangent bundle, h is the determinant of the metric g on X, which is a real function. Taking conjugates, we then get $\nabla^{1,0} = (\det g)^{-1}\partial(\det g) = \partial \log\det g$. Then

$$F_\nabla = \nabla^{1,0}(\nabla^{1,0}) + \overline{\partial}(\nabla^{1,0}) = \partial^2 \log\det g + \overline{\partial}\partial\log\det g = \overline{\partial}\partial\log\det g.$$

Switching ∂ and $\overline{\partial}$, we have

$$\operatorname{Ric}\omega = \sqrt{-1}F_\nabla = -\sqrt{-1}\partial\overline{\partial}\log\det g$$

as promised.

This formula makes it clear that $\operatorname{Ric}\omega$ is a real closed $(1,1)$-form. Let g' be another choice of Hermitian metric with corresponding fundamental form ω'. Then

$$\operatorname{Ric}\omega' - \operatorname{Ric}\omega = -\sqrt{-1}\partial\overline{\partial}\log\left(\frac{\det g'}{\det g}\right)$$

is an exact form, since $\frac{\det g'}{\det g}$ is a globally defined function. Thus $[\operatorname{Ric}\omega]$ is independent of ω. $\square$

Remark. Note that since the Chern connection ∇ on the tangent bundle of X is the complexification of the Levi-Civita connection on the underlying real manifold, the curvature of ∇ is the Riemann curvature of the real manifold, and so the curvature of the Chern connection on $K_X^{-1} = \det TX$ does indeed correspond to the Ricci curvature of the underlying real manifold by the identity $\log\det g = \operatorname{tr}\log g$, since the Ricci curvature is the trace of Riemann curvature.

Definition 2.9. The *first Chern class* of a Kähler manifold (X, ω) is the real cohomology class

$$c_1(X) = \frac{1}{2\pi}[\operatorname{Ric}\omega] \in H^2(X, \mathbb{R}).$$

Remark. Chern classes are most generally and properly defined on general vector bundles. When one speaks of the first Chern class of a *manifold*, this means the first Chern class of the tangent bundle. The Ricci form represents the first Chern class because it is the curvature form of the top exterior power of the tangent bundle, i.e. K_X^{-1}, and the first Chern class is insensitive to taking top exterior powers (see e.g. [GH78, §3.3]).

The factor of 2π actually guarantees that $c_1(X)$ is integral, i.e. lies in $H^2(X, \mathbb{Z})$, which is interesting but makes little difference to us. What *is* important to us is stating the Einstein condition:

Definition 2.10. A Kähler metric ω is *Kähler-Einstein* if there exists a real constant λ with

$$\operatorname{Ric}\omega = \lambda\omega.$$

A Kähler manifold (X, ω) is Kähler-Einstein if there exists a Kähler-Einstein metric $\omega' \in [\omega]$.

Two immediate things to note about this are the following: first, the scalar curvature of X is the trace of the Ricci form, and the trace of the metric is the dimension, so Kähler-Einstein manifolds have constant scalar curvature $\lambda \dim X$. Second, the Einstein condition also means that the first Chern class of a Kähler-Einstein manifold must be a definite class, since $\operatorname{Ric}\omega$ is proportional to the metric, which is positive definite. Hence the question of whether a given Kähler manifold admits a Kähler-Einstein metric can be split into cases where $c_1(X)$ is assumed to be positive, negative or zero. The latter two cases are completely solved:

Aubin-Yau theorem. *[Aub76, Yau78] Let* (X, ω) *be a compact Kähler manifold with* $c_1(X) \leq 0$*. Then there exists a unique Kähler-Einstein metric* $\omega' \in [\omega]$*.*

Hence we are interested in the case $c_1(X) > 0$, which is considerably more difficult. In this case such metrics do not always exist:

Matsushima's obstruction. *[Mat57] Let* X *be a compact Kähler manifold with* $c_1(X) > 0$*. If* X *admits a Kähler-Einstein metric, then the holomorphic automorphism group* $\operatorname{Aut} X$ *is reductive.*

Example. Let X be the projective plane blown up at a point. Then X is projective, which we have shown means that X is a compact Kähler manifold. The first Chern class of X is the divisor class of the anticanonical line bundle on X. The latter is $3H - E$ where H is a hyperplane class and E is the exceptional divisor of the blow-up. This is an ample class and hence by the Kodaira embedding theorem it is positive, i.e. $c_1(X) > 0$.

Since $\operatorname{Aut} \mathbb{P}^2 = \mathrm{PGL}_3$, the automorphism group of X is the subgroup of PGL_3 consisting of matrices fixing a point, i.e. of the form (assuming $p = [0 : 0 : 1]$):

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix}.$$

The map to GL_2 given by sending each element to its top-left 2×2 square realises this group as $\mathbb{C}^2 \rtimes \mathrm{GL}_2(\mathbb{C})$, which has nontrivial unipotent radical $\mathbb{C}^2$ and hence is not reductive. Therefore by Matsushima's obstruction, X does not admit a Kähler-Einstein metric. A similar calculation shows the same result for $\mathbb{P}^2$ blown up at two points.

In dimension two, the above are known to be the only counterexamples: specifically, a compact Kähler surface with positive first Chern class is isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{P}^2$ blown up at $0 \leq k \leq 8$ general points, and Tian-Yau [TY87] proved that all such manifolds admit Kähler-Einstein metrics except when $k = 1$ or $k = 2$.

In the face of obstructions such as Matsushima's, conditions guaranteeing the existence of Kähler-Einstein metrics in the $c_1(X) > 0$ case were searched for. In [Tia97], Tian proved that the Matsushima condition is not sufficient for the existence of such metrics, i.e. that there even exist Kähler manifolds with $c_1(X) > 0$ whose automorphism group is reductive which still do not admit KE metrics. In the same paper, Tian conjectured that the existence of KE metrics is equivalent to a certain stability condition on the manifold, which we discuss in the next section.

2.2 K-Stability

2.2.1 Theorems of Kodaira and Chow

The notion of K-stability was introduced as an attempt to translate the problem of existence of Kähler-Einstein metrics into the language of algebraic geometry, through what became known as the *Yau-Tian-Donaldson conjecture*. We thus begin this discussion by addressing which algebraic varieties correspond to the compact Kähler manifolds with positive first Chern class. To begin with, note that any smooth complex projective variety X is automatically a projective complex manifold, so in particular Kähler. Conversely, we have:

Chow's theorem. *[Cho49] Any closed submanifold of complex projective space is a smooth projective variety.*

Now it remains to interpret the condition $c_1(X) > 0$. Consider the following:

Definition 2.11. A line bundle L on a complex manifold is called *positive* if its first Chern class can be represented by a positive-definite form.

So in particular, as we have seen, $c_1(X) > 0$ if and only if the anticanonical bundle K_X^{-1} is positive. Recall that a line bundle L is called *very ample* if it is generated by global sections $s_0, \ldots, s_n \in H^0(X, L)$ and the map $\varphi_L \colon X \to \mathbb{P}^n, p \mapsto [s_0(p) : \ldots : s_n(p)]$ is a closed embedding, and L is called *ample* if some tensor power of L is very ample. The following theorem of Kodaira completes our discussion:

Kodaira embedding theorem. *[Kod54] A holomorphic line bundle $L \to X$ over a complex manifold is positive if and only if it is ample.*

Definition 2.12. A *Fano variety* is a complete, normal variety with ample anticanonical bundle.

Combining the above results, we see that 'compact Kähler manifold with positive first Chern class' and 'smooth complex Fano variety' refer to essentially the same concept. Therefore any algebro-geometric condition equivalent to the existence of a Kähler-Einstein metric should be applied to smooth complex Fano varieties.

2.2.2 K-Stability and Its Equivariant Version

There are various notions of 'stability' in algebraic geometry which usually take the form of a numerical criterion associated to some family of deformations of a scheme or variety, in the following sense: suppose we have schemes X, Y and S, a flat morphism of schemes $f: X \to S$ and a point $s \in S$ such that the fibre X_s is isomorphic to Y. Then we can regard the fibres of f as constituting a family of deformations of Y parameterised by the points of S. The flatness condition means that various numerical invariants such as dimension, genus etc. are constant along the fibres [Har77, §III.9]. For K-stability, the relevant deformations are defined as follows by Donaldson [Don18]:

Definition 2.13. Let (X, L) be a complex polarised variety and let $m > 0$. A *test configuration* for (X, L) of *exponent* m consists of:

- A flat morphism of schemes $\pi: \mathcal{X} \to \mathbb{A}^1_{\mathbb{C}}$;
- A π-relatively ample line bundle $\mathcal{L} \to \mathcal{X}$;
- A $\mathbb{C}^\times$ action on $\mathcal{X}, \mathcal{L}$;

such that π and the bundle map $\mathcal{L} \to \mathcal{X}$ are $\mathbb{C}^\times$-equivariant for the standard action of $\mathbb{C}^\times$ on $\mathbb{A}^1_{\mathbb{C}}$ by multiplication, and for some (and hence all by equivariance) $t \neq 0$ in $\mathbb{A}^1_{\mathbb{C}}$, the pair $(X_t, L_t) := (\pi^{-1}(t), \mathcal{L}|_{\pi^{-1}(t)})$ is isomorphic to $(X, L^{\otimes m})$. We also require that the *central fibre* X_0 is irreducible.

We call $(\mathcal{X}, \mathcal{L})$ a *product configuration* if $\mathcal{X} \cong X \times \mathbb{A}^1$ and a *trivial configuration* if it is a product configuration and the $\mathbb{C}^\times$-action is trivial on X.

Note that since $0 \in \mathbb{A}^1_{\mathbb{C}}$ is fixed by the standard $\mathbb{C}^\times$ action, the morphism π induces a $\mathbb{C}^\times$ action on the central fibre X_0 and the line bundle L_0. Also note that Fano varieties are polarised by their anticanonical bundle, so the notion of test configuration makes sense in this case.

In the general case of a projective scheme Z with an ample line bundle Λ, we can consider the vector spaces $H^k = H^0(Z, \Lambda^{\otimes k})$ of global sections of the tensor powers of Λ. Let $d_k := \dim H^k$. For k large enough that $\Lambda^{\otimes k}$ is very ample, the d_k are known to be given by a Hilbert polynomial of degree $n = \dim Z$. Now suppose there is a $\mathbb{C}^\times$-action on the pair (Z, Λ). This induces a $\mathbb{C}^\times$ action on each H^k. Let w_k be the sum of the weights of this action, or equivalently the weight on the top exterior power. Then for k large enough, w_k is also given by a polynomial, this being of degree $n+1$ [Don18]. Now set $F(k) = w_k/kd_k$, so that there is an expansion for large k given by:

$$F(k) = F_0 + F_1 k^{-1} + F_2 k^{-2} + \dots.$$

Definition 2.14. The *Donaldson-Futaki invariant* of (Z, Λ) is the coefficient F_1 in the above expansion. For a test configuration $(\mathcal{X}, \mathcal{L})$ of a polarised variety (X, L), we define $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ to be the Donaldson-Futaki invariant of the central fibre (X_0, L_0).

Definition 2.15. A polarised variety (X, L) is:

- *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for every test configuration $(\mathcal{X}, \mathcal{L})$ on (X, L);
- *K-polystable* if it is K-semistable and $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ only for product configurations;
- *K-stable* if it is K-semistable and $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ only for trivial configurations;
- K-unstable if it is not K-semistable.

An important result of Li-Xu immediately allows us to restrict the set of test configurations we need to check in order to verify K-(semi/poly)stability:

Definition 2.16. A test configuration $(\mathcal{X}, \mathcal{L})$ for a polarised variety (X, L) is called *special* if the central fibre X_0 is normal.

Li-Xu theorem. *[LX14] For a Fano variety $(X, -K_X)$, K-(poly/semi)stability can be verified by checking the Donaldson-Futaki invariant of only the special test configurations.*

We can now state the theorem which allows us to investigate the Kähler-Einstein metric problem through algebraic-geometric means:

Chen-Donaldson-Sun theorem. *[CDS15] A smooth complex Fano variety X admits a Kähler-Einstein metric if and only if (X, K_X^{-1}) is K-polystable.*

This is the main result leading to the interest in K-(poly/semi)stability of Fano varieties. Unfortunately, since there are generally infinitely many test configurations for a given polarised variety, it is very difficult to check K-stability in the general case, even accounting for the Li-Xu theorem.

The α-invariant of Tian [Tia87] was for a long time one of the only practical methods to check K-stability. Recently, work of Abban-Zhuang [AZ20, AZ21] has provided other more powerful methods of verification. Otherwise, most progress on this front has come from the equivariant perspective due to the work of Datar-Székelyhidi [DS15]. They showed that if (X, L) comes with the action of a reductive algebraic group G, there exists a variant of K-stability for (X, L) which is easier to check and still guarantees the existence of a Kähler-Einstein metric. This has allowed concrete conditions for K-stability, and thus the existence of Kähler-Einstein metrics, to be found in various new contexts.

Definition 2.17. Let X be a G-variety, and let $\pi\colon L \to X$ be a line bundle on X. We say that L is *G-linearised* if there is a G-action on L such that π is G-equivariant and the map $\pi^{-1}(x) \to \pi^{-1}(g \cdot x)$ induced on the fibres is linear for all $g \in G$ and all $x \in X$.

Definition 2.18. Let G be a reductive algebraic group and let (X, L) be a polarised variety with a G-action on X such that L is G-linearised. A test configuration $(\mathcal{X}, \mathcal{L})$ of exponent m is *G-equivariant* if there is a G-action on $(\mathcal{X}, \mathcal{L})$ which commutes with the $\mathbb{C}^\times$ action and such that the isomorphisms between $(X, L^{\otimes m})$ and (X_t, L_t) for $t \neq 0$ are G-equivariant. Then (X, L) is *equivariantly K-(poly/semi)stable* if it is K-(poly/semi)stable with respect to G-equivariant special test configurations.

The main result of Datar-Székelyhidi is the following:

Datar-Székelyhidi theorem. *[DS15] Let G be a reductive algebraic group and let X be a smooth complex Fano G-variety. Then (X, K_X^{-1}) is equivariantly K-polystable if and only if X admits a Kähler-Einstein metric.*

Remark. We should mention that the result of Datar-Székelyhidi has been generalised to the singular case when G is finite by Liu-Zhu [LZ20] and for general reductive groups by Zhuang [Zhu21]. Specifically, Zhuang uses a purely algebraic argument showing (among other things) that K-polystability of a log Fano pair (X, Δ) is equivalent to G-equivariant K-polystability when G is reductive.

The equivariant perspective has been fruitful in the recent study of K-stability for Fano varieties, since many natural and highly symmetric classes of varieties can be equipped with reductive group actions which often facilitate their analysis by simple combinatorial techniques.

For toric Fano varieties, T-equivariance means that the only test figuration which needs to be checked is the trivial one, making it very easy to prove the existence of a KE metric. Although this result had already been proved by Wang and Zhu [WZ04] using a different method, the new approach is easier and more adaptable to different circumstances. For example, in the case of complexity one torus actions, Ilten and Süß [IS17] found new examples of K-polystable Fano T-varieties for which the existence of a KE metric was not previously known by any other method. Similarly, Delcroix [Del16] characterised K-stability for spherical varieties (i.e. G-varieties of complexity zero), generalising the result of Wang and Zhu significantly beyond what had previously been known.

In both cases just mentioned, equivariant K-stability was shown to be equivalent to much more easily checkable criteria expressed in terms of combinatorial objects associated to the group action. In the next section we discuss some general theory of reductive groups and their actions which facilitate these types of description.

2.3 Representations and Actions of Reductive Groups

Here we recall some basic results on the properties of reductive group actions on algebraic varieties and the representation theory induced by such actions. The material and proofs will largely follow [Hum75, PV94, Tim11]. By *algebraic group* we will mean, unless otherwise specified, a connected affine algebraic group over a fixed algebraically closed field $\Bbbk$ of characteristic 0. All varieties are assumed to be integral over $\Bbbk$.

2.3.1 Homogeneous Spaces and Embeddings

We will for the remainder of this thesis be considering algebraic group actions on varieties, especially those which can be viewed as embeddings, or compactifications, of homogeneous spaces.

Definition 2.19. An *algebraic G-action* on a variety X is a morphism $G \times X \to X$,

$(g, x) \mapsto g \cdot x$ such that $g \cdot (h \cdot x) = (gh) \cdot x$ and $e \cdot x = x$ for all $g, h \in G$ and all $x \in X$, where $e \in G$ is the identity. In this case we call X a *G-variety.* A morphism $\varphi\colon X \to Y$ of G-varieties is *equivariant*, and called a *G-morphism*, if $\varphi(g \cdot x) = g \cdot \varphi(x)$ for all $x \in X$ and all $g \in G$. A G-variety X is *homogeneous* if the G-action on X is transitive, and a *homogeneous space* for G is a pair (X, x) consisting of a homogeneous G-variety X and a distinguished basepoint $x \in X$. A morphism of homogeneous spaces is a G-morphism which preserves basepoints.

Many of the points of interest and results to follow will be particularly concerned with the orbit structure of a given G-variety X, so let us address the basic properties of these orbits:

Proposition 2.2. *Let X be a G-variety. Then each G-orbit in X is a smooth, locally closed subvariety, whose boundary is a union of orbits of strictly lower dimension. Also, orbits of minimal dimension are closed, so in particular X contains a closed orbit.*

Proof. [Hum75, Prop 8.3] Let $x \in X$ and consider the G-orbit $G \cdot x$. The orbit map $G \to X$, $g \mapsto g \cdot x$ is the composite of the inclusion $G \to G \times \{x\}$ and the action morphism, so is itself a morphism. The image $G \cdot x$ is therefore a constructible subset of X, so contains an open subset U of its closure. Since G acts transitively on $G \cdot x$, we have $G \cdot x = \bigcup_{g \in G} g \cdot U$, so $G \cdot x$ is in fact open in its closure and hence is a locally closed subvariety. The existence of a nonsingular point in $G \cdot x$ ensures that it is smooth, since the action carries nonsingular points to nonsingular points. Now since $G \cdot x$ is dense in its closure, its boundary $\overline{G \cdot x} \setminus G \cdot x$ has strictly lower dimension than $G \cdot x$, and must be a union of orbits since it is G-stable. Finally, the boundary of an orbit of minimal dimension must then be empty, so these orbits are indeed closed. □

Each orbit $G \cdot x$ in X gives rise to a homogeneous G-space $(G \cdot x, x)$, and if G_x is the stabiliser of x in G, then the orbit map induces a bijection $G/G_x \to G \cdot x$ which maps G_x to x. It would therefore be desirable to endow G/G_x with the structure of a variety such that this bijection is in fact an isomorphism of varieties; then every homogeneous space (X, x) for G could be viewed in the form $(G/G_x, G_x)$. This would allow us to analyse the possible homogeneous spaces for G entirely in terms of G and its closed subgroups. With a preliminary result to follow, we will demonstrate exactly this construction.

Chevalley's theorem. *Let H be a closed subgroup of G. Then there is a rational G-module V on which G acts and a line $L \subseteq V$ such that H is the stabiliser of L.*

Proof. [Hum75, Thm 11.2] Let I be the ideal of functions in $\Bbbk[G]$ vanishing on H. Then I is finitely generated since $\Bbbk[G]$ is Noetherian, so there is a finite dimensional subspace $U \subseteq \Bbbk[G]$ which generates I. Let G act on $\Bbbk[G]$ by translation of functions. Then I is H-stable, so we may assume that U is H-stable as well. Then U is contained in a finite dimensional G-stable subspace $W \subseteq \Bbbk[G]$ (here W is spanned by the G-translates of a finite basis of U), since V is a *rational* G-module, and H is the stabiliser of U under this action. Now let $n = \dim U$ and pass to the n^{th} exterior power: then H is the stabiliser of the line $L := \wedge^n U$ in the finite dimensional G-vector space $V := \wedge^n W$. □

This theorem allows us to view H as the stabiliser of the point $x \in \mathbb{P}(V)$ corresponding to L, and thus identify (as sets) the coset space G/H with the orbit $G \cdot x \subseteq \mathbb{P}(V)$. We will define the variety G/H first abstractly, following [Tim11], then use Chevalley's theorem to prove it has the properties we want.

Definition 2.20. Let H be a closed subgroup of G and let $q \colon G \to G/H$ be the coset map. The *geometric quotient* of G by H is the coset space G/H equipped with the quotient topology induced by q and the sheaf of functions $\mathcal{O}_{G/H} = q_* \mathcal{O}_G^H$, where $\mathcal{O}_G^H$ are the H-invariant regular functions on G under the H-action by right translation.

Theorem 2.2. *For any closed subgroup H of G, $(G/H, \mathcal{O}_{G/H})$ is a smooth quasiprojective variety.*

Proof. [Tim11, Thm 1.3] Using Chevalley's theorem, pick a finite dimensional G-vector space V such that H is the stabiliser of a line L. Then under the induced G-action on $\mathbb{P}(V)$, H stabilises the point x corresponding to L, so the orbit map $\pi \colon G \to G \cdot x$ factors through q and a bijection $\bar{\pi} \colon G/H \to G \cdot x$, which is continuous by the universal property of the quotient topology. Since π is a morphism, we have maps $\mathcal{O}_{G \cdot x}(U) \to \mathcal{O}_G(\pi^{-1}(U))$ for each open $U \subseteq G \cdot x$, and since π is constant on cosets of H, we can view these as mapping into $\mathcal{O}_G(\pi^{-1}(U))^H = \mathcal{O}_{G/H}(\bar{\pi}^{-1}(U))$. Hence $\bar{\pi}$ is a morphism of schemes. If we can show that $\bar{\pi}$ is an isomorphism, we will be done, since $G \cdot x$ is a smooth, locally closed subset of $\mathbb{P}(V)$ by Proposition 2.2.

By [Har77, Cor III.10.7], since π is a morphism of varieties over $\Bbbk$ and G is smooth, there is a nonempty open subset $W \subseteq G \cdot x$ such that $\pi \colon \pi^{-1}(W) \to W$ is a smooth morphism. Since π is G-equivariant, it is smooth on all G-translates of W, which cover $G \cdot x$, so π is a smooth morphism. Now smooth morphisms are in particular open [Har77, Ex. III.9.1], and since the quotient $G \to G/H$ is open, it follows that $\bar{\pi}$ must also be open. Hence $\bar{\pi}$ is an open continuous bijection, i.e. a homeomorphism.

It remains to show the isomorphism of sheaves $\mathcal{O}_{G \cdot x} \cong \bar{\pi}_* \mathcal{O}_{G/H}$. We first note that $\bar{\pi}_* \mathcal{O}_{G/H} = \bar{\pi}_*(q_* \mathcal{O}_G^H) = \pi_* \mathcal{O}_G^H$. Now π is a surjective morphism of varieties, so induces an injective morphism $\pi^{\#} \colon \mathcal{O}_{G \cdot x} \to \pi_* \mathcal{O}_G$, whose image is a subsheaf of $\pi_* \mathcal{O}_G^H$ since regular functions on $G \cdot x$ are H-invariant. Let $U \subseteq G \cdot x$ be open and let $f \in \pi_* \mathcal{O}_G^H(U) = \mathcal{O}_G(\pi^{-1}(U))^H$. We need to show that $f = \pi^{\#}(h)$ for some $h \in \mathcal{O}_{G \cdot x}(U)$.

Let $\varphi \colon G \dashrightarrow G \cdot x \times \mathbb{A}^1$ be the rational map given by $\varphi(g) = (\pi(g), f(g))$ and let Z be the closure of $\varphi(G)$. Now $\varphi(G)$ is constructible, so contains an open dense subset $O \subseteq Z$, and the projection $p_1 \colon Z \to G \cdot x$ is injective when restricted to O by H-invariance of f. An injective morphism in characteristic 0 is birational, so we get an isomorphism of function fields $\Bbbk(U) \cong \Bbbk(Z)$. On the other hand, f is the pullback under φ of the projection $p_2 \colon Z \to \mathbb{A}^1$, that is, $f = \varphi^{\#}(p_2)$. Now since p_2 is a rational function on Z, we have $p_2 = p_1^{\#}(h)$ for some rational function $h \in \Bbbk(U)$. Thus $f = \varphi^{\#}(p_1^{\#}(h)) = \pi^{\#}(h)$. Then if h has a pole on U, f has a pole on $\pi^{-1}(U)$, which is impossible, so h is a regular function, i.e. $h \in \mathcal{O}_{G \cdot x}(U)$ as required. Hence we are done. □

It is immediate from the definition of the variety G/H that it satisfies the following universal property: for any morphism $\varphi \colon G \to X$ which is constant on cosets of H, there is a unique morphism $\bar{\varphi} \colon G/H \to X$ such that $\bar{\varphi} \circ q = \varphi$, where $q \colon G \to G/H$ is the quotient map. In particular, for a homogeneous space (X, x) with $G_x \supseteq H$, the orbit map $\pi \colon G \to X$ factors through $\bar{\pi} \colon G/H \to X$, and $\bar{\pi}$ is an isomorphism if and only if $G_x = H$. This specifies G/H uniquely.

Definition 2.21. An *embedding* of a homogeneous space G/H is a G-variety X containing a dense open orbit isomorphic to G/H.

Embeddings of a fixed homogeneous space G/H are then by definition birational to each other, and much of the theory to follow will concern classifying up to isomorphism the embeddings of a given homogeneous space. We first need to develop some representation theory.

2.3.2 Representation Theory of Reductive Groups

For the most part, we will be considering actions of specifically *reductive* algebraic groups on varieties, because these groups have good representation-theoretic properties, which we discuss here.

Recall that any connected algebraic group G acts on its own co-ordinate ring $\Bbbk[G]$ by right translation of functions: for $g, h \in G$ and $f \in \Bbbk[G]$, $(g \cdot f)\colon h \mapsto f(hg)$. We call $g \in G$ *unipotent* if right translation by g is a unipotent linear map when restricted to any finite dimensional subspace of $\Bbbk[G]$, and call a subgroup of G unipotent if all of its elements are unipotent in this sense. Note that unipotent groups are nilpotent [Hum75, Cor 17.5].

Definition 2.22. An algebraic group G is *reductive* if it contains no nontrivial connected normal unipotent subgroup and *semisimple* if it contains no nontrivial connected normal solvable subgroup.

Example. Examples of reductive groups include GL_n, $(\Bbbk^\times)^n$ and SL_n, which is semisimple. Non-reductive groups include any unipotent group, e.g. $(\Bbbk^+)^n$.

A *Borel subgroup* of an algebraic group G is a maximal closed connected solvable subgroup. A *torus* is an algebraic group isomorphic to $(\Bbbk^\times)^n$ for some n. Borel subgroups are conjugate in any algebraic group: we usually fix a particular Borel subgroup $B \subseteq G$, which is isomorphic to $U \rtimes T$ for some maximal unipotent subgroup $U \subseteq G$ and maximal torus $T \subseteq G$. If a Borel subgroup B contains the maximal torus T, there is a unique *opposite* Borel subgroup B^- with $B \cap B^- = T$. There is also a corresponding opposite maximal unipotent subgroup U^-.

We will denote the Lie algebra of an algebraic group by the corresponding lowercase Fraktur character, so $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{b}$ of B, and so on. The Lie algebra of a connected, reductive group G with a maximal torus T has a root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

where Δ is the root system with respect to T and $\mathfrak{g}^\alpha$ are the root spaces. The choice of a Borel subgroup B with maximal unipotent subgroup U determines a set of positive roots Δ_+, namely the α such that $\mathfrak{g}^\alpha$ span $\mathfrak{u}$. Finally, ρ will denote half the sum of the positive roots.

We will use freely the fact that any algebraic group G embeds in $\mathrm{GL}(V)$ for some finite-dimensional vector space V [Hum75, Thm 8.6]. In this case, we will choose B, U and T to be the upper triangular, upper unitriangular and diagonal matrices in the image of G, respectively.

Definition 2.23. A *rational G-module* is a $\Bbbk$-vector space V equipped with a linear action of G such that V is a union of finite dimensional G-stable subspaces.

We will throughout speak of 'G-modules' with rationality assumed. One of the most useful aspects of reductivity is the following:

Theorem 2.3. *An algebraic group G is reductive if and only if every G-module V is completely reducible, i.e. if and only if every G-stable submodule W of V has a G-stable complement W' with $V = W \oplus W'$.*

Remark. Note the above theorem uses our assumption that $\Bbbk$ is of characteristic 0 essentially.

Lemma 2.1. *Let G be a unipotent algebraic group and let V be a non-zero G-module. Then V contains a non-zero fixed point under G.*

Proof. [Bri10, Ex. 1.22] If V contains a nonzero G-stable submodule W, then a fixed point in W is also a fixed point in V, so we can assume that V is simple. We will prove that $V \cong \Bbbk$ and G acts trivially on V, so in particular V contains a fixed point.

Suppose G acts nontrivially on V. Consider the subgroup $G_V := \bigcap_{v \in V} G_v$, which is normal. The quotient G/G_V acts on V with no fixed points, and $(gG_V) \cdot v = g \cdot v$ for all $g \in G$ and $v \in V$. Now G is nilpotent, and thus so is G/G_V, so the centre $Z(G/G_V)$ is nontrivial, and thus acts nontrivially on V. Hence $Z(G)$ itself contains some element which acts nontrivially. Since V is simple, by Schur's lemma each $g \in Z(G)$ acts via scalar multiplication $g \cdot v = \chi(g)v$ for some morphism of algebraic groups $\chi \colon Z(G) \to \Bbbk^\times$. But $Z(G) \cong \Bbbk^n$ for some n, being isomorphic to a closed subgroup of the upper unitriangular matrices, so any such morphism is constant. Hence $Z(G)$ acts trivially, a contradiction. $\square$

Proof of Theorem 2.3. [Mil17, Thm 22.42] Suppose every G-module is completely reducible, and let U be a connected normal unipotent subgroup of G. We can choose a non-zero finite-dimensional G-module V such that G embeds as a closed subgroup of $\mathrm{GL}(V)$. Then $V^U \subseteq V$ is non-zero by the lemma, and normality of U ensures that V^U is G-stable. Thus by complete reducibility of V, V^U admits a G-stable complement W in V. Then W is a U-module with $W^U = 0$, so $W = 0$. Thus $V^U = V$, i.e. U fixes V pointwise. Since U was embedded injectively into $GL(V)$, it follows that U is trivial and hence G is reductive.

For the converse, we assume the following fact: a reductive group G decomposes as $G = Z \cdot G'$, the product of the identity component of its centre, which is a torus, and its derived subgroup, which is semisimple (see e.g. [Hum75, Thm. 27.5]). Now let V be a nonzero G-module. All T-modules for a torus T are completely reducible, so V is completely reducible when viewed as a Z-module. Write $V = \bigoplus_i V_i$ where V_i are the simple submodules of V. Then since Z and G' commute, each V_i is G'-stable. Since G' is semisimple in characteristic 0, its Lie algebra $\mathfrak{g}'$ is semisimple [Hum75, Th 13.5], so any $\mathfrak{g}'$-module is completely reducible by Weyl's theorem. But each V_i is such a module, so we have $V_i = \bigoplus_j V_{ij}$ for simple submodules V_{ij} of V_i, and so V is completely reducible. □

Corollary 2.1. *Let* $\mathrm{Irr}\, G$ *denote the set of isomorphism classes of simple* G*-modules. Any* G*-module* M *admits a decomposition*

$$M \cong \bigoplus_{V \in \mathrm{Irr}\, G} \mathrm{Hom}^G(V, M) \otimes V.$$

Let $G \times G$ *act on* G *by* $(g, h) \cdot x = gxh^{-1}$ *for* $g, h, x \in G$*. Then there is a decomposition of* $G \times G$*-modules*

$$\Bbbk[G] \cong \bigoplus_{V \in \mathrm{Irr}\, G} V^* \otimes V.$$

Proof. [Bri10, Lemma 2.1] By complete reducibility, M is a direct sum of simple G-modules, so we may assume that M itself is simple. Then by Schur's lemma, $\mathrm{Hom}^G(V, M) \cong \Bbbk$ if $V \cong M$ and is zero otherwise. The first claim follows.

The second claim follows from the first provided that $\mathrm{Hom}^G(V, \Bbbk[G]) \cong V^*$. For $f \in \mathrm{Hom}^G(V, \Bbbk[G])$, let $\varphi(f) \in V^*$ be the function $v \mapsto f(v)(e_G)$. For $h \in V^*$, let $\psi(h)$ be the function $V \to \Bbbk[G]$ given by $v \mapsto (g \mapsto h(g \cdot v))$. An easy check shows that φ and ψ provide the required isomorphism. □

Definition 2.24. A *rational G-algebra* is a rational G-module A with the structure of a commutative associative algebra where the G-action is by algebra automorphisms.

Another useful property of reductive groups is the following:

Theorem 2.4. *Let G be a reductive group and let A be a rational G-algebra. Then A^G is finitely generated if A is finitely generated.*

Proof. [PV94, Thm 3.6] Since G is reductive, A is completely reducible, so the G-submodule A^G of invariants has a complement A_G, which can be characterised as the sum of all nontrivial G-submodules of A. Let $R_A\colon A \to A^G$ be the projection to A^G. Then R_A has the following properties:

(i) if $\varphi\colon A \to B$ is a G-morphism of rational G-algebras, $\varphi \circ R_A = R_B \circ \varphi$;

(ii) if $\varphi(A) = B$, then $\varphi(A^G) = B^G$;

(iii) for all $a \in A^G$ and $b \in A$, $R_A(ab) = aR_A(b)$.

Note that (ii) follows directly from (i), and by viewing multiplication by a as a G-endomorphism of A, so does (iii). For (i), equivariance gives $\varphi(A^G) \subseteq B^G$ and $\varphi(A_G) \subseteq B_G$, from which the claim follows.

Let U be a finite dimensional G-subspace which generates A and let SU be the symmetric algebra of U. The action of G on U naturally extends to an action on SU, and the inclusion $U \hookrightarrow A$ induces a surjective G-morphism $\pi\colon SU \to A$. Then by (ii) we have $\pi((SU)^G) = A^G$, so it suffices to prove that $(SU)^G$ is finitely generated. Since SU has a natural G-grading, we can replace A with SU and assume that $A = \bigoplus_{n\geq 0} A_n$ is a G-graded G-algebra with $A_0 = \mathbb{k}$.

Let I be the homogeneous ideal $\bigoplus_{n>0} A_n^G$ of A, which by the Hilbert basis theorem is finitely generated, say by $b_1, \dots, b_m$, which we can assume to be homogeneous. It remains to prove that each A_n^G is contained in $B = \mathbb{k}[b_1, \dots, b_m]$. Let $a \in A_n^G$ for some $n > 0$. Then $a \in I$ so we can write $a = \sum_{i=1}^m a_i b_i$, assuming without loss of generality that each a_i is homogeneous of degree $n - \deg b_i$. Using (iii) we have $a = R_A(a) = R_A\left(\sum_{i=1}^m a_i b_i\right) = \sum_{i=1}^m R_A(a_i) b_i$. Now proceed by induction on n: if $n = 1$, the a_i are scalars and $a \in B$. If $A_k^G \subseteq B$ for all $k < n$, then the $R_A(a_i)$ lie in B, so a does too. Hence we are done. $\square$

Corollary 2.2. *If X is an affine G-variety, then $\Bbbk[X]^G$ is finitely generated.*

Remark. In fact, this property characterises reductive groups, in a sense. Specifically, let G be an algebraic group. It is a theorem of Popov that if $\Bbbk[X]^G$ is finitely generated for every affine G-variety X, then G is reductive [PV94, Thm. 3.8].

The next result demonstrates why the representation of a Borel subgroup $B \subseteq G$ induced by a G-action is also important to keep track of.

Lie-Kolchin theorem. *Let a connected solvable algebraic group G act on a finite dimensional vector space V. Then there exists $v \in V$ which is a simultaneous eigenvector for all $g \in G$.*

Proof. [Spr98, Thm 6.3.1] We proceed by induction on d, the derived length of G, and $n = \dim V$. If V contains a nonzero proper G-stable submodule W, then $\dim W < n$ so induction gives us a common eigenvector $v \in W$ for G, hence a common eigenvector in V. So we may assume V is irreducible.

Let G' be the derived subgroup of G, which is a connected solvable normal subgroup of G with derived length $d - 1$. Induction then gives a common eigenvector of G' in V. Let V' be the subspace of V spanned by all such vectors. It follows from normality of G' that V' is G-stable, so $V' = V$ by irreducibility.

Now G' acts on V via scalar multiplication, and since commutators have determinant 1, these scalars must be n^{th} roots of unity, of which $\Bbbk$ contains finitely many. So G' is finite and connected, hence trivial. Then G is commutative, whence the result follows from the Jordan decomposition. □

This leads naturally to the notions of characters and highest weights, which further simplify the study of B-actions.

Definition 2.25. A *character* of an algebraic group G is a morphism of algebraic groups $\chi\colon G \to \Bbbk^\times$. The set of characters of G forms an abelian group $\mathfrak{X}(G)$ under pointwise multiplication, called the *character group* of G.

Example. The character group of a torus $T = (\Bbbk^\times)^n$ is free abelian of rank n. The character group of a unipotent group is trivial.

Example. Let G be a connected algebraic group. Then a Borel subgroup $B \subseteq G$ is isomorphic to $U \rtimes T$ for a maximal unipotent subgroup U and maximal torus T of G. Since characters of B are trivial on U, $\mathfrak{X}(B) = \mathfrak{X}(T)$ is free abelian of rank $\dim T$.

Theorem 2.5. *Let M be a simple G-module. The subspace M^U is a line where B acts via a character χ_M, and M is uniquely determined up to isomorphism by χ_M.*

Proof. [Bri10, Thm 2.4] By the Bruhat decomposition (see [Hum75, Prop. 28.5]), the product $U^-B = B^-B = B^-U$ is open in G. Let $v \in M^U$ be nonzero and using the Lie-Kolchin theorem, choose $f \in M^*$ to be a nonzero B^--eigenvector of weight λ. Consider the U-invariant map $G \to \Bbbk$, $g \mapsto f(g \cdot v)$. It is nonzero, since M is simple and spanned by $G \cdot v$. It is also, by choice of f, a B^--eigenvector in $\Bbbk[G]$. Then given any other nonzero $v' \in M^U$, the map

$$g \mapsto \frac{f(g \cdot v)}{f(g \cdot v')}$$

is a nonzero rational function on G, invariant on the open subset $B^- \times U$, hence constant. Then there is some $t \in \Bbbk^\times$ with $f(g \cdot (v - tv')) = 0$ for all $g \in G$. It follows, since f is nonzero, that $v = tv'$ and hence M^U is indeed a line. That B then acts via a character χ follows from Lie-Kolchin.

It follows that f is a B-eigenvector of weight $-\chi = \lambda$, seen by restricting to $B \cap B^- = T$. It also follows that $(M^*)^{U^-}$ is a line spanned by f. Now by Corollary 2.1 we have a decomposition of $T \times G$-modules

$$\Bbbk[G]^{U^-} \cong \bigoplus_{V \in \operatorname{Irr} G} (V^*)^{U^-} \otimes V \cong \bigoplus_{V \in \operatorname{Irr} G} V.$$

This identifies M with the T-eigenspace of $\Bbbk[G]^{U^-}$ where T acts with weight χ, and uniqueness of M follows from uniqueness of T-eigenspaces. □

We call χ_M the *highest weight* of M. The simple module with highest weight χ will be denoted V_χ. Characters $\chi \in \mathfrak{X}(B)$ with $V_\chi \neq 0$ are called *dominant weights* and we let $\mathfrak{X}^+$ be the set of these. It follows that the nonzero simple G-modules are classified up to isomorphism by the dominant weights of G.

For an arbitrary G-module M and $\chi \in \mathfrak{X}(B)$, let

$$M_\chi^{(B)} = \{m \in M \mid b \cdot m = \chi(b)m \text{ for all } b \in B\}.$$

This is the set of *highest weight vectors* of weight χ. The full set of highest weight vectors is $M^{(B)} = \bigcup_{\chi\in\mathfrak{X}^+} M^{(B)}_\chi$.

Proposition 2.3. *For any G-module M, there is a decomposition*

$$M \cong \bigoplus_{\chi\in\mathfrak{X}^+} M^{(B)}_\chi \otimes V_\chi.$$

Proof. By Corollary 2.1 and Theorem 2.5 it suffices to show that $M^{(B)}_\chi \cong \mathrm{Hom}^G(V_\chi, M)$. By Schur's lemma a G-morphism $\varphi\colon V_\chi \to M$ is either 0 or an isomorphism onto a G-submodule of M containing $M^{(B)}_\chi$, so φ is determined by the image $\varphi(1) \in M^{(B)}_\chi$ when it is restricted to the line V^U_χ and any element of $M^{(B)}_\chi$ determines such a morphism. □

We can use the proposition above to calculate dimensions of (finite-dimensional) G-modules:

Corollary 2.3. *Let $m_\chi(M)$ denote $\dim M^{(B)}_\chi$. If M is a finite-dimensional G-module, we have*

$$\dim M = \sum_{\chi\in\mathfrak{X}^+} m_\chi(M) \prod_{\alpha^\vee\in\Delta^\vee_+} \left(1 + \frac{(\chi,\alpha^\vee)}{(\rho,\alpha^\vee)}\right).$$

Proof. The product is the Weyl dimension formula for $\dim V_\chi$ (see e.g. [Hum72, §24.3]). The result then immediately follows from Proposition 2.3. □

2.3.3 Group Actions on Varieties

We can apply the results of the previous subsection to the action of a reductive group G on an algebraic variety X. If X is affine, we have seen that $\Bbbk[X]$ is a rational G-algebra with a finitely generated subalgebra of invariants. For any variety X, there is also an action on the function field $\Bbbk(X)$, although this is not in general a rational G-algebra. We will call a highest weight vector in $\Bbbk[X]$ or $\Bbbk(X)$ a *B-eigenfunction* or *semi-invariant*. Let $\Lambda(X) \subseteq \mathfrak{X}(B)$ be the set of all weights of rational B-eigenfunctions on X: it is a sublattice of the character group called the *weight lattice* of X. We define the *rank* $r_G(X)$ of X to be $\mathrm{rk}\,\Lambda$. The rank and weight lattice are G-birational invariants of X. Another particularly important birational invariant of the G-action is the complexity:

Definition 2.26. Let G be an algebraic group and let X be a G-variety. The *complexity* $c_G(X)$ of the G-action on X is the minimal codimension of a B-orbit on X.

The following theorem of Rosenlicht allows an alternative characterisation:

Rosenlicht's theorem. *[Ros56] Let G be a connected algebraic group acting on a variety X. Then there exists a nonempty open G-stable subset $X_0 \subseteq X$, a variety X_0/G* (a geometric quotient) *and a surjective morphism $\pi\colon X_0 \to X_0/G$ whose fibres are exactly the G-orbits in X_0, such that X_0/G has the quotient topology induced by π, and $\Bbbk(X_0/G) \cong \Bbbk(X_0)^G$.*

Proposition 2.4. *For a G-variety X, $c_G(X) = \operatorname{trdeg}_{\Bbbk} \Bbbk(X)^B$.*

Proof. Using Rosenlicht's theorem, let $X_0 \subseteq X$ be nonempty, open and B-stable with geometric quotient $Y_0 = X_0/B$. Since $\Bbbk(Y_0) = \Bbbk(X_0)^B = \Bbbk(X)^B$ we have:

$$\operatorname{trdeg} \Bbbk(X)^B = \dim Y_0 = \dim X - \max_{x \in X} \dim B \cdot x = c_G(X)$$

as required. □

Example. Let $G = \mathbb{C}^\times$ act on $X = \mathbb{A}^2$ by $t\cdot(x, y) = (tx, t^{-1}y)$. This orbits of this action are the origin $\{(0, 0)\}$, the punctured axes $\{(x, 0) \mid x \in \mathbb{C}^\times\}$ and $\{(0, y) \mid y \in \mathbb{C}^\times\}$, and the conics $\{(x, y) \mid xy = a\}$ for each $a \in \mathbb{C}^\times$.

Since $G = B$ in this case, the existence of maximal orbits of codimension 1 means that this is a complexity one action.

On the other hand, as per Rosenlicht's theorem, this variety has an open subset $X_0 = \mathbb{A}^2 \setminus \{xy = 0\}$ admitting a geometric quotient $X_0/B = \mathbb{A}^1 \setminus \{0\}$, via the morphism $\pi\colon X_0 \to \mathbb{P}^1$, $(x, y) \mapsto xy$. This quotient, defined by the B-invariant rational function xy on X has dimension 1, and function field $\Bbbk(xy)$ with transcendence degree 1 over $\Bbbk$. This demonstrates the connection between the complexity, the field of B-invariants and geometric B-quotients.

A helpful result of Knop [Kno95] shows that complexity, rank and the weight lattice behave well when considering G-stable subvarieties:

Theorem 2.6. *Let X be a normal G-variety and let $Y \subseteq X$ be a closed irreducible G-subvariety. We can extend rational B-eigenfunctions from Y to X, so $c_G(Y) \leq c_G(X)$, $r_G(Y) \leq r_G(X)$ and $\Lambda(Y) \subseteq \Lambda(X)$.*

Lemma 2.2. *Let X be an affine G-variety. Any rational semi-invariant $f \in \Bbbk(X)^{(B)}$ can be written as a quotient of regular semi-invariants.*

Proof. [PV94, Thm 3.3] Write $f = \frac{p}{q}$ with $p, q \in \Bbbk[X]$. The B-module spanned by $B \cdot q$ is finite dimensional, so by Lie-Kolchin contains a B-eigenfunction $q' = \sum_i a_i(b_i \cdot q)$ for some $a_i \in \Bbbk$ and $b_i \in B$. Then $fq' = \sum_i a_i(b_i \cdot p) := p' \in \Bbbk[X]^{(B)}$ and $f = \frac{p'}{q'}$. □

Proof of Theorem 2.6. [Tim11, Thm 5.7] Since complexity, rank and the weight lattice are birational invariants, we may assume that X is projective. Let $\hat{X}$, $\hat{Y}$ be the affine cones over X and Y. The G-action on X lifts to a $\hat{G} = G \times \Bbbk^\times$-action on $\hat{X}$, where $\Bbbk^\times$ acts by homotheties, and $\hat{Y}$ is $\hat{G}$-stable. A rational semi-invariant $f \in \Bbbk(Y)^{(B)}$ pulls back to $\Bbbk(\hat{Y})^{(\hat{B})}$, where $\hat{B} = B \times \Bbbk^\times$. By the above Lemma, $f = \frac{p}{q}$, where $p, q \in \Bbbk[\hat{Y}]^{(\hat{B})}$ are homogeneous. Let $p', q' \in \Bbbk[\hat{X}]^{(\hat{B})}$ be extensions of p, q to $\hat{X}$, so that $\tilde{f} = \frac{p'}{q'}$ pushes forward to a rational B-eigenfunction on X restricting to f on Y.

Since any B-eigenfunction on Y extends to one on X of the same weight, we get $\Lambda(Y) \subseteq \Lambda(X)$ and hence $r_G(Y) \leq r_G(X)$. Furthermore, the above proves that $\Bbbk(Y)^B$ is the residue field of $\mathcal{O}^B_{X,Y}$ from which follows $c_G(Y) = \operatorname{trdeg} \Bbbk(Y)^B \leq \operatorname{trdeg} \Bbbk(X)^B = c_G(X)$. □

The invariants, semi-invariants and weight lattice of a G-variety have an important relationship which will be used to good effect later on:

Proposition 2.5. *[Tim11, §13.1] Let X be a G-variety, let $K = \Bbbk(X)$ and let Λ be the weight lattice of X. There is a split exact sequence of abelian groups*

$$0 \longrightarrow (K^B)^\times \longrightarrow K^{(B)} \longrightarrow \Lambda \longrightarrow 0.$$

Proof. The first map is an inclusion, since a nonzero B-invariant function is just a semi-invariant of weight 1. The second map sends a semi-invariant f to its weight $\chi \in \Lambda$. This map is surjective by definition of Λ and has kernel $(K^B)^\times$. Hence the sequence is exact. Since Λ is free, the sequence splits under a map $e\colon \Lambda \to K^{(B)}$, given by sending a basis weight χ to some eigenfunction with weight χ. □

2.3.4 G-Linearisation of Line Bundles

We will need various results on line bundles on G-varieties, which we collect here, following [Pop74, KKV89, KKLV89]. Recall:

Definition 2.27. Let G be an algebraic group, let X be a G-variety, and let L be a line bundle on X. A *G-linearisation* of L is a G-action on L such that:

(i) the natural projection $L \to X$ is G-equivariant;

(ii) the induced maps $L_x \to L_{g\cdot x}$ are linear for every $x \in X$ and $g \in G$.

A line bundle with a G-linearisation is called a *G-line bundle*. We denote the group of isomorphism classes of G-line bundles by $\operatorname{Pic}_G X$.

Proposition 2.6. *Let H be a closed subgroup of a connected algebraic group G. Then $\operatorname{Pic}_G(G/H)$ is isomorphic to the character group $\mathfrak{X}(H)$. In particular, every G-linearisable line bundle on G is trivial. The kernel of the forgetful homomorphism $\operatorname{Pic}_G(G/H) \to \operatorname{Pic}(G/H)$ consists of characters corresponding to different G-linearisations of the trivial line bundle $G/H \times \Bbbk$, and these are exactly the characters of H obtained by restricting characters of G.*

Proof. [KKV89, §3] Let $\chi \in \mathfrak{X}(H)$ be a character and define a line bundle L_χ as follows: let H act on $G \times \Bbbk$ by $h \cdot (g, x) = (gh^{-1}, \chi(h)x)$, and take the quotient $(G \times \Bbbk)/H$, i.e. equip this set with the quotient topology and the direct image of the sheaf of H-invariant functions. Let G act on $G \times \Bbbk$ by $g' \cdot (g, x) = (g'g, x)$. It is easy to see that this commutes with the H-action, so induces a G-action on L_χ, linear on fibres. Hence we have a homomorphism $\mathfrak{X}(H) \to \operatorname{Pic}_G(G/H)$.

Conversely, let L be a G-linearised line bundle on G/H. Then H acts on the fibre $L_H \cong \Bbbk$ over $eH \in G/H$ linearly, and hence by a character χ. Hence the projection $G \times L_H \to L$ induces an isomorphism $L \cong L_\chi$, giving our isomorphism $\mathfrak{X}(H) \cong \operatorname{Pic}_G(G/H)$. Taking $H = \{e\}$ gives triviality of $\operatorname{Pic}_G G$.

Now suppose L_χ is a G-linearisation of the trivial line bundle $G/H \times \Bbbk$. Then the fibrewise map $L_{\chi,gH} \to L_{\chi,g'gH}$ is an action of G on $\Bbbk$, i.e. given by a character of G, and hence χ must be a restriction to H of that character. Likewise if we assume χ is the restriction of a character of G, then the map $G \times \Bbbk \to G/H \times \Bbbk$, $(g, x) \to (gH, \chi(g)x)$ induces a G-isomorphism of L_χ with $G/H \times \Bbbk$. □

Theorem 2.7. *[KKLV89, Prop 4.5, 4.6] Let G be a connected affine algebraic group. Then $\operatorname{Pic} G$ is finite and there exists a finite covering $\tilde{G} \to G$ of algebraic groups such that $\operatorname{Pic} \tilde{G} = 0$.*

Proposition 2.7. *[KKLV89, Prop 2.4] Let L be a line bundle on a normal G-variety X. There is an integer $n > 0$ such that $L^{\otimes n}$ is G-linearisable. We can in fact take n*

to be the order of Pic G*. In particular, if* G *is factorial, i.e.* Pic $G = 0$*, then any line bundle on* X *is* G*-linearisable.*

Using Theorem 2.7 and Proposition 2.7, we may assume from now on that all line bundles on normal G-varieties are G-linearised.

Lemma 2.3. *Let* L *be a* G*-linearised line bundle on a normal* G*-variety* X*. Then there is a* G*-action making* $H^0(X, L)$ *a rational* G*-module.*

Proof. For a section $s\colon X \to L$, the action is given by $(g \cdot s)(x) = g \cdot (s(g^{-1} \cdot x))$. See [KKLV89, Lemma 2.5] for details. □

We can use this property to find an open cover of any normal G-variety by G-invariant quasiprojective subsets, by the following theorem of Sumihiro [Sum74, Lemma 8]:

Sumihiro's theorem. *Let* G *be an algebraic group and let* X *be a normal* G*-variety. Any point* $x \in X$ *has an open* G*-stable neighbourhood* U *which admits a locally closed* G*-equivariant embedding* $U \hookrightarrow \mathbb{P}(V)$ *for some* G*-module* V*.*

Proof. [Tim11, Thm C.7] Let D be an effective divisor on X such that $X \setminus D$ is an affine open neighbourhood of x. Let $U_0 = X \setminus \bigcap_{g\in G} gD$. Then U_0 is a G-stable affine open neighbourhood of x, and its complement is an effective Cartier divisor. Let L be the corresponding line bundle, and choose a section $\sigma_0 \in H^0(X, L)$ such that $U_0 = X_{\sigma_0}$. Then for some $d_i \geq 0$ and $\sigma_i \in H^0(X, L^{\otimes d_i})$ we have:

$$\Bbbk[U_0] = \bigcup_{d\geq 0} H^0(X, L^{\otimes d})/\sigma_0^d = \Bbbk\left[\frac{\sigma_1}{\sigma_0^{d_1}}, \dots, \frac{\sigma_m}{\sigma_0^{d_m}}\right].$$

We can replace L by some power and assume both that all $d_i = 1$ and by Proposition 2.7 that L is G-linearised. By Lemma 2.3, the σ_i lie in some finite dimensional G-stable submodule $M \subseteq H^0(X, L)$. The induced G-equivariant rational map $\varphi\colon X \dashrightarrow \mathbb{P}(M^*)$ is then an embedding of U_0 onto a locally closed subvariety of $\mathbb{P}(M^*)$. □

Corollary 2.4. *Let* Y *be a closed* G*-stable subvariety of a normal* G*-variety* X*. Then there is an open affine* B*-stable subset* $X_0 \subseteq X$ *intersecting* Y*.*

Proof. [Tim97, Lemma 1.1] Sumihiro's theorem allows us to assume that $X \subseteq \mathbb{P}(V)$ is quasiprojective. Consider the boundary $Z := \overline{X} \setminus X$ of X in $\mathbb{P}(V)$ and denote by $\hat{X}, \hat{Y}$

and $\hat{Z}$ the affine cones in V over $\overline{X}, \overline{Y}$ and Z respectively. Let g_1 be a homogeneous B-eigenfunction on $\hat{Y} \sqcup \hat{Z}$ which vanishes on $\hat{Z}$ but not on $\hat{Y}$. By Theorem 2.6 we can extend g_1 to a B-eigenfunction g_0 on $\hat{X}$. Then the set $X_0 := \{x \in \overline{X} \mid g_0(x) \neq 0\}$ is affine and open, is B-stable since g_0 is a B-eigenfunction, and intersects Y by construction of g_1. □

2.4 Combinatorial Description of G-Varieties

2.4.1 Valuations

As much of the theory to follow concerns the analysis of the interactions of valuations on the function field of a G-variety and the G-action itself, we begin by collating some useful theory on this topic, most of which can be found in [Tim11, §19, Appendix B].

Let $\Bbbk$ be an algebraically closed field of characteristic zero, and let K be a finitely generated field extension of $\Bbbk$. When K is acted on by an algebraic group G (assumed to be connected and reductive over $\Bbbk$), fix a Borel subgroup B and let K^B and $K^{(B)}$ denote the invariant and semi-invariant elements, respectively. Write $K^{(B)}_\chi$ for the set of semi-invariants of a specific weight $\chi \in \mathfrak{X}(B)$.

We call a normal $\Bbbk$-variety X with $\Bbbk(X) = K$ a *model* of K, and a *G-model* if X has a G-action compatible with the G-action on K. Let $\mathcal{D}$ denote the set of non-G-stable prime divisors on all G-models of K and let $\mathcal{D}^B$ denote the subset of B-stable divisors (called *colours*). Finally, let $K_B \subseteq K$ denote the subalgebra of functions with B-stable divisor of poles on some G-model of X.

For our purposes, a *valuation* of K means the following:

Definition 2.28. A *valuation* of $K/\Bbbk$ is a function $\nu\colon K \to \mathbb{Q} \cup \{\infty\}$ such that:

- $\nu(K^*) \cong \mathbb{Z}$ or $\{0\}$;
- $\nu(\Bbbk^*) = \{0\}$;
- $\nu(f) = \infty$ if and only if $f = 0$;

and for all $f, g \in K^*$:

- $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$;

- $\nu(fg) = \nu(f) + \nu(g)$.

An immediate result from the definition which will come in useful later is the following:

Lemma 2.4. *Let ν be a valuation on K and let $f, g \in K$ with $\nu(f) \neq \nu(g)$. Then $\nu(f+g) = \min\{\nu(f), \nu(g)\}$.*

Proof. We already have that $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$. We can assume without loss of generality that $\nu(f) > \nu(g)$, giving $\nu(f+g) \geq \nu(g)$. Suppose further that $\nu(f+g) > \nu(g)$. Then we have:

$$\nu(g) = \nu(f+g-f) \geq \min\{\nu(f+g), \nu(-f)\} = \min\{\nu(f+g), \nu(f)\} > \nu(g)$$

a contradiction. Hence $\nu(f+g) = \nu(g)$. □

A valuation ν defines a *valuation ring* $\mathcal{O}_\nu = \{f \in K \mid \nu(f) \geq 0\}$, which is a local ring with maximal ideal $\mathfrak{m}_\nu = \{f \in K \mid \nu(f) > 0\}$, residue field $\Bbbk(\nu) = \mathcal{O}_\nu/\mathfrak{m}_\nu$ and fraction field K. In particular $\mathcal{O}_\nu$ is a discrete valuation ring, so upon choosing an element $g \in \mathfrak{m}_\nu$ (a *uniformising parameter*), any element $f \in K$ can be written in the form $f = ug^k$ for some uniquely determined unit $u \in \mathcal{O}_\nu$ and integer k. Then $\nu(f) = \nu(u) + k\nu(g) = k\nu(g)$, since u is a unit.

Proposition 2.8. *Let ν be a valuation of K with valuation ring $\mathcal{O}_\nu$. Then $\mathcal{O}_\nu$ is a maximal subring of K and determines ν up to proportionality.*

Proof. Suppose $\mathcal{O}_\nu \subsetneq A \subset K$. Choose a uniformising parameter $t \in \mathfrak{m}_\nu$ and let $g \in A \setminus \mathcal{O}_\nu$, so that we can write $g = ut^{-k}$ for some $u \in \mathcal{O}_\nu^\times$ and $k \geq 1$. Since $u^{-1} \in A$, $u^{-1}g = t^{-k} \in A$. Since $t \in A$, A contains all powers of t and hence all elements of the form ut^k for $u \in \mathcal{O}_\nu^\times$, i.e. $A = K$.

For the second claim, suppose $\mathcal{O}_\nu = \mathcal{O}_{\nu'}$ and choose a uniformising parameter $t \in \mathfrak{m}_\nu = \mathfrak{m}_{\nu'}$. Let $f = ut^k \in K$ where u is a unit in $\mathcal{O}_\nu$ and $k \in \mathbb{Z}$. We have $\nu(f) = k\nu(t)$ and $\nu'(f) = k\nu'(t)$, so $\nu(f)/\nu'(f) = \nu(t)/\nu'(t) = c \in \mathbb{Q}_+$. The first equality shows that the right hand side is independent of f and hence $\nu = c\nu'$. □

Geometric Valuations

The principal examples of valuations in algebraic geometry are those given by the order of vanishing of a rational function along a prime divisor.

Definition 2.29. Let D be a prime divisor on a normal variety X over $\Bbbk$ with local ring $\mathcal{O}_{X,D}$ and let $f \in \Bbbk(X)^*$. We can write $f = g/h$ where $g, h \in \mathcal{O}_{X,D}$. Then the *order of vanishing* of f along D is the difference of the lengths of the modules $\mathcal{O}_{X,D}/(g)$ and $\mathcal{O}_{X,D}/(h)$. This defines a valuation ν_D of $K = \Bbbk(X)$.

Hence we make the following definition:

Definition 2.30. A valuation ν on $K/\Bbbk$ is *geometric* if there exists a model X of K and a prime divisor $D \subseteq X$ such that $\nu(f) = c \cdot \nu_D(f)$ for some $c \in \mathbb{Q}_+$ and all $f \in K$.

Definition 2.31. Let X be a model of K. A closed subvariety Y of X is a *centre* of ν on X if $\mathcal{O}_\nu$ dominates $\mathcal{O}_{X,Y}$ (i.e. $\mathcal{O}_\nu \supseteq \mathcal{O}_{X,Y}$ and $\mathfrak{m}_\nu \supseteq \mathfrak{m}_{X,Y}$).

Proposition 2.9. *A prime divisor $D \subseteq X$ is a centre for its own geometric valuation ν_D.*

Proof. Let $f \in K^*$ and write $f = g/h = \delta^{n-m}\bar{g}/\bar{h}$ where δ is a uniformising parameter and $\bar{g}, \bar{h} \in \mathcal{O}_{X,D}$ are units. Then $\nu_D(f) = n-m$, and $f \in \mathcal{O}_{X,D}$ if and only if $n-m \geq 0$, i.e. if and only if $f \in \mathcal{O}_{\nu_D}$, and likewise $f \in \mathfrak{m}_{X,D}$ if and only if $n-m > 0$, i.e. $f \in \mathfrak{m}_{\nu_D}$. Hence $\mathcal{O}_{\nu_D} = \mathcal{O}_{X,D}$ and $\mathfrak{m}_{\nu_D} = \mathfrak{m}_{X,D}$. □

Proposition 2.10. *If $\varphi \colon X \to X'$ is a dominant morphism and ν has centre Y on X, then the restriction ν' of ν to $K' = \Bbbk(X')$ has centre $Y' = \overline{\varphi(Y)} \subseteq X'$.*

Proof. Since φ is dominant, it induces an inclusion $\varphi^* \colon K' \hookrightarrow K$, so the description of ν' makes sense. Now $\varphi(Y)$ is an open subset of Y' and hence has the same local ring, and φ^* induces inclusions $\mathcal{O}_{X',\varphi(Y)} \subseteq \mathcal{O}_{X,Y}$, $\mathfrak{m}_{X',\varphi(Y)} \subseteq \mathfrak{m}_{X,Y}$. Hence $\mathcal{O}_\nu$ dominates $\mathcal{O}_{X',Y'}$. □

Valuative Criterion of Separation. *A normal prevariety X is separated if and only if any geometric valuation of $K = \Bbbk(X)$ has at most one centre on X.*

Proof. See [Tim11, Appendix B] □

Valuative Criterion of Properness. *A morphism $\varphi \colon X' \to X$ of normal varieties is proper if and only if any geometric valuation of $K' = \Bbbk(X')$ has a centre on X' whenever the restriction to K has a centre on X.*

Proof. See [Tim11, Appendix B] □

Valuative Criterion of Completeness. *A normal variety X is complete if and only if any geometric valuation ν on K has a centre on X.*

Proof. Completeness is properness over a field, and every valuation has centre on $\operatorname{Spec} \Bbbk$. □

Proposition 2.11. *Let X be an affine model of K and let ν be a valuation on K. Then ν has a centre $Y \subseteq X$ if and only if $\nu|_{\Bbbk[X]} \geq 0$, in which case $\mathcal{I}(Y) = \Bbbk[X] \cap \mathfrak{m}_\nu$.*

Proof. If $\nu|_{\Bbbk[X]} \geq 0$, then the subvariety $Y = \mathcal{Z}(\Bbbk[X] \cap \mathfrak{m}_\nu)$ is a centre for ν. Conversely, if $Y \subseteq X$ is a centre for ν, then $\Bbbk[X] \subseteq \mathcal{O}_{X,Y} \subseteq \mathcal{O}_\nu$ so $\nu|_{\Bbbk[X]} \geq 0$ as required. Note that $\mathcal{I}(Y) \subseteq \mathfrak{m}_{X,Y} \subseteq \mathfrak{m}_\nu$, so $\mathcal{I}(Y) \subseteq \Bbbk[X] \cap \mathfrak{m}_\nu$. Let $f \in (\Bbbk[X] \cap \mathfrak{m}_\nu) \setminus \mathcal{I}(Y)$. Then f is invertible in $\mathcal{O}_{X,Y} \subseteq \mathcal{O}_\nu$. But $f \in \mathfrak{m}_\nu$ cannot be invertible in $\mathcal{O}_\nu$, so we have a contradiction. Thus $\mathcal{I}(Y) = \Bbbk[X] \cap \mathfrak{m}_\nu$ as required. □

Proposition 2.12. *A valuation $\nu \neq 0$ on K is geometric if and only if* $\operatorname{trdeg} \Bbbk(\nu) = \operatorname{trdeg} K - 1$.

Proof. [Tim11, Prop B.7] Suppose ν is geometric, i.e. $\nu = \nu_D$ up to multiples for some prime divisor D on some model X of K. Then $\Bbbk(\nu) = \Bbbk(D)$ has transcendence degree $\operatorname{trdeg} \Bbbk(D) = \dim D = \dim X - 1 = \operatorname{trdeg} K - 1$.

Let $n = \operatorname{trdeg} K$ and suppose that the residues of $f_1, \dots, f_{n-1} \in \mathcal{O}_\nu$ form a transcendence basis of $\Bbbk(\nu)$ over $\Bbbk$. Choose some nonzero $f_n \in \mathfrak{m}_\nu$. Then $f_1, \dots, f_n$ is a transcendence basis for K: indeed a nontrivial algebraic relation $g(f_1, \dots, f_{n-1}) = f_n$ over $\Bbbk$ gives rise to a relation $g(\bar{f}_1, \dots, \bar{f}_{n-1}) = 0$ in $\Bbbk(\nu)$, a contradiction. Now let X be the affine variety defined by the integral closure of $\Bbbk[f_1, \dots, f_n]$ in K. Since $\mathcal{O}_\nu$ is integrally closed and contains $f_1, \dots, f_n$, we have $\Bbbk[X] \subseteq \mathcal{O}_\nu$, so $\nu|_{\Bbbk[X]} \geq 0$ and hence by Proposition 2.11, ν has a centre Y on X. Now $f_1, \dots, f_{n-1} \in \Bbbk[Y]$ are algebraically independent, so Y is a prime divisor. Finally, since $\mathcal{O}_\nu = \mathcal{O}_{X,Y}$, we have $\nu = \nu_Y$ up to a constant, so ν is geometric. □

The above theorem allows us to show that not all discrete valuations are geometric:

Example. [Spi90, Ex 2.8] Let $K = \Bbbk(x, y) = \Bbbk(\mathbb{A}^2)$ be the function field of the affine plane. Let $q(t) \in \Bbbk[[t]]$ be a formal power series not algebraic over $\Bbbk[t]$ (e.g. the exponential series over $\mathbb{C}$). Let ν_t be the t-adic valuation on $\Bbbk((t))$, i.e. $\nu_t(q(t))$ is the degree of the first non-zero term in $q(t)$. We can embed $K = \Bbbk(x, y) \hookrightarrow \Bbbk((t))$ by mapping $x \mapsto t$ and $y \mapsto q(t)$. Now we pull back ν_t under this embedding to a valuation ν_q on K. Then the value group is $\mathbb{Z}$ and every non-constant element of $\Bbbk[x, y]$ has positive value. It follows that the residue field of ν_q is $\Bbbk$, i.e. $\operatorname{trdeg}_{\Bbbk} \Bbbk(\nu_q) = 0$. The above theorem shows that ν_q is therefore not geometric.

Theorem 2.8. *Let $K' \subseteq K$ be a subfield containing $\Bbbk$. Then:*

(i) *If ν is a geometric valuation of K, then $\nu' = \nu|_{K'}$ is geometric.*

(ii) *Any geometric valuation ν' of K' extends to a geometric valuation ν of K.*

Proof. [Tim11, Prop B.8] First, if ν is geometric, take $f_1, \dots, f_m \in \mathcal{O}_\nu$ whose residues in $\Bbbk(\nu)$ form a transcendence basis of $\Bbbk(\nu)$ over $\Bbbk(\nu')$. Then the f_i are algebraically independent over K': indeed if not, one can take an algebraic dependence over $\mathcal{O}_{\nu'}$ with some term not in $\mathfrak{m}_{\nu'}$, which when passing to $\Bbbk(\nu')$ contradicts the assumption that these residues form a transcendence basis. Hence $m = \operatorname{trdeg}_{\Bbbk(\nu')} \Bbbk(\nu) \leq \operatorname{trdeg}_{K'} K$, giving:

$$\operatorname{trdeg} \Bbbk(\nu') = \operatorname{trdeg} \Bbbk(\nu) - m \geq \operatorname{trdeg} K - 1 - \operatorname{trdeg}_{K'} K = \operatorname{trdeg} K' - 1.$$

Since $\operatorname{trdeg} \Bbbk(\nu')$ cannot exceed $\operatorname{trdeg} K' - 1$, we have equality, and hence by Theorem 2.8, ν' is geometric.

Now if ν' is geometric, we can take a complete normal variety X' with a prime divisor D' such that ν' is proportional to $\nu_{D'}$. Now let X be any complete model of K and note that since $\Bbbk(X') = K' \subseteq K = \Bbbk(X)$, there is a rational map $\varphi\colon X \dashrightarrow X'$. Then the graph of φ is a locally closed subvariety of $X \times X'$: let Z be the normalisation of its closure. Then Z is a complete normal variety with $\Bbbk(Z) = K$, and we can take a divisor $D \subseteq Z$ to be a component of the preimage of D' which maps onto D', so that $\nu = \nu_D$ extends ν'. □

G-Valuations

Since this thesis concerns varieties equipped with reductive group actions, we also need to consider how these geometric valuations behave with respect to an action. With this in mind, consider the following:

Definition 2.32. Let a reductive group G act birationally on K, i.e. such that K is the function field of some G-variety X. A valuation ν of K is called *G-invariant* if $\nu(g \cdot f) = \nu(f)$ for all $g \in G$ and all $f \in K$. A *G-valuation* is a G-invariant geometric valuation. We denote by $\mathcal{V} = \mathcal{V}(K)$ the set of all G-valuations of K.

Our first result, due to Sumihiro, shows that we can approximate arbitrary geometric valuations using G-valuations:

Sumihiro's approximation. *[Sum74, §4] Let ν be a geometric valuation of K. Then there exists a G-valuation $\bar{\nu}$ such that for any $f \in K$ we have $\bar{\nu}(f) = \nu(gf)$ for general $g \in G$.*

Proof. [Tim11, Prop 19.2] Let X be a model of K with prime divisor D such that $\nu = \nu_D$. Then $\nu' = \nu_{G\times D}$ is a geometric valuation on $\Bbbk(G\times X)$. From any $f \in \Bbbk(G\times X)$ and some fixed $g \in G$, we can obtain a rational function $f(g, -) \in \Bbbk(X)$ to which we can apply ν. The order of vanishing of f along $G \times D$ should only be different from the order of $f(g, -)$ along D for choices of g in some closed subset. Hence for general $g \in G$ we have $\nu'(f) = \nu(f(g, -))$. Now the action of G on X (a morphism $G \times X \to X$) induces an embedding $\Bbbk(X) \hookrightarrow \Bbbk(G \times X)$, so we can define $\bar{\nu} = \nu'|_{\Bbbk(X)}$. Then $\bar{\nu}$ is geometric by Theorem 2.8, it is G-invariant by construction, and we have $\bar{\nu}(f) = \nu(gf)$ for general $g \in G$ by the above argument. □

Corollary 2.5. *Let $K' \subseteq K$ be a G-subfield. The restriction of a G-valuation of K to K' is a G-valuation, and any G-valuation of K' can be extended to a G-valuation of K.*

Proof. [Tim11, Cor 19.6] Let ν be a G-valuation of K. By Theorem 2.8, $\nu' = \nu|_{K'}$ is geometric, and it clearly remains G-invariant, proving the first claim. Now let ν' be some G-valuation of K' and extend it to K using Theorem 2.8. We obtain a geometric valuation ν on K, and Sumihiro's approximation tells us that there exists a G-valuation $\bar{\nu}$ of K with $\bar{\nu}(f) = \nu(gf)$ for general $g \in G$ and any $f \in K$. In particular if $f \in K'$ then $\bar{\nu}(f) = \nu(gf) = \nu'(gf) = \nu'(f)$, so $\bar{\nu}$ extends ν' and we are done. □

Corollary 2.6. *[Tim11, Cor 19.7] Let X be a G-model of K and let L be a G-line bundle on X. For any G-valuation ν of K, any $\sigma, \eta \in H^0(X, L)$ with $\eta \neq 0$ and any $g \in G$, we have $\nu(\sigma/\eta) = \nu(g\sigma/\eta)$.*

Proof. Let $R = \bigoplus_{n\geq 0} H^0(X, L^{\otimes n})$. Then R is a rational G-algebra because L is a G-line bundle. The quotient field of R consists of functions which can be represented by a ratio of sections of some $L^{\otimes n}$ and, since $\eta \neq 0$, we can write this field as $K'(\eta)$ for some subfield $K' \subseteq K$. Now by Corollary 2.5 we can extend ν from K' to R. Then we have $\nu(\sigma/\eta) = \nu(\sigma) - \nu(\eta) = \nu(g\sigma) - \nu(\eta) = \nu(g\sigma/\eta)$ since ν is now a G-valuation on R. □

Theorem 2.9. *Any G-valuation is proportional to ν_D for a G-stable prime divisor D on some G-model X of K.*

Proof. [Tim11, Prop 19.8] Let ν be a G-valuation of K and choose $f_1, \ldots, f_s \in \mathcal{O}_\nu$ whose residues generate $\Bbbk(\nu)$. Let X be a normal projective G-model of K and let L be a G-line bundle on X such that $f_i = \sigma_i/\sigma_0$ for some $\sigma_i, \sigma_0 \in H^0(X, L)$. Let $M \subseteq H^0(X, L)$ be the G-submodule generated by $\sigma_0, \ldots, \sigma_s$.

We obtain a rational map $\varphi\colon X \dashrightarrow \mathbb{P}(M^*)$ which is G-equivariant by construction. Taking the closure of the graph of φ and its normalisation, we obtain a normal projective G-model with a G-equivariant *morphism* φ to $\mathbb{P}(M^*)$, so we can replace X with this variety. We can assume that $\nu|_{\Bbbk(X')}$ has a centre Y on $X' = \varphi(X)$ since this variety is complete. Since $\nu(f_i) \geq 0$, by Corollary 2.6 $\nu(g \cdot \sigma_i/\sigma_0) = \nu(\sigma_i/\sigma_0) \geq 0$ for all $g \in G$ and all i, so ν is non-negative on M/σ_0.

Hence Y intersects the affine chart X'_{σ_0}, and so $f_1, \ldots, f_s \in \mathcal{O}_{X',Y}$. Pulling back, this means that if D is the centre of ν on X, then $f_1, \ldots, f_s \in \mathcal{O}_{X,D}$, and since the residues of these generate $\Bbbk(\nu)$, D has codimension 1 and is therefore a divisor. □

Lemma 2.5. *Let $A \subseteq K$ be a rational G-algebra. Then for any $\nu \in \mathcal{V}$ and any $f \in A$, if $M \subseteq A$ is the G-submodule generated by f, we have $\nu(f) = \min_{\tilde{f}\in M^{(B)}} \nu(\tilde{f})$.*

Proof. [Tim11, Lemma 19.10] As M is generated by f, we have $\nu(g) \geq \nu(f)$ for any $g \in M$, so we just need to show that there exists $\tilde{f} \in M^{(B)}$ with equality. By Proposition 2.3, $M^{(B)}$ generates M, and hence $\nu(f) \geq \min_{\tilde{f}\in M^{(B)}} \nu(\tilde{f})$ and we are done. □

The following lemma of Knop [Kno93, Cor 3.5] allows us to replace many functions with B-eigenfunctions, and will be used frequently:

Knop's Lemma. *For any G-valuation ν of K and any rational function $f \in K_B$ there exists $\tilde{f} \in K^{(B)}$ such that:*

- $\nu(\tilde{f}) = \nu(f)$;
- $w(\tilde{f}) \geq w(f)$ *for all* G*-valuations* w *of* K;
- $\nu_D(\tilde{f}) \geq \nu_D(f)$ *for all* $D \in \mathcal{D}^B$.

Proof. [Tim11, Lemma 19.12] Let X be a G-model of K and let $\delta = \operatorname{div}_\infty f$ be the (B-stable) divisor of poles of f on X. Then δ is an effective divisor, so $H^0(X, \mathcal{O}(\delta))$ contains a canonical section η_δ. Let $\sigma = f\eta_\delta \in H^0(X, \mathcal{O}(\delta))$. Extend all G-valuations of K to the rational G-algebra $R = \bigoplus_{n\geq 0} H^0(X, \mathcal{O}(n\delta))$ and consider the G-submodule $M \subseteq H^0(X, \mathcal{O}(\delta))$ generated by σ. For any B-eigensection $\tilde{\sigma} \in M^{(B)}$, set $\tilde{f} = \tilde{\sigma}/\eta_\delta$. Then $\nu_D(\tilde{f}) \geq \nu_D(f)$ for all $D \in \mathcal{D}^B$, $w(\tilde{f}) \geq w(f)$ for all $w \in \mathcal{V}$, and by Lemma 2.5, there exists $\tilde{\sigma}$ such that $\nu(\tilde{f}) = \nu(f)$. □

Corollary 2.7. *A* G*-valuation of* K *is uniquely determined by its restriction to* $K^{(B)}$.

Proof. [Tim11, Cor 19.13] By Corollary 2.4 any G-model X of K admits a B-stable affine open subset X_0. Let $f \in K = \Bbbk(X)$ be regular on X_0. Then the divisor of poles of f lies in $Y = X \setminus X_0$. If $\operatorname{codim} Y \geq 2$, then by normality f is regular on X, so its (empty) divisor of poles is B-stable, and $f \in K_B$. If $\operatorname{codim} Y = 1$, then Y is a union of B-stable prime divisors, one of which must be the divisor of poles of f, hence $f \in K_B$ here too. It follows that K is the quotient field of K_B. Now if ν_1, ν_2 are G-valuations which differ on K, they must in fact differ on K_B, so we can choose $f \in K_B$ with $\nu_1(f) < \nu_2(f)$. Then Knop's Lemma gives $f' \in K^{(B)}$ with $\nu_1(f') < \nu_2(f')$ and we are done. □

Corollary 2.8. *A* G*-valuation* ν *of* K *is determined by its restriction to* K^B *and a functional on the weight lattice* Λ.

Proof. By Corollary 2.7, ν is determined by its restriction to $K^{(B)}$. The splitting of the exact sequence in Proposition 2.5 shows that this restriction is determined by its value on K^B and a functional $\ell_\nu \colon \chi \mapsto \nu(e(\chi))$ on Λ, where $e \colon \Lambda \to K^{(B)}$ is the chosen splitting of the exact sequence. □

2.4.2 Luna-Vust Theory

Overview

The well-known combinatorial description of toric varieties in terms of fans, and indeed the more general description of spherical varieties with coloured fans turn out to be special cases of a very general theory of embeddings of homogeneous spaces, first described by Luna and Vust [LV83] and later developed by Knop [Kno93] and Timashev [Tim97]. We will develop this theory here.

We keep the set-up of the last subsection: K is a finitely generated field extension of $\Bbbk$ with a birational action of a connected and reductive algebraic group G with a fixed Borel subgroup B, maximal unipotent subgroup U and maximal torus T. This fixes a G-birational class of G-models of K, and the theory uses properties of B-eigenfunctions and G-valuations on K to classify these G-models up to G-isomorphism. The classification of G-birational classes of G-models is achieved using Galois cohomology, see [PV94, §2]. The Luna-Vust theory classifies G-models in terms of *coloured data* associated to *G-germs*, which give local information around closed G-stable subvarieties, and *B-charts*, which are open affine B-stable subvarieties.

The key idea in this approach is that all G-models of K can be glued together into a scheme $\mathbb{X} = \mathbb{X}(K)$, which allows us to study them all at once. The points of $\mathbb{X}$ correspond to localisations of finitely generated $\Bbbk$-algebras whose quotient field is K, i.e. all possible local rings of points in G-models of K. Thus any G-model X can be considered as a subset of $\mathbb{X}$ consisting of the points corresponding to its local rings.

Since G acts on K birationally, this action moves around the local subrings of K, i.e. the points of $\mathbb{X}$. So G acts on $\mathbb{X}$ as a set, but this is not an action in the category of schemes. The action map $G \times \mathbb{X} \to \mathbb{X}$ is rational but not necessarily a morphism.

Example. [Tim11, Ex 12.1] Let $G = \Bbbk$ act on $\mathbb{A}^1$ by translations, so that we have a birational action of G on $K = \Bbbk(t)$. Let X be the cuspidal curve given by the equation $y^2 = x^3$. Setting $t = y/x$ shows that X is also a model of K. Consider the singular point $x_0 = (0, 0)$ on X. Its local ring $\mathcal{O}_{x_0}$ consists of rational functions in $x = t^2$ and $y = t^3$ whose denominator has nonzero constant term.

Let $\mathbb{X} = \mathbb{X}(K)$ and let $\alpha\colon G \times \mathbb{X} \to \mathbb{X}$ be the action map. We have an induced map $\alpha^*\colon K \to \Bbbk(G \times \mathbb{X})$. Then, letting u be a co-ordinate on $G = \Bbbk$, we have that

$\alpha^*(t) = u + t$. Then $\alpha^*(t^d) = u^d + u^{d-1}t + \ldots + t^d$. This function is not defined at $0 \times x_0 \in G \times \mathbb{X}$ because t is not defined at x_0. It follows that the action map α is not a morphism of schemes, but it is regular away from x_0. Indeed all other points of $\mathbb{X}$ are identified via the normalisation $X \to \mathbb{A}^1$ with points of $\mathbb{A}^1$ where G acts regularly.

In the above example we see that there is a maximal G-stable open subset of $\mathbb{X}$ on which G acts as an algebraic group. This is the case in general [Tim11, Prop 12.1] and we denote this subset by $\mathbb{X}_G$. All of the rest of the theory takes place in $\mathbb{X}_G$, and all G-models are realised as G-stable Noetherian separated open subsets of $\mathbb{X}_G^{\text{norm}}$, the subset of normal points in $\mathbb{X}_G$.

The local ring of a closed subvariety determines much of the local geometry in a neighbourhood of that subvariety, and for a G-stable closed subvariety $Y \subseteq X$ of some G-model X, the local ring $\mathcal{O}_{X,Y}$ corresponds to some G-fixed point or G-stable subvariety of $\mathbb{X}_G$. Since they determine local properties of G-models, we refer to such stable points in $\mathbb{X}_G$ as *G-germs*, in analogy with the germ of a function.

Definition 2.33. A *G-germ* of K is a point of $\mathbb{X}_G$ fixed by G, or a G-stable subvariety of $\mathbb{X}_G$. We denote the set of G-germs in $\mathbb{X}_G$ by ${}_G\mathbb{X}$, and similarly if $X \subseteq \mathbb{X}_G$ is an open subset, the set of G-germs contained in X is denoted ${}_GX$. A *geometric realisation* of a G-germ is a G-model X such that the G-germ is contained in ${}_GX$, i.e. the G-germ intersects X in a G-stable subvariety Y.

Any G-germ admits a geometric realisation: we can find an affine open neighbourhood $X_0 \subseteq \mathbb{X}_G$ of the G-germ, and then $X := G \cdot X_0$ is a G-model realising the germ. The idea now is that we can determine an open subset $X \subseteq \mathbb{X}$ by its G-germs and use properties of these G-germs to prove facts about X.

For example, the *support* $\mathcal{S}_Y$ of a G-germ Y is the set of G-valuations which have centre on Y in any geometric realisation of Y. The support of any G-germ Y is non-empty: take a geometric realisation X of Y, then let $\tilde{X}$ be the normalisation of the blow up of X along Y. Take the preimage in $\tilde{X}$ of the exceptional divisor of the blow up. Any of its irreducible components D then defines a valuation ν_D which has centre on Y since the normalised blow-up gives a dominant morphism $D \to Y$.

We have a valuative criterion of separation:

Theorem 2.10. *A G-stable open subset $X \subseteq \mathbb{X}_G$ is separated if and only if the supports of all its G-germs are disjoint.*

Proof. [Tim11, Tjm 12.11] Suppose X is not separated and let Δ be the diagonal in $X \times X$. Then $\overline{\Delta}$ is a G-model of K. The two projections $\pi_i \colon X \times X \to X$ induce birational G-equivariant morphisms $\overline{\Delta} \to X$. Now let $Y \subseteq \overline{\Delta} \setminus \Delta$ be a G-orbit (such an orbit exists since Δ is G-stable in $\overline{\Delta}$). Then $Y_i = \pi_i(Y)$ for $i = 1, 2$ are distinct G-orbits (hence distinct G-germs) on X, but by Proposition 2.10 and the remarks above we have $\mathcal{S}_{Y_1} \cap \mathcal{S}_{Y_2} \supseteq \mathcal{S}_Y \neq \emptyset$.

If X is separated, then by the Valuative Criterion of Separation, any valuation on X has at most one centre, so supports of G-germs on X are disjoint. □

This tells us that a G-model X of K is determined by a Noetherian open subset ${}_GX \subseteq {}_G\mathbb{X}$ of germs with disjoint support. Hence X can be covered by finitely many G-translates of local neighbourhoods of G-germs. We want to find such a cover consisting of particularly simple and easy to work with sets, and this is where *B-charts* are useful. We will define these in the next subsection after the following discussion on morphisms.

Suppose $K' \subseteq K$ is a subfield containing $\Bbbk$. The inclusion induces a dominant rational map $\varphi \colon \mathbb{X} \dashrightarrow \mathbb{X}' = \mathbb{X}(K')$. Taking models $X \subseteq \mathbb{X}$ and $X' \subseteq \mathbb{X}'$, φ restricts to a dominant rational map $X \dashrightarrow X'$.

Proposition 2.13. *Let K' now be a G-subfield of K, and let X' and X be G-models of K' and K respectively. The rational map $\varphi \colon X \dashrightarrow X'$ is regular if and only if for any G-germ $Y \subseteq X$, there exists a G-germ $Y' \subseteq X'$ such that $\mathcal{O}_{X,Y}$ dominates $\mathcal{O}_{X',Y'}$.*

Proof. [Tim11, Prop 12.12] If φ is regular, we can set $Y' = \overline{\varphi(Y)}$. Then Y' is G-stable since φ is G-equivariant by virtue of K' being a G-subfield, and $\mathcal{O}_{X,Y}$ dominates $\mathcal{O}_{X',Y'}$ by Proposition 2.10.

Conversely, if Y is a G-germ of X and Y' a G-germ of X' such that $\mathcal{O}_{X,Y}$ dominates $\mathcal{O}_{X',Y'}$, then there exist finitely generated subalgebras $A \subseteq A'$ in K such that $\mathcal{O}_{X,Y}$ and $\mathcal{O}_{X',Y'}$ are localisations of A and A'. Taking further localisations of A and A' allows us to assume that $X_0 = \operatorname{Spec} A$ and $X_0' = \operatorname{Spec} A'$ are open subsets of X and X' respectively. Then φ restricts to a regular map $X_0 \to X_0'$, and hence by equivariance to a regular map $G \cdot X_0 \to G \cdot X_0'$. Assuming we can do this with every G-germ Y of X, the sets $G \cdot X_0$ cover X and the induced maps glue to a regular map $X \to X'$. □

The inclusion $K' \subseteq K$ allows us to restrict valuations of K to K'. By Corollary 2.5, this induces a surjective map $\varphi_*\colon \mathcal{V}(K) \to \mathcal{V}(K')$. Then if $\varphi\colon X \to X'$ is a morphism, the above Proposition says that for any G-germ $Y \subseteq X$ there is a G-germ $Y' = \overline{\varphi(Y)} \subseteq X'$, and we have $\varphi_*(\mathcal{S}_Y) \subseteq \mathcal{S}_{Y'}$. This gives a valuative criterion of properness:

Theorem 2.11. *A morphism $\varphi\colon X \to X'$ of G-models is proper if and only if*

$$\bigcup_{Y \subseteq X} \mathcal{S}_Y = \varphi_*^{-1}\left(\bigcup_{Y' \subseteq X'} \mathcal{S}_{Y'}\right)$$

where Y and Y' run over all G-germs in X and X' respectively.

Proof. [Tim11, Thm 12.13] The 'only if' part of our theorem follows immediately from the Valuative Criterion of Properness, since the right hand side consists exactly of those G-valuations of K having centre on K'.

Now suppose the equality holds. We will construct a proper G-morphism $\bar{\varphi}\colon \overline{X} \to X'$ with X contained as an open subset in $\overline{X}$. Then if $\overline{X} = X$, φ is proper. If φ is not proper, then $\overline{X} \setminus X$, being non-empty, contains a G-orbit Y_0. Then, setting $Y' = \bar{\varphi}(Y_0)$, we have $\varphi_*(\mathcal{S}_{Y_0}) \subseteq \mathcal{S}_{Y'}$. Then $\mathcal{S}_{Y_0}$ has non-empty intersection with $\varphi_*^{-1}\left(\bigcup_{Y' \subseteq X'} \mathcal{S}_{Y'}\right) = \bigcup_{Y \subseteq X} \mathcal{S}_Y$. By Theorem 2.10, this contradicts separatedness of $\overline{X}$.

The morphism $\bar{\varphi}$ is constructed as follows: by [Sum74, Thm 3], there exist G-equivariant completions $\overline{X}$, $\overline{X}'$ of X and X'. Then φ gives a rational map $\overline{X} \dashrightarrow \overline{X}'$. Replace $\overline{X}$ by the closure of the graph of this rational map, and let $\bar{\varphi}$ be the projection to $\overline{X}'$. Then $\bar{\varphi}$ is proper since $\overline{X}$ and $\overline{X}'$ are complete. Finally, replace $\overline{X}$ with $\bar{\varphi}^{-1}(X')$, so that $\bar{\varphi}$ is the required proper morphism $\overline{X} \to X'$. □

B-charts

Definition 2.34. A *B-chart* is a B-stable affine open subset of $\mathbb{X}_G^{\mathrm{norm}}$, or a B-stable affine open subset of a particular G-model X.

By Corollary 2.4, any G-germ $Y \in {}_G X$ admits a B-chart $X_0 \subseteq X$ intersecting Y, so we can cover any G-model by the G-translates of finitely many B-charts. We will use the behaviour of these B-charts and their G-translates to distinguish G-models of K using combinatorial data. Let X be a G-model of K, and recall that $\mathcal{D}$ is the set of prime divisors of X which are not G-stable, and $\mathcal{D}^B \subseteq \mathcal{D}$ is the subset of B-stable divisors. Then $\mathcal{D}^B$ does not depend on the choice of G-model X. We also have the set $\mathcal{V}$ of G-valuations of K, and the pair $(\mathcal{V}, \mathcal{D}^B)$ is called the *coloured equipment* of K. Throughout we will identify prime divisors on G-models of K with their respective geometric valuations.

The next step is to associate *coloured data* to B-charts and G-models in order to classify them. Let X_0 be a B-chart, so $\Bbbk[X_0]$ is an integrally closed finitely generated domain, so in particular a Krull ring (see [Mat86, §12]). Then $\Bbbk[X_0]$ is the intersection in its fraction field of all its localisations at prime ideals of height 1, i.e. the local rings of the prime divisors of X_0. These prime divisors are either G-stable, so their valuation lies in $\mathcal{V}$, B-stable but not G-stable, so they lie in $\mathcal{D}^B$, or neither G-stable nor B-stable. Hence there are subsets $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{R} \subseteq \mathcal{D}^B$ and $\tilde{\mathcal{R}} = \mathcal{R} \sqcup \mathcal{D} \setminus \mathcal{D}^B$ such that

$$\Bbbk[X_0] = \bigcap_{w \in \mathcal{W}} \mathcal{O}_w \cap \bigcap_{D \in \tilde{\mathcal{R}}} \mathcal{O}_{\nu_D}.$$

It follows that the sets $\mathcal{W}$ and $\mathcal{R}$ determine the B-chart X_0 uniquely, and we call $(\mathcal{W}, \mathcal{R})$ the *coloured data* of X_0. Think of the coloured data of a B-chart as the collection of B- and G-stable divisors it intersects on a given G-model. Some other B-chart X_1 will have coloured data which differ from that of X_0 by finitely many elements. Call a pair $(\mathcal{W}, \mathcal{R}) \subseteq (\mathcal{V}, \mathcal{D}^B)$ *admissible* if it corresponds to some B-chart of $\mathbb{X}_G$. Then the sets $\mathcal{W} \sqcup \mathcal{R}$ for all admissible pieces of coloured data $(\mathcal{W}, \mathcal{R})$ lie in one equivalence class **CD** of $\mathcal{P}(\mathcal{V} \sqcup \mathcal{D}^B)$ under the equivalence relation "differ by finitely many elements". Now we want to find conditions on such subsets which guarantee admissibility.

For any pair $\mathcal{W} \sqcup \mathcal{R} \in \mathbf{CD}$, define the $\Bbbk$-algebra

$$\mathcal{A} = \mathcal{A}(\mathcal{W}, \mathcal{R}) := \bigcap_{w \in \mathcal{W}} \mathcal{O}_w \cap \bigcap_{D \in \tilde{\mathcal{R}}} \mathcal{O}_{\nu_D}.$$

Then $\mathcal{A}$ is a Krull ring: we must check that for all $f \in K^*$, $\nu(f) \neq 0$ for only finitely many $\nu \in \mathcal{W} \sqcup \mathcal{R}$. This is true since by definition of **CD**, $\mathcal{W} \sqcup \mathcal{R}$ differs from the coloured data of a B-chart by only finitely many elements, and the condition holds in this case.

As an example, consider the ring $\mathcal{A}(\emptyset, \emptyset)$: then $\tilde{R} = \mathcal{D} \setminus \mathcal{D}^B$ and $\mathcal{A}$ is the intersection of all valuation rings corresponding to all non-B-stable divisors. Thus $f \in \mathcal{A}$ if and only if $\nu_D(1/f) \leq 0$ for all non-B-stable divisors D, i.e. the divisor of poles of f is B-stable, and we see that $\mathcal{A} = K_B$.

We introduce some notation: for $f \in K$ and a subset $\mathcal{V}_0 \subseteq \mathcal{W} \sqcup \mathcal{R}$, we write $\langle \mathcal{V}_0, f \rangle \geq 0$ to mean $\nu(f) \geq 0$ for all $\nu \in \mathcal{V}_0$, where I identify divisors $D \in \mathcal{R}$ with their respective valuations ν_D.

Now $\operatorname{Spec} \mathcal{A}$ is a B-chart if and only if (1): $\operatorname{Quot} \mathcal{A} = K$ and (2): $\mathcal{A}$ is finitely generated. We say that ν is *essential for* $\mathcal{A}$ if removing $\mathcal{O}_\nu$ from the intersection defining $\mathcal{A}$ results in a different ring.

Proposition 2.14. *All valuations from $\tilde{\mathcal{R}}$ are essential for $\mathcal{A}$.*

Proof. [Tim11, Prop 13.7] We construct a function f lying in every valuation ring $\mathcal{O}_\nu$, $\nu \in \mathcal{W} \sqcup \tilde{\mathcal{R}}$ except for one, so that removing this one adds f to $\mathcal{A}$.

Let X be a smooth G-model of K. Let $D \in \tilde{\mathcal{R}}$ and consider the G-line bundle $L = \mathcal{O}_X(D)$. Let $\eta \in H^0(X, L)$ be a section with $\operatorname{div} \eta = D$. Now D is not G-stable, so choose $g \in G$ with $gD \neq D$, and let $f = g \cdot \eta / \eta$.

Since $\operatorname{div} g \cdot \eta = gD \neq D$, we have $\nu_D(f) = \nu_D(g \cdot \eta) - \nu_D(\eta) = 0 - 1 = -1$, so $f \notin \mathcal{O}_{\nu_D}$.

Since $\nu_{D'}(\eta) = 0$ for all $D' \neq D$, we have $\langle \tilde{\mathcal{R}} \setminus D, f \rangle \geq 0$, and $f \in \mathcal{O}_{\nu_{D'}}$ for all $D' \in \tilde{\mathcal{R}} \setminus D$.

Finally, Corollary 2.6 gives us $w(g \cdot \eta/\eta) = w(\eta/\eta) = 0$, i.e. $f \in \mathcal{O}_w$, for all $w \in \mathcal{W}$.

Hence $f \in \mathcal{O}_\nu$ for all $\nu \in \mathcal{W} \sqcup \tilde{\mathcal{R}} \setminus D$ but $f \notin \mathcal{O}_{\nu_D}$. Then $f \notin \mathcal{A}$ but $f \in \mathcal{A} \setminus \mathcal{O}_{\nu_D}$, so ν_D is essential for $\mathcal{A}$. □

Now assuming (1) and (2), and in light of the above proposition, the data $(\mathcal{W}, \mathcal{R})$ corresponds exactly to the B-stable and G-stable divisors of the B-chart $\operatorname{Spec} \mathcal{A}$ if and only if all $w \in \mathcal{W}$ are essential for $\mathcal{A}$. Thus coloured data corresponding to B-charts can be characterised as follows:

Theorem 2.12. *Let $\mathcal{A} = \mathcal{A}(\mathcal{W}, \mathcal{R})$ as above. Then:*

(i) $\operatorname{Quot} \mathcal{A} = K$ *if and only if*

(C): *for all finite subsets $\mathcal{V}_0 \subseteq \mathcal{W} \sqcup \mathcal{R}$, there exists $f \in K^{(B)}$ such that $\langle \mathcal{V}_0, f \rangle > 0$ and $\langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0$;*

(ii) $\mathcal{A}$ *is finitely generated if and only if*

(F): $\mathcal{A}^U := \Bbbk[f \in K^{(B)} \mid \langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0]$ *is finitely generated;*

(iii) *A valuation $w \in \mathcal{W}$ is essential for $\mathcal{A}$ if and only if*

(W): *there exists $f \in K^{(B)}$ such that $\langle \mathcal{W} \sqcup \mathcal{R} \setminus \{w\}, f\rangle \geq 0$ and $w(f) < 0$.*

Then $(\mathcal{W}, \mathcal{R})$ are the coloured data of a B-chart if and only if (C) and (F) are satisfied, and the data corresponds exactly to the G- and B-stable divisors of this B-chart if and only if (W) is satisfied for all $w \in \mathcal{W}$.

Proof. [Tim11, Thm 13.8] (i): Suppose $\operatorname{Quot} \mathcal{A} = K$. If $\mathcal{V}_0 = \{\nu_1, \ldots, \nu_n\}$ and we have $f_i \in K^{(B)}$ with $\nu_i(f_i) > 0$ and $\langle \mathcal{W} \sqcup \mathcal{R}, f_i\rangle \geq 0$, then $f = f_1 \cdots f_n$ has $\langle \mathcal{W} \sqcup \mathcal{R}, f\rangle \geq 0$ and $\nu_i(f) = \nu_i(f_i) + \sum_{j\neq i} \nu_i(f_j) \geq \nu_i(f_i) > 0$. Hence we can assume $\mathcal{V}_0 = \{\nu\}$.

Now choose $f \in \mathcal{A}$ with $\nu(f) > 0$ and $\langle \mathcal{W} \sqcup \mathcal{R}, f\rangle \geq 0$. Since $\mathcal{A} \subseteq K_B = \mathcal{A}(\emptyset, \emptyset)$, we can apply Knop's Lemma to obtain f' in $\mathcal{A}^{(B)}$ with $\nu(f') = \nu(f) > 0$ and $\langle \mathcal{W} \sqcup \mathcal{R}, f'\rangle \geq 0$, as required.

Conversely, assume (C) holds and let $h \in K_B$. Let $\mathcal{V}_0 = \{\nu \in \mathcal{W} \sqcup \mathcal{R} \mid \nu(h) < 0\}$. By (C) there exists $f \in K^{(B)}$ with $\langle \mathcal{V}_0, f\rangle > 0$, so we can take $n = \max_{\nu \in \mathcal{V}_0}\{-\nu(h)\}$, giving $hf^n \in \mathcal{A}$. Hence $K_B \subseteq \operatorname{Quot} \mathcal{A}$, and since we have $\mathcal{A} \subseteq K_B$ and $\operatorname{Quot} K_B = K$, this gives $\operatorname{Quot} \mathcal{A} = K$.

(ii): We first show that the definition in the Theorem does indeed give $\mathcal{A}^U$. Note that $\{f \in K^{(B)} \mid \langle \mathcal{W} \sqcup \mathcal{R}, f\rangle \geq 0\}$ is $\mathcal{A}^{(B)} = \mathcal{A} \cap K^{(B)}$. Now $U \subseteq B$ implies $\mathcal{A}^{(U)} \supseteq \mathcal{A}^{(B)}$, but since U has no nontrivial characters, we have $\mathcal{A}^U = \mathcal{A}^{(U)}$, hence it follows that $\mathcal{A}^U \supseteq \Bbbk[\mathcal{A}^{(B)}]$. Likewise if f is U-invariant, then B acts on f the same as T does, and hence f is B semi-invariant, i.e. $f \in \Bbbk[\mathcal{A}^{(B)}]$.

To prove the claim, let X be a smooth G-model of K and choose an effective divisor D on X with support $\mathcal{D}^B \setminus \mathcal{R}$ and associated line bundle L. Take a G-linearisation of L and choose a section $\eta \in H^0(X, L)^{(B)}$. Let

$$R_n = \{\sigma \in H^0(X, L^{\otimes n}) \mid \sigma/\eta^n \in \mathcal{A}\}$$

and consider the graded algebra $R = \bigoplus_{n\geq 0} R_n$.

By construction we have $\mathcal{A} = \bigcup_{n\geq 0} \eta^{-n} R_n$, which shows that $\mathcal{A} \subseteq \operatorname{Quot} R$, giving $K \subseteq \operatorname{Quot} R$. Then by Theorem 2.8, we can extend G-valuations of K to $\operatorname{Quot} R$. In particular, extending valuations in $\mathcal{W}$ to $\operatorname{Quot} R$, we can write

$$R_n = \{\sigma \in H^0(X, L^{\otimes n}) \mid w(\sigma) \geq nw(\eta)\ \forall w \in \mathcal{W}\}$$

since $\sigma/\eta^n \in \mathcal{A}$ implies $w(\sigma/\eta^n) \geq 0$. This definition makes it clear that each R_n is G-stable, hence so is R.

We use the fact that for connected reductive G, a rational G-algebra S is finitely generated if and only if S^U is finitely generated [Tim11, Thm. D.5(1)]. In our case, $\mathcal{A}$ is not a rational G-algebra, but we can construct related G-algebras as follows.

Suppose $\mathcal{A}$ is finitely generated by $f_1, \ldots, f_m$. Let S_n be the G-submodule of R_n generated by $\eta^n f_1, \ldots, \eta^n f_m$. Then $S = \bigoplus_{n\geq 0} S_n$ is a finitely generated graded G-algebra, so S^U is finitely generated. Then $\mathcal{A}^U = \bigcup_{n\geq 0} \eta^{-n} S_n^U$ is also finitely generated.

Finally, if $\mathcal{A}^U$ is finitely generated, say by $f_1, \ldots, f_m$, let S_n be the G-submodule of R_n generated by functions $\eta^n f$, where f runs over all G-translates of the f_i, and let $S = \bigoplus_{n\geq 0} S_n$. Then S is an integrally closed G-subalgebra of R, $\eta \in S_1$ and $\operatorname{Quot} S = \operatorname{Quot} R$. We have by construction $\mathcal{A}^U = \bigcup_{n\geq 0} \eta^{-n} S_n^U$, so S^U is finitely generated, and hence so is S. The algebra $\mathcal{A}' = \bigcup_{n\geq 0} \eta^{-n} S_n$ is finitely generated and has the form $\mathcal{A}' = \mathcal{A}(\mathcal{W}', \mathcal{R})$ for some $\mathcal{W}' \supseteq \mathcal{W}$. We also have $\mathcal{A}^U = (\mathcal{A}')^U$ by construction.

Let $f \in \mathcal{A}$ and $w \in \mathcal{W}'$. If $w(f) < 0$, then we can replace f with a B-eigenfunction which must then lie in $\mathcal{A}^{(B)} \backslash (\mathcal{A}')^{(B)}$, a contradiction since $\mathcal{A}^{(B)} = \mathcal{A}^U = (\mathcal{A}')^U = (\mathcal{A}')^{(B)}$. Hence $w(f) \geq 0$ for all $w \in \mathcal{W}$, from which it follows that $\mathcal{A} = \mathcal{A}'$ is finitely generated.

(iii): If (W) holds, then it is clear that $f \notin \mathcal{A}$ but $f \in \mathcal{A} \setminus \{\mathcal{O}_w\}$, so w is essential for $\mathcal{A}$. Now suppose w is essential. Then there exists $f \in K$ with $\langle \mathcal{W} \sqcup \mathcal{R} \setminus \{w\}, f \rangle \geq 0$ and $w(f) < 0$. By Knop's Lemma, we can replace f with a B-eigenfunction to obtain (W).

The final claim now follows from discussions above. □

G-germs

As above for B-charts, we now describe G-germs in terms of the coloured equipment $(\mathcal{V}, \mathcal{D}^B)$ of K. Let $Y \in {}_G\mathbb{X}^{\mathrm{norm}}$ be a G-germ. The *coloured data* of Y consists of the sets $\mathcal{V}_Y \subseteq \mathcal{V}$, $\mathcal{D}^B_Y \subseteq \mathcal{D}^B$ of valuations corresponding to G- and B-stable prime divisors containing Y on any geometric realisation X of Y.

As mentioned previously, any G-germ intersects a B-chart, so consider a geometric realisation $Y \subseteq X$ of a given G-germ and a B-chart $X_0 \subseteq X$ intersecting Y. Then $Y_0 := Y \cap X_0$ is the centre of any valuation $\nu \in \mathcal{S}_Y$, and conversely if a G-valuation

$\nu \in \mathcal{V}$ is non-negative on $\mathcal{A} = \Bbbk[X_0]$, then it determines a G-germ intersecting X_0. If $(\mathcal{W}, \mathcal{R})$ are the coloured data of X_0, then $(\mathcal{V}_Y, \mathcal{D}_Y^B) \subseteq (\mathcal{W}, \mathcal{R})$. We then have:

Proposition 2.15. *Let Y be a G-germ with coloured data $(\mathcal{V}_Y, \mathcal{D}_Y^B)$ and support $\mathcal{S}_Y$. Then:*

(i) *Y is uniquely determined by $(\mathcal{V}_Y, \mathcal{D}_Y^B)$, and:*

(ii) *A G-valuation ν is in $\mathcal{S}_Y$ if and only if*
(S): *for all $f \in K^{(B)}$, $\langle \mathcal{V}_Y \sqcup \mathcal{D}_Y^B, f\rangle \geq 0$ implies $\nu(f) \geq 0$, and strictness of any of the lefthand inequalities implies strictness of the righthand inequality.*

Proof. [Tim11, Prop 14.1] Let X be a geometric realisation of the G-germ Y and let $X_0 \subseteq X$ be a B-chart intersecting Y.

We claim that for all $f \in K^{(B)}$, we have $f \in \mathcal{O}_{X,Y}^{(B)} \iff \langle \mathcal{V}_Y \sqcup \mathcal{D}_Y^B, f\rangle \geq 0$ and $f \in \mathfrak{m}_{X,Y}^{(B)}$ if and only if one of the inequalities is strict. Indeed, $f \in \mathcal{O}_{X,Y}^{(B)}$ if and only if f is regular in a neighbourhood of Y. Now the divisor of poles of f is B-stable, so f is regular on all non-B-stable divisors, and f is regular on all other prime divisors containing Y if and only if $\langle \mathcal{V}_Y \sqcup \mathcal{D}_Y^B, f\rangle \geq 0$. The claim on $\mathfrak{m}_{X,Y}$ follows from the fact that the divisor of zeroes of f is B-stable, so its valuation lies in $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$.

(ii): Now let $\nu \in \mathcal{S}_Y$, i.e. ν has centre on Y. Then $\mathcal{O}_\nu$ dominates $\mathcal{O}_{X,Y}$. By the above claim and this fact, we have $\langle \mathcal{V}_Y \sqcup \mathcal{D}_Y^B, f\rangle \geq 0 \iff f \in \mathcal{O}_{X,Y}^{(B)} \implies f \in \mathcal{O}_\nu \implies \nu(f) \geq 0$, and there exists $\nu' \in \mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ with $\nu'(f) > 0 \iff f \in \mathfrak{m}_{X,Y}^{(B)} \implies f \in \mathfrak{m}_\nu$, i.e. (S) holds.

Conversely, let $\nu \in \mathcal{V}$ and assume assume (S) holds for ν. Suppose there exists $f \in \mathcal{O}_{X,Y}$ such that $\nu(f) < 0$. We can apply Knop's Lemma to replace f by a B-eigenfunction, but then $f \in K^{(B)}$, $\langle \mathcal{V}_Y \sqcup \mathcal{D}_Y^B, f\rangle \geq 0$, $\nu(f) < 0$, contradicting (S). Hence $\mathcal{O}_\nu \supseteq \mathcal{O}_{X,Y} \supseteq \Bbbk[X_0]$, and ν has some centre $Y' \supseteq Y$ on X. It remains to show that $Y' = Y$. If $Y' \neq Y$, then $\mathfrak{m}_\nu \not\supseteq \mathfrak{m}_{X,Y}$, so we can choose $f \in \mathfrak{m}_{X,Y}$ such that $\nu(f) = 0$. Replacing f with a B-eigenfunction again, note that $f \in \mathfrak{m}_{X,Y}^{(B)}$, so by the previous claim there is $\nu' \in \mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ with $\nu'(f) > 0$, contradicting (S). Hence $Y = Y'$ and we are done.

(i): The G-germ Y is determined by its local ring $\mathcal{O}_{X,Y}$. Let $\mathcal{A} = \mathcal{A}(\mathcal{V}_Y, \mathcal{D}_Y^B)$, so that we have $\Bbbk[X_0] \subseteq \mathcal{A} \subseteq \mathcal{O}_{X,Y}$. Now $\mathcal{O}_{X,Y}$ is the localisation of $\mathcal{A}$ in the ideal $\mathcal{I}_Y = \mathcal{A} \cap \mathfrak{m}_{X,Y}$, hence determined in $\mathcal{A}$ by $\mathcal{I}_Y$. In turn, $\mathcal{I}_Y$ is determined by the fact that $\nu > 0$ on $\mathcal{I}_Y$ for all $\nu \in \mathcal{S}_Y$. Finally, the equivalence of $\nu \in \mathcal{S}_Y$ with condition (S) shown above demonstrates that $\mathcal{S}_Y$ is determined by the coloured data $(\mathcal{V}_Y, \mathcal{D}_Y^B)$, and hence so is Y. □

The last step in classifying G-models is to determine which G-germs are contained in a given B-chart $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$, since a G-model X is determined by the G-germs in ${}_G X$. That is, we want conditions on elements of $\mathcal{W} \sqcup \mathcal{R}$ which determine whether they are contained in $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$. These are:

Theorem 2.13. *Let X_0 be a B-chart with coloured data $(\mathcal{W}, \mathcal{R})$ and let Y be a G-germ with coloured data $(\mathcal{V}_Y, \mathcal{D}_Y^B)$. Then:*

(i) *A G-valuation $\nu \in \mathcal{V}$ has a centre on X_0 if and only if*
(V): *for all $f \in K^{(B)}$, $\langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0$ implies $\nu(f) \geq 0$;*

(ii) *Let $\nu \in \mathcal{S}_Y$. Then a G-valuation $w \in \mathcal{W}$ belongs to $\mathcal{V}_Y$ if and only if*
(V′): *for all $f \in K^{(B)}$, $\langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0$ and $\nu(f) = 0$ imply $w(f) = 0$;*

(iii) *A divisor $D \in \mathcal{R}$ belongs to $\mathcal{D}_Y^B$ if and only if*
(D): *for all $f \in K^{(B)}$, $\langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0$ and $\nu(f) = 0$, imply $\nu_D(f) = 0$.*

Proof. [Tim11, Thm 14.2] (i): By Proposition 2.11, $\nu \in \mathcal{V}$ has a centre on X_0 if and only if $\nu|_{\mathcal{A}} \geq 0$. Supposing this is the case, let $f \in K^{(B)}$ satisfy $\langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0$, that is $f \in \mathcal{O}_\nu$ for all $\nu \in \mathcal{W} \sqcup \mathcal{R}$. Since f has B-stable divisor of poles, we also have $f \in \mathcal{O}_{\nu_D}$ for all $D \in \mathcal{D} \setminus \mathcal{D}^B$. Hence $f \in \mathcal{A}$ and $\nu(f) \geq 0$ as required.

Now assume (V) holds for some $\nu \in \mathcal{V}$ and suppose there exists $f \in \mathcal{A}$ with $\nu(f) < 0$. By virtue of being in $\mathcal{A}$, we have $\langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0$, and $f \in K_B$. Use Knop's Lemma to replace f with a B-eigenfunction, we obtain a contradiction with (V). So $\nu(f) > 0$ and ν has centre on X_0.

(ii): Let $w \in \mathcal{W}$ be the valuation of a G-stable prime divisor $D \subseteq X_0$, and suppose $w \in \mathcal{V}_Y$, i.e. $Y \subseteq D$. Let $f \in \mathcal{A}$, so $\langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0$. If $\nu(f) = 0$, i.e. f does not vanish on Y, then f also does not vanish on $D \supseteq Y$, i.e. $w(f) = 0$. Now restricting to $\mathcal{A}^{(B)}$ and noting that $K^{(B)}$ is its quotient field, (V′) follows.

Now assume (V′) and let $w \in \mathcal{W}$ be as above. Suppose $w \notin \mathcal{V}_Y$, i.e. D does not contain Y. Then there is $f \in \mathcal{A}$ vanishing on D but not in Y, i.e. $w(f) > 0$

but $\nu(f) = 0$. Replacing f by a B-eigenfunction using Knop's Lemma, we obtain a contradiction with (V′), so $w \in \mathcal{V}_Y$.

(iii): This is identical to (ii) but with B-stable divisors instead of G-stable, which makes no difference to the proof. □

G-models

In full, the Luna-Vust theory allows us to classify G-models of a function field K as follows [Tim11, §14.2]:

- Take a finite collection $(\mathcal{W}_i, \mathcal{R}_i)$ of coloured data satisfying (C) and (F), and reduce the $\mathcal{W}_i$ if necessary so that they satisfy (W). These define finitely many B-charts X_i;
- Using (V), (V′) and (D), compute the coloured data $(\mathcal{V}_Y, \mathcal{D}_Y^B)$ of all G-germs intersecting the X_i from $(\mathcal{W}_i, \mathcal{R}_i)$;
- Compute the supports of these G-germs from the coloured data $(\mathcal{V}_Y, \mathcal{D}_Y^B)$ using (S);
- If and only if the supports are disjoint, the G-models $X_i' := G \cdot X_i$ can be glued together into a G-model X. Then X is uniquely determined among G-models by the coloured data of its G-germs, and all G-models arise in this way.

Remark. [Tim11, 14.3] Here is a good place to remark that the collection X_i of B-charts covering a G-model is not uniquely determined, and a given G-germ may intersect many B-charts. Indeed, suppose X is a G-model, Y is a G-germ in X, ν is a valuation in $\mathcal{S}_Y$, and X_0 is a B-chart intersecting Y with coloured data $(\mathcal{W}, \mathcal{R})$. Take some $f \in \mathcal{A}^{(B)}$ with $\nu(f) = 0$. Then the open subset of X_0 given by localising in f is a new B-chart intersecting Y. Its coloured data is obtained from $(\mathcal{W}, \mathcal{R})$ by removing the finite subsets $(\mathcal{W}_f, \mathcal{R}_f)$ of valuations which are positive on f. These subsets lie in $\mathcal{W} \setminus \mathcal{V}_Y$ and $\mathcal{R} \setminus \mathcal{D}_Y^B$, and in particular if these two sets are finite, then Y admits a *minimal* B-chart X_Y with coloured data $\mathcal{W} = \mathcal{V}_Y$, $\mathcal{R} = \mathcal{D}_Y^B$.

We can determine some facts about a G-model from the coloured data quite easily. For example:

Proposition 2.16. *Affine G-models are in bijection with coloured data $(\mathcal{W}, \mathcal{D}^B)$ satisfying* (C),(F) *and* (W).

Proof. Any affine G-model is in particular also a B-chart. Indeed, the affine G-models are exactly the G-stable B-charts. Given an arbitrary B-chart $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$, let $X = G \cdot X_0$. Then $X \setminus X_0$ is B-stable but not G-stable, hence a union of colours. These colours must be exactly those in $\mathcal{D}^B \setminus \mathcal{R}$, since they don't intersect X_0. A B-chart is G-stable if and only if $X = X_0$, i.e. $\mathcal{R} = \mathcal{D}^B$, from which the claim follows. □

Proposition 2.17. *A G-model X is complete if and only if $\bigcup_Y \mathcal{S}_Y = \mathcal{V}$, where Y runs over all G-germs in X.*

Proof. This follows immediately from Theorem 2.11, since X is complete if and only if it is proper over $\operatorname{Spec} \Bbbk$, and $\mathcal{V}(\Bbbk)$ is a point supporting the unique G-germ of $\operatorname{Spec} \Bbbk$. □

Chapter 3

Combinatorial Description of Smooth Fano SL_2-Threefolds

3.1 Varieties of Complexity One

3.1.1 Hyperspace, Divisors and Functionals

We can now approach the topic of finding a combinatorial description of complexity one G-varieties. This work was mainly completed by Timashev, and this section largely follows [Tim11, Tim97].

Hyperspace

Now we assume that X is a normal G-variety of complexity one, where G as always is connected and reductive. Recall that by Corollary 2.8, a G-valuation of $K = \Bbbk(X)$ is determined by a functional on Λ and the restriction to K^B.

Since K^B has transcendence degree 1, it is the function field common to a birational class of curves which contains a unique smooth projective curve C. The local rings $\mathcal{O}_{C,p}$ at points $p \in C$ are regular local rings of dimension 1, i.e. discrete valuation rings. Hence to any point $p \in C$ is associated a valuation ν_p of K^B, the order of vanishing of a rational function at p. These and their positive rational multiples constitute all geometric valuations of K^B, so the restriction to K^B of a G-valuation ν of K must be of the form $h\nu_p$ for some $h \in \mathbb{Q}_{\geq 0}$ and some $p \in C$, since the restriction of a geometric valuation to a subfield is itself geometric (Theorem 2.8). If $h = 0$, then the point

p is irrelevant and the restriction to K^B is trivial. We call such valuations *central*. Otherwise the pair (p, h) uniquely determines $\nu|_{K^B}$.

Hence G-valuations $\nu \in \mathcal{V}$ are described by triples (p, h, ℓ) where $p \in C$, $h \in \mathbb{Q}_{\geq 0}$ and $\ell \in \mathrm{Hom}(\Lambda, \mathbb{Q}) = \mathcal{Q}$. We thus view G-valuations as elements of the space $\bigcup_{p \in C}\{p\} \times \mathbb{Q}_{\geq 0} \times \mathcal{Q}$, and these valuations are uniquely determined up to the equivalence relation defined by letting $(p, h, \ell) \sim (p', h', \ell')$ if and only if we have equality or $h = h' = 0$ and $\ell = \ell'$. We define the *hyperspace* of K to be

$$\mathcal{H} := \bigcup_{p \in C}(\{p\} \times \mathbb{Q}_{\geq 0} \times \mathcal{Q})/ \sim .$$

By the previous discussion, the set $\mathcal{V}$ of G-valuations of K is embedded in $\mathcal{H}$, and we can also map $\mathcal{D}^B$ into $\mathcal{H}$ (not necessarily injectively) by sending a divisor to the point of $\mathcal{H}$ corresponding to its associated valuation. We will refer to this map by κ, and we call the collection $(\mathcal{H}, \mathcal{V}, \mathcal{D}^B, \kappa)$ the *coloured hyperspace* of K. We know from before that in the Luna-Vust theory we classify G-varieties using the coloured equipment $(\mathcal{V}, \mathcal{D}^B)$ of K, and we use the map κ and the hyperspace $\mathcal{H}$ to describe this data combinatorially, with certain types of cones and fans sitting inside the hyperspace. We also denote by $\mathcal{H}_p, \mathcal{V}_p$, etc. the subsets of $\mathcal{H}$, $\mathcal{V}$, etc. corresponding to a specific choice of $p \in C$, and will call $\mathcal{H}_p$ the *slice of hyperspace* corresponding to p. Finally, the subset of $\mathcal{H}$ of all points with $h = 0$ is called the *central hyperplane* and denoted $\mathcal{Z}$.

Splitting Maps

We have seen that the exact sequence in Proposition 2.5 splits under a map $e \colon \Lambda \to K^{(B)}$, but (in complexity one at least) this map is not canonical. Hence neither are the maps sending $\mathcal{V}$ and $\mathcal{D}^B$ to hyperspace, so it is worth clearing up what happens when a different splitting map is chosen. Suppose $e' \colon \Lambda \to K^{(B)}$ is a different splitting and $\nu \in \mathcal{V}$ corresponds to (p, h, ℓ) under e and (p', h', ℓ') under e'. The choice of splitting has no effect on p, h since these are determined by the restriction to K^B, but the functionals ℓ, ℓ' will be different: for $\chi \in \Lambda$ we have

$$\ell'(\chi) - \ell(\chi) = \nu(e'(\chi)) - \nu(e(\chi)) = \nu(e'(\chi)/e(\chi)) = h\nu_p(e'(\chi)/e(\chi)).$$

Hence we see that the change in splitting corresponds to a linear 'co-ordinate change' $(p, h, \ell) \mapsto (p, h, \ell + h\ell_p)$, where $\ell_p(\chi) := \nu_p(e'(\chi)/e(\chi))$. Since C is a smooth projective curve, the principal divisor corresponding to the rational function $e'(\chi)/e(\chi)$ has degree zero, so we have $0 = \sum_{p\in C} \nu_p(e'(\chi)/e(\chi)) = \sum_{p\in C} \ell_p(\chi)$ for all $\chi \in \Lambda$, hence $\sum_{p\in C} \ell_p = 0$. When $C = \mathbb{P}^1$, which will hold in most cases of interest to us, it is conversely true that any such collection of integral shifts $\ell_p \in \mathrm{Hom}(\Lambda, \mathbb{Z})$ with $\sum_{p\in C} \ell_p = 0$ defines a passage from one splitting to another by taking e' such that $\nu_p(e'(\chi)/e(\chi)) = \ell_p(\chi)$.

Quasihomogeneous and One-Parameter Cases

There are two distinct types of variety which arise in complexity one. Note that in complexity zero, K^B and K^G are necessarily the same since $K^G \subseteq K^B = \Bbbk$. In complexity one, with $\operatorname{trdeg} K^B = 1$, there are two possibilities for K^G which correspond to different orbit structures in G-models:

- In the *quasihomogeneous case*, where $K^G = \Bbbk$, there is an open G-orbit in any G-model X of K, so X is an embedding of a homogeneous space G/H. This is a minimal model for the G-birational class determined by K and its G-action.

- In the *one-parameter* case, where $K^G = K^B = \Bbbk(C)$, there is a family (parameterised by C) of spherical G-orbits of codimension 1 in any G-model of K.

Remark. The two cases given above do indeed exhaust all possible relationships between K^G and K^B in complexity one. *A priori*, it is possible that $K^G \hookrightarrow K^B$ is a finite extension (of degree at least 2) rather than an equality. Suppose this is the case. Then K^G is the function field of some smooth projective curve C' and we will have a finite morphism $f \colon C \to C'$ of degree $d > 1$. We also have rational quotient maps from X to C and C' and these should commute with f and separate orbits. However if we pull back some $p \in C'$ to C and then to X, the fibre will contain two different orbits, while if we pull it back directly to X it should only contain one. Hence $K^G = \Bbbk$ or $K^G = K^B$.

We will primarily be interested in the quasihomogeneous case, and we assume $K^G = \Bbbk$ from now on.

Theorem 3.1. *In the quasihomogeneous case, the smooth projective curve C such that $K^B = \Bbbk(C)$ is in fact $\mathbb{P}^1$.*

Lemma 3.1. *Let G be a connected algebraic group over $\Bbbk$. Then G is rational as a variety.*

Proof. Assume G is reductive. Then the Bruhat decomposition [Hum75, Prop 28.5] tells us that G has an open subset isomorphic to U^-TU, where U^- is the opposite maximal unipotent subgroup to U relative to T. Now U^- and U are isomorphic to affine spaces and T is a product of open subsets of affine spaces, so these are all rational varieties. Hence the open Bruhat cell of G is also rational, and thus so is G.

Now let G be connected but not necessarily reductive. By the Levi decomposition [Hum75, Thm 30.2], G is a semidirect product of a reductive group L and the unipotent radical $R_u(G)$. By the above, both L and $R_u(G)$ are rational, and hence again so is G. □

Proof of Theorem 3.1. [Tim11, §16.2] Since we are in the quasihomogeneous case, $K = \Bbbk(G/H)$ for some H. Then $\Bbbk(C) \subseteq \Bbbk(G/H)$ so we get a dominant rational map $G/H \dashrightarrow C$, which gives a dominant rational map $G \dashrightarrow C$ via the quotient. Since G is rational, C is unirational and hence $C = \mathbb{P}^1$ by Lüroth's theorem [Har77, Ex IV.2.5.5]. □

Regular, Subregular and Central Elements

Any G-model X has an associated dominant rational B-quotient map $\pi\colon X \dashrightarrow C$, arising from the inclusion $\Bbbk(C) \subseteq K$, which separates general B-orbits (since $K^B = \Bbbk(C)$). This means there is a one-parameter family of B-stable prime divisors $D_p \subseteq \pi^*(p)$ in X parameterised by points $p \in C_0 = \pi(X)$, an open subset of C.

The choice of splitting map $e\colon \Lambda \to K^{(B)}$ marks colours lying in $\operatorname{div} e(\chi)$ for $\chi \in \Lambda$ out as distinguished: colours lying outside any $\operatorname{div} e(\chi)$ have $\ell = 0$, and those lying inside any will have $\ell = \nu_D(e(\chi)) \neq 0$. Reducing C_0 if necessary to remove all (finitely many) points whose pullbacks lie in some $\operatorname{div} e(\chi)$, it follows that $\ell = 0$ for all but finitely many colours.

In the quasihomogeneous case, the rational B-quotient maps to $\mathbb{P}^1$, so is determined by a one-dimensional linear system of colours, i.e. there is a line bundle $\mathcal{L}$ on G/H and a two-dimensional space M of B-eigensections of $\mathcal{L}$ which defines homogeneous co-ordinates on $\mathbb{P}^1 = \mathbb{P}(M^*)$. Elements of M correspond to equations defining a point of $\mathbb{P}^1$ and hence in turn a B-divisor on X by pulling back under π.

Any G-line bundle on G/H is determined by a character $\chi \in \mathfrak{X}(H)$ [Tim11, §2.1]: each is of the form $\mathcal{L}_\chi = G \star_H \Bbbk_\chi = (G \times \Bbbk_\chi)/H$ where $\Bbbk_\chi$ is $\Bbbk$ with the H-action determined by χ (see section 2.3.4). Then the global sections of this line bundle are given by H-eigenfunctions on G with weight $-\chi$, i.e. $H^0(G/H, \mathcal{L}(\chi)) = \Bbbk[G]^{(H)}_{-\chi}$. Now in our case a B-divisor on G/H determines a line bundle $\mathcal{L}$ which can be in turn determined by a B-eigensection, uniquely up to multiplication by an invertible function, that is a nonzero multiple of a character of G. Hence $\mathcal{D}^B(G/H)$ corresponds bijectively to generators of $\Bbbk[G]^{(B\times H)}/\Bbbk^*\mathfrak{X}(G)$, a multiplicative semigroup.

Returning to the linear system defining π, we now see that the line bundle $\mathcal{L}$ must be in the form $\mathcal{L}_{-\chi}$ and the space M, consisting of B-eigensections of $\mathcal{L}_\chi$, must have the form $\Bbbk[G]^{(B\times H)}_{(\lambda,\chi)}$ for some $\lambda \in \mathfrak{X}(B)$. Irreducible elements of $M = \Bbbk[G]^{(B\times H)}_{(\lambda,\chi)}$ and the corresponding divisors are called *regular*. A regular colour is of the form $D_p = \pi^*(p)$ for some $p \in \mathbb{P}^1$ and has h-coordinate 1 in hyperspace.

There will also in general be a set of one-dimensional spaces $\Bbbk[G]^{(B\times H)}_{(\lambda_i,\chi_i)}$ containing eigenfunctions which correspond to different B-divisors. Irreducible elements of these spaces which divide regular semi-invariants are called *subregular*, as are the corresponding B-divisors, which must then occur inside the element of the linear system defined by the regular semi-invariant in question. Suppose a subregular colour D_i is defined by $\eta_i \in \Bbbk[G]^{(B\times H)}_{(\lambda_i,\chi_i)}$ where η_i divides $\eta \in M$, and $\operatorname{div}\eta = \pi^*(p)$. Then D_i is represented in hyperspace by (p, h_i, ℓ_i), where $h_i > 1$ is the multiplicity of η_i in η or equivalently of D_i in $\operatorname{div}\eta$.

Elements in $\Bbbk[G]^{(B\times H)}_{(\lambda_i,\chi_i)}$ which do not divide regular semi-invariants are called *central* because the valuation given by the corresponding B-divisor is central in the sense defined earlier. Hence the corresponding colours lie at points $(0, \ell)$ in the central hyperplane of the hyperspace.

Proposition 3.1. *For any $p \in \mathbb{P}^1$, only finitely many colours are mapped to $\mathcal{H}_p$. In particular there are only finitely many central colours.*

Proof. [Tim11, Lemma 20.4] Take a B-chart $X_0 \subseteq X$ small enough that there is a geometric quotient $\pi\colon X_0 \to X_0/B$. Then X_0/B is a smooth rational curve so birational to $\mathbb{P}^1$. Hence ν_p has centre on X_0/B for $p \in \mathbb{P}^1$. If a colour D goes to $\mathcal{H}_p$ in hyperspace, i.e. $\nu_D|_{K^B} = h\nu_p$, then either $D = \pi^{-1}(D_0)$, where D_0 is the centre of ν_p on X_0/B, or D is an irreducible component of $X \setminus X_0$. In all there are finitely many such D.

A central colour lies in every $\mathcal{H}_p$, so by the above there can only be finitely many. □

Since there are finitely many central colours, and there can also only be finitely many subregular colours, it follows that all but finitely many colours are regular, i.e. have $h = 1$. Most colours are regular and non-distinguished, so lie at the points $\varepsilon_p := (p, 0, 1)$ in hyperspace.

Linear Functionals on Hyperspace

We now want to define linear functionals, cones etc. on the hyperspace $\mathcal{H}$. Specifically we want to interpret semi-invariants as linear functionals on $\mathcal{H}$.

Definition 3.1. A *linear functional* on $\mathcal{H}$ is a function $\varphi\colon \mathcal{H} \to \mathbb{Q}$ such that $\varphi_p := \varphi|_{\mathcal{H}_p}$ is a $\mathbb{Q}$-linear functional on $\mathcal{H}_p$ for all $p \in \mathbb{P}^1$ and $\sum_{p\in\mathbb{P}^1} \langle \varepsilon_p, \varphi_p\rangle = 0$ where ε_p is the point $(p, \ell, h) = (p, 0, 1) \in \mathcal{H}_p$. We denote by $\mathcal{H}^*$ the space of linear functionals on $\mathcal{H}$ and for $\varphi \in \mathcal{H}^*$ we define the *kernel* $\ker\varphi$ to be $\bigcup_{p\in\mathbb{P}^1} \ker\varphi_p$.

Now suppose $f \in K^{(B)}$ is a semi-invariant, so we can write $f = f_0 e_\lambda$ for some $f_0 \in K^B$ and some $\lambda \in \Lambda$. Then f determines a linear functional φ on hyperspace as follows: for $p \in \mathbb{P}^1$ and $q = (\ell, h) \in \mathcal{H}_p$ let $\langle q, \varphi_p\rangle = h\nu_p(f_0) + \langle \ell, \lambda\rangle$. Then $\sum_{p\in\mathbb{P}^1} \langle \varepsilon_p, \varphi_p\rangle = \sum_{p\in\mathbb{P}^1} \nu_p(f_0)$ is the degree of the principal divisor (f_0) on $\mathbb{P}^1$, i.e. 0. Hence the collection $\{\varphi_p\}_{p\in\mathbb{P}^1}$ does indeed constitute a linear functional on $\mathcal{H}$.

Proposition 3.2. *A semi-invariant $f \in K^{(B)}$ is determined up to scalar multiples by its corresponding functional φ. Also, any functional φ on hyperspace has a multiple which is the functional corresponding to a semi-invariant.*

Proof. Let $f = f_0 e_\lambda$ and $f' = f_0' e_{\lambda'}$ be semi-invariants, and suppose they define the same functional φ on $\mathcal{H}$. Then for any $p \in \mathbb{P}^1$ and any $q = (\ell, h) \in \mathcal{H}_p$ we have $\langle q, \varphi\rangle = h\nu_p(f_0) + \langle \ell, \lambda\rangle = h\nu_p(f_0') + \langle \ell, \lambda'\rangle$.

Taking $h = 0$ we get $\langle \ell, \lambda\rangle = \langle \ell, \lambda'\rangle$ for all $\ell \in \mathcal{Q}$, so $\lambda = \lambda'$. Taking $\ell = 0$ and $h = 1$ gives $\nu_p(f_0) = \nu_p(f_0')$, so $\nu_p(f_0/f_0') = 0$ for all $p \in \mathbb{P}^1$, i.e. f_0/f_0' is a constant.

Likewise, any linear functional $\{\varphi_p\}_{p\in\mathbb{P}^1}$ on $\mathcal{H}$ can be associated with a semi-invariant rational function. Indeed, since $\sum_{p\in\mathbb{P}^1} \langle \varepsilon_p, \varphi_p\rangle = 0$, the divisor $\sum_{p\in\mathbb{P}^1} \langle \varepsilon_p, \varphi_p\rangle \cdot p$ on $\mathbb{P}^1$ is principal, i.e. it is the divisor of some $f_0 \in \Bbbk(\mathbb{P}^1) = K^B$.

For $q = (\ell, h) \in \mathcal{H}_p$, write $q = h\varepsilon_p + (\ell, 0)$, so we have $\langle q, \varphi_p \rangle = h\nu_p(f_0) + \langle (\ell, 0), \varphi_p \rangle$. Now $\langle (\ell, 0), \varphi_p \rangle \in \mathbb{Q}$, and $\ell \colon \Lambda \to \mathbb{Q}$, so there is some integer multiple k such that we can choose $\lambda \in \Lambda$ with $\langle \ell, \lambda \rangle = k\langle (\ell, 0), \varphi_p \rangle$. Then $\varphi_p(q) = h\nu_p(f_0) + \frac{1}{k}\langle \ell, \lambda \rangle$, i.e. $k\varphi$ is the functional associated to the semi-invariant $f = f_0^k e_\lambda \in K^{(B)}$. □

The 'up to scalar multiples' caveats turn out to be unimportant since we will only be interested in whether a linear functional is nonnegative or not at any given point, as this will determine which points lie in certain cones and which do not. Likewise it makes no difference if a semi-invariant f is replaced by f^n.

Next we see that linear functionals split into two distinct types, determined by their behaviour on the central hyperplane $\mathcal{Z} \subseteq \mathcal{H}$.

Proposition 3.3. *Let $f \in K^B$. The corresponding functional φ vanishes on $\mathcal{Z}$ and on $\mathcal{H}_p$ for all but finitely many $p \in \mathbb{P}^1$. The remaining p split into points for which φ is positive on $\mathcal{H}_p \setminus \mathcal{Z}$ and points for which φ is negative on $\mathcal{H}_p \setminus \mathcal{Z}$. Conversely, any such functional is determined by its positive and negative half-spaces $\mathcal{H}_p$ and a collection of numbers $d_p = \langle \varepsilon_p, \varphi_p \rangle$ with $\sum d_p = 0$. We call these φ* functionals of type I.

Proof. [Tim97, §3.1] Remembering that $K^B = \Bbbk(\mathbb{P}^1)$, if $p \in \mathbb{P}^1$ and $q = (\ell, h) \in \mathcal{H}_p$, we have $\langle q, \varphi_p \rangle = h\nu_p(f)$. From this it is clear that φ vanishes on $\mathcal{Z}$, where $h = 0$, and φ is nonzero only on slices $\mathcal{H}_p$ of hyperspace corresponding to $p \in \operatorname{Supp}(f)$. There are finitely many such points, and φ is positive on $\mathcal{H}_p$ when p is a zero of f, and negative when p is a pole.

Being given a finite collection of positive and negative half-spaces $\mathcal{H}_p$ amounts by the discussion above to being given the points $p \in \mathbb{P}^1$ on which a divisor D on $\mathbb{P}^1$ is supported. The numbers d_p then tell us the multiplicity of p in D, and the condition $\sum d_p = 0$ tells us $D = (f)$ for some $f \in \Bbbk(\mathbb{P}^1)$. □

Proposition 3.4. *Functionals φ corresponding to semi-invariants $f \notin K^B$ do not vanish on $\mathcal{Z}$ and are determined by the hyperplanes* $\ker \varphi_p \subseteq \mathcal{H}_p$. *These hyperplanes have common intersection in $\mathcal{Z}$ and all but finitely many of them contain $\varepsilon_p = (p, 0, 1) \in \mathcal{H}_p$.*

These functionals also satisfy the following 'balancing condition of inclination to the vertical': for any $\ell \in \mathcal{Z} \setminus$ $\ker \varphi$ *and any $p \in \mathbb{P}^1$,* $\ker \varphi_p$ *meets the line $\varepsilon_p + \mathbb{Q}\ell$ at a unique point $q_p = (\ell_p, 1)$. The condition is that $\sum \ell_p = 0$.*

Conversely, any collection of hyperplanes in the $\mathcal{H}_p$ having common intersection in $\mathcal{Z}$ and satisfying the above balancing condition determine up to proportionality a functional of this type, once the positive and negative subspaces in each $\mathcal{H}_p$ are specified. We call these functionals of type II.

Proof. [Tim97, Remark 3.1] Let $f = f_0 e_\lambda$, with $f_0 \in K^B$ and $\lambda \neq 1$. Then for $q = (\ell, h) \in \mathcal{H}_p$, $\langle q, \varphi_p \rangle = h\nu_p(f_0) + \langle \ell, \lambda \rangle$. Taking $h = 0$ gives $\varphi|_{\mathcal{Z}}(\ell) = \langle \ell, \lambda \rangle \neq 0$ for $\ell \neq 0$, since $\lambda \neq 1$. Since φ must be a function on $\mathcal{H}$, each φ_p must agree on $\mathcal{Z}$, so in particular each $\ker \varphi_p$ must intersect $\mathcal{Z}$ at the same points. For $\varepsilon_p \in \mathcal{H}_p$ we have $\langle \varepsilon_p, \varphi_p \rangle = \nu_p(f_0)$, which is zero for all but finitely many p, so $\varepsilon_p \in \ker \varphi_p$ for all but finitely many p, as required.

As for the balancing condition, since $\langle q_p, \varphi_p \rangle = 0$ for all $p \in \mathbb{P}^1$, we have

$$0 = \sum_{p \in \mathbb{P}^1} \langle q_p, \varphi_p \rangle = \sum_{p \in \mathbb{P}^1} \langle \varepsilon_p, \varphi_p \rangle + \sum_{p \in \mathbb{P}^1} \langle \ell_p, \lambda \rangle = \sum_{p \in \mathbb{P}^1} \langle \ell_p, \lambda \rangle = \langle \sum_{p \in \mathbb{P}^1} \ell_p, \lambda \rangle,$$

which gives $\sum_{p \in \mathbb{P}^1} \ell_p = 0$ since $\lambda \neq 1$.

Now suppose we are given a collection L_p of subspaces in each $\mathcal{H}_p$, all but finitely many of which contain ε_p, which have common intersection K in $\mathcal{Z}$ and which satisfy the balancing condition of inclination to the vertical. Choose $\ell \in \mathcal{Z} \setminus K$ and $\lambda \neq 1$ in Λ such that $\langle \ell, \lambda \rangle = 1$. For each $p \in \mathbb{P}^1$, find the points $q_p = (\ell_p, 1)$ lying in each $(\varepsilon_p + \mathbb{Q}\ell) \cap L_p$. Then all but finitely many ℓ_p are 0, and $\sum_{p \in \mathbb{P}^1} \ell_p = 0$. Set $d_p = -\langle \ell_p, \lambda \rangle$ for each $p \in \mathbb{P}^1$. Then all but finitely many d_p are 0 and $\sum_{p \in \mathbb{P}^1} d_p = 0$ as well. Hence the divisor $D = \sum_{p \in \mathbb{P}^1} d_p \cdot p$ is prinicpal, so equal to (f_0) for some $f_0 \in \Bbbk(\mathbb{P}^1)$. Then the functional φ corresponding to $f_0 e_\lambda \in K^{(B)}$ has $\ker \varphi_p = L_p$ for all $p \in \mathbb{P}^1$, $d_p = \langle \varepsilon_p, \varphi_p \rangle$ and so on.

This process determines φ up to proportionality since ℓ (hence ℓ_p and hence d_p) are all determined up to proportionality as well. □

3.1.2 Hypercones and Hyperfans

B-Charts and Hypercones

We are now ready to begin interpreting the conditions of the Luna-Vust theory in terms of how the coloured data of a G-model appears in the hyperspace. We will start with the properties of B-charts, so let $(\mathcal{W}, \mathcal{R}) \in \mathbf{CD}$, let $\mathcal{A} = \mathcal{A}(\mathcal{W}, \mathcal{R})$ and let $X_0 = \operatorname{Spec} \mathcal{A}$.

Proposition 3.5. *Condition* (F) *holds for all* $(\mathcal{W}, \mathcal{R}) \in \mathbf{CD}$.

Proof. [Tim97, Prop 3.1] Recall that we must show finite generation of $\mathcal{A}^U = \Bbbk(\mathcal{A}^{(B)}) = \Bbbk[f \in K^{(B)} \mid \langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0]$. We know that $K^{(B)} = K^B \cdot e(\Lambda)$, and furthermore we know from our discussion of (sub)regular and central colours that all but finitely many valuations from $\mathcal{W} \sqcup \mathcal{R}$ (say all but those in a finite subset $\mathcal{W}_0 \sqcup \mathcal{R}_0$) correspond to regular colours, i.e. are equal to 0 on $e(\Lambda)$ and are given by ν_p on $K^B = \Bbbk(\mathbb{P}^1)$ for p lying in some open subset $\mathbb{P}^1_0$ of $\mathbb{P}^1$. Hence the condition $\langle \mathcal{W} \sqcup \mathcal{R}, f \rangle \geq 0$ requires that f is regular on $\mathbb{P}^1_0$ and we can write $\mathcal{A}^{(B)} \subseteq \Bbbk[\mathbb{P}^1_0] \cdot e(\Lambda)$. We may even reduce $\mathbb{P}^1_0$ if necessary such that we can assume that a unique element of $\mathcal{W} \sqcup \mathcal{R}$ falls into $\mathcal{H}_p$ for each $p \in \mathbb{P}^1_0$.

Now consider $\mathbb{P}^1$ as $\mathbb{P}(\mathbb{A}^2)$ so that $\Bbbk(\mathbb{P}^1) = \Bbbk(\mathbb{A}^2)_{\deg=0}$, let $\mathbb{A}^2_0$ be the open subset of $\mathbb{A}^2$ corresponding to $\mathbb{P}^1_0$ and let g_p be the equation of the line in $\mathbb{A}^2$ corresponding to $p \in \mathbb{P}^1$. Then the algebra $\mathcal{A}^U = \Bbbk[\mathcal{A}^{(B)}] \subseteq \Bbbk[\mathbb{A}^2_0] \otimes \Bbbk[e(\Lambda)]$ is generated by the semigroup

$$\mathcal{A}^{(B)} = \{f = f_0 e_\lambda \mid f_0 \in \Bbbk[\mathbb{A}^2_0],\ \deg f_0 = 0,\ \lambda \in \Lambda,\ \langle \mathcal{W}_0 \sqcup \mathcal{R}_0, f \rangle \geq 0\}.$$

Let $\hat{\mathcal{A}}^{(B)}$ be the group obtained by allowing homogeneous f of arbitrary degree in the above description of $\mathcal{A}^{(B)}$. Then $\hat{\mathcal{A}}^{(B)}$ generates a B-algebra $\hat{\mathcal{A}}$ such that $\mathcal{A}^U = \hat{\mathcal{A}}_{\deg=0} = (\hat{\mathcal{A}})^{\Bbbk^\times}$. By Hilbert's theorem on invariants it will then suffice to show that $\hat{\mathcal{A}}$ is finitely generated.

For homogeneous $f_0 \in \Bbbk[\mathbb{A}^2_0]$, write $f_0 = \prod_{p \notin \mathbb{P}^1_0} g_p^{k_p} \cdot q$ where $q \in \Bbbk[\mathbb{A}^2]$ is a homogeneous polynomial coprime to all g_p, and k_p are integers. Now we want to rewrite the condition $\langle \mathcal{W}_0 \sqcup \mathcal{R}_0, f \rangle \geq 0$. For $\nu \in \mathcal{W}_0 \sqcup \mathcal{R}_0$ mapping into $\mathcal{H}_p$, we have

$$\nu(f) = \nu(f_0) + \nu(e_\lambda) = h_\nu \nu_p(f_0) + \langle \ell_\nu, \lambda \rangle = h_\nu k_p + \langle \ell_\nu, \lambda \rangle,$$

so the condition $\langle \mathcal{W}_0 \sqcup \mathcal{R}_0, f \rangle \geq 0$ reduces to a finite set of inequalities $h_\nu k_p + \langle \ell_\nu, \lambda \rangle \geq 0$. These inequalities determine a finitely generated subsemigroup in the group

$$\Big\{ \prod_{p \notin \mathbb{P}^1_0} g_p^{k_p} \cdot e_\lambda \mid k_p \in \mathbb{Z}, \lambda \in \Lambda \Big\}.$$

This subsemigroup, together with the co-ordinate functions on $\mathbb{A}^2$, generates the algebra $\hat{\mathcal{A}}$, so we are done. □

Now as with the complexity zero case, we want to form cones in $\mathcal{H}$ from the coloured data $(\mathcal{W}, \mathcal{R})$, but since the hyperspace is not a finite dimensional vector space, we have to be more careful about our definitions. With that in mind, let $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$ be the subset of $\mathcal{H}$ consisting of points q where $\langle q, \varphi\rangle \geq 0$ for all functionals φ with $\langle \mathcal{W} \sqcup \mathcal{R}, \varphi\rangle \geq 0$. We can view this as something like a 'double dual' in $\mathcal{H}$ of the set $\mathcal{W} \cup \kappa(\mathcal{R})$. It is not a convex cone in the usual sense, but $\mathcal{K} = \mathcal{C} \cap \mathcal{Z}$ is a cone in $\mathcal{Z}$, and for each $p \in \mathbb{P}^1$, $\mathcal{C}_p = \mathcal{C} \cap \mathcal{H}_p$ is a cone in $\mathcal{H}_p$.

Before we move on to describing the properties of the cones $\mathcal{K}$ and $\mathcal{C}_p$, we introduce a typology of B-charts which will be necessary to keep track of.

Proposition 3.6. *B-charts split into two types. Those we will call* type I *have non-trivial B-invariants in their algebra of functions, i.e. $\mathcal{A}^B \neq \Bbbk$. In this case, there is some $p \in \mathbb{P}^1$ for which no elements of $\mathcal{W} \sqcup \mathcal{R}$ map to $\mathcal{H}_p \setminus \mathcal{Z}$, or equivalently, some $p \in \mathbb{P}^1$ with $\mathcal{C}_p \subseteq \mathcal{K}$.*

A B-chart is of type II *if $\mathcal{A}^B = \Bbbk$. For B-charts of type II, all $\mathcal{H}_p$ contain non-central elements of $\mathcal{W} \sqcup \mathcal{R}$.*

Proof. [Tim97, Remark 3.1] Clearly any B-chart is either of type I or of type II and never both. Suppose $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$ is a B-chart of type I, i.e. $\mathcal{A}^B \neq \Bbbk$. Let $f \in \mathcal{A}^B$ be a nonconstant B-invariant with $\langle \mathcal{W} \sqcup \mathcal{R}, f\rangle \geq 0$. Since $f \in K^B = \Bbbk(\mathbb{P}^1)$, let $p \in \mathbb{P}^1$ be a pole of f. For any $\nu \in \mathcal{W} \sqcup \mathcal{R}$ with $\nu|_{K^B} = h\nu_p$ and $h \geq 0$, we have $\nu(f) = h\nu_p(f) \geq 0$ since $f \in \mathcal{A}$, but $h\nu_p(f) \leq 0$ since $\nu_p(f) < 0$, so $h = 0$ and ν lies in $\mathcal{Z}$. Hence $\mathcal{H}_p \setminus \mathcal{Z}$ contains no valuations from $\mathcal{W} \sqcup \mathcal{R}$.

Now suppose X_0 is of type II, i.e. $\mathcal{A}^B = \Bbbk$. For any $p \in \mathbb{P}^1$, we can choose nonconstant $f \in \Bbbk(\mathbb{P}^1) = K^B$ with $\nu_p(f) < 0$. Since $f \notin \mathcal{A}^B$, there must be $\nu \in \mathcal{W} \sqcup \mathcal{R}$ with $\nu(f) < 0$. Hence $\nu|_{K^B} = h\nu_p$ with $h \neq 0$, i.e. ν lies in $\mathcal{H}_p \setminus \mathcal{Z}$. □

First, we can describe the cones $\mathcal{C}_p$ for a B-chart of type I:

Lemma 3.2. *Let $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$ be a B-chart of type I. Then:*

(i) *Any collection $\{\varphi_p\}_{p \in \mathbb{P}^1}$ of linear functionals on each $\mathcal{H}_p$ which agree on $\mathcal{Z}$ can be made into a functional φ on $\mathcal{H}$ satisfying the balancing condition $\sum_{p \in \mathbb{P}^1} \langle \varepsilon_p, \varphi_p\rangle =$*

0. *The resulting functional φ can be chosen to have the same positive and negative half-spaces with respect to $\mathcal{W} \sqcup \mathcal{R}$ as the original collection of functionals;*

(ii) *If $\mathcal{H}_{p_0}$ contains no noncentral elements of $\mathcal{W} \sqcup \mathcal{R}$, then for $p \neq p_0$, any functional φ_p on $\mathcal{H}_p$ which is non-negative on elements of $\mathcal{W} \sqcup \mathcal{R}$ can be extended to a functional φ on all of $\mathcal{H}$ which is also non-negative on $\mathcal{W} \sqcup \mathcal{R}$;*

(iii) *Let $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$. For $p \in \mathbb{P}^1$, the cone $\mathcal{C}_p = \mathcal{C} \cap \mathcal{H}_p$ is generated by elements of $\mathcal{W} \sqcup \mathcal{R}$ falling in $\mathcal{H}_p$.*

Proof. [Tim97, Lemma 3.1] (i): Let p_0 be a point where $\mathcal{H}_{p_0}$ contains no noncentral elements of $\mathcal{W} \sqcup \mathcal{R}$. We can vary φ_{p_0} such that it keeps the same values on $\mathcal{Z}$ but we have $\langle \varepsilon_{p_0}, \varphi_{p_0} \rangle = -\sum_{p \neq p_0} \langle \varepsilon_p, \varphi_p \rangle$. Then the new collection $\{\varphi_p\}_{p \in \mathbb{P}^1}$ defines a functional φ on $\mathcal{H}$, and the positive/negative half-spaces are the same since $\mathcal{H}_{p_0} \setminus \mathcal{Z}$ is disjoint from $\mathcal{W} \sqcup \mathcal{R}$.

(ii): Restricting φ_p to $\mathcal{Z}$ gives a linear functional on the common hyperplane of all $\mathcal{H}_q$, $q \in \mathbb{P}^1$. Hence we can define linear functionals φ_q on every $\mathcal{H}_q$ by extending $\varphi_p|_{\mathcal{Z}}$ and we are free to require these functionals to be non-negative on $\mathcal{W} \sqcup \mathcal{R}$. Now by (i) we can force these functionals to satisfy the balancing condition by varying φ_{p_0}, without effecting non-negativity on $\mathcal{W} \sqcup \mathcal{R}$.

(iii): If p_0 is a point where $\mathcal{H}_{p_0}$ contains no non-central elements of $\mathcal{W} \sqcup \mathcal{R}$, then the cone generated by elements of $\mathcal{W} \sqcup \mathcal{R}$ falling in $\mathcal{H}_{p_0}$ is the cone generated by the central valuations. Hence if the statement holds true for other p, it must be true for p_0, since the central valuations lie in every $\mathcal{H}_p$.

For any other $p \in \mathbb{P}^1$, let $\mathcal{C}'$ be the cone in $\mathcal{H}_p$ generated by the elements of $\mathcal{W} \sqcup \mathcal{R}$ falling into $\mathcal{H}_p$. Then $q \in \mathcal{C}'$ if and only if for all functionals φ_p on $\mathcal{H}_p$ non-negative on elements of $\mathcal{W} \sqcup \mathcal{R}$ falling into $\mathcal{H}_p$, we have $\varphi_p(q) \geq 0$. By (ii) any such functional corresponds to a functional φ on all of $\mathcal{H}$ which is non-negative on $\mathcal{W} \sqcup \mathcal{R}$. Likewise, any functional φ on $\mathcal{H}$ non-negative on $\mathcal{W} \sqcup \mathcal{R}$ defines a functional φ_p on $\mathcal{H}_p$ non-negative on elements of $\mathcal{W} \sqcup \mathcal{R}$ falling into $\mathcal{H}_p$. It follows that $\mathcal{C}' = \mathcal{C}_p$. □

The cones $\mathcal{C}_p$ for B-charts of type II have a different description:

Proposition 3.7. *Let $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$ be a B-chart of type II. Then:*

(i) *Any functional non-negative on $\mathcal{W} \sqcup \mathcal{R}$ must be of type II;*

(ii) *Non-central elements of* $\mathcal{W} \sqcup \mathcal{R}$ *generate a collection* $\{\varepsilon_p + \mathcal{P}_p\}_{p\in\mathbb{P}^1}$ *of polytopes in* $\mathcal{H}$*, where* $\mathcal{P}_p \subseteq \mathcal{Z}$*, and we have* $\mathcal{P}_p = \{0\}$ *for all but finitely many* p*;*

(iii) *Let* $\mathcal{P} = \sum_{p\in\mathbb{P}^1} \mathcal{P}_p$*. Then for any functional* φ_p *on* $\mathcal{H}_p$ *non-negative on* $\mathcal{P}$ *and elements of* $\mathcal{W} \sqcup \mathcal{R}$ *falling in* $\mathcal{H}_p$ *for some given* $p \in \mathbb{P}^1$*, there is a corresponding functional on* $\mathcal{H}$ *non-negative on* $\mathcal{W} \sqcup \mathcal{R}$*.*

(iv) *Let* $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$*. The cones* $\mathcal{C}_p = \mathcal{C} \cap \mathcal{H}_p$ *are generated by elements of* $\mathcal{W} \sqcup \mathcal{R}$ *falling into* $\mathcal{H}_p$ *and by the polytope* $\mathcal{P} = \sum_{p\in\mathbb{P}^1} \mathcal{P}_p$*.*

Proof. [Tim97, Lemma 3.2] (i) For any functional φ of type I, there are $p \in \mathbb{P}^1$ for which φ_p takes negative values on all of $\mathcal{H}_p \setminus \mathcal{Z}$, so φ cannot be non-negative on $\mathcal{W} \sqcup \mathcal{R}$ since it contains valuations mapping to $\mathcal{H}_p \setminus \mathcal{Z}$ for any p.

(ii): Fix $p \in \mathbb{P}^1$ and consider the affine hyperplane $\varepsilon_p + \mathcal{Z} \subseteq \mathcal{H}_p$. Any non-central element $q \in \mathcal{H}_p$ is uniquely determined by its h co-ordinate and the point of intersection $\mathbb{Q}_{\geq 0}q \cap (\varepsilon_p + \mathcal{Z})$. Since we are only interested in elements of $\mathcal{W} \sqcup \mathcal{R}$ up to multiples, we can in fact determine elements in this set entirely by this point of intersection, effectively setting $h = 1$.

Hence non-central elements of $\mathcal{W} \sqcup \mathcal{R}$ generate a polytope $\varepsilon_p + \mathcal{P}_p$ in $\mathcal{H}_p$ for each p, given by the convex hull of the points of intersection with $\varepsilon_p + \mathcal{Z}$ of noncentral valuations in $\mathcal{W} \sqcup \mathcal{R}$ which land in $\mathcal{H}_p$. Since for all but finitely many p, the only noncentral valuation from $\mathcal{W} \sqcup \mathcal{R}$ lying in $\mathcal{H}_p$ is ε_p, it follows that $\mathcal{P}_p = \{0\}$ for all but finitely many p.

(iii): By the above, the Minkowski sum $\sum_{p\in\mathbb{P}^1} \mathcal{P}_p$ makes sense, since $\mathcal{P}_p = \{0\}$ for almost all p.

To obtain such a functional, define ψ_p on $\mathcal{H}_p$ as the result of rotating $\ker \varphi_p$, without changing $\varphi_p|_{\mathcal{Z}}$, so that it becomes supporting for $\varepsilon_p + \mathcal{P}_p$. Then $\psi_p|_{\mathcal{Z}} = \varphi_p|_{\mathcal{Z}}$, and $\ker \psi_p$ is a supporting hyperplane of $\varepsilon_p + \mathcal{P}_p$. For $q \neq p$, we can now define ψ_q by extending $\psi_p|_{\mathcal{Z}}$ to $\mathcal{H}_q$ in such a way that it is supporting for $\varepsilon_q + \mathcal{P}_q$ for every q. Now for all q we have

$$\langle \varepsilon_q + \mathcal{P}_q, \psi_q \rangle \geq 0 \implies \sum_{q\in\mathbb{P}^1} \langle \varepsilon_q, \psi_q \rangle + \langle \mathcal{P}, \psi \rangle \geq 0.$$

Since $\ker \psi_q$ is supporting for every $\varepsilon_q + \mathcal{P}_q$, equality is achieved at certain points in each of the above inequalities. It follows that $\sum_{q\in\mathbb{P}^1} \langle \varepsilon_q, \psi_q \rangle \leq 0$. By moving the hyperplanes

$\ker \psi_q$ for certain $q \neq p$ away from the ones supporting $\varepsilon_q + \mathcal{P}_q$, we can increase $\langle \varepsilon_q, \psi_q \rangle$ without changing the inequality $\langle \varepsilon_q + \mathcal{P}_q, \psi_q \rangle \geq 0$, thus giving $\sum_{q \in \mathbb{P}^1} \langle \varepsilon_q, \psi_q \rangle = 0$ as required.

The resulting functional ψ is non-negative on central elements of $\mathcal{W} \sqcup \mathcal{R}$ because φ_p was, and we chose ψ to agree with φ_q on $\mathcal{Z}$. It is non-negative on non-central elements of $\mathcal{W} \sqcup \mathcal{R}$ by virtue of being non-negative on $\varepsilon_q + \mathcal{P}_q$ for all q, since these polytopes are generated by those elements.

(iv) If φ is a functional on $\mathcal{H}$ non-negative on $\mathcal{W} \sqcup \mathcal{R}$, then since $\varepsilon_p + \mathcal{P}_p$ is a convex hull of non-negative rational multiples of points in $\mathcal{W} \sqcup \mathcal{R}$, we have $\langle \varepsilon_p + \mathcal{P}_p, \varphi_p \rangle \geq 0$ for any $p \in \mathbb{P}^1$. Then

$$0 \leq \sum_{p \in \mathbb{P}^1} \langle \varepsilon_p + \mathcal{P}_p, \varphi_p \rangle = \sum_{p \in \mathbb{P}^1} \langle \varepsilon_p, \varphi_p \rangle + \sum_{p \in \mathbb{P}^1} \langle \mathcal{P}_p, \varphi_p \rangle = \langle \mathcal{P}, \varphi \rangle.$$

Hence the cone generated by $\mathcal{P}$ is contained in $\mathcal{C}_p$ for all $p \in \mathbb{P}^1$. Likewise by arguments in the type I case, the cones in each $\mathcal{H}_p$ generated by elements of $\mathcal{W} \sqcup \mathcal{R}$ falling into $\mathcal{H}_p$ are contained in $\mathcal{C}_p$ for all $p \in \mathbb{P}^1$. The reverse inclusion is given by (iii). □

Having shown how $\mathcal{C}$ is generated for B-charts of both types, we are ready to interpret the remaining conditions from the Luna-Vust theory in terms of $\mathcal{C}$.

Proposition 3.8. *For any B-chart $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$ and $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$, the condition* (C) *of the Luna-Vust theory says that each $\mathcal{C}_p$, $p \in \mathbb{P}^1$ is strictly convex, and $0 \notin \kappa(\mathcal{R})$. For B-charts of type II, we additionally have $0 \notin \mathcal{P}$.*

Proof. [Tim97, §3.2] Since each $\mathcal{C}_p$ lies in a half-space, these cones are all strictly convex if and only if $\mathcal{K}$ is strictly convex. By Proposition 3.1, only finitely many elements of $\mathcal{W} \sqcup \mathcal{R}$ fall in $\mathcal{Z}$. Then (C) holds if and only if for any finite subset of $\mathcal{W} \sqcup \mathcal{R}$, there is a functional φ non-negative on $\mathcal{W} \sqcup \mathcal{R}$ and positive on this subset.

Assume (C). Since there are finitely many central valuations mapping to $\mathcal{K}$, there is a functional non-negative on $\mathcal{W} \sqcup \mathcal{R}$ and positive on $\mathcal{K}$. This is impossible if $\mathcal{K}$ contains a subspace of $\mathcal{Q}$. Hence $\mathcal{K}$ is strictly convex and hence so are all $\mathcal{C}_p$.

Likewise, there are finitely many elements of $\mathcal{R}$ falling in any particular $\mathcal{H}_p$, so there is a functional positive on $\kappa(\mathcal{R}) \cap \mathcal{H}_p$ for any p, so $0 \notin \kappa(\mathcal{R})$.

Finally, if X_0 is of type II, then all but finitely many $\mathcal{P}_p$ are $\{0\}$, so $\mathcal{P}$ is in fact generated by a finite set of elements of $\mathcal{W} \sqcup \mathcal{R}$. Hence there is a functional positive on $\mathcal{P}$, so $0 \notin \mathcal{P}$.

Now suppose strict convexity of $\mathcal{K}$ is given, $0 \notin \kappa(\mathcal{R})$ and, if X_0 is of type II, $0 \notin \mathcal{P}$. Let $\mathcal{V}_0$ be a finite subset of $\mathcal{W} \sqcup \mathcal{R}$, and suppose without loss of generality that elements of $\mathcal{V}_0$ all fall in $\mathcal{H}_p$ for some $p \in \mathbb{P}^1$. Take a functional on $\mathcal{Z}$ which is positive on $\mathcal{K} \setminus \{0\}$ and extend it to functionals φ_q on $\mathcal{H}_q$ for all $q \in \mathbb{P}^1$. If X_0 is of type I, the φ_q are arbitrary, and if X_0 is of type II, we choose φ_q to be supporting functionals of the polytopes $\varepsilon_q + \mathcal{P}_q$ as in Proposition 3.7 (iii).

In type I, choose p_0 with $\mathcal{H}_{p_0}$ containing no non-central valuations. We increase φ_p so that it is positive on $\mathcal{V}_0$, and vary φ_{p_0} to obtain the balancing condition.

In type II, we have $\sum_{q \in \mathbb{P}^1} \langle \varepsilon_q, \varphi_q \rangle \leq 0$, so at the same time we increase $\langle \varepsilon_p, \varphi_p \rangle$ to make φ positive on $\mathcal{V}_0$ and satisfy the balancing conditon. Either way, (C) holds. □

Proposition 3.9. *Condition* (W) *of the Luna-Vust theory says that elements* $w \in \mathcal{W}$ *are the generators of the edges of the cones* $\mathcal{C}_p$ *which do not intersect* $\kappa(\mathcal{R})$ *or (for charts of type II)* $\mathcal{P}$*.*

Proof. [Tim97, §3.2] Assume (W). If there is a functional φ non-negative on $\mathcal{C}$ but negative on w, w generates an edge of $\mathcal{C}_p$ for whichever p satisfies $w|_{K^B} = h\nu_p$. These edges cannot intersect $\mathcal{R}$ since $\langle \mathcal{R}, \varphi \rangle \geq 0$ by assumption. If X_0 is of type II, then for the edge generated by w to intersect $\mathcal{P}$, it is necessary that $w \in \mathcal{K}$. But then $\mathcal{W} \setminus \{w\} \sqcup \mathcal{R}$ generates the same polytope $\mathcal{P}$ as $\mathcal{W} \sqcup \mathcal{R}$, and $\langle \mathcal{W} \setminus \{w\} \sqcup \mathcal{R}, \varphi \rangle \geq 0$ implies $\langle \mathcal{P}, \varphi \rangle \geq 0$ (by Proposition 3.7 (iii)). Hence $\mathcal{P}$ does not intersect the edge generated by w either.

Conversely, let $\mathbb{Q}_{\geq 0} w$ be an edge of $\mathcal{C}_p$ not intersecting $\kappa(\mathcal{R})$ (or $\mathcal{P}$ in type II). Choose a functional φ_p on $\mathcal{H}_p$ which is negative on w, non-negative on elements of $\mathcal{W} \setminus \{w\} \sqcup \mathcal{R}$ falling in $\mathcal{H}_p$, and for type II, non-negative on $\mathcal{P}$. Now by applying either Lemma 3.2(ii) in type I or Proposition 3.7(iii) in type II to φ_p and $\mathcal{W} \setminus \{w\} \sqcup \mathcal{R}$, we obtain a functional φ on $\mathcal{H}$ negative on w and non-negative on $\mathcal{W} \setminus \{w\} \sqcup \mathcal{R}$. Hence $\mathcal{W}$ can be recovered from $\mathcal{C}$. □

Definition 3.2. A *hypercone* in $\mathcal{H}$ is a union $\mathcal{C} = \bigcup_{p \in \mathbb{P}^1} \mathcal{C}_p$ of finitely generated convex cones $\mathcal{C}_p = \mathcal{C} \cap \mathcal{H}_p$ such that:

(i) $\mathcal{C}_p = \mathcal{K} + \mathbb{Q}_{\geq 0} \varepsilon_p$ for all but finitely many p, where $\mathcal{K} = \mathcal{C} \cap \mathcal{Z}$;

(ii) Either:

(I) there exists $p \in \mathbb{P}^1$ with $\mathcal{C}_p = \mathcal{K}$, or;

(II) the polytope $\mathcal{P} = \sum_{p \in \mathbb{P}^1} \mathcal{P}_p$ is non-empty, where the $\mathcal{P}_p$ are defined by $\varepsilon_p + \mathcal{P}_p = \mathcal{C}_p \cap (\varepsilon_p + \mathcal{Z})$.

The hypercone $\mathcal{C}$ is called *strictly convex* if every $\mathcal{C}_p$ is strictly convex and $0 \notin \mathcal{P}$.

Definition 3.3. A *coloured hypercone* in $\mathcal{H}$ is a pair $(\mathcal{C}, \mathcal{R})$ such that $\mathcal{R} \subseteq \mathcal{D}^B$, $0 \notin \mathcal{R}$, and $\mathcal{C}$ is a strictly convex hypercone in $\mathcal{H}$ generated by $\kappa(\mathcal{R})$, a finite subset $\mathcal{W} \subseteq \mathcal{V}$, and (if $\mathcal{C}$ is of type II) the polytope $\mathcal{P}$.

Theorem 3.2. *B-charts correspond bijectively to coloured hypercones in $\mathcal{H}$ of the corresponding type.*

Proof. [Tim97, Thm 3.1] By Proposition 3.8 and Proposition 3.9, if $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$ is a B-chart, then $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$ is a coloured hypercone of the corresponding type. Furthermore from the same results, given a coloured hypercone $(\mathcal{C}, \mathcal{R})$, we can construct $\mathcal{W} \subseteq \mathcal{V}$ such that $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$ corresponds to a B-chart. □

G-Germs and Supported Hypercones

Having obtained a full description of the B-charts, we move on to describing how G-germs behave in hyperspace.

Proposition 3.10. *We say that a G-germ Y is of* type I *if it admits a B-chart of type I, and of* type II *if it does not. A G-germ Y is of type I if and only if $\mathcal{V}_Y \sqcup \mathcal{D}^B_Y$ is finite, and of type II if and only if it admits a minimal B-chart.*

Proof. [Tim97, §3.3] Suppose $\mathcal{V}_Y \sqcup \mathcal{D}^B_Y$ is finite and choose a B-chart $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$ intersecting Y. There are infinitely many $p \in \mathbb{P}^1$ such that $\mathcal{H}_p$ contains no elements of $\mathcal{V}_Y \sqcup \mathcal{D}^B_Y$. Choose one such p with the additional property that $\mathcal{C} = \mathcal{C}(\mathcal{W}, \mathcal{R})$ is generated by $\mathcal{K}$ and $\mathbb{Q}_{\geq 0}\varepsilon_p$, and remove any valuation mapping to ε_p from $\mathcal{W} \sqcup \mathcal{R}$. The resulting coloured data define a B-chart of type I intersecting Y.

Conversely, suppose Y admits a B-chart $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$ of type I, i.e. $\mathcal{A}^B \neq \Bbbk$. Let f be a nonconstant element of $\mathcal{A}^B \subseteq K^B$. Then f defines a functional φ of type I on hyperspace, i.e. φ vanishes on $\mathcal{Z}$ and on all but finitely many $\mathcal{H}_p$, and the remaining $\mathcal{H}_p$ are split into those on which φ is positive or negative. Since $f \in \mathcal{A}$, we have

$\langle \mathcal{W} \sqcup \mathcal{R}, \varphi \rangle \geq 0$. In particular, there are only finitely many elements of $\mathcal{W} \sqcup \mathcal{R}$ on which φ is positive. Hence we take $f \in \mathcal{A}^B \cap \mathcal{I}_Y$. Then φ is positive on any divisor containing Y, hence there can only be finitely many such divisors, and $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ is finite.

We see then that a G-germ is of type II if $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ is infinite. Suppose Y admits a minimal B-chart, i.e. $X_Y = \operatorname{Spec} \mathcal{A}(\mathcal{V}_Y, \mathcal{D}_Y^B)$ is a B-chart. Then $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ must be infinite since any B-chart must intersect infinitely many colours, hence Y is of type II.

Now suppose Y is of type II. Then $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ must contain a non-central valuation mapping to $\mathcal{H}_p$ for every $p \in \mathbb{P}^1$, or it would admit a B-chart of type I. Then $\mathcal{V} \setminus \mathcal{V}_Y$ and $\mathcal{R} \setminus \mathcal{D}_Y^B$ must be finite, so $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{V}_Y, \mathcal{D}_Y^B)$ is a minimal B-chart for Y. □

Proposition 3.11. *Let Y be a G-germ with coloured data $(\mathcal{V}_Y, \mathcal{D}_Y^B)$ and let $\mathcal{C}_Y = \mathcal{C}(\mathcal{V}_Y, \mathcal{D}_Y^B)$ be the bidual set in $\mathcal{H}$ to $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$, as in the definition of the hypercone corresponding to a B-chart. If Y is of type I, then $\mathcal{C}_Y$ is a coloured cone in some $\mathcal{H}_p$. If Y is of type II, then $\mathcal{C}_Y$ is a coloured hypercone of type II.*

Proof. [Tim97, §3.3] If Y is of type I, we have seen that $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ is finite. For $\mathcal{C}_Y$ to be a coloured cone in some $\mathcal{H}_p$, the only thing we need to check is that all non-central elements of $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ fall in the same slice of the hyperspace.

Let X_0 be a B-chart of type I intersecting Y and let $X = G \cdot X_0$ be the corresponding G-model. Consider the rational B-quotient $\pi \colon X \dashrightarrow \mathbb{P}^1$. For any B-stable $D \supseteq Y$, either D is central or $\pi(D)$ is a point p_D. Hence for all non-central D whose valuations lie in $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$, we have $\pi(Y) \subseteq \pi(D) = \{p_D\}$. Now either $\pi(Y)$ is empty, i.e. Y lies outside the domain of definition of π, in which case $\mathcal{V}_Y \sqcup \mathcal{D}_Y^B$ consists entirely of central elements, or $\pi(Y)$ is nonempty, in which case $\emptyset \neq \pi(Y) \subseteq \cap_D \{p_D\}$, and all p_D must be the same point.

If Y is of type II, then it admits a minimal B-chart $X_Y = \operatorname{Spec} \mathcal{A}(\mathcal{V}_Y, \mathcal{D}_Y^B)$ of type II. Then $\mathcal{C}_Y$ is the coloured hypercone of type II corresponding to X_Y and we are done. □

Definition 3.4. Let $\mathcal{C}$ be a hypercone of type II in $\mathcal{H}$. The *relative interior* of $\mathcal{C}$ is the union $\bigcup_{p \in \mathbb{P}^1} \operatorname{relint} \mathcal{C}_p \cup \operatorname{relint} \mathcal{K}$.

Proposition 3.12. *Let Y be a G-germ with coloured data $(\mathcal{V}_Y, \mathcal{D}_Y^B)$. Condition* (S) *of the Luna-Vust theory means that a valuation $\nu \in \mathcal{V}$ is in $\mathcal{S}_Y$ if and only if $\nu \in$*

relint $\mathcal{C}_Y \cap \mathcal{V}$.

Proof. [Tim11, §16.4] Condition (S) says that $\nu \in \mathcal{S}_Y$ if and only if for any functional φ on $\mathcal{H}$, $\langle \mathcal{V}_Y \sqcup \mathcal{D}_Y^B, \varphi \rangle \geq 0$ implies $\langle \nu, \varphi \rangle \geq 0$, and if $>$ appears on the left, it appears on the right. The first condition is by definition equivalent to $\nu \in \mathcal{C}_Y$, and the second forces $\nu \in \operatorname{relint} \mathcal{C}_Y$. The claim follows. □

Definition 3.5. Let $\mathcal{C}$ be a hypercone in $\mathcal{H}$. A *face* of $\mathcal{C}$ is a face of some $\mathcal{C}_p$ not intersecting $\mathcal{P}$. A *hyperface* of $\mathcal{C}$ is a hypercone $\mathcal{C}' = \mathcal{C} \cap \ker \varphi$ for some functional φ on $\mathcal{H}$ with $\langle \mathcal{C}, \varphi \rangle \geq 0$. We call φ a *supporting functional* for the face $\mathcal{C}'$.

A *(hyper)face* of a coloured hypercone $(\mathcal{C}, \mathcal{R})$ is a coloured (hyper)cone $(\mathcal{C}', \mathcal{R}')$ where $\mathcal{C}'$ is a (hyper)face of $\mathcal{C}$ and $\mathcal{R}' = \mathcal{R} \cap \kappa^{-1}(\mathcal{C}')$.

Lemma 3.3. *Let $(\mathcal{C}, \mathcal{R})$ be a coloured hypercone and let $\nu \in \mathcal{C} \setminus \operatorname{relint} \mathcal{C}$ be nonzero. Then there is a unique face or hyperface $(\mathcal{C}', \mathcal{R}')$ of $(\mathcal{C}, \mathcal{R})$ containing ν in its relative interior, given as the intersection of all hyperfaces of $\mathcal{C}$ containing ν.*

Proof. [Tim11, Lemma 16.17] Since the relative interior is $\mathcal{C}$ with all (hyper)faces removed, clearly ν lies in some (hyper)face of $\mathcal{C}$. It cannot lie in every (hyper)face, since $\{0\}$ is a face, so it must lie in the relative interior of some (hyper)face, which must then be unique.

Hence let $\mathcal{C}'$ be the (hyper)face containing ν in its relative interior. Let φ be a functional on $\mathcal{H}$ such that $\langle \mathcal{C}, \varphi \rangle \geq 0$ and $\langle \nu, \varphi \rangle = 0$, i.e. ν lies in the hyperface $\mathcal{C} \cap \ker \varphi$ of $\mathcal{C}$. Then $\langle \mathcal{C}', \varphi \rangle = 0$ since otherwise, $\nu \in \operatorname{relint} \mathcal{C}'$ would imply $\langle \nu, \varphi \rangle > 0$, a contradiction. Hence $\mathcal{C}' \subseteq \mathcal{C} \cap \ker \varphi$ for any such φ, i.e. $\mathcal{C}'$ is contained in the intersection of all hyperfaces containing ν. If $\mathcal{C}$ is itself a hyperface, then since it contains ν itself, the reverse inclusion must hold.

Assuming now then that $\mathcal{C}'$ is a face, the reverse inclusion follows from the claim: for any $q \in \mathcal{C} \setminus \mathcal{C}'$, there is a functional φ on $\mathcal{H}$ such that $\langle \mathcal{C}, \varphi \rangle \geq 0$, $\langle \nu, \varphi \rangle = 0$ and $\langle q, \varphi \rangle > 0$.

Say $\mathcal{C}' \subseteq \mathcal{C}_p$ for some $p \in \mathbb{P}^1$. Then take a functional φ_p on $\mathcal{H}_p$ such that $\langle \mathcal{C}_p, \varphi_p \rangle \geq 0$ and $\mathcal{C}' = \mathcal{C} \cap \ker \varphi_p$. Now take $\varphi_p|_{\mathcal{Z}}$ and extend it to every $\mathcal{H}_p$, using Lemma 3.2(i) if $\mathcal{C}$ is of type I or Proposition 3.7(iii) if $\mathcal{C}$ is of type II to ensure that the resulting collection defines a functional φ on $\mathcal{H}$ non-negative on $\mathcal{C}$. Then $\langle \mathcal{C}, \varphi \rangle \geq 0$, $\langle \nu, \varphi \rangle = 0$

by construction, and if $q \in \mathcal{H}_x$ we can increase φ_x to give $\langle q, \varphi_x \rangle > 0$ while retaining the balancing condition by decreasing some other φ_y. □

Proposition 3.13. *Let $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$ be a B-chart, let $\mathcal{C}$ be the associated coloured hypercone and let Y be a G-germ with coloured data $(\mathcal{V}_Y, \mathcal{D}_Y^B)$. Then:*

(i) *Condition* (V) *of the Luna-Vust theory means that $\nu \in \mathcal{V}$ has centre on X_0 if and only if ν lies in $\mathcal{C}$.*

(ii) *Conditions* (V′) *and* (D) *of the Luna-Vust theory mean that elements of $\mathcal{W}$ and $\mathcal{R}$ lie in $\mathcal{V}_Y$ and $\mathcal{D}_Y^B$ if and only if they lie in $\mathcal{C}_Y$, which is a face of $\mathcal{C}$ if Y is of type I or a hyperface of type II if Y is of type II.*

Proof. [Tim11, §16.4] (i) Condition (V) says that ν has centre on X_0 if and only if $\langle \mathcal{W} \sqcup \mathcal{R}, \varphi \rangle \geq 0$ implies $\langle \nu, \varphi \rangle \geq 0$ for any functional φ, which is the defining property of elements of $\mathcal{C}$.

(ii) Condition (V′) says that if $\nu \in \mathcal{S}_Y$, then $w \in \mathcal{W}$ lies in $\mathcal{V}_Y$ if and only if for any functional φ on $\mathcal{H}$ with $\langle \mathcal{W} \sqcup \mathcal{R}, \varphi \rangle \geq 0$ and $\langle \nu, \varphi \rangle = 0$, we have $\langle w, \varphi \rangle = 0$. We know that $\nu \in \mathcal{S}_Y$ means ν is in the relative interior of $\mathcal{C}_Y$ which by the above Lemma means that $\mathcal{C}_Y$ is the unique (hyper)face of $\mathcal{C}$ containing ν in its relative interior, being a face if Y is of type I or a hyperface if Y is of type II. Then (V′) says that $w \in \mathcal{V}_Y$ if and only if $w \in \mathcal{C}_Y$. Condition (D) says the same for B-divisors, and the proof is identical. □

Definition 3.6. Let $\mathcal{C}$ be a hypercone of type II in $\mathcal{H}$. We say that $\mathcal{C}$ is *supported* if $\operatorname{relint} \mathcal{C} \cap \mathcal{V}$ is non-empty.

Theorem 3.3. *G-germs of type I are in bijection with supported coloured cones in $\mathcal{H}$, and G-germs of type II are in bijection with supported coloured hypercones of type II in $\mathcal{H}$. Inclusion of G-germs in each other corresponds to opposite inclusions of the respective (hyper)cones as (hyper)faces of each other.*

Proof. [Tim97, Thm 3.2] For G-germs of either type, we know the corresponding coloured (hyper)cones must be supported by (S), since $\mathcal{S}_Y$ is nonempty and consists of valuations mapping to $\operatorname{relint} \mathcal{C}_Y \cap \mathcal{V}$.

Let Y be a G-germ of type I with corresponding coloured cone $(\mathcal{C}_Y, \mathcal{D}_Y^B)$. Choose a B-chart of type I intersecting Y with coloured hypercone $(\mathcal{C}, \mathcal{R})$. Then we know that

$\mathcal{W}$ can be recovered from $\mathcal{C}$ as the set of generators of edges not intersecting $\kappa(\mathcal{R})$. Then $\mathcal{V}_Y$ is recovered as the subset of $\mathcal{W}$ of generators of edges included in the face $\mathcal{C}_Y$ of $\mathcal{C}$. Hence the correspondence of G-germs of type I with supported coloured cones is bijective.

Since G-germs of type II admit minimal B-charts, and B-charts of type II correspond bijectively to coloured hypercones of type II, we just need to prove that any B-chart of type II whose coloured hypercone is supported corresponds to a unique G-germ of type II. Indeed let X_0 be a B-chart with coloured hypercone $\mathcal{C}$ which is supported. Then $\operatorname{relint} \mathcal{C} \cap \mathcal{V}$ is a non-empty finite set of G-valuations and hence corresponds to the support of a G-germ Y. Since supports of distinct G-germs must be disjoint, Y is determined by $\mathcal{C}$.

If $Y_1 \subseteq Y_2$ are G-germs of any type, then $(\mathcal{V}_{Y_2}, \mathcal{D}^B_{Y_2}) \subseteq (\mathcal{V}_{Y_1}, \mathcal{D}^B_{Y_1})$, so $\mathcal{C}_{Y_2}$ is contained in $\mathcal{C}_{Y_1}$. Let $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R})$ be a B-chart with coloured hypercone $\mathcal{C}$ intersecting Y_2, and hence Y_1. Then $\mathcal{C}_{Y_1}$ and $\mathcal{C}_{Y_2}$ are both (hyper)faces of $\mathcal{C}$, so $\mathcal{C}_{Y_2}$ is a (hyper)face of $\mathcal{C}_{Y_1}$. The converse implication is immediate. □

G-models and Hyperfans

We know that G-models are determined by their G-germs, which lie in a finite collection of B-charts. The supports of the G-germs must be disjoint, and inclusions of G-germs must be kept track of. In this spirit we make the following definition:

Definition 3.7. A *coloured hyperfan* in $\mathcal{H}$ is a collection of supported coloured cones and supported coloured hypercones of type II in $\mathcal{H}$, obtained as the set of all supported (hyper)faces of a finite collection of coloured hypercones, subject to the condition that the relative interiors of these (hyper)faces are disjoint inside $\mathcal{V}$.

Then by Theorem 3.2, Proposition 3.12 and Theorem 3.3, we have proved [Tim97, Thm 3.3]:

Theorem 3.4. *G-models of K are in bijection up to isomorphism with coloured hyperfans in $\mathcal{H}$.*

Applying Proposition 2.16 and Proposition 2.17, we obtain:

Corollary 3.1. *Affine G-models of K are in bijection with coloured hypercones of the form $(\mathcal{C}, \mathcal{D}^B)$.*

Corollary 3.2. *A G-model is complete if and only if its coloured hyperfan covers $\mathcal{V}$.*

Proposition 3.14. *We say that a G-model X is of* type I *if all of its G-germs are of type I, and of* type II *if it contains any G-germ of type II. For any G-model X, there exists a G-model $\check{X}$ of type I and a proper birational morphism $\varphi\colon \check{X} \to X$.*

Proof. [Tim11, §16.6] We may assume without loss of generality that X contains only one G-germ Y of type II. Consider the supported coloured hypercone of type II $\mathcal{C}_Y \subseteq \mathcal{H}$ corresponding to Y: for each $p \in \mathbb{P}^1$, the slice $\mathcal{C}_{Y,p} = \mathcal{C}_Y \cap \mathcal{H}_p$ of $\mathcal{C}_Y$ is a supported coloured cone in $\mathcal{H}_p$. Let $\check{X}$ be the G-model corresponding to the coloured hyperfan obtained as the collection of the coloured cones corresponding to all G-germs of X of type I and all $\mathcal{C}_{Y,p}$. Then $\check{X}$ is a G-model of $K = \Bbbk(X)$ of type I.

We have a birational map $\varphi\colon \check{X} \dashrightarrow X$ since they have the same function field. By Proposition 2.13, this is a morphism if and only if for any G-germ $\check{Z} \subseteq \check{X}$, there exists a G-germ $Z \subseteq X$ such that $\mathcal{O}_{\check{X},\check{Z}}$ dominates $\mathcal{O}_{X,Z}$. This is the case since every G-germ of $\check{X}$ is either also a G-germ of X or one of the $\check{Y}_p$ corresponding to $\mathcal{C}_{Y,p}$. In this case the coloured data $(\mathcal{V}_{\check{Y}_p}, \mathcal{D}^B_{\check{Y}_p})$ are contained in $(\mathcal{V}_Y, \mathcal{D}^B_Y)$ and so $\mathcal{O}_{\check{X},\check{Y}_p}$ dominates $\mathcal{O}_{X,Y}$ for all $p \in \mathbb{P}^1$.

Finally, Theorem 2.11 says that φ is proper if and only if $\bigcup_{\check{Z}\subseteq\check{X}} \mathcal{S}_{\check{Z}} = \varphi_*^{-1}(\bigcup_{Z\subseteq Y} \mathcal{S}_Z)$, where φ_* is the restriction map $\mathcal{V}(\check{K}) \to \mathcal{V}(K)$. Since $\check{K} = K$, φ_* is the identity, so we just need $\bigcup_{\check{Z}\subseteq\check{X}} \mathcal{S}_{\check{Z}} = \bigcup_{Z\subseteq X} \mathcal{S}_Z$. This holds since we either have $\check{Z} = Z$ or $\check{Z} = \check{Y}_p$ and then $\mathcal{S}_Y = \operatorname{relint} C_Y \cap \mathcal{V} = \bigcup_{p\in\mathbb{P}^1} \operatorname{relint} \mathcal{C}_{Y,p} \cap \mathcal{V}_p = \bigcup_{p\in\mathbb{P}^1} \mathcal{S}_{\check{Y}_p}$. □

3.2 Homogeneous SL_2-Spaces

We can now apply the theory from the previous sections to calculate the coloured hyperspace of a complexity one homogeneous space, and we explain how we can use one calculated example to simplify the calculation of others. Using these techniques we calculate the coloured hyperspace for every homogeneous complexity-one SL_2-space. First, we outline a theoretical technique for calculating the position of G-valuations in the hyperspace.

3.2.1 The Method of Formal Curves

Let G/H be a complexity one homogeneous space with function field K, G-valuations $\mathcal{V}$ and so on. The *method of formal curves* is a way to calculate the position of valuations in $\mathcal{V}$ in the hyperspace $\mathcal{H}$ of G/H by restricting functions $f \in K$ to formal curves approaching the associated divisors. We explain this method following [Tim11, §24], mainly without proofs.

Specifically, let $\mathcal{V}^1 \subseteq \mathcal{V}$ be the subset of G-valuations of K such that $\Bbbk(\nu)^G = \Bbbk$. Geometrically, given an embedding $G/H \to X$, if D is a G-stable divisor of X and ν is proportional to ν_D, then $\nu \in \mathcal{V}^1$ if and only if D contains a dense G-orbit: indeed if this is the case, then $\mathcal{O}_\nu = \mathcal{O}_{X,D}$, $\mathfrak{m}_\nu = \mathfrak{m}_{X,D}$, so $\Bbbk(\nu) = \Bbbk(D)$. Then $\Bbbk(D)^G = \Bbbk(\nu)^G = \Bbbk$ if and only if D contains a homogeneous space as a dense orbit.

The method of formal curves computes valuations in the subset $\mathcal{V}^1$, and the following fact shows that this is essentially sufficient to calulate $\mathcal{V}$.

Proposition 3.15. *When G/H is of complexity one, $\mathcal{V}^1$ contains all non-central valuations in $\mathcal{V}$.*

Proof. By [Tim11, Prop. 21.3], a nonzero G-valuation ν is central if and only if $c_G(\Bbbk(\nu)) = c_G(K) = 1$ and non-central if and only if $c_G(\Bbbk(\nu)) = c_G(K) - 1 = 0$. Thus a non-central valuation on G/H has $\Bbbk(\nu)^G \subseteq \Bbbk(\nu)^B = \Bbbk$. □

Given this, the central valuations can be calculated as 'limits' of the non-central ones, in a certain sense. We move on now to describe the method itself.

Definition 3.8. Let X be a $\Bbbk$-scheme and let A be a ring. We call a $\Bbbk$-morphism $\operatorname{Spec} A \to X$ an *A-point* of X and denote the set of A-points of X by $X(A)$.

Now suppose X is a $\Bbbk$-variety and A is a local $\Bbbk$-algebra. Let χ be an A-point of X, i.e. $\chi\colon \operatorname{Spec} A \to X$. The (unique) closed point of A is mapped by χ to the generic point of a closed subvariety $Y \subseteq X$, which we call the *centre* of χ.

Definition 3.9. Let X be a $\Bbbk$-variety. A *germ of a curve* in X is a pair (χ, θ_0), whereΘ is a smooth projective curve, $\chi \in X(\Bbbk(\Theta))$, and $\theta_0 \in \Theta$. That is, $\chi\colon \Theta \to X$ is a rational map from a smooth projective curve and $\theta_0 \in \Theta$ is a basepoint. The germ is *convergent* if $\chi \in X(\mathcal{O}_{\Theta,\theta_0})$, i.e. χ is regular at θ_0, in which case the point $x_0 = \chi(\theta_0) \in X$ is then *limit* of the curve.

The idea is that for $f \in K$ and $\nu \in \mathcal{V}^1$, if ν is proportional to ν_D for a divisor D containing a dense G-orbit, we take a curve Θ meeting D transversally at a point $x_0 \in D$ with $\overline{G \cdot x_0} = D$ and calculate $\nu(f)$ by restricting to Θ. In terms of germs, take a germ of a curve $\chi\colon \Theta \dashrightarrow G/H$ which converges to $x_0 \in D$. Then we have $\nu_D(f) = \nu_{\theta_0}(\chi^*(gf))$ for general $g \in G$, modulo positive multiples given by the intersection multiplicity of Θ and D at x_0.

It turns out that the calculations are easier using the related notion of a formal germ in D:

Definition 3.10. A *germ of a formal curve* (or *formal germ*) in X is a $\Bbbk(\!(t)\!)$-point of X. A $\Bbbk[[t]]$-point is a *convergent formal germ* and its centre $x_0 \in X$ is the *limit* of the formal germ.

Any germ of a curve $\theta_0 \in \Theta \dashrightarrow X$ defines a formal germ as follows: take the completion $\widehat{\mathcal{O}}_{\Theta,\theta_0}$ of the local ring of Θ at θ_0 and a uniformising parameter t in this ring. This gives $\widehat{\mathcal{O}}_{\Theta,\theta_0} \cong \Bbbk[[t]]$ and $\Bbbk(\Theta) \subseteq \Bbbk(\!(t)\!)$, inducing a map $\operatorname{Spec} \Bbbk(\!(t)\!) \to \operatorname{Spec} \Bbbk(\Theta) \to X$, i.e. a formal germ in X. Clearly a convergent germ of a curve induces a convergent formal germ with the same limit. There is a sense (see [Tim11, Thm A.16]) in which 'almost all' formal germs are induced by germs of formal curves.

In the formal germ perspective, the calculation of ν then works as follows: given a formal germ $x(t) \in G/H(\Bbbk(\!(t)\!))$ induced by a germ of a curve (χ, θ_0), we have $\nu_{\theta_0}(\chi^*(gf)) = \nu_{x(t)}(f) := \operatorname{ord}_t f(g \cdot x(t))$, where g is a general point of G.

Theorem 3.5. *For any $x(t) \in G/H(\Bbbk(\!(t)\!))$, the formula $\nu_{x(t)}(f) = \operatorname{ord}_t f(g \cdot x(t))$ defines a valuation $\nu_{x(t)} \in \mathcal{V}^1$, and any $\nu \in \mathcal{V}^1$ is proportional to some $\nu_{x(t)}$. If $X \supseteq G/H$ is a G-model of K and $Y \subseteq X$ is the centre of ν, then $x(t) \in X(\Bbbk[[t]])$ and $Y = \overline{G \cdot x(0)}$.*

Proof. [Tim11, Thm 24.2] □

Hence we can calculate $\mathcal{V}^1$ by considering only formal germs. Now we see a series of useful results which shrink the number of formal germs we actually have to consider to calculate all valuations in $\mathcal{V}^1$.

Lemma 3.4. *For any $g(t) \in G(\Bbbk[[t]])$ and any $x(t) \in G/H(\Bbbk(\!(t)\!))$, we have $\nu_{g(t)x(t)} = \nu_{x(t)}$.*

Proof. [Tim11, Lemma 24.3] □

Lemma 3.5. *For any $x(t) \in G/H(\Bbbk((t)))$ there exists $n \in \mathbb{N}$ such that $x(t^n) = g(t) \cdot H$ for some $g(t) \in G(\Bbbk((t)))$.*

Proof. [Tim11, Lemma 24.5] □

Since $\nu_{x(t^n)} = n \cdot \nu_{x(t)}$ and we only care about valuations up to positive multiples, the two above lemmas mean that we can describe $\mathcal{V}^1$ using only formal germs in $G(\Bbbk((t)))$, considered up to left multiplication by $G(\Bbbk[[t]])$ and right multiplication by $H(\Bbbk((t)))$. One final decomposition further reduces the number of formal germs to consider:

Iwasawa Decomposition. *There is a decomposition $G(\Bbbk((t))) = G(\Bbbk[[t]]) \cdot \mathfrak{X}^*(T) \cdot U(\Bbbk((t)))$ where T is a maximal torus in G, U is a maximal unipotent subgroup and $\mathfrak{X}^*(T)$ is regarded as a subset of $T(\Bbbk((t)))$.*

Combining all of the above restrictions, we can restate the Theorem above as:

Corollary 3.3. *Every $\nu \in \mathcal{V}^1$ is proportional to $\nu_{g(t)}$ for $g(t) \in \mathfrak{X}^*(T) \cdot U(\Bbbk((t)))$.*

Proof. [Tim11, Cor 24.6] □

We can now use this result to greatly simplify the calculations of non-central G-valuations for any homogeneous space.

3.2.2 SL_2

Now we will caluclate the coloured data of the homogeneous space SL_2, as done by Timashev [Tim97, §5].

Consider the action of $G = \mathrm{SL}_2$ on itself by left multiplication of matrices, let $K = \Bbbk(G)$ and choose subgroups B, U and T of G consisting of upper triangular, upper unitriangular and diagonal matrices, respectively. The action gives the homogeneous space $G/H = \mathrm{SL}_2/\{e\}$. The B-orbit of the identity is B itself, which is a maximal orbit of codimension 1, so this is a complexity one homogeneous space. Note that $\mathfrak{X}(B) = \mathbb{Z}\alpha$ where α is the character $\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right) \mapsto a$. For $g = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$, the functions $g \mapsto z$ and $g \mapsto w$ are semi-invariant of weight α, so $\Lambda = \mathbb{Z}\alpha$ as well. These semi-invariants generate the space $M = \Bbbk[G]^{(B)}_\alpha$, and $K^B = \Bbbk(z/w)$. Fix a splitting $e \colon \Lambda \to K^{(B)}$ given by $e_\alpha = z$. Then all semi-invariants are of the form fe^k_α where $f \in K^B$ and $k \in \mathbb{Z}$.

We are interested in G-valuations and valuations of colours of K, which are determined by their restrictions to $K^{(B)}$, which are in turn determined by their restrictions to K^B and a functional $\ell\colon \Lambda \to \mathbb{Q}$.

The rational B-quotient map is determined by the invariant z/w and thus looks like $\pi\colon \mathrm{SL}_2 \to \mathbb{P}^1$, $g \mapsto [z : w]$. The regular semi-invariants thus lie in the space M generated by z and w. Since no other semi-invariants can divide z or w, there are no subregular semi-invariants. Likewise there are also no central colours. The fibre of a point $p = [\alpha\colon \beta] \in \mathbb{P}^1$ is the regular B-divisor $D_p = \mathcal{Z}(\beta z - \alpha w)$, and all B-stable divisors are of this form. The chosen splitting e marks out the point $\infty = [0 : 1]$ with $D_\infty = \mathcal{Z}(e_\alpha)$ as distinguished. As discussed in Section 3.4, non-distinguished regular colours D_p for $p \neq \infty$ sit at $(p, \ell, h) = (p, 0, 1) \in \mathcal{H}_p$, and D_∞ has $\ell = \nu_\infty(e_\alpha) = 1$, so sits at $(\infty, 1, 1) \in \mathcal{H}_\infty$.

We can calculate $\mathcal{V}^1 \subseteq \mathcal{V}$ using Corollary 3.3: fix $m \in \mathbb{Z}$ and $u(t) \in \Bbbk(\!(t)\!)$ and let

$$x(t) = \begin{pmatrix} t^m & u(t) \\ 0 & t^{-m} \end{pmatrix} \in \mathfrak{X}^*(T) \cdot U(\Bbbk(\!(t)\!))$$

where $\operatorname{ord}_t u(t) = n \leq -m$. Then any non-central G-valuation is proportional to $\nu_{x(t)}$, where $\nu_{x(t)}(f) = \operatorname{ord}_t(f(g \cdot x(t)))$ for any $f \in K^{(B)}$ and generic $g \in G$. Let $p = [\alpha : \beta]$ and

$$d_p = \nu_{x(t)}(\beta z - \alpha w) = \operatorname{ord}_t((\beta t^m - \alpha u(t))z - \alpha t^{-m} w).$$

The value of d_p is constant along $\mathbb{P}^1$ except at the distinguished point, where it jumps by some $h \in \mathbb{Q}_{\geq 0}$, so that ν is represented in hyperspace by (x, ℓ, h), where $\ell = \nu_{x(t)}(e_\alpha)$.

Note that for any p, $d_p \in [m, -m]$. Now suppose that $m \leq n$. We have $d_p \geq \min\{\operatorname{ord}_t(\beta t^m - \alpha u(t)), -m\} = \operatorname{ord}_t(\beta t^m - \alpha u(t)) \geq \min\{m, n\} = m$. Now if $d_p > m$, we have $d_p \in (m, -m]$. Otherwise $d_p = m$. Since h is the difference between the maximum possible value of d_p (which is $-m$) and the minimum, which we see lies in the interval $[m, -m)$, we have $h \in (0, -2m]$. Finally, ℓ is given by the value of d_p, which at non-distinguished points is m and at the distinguished point is $m + h$.

In the case $m > n$, we have $d_p = n$ when $\alpha \neq 0$, and when $\alpha = 0$ (at the distinguished point), we have $d_p = m$. Hence $h = m - n$, $\ell = n$ for nondistinguished points and $\ell = n + h$ for ∞.

In either case we have $h > 0$ and the possible (ℓ, h) are defined for $p \neq \infty$ by the inequality $2\ell + h \leq 0$, and for $p = \infty$ by $2\ell - h \leq 0$. Re-including the central valuations allows $h \geq 0$. Thus we have valuation cones $\mathcal{V}_p = \{(\ell, h) \in \mathcal{H}_p \mid 2\ell + h \leq 0, h \geq 0\}$ for $p \neq \infty$, and $\mathcal{V}_\infty = \{(\ell, h) \in \mathcal{H}_\infty \mid 2\ell - h \leq 0, h \geq 0\}$. The picture of the hyperspace is thus:

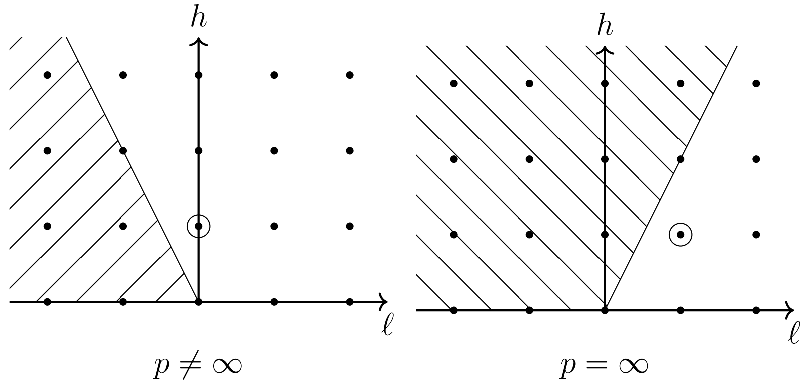


Figure 1: Coloured data of SL_2

where dashed areas denote the valuation cones and colours are denoted by unfilled circles.

3.2.3 Calculating Hyperspace of Quotients

Having calculated above the coloured data for the homogeneous space $SL_2/\{e\}$, it would be useful to be able to exploit this knowledge to make it easier to find the data for other homogeneous SL_2-spaces, rather than starting from scratch. Thankfully this is possible as follows [Tim97, §2.3]: suppose we know the structure of the hyperspace associated to a field $K = \Bbbk(G/H)$ and we want to calculate the data associated to the homogeneous space $G/\overline{H}$, where $\overline{H}$ contains H as a normal subgroup of finite index. Then $G/\overline{H} = (G/H)/(\overline{H}/H)$, so letting $A = \overline{H}/H$ we are in a situation where the finite group A acts on K and we want to find the coloured data for $\overline{K} = \Bbbk(G/\overline{H}) = K^A$ using the known data for K.

First, the lattice $\overline{\Lambda}$ of weights of B-eigenfunctions in $\overline{K}$ will be a sublattice of Λ of finite index (since A is finite). This has no effect on $\mathcal{Q} = \mathrm{Hom}(\Lambda, \mathbb{Q})$, i.e. $\overline{\mathcal{Q}} = \mathcal{Q}$, but there will be more integral points: $\mathrm{Hom}(\overline{\Lambda}, \mathbb{Z}) \subseteq \overline{\mathcal{Q}}$ has more points than $\mathrm{Hom}(\Lambda, \mathbb{Z}) \subseteq \mathcal{Q}$ since Hom reverses sublattice inclusions.

Now in the case of K we will have chosen a splitting $e\colon \Lambda \to K^{(B)}$, so for each $\lambda \in \overline{\Lambda}$ we will have a semi-invariant $e_\lambda \in K^{(B)}$, but we need to pass to some $\overline{e}_\lambda \in \overline{K}^{(B)} = K^{(B)\times A}$. As explained above, the choice of a splitting e can be controlled by making a balanced collection of integral shifts, i.e. choosing $\ell_x \in \mathrm{Hom}(\overline{\Lambda}, \mathbb{Z})$ with $\sum_{x\in\mathbb{P}^1} \ell_x = 0$ and defining $\overline{e}$ such that $\nu_x(\overline{e}(\lambda)/e(\lambda)) = \ell_x(\lambda)$ for each x. This can be done in such a way as to obtain A-invariant functions $\overline{e}_\lambda$ for each $\lambda \in \overline{\Lambda}$.

Finally, the group A will also act on $\mathbb{P}^1$ via the B-quotient π. This causes points $p \in \mathbb{P}^1$ to be permuted and hence also the sets $\mathcal{Q}_p, \mathcal{V}_p$ in $\mathcal{H}$ will be permuted accordingly. We take the geometric quotient $\mathbb{P}^1 \to \mathbb{P}^1/A = \overline{\mathbb{P}^1}$ (which is always well-defined for finite A) which identifies points $p \in \mathbb{P}^1$ within the same A-orbit, and hence identifies the slices $\mathcal{H}_p$ of hyperspace for points in the same orbit. Since $\overline{K} \subseteq K$, every G-valuation of $\overline{K}$ extends to a G-valuation of K (Corollary 2.5), so the sets $\mathcal{V}_p$ are also identified as $\overline{\mathcal{V}}_{\overline{p}}$ when we 'factorise' by the A-action.

Because of this identification of points in the same orbit, we must always ensure before factorising that the slices $\mathcal{H}_p$ of hyperspace corresponding to points in the same A-orbit align with each other, i.e. the colours, valuation cones etc. look the same within the slices corresponding to different points.

The final adjustment required is as follows: the action of A on $\mathbb{P}^1$ will in general have a kernel A_*, i.e. the intersection of all point stabilisers may not be trivial. Likewise certain points in $\mathbb{P}^1$ will have nontrivial stabilisers. This means that the h-coordinate in hyperspace for elements of $\overline{\mathcal{H}}_{\overline{p}}$ where $\overline{p} \in \overline{\mathbb{P}^1}$ will be $[A_p : A_*]$ times the h-coordinate corresponding to any of its preimages $p \in \mathbb{P}^1$.

3.2.4 Finite Subgroups of SL_2

We now want to calculate the coloured data for the remaining homogeneous complexity-one SL_2-spaces. These must all be of the form SL_2/H for H finite, so we first discuss the finite subgroups of SL_2.

Let $\Bbbk = \mathbb{C}$, so that we can identify $\mathbb{P}^1$ with the Riemann sphere S^2. Then $\mathrm{SU}_2 \subseteq \mathrm{SL}_2$ acts on $\mathbb{P}^1$ via the double cover $\mathrm{SU}_2 \to \mathrm{SO}_3$. We can thus realise finite subgroups of SL_2 as pullbacks under this double cover of finite rotation groups of the sphere.

Choose any diameter of S^2. Then rotations of the sphere about this diameter by an angle of $2\pi/n$ generate a cyclic group $\mathbb{Z}_n$ of order n in SO_3 for any $n \in \mathbb{N}$. For odd

n, $\mathbb{Z}_n$ does not contain $-I$, so in this case its preimage in SL_2 is still $\mathbb{Z}_n$. For even n we get $\mathbb{Z}_{n/2} \subseteq \mathrm{SL}_2$. Either way this realises $\mathbb{Z}_n$ as a subgroup of SL_2 for every $n \in \mathbb{N}$.

Now suppose a regular n-gon is inscribed in S^2. Then its symmetry group D_n is a subgroup of SO_3 of order $2n$, and the preimage of this group in $\mathrm{SU}_2 \subseteq \mathrm{SL}_2$ is a subgroup of order $4n$. This group is the *binary dihedral group* $\tilde{D}_n$.

Similarly, if a regular polyhedron is inscribed in S^2, then the preimage in SU_2 of its rotational symmetry group in SO_3 is a finite subgroup of SL_2. Hence SL_2 contains the *binary tetrahedral group* $\tilde{T}$ of order 24, the *binary cubic group* $\tilde{C}$ of order 48 and the *binary icosahedral group* $\tilde{I}$ of order 120. Note that due to duality the binary octahedral group and binary dodecahedral group are isomorphic to $\tilde{C}$ and $\tilde{I}$ respectively.

It is well-known (see e.g. [PV94, §0.13]) that up to conjugation any finite subgroup $H \subseteq \mathrm{SL}_2$ is isomorphic either to a cyclic group $\mathbb{Z}_n$, a binary dihedral group $\tilde{D}_n$ or one of the three distinct binary polyhedral groups. In the remainder of this section we will describe their hyperspaces and hence (since $c(\mathrm{SL}_2/H) = 1$ only if H is finite) classify the homogeneous SL_2-spaces of complexity one, again following the same calculation by Timashev [Tim97, §5].

3.2.5 $SL_2/\mathbb{Z}_m$

Consider the homogeneous space $G/H = \mathrm{SL}_2/\mathbb{Z}_m$ for $m \in \mathbb{Z}$. Here H embeds into SL_2 as the subgroup generated by $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$, where ε is a primitive m^{th} root of unity. We have $\mathfrak{X}(H) = \mathbb{Z}_m$ by identifying characters $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right) \mapsto \varepsilon^k$ with $k \in \mathbb{Z}_m$, and we write these characters as ε^k with multiplicative notation. We will find the coloured data for this space using the previous example of $\mathrm{SL}_2/\{e\}$ and the techniques outlined in section 3.2.3

The functions z, w from the earlier case do not lie in $K = \Bbbk(G)^H$ since they are not H-invariant, but they do lie in $\Bbbk[G]^{(B \times H)}$, having weights (α, ε) and $(\alpha, \varepsilon^{-1})$, respectively. Hence they are subregular semi-invariants and still define B-divisors, but they cannot be regular as their H-weights are different. To obtain regular semi-invariants and hence our B-quotient map, we can simply take powers of these functions which have equal weights. To that end, set $\overline{m} = m$ for odd m and $\overline{m} = m/2$ for even m. Then $z^{\overline{m}}, w^{\overline{m}}$ both have weights $(\overline{m}\alpha, \varepsilon^{\overline{m}})$. These regular semi-invariants generate the space $M = \Bbbk[G]^{(B \times H)}_{(\overline{m}\alpha, \varepsilon^{\overline{m}})}$ and define the B-quotient map π. Then we can obtain Λ

and choose our splitting as follows: let $F \in M$ be an arbitrary regular semi-invariant. Then

$$\Lambda = \begin{cases} \mathbb{Z}\alpha & \text{m is odd, } e_\alpha = \frac{F}{(zw)^{\frac{m-1}{2}}} \\ \mathbb{Z}\cdot 2\alpha & \text{m is even, } e_{2\alpha} = \frac{F^2}{(zw)^{\frac{m}{2}-1}}. \end{cases}$$

Hence for even m, $\mathrm{Hom}(\Lambda, \mathbb{Z}) = \frac{1}{2}\mathbb{Z}$, while for odd m it is $\mathbb{Z}$ as before.

Now the action of H on $\mathbb{P}^1$ via the quotient $\pi\colon G \dashrightarrow \mathbb{P}^1$ can be shown to have the form $\varepsilon^k \cdot [\alpha : \beta] = [\varepsilon^k\alpha : \varepsilon^{-k}\beta]$, so the kernel H_* of the action is as follows:

$$H_* = \begin{cases} \{e\} & \text{m odd} \\ \langle \varepsilon^{\overline{m}} \rangle & \text{m even.} \end{cases}$$

This corresponds to the stabiliser H_p of a general point $p \in \mathbb{P}^1$. The specific points $0 = [1:0]$ and $\infty = [0:1]$ are fixed by H, so for these points we have $[H_p\colon H_*] = \overline{m}$, and the jump for these points will be multiplied by $\overline{m}$. All other points have $H_p = H_*$, so there is no effect for them.

The result of all of these considerations is that for the distinguished point p_1 and for a general point $p \neq 0, \infty$, the valuation cones and the positions of the colours in hyperspace are the same as for $\mathrm{SL}_2/\{e\}$ as described above, except that for even m there are twice as many lattice points on the ℓ-axis.

One last issue arises: upon factorising the hyperspace by H, the valuation cone for the distinguished point in $\mathbb{P}^1$ becomes identified with those of the other points in its orbit, so we must perform shifts to these cones $\ell \mapsto \ell + h\ell_p$ for some $p \in \mathbb{P}^1$ in order for them to line up, and these shifts must balance. The orbit of the distinguished point contains $|H|/|H_{p_1}| = \overline{m}$ points, and selecting $\ell_p = 1$ for each $p \in H \cdot p_1$ shifts the cone $\mathcal{V}_p$ defined by $2\ell + h \leq 0$ to line up with the cone $\mathcal{V}_{p_1}$ defined by $2\ell - h \leq 0$. Now we have shifted $(\overline{m} - 1)$ cones by 1 and we must balance these shifts out. Hence we shift the two cones $\mathcal{V}_0$ and $\mathcal{V}_\infty$ each by $(\overline{m} - 1)/2$ in the opposite direction. To begin with, these cones were defined by $2\ell + h \leq 0$. After the shift, we have $2\ell + \overline{m}h \leq 0$, but then we remember to scale the h-axis by $\overline{m}$, obtaining again cones defined by $2\ell + h \leq 0$.

The only difference then between the cones for a general point and the cones for $0, \infty$ is that the colours for the general point have $(\ell, h) = (0, 1)$ while after the shifts and scaling, the colours D_0, D_∞ sit at $(\ell, h) = (\frac{1-\overline{m}}{2}, \overline{m})$. Hence (after identifying points with the same orbit in $\mathbb{P}^1$) the hyperspace is given by:

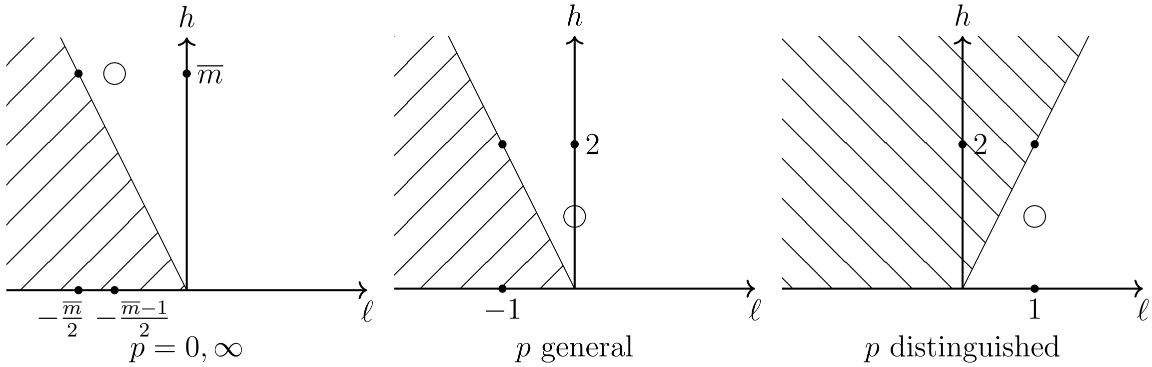


Figure 2: Coloured data of $\mathrm{SL}_2/\mathbb{Z}_m$

where $\mathcal{Q} = \Lambda^*$, the ℓ-axis, is given by $\mathbb{Z}$ if m is odd and $\frac{1}{2}\mathbb{Z}$ if m is even.

3.2.6 Binary Dihedral and Polyhedral Groups

The binary dihedral and binary polyhedral groups each have centre $\mathbb{Z}_2$ arising from the pullback under the double cover $\mathrm{SU}_2 \to \mathrm{SO}_3$, and in each case the quotient by the centre is the usual dihedral or polyhedral group. Hence for all of these groups $\tilde{H}$ ($H = D_n, T, C, I$), we can calculate the hyperspace for $\mathrm{SL}_2/\tilde{H} = (\mathrm{SL}_2/\mathbb{Z}_2)/H$ from that of $\mathrm{SL}_2/\mathbb{Z}_2$ using factorisation by H. In this case the action of $\tilde{H}$ on $S^2 = \mathbb{P}^1$ is also obtained by factorisation: we have $\mathbb{P}^1/\tilde{H} = (\mathbb{P}^1/\mathbb{Z}_2)/H = \mathbb{P}^1/H$ since $\mathbb{Z}_2$ acts trivially on $\mathbb{P}^1$.

Now the action of H on $\mathbb{P}^1$ is just the usual action of the rotational symmetry group of a regular polytope inscribed in the sphere. There will be three distinguished orbits p_v, p_e and p_f of H on $\mathbb{P}^1$ consisting of the vertices, the projections of the edge midpoints onto S^2 and the projections of the face centres onto S^2. For D_n, we view the centre of the n-gon as a face centre with projection to S^2 consisting of the two endpoints of the line segment passing through the centre of the sphere at right angles to the plane containing the n-gon.

These distinguished orbits have orders corresponding to the number of vertices, edges and faces of the polytope in question and each point p within such an orbit has a stabiliser of order $|H|/|H \cdot p|$. Each orbit then has a corresponding subregular semi-invariant f_v, f_e, f_f of multiplicity equal to the order of the corresponding stabiliser and B-weight equal to $|H \cdot p|\alpha$.

To calculate the hyperspace of $\mathrm{SL}_2/\tilde{H}$, we start with the hyperspace of $\mathrm{SL}_2/\mathbb{Z}_2$, which has one distinguished point and the exact same appearance for all other points. To form a correct hyperspace for $\mathrm{SL}_2/\tilde{H}$ we must perform a series of balanced integral shifts (as described in Section 3.1.1) so that the slices of hyperspace for points in the same distinguished H-orbit look the same, since these points are identified on $\mathbb{P}^1/H$. After this, the h co-ordinates in each slice must be scaled by the multiplicity of the corresponding subregular semi-invariant. Points lying outside the three distinguished orbits do not have their slices of hyperspace affected at all.

The distinguished point p_0 of $\mathbb{P}^1/\mathbb{Z}_2$ may lie in any orbit of H on $\mathbb{P}^1$ and we are free to choose which one at will, since different choices are related by some series of balanced integral shifts. However, we can perform our calculations more systematically by choosing $p_0 \in p_v$, as follows:

Suppose the distinguished orbits p_v, p_e and p_f have orders v, e and f respectively, and that $p_0 \in p_v$. We exploit the fact that $v - e + f = 2$. To begin with, the slice of hyperspace corresponding to p_0 lies two units to the right relative to that of a general point, so we take a general point $p_1 \in p_v$ and shift its slice by one unit to the right, then shift the slice of p_0 one unit to the left, so they now align. These shifts balance, and there are $v - 2$ remaining points in p_v which we must also shift by one unit to the right. Now we have performed a total shift of $v-2$ units to the right on the hyperspace. But $v - 2 = e - f$, and the remaining orbits p_e and p_f consist entirely of general points, so we shift all points in p_e one unit left and all points in p_f by one unit right, and everything balances.

Finally, we scale the three distinguished slices of hyperspace by the respective multiplicities of f_v, f_e and f_f to obtain the hyperspace of $\mathrm{SL}_2/\tilde{H}$.

3.2.7 $SL_2/\tilde{D}_n$

The binary dihedral group $\tilde{D}_n$ is an extension of the dihedral group D_n of order $2n$ by $\mathbb{Z}_2$, so has order $4n$. It is given by generators and relations by $\langle x, r \mid x^{2n} = 1, r^2 = x^n, r^{-1}xr = x^{-1}\rangle$, where x corresponds to rotations of the n-gon by $2\pi/n$ about its centre, and r corresponds to reflections of the n-gon obtained by rotating the sphere by π about the axis joining either opposite vertices, opposite face midpoints, or a vertex and its opposite face midpoint. We can thus realise $\tilde{D}_n$ as a subgroup of SL_2 by choosing $x = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$ for ε a primitive $2m^{\text{th}}$ root of unity and $r = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Adding the relation $xr = rx$ into those defining $\tilde{D}_n$, we see that the derived subgroup is $\langle x^2 \rangle$, of order n, and hence the abelianisation is of order 4, isomorphic to $\mathbb{Z}_4$ for odd n and to $\mathbb{Z}_2^2$ for even n. Since $\mathfrak{X}(G) = \mathfrak{X}(G/[G,G])$ for any G, we calculate characters of $\tilde{D}_n$ on the abelianisation. Characters of $\tilde{D}_n$ are determined by the images of x and r in $\Bbbk^*$. After the quotient by x^2, the image of x must be ± 1. For odd n, $x^2 = 1$ forces $r^2 = x$, so the possible images $(\chi(x), \chi(r))$ are $(1,1), (1,-1), (-1,i), (-1,-i)$. For even n, the quotient forces $r^2 = 1$, so we get characters $(\chi(x), \chi(r)) = (1,1), (1,-1), (-1,1), (-1,-1)$.

By the discussions of Section 3.2.6, there are subregular semi-invariants f_v, f_e and f_f of multplicities $2, 2$ and n respectively, and hence respective B-weights $n\alpha, n\alpha$ and 2α. Using the fact that $\Bbbk[G]_\alpha^{(B)}$ is generated by z, w, we can form $(B \times \tilde{D}_n)$ semi-invariants by choosing combinations of these which are semi-invariant with respect to $\tilde{D}_n$. One can check that $f_v = z^n + (-iw)^n$, $f_e = z^n - (-iw)^n$ and $f_f = zw$ have the required B-weights and are $\tilde{D}_n$ semi-invariant, with respective weights $(-1, i^n), (-1, -i^n)$ and $(1, -1)$. Therefore the regular semi-invariants have $(B \times H)$ biweights given by the biweights of f_v^2, f_e^2 and f_f^n, i.e. they fill the space $M = \Bbbk[G]^{(B \times H)}_{(2n\alpha, (1, (-1)^n))}$.

Thus we have $\Lambda = \mathbb{Z} \cdot 2\alpha$, we can choose a $(B \times H)$-invariant function by taking the quotient of any two elements of M, and we can choose an H-invariant splitting function $e_{2\alpha} = f_v f_e / f_f^{n-1}$ of B-weight 2α.

Now we calculate the hyperspace, slightly diverging from Section 3.2.6. Let the distinguished point p_0 of $\mathbb{P}^1/\mathbb{Z}_2$ lie in p_v, and starting with the hyperspace for $\mathrm{SL}_2/\mathbb{Z}_2$, shift the slice corresponding to p_0 one unit to the left and the slices of the remaining points in p_v one unit to the right. Next, shift the slices for the n points in p_e by one unit to the right and the slices for the 2 points in p_f by $n-1$ units to the left. The shifts all balance, so it remains only to scale the h co-ordinates in the slices p_v, p_e and p_f by $2, 2$ and n respectively. Hence the hyperspace looks like:

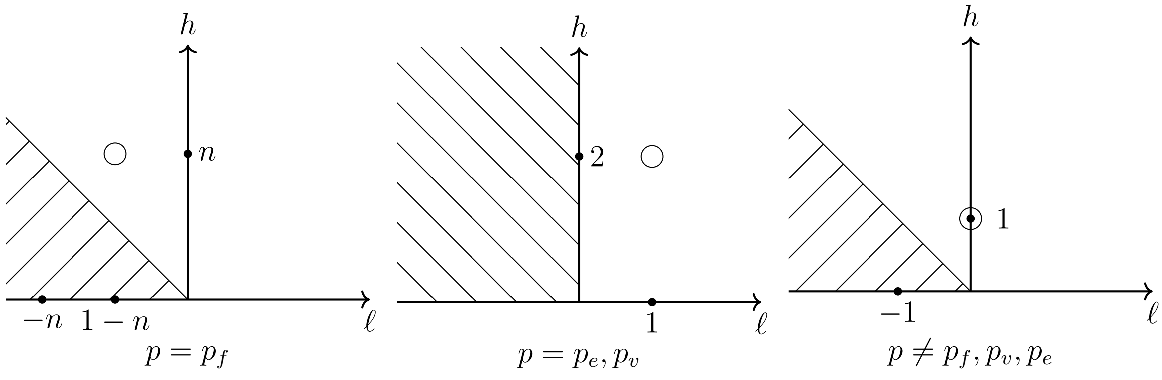


Figure 3: Coloured data of $\mathrm{SL}_2/\tilde{D}_n$

3.2.8 $SL_2/\tilde{T}$

The binary tetrahedral group $\tilde{T} \subseteq \mathrm{SL}_2$ is an extension of the tetrahedral group T of order 12 by $\mathbb{Z}_2$, so has order 24. The tetrahedral group is generated by rotations of the sphere by $2\pi/3$ about the axis joining a vertex and the projection onto the opposite face midpoint, and rotations by π about two orthogonal axes joining opposite edge midpoints. Hence T is a semidirect product of D_2 and $\mathbb{Z}_3$ with $\mathbb{Z}_3$ acting on the normal subgroup D_2 by conjugation. Pulling back to $\tilde{T}$ we see that it is thus a semidirect product of $\tilde{D}_2$ and $\mathbb{Z}_3$. We can realise $\tilde{T}$ as the subgroup of SL_2 generated by $x = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $r = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\omega = -\frac{1}{2}\left(\begin{smallmatrix} 1+i & -1+i \\ 1+i & 1-i \end{smallmatrix}\right)$.

It follows that $\tilde{D}_2$ is the derived subgroup of $\tilde{T}$ and hence the abelianisation is $\mathbb{Z}_3$, generated by the image of ω. Hence characters of $\tilde{T}$ are determined by the image $\varepsilon^k = \chi(\omega)$ where ε is a primitive cube root of unity and $k = 0, 1, 2$.

The tetrahedron has 4 vertices, 6 edges and 4 faces, so the subregular semi-invariants f_v, f_e and f_f have respective multiplicities $3, 2, 3$ and B-weights $4\alpha, 6\alpha$ and 4α. The $\tilde{T}$-weights are determined by the action of ω: this leaves the symmetries of the edge midpoints unaffected and affects the vertex and face symmetries oppositely, so the subregular semi-invariants f_v, f_e and f_f have $\tilde{T}$-weights $\varepsilon, 1$ and ε^{-1}. The regular semi-invariants are then generated by f_v^3, f_e^2 and f_f^3 and fill the space $M = \Bbbk[G]^{(B\times H)}_{(12\alpha,1)}$.

The weight lattice is therefore $\Lambda = \mathbb{Z} \cdot 2\alpha$, a quotient of any two elements of M gives a $(B \times H)$-invariant, and the splitting $e_{2\alpha} = f_v f_f / f_e$ gives an H-invariant function of B-weight 2α.

Now we follow Section 3.2.6 to calculate the hyperspace from that of $\mathrm{SL}_2/\mathbb{Z}_2$. We let the distinguished point $p_0 \in \mathbb{P}^1/\mathbb{Z}_2$ lie in p_v, and shift its slice of hyperspace one unit to the right, and the slices for the remaining 3 points in p_v one unit to the left. Then p_e and p_f contain non-distinguished points, so we shift those in p_e to the left and those in p_f to the right. Then all shifts balance, and it remains to scale the h co-ordinate in the slices of p_v, p_e and p_f by factors of $3, 2$ and 3, respectively. Hence the hyperspace looks like:

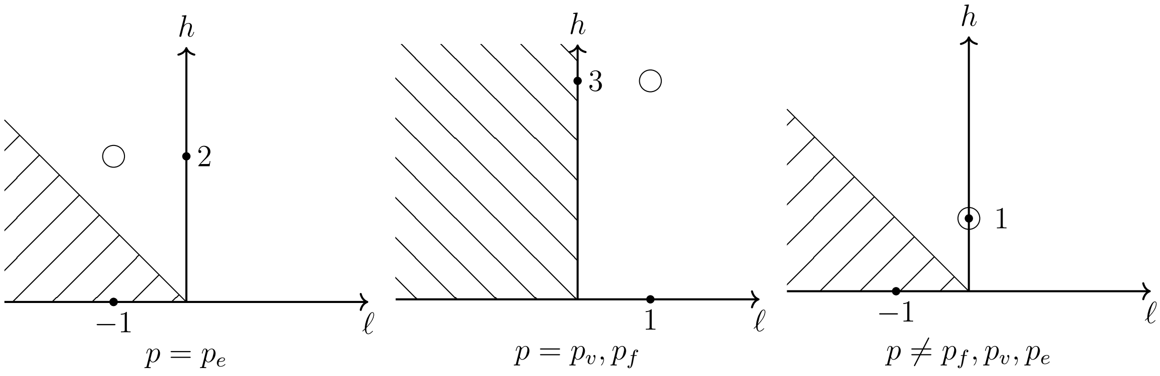


Figure 4: Coloured data of $\mathrm{SL}_2/\tilde{T}$

3.2.9 $SL_2/\tilde{C}$

The binary cubic group $\tilde{C} \subseteq \mathrm{SL}_2$ is an extension of the cubic group C of order 24 by $\mathbb{Z}_2$ and hence has order 48. Note that there are two ways a tetrahedron can be inscribed in a cube, given by aligning the 6 edges of the tetrahedron with the diagonals of the 6 faces of the cube, with the two diagonals of each face giving the two different inscriptions. These inscriptions are related by the rotation of the cube by $\pi/2$ along an axis joining two opposite vertices, and all symmetries of the cube are given by composition of a tetrahedral symmetry and one of these rotations. Hence C is a semidirect product of T and $\mathbb{Z}_2$, and $\tilde{C}$ is a semidirect product of $\tilde{T}$ and $\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting on $\tilde{T}$ by conjugation. We can realise $\tilde{C}$ as the subgroup of SL_2 generated by $y = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1+i & 0 \\ 0 & 1-i \end{smallmatrix}\right)$ and the generators r and ω from the above description of $\tilde{T}$, noting that y^2 is the third generator x of $\tilde{T}$.

This means that $\tilde{T}$ is the derived subgroup of $\tilde{C}$, and the abelianisation is $\mathbb{Z}_2$. Hence characters of $\tilde{C}$ are determined by their value ± 1 on y.

The 8 vertices, 12 edges and 6 faces of the cube mean that the subregular semi-invariants f_v, f_e and f_f have respective multiplicities $3, 2, 4$ and B-weights $8\alpha, 12\alpha, 6\alpha$. Since $\tilde{C}$-weights are determined by the action of y, which fixes vertices and swaps edge and face midpoints, the subregular semi-invariants have respective $\tilde{C}$-weights 1, -1 and -1. The regular semi-invariants are then generated by f_v^3, f_e^2 and f_f^4 and fill the space $M = \Bbbk[G]^{(B\times H)}_{(24\alpha,1)}$.

The weight lattice is then $\mathbb{Z} \cdot 2\alpha$, a quotient of any two elements of M gives a $(B \times H)$-invariant, and the splitting $e_{2\alpha} = f_v f_f / f_e$ gives an H-invariant function of weight 2α.

Following Section 3.2.6, we calculate the hyperspace from that of $\mathrm{SL}_2/\mathbb{Z}_2$. Let the distinguished point $p_0 \in \mathbb{P}^1/\mathbb{Z}_2$ lie in p_v, and shift its slice of hyperspace by one unit to the right, and shift the slices of the 7 other points in p_v by one unit to the left. Then p_e and p_f consist of non-distinguished points, so we shift those in p_e by one unit to the left and those in p_f by one unit to the right. Then all shifts balance, and we scale the h co-ordinate in the slices p_v, p_e and p_f by factors of 3, 2 and 4, respectively, giving:

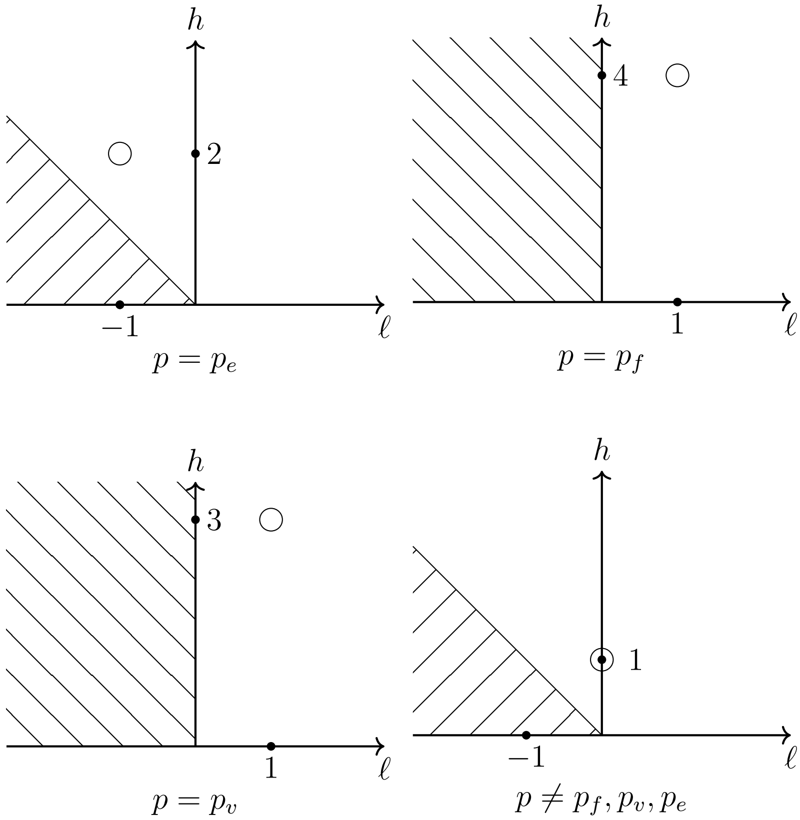


Figure 5: Coloured data of $\mathrm{SL}_2/\tilde{C}$

3.2.10 $SL_2/\tilde{I}$

The binary icosahedral group $\tilde{I} \subseteq \mathrm{SL}_2$ is an extension of the icosahedral group I of order 60 by $\mathbb{Z}_2$, so has order 120. It is a perfect group, i.e. equal to its derived subgroup, so its abelianisation is trivial and it has no nontrivial characters.

The 12 vertices, 30 edges and 20 faces of the icosahedron mean that the subregular semi-invariants f_v, f_e and f_f have respective multiplicities 5, 2 and 3, and B-weights $12\alpha, 30\alpha$ and 20α. The regular semi-invariants are generated by f_v^5, f_e^2 and f_v^3, filling $M = \Bbbk[G]^{(B)}_{60\alpha}$. The weight lattice is $\Lambda = \mathbb{Z} \cdot 2\alpha$, an invariant function is obtained as a quotient of any two elements of M, and the splitting $e_{2\alpha} = f_v f_f / f_e$ gives a B semi-invariant of weight 2α.

Following Section 3.2.6 we calculate the hyperspace from that of $\mathrm{SL}_2/\mathbb{Z}_2$. Let the distinguished point $p_0 \in \mathbb{P}^1/\mathbb{Z}_2$ lie in p_v, and shift its slice of the hyperspace by one unit to the left, and shift the slices corresponding to the other 11 points in p_v by one unit to the right. Then shift the slices of all 30 points in p_e by one unit to the left and the slices of the 20 points in p_f by one unit to the right. These shifts balance, so we scale the h co-ordinates in the slices p_v, p_e and p_f by $5, 2$ and 3, respectively, giving:

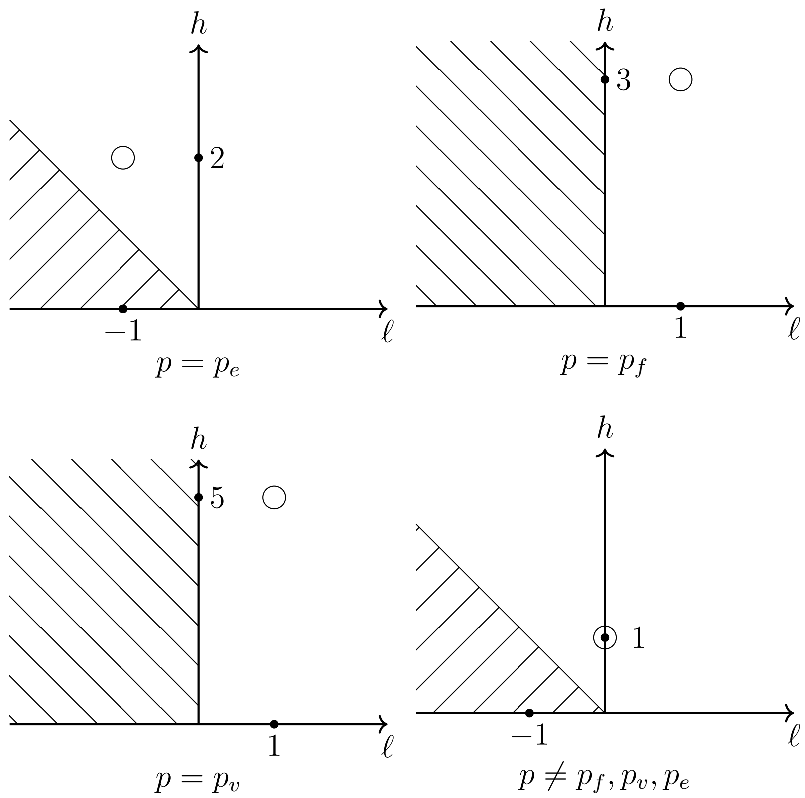


Figure 6: Coloured data of $\mathrm{SL}_2/\tilde{I}$

3.3 Smooth Fano SL_2-Threefolds

Recently Cheltsov, Przyjalkowski and Shramov [CPS19] classified all smooth Fano threefolds with infinite automorphism groups, explicitly listing these groups (specifically their connected components) in a table. As my research into K-stability focusses on smooth Fano varieties, and since it is clear that for a threefold to admit a complexity one group action, it must have an infinite automorphism group, the varieties in this table provide useful examples for me. Indeed for my purposes the simplest non-trivial complexity one G-varieties to work with involve SL_2 acting with finite point stabilisers on a smooth Fano threefold. These varieties are the subject of much of the rest of this thesis.

3.3.1 Existence of SL_2-Actions

The first step is to identify within the CPS list those threefolds admitting a (quasi-homogeneous) SL_2-action of complexity one. An algebraic action of SL_2 on a variety X is a morphism (of algebraic groups) $\varphi\colon \mathrm{SL}_2 \to \operatorname{Aut} X$. If X is a threefold, then for this action to be quasihomogeneous, there must be $x \in X$ with $\dim G \cdot x = 3$. Since $G \cdot x \cong G/G_x$, we have $\dim G \cdot x = \dim G - \dim G_x = 3 - \dim G_x$. Hence there must be at least one point with $\dim G_x = 0$, and since $\ker \varphi = \bigcap_{x \in X} G_x$, it follows that $\ker \varphi$ must be finite. Thus by the first isomorphism theorem, we require that $\operatorname{Aut} X$ contains a subgroup isomorphic to SL_2/H for some finite normal subgroup H of SL_2. Any finite normal subgroup of a connected group (like SL_2) lies in the centre [Hum75, Ex. 7.11], which in this case is $\mathbb{Z}_2$. Thus we have:

Theorem 3.6. *Let X be a smooth Fano threefold. If X admits a quasihomogeneous SL_2 action of complexity one,* $\operatorname{Aut} X$ *contains a subgroup isomorphic to SL_2 or PGL_2.*

This allows us to eliminate immediately from the CPS list any variety with automorphism group having dimension less than 3, for example.

Although in the long-term it will be useful to calculate combinatorial data for varieties which are known *not* to be K-stable, in order to see what differences these data have to that of the K-stable varieties, for now we will concentrate on varieties for which there is a chance of K-stability. CPS explicitly specify (in Corollary 1.5) the varieties on their list which have non-reductive automorphism groups, and thus cannot be K-stable by the Matsushima obstruction.

Finally, if I further discount toric varieties and those varieties admitting the action of a 2-torus (listed by Süß in [Süß14]), since their K-stability has been checked in [IS17], the list is considerably shortened. Hence below are the smooth Fano threefolds admitting an SL_2-action which have reductive automorphism groups and do not admit the action of a 2-torus or 3-torus. They are listed by the number used by CPS to identify the variety, a short description, and the connected component of the automorphism group:

1.10	V_{22}, a zero locus of three sections of the rank 3 vector bundle $\bigwedge^2 \mathcal{Q}$, where $\mathcal{Q}$ is the universal quotient bundle on $\mathrm{Gr}(3,7)$	PGL_2
1.15	V_5, a section of $\mathrm{Gr}(2,5) \subseteq \mathbb{P}^9$ by a linear subspace of codimension 3	PGL_2
2.21	The blow up of a quadric threefold $Q \subseteq \mathbb{P}^4$ along a twisted quartic curve	PGL_2
2.27	The blow up of $\mathbb{P}^3$ along a twisted cubic curve	PGL_2
3.13	The blow up of a divisor $W \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$ along a curve of bidegree $(2,2)$ which is mapped to irreducible conics by the natural projections to $\mathbb{P}^2$	PGL_2
3.17	A divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1,1,1)$, or a blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve of bidegree $(1,1)$	PGL_2
4.6	The blow up of $\mathbb{P}^3$ along a disjoint union of three lines	PGL_2

Table 3.1: Smooth Fano SL_2-threefolds with reductive automorphism groups and no 2- or 3-torus action

(Note: the cases 1.10, 2.21, 3.13 consist of families of varieties, only some of which have the listed automorphism group.)

3.3.2 SL_2-Actions on Symmetric Powers

Many of the SL_2-actions on threefolds to be considered here will be induced by an SL_2-action on $\mathbb{P}^n$ for some n, often in the case where $\mathbb{P}^n$ is realised as the projectivisation of a symmetric power of $\Bbbk^2$. We describe these actions and some of their properties here, and will use the results throughout the remainder of this section.

Proposition 3.16. *Let $G = SL_2$. Fix a Borel subgroup B consisting of the upper triangular matrices in G. Then $\mathfrak{X}(B) = \mathbb{Z}\alpha$, where α is the character $\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right) \mapsto a$. The dominant weights are the non-negative integer multiples of α, and the simple G-module of highest weight $n\alpha$ can be realised as the space $S^n\mathbb{k}^2 = \mathbb{k}[x, y]_n$ of homogeneous degree n polynomials in 2 variables, where G acts by linear change of variables.*

Proof. The G-module $\mathbb{k}[x, y]_n$ is indeed simple: if not, it decomposes as a direct sum of G-submodules by complete reduciblity (Theorem 2.3). Each of these G-submodules must contain a nonzero U-invariant by Lemma 2.1. But the only U-invariants in $\mathbb{k}[x, y]_n$ are scalar multiples of y^n, and complementary submodules cannot both nontrivially intersect a single line. A simple check shows that B acts on $\mathbb{k}y^n$ with weight $n\alpha$. □

Proposition 3.17. *By the above, G acts on $\mathbb{P}^n = \mathbb{P}(S^n\mathbb{k}^2)$. Under this action, the rational normal curve $Z \subseteq \mathbb{P}^n$ defined as the image of $\mathbb{P}^1 = \mathbb{P}(\mathbb{k}^2)$ under the degree n Veronese map, is a G-orbit.*

Proof. Since G acts transitively on $\mathbb{P}^1$ under the standard linear action, it suffices to show that ν_n is G-equivariant. But ν_n maps $[x : y]$ to $[x^n : x^{n-1}y : \ldots : xy^{n-1} : y^n]$ and since the G-action on $\mathbb{P}^n$ is defined by the same linear changes of the variables x, y as the action on $\mathbb{P}^1$, equivariance follows immediately. □

Proposition 3.18. *Let $\mathbb{P}^n = \mathbb{P}(S^n\mathbb{k}^2)$ $(n \geq 2)$ have homogeneous co-ordinates z_k, $0 \leq k \leq n$. Then (where they are regular) z_n is a B-semi-invariant of weight $n\alpha$ and $z_{n-2}z_n - z_{n-1}^2$ is a semi-invariant of weight $(2n - 4)\alpha$.*

Proof. Since z_n corresponds to $y^n \in S^n\mathbb{k}^2$, it is a semi-invariant of weight $n\alpha$ by Proposition 3.16. For $z_{n-2}z_n - z_{n-1}^2$, note that $\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right) \in B$ maps the co-ordinate function z_k to the linear polynomial given by making the replacement $x^{n-m}y^m \mapsto z_m$ in the expression $(ax + by)^{n-m}y^m/a^m$. It is then straightforward to check the claim. □

Proposition 3.19. *Each co-ordinate function z_k on $\mathbb{P}^n = S^n\mathbb{k}^2$ is a T-eigenvector of weight $(2k - n)\alpha$, where $T \subseteq B$ is a maximal torus. It follows that a homogeneous B-eigenfunction of degree d must be a linear combination of monomials $z_{k_1} \cdots z_{k_d}$ with $\sum_{i=1}^d k_i = \frac{1}{2}(m + dn)$. In particular, a G-invariant divisor of degree d in $\mathbb{P}^n$ must be defined by a linear combination of such monomials with $\sum_{i=1}^d k_i = \frac{dn}{2}$.*

Proof. An element $\left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right) \in T$ acts on z_k by $z_k \mapsto (ax)^{n-k}(y/a)^k \mapsto a^{n-2k}z_k$, so z_k has weight $(2k-n)\alpha$. The B-weight of a B-eigenfunction must equal the T-weight of the same function, which must in turn equal the weight of any of its individual terms. Hence for a homogeneous polynomial of degree d, constructed from monomials $z_{k_1} \cdots z_{k_d}$, to be a B-eigenfunction of weight $m\alpha$ we must have $\sum_{i=1}^{d}(2k_i - n) = m$, or $\sum_{i=1}^{d} k_i = \frac{1}{2}(m + dn)$. Finally, a G-invariant divisor must be defined by a G-semi-invariant homogeneous polynomial. Since G (being a perfect group) has no nontrivial characters, such a polynomial must in particular have B-weight 0, from which the final claim follows. □

3.3.3 Blow-up of $\mathbb{P}^3$ Along Three Lines (4.6)

Hyperfan of $\mathbb{P}^3$

Let $G = \mathrm{SL}_2$ act on $X = \mathbb{P}(M_2(\Bbbk)) \cong \mathbb{P}^3$ by left multiplication of matrices. Then the orbit $B \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is $\left\{ \left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{P}^3 \right\}$. This point has stabiliser $\mathbb{Z}_2$ and the orbit therefore has dimension $\dim B = 2$. This must be a maximal orbit and so this action has complexity one. The G-orbit of the same point is $\left\{ \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathbb{P}^3 \mid xw - yz \neq 0 \right\} = \mathrm{PGL}_2$. This is an open subset of $\mathbb{P}^3$, so we are in the quasihomogeneous case, and $\mathbb{P}^3$ is an embedding of $\mathrm{PGL}_2 = \mathrm{SL}_2/\mathbb{Z}_2 = G/H$. The degenerate matrices constitute a G-stable prime divisor $D = \mathcal{Z}(xw - yz)$ of $\mathbb{P}^3$, which contains all of its closed orbits.

There is a family of colours parameterised by points in $\mathbb{P}^1$: namely, for $p = [\alpha : \beta] \in \mathbb{P}^1$, the divisor $D_p = \mathcal{Z}(\beta z - \alpha w)$ is a colour.

Coloured Hyperspace

We know from 3.2.5 that the weight lattice Λ of $\mathrm{SL}_2/\mathbb{Z}_2$ is $\mathbb{Z} \cdot 2\alpha$, so we identify its dual $\mathcal{Q}$ with $\frac{1}{2}\mathbb{Z}$, and the field of B-invariants can be generated by z/w. In the case of $\mathrm{SL}_2/\mathbb{Z}_2$, to keep the convention of our analysis of SL_2, we would choose a distinguished semi-invariant $e_{2\alpha} = z^2 \in \Bbbk[G/H]^{(B)}_{2\alpha}$. This corresponds to the rational function $F = z^2/(xw - yz)$ on $\mathbb{P}^3$. It is still a semi-invariant of weight 2α and we will set $e_{2\alpha} = F$.

In the hyperspace, the choice of $e_{2\alpha}$ means that $\infty = [0 : 1]$ is distinguished, since its pullback under the B-quotient map is the divisor of F. For all other points, the valuation cone is defined by $\ell+h \leq 0$ (not $2\ell+h$ as before since we identify $\mathcal{Q}$ with $\frac{1}{2}\mathbb{Z}$), and the colour D_p lies at $(0,1)$ in $\mathcal{H}_p$. For $p=\infty$, the valuation cone is $\ell - h \leq 0$, and the colour D_p lies at $(2,1)$. Indeed, we have $\ell_{D_\infty} = \nu_{D_\infty}(e_{2\alpha}) = \nu_z(z^2/(xw-yz)) = 2$.

Let $D := \mathcal{Z}(xw-yz) \subseteq \mathbb{P}^3$ be the divisor of degenerate matrices, with associated valuation ν_D. This valuation is G-invariant and geometric, so lies in $\mathcal{V}$. To locate ν_D in the hyperspace $\mathcal{H}$, first note that $\nu_D(z/w) = 0$, so ν_D has trivial restriction to K^B and is thus central. We also have $\nu_D(e_{2\alpha}) = \nu_D(z^2/(xw-yz)) = -1$, so ν_D sits at $(-1,0)$ in the centre of the hyperspace. Since any positive rational multiple of ν_D is also a G-valuation, the cone of central valuations $\mathcal{K} = \mathcal{V} \cap \mathcal{Z}$ is $\mathbb{Q}_{\leq 0}$.

Coloured Data

As has been noted, all closed G-orbits in X lie in the divisor of degenerate matrices D. Since we are projectivising 2×2 matrices, D must consist exclusively of (projectivisations of) rank 1 matrices. It is not difficult to check that each closed orbit consists of matrices with a given kernel, so they are parameterised by $\mathbb{P}^1$. For $p \in \mathbb{P}^1$ we write Y_p for the closed orbit of matrices whose kernel is the line in $\Bbbk^2$ represented by p. Each colour D_p contains the closed orbit Y_p and this orbit is contained in no other B-divisor, so the coloured data of the G-germs in X are as follows: $\mathcal{V}_{Y_p} = \{\nu_D\}$, $\mathcal{D}^B_{Y_p} = \{D_p\}$; $\mathcal{V}_D = \{\nu_D\}, \mathcal{D}^B_D = \emptyset$.

Thus for $p \neq \infty$, the minimal G-germ Y_p corresponds to the coloured cone in $\mathcal{H}_p$ spanned by the colour D_p at $(0,1)$ and the G-divisor D at $(-1,0)$, i.e. it is the upper-left quarter-plane. Similarly for $p=\infty$ the coloured cone is spanned by $D_\infty = (2,1)$ and D. We can see that for any p, the coloured cones spanned by the minimal G-germs cover $\mathcal{V}_p$ entirely, in accordance with completeness of $\mathbb{P}^3$. Hence the coloured hyperfan of $\mathbb{P}^3$ looks as follows:

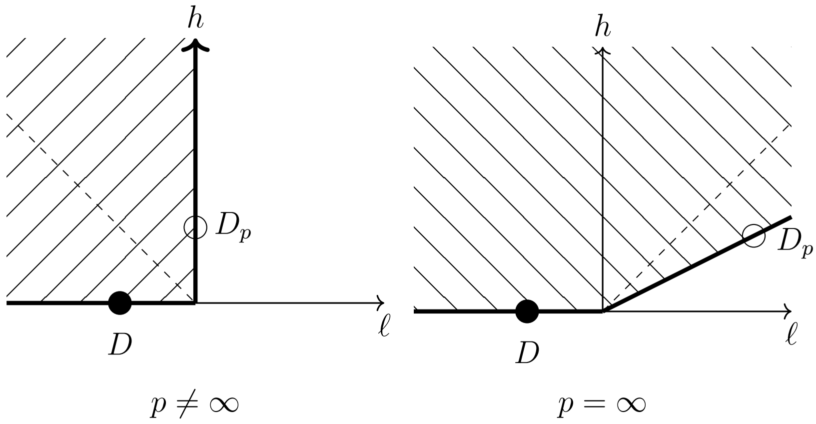


Figure 7: Coloured hyperfan of $\mathbb{P}^3$ (linear action)

where filled circles represent G-divisors, unfilled circles represent colours, thick lines indicate rays spanned by G-germs and B-divisors, hatched areas show the coloured cones generated by minimal G-germs, and dashed lines show the boundaries of the valuation cones.

Blow-up of One Line

The closed G-orbits $Y_p \subseteq \mathbb{P}^3$, where $p = [\alpha\colon \beta] \in \mathbb{P}^1$, consist of matrices whose kernel is the line in $\Bbbk^2$ represented by p. That is, $Y_p = \mathcal{Z}(\beta x - \alpha y, \beta z - \alpha w)$. Each of these closed orbits is a G-stable line in $\mathbb{P}^3$, and they are mutually disjoint. We will obtain our example by blowing up three of these lines, which can be chosen arbitrarily. First we investigate what happens to the coloured data and hyperspace after one blow-up, and the rest follows easily.

Let $0 = [1 : 0] \in \mathbb{P}^1$ and consider $Y_0 = \mathcal{Z}(y, w) \subseteq \mathbb{P}^3$. Let $X := \mathrm{Bl}_{Y_0}(\mathbb{P}^3) = \mathcal{Z}(yv - wu) \subseteq \mathbb{P}^1 \times \mathbb{P}^3$. Note that under the blow-up, the colours D_p and closed orbits Y_p of $\mathbb{P}^3$ where $p \neq 0$ pull back isomorphically to X, and since Y_0 is G-stable, the blow-up is equivariant.

The exceptional divisor of this blow-up is $E_0 = \mathcal{Z}(y, w) \subseteq \mathbb{P}^1 \times \mathbb{P}^3$ and the strict transform of the divisor of degenerate matrices is $\tilde{D} = \mathcal{Z}(xw - yz, uz - vx, yv - uw) \subseteq \mathbb{P}^1 \times \mathbb{P}^3$. These are the only G-stable prime divisors in X, and their intersection is the curve $\mathcal{Z}(uz - vx, y, w)$. Together $\tilde{D}, E_0$, their intersection and the closed orbits Y_p $(p \neq 0)$ constitute all G-germs of X. Meanwhile, the colours of X are the colours D_p $(p \neq 0)$ of $\mathbb{P}^3$ and the strict transform $\tilde{D}_0 = \mathcal{Z}(w, v)$.

Hence the coloured data of the G-germs of X are as follows: for $p \neq 0$ we have $\mathcal{V}_{Y_p} = \{\nu_{\tilde{D}}\}$, $\mathcal{D}^B_{Y_p} = \{\tilde{D}_p\}$. Then also $\mathcal{V}_{\tilde{D}\cap E_0} = \{\nu_{\tilde{D}}, \nu_{E_0}\}$, $\mathcal{D}^B_{\tilde{D}\cap E_0} = \emptyset$, $\mathcal{V}_{\tilde{D}} = \{\nu_{\tilde{D}}\}$, $\mathcal{D}^B_{\tilde{D}} = \emptyset$, $\mathcal{V}_{E_0} = \{\nu_{E_0}\}$, $\mathcal{D}^B_{E_0} = \emptyset$.

The set-up of the hyperspace is unchanged from the example of $\mathbb{P}^3$: Λ is generated by 2α, $\mathcal{Q}$ is identified with $\frac{1}{2}\mathbb{Z}$, we choose the splitting $e_{2\alpha} = z^2/(xw - yz)$ (marking ∞) as the distinguished point), and the valuation cones are defined by $\ell + h \leq 0$ for $p \neq \infty$ and $\ell - h \leq 0$ for $p = \infty$. It remains to locate the colours and G-divisors.

The central divisor $\tilde{D}$ still sits at $(-1, 0)$ in every section of the hyperspace, as before. For $p = 0$, the colour $\tilde{D}_0$ sits at $(0, 1)$ and the G-divisor E_0 sits at $(-1, 1)$. The coloured cone defined by the minimal G-germ $\tilde{D} \cap E_0$ spans the rays defined by $\tilde{D}$ and E_0. For $p \neq 0$, the colour $\tilde{D}_p$ sits at $(0, 1)$ as before and the cone defined by $\tilde{Y}_p$ spans the rays defined by D and $\tilde{D}_p$. Again we see that the coloured cones defined by the various G-germs all cover the valuation cone in each slice of the hyperspace, as required by completeness of X. Thus the coloured hyperfan looks like:

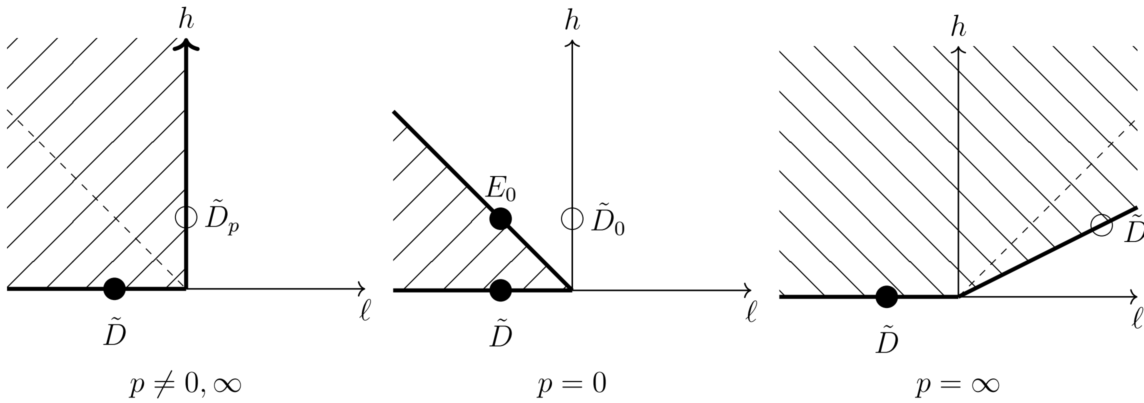


Figure 8: Coloured hyperfan of the blow-up of $\mathbb{P}^3$ along one line

Blow-up of Three Lines

Now we can go back to $\mathbb{P}^3$, choose three arbitrary non-distinguished points (say $q, r, s \in \mathbb{P}^1 \setminus \{\infty\}$) and successively blow up their corresponding G-orbits, in this case Y_q, Y_r and Y_s. From the calculations above for the first blow up it is clear what happens as far as G-germs and the hyperspace are concerned: the slices of hyperspace corresponding to all points $p \neq q, r, s$ will be unchanged from their description above, and in each of the slices corresponding to q, r, s there will be a new G-divisor (the exceptional divisor of the blow-up) sitting at $(-1, 1)$, while the corresponding colour does not move from $(0, 1)$. The minimal G-germs are Y_p for $p \neq q, r, s$ and $\tilde{D} \cap E_p$ for $p = q, r, s$. The former define coloured cones bounded by $\tilde{D}$ and $\tilde{D}_p$ in $\mathcal{H}_p$ ($p \neq q, r, s$) and the latter define coloured cones bounded by $\tilde{D}$ and E_p in $\mathcal{H}_p$ ($p = q, r, s$). Hence we get the following coloured hyperfan:

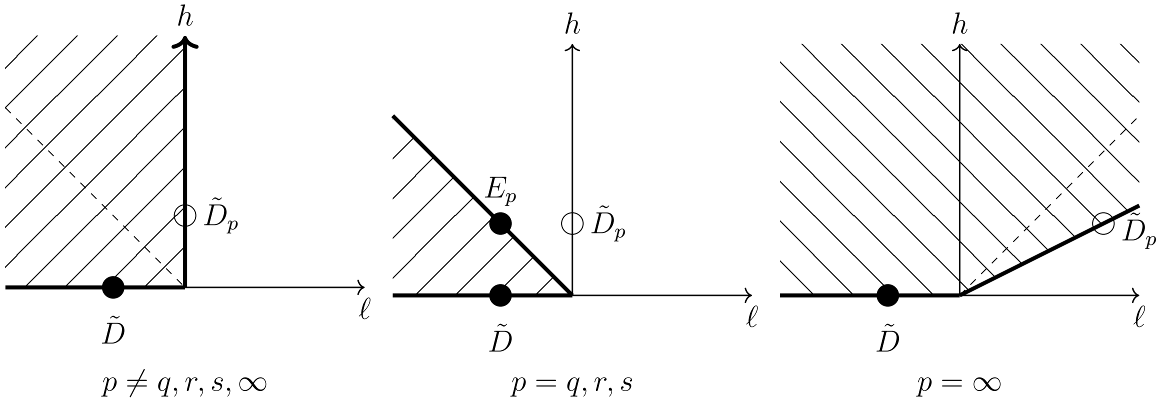


Figure 9: Coloured hyperfan of the blow-up of $\mathbb{P}^3$ along three lines

3.3.4 Blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ (3.17)

Orbits and G-germs Before the Blow-up

Let $G = \mathrm{SL}_2$ act on $\mathbb{P}^1 \times \mathbb{P}^2$, linearly on the first factor and quadratically on the second. The G-orbit of the point $P = ([1 : 0], [1 : 0 : 1])$ is

$$\{([a : c], [a^2 + b^2 : ac + bd : c^2 + d^2]) \mid ad - bc = 1\}.$$

The stabiliser G_P is $\mathbb{Z}_4$, so this orbit is open, and the B-orbit is easily checked to be 2 dimensional. Hence $\mathbb{P}^1 \times \mathbb{P}^2$ is a complexity one G-variety and an embedding of $G/H = \mathrm{SL}_2/\mathbb{Z}_4$.

The divisors $\Delta = \mathcal{Z}(x_0^2 z_2 + x_1^2 z_0 - 2x_0 x_1 z_1)$ and $F = \mathcal{Z}(z_0 z_2 - z_1^2)$ are G-stable. Consider the G-stable curve $C = \mathcal{Z}(x_1 z_0 - x_0 z_1, x_1 z_1 - x_0 z_2) \subseteq \mathbb{P}^1 \times \mathbb{P}^2$. Note that $C = F \cap \Delta$. The orbits on $\mathbb{P}^1 \times \mathbb{P}^2$ are as follows: C is itself a closed orbit, then $\Delta \setminus C$ and $F \setminus C$ are orbits, and the open orbit described above is $\mathbb{P}^1 \times \mathbb{P}^2 \setminus (F \cup \Delta)$. This is shown in detail by calculations after the blow-up in a later subsection. Hence the proper G-germs of $\mathbb{P}^1 \times \mathbb{P}^2$ are C, F and Δ.

(Semi-) Invariant Functions

Recall that for $\mathrm{SL}_2/\mathbb{Z}_4$, the weight lattice Λ is generated by 2α, $\mathcal{Q} = \Lambda^*$ is identified with $\frac{1}{2}\mathbb{Z}$, the field of invariants is generated by z^2/w^2 and we chose a semi-invariant regular function F from the module $M = \Bbbk[G]^{(B\times H)}_{(2\alpha,\varepsilon^2)}$ spanned by z^2 and w^2 to give a splitting $e_{2\alpha} = F^2/(zw)$.

On $\mathbb{P}^1 \times \mathbb{P}^2$, the function $f_0 = g_0/h_0 = x_1^2(z_0z_2 - z_1^2)/(x_0z_2 - x_1z_1)^2$ is B-invariant and under the isomorphism $G/H \cong \mathbb{P}^1 \times \mathbb{P}^2 \setminus (F \cup \Delta)$ corresponds to the invariant z^2/w^2. Hence this function defines a rational B-quotient map $\pi\colon \mathbb{P}^1 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $P \mapsto [g_0(P) : h_0(P)]$.

For $p = [\alpha : \beta] \in \mathbb{P}^1$, the pullbacks $\pi^*(p) = \mathcal{Z}(\beta g_0 - \alpha h_0)$ define a family of regular colours $D_p = \pi^*(p)$ except at three points:

$$p = \infty = [0 : 1] : \pi^*(p) = \mathcal{Z}(x_1^2(z_0z_2 - z_1^2)) = F \cup \mathcal{Z}(x_1^2) = F \cup D_\infty,$$

$$p = -1 = [1 : -1] : \pi^*(p) = \mathcal{Z}(z_2(x_0^2z_2 + x_1^2z_0 - 2x_0x_1z_1)) = \Delta \cup \mathcal{Z}(z_2) = \Delta \cup D_{-1},$$

$$p = 0 = [1 : 0] : \pi^*(p) = \mathcal{Z}((x_0z_2 - x_1z_1)^2) = D_0.$$

Note that D_∞ and D_0 correspond to points in $\mathbb{P}^1$ of multiplicity 2, i.e. they are subregular colours and thus have h-coordinate 2 in hyperspace, in accordance with the calculation of the hyperspace of $\mathrm{SL}_2/\mathbb{Z}_4$ in Section 3.2.5.

Now we choose as the splitting semi-invariant

$$e_{2\alpha} = \frac{z_2^2(x_0^2z_2 + x_1^2z_0 - 2x_0x_1z_1)}{x_1(z_0z_2 - z_1^2)(x_0z_2 - x_1z_1)}.$$

This corresponds in the homogeneous space to the function $(z^2 + w^2)^2/(zw)$, and hence to the choice $F = z^2 + w^2 \in M$ and thus marks out the point $-1 \in \mathbb{P}^1$ as distinguished.

Coloured Data and Hyperfan

We first note here that the curve C is contained in the G-divisors Δ and F, and in every colour D_p for $p \neq -1, \infty$, i.e. in colours lying over points in $\mathbb{P}^1$ whose pullback does not contain Δ or F. Hence the coloured data of C is: $\mathcal{V}_C = \{\nu_\Delta, \nu_F\}$, $\mathcal{D}^B_C = \{D_p \mid p \neq -1, \infty\}$, and the remaining coloured data is $\mathcal{V}_\Delta = \{\nu_\Delta\}, \mathcal{V}_F = \{\nu_F\}$, $\mathcal{D}^B_\Delta = \mathcal{D}^B_F = \emptyset$. We see from this that C defines a coloured hypercone of type II in $\mathcal{H}$.

In accordance with Section 3.2.5, the subregular colours D_0, D_∞ sit at $(-1, 2)$ in their respective slices of hyperspace, the colour D_{-1} sitting over the distinguished point sits at

$(2,1)$, and the non-distinguished regular colours D_p for $p \neq 0, -1, \infty$ lie at $(0,1)$. Finally, the G-divisors F and Δ go to $(p, \ell, h) = (\infty, -1, 1)$ and $(p, \ell, h) = (-1, 1, 1)$ respectively. Then the polytope defined by C is $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_\infty + \mathcal{P}_{-1} = \{-1/2\} + \{-1\} + \{1\} = \{-1/2\}$. Hence the coloured hyperfan of $\mathbb{P}^1 \times \mathbb{P}^2$ is as follows:

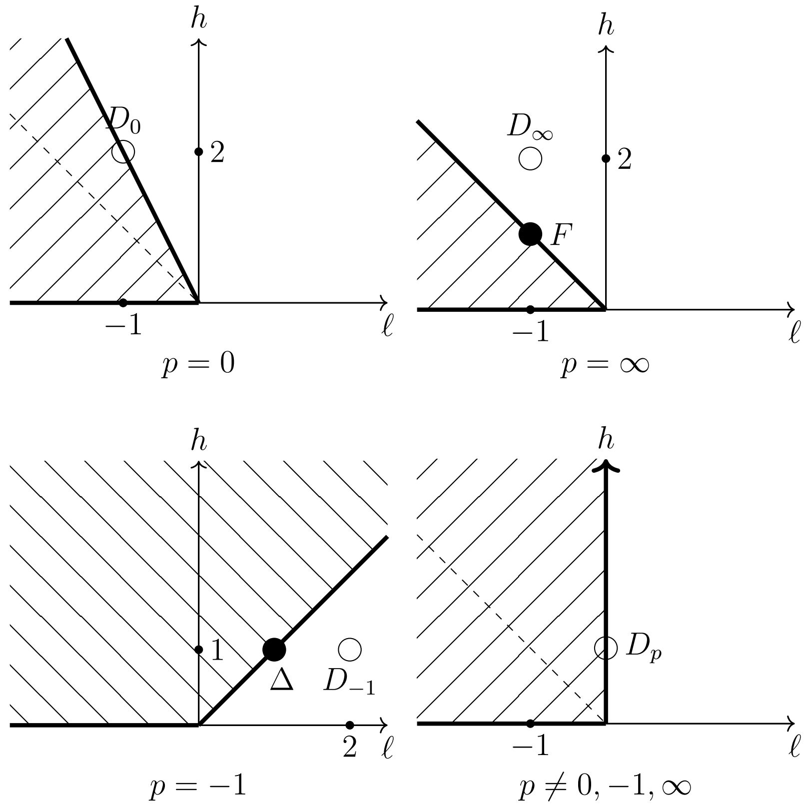


Figure 10: Coloured hyperfan of $\mathbb{P}^1 \times \mathbb{P}^2$

Blow-up

We now blow up C to obtain the variety $X = \mathcal{Z}(x_0y_0z_2 + x_1y_1z_0 - x_0y_1z_1 - x_1y_0z_1) \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. Then X contains the G-stable divisors $\tilde{\Delta} = \mathcal{Z}(x_0y_1 - x_1y_0)$, the strict transform of the divisor Δ defined above, $E = \mathcal{Z}(x_1z_0 - x_0z_1, x_1z_1 - x_0z_2)$, the exceptional divisor of the blow-up, and $\tilde{F} = \mathcal{Z}(y_1z_0 - y_0z_1, y_0z_2 - y_1z_1)$, the strict transform of the above divisor $\tilde{F}$. Let $D = E \cup \tilde{F}$.

Orbits and G-germs After the Blow-up

Claim: the G-orbits on X are $X \setminus (D \cup \tilde{\Delta})$, which is open, $\tilde{\Delta} \setminus (D \cap \tilde{\Delta})$, $E \setminus (E \cap \tilde{\Delta})$, $\tilde{F} \setminus (\tilde{F} \cap \tilde{\Delta})$ and $D \cap \tilde{\Delta}$, which is closed. We note that $D \cap \tilde{\Delta} = E \cap \tilde{\Delta} = \tilde{F} \cap \tilde{\Delta} = E \cap \tilde{F}$. Thus the G-germs of X are $\tilde{\Delta}$, E, $\tilde{F}$ and $D \cap \tilde{\Delta}$, with the latter being minimal.

Proposition 3.20. *The open* SL_2*-orbit on* X *is* $X \setminus (D \cup \tilde{\Delta})$.

Proof. For $(p, q) \in \mathbb{P}^1 \times \mathbb{P}^1$, let $X_{p,q} = X \cap (\{(p, q)\} \times \mathbb{P}^2)$. Since the torus $\Bbbk^* \subseteq \mathrm{SL}_2$ fixes $0 := [0 : 1], \infty := [1 : 0] \in \mathbb{P}^1$, it must also leave $X_{0,\infty}$ stable. We claim that $\Bbbk^*$ acts transitively on $X_{0,\infty} \setminus D_{0,\infty}$:

Indeed, suppose $Q = (0, \infty, [q_0 : q_1 : q_2]) \in X_{0,\infty} \setminus D_{0,\infty}$. The equation for X demands that $q_1 = 0$, and this means that the equations for $E, \tilde{F}$ reduce to $q_0 = 0$ and $q_2 = 0$, respectively. Hence we must have $Q = (0, \infty, [q_0 : 0 : q_2])$, with $q_0 q_2 \neq 0$. Now consider $P = (0, \infty, [1 : 0 : 1])$, whose image under $A = \left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right) \in \Bbbk^*$ is $(0, \infty, [a^2 : 0 : 1/a^2]) = (0, \infty, [a^4 : 0 : 1])$. By setting a to be any fourth root of q_0/q_2, we thus have that $Q = A \cdot P$, proving the claim.

Now let $S = (p, q, r) \in X \setminus (D \cup \tilde{\Delta})$. Since $p, q \in \mathbb{P}^1$ are distinct, there exists $M \in \mathrm{SL}_2$ with $M \cdot S = (0, \infty, M \cdot r) \in X_{0,\infty} \setminus D_{0,\infty}$. Now by the above there exists $A \in \Bbbk^*$ with $A \cdot (M \cdot S) = P$, hence $\mathrm{SL}_2 \cdot P = X \setminus (D \cup \tilde{\Delta})$ as promised. □

Proposition 3.21. $\tilde{\Delta} \setminus (D \cap \tilde{\Delta})$ *is an* SL_2*-orbit on* X.

Proof. First, note that the Borel subgroup $B \subseteq \mathrm{SL}_2$ fixes $\infty \in \mathbb{P}^1$, which we use to show that B acts transitively on $X_{\infty,\infty} \setminus D_{\infty,\infty}$. The equations for X and D here reduce to $z_2 = 0$, $z_1 = 0$ respectively. Let $P = (\infty, \infty, [0 : 1 : 0]) \in X_{\infty,\infty} \setminus D_{\infty,\infty}$, so that $A = \left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right) \cdot P = (\infty, \infty, [2ab : 1 : 0])$. If $r = [r_0 : r_1 : 0] \in X_{\infty,\infty} \setminus D_{\infty,\infty}$, then setting $a = 1, b = r_0/2r_1$ gives $A \cdot P = (\infty, \infty, r)$, so we are done. Now for any $Q = (p, p, q) \in \tilde{\Delta} \setminus (D \cap \tilde{\Delta})$ there exists $M \in \mathrm{SL}_2$ with $M \cdot Q = (\infty, \infty, M \cdot q)$, so there exists $A \in B$ such that $A \cdot M \cdot Q = P$ as required. □

Proposition 3.22. $E \setminus (E \cap \tilde{\Delta})$ *and* $\tilde{F} \setminus (\tilde{F} \cap \tilde{\Delta})$ *are* SL_2*-orbits on* X.

Proof. Let $Q = (p, q, r) \in E \setminus (E \cap \tilde{\Delta})$. As above, there exists $M \in \mathrm{SL}_2$ with $M \cdot Q = (0, \infty, M \cdot r) \in D_{0,\infty}$. Now since E is SL_2 stable, we have in fact that $M \cdot Q \in (E)_{0,\infty}$, which is a singleton. Hence $E \setminus (E \cap \tilde{\Delta})$ is an SL_2-orbit, and the case for $\tilde{F}$ is symmetric. □

Proposition 3.23. *The final* SL_2*-orbit on* X *is* $\tilde{\Delta} \cap D$. *Hence the* G*-orbits on* X *are* $D \cup \tilde{\Delta}$, $\tilde{\Delta} \setminus (D \cap \tilde{\Delta})$, $E \setminus (E \cap \tilde{\Delta})$, $\tilde{F} \setminus (\tilde{F} \cap \tilde{\Delta})$ *and* $D \cap \tilde{\Delta}$

Proof. Let $Q = (p, p, q) \in \tilde{\Delta} \cap D$, noting that assuming the equations for $\tilde{\Delta}$, the equations for E and $\tilde{F}$ become the same, i.e. $\tilde{\Delta} \cap E = \tilde{\Delta} \cap \tilde{F}$. It follows that $D_{\infty,\infty}$ is a singleton, say P. We can as before choose $M \in \mathrm{SL}_2$ such that $M \cdot Q \in D_{\infty,\infty}$, showing that $\tilde{\Delta} \cap D = \mathrm{SL}_2 \cdot P$.

Now it is clear that the orbits described so far cover X, so they must constitute an exhaustive list. □

Hyperfan of X

As in previous examples, blowing up the curve C does not change the position of any divisor in hyperspace, but it adds the new G-divisor E. Taking the same invariant and semi-invariant rational functions used above (i.e. f_0 and $e_{2\alpha}$) we see that E sits over $0 \in \mathbb{P}^1$ and lies at $(p, \ell, h) = (0, -1, 1) \in \mathcal{H}$. The colour $\tilde{D}_0$ no longer contains the minimal G-germ $E \cap \tilde{\Delta}$, which now has coloured data $\mathcal{V}_{E\cap\tilde{\Delta}} = \{\nu_E, \nu_{\tilde{F}}, \nu_{\tilde{\Delta}}\}$, $\mathcal{D}^B_{E\cap\tilde{\Delta}} = \{\tilde{D}_p \mid p \neq -1, 0, \infty\}$.

We thus have $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_\infty = \mathcal{P}_{-1} = \{-1\} + \{-1\} + \{1\} = \{-1\}$. Hence all things considered the coloured hyperfan for X looks like:

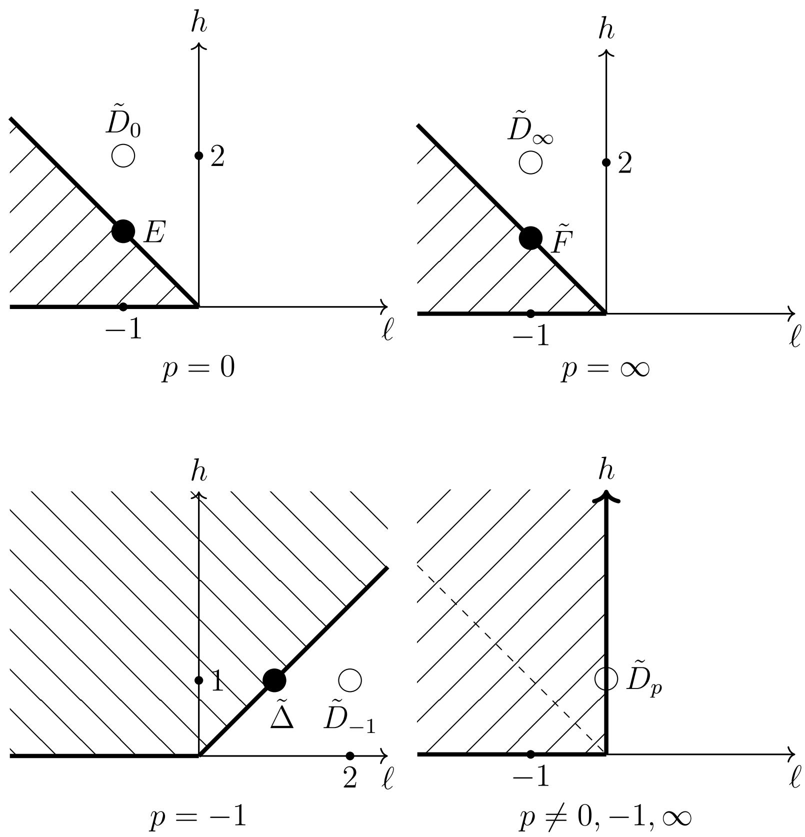


Figure 11: Coloured hyperfan of the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along C

3.3.5 Blow-up of the Divisor W in $\mathbb{P}^2 \times \mathbb{P}^2$ (3.13)

Definition and Structure of W

Let $G = \mathrm{SL}_2$ act diagonally on $\mathbb{P}^2 \times \mathbb{P}^2$ with the action on each factor $\mathbb{P}^2 = \mathbb{P}(S^2\Bbbk^2)$ described in Section 3.3.2. Then the divisor $W = \mathcal{Z}(x_0y_2 - 2x_1y_1 + x_2y_0)$ is G-stable. The point $P = ([1 : 0 : 1], [0 : 1 : 0]) \in W$ has a 2 dimensional B-orbit, and a G-stabiliser of order 8 generated by $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, so let $H = G_P = \tilde{D}_2$, the binary dihedral group of order 8. Then we see that W is a quasihomogeneous complexity-one G-variety containing the homogeneous space G/H.

Now consider the conic $C = \mathcal{Z}(x_0x_2 - x_1^2) \subseteq \mathbb{P}^2$. It is G-stable under the action induced from $S^2\Bbbk^2$ by Proposition 3.17, and hence the divisors $C \times \mathbb{P}^2$ and $\mathbb{P}^2 \times C$ on $\mathbb{P}^2 \times \mathbb{P}^2$ are also G-stable. Let E_∞ and E_0 respectively be the intersections of these divisors with W. Their union is the complement of G/H in W and their intersection is a G-stable curve $Z = (C \times C) \cap W$. The equations of W also force $Z = \operatorname{diag}(\mathbb{P}^2 \times \mathbb{P}^2) \cap W$.

Thus the G-germs of W are exactly Z, E_0 and E_∞, with the latter two containing the former, which is minimal.

(Semi-) Invariant Functions

For $G/H = \mathrm{SL}_2/\tilde{D}_2$, the weight lattice is $\Lambda = \mathbb{Z}(2\alpha)$, $\mathcal{Q} = \Lambda^*$ is identified with $\frac{1}{2}\mathbb{Z}$ and the field of invariants is generated by f_f^2/f_v^2. In W, the equations defining E_0 and E_∞ are invariant, and x_2, y_2 are B-semi-invariant of weight 2α. Hence $f = y_2^2(x_0x_2 - x_1^2)/x_2^2(y_0y_2 - y_1^2)$ is invariant, and one can check that it does indeed correspond to f_f^2/f_v^2 on the open orbit.

Now f defines a B-quotient $\pi\colon W \dashrightarrow \mathbb{P}^1$, $P \mapsto [y_2^2(x_0x_2 - x_1^2) : x_2^2(y_0y_2 - y_1^2)]$. The pullback of $p = [\alpha : \beta] \in \mathbb{P}^1$ is $\mathcal{Z}(\beta y_2^2(x_0x_2 - x_1^2) - \alpha x_2^2(y_0y_2 - y_1^2))$ and defines a regular colour for all p except for the following:

$$p = [1 : 0] = 0 : \pi^*(p) = \mathcal{Z}(x_2^2(y_0y_2 - y_1^2)) = \mathcal{Z}(x_2^2) \cup \mathcal{Z}(y_0y_2 - y_1^2) = D_0 \cup E_0$$

$$p = [0 : 1] = \infty : \pi^*(p) = \mathcal{Z}(y_2^2(x_0x_2 - x_1^2)) = \mathcal{Z}(y_2^2) \cup \mathcal{Z}(x_0x_2 - x_1^2) = D_\infty \cup E_\infty$$

$$p = [-1 : 1] = -1 : \pi^*(p) = \mathcal{Z}(-(x_1y_2 - x_2y_1)^2) = D_{-1}.$$

Here we see three subregular colours (D_0, D_∞, D_{-1}) of multiplicity 2 corresponding to the three subregular semi-invariants on G/H, and the two G-divisors E_0, E_∞ defining the minimal G-germ Z. One can check that every colour except D_0, D_∞ contains Z.

The function $x_2y_2/(x_1y_2 - x_2y_1)$ is semi-invariant of weight 2α and corresponds to f_ef_v/f_f in the homogeneous space, so we choose this as our splitting $e_{2\alpha}$. Its divisor is $D_0 + D_\infty - D_{-1}$, so these points are distinguished by it.

Hyperfan Before the Blow-up

We recall that the G-germs of W are E_0, E_∞ and Z, with the latter being minimal. The coloured data are thus $\mathcal{V}_Z = \{\nu_{E_0}, \nu_{E_\infty}\}$, $\mathcal{D}^B_Z = \{D_p \mid p \neq 0, \infty\}$ and $\mathcal{V}_{E_i} = \{\nu_{E_i}\}$, $\mathcal{D}^B_{E_i} = \emptyset$ for $i = 0, \infty$. Thus Z defines a supported coloured hypercone of type II in $\mathcal{H}$.

From our choice of invariant and splitting semi-invariant, the G-divisors and colours map to the following points in hyperspace: $E_0 \mapsto (0, 0, 1)$, $D_0 \mapsto (0, 1, 2)$, $E_\infty \mapsto (\infty, 0, 1)$, $D_\infty \mapsto (\infty, 1, 2)$, $D_{-1} \mapsto (-1, -1, 2)$ and $D_p \mapsto (p, 0, 1)$ for $p \neq 0, \infty, -1$. Therefore the polytope defined by Z is given by $\mathcal{P}_{-1} = \{-1/2\}$, $\mathcal{P}_p = \{0\}$ for $p \neq -1$, so $\mathcal{P} = \{-1/2\}$. Hence the coloured hyperfan of W looks like:

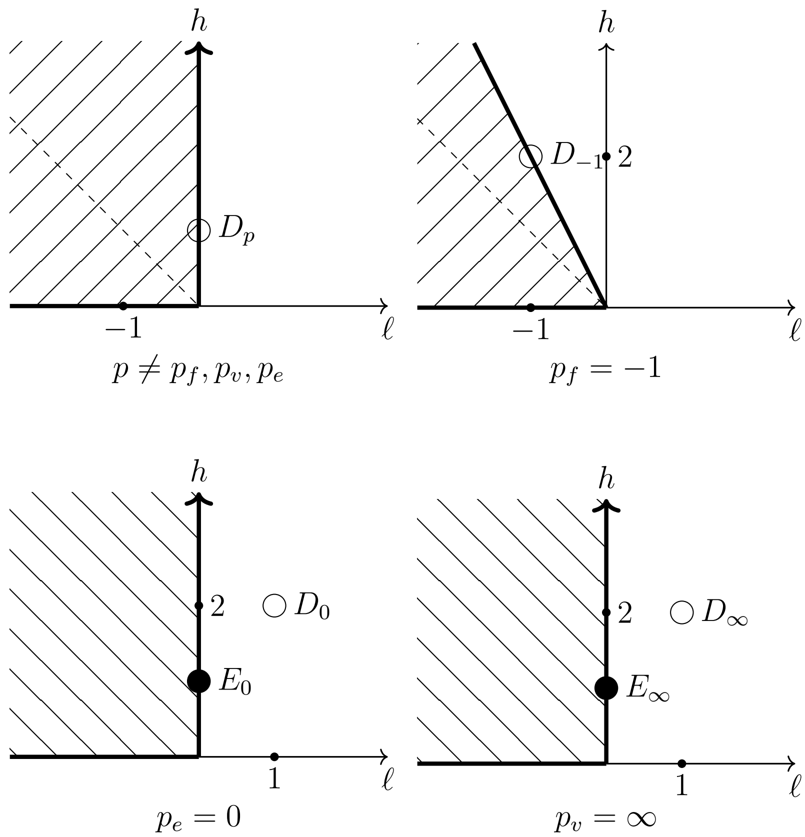


Figure 12: Coloured hyperfan of the divisor W on $\mathbb{P}^2 \times \mathbb{P}^2$

Blow-up

To obtain the variety we want, we blow up W along Z. Since Z defines a hypercone of type II, it has a minimal B-chart $U = W \setminus (D_0 \cup D_\infty)$. We will simplify matters by blowing up this chart instead.

Hence consider U as an affine chart of W, i.e. we set $x_2 = y_2 = 1$ in W to obtain $W \cap U = \mathcal{Z}(x_0 - 2x_1y_1 + y_0) \subseteq \mathbb{A}^4$. Eliminate $x_0 = 2x_1y_1 - y_0$ so that $W \cap U = \operatorname{Spec} \Bbbk[x_1, y_0, y_1] = \mathbb{A}^3$. Then $E_0 \cap U = \mathcal{Z}(y_0 - y_1^2) \subseteq \mathbb{A}^3$, $E_\infty \cap U = \mathcal{Z}(2x_1y_1 - y_0 - x_1^2) \subseteq \mathbb{A}^3$ and $Z \cap U = \mathcal{Z}(y_0 - y_1^2, 2x_1y_1 - y_0 - x_1^2) = \mathcal{Z}(y_1 - x_1, y_0 - y_1^2) \subseteq \mathbb{A}^3$.

Now take $X = \operatorname{Bl}_{U \cap Z}(W \cap Z) = \mathcal{Z}(z_0(y_0 - y_1^2) - z_1(y_1 - x_1)) \subseteq \mathbb{A}^3 \times \mathbb{P}^1$. The exceptional divisor is $E = \mathcal{Z}(y_0 - y_1^2, y_1 - x_1)$, and we have strict transforms $\tilde{E}_0 = \mathcal{Z}(y_0 - y_1^2, z_1)$ and $\tilde{E}_\infty = \mathcal{Z}(z_0(x_1 - y_1) - z_1, 2x_1y_1 - y_0 - x_1^2)$ of the G-divisors from downstairs. Any two of these three G-divisors intersect in the curve $Y = \mathcal{Z}(y_0 - y_1^2, y_1 - x_1, z_1)$, which is hence the unique minimal G-germ of the blow-up.

The invariant rational function on W becomes $f = (2x_1y_1 - y_0 - x_1^2)/(y_0 - y_1^2)$ on $U \cap W$ and hence also on X. Under the induced B-quotient π to $\mathbb{P}^1$ we see that $\pi^*([1 : -1]) = \mathcal{Z}(x_1 - y_1, y_0 - y_1^2) \cup \mathcal{Z}(x_1 - y_1, z_0) = E \cup \tilde{D}_{-1}$. Hence E sits in the slice of hyperspace corresponding to $-1 \in \mathbb{P}^1$, and the colour $\tilde{D}_{-1}$ does not contain the minimal G-germ Y, as D_{-1} did before the blow-up.

Choosing a uniformising element $\delta = (2x_1y_1 - y_0 - x_1^2 + (y_0 - y_1^2))/(y_0 - y_1^2) = -(y_1 - x_1)^2/(y_0 - y_1^2)$ of the DVR corresponding to -1 and taking an affine chart $z_0 = 1$ of X, a simple calculation shows that $h_E = \nu_E(\delta) = 1$.

Likewise, the splitting semi-invariant $e_{2\alpha}$ from above becomes $1/(x_1 - y_1)$ on X, giving $\ell_E = \nu_E(e_{2\alpha}) = -1$. Hence $E \mapsto (-1, -1, 1) \in \mathcal{H}$. The positions in hyperspace of all other G-divisors and colours remain as always unchanged by the blow-up.

Thus the curve Y has coloured data $\mathcal{V}_Y = \{\nu_E, \nu_{\tilde{E}_0}, \nu_{\tilde{E}_\infty}\}$, $\mathcal{D}_Y^B = \{D_p \mid p \neq 0, \infty, -1\}$. It defines a supported coloured hypercone of type II in $\mathcal{H}$ with associated polytope $\mathcal{P} = \mathcal{P}_{-1} = \{-1\}$. Hence X (and therefore the variety $\operatorname{Bl}_Z W$) has the following coloured hyperfan:

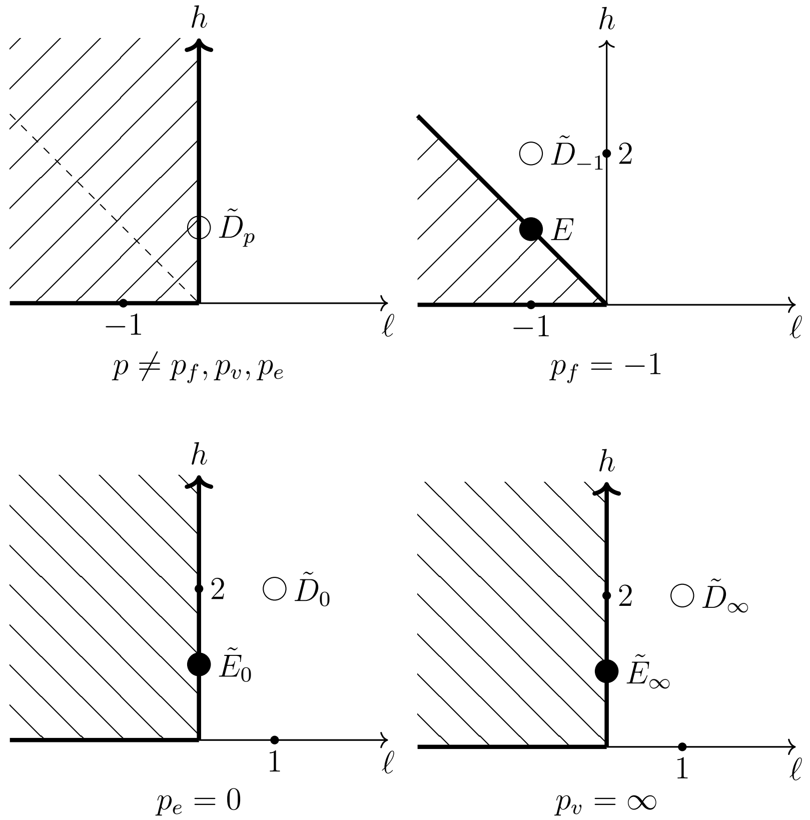


Figure 13: Coloured hyperfan of the blow-up of W along Z

3.3.6 Blow-up of $\mathbb{P}^3$ Along the Twisted Cubic (2.27)

Homogeneous Space and Structure

Let $G = \mathrm{SL}_2$ act on $\mathbb{P}^3 = \mathbb{P}(S^3\Bbbk^2)$ as in Section 3.3.2. The point $P = [1:0:0:1]$ has as its G-stabiliser the binary dihedral group $H = \tilde{D}_3$ of degree 3 and order 12. The same point has B-stabiliser equal to the group of sixth roots of unity, so $\dim B \cdot P = 2$. Hence $G \cdot P$ is the homogeneous space G/H, realising $\mathbb{P}^3$ (in a different way to previous examples) as a quasihomogeneous complexity-one G-variety.

Let Z be the rational normal curve of degree 3 in $\mathbb{P}^3$, i.e. $Z = \mathcal{Z}(x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2)$. Then Z is a closed orbit in $\mathbb{P}^3$ by Proposition 3.17. We will see that Z is the unique minimal G-germ of $\mathbb{P}^3$, and is contained in its unique G-divisor.

(Semi-) Invariant Functions

By Section 3.2.7, there are semi-invariant functions f_v, f_e, f_f in $\Bbbk[G]^{(B\times H)}$ of respective biweights $(3\alpha, (-1,3)), (3\alpha, (-1,1))$ and $(2\alpha, (1,2))$.

In $\mathbb{P}^3$, x_3 is semi-invariant of B-weight 3α, and $x_1x_3 - x_2^2$ is semi-invariant of B-weight 2α by Proposition 3.18. Hence the latter corresponds to f_f. On G/H, we see that, acting with $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) \in H$ on the right, x_3 has H-weight $(-1,3)$, so x_3 corresponds to f_v Finally, $2x_2^3 - 3x_1x_2x_3 + x_0x_3^2$ has B-weight 3α and thus by process of elimination it corresponds to f_e.

Now the function f_f^3/f_e^2 is B-invariant, so gives the B-quotient map π to $\mathbb{P}^1$, i.e. $\pi(P) = [f_f^3(P) : f_e^2(P)]$ for $P \in \mathbb{P}^3$. This defines a family of regular colours $D_p = \pi^*(p)$ in $\mathbb{P}^3$ for all $p = [\alpha : \beta] \in \mathbb{P}^1$ except:

$$p = 0 : \pi^*(p) = \mathcal{Z}(f_e^2) = D_0,$$

$$p = \infty : \pi^*(p) = \mathcal{Z}(f_f^3) = D_\infty,$$

$$p = -4 : \pi^*(p) = \mathcal{Z}(f_v^2) \cup \mathcal{Z}(3x_1^2x_2^2 - 4x_1^3x_3 - x_0^2x_3^2 - 4x_0x_2^3 + 6x_0x_1x_2x_3) = D_{-4} \cup F$$

where F is a G-divisor. The subregular colours D_0, D_∞ and D_{-4} have multiplicities 2, 3 and 2, respectively, in accordance with the pictures above.

Finally, we choose a splitting $e_{2\alpha} = f_v f_e / f_f^2 \in K_{2\alpha}^{(B)}$.

Coloured Data and Hyperfan of $\mathbb{P}^3$

The minimal G-germ is contained in F and in every colour except D_{-4}, so has coloured data $\mathcal{V}_Z = \{\nu_F\}, \mathcal{D}_Z^B = \{D_p \mid p \neq -4\}$. It therefore defines a supported coloured hypercone of type II in $\mathcal{H}$.

By our choice of splitting we see that $D_0 \mapsto (\ell, h) = (1,2)$ in hyperspace, $D_\infty \mapsto (-2,3)$, $D_{-4} \mapsto (1,2)$, $F \mapsto (0,1)$ and as always the regular colours $D_p \mapsto (0,1)$ for $p \neq 0, \infty, -4$.

Therefore the polytope defined by Z is given by $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_\infty + \mathcal{P}_{-4} = \{1/2 - 1/3 + 0\} = \{-1/6\}$.

Hence the coloured hyperfan of $\mathbb{P}^3$ looks like:

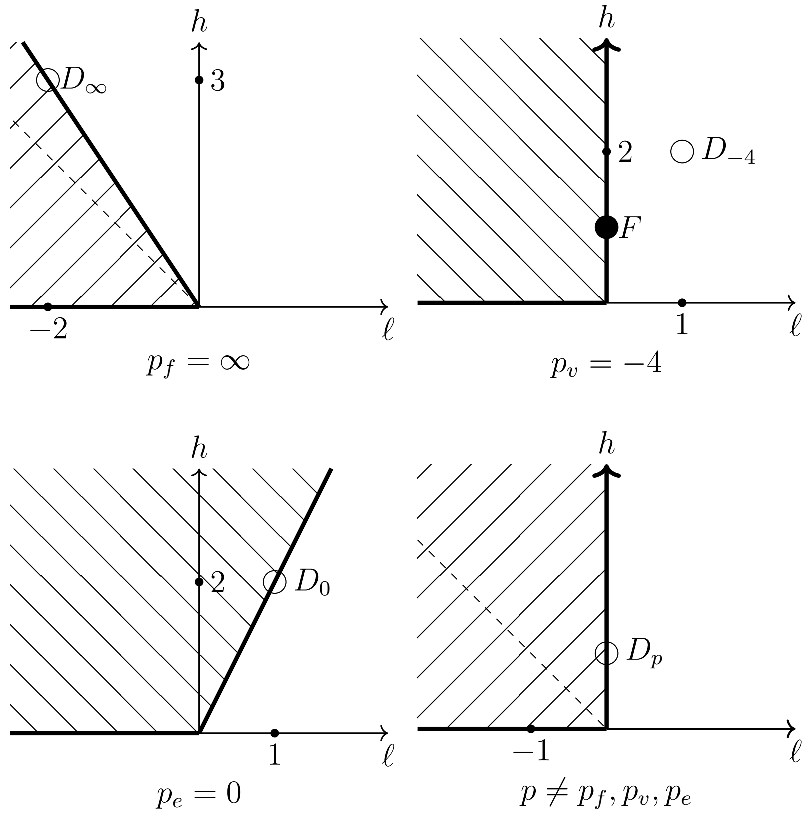


Figure 14: Coloured hyperfan of $\mathbb{P}^3$ (cubic action)

Blow-up

Now to get the variety we need, we blow up Z. To simplify what happens, we will take affine charts. Take the minimal B-chart for Z, which is $U_Z = \mathbb{P}^3 \setminus \mathcal{Z}(x_3)$. By setting $x_3 = 1$, we get $U_Z \cap \mathbb{P}^3 = \mathbb{A}^3 = \operatorname{Spec} \Bbbk[x_0, x_1, x_2]$. Then $Z \cap U_Z$ becomes $\mathcal{Z}(x_0 - x_1x_2, x_1 - x_2^2)$.

Now $X = \mathrm{Bl}_{Z\cap U_Z}(\mathbb{A}^3) = \mathcal{Z}(z_0(x_1 - x_2^2) - z_1(x_0 - x_1x_2)) \subseteq \mathbb{A}^3 \times \mathbb{P}^1$. The exceptional divisor is $E = \mathcal{Z}(x_1 - x_2^2, x_0 - x_1x_2)$. Take another affine chart V defined by $z_1 = 1$. Using the equation for X we can eliminate x_0 to obtain $X \cap V = \mathbb{A}^3 = \operatorname{Spec} \Bbbk[x_1, x_2, z_0]$. Now $E \cap V = \mathcal{Z}(x_1 - x_2^2)$.

On $V \cap X$, the B-invariant above becomes $(x_2^2 - x_1)/(2x_2(1 + z_0))^2$, and the splitting semi-invariant becomes $2x_2(1 + z_0)/(x_1 - x_2^2)$. Hence E sits over ∞ in hyperspace and is mapped to $(-1, 1) \in \mathcal{H}_\infty$.

Since the blow-up is an isomorphism away from Z, nothing else in hyperspace moves from its previous position. There is a new minimal G-germ $Y = E \cap \tilde{F}$ with coloured data $\mathcal{V}_Y = \{\nu_E, \nu_{\tilde{F}}\}, \mathcal{D}_Y^B = \{D_p \mid p \neq \infty, -4\}$. The coloured hyperfan for X is thus:

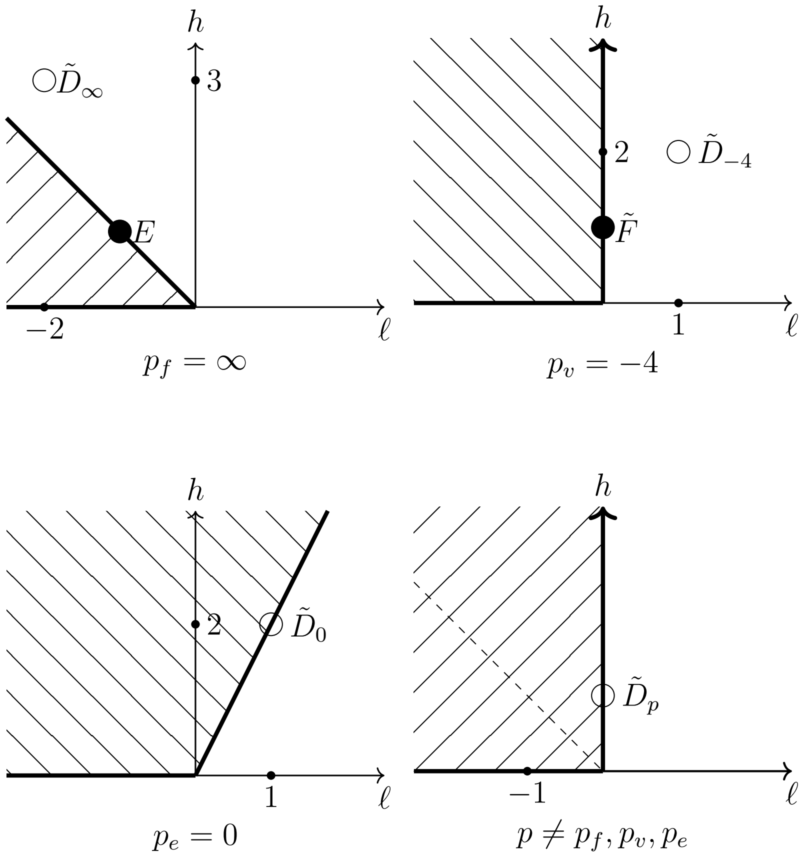


Figure 15: Coloured hyperfan of the blow-up of $\mathbb{P}^3$ along the twisted cubic

3.3.7 Blow-up of the Quadric Threefold (2.21)

Homogeneous Space and Structure

Let $G = \mathrm{SL}_2$ act on $\mathbb{P}^4 = \mathbb{P}(S^4\Bbbk^2)$ as in Section 3.3.2. Then the quadric hypersurface $Q = \mathcal{Z}(3x_2^2 - 4x_1x_3 + x_0x_4)$ is a smooth, G-stable threefold. The stabiliser G_P of the point $P = [0 : 1 : 0 : 0 : 1] \in Q$ is the binary tetrahedral group $\widetilde{T}$, hence the orbit $G \cdot P$ is 3 dimensional and thus open. The same point has the group of sixth roots of unity as its B-stabiliser, so has a 2-dimensional B orbit, so that Q is a quasihomogeneous complexity one G-variety containing the homogeneous space $G/H = \mathrm{SL}_2/\tilde{T}$.

Let Z be the rational normal curve of degree 4 in $\mathbb{P}^4$, i.e. $Z = \mathcal{Z}(x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2)$. Then Z is a closed G-orbit in Q by Proposition 3.17. We will see that Z is the unique minimal G-germ in Q, and is contained in a unique G-divisor.

(Semi-) Invariant Functions

We know from Section 3.2.8 that there are semi-invariant regular functions f_e, f_f, f_v in $\Bbbk[G]^{(B\times H)}$ of respective biweights $(6\alpha, 1), (4\alpha, \varepsilon^{-1})$ and $(4\alpha, \varepsilon)$, where ε is a primitive cube root of unity.

On Q, x_4 and $x_2x_4 - x_3^2$ have B-weight 4α by Proposition 3.18, and checking on the homogeneous space $G/H = G\cdot[0:1:0:0:1]$ we see that acting by $\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{smallmatrix}\right)$ on the right, they have H-weights ε^{-1} and ε respectively. Hence $x_4 = f_f$, $x_2x_4 - x_3^2 = f_v$. Finally, the function $2x_3^3 + x_1x_4^2 - 3x_2x_3x_4$ is B-semi-invariant of weight 6α and H-invariant, so this is f_e.

Now f_v^3/f_e^2 is a B-invariant rational function on Q, so the B-quotient map π is given by $\pi(P) = [f_v^3(P) : f_e^2(P)]$ for $P \in Q$, and defines a family of regular colours $D_p = \pi^*(p)$ for all $p = [\alpha : \beta] \in \mathbb{P}^1$ except:

$$p = 0 : \pi^*(p) = \mathcal{Z}(f_e^2) = D_0,$$

$$p = \infty : \pi^*(p) = \mathcal{Z}(f_v^3) = D_\infty,$$

$$p = -4 : \pi^*(p) = \mathcal{Z}(f_f^3) \cup \mathcal{Z}(4x_2^3 + x_1^2x_4 + x_0x_3^2 - 6x_1x_2x_3) = D_{-4} \cup F$$

where F is a G-invariant divisor. Note that the subregular colours D_0, D_∞ and D_{-4} have multiplicities 2, 3 and 3, respectively, as we should expect from the hyperspace of G/H.

Finally, we choose as a splitting semi-invariant $e_{2\alpha} = f_vf_f/f_e \in K_{2\alpha}^{(B)}$.

Coloured Data and Hyperfan of Q

The minimal G-germ Z is contained in F and in every colour except D_{-4}, so has coloured data $\mathcal{V}_Z = \{\nu_F\}$, $\mathcal{D}_Z^B = \{D_p \mid p \neq -4\}$. Hence it defines a supported coloured hypercone of type II in $\mathcal{H}$.

Our choice of invariant and splitting functions mean that D_0 sits at $(\ell, h) = (-1, 2)$ in hyperspace, $D_{-4}, D_\infty \mapsto (1, 3)$, $F \mapsto (0, 1)$ and as always for regular colours we have $D_p \mapsto (0, 1)$ for $p \neq 0, \infty, -4$.

Hence the polytope defined by Z is given by $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_\infty + \mathcal{P}_{-4} = \{-1/2 + 1/3 + 0\} = \{-1/6\}$. Therefore the full coloured hyperfan defined by Q is as follows:

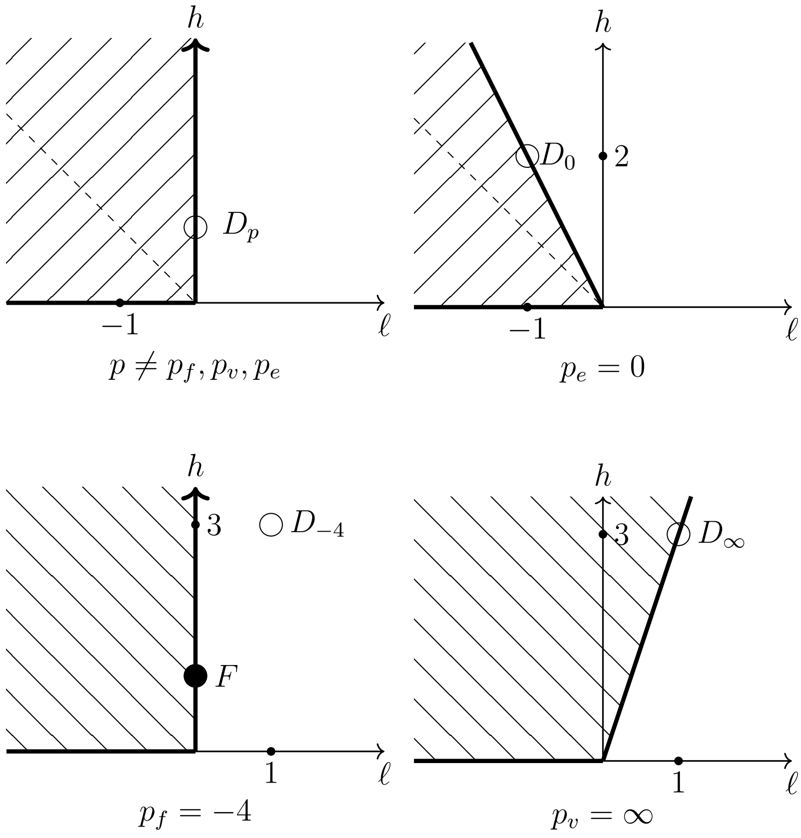


Figure 16: Coloured hyperfan of the quadric threefold Q

Blow-up

The variety we want is obtained by blowing up Z. We calculate the effect on the hyperfan by blowing up in an affine chart. Since Z defines a coloured hypercone of type II, it has a minimal B-chart U_Z with the same coloured data. Indeed Z is contained in every colour except $D_{-4} = \mathcal{Z}(x_4)$, so $U_Z = \mathbb{P}^4 \setminus \mathcal{Z}(x_4)$. Setting $x_4 = 1$ allows us to eliminate x_0 using the equation for Q, so $U_Z \cap Q \cong \mathbb{A}^3 = \operatorname{Spec} \Bbbk[x_1, x_2, x_3]$. Then $U_Z \cap Z = \mathcal{Z}(x_1 - x_2x_3, x_2 - x_3^2)$.

Now $X = \mathrm{Bl}_{U_Z \cap Z}(\mathbb{A}^3) = \mathcal{Z}(z_0(x_2 - x_3^2) - z_1(x_1 - x_2x_3)) \subseteq \mathbb{A}^3 \times \mathbb{P}^1$. The exceptional divisor E is given by $\mathcal{Z}(x_1 - x_2x_3, x_2 - x_3^2)$. Now we take another affine chart V by setting $z_1 = 1$, which allows us to eliminate x_1 using the equation for X. Thus $V \cap X = \mathbb{A}^3 = \operatorname{Spec} \Bbbk[x_2, x_3, z_0]$, and $E \cap V = \mathcal{Z}(x_2 - x_3^2)$.

On $V \cap X$, the invariant function f_v^3/f_e^2 becomes $(x_2 - x_3^2)/(z_0 - 2x_3)^2$, and the splitting semi-invariant is $1/(z_0 - 2x_3)$, so we see that $E \mapsto (p, \ell, h) = (\infty, 0, 1)$ in hyperspace.

Since the blow-up is an isomorphism away from Z, all other colours and G-divisors lie at the same points in hyperspace as before. The blow-up introduces a new minimal G-germ $Y = E \cap \tilde{F}$ which must have coloured data $\mathcal{V}_Y = \{\nu_E, \nu_{\tilde{F}}\}$, $\mathcal{D}^B_Y = \{D_p \mid p \neq \infty, -4\}$. Hence the coloured hyperfan for X looks like:

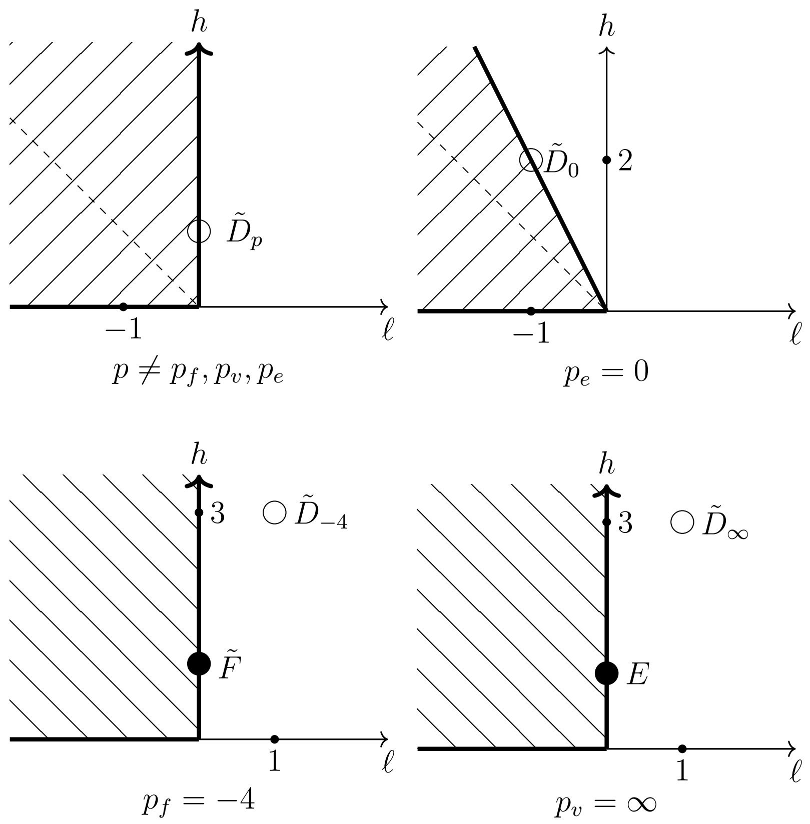


Figure 17: Coloured hyperfan of the blow-up of Q along Z

3.3.8 V_5 (1.15)

Description and Homogeneous Space

Let $G = \mathrm{SL}_2$ act on $\mathbb{k}^2$ with the standard linear action, and hence on $\mathbb{P}^6 = \mathbb{P}(S^6\mathbb{k}^2)$. By [Fur92], we can realise V_5 as the closure in $\mathbb{P}^6$ of the G-orbit of the point $P = [0 : 1 : 0 : 0 : 0 : -1 : 0]$. This explicitly shows that V_5 is a quasihomogeneous G-variety of complexity one. The stabiliser of P in G contains the matrices y, r and ω which, in Section 3.2.9, were shown to generate the binary cubic group $H = \tilde{C}$. Since no other finite subgroup of G contains $\tilde{C}$, we see that $G \cdot P \cong G/H$. Hence there are three subregular semi-invariants f_v, f_e and f_f of respective $(B \times H)$ biweights $(8\alpha, 1), (12\alpha, -1)$ and $(6\alpha, -1)$ and multiplicities $3, 2$ and 4.

The description of V_5 in [Fur92] realises it as the subvariety of $\mathbb{P}^6$ defined by the equations

$$\begin{aligned} x_0x_4 - 4x_1x_3 + 3x_2^2 &= 0, \\ x_0x_5 - 3x_1x_4 + 2x_2x_3 &= 0, \\ x_0x_6 - 9x_2x_4 + 8x_3^2 &= 0, \\ x_1x_6 - 3x_2x_5 + 2x_3x_4 &= 0, \\ x_2x_6 - 4x_3x_5 + 3x_4^2 &= 0. \end{aligned}$$

It is clear from the above then that the rational normal curve Z of degree 6 defined as the image of the Veronese embedding $\nu_6 \colon \mathbb{P}^1 \to \mathbb{P}^6$ lies inside V_5 as a minimal G-germ. We will show later that Z is in fact the unique minimal G-germ of V_5, contained in a unique G-stable divisor.

(Semi)-Invariant Functions

One can find semi-invariants of the correct B-weights by using a torus action, under which each co-ordinate function has a given weight. In this case x_k has weight $-6+2k$. It is easy to see immediately that x_6 is B semi-invariant of weight 6, so must be the subregular semi-invariant f_f. Likewise $x_4x_6 - x_5^2$ is B semi-invariant of weight 8, so represents f_v. Now from Section 3.2.9, we know that f_vf_f/f_e is a rational function, so has degree 0. Hence f_e must have degree 3 and B- (hence T-) weight 12α, and therefore must be a linear combination of the monomials $x_3x_6^2, x_4x_5x_6$ and x_5^3. A simple check shows that $f_e = x_3x_6^2 - 3x_4x_5x_6 + 2x_5^3$ suffices.

Hence f_v^3/f_e^2 is an invariant rational function defining a rational B-quotient $\pi \colon V_5 \dashrightarrow \mathbb{P}^1$, $P \mapsto [f_v^3(P) : f_e^2(P)]$. The pullback of $p = [\alpha : \beta] \in \mathbb{P}^1$ defines a regular colour for all p except the following:

$$p = 0 : \pi^*(p) = \mathcal{Z}(f_e^2) = D_0;$$

$$p = \infty : \pi^*(p) = \mathcal{Z}(f_v^3) = D_\infty;$$

$$p = -4 : \pi^*(p) = \mathcal{Z}(x_6^4) \cup \mathcal{Z}(x_1x_5 + 3x_3^2 - 4x_2x_3) = D_{-4} \cup F;$$

where F is then a unique G-divisor containing the G-germ Z. Likewise it is straightforward that Z is contained in every colour except D_{-4}.

We choose as mentioned the semi-invariant splitting function $e_{2\alpha} = f_v f_f / f_e$ of B-weight 2α, and the colours D_0, D_∞ and D_{-4} are distinguished by it.

Coloured Hyperfan

We already know from previous discussions and our choice of splitting that regular colours D_p for $p \neq 0, \infty - 4$ are mapped to points $(p, 0, 1)$ of hyperspace, and that $D_0 \mapsto (0, -1, 2)$, $D_\infty \mapsto (\infty, 1, 3)$ and $D_{-4} \mapsto (-4, 1, 4)$. Finally, it is easy to check on the B-chart $V_5 \setminus \mathcal{Z}(x_6)$ that F is mapped to $(-4, 0, 1)$.

The coloured data of the minimal G-germ Z is $\mathcal{V}_Z = \{\nu_F\}$, $\mathcal{D}_Z^B = \{D_p \mid p \neq -4\}$. Hence Z is a G-germ of type II and defines a supported coloured hypercone of type II generated by its coloured data and the polytope $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_\infty + \mathcal{P}_{-4} = \{-1/2\} + \{1/3\} + \{0\} = \{-1/6\}$. Hence the coloured hyperfan of V_5 is as follows:

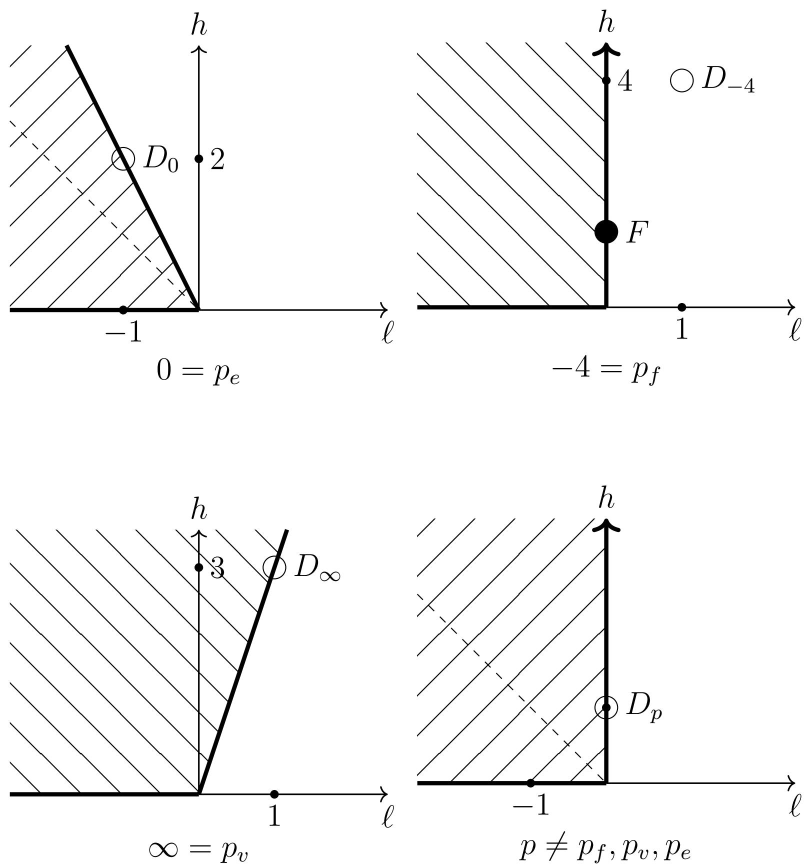


Figure 18: Coloured hyperfan of V_5

3.3.9 V_{22} (1.10)

Description and Homogeneous Space

Let $G = \mathrm{SL}_2$ act on $\mathbb{k}^2$ with the standard linear action, and hence on $\mathbb{P}^{12} = \mathbb{P}(S^{12}\mathbb{k}^2)$. By [Fur92], we can realise V_{22} as the closure in $\mathbb{P}^{12}$ of the G-orbit of the point

$$P = [0 : 1 : 0 : 0 : 0 : 0 : 11 : 0 : 0 : 0 : 0 : 1 : 0].$$

This explicitly shows that V_{22} is a quasihomogeneous G-variety of complexity one. The stabiliser of P in G contains $\left(\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{smallmatrix}\right)$ for ε a primitive tenth root of unity. Furthermore, the function x_{12} on V_{22} is a B-eigenfunction of weight 12α. The only finite subgroup H of SL_2 containing an element of order 10 and such that SL_2/H has a semi-invariant of weight 12α is $\tilde{I}$, the binary icosahedral group. Hence V_{22} is an embedding of $\mathrm{SL}_2/\tilde{I}$.

Then by 3.2.10 there are three subregular semi-invariants f_v, f_e and f_f of respective B-weights $12\alpha, 30\alpha$ and 20α and multiplicities $5, 2$ and 3. Note that $\tilde{I}$ has no weights.

The description of V_{22} in [Fur92] realises it as the subvariety of $\mathbb{P}^{12}$ defined by the equations

$$\sum_{\lambda=0}^{\rho} \binom{8}{\lambda}\binom{8}{\rho-\lambda}(x_\lambda x_{\rho+4-\lambda} - 4x_{\lambda+1}x_{\rho+3-\lambda} + 3x_{\lambda+2}x_{\rho+2-\lambda}) = 0$$

for $0 \leq \rho \leq 16$. It is easy to check using these equations that the rational normal curve Z of degree 12, defined as the image of the Veronese embedding $\nu_{12}\colon \mathbb{P}^1 \to \mathbb{P}^{12}$, lies inside V_{22} as a minimal G-germ. We will see that Z is in fact the unique minimal G-germ of V_{22}, contained in a unique G-stable divisor.

(Semi)-Invariant Functions

One can find semi-invariants of the correct B-weights by using a torus action, under which each co-ordinate function has a given weight. In this case x_k has weight $(-12 + 2k)\alpha$. We have mentioned that x_{12} is B semi-invariant of weight 12α, so must be the subregular semi-invariant f_v. Likewise $x_{10}x_{12} - x_{11}^2$ is B semi-invariant of weight 20α, so represents f_f. Now from Section 3.2.10, we know that $f_v f_f / f_e$ is a rational function, so has degree 0. Hence f_e must have degree 3 and B- (hence T-) weight 30α, and therefore must be a linear combination of the monomials $x_{10}x_{11}x_{12}, x_{11}^3$ and $x_9x_{12}^2$. A simple check shows that $f_e = 3x_{10}x_{11}x_{12} - 2x_{11}^3 - x_9x_{12}^2$ suffices.

Hence f_f^3/f_e^2 is an invariant rational function defining a rational B-quotient $\pi\colon V_{22} \dashrightarrow \mathbb{P}^1$, $P \mapsto [f_f^3(P) : f_e^2(P)]$. The pullback of $p = [\alpha : \beta] \in \mathbb{P}^1$ defines a regular colour for all p except the following:

$$p = 0 : \pi^*(p) = \mathcal{Z}(f_e^2) = D_0;$$

$$p = \infty : \pi^*(p) = \mathcal{Z}(f_v^3) = D_\infty.$$

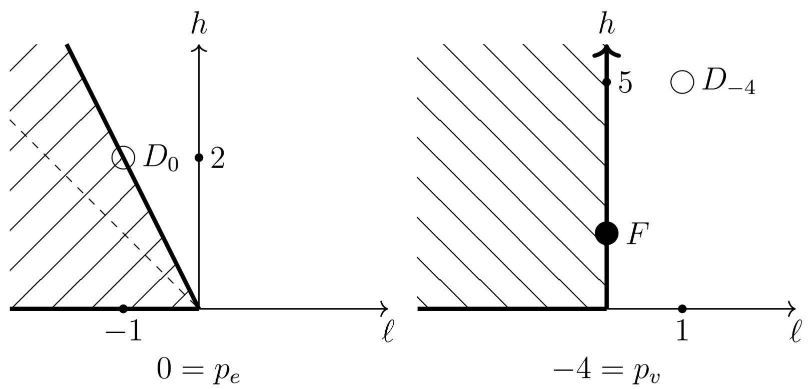
We also see that the subregular colour $\mathcal{Z}(f_v)$ lies over $p = -4 = [1 : -4]$, so we write $D_{-4} = \mathcal{Z}(f_v)$, recalling that this colour has multiplicity 5.

We choose as mentioned the semi-invariant splitting function $e_{2\alpha} = f_v f_f / f_e$ of B-weight 2α, and the colours D_0, D_∞ and D_{-4} are distinguished by it.

Coloured Hyperfan

We already know from previous discussions and our choice of splitting that regular colours D_p for $p \neq 0, \infty - 4$ are mapped to points $(p, 0, 1)$ of hyperspace, and that $D_0 \mapsto (0, -1, 2)$, $D_\infty \mapsto (\infty, 1, 3)$ and $D_{-4} \mapsto (-4, 1, 5)$.

Consider the coloured data of the minimal G-germ Z: we know that $\mathcal{D}_Z^B = \{D_p \mid p \neq -4\}$. This is infinite, so Z is a G-germ of type II and defines a coloured hypercone generated by its coloured data and the polytope $\mathcal{P}$. However, the slice $\mathcal{H}_{-4}$ does not contain a non-central element of $\mathcal{D}_Z^B$, and hence must contain a non-central element of $\mathcal{V}_Z$ since Z is of type II. Hence there must be a G-divisor $F \subseteq V_{22}$ containing Z and lying over $-4 \in \mathbb{P}^1$. In the slice $\mathcal{H}_{-4}$, $\mathcal{C}_Z$ is then generated by $\mathcal{P}$, which is central, and F, and by completeness it must cover the valuation cone. Hence we must have $F \mapsto (-4, 0, h)$ for some $h \geq 0$, and the hyperfan is the same for any such h, so take $h = 1$. Then $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_\infty + \mathcal{P}_{-4} = \{-1/2\} + \{1/3\} + \{0\} = \{-1/6\}$. Hence the coloured hyperfan of V_{22} is as follows:



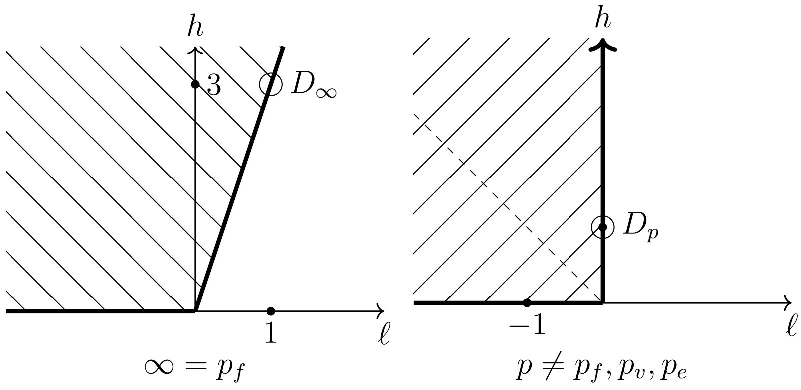


Figure 19: Coloured hyperfan of V_{22}

Chapter 4

β-Invariant and K-Stability

Here we discuss an invariant introduced by Fujita [Fuj16] and Li [Li17] which they have shown to have an intimate connection to K-stability. In the first section, we discuss some general properties of divisors on G-varieties of complexity one which will be needed later on. Then we define the β-invariant of Fujita and Li and use it to state the main results of this thesis, namely the K-polystability of the varieties described in the previous chapter. We prove these results in the remaining sections.

4.1 Divisors on Complexity One G-Varieties

We now begin to study the properties of divisors on complexity-one G-varieties, following [Tim00]. Throughout this section X is a normal but possibly singular variety unless otherwise specified. Helpfully, we can reduce everything to B-stable divisors:

Proposition 4.1. *Let a connected solvable algebraic group B act on a normal variety X. Then any Weil divisor on X is linearly equivalent to a B-stable one.*

Proof. [Tim11, Prop 17.1] We may assume that X is smooth, so that any Weil divisor δ on X is Cartier. We may also assume that δ is the difference of two effective divisors, and hence we only need to prove the claim for effective δ. Since δ is Cartier, it has a corresponding line bundle $\mathcal{O}(\delta)$, and we may also assume that $\mathcal{O}(\delta)$ is B-linearised by Proposition 2.7. Hence the space $H^0(X, \mathcal{O}(\delta))$ is a B-module by Lemma 2.3, and since δ is effective, it is non-empty. Then by the Lie-Kolchin theorem, $H^0(X, \mathcal{O}(\delta))$ contains a nonzero B-eigensection σ. Then $\operatorname{div} \sigma$ is B-stable and linearly equivalent to δ. □

4.1.1 Cartier Divisors

Next we want to investigate conditions which guarantee that a divisor is Cartier. We will assume that the associated line bundle to any Cartier divisor is G-linearised (see section 2.3.4).

Lemma 4.1. *Any prime divisor $D \subseteq X$ which does not contain a G-orbit is Cartier and generated by global sections.*

Proof. See [Tim11, Lemma 17.3]. □

Theorem 4.1. *Let δ be a divisor on X and assume by Proposition 4.1 that δ is B-stable. Then δ is Cartier if and only if for any G-germ Y of X, there exists $f_Y \in K^{(B)}$ such that each prime divisor D containing Y occurs in δ with multiplicity $\nu_D(f_Y)$.*

Proof. [Tim11, Thm 17.4] We are essentially proving that δ is Cartier if and only if it is locally principal at generic points of G-germs. Also note that it suffices to verify the condition at G-orbits since every G-germ is a union of G-orbits.

Suppose δ is locally principal at generic points of G-germs. Choose a G-orbit $Y \subseteq X$ and replace δ with the linearly equivalent B-stable divisor $\delta - \operatorname{div} f_Y$. Then no component of δ contains Y since we have removed the term $\nu_D(f_Y) \cdot [D]$ from δ for each D containing Y.

Take a B-chart $X_0 = \operatorname{Spec} \mathcal{A}(\mathcal{W}, \mathcal{R}) \subseteq X$ intersecting Y. Let $D_1, \dots, D_n$ be the components of δ intersecting X_0. Then the D_i are B- or G-stable, since δ is, and they do not contain Y. Then $\nu_{D_i} \in \mathcal{W} \setminus \mathcal{V}_Y$ or $D_i \in \mathcal{R} \setminus \mathcal{D}_Y^B$ depending on if D_i is G-stable or not. Hence by conditions (V′) or (D) of the Luna-Vust theory, there exist $f_i \in \mathcal{A}^{(B)}$ such that $f_i|_Y \neq 0$ but $\nu_{D_i}(f_i) > 0$, for each i. Hence we can localise X_0 along $f_1, \dots, f_n$ and thereby assume that no component of δ intersects X_0.

Then by the proof of Lemma 4.1, δ is Cartier on $G \cdot X_0$, and hence also on X.

Conversely, assume δ is Cartier. Let $Y \subseteq X$ be a G-germ, and use Sumihiro's theorem to find an open quasiprojective G-stable subset $X_0 \subseteq X$ intersecting Y. Then the restriction of δ to X_0 can be represented by the difference of two globally generated divisors, so we can verify the condition assuming that δ is globally generated. Furthermore, we can replace X with X_0 since any prime divisor $D \supseteq Y$ intersects X_0 in a prime divisor containing $X_0 \cap Y$.

Then $H^0(X, \mathcal{O}(\delta))$ is a G-module and since $\mathcal{O}(\delta)$ is generated by global sections, there exists a section σ which does not vanish on Y. We can take σ to be a B-eigensection by Lie-Kolchin. Then δ is principal on the B-stable open subset X_σ where σ does not vanish, and this set contains Y. Hence we can take f_Y to be the equation of σ on X_σ. □

Corollary 4.1. *A Cartier divisor δ on a G-model X is determined by the following data:*

(i) *a collection $\{f_Y\}$ of B-eigenfunctions for each G-germ $Y \subseteq X$ such that $\nu(f_{Y_1}) = \nu(f_{Y_2})$ and $\nu_D(f_{Y_1}) = \nu_D(f_{Y_2})$ for all $\nu \in \mathcal{V}_{Y_1} \cap \mathcal{V}_{Y_2}$ and all $D \in \mathcal{D}^B_{Y_1} \cap \mathcal{D}^B_{Y_2}$;*

(ii) *a collection of integers m_D for each $D \in \mathcal{D}^B \setminus \bigcup_{Y \subseteq X} \mathcal{D}^B_Y$ (m_D being the multiplicity of D in the divisor), only finitely many of which are nonzero.*

If X is quasihomogeneous of complexity one, each f_Y determines up to scalar multiples (and is up to powers determined by) a linear functional φ_Y on the coloured cone or hypercone $\mathcal{C}_Y$ such that $\varphi_{Y_1}|_{\mathcal{C}_{Y_2}} = \varphi_{Y_2}$ whenever $\mathcal{C}_{Y_2}$ is a face of $\mathcal{C}_{Y_1}$, that is, whenever Y_2 contains Y_1. Then the functionals φ_Y paste together to a piecewise linear function on $\bigcup_{Y \subseteq X} \mathcal{C}_Y \cap \mathcal{V}$, which we call a *piecewise linear function* on the coloured hyperfan $\mathcal{F}_X = \{\mathcal{C}_Y \mid Y \subseteq X\}$ of X. Then Cartier divisors on X correspond to these piecewise linear functions.

4.1.2 Globally Generated and Ample Divisors

Proposition 4.2. *Let δ be a Cartier divisor on X given by $\{f_Y\}$, $\{m_D\}$ as above. Then:*

(i) *δ is globally generated if and only if f_Y can be chosen such that for any G-germ $Y \subseteq X$, we have:*

  (a) *for any other G-germ $Y' \subseteq X$ and every B-stable divisor D containing Y', $\nu_D(f_Y) \leq \nu_D(f_{Y'})$;*

  (b) *for any $D \in \mathcal{D}^B \setminus \bigcup_{Y' \subseteq X} \mathcal{D}^B_{Y}$, $\nu_D(f_Y) \leq m_D$.*

(ii) *δ is ample if and only if, after replacing δ with some positive multiple, f_Y can be chosen such that for any G-germ $Y \subseteq X$, there exists a B-chart $X_0 \subseteq X$*

intersecting Y such that (a) and (b) hold, and the inequalities therein are strict if and only if $D \cap X_0 = \emptyset$.

Proof. [Tim11, Thm 17.18] (i): We first note that δ is globally generated if and only if for any G-germ $Y \subseteq X$, there exists $\eta \in H^0(X, \mathcal{O}(\delta))$ with $\eta|_Y \neq 0$. Assume this is the case, and take η to be a B-eigensection. Then there is $f \in K^{(B)}$ such that $\operatorname{div} f + \delta \geq 0$, and no $D \supseteq Y$ appears in $\operatorname{div} f + \delta$ with positive multiplicity. Since $D \supseteq Y$ appears in $\operatorname{div} f + \delta$ with multiplicity zero, it occurs in δ with multiplicty $-\nu_D(f) = \nu_D(f^{-1})$, so we can replace f_Y with f^{-1}.

If Y' is another G-germ and D a B-stable divisor containing Y, then D has multiplicity $\nu_D(f_{Y'})$ in δ, and non-negative multiplicity in $\delta + \operatorname{div} f = \delta - \operatorname{div} f_Y$, so $\nu_D(f_{Y'}) - \nu_D(f_Y) \geq 0$, giving (a). A similar argument gives (b).

Now assuming (a) and (b), for any G-germ $Y \subseteq X$, the global section $f_Y^{-1} \in H^0(X, \mathcal{O}(\delta))$ does not vanish at Y, so δ is globally generated.

(ii): Suppose δ is ample. Then we can replace δ with a multiple and assume it is very ample. Take a G-submodule $M \subseteq H^0(X, \mathcal{O}(\delta))$ determining a locally closed G-equivariant embedding $X \to \mathbb{P}(M^*)$. Now let $Y \subseteq X$ be a G-germ. We can choose a homogeneous B-semi-invariant polynomial on $\mathbb{P}(M^*)$, i.e. an element of $H^0(X, \mathcal{O}(\delta)^{\otimes N})^{(B)}$ for some N, which vanishes on $\overline{X} \setminus X$ but not on Y. Replacing δ with $N\delta$, we can assume there is a section $\eta \in H^0(X, \mathcal{O}(\delta))^{(B)}$ with the same properties. Then $X_\eta \subseteq X$ is a B-chart intersecting Y and there exists $f \in K^{(B)}$ with $\operatorname{div} f + \delta = \operatorname{div} \eta \geq 0$.

Replacing f_Y with f^{-1} as above, we get (a) and (b). Now let $Y' \subseteq X$ be another G-germ and let D be a divisor containing Y. Then $D \cap X_0$ is empty if and only if η vanishes on D, i.e. D occurs in $\operatorname{div} \eta = \operatorname{div} f + \delta$ with positive multiplicity. But the multiplicity of D in $\operatorname{div} f + \delta$ is $\nu_D(f_{Y'}) - \nu_D(f_Y)$, so (c) holds.

Assuming (a), (b) and (c), we will show that a multiple of δ is very ample. Let $Y \subseteq X$ be a G-germ. By the above there is $\eta \in H^0(X, \mathcal{O}(\delta))^{(B)}$ determined by f_Y^{-1} such that $X_\eta \subseteq X$ is a B-chart intersecting Y. We can choose finitely many η_i defining B-charts X_i such that $G \cdot X_i$ cover X. Then

$$\Bbbk[X_i] = \bigcup_{n \geq 0} \eta_i^{-n} H^0(X, \mathcal{O}(\delta)^{\otimes n}) = \Bbbk\left[\frac{\sigma_{i,1}}{\eta_i^{n_i}}, \ldots, \frac{\sigma_{i,s_i}}{\eta_i^{n_i}}\right]$$

where $n_i, s_i \in \mathbb{N}$ and $\sigma_{i,j} \in H^0(X, \mathcal{O}(\delta)^{\otimes n_i})$. Replacing δ by a multiple, we can assume

$n_i = 1$.

Take the finite dimensional G-submodule $M \subseteq H^0(X, \mathcal{O}(\delta))$ generated by the η_i and $\sigma_{i,j}$. The corresponding rational map $\varphi\colon X \dashrightarrow \mathbb{P}(M^*)$ is G-equivariant and defined on X_i, hence on $G \cdot X_i$, hence on all of X. Also, $\varphi^{-1}(\mathbb{P}(M^*)_{\eta_i}) = X_i$, and $\varphi|_{X_i}$ is a closed embedding into $\mathbb{P}(M^*)_{\eta_i}$ for each i. Hence φ is a locally closed embedding of X into $\mathbb{P}(M^*)$, and δ is very ample. $\square$

4.1.3 Global Sections

Let $\mathcal{B}(X)$ be the set of all B-stable prime divisors on X, including the G-stable ones. Let $\delta = \sum_{D \in \mathcal{B}(X)} m_D D$ be a B-stable Cartier divisor, and let $\eta_\delta \in H^0(X, \mathcal{O}(\delta))^{(B)}$ be the respective rational B-eigensection (i.e. $\operatorname{div} \eta_\delta = \delta$). We have

$$H^0(X, \mathcal{O}(\delta))^{(B)} = \{f\eta_\delta \mid f \in K^{(B)}, \operatorname{div} f + \delta \geq 0\}.$$

The B-weight of an arbitrary B-eigensection $\sigma = f\eta_\delta$ is $\lambda + \lambda_\delta$, where λ is the weight of f and λ_δ is the weight of η_δ. The latter is determined up to a character of G and can be calculated as follows: let Y be a G-orbit intersecting δ and pull $Y \cap \delta$ back to G under the orbit map, giving a divisor $\tilde{\delta}$ on G. Since we assume G to be factorial, $\tilde{\delta}$ is principal, defined by a rational function $F \in \Bbbk(G)^{(B)}$. Then λ_δ is the B-weight of F.

It follows that

$$H^0(X, \mathcal{O}(\delta))^{(B)}_{\lambda+\lambda_\delta} \cong \{f \in K^{(B)}_\lambda \mid \operatorname{div} f + \delta \geq 0\} \cong \{f \in K^B \mid \operatorname{div} f + \operatorname{div} e_\lambda + \delta \geq 0\}.$$

We want to calculate the calculate the dimension of the space $H^0(X, \mathcal{O}(\delta))$ of global sections of δ. Recall from Corollary 2.3 that, setting $m_\lambda(\delta) = \dim H^0(X, \mathcal{O}(\delta))^{(B)}_{\lambda+\lambda_\delta}$ for brevity, we have

$$\dim H^0(X, \mathcal{O}(\delta)) = \sum_{\lambda \in \Lambda} m_\lambda(\delta) \prod_{\alpha^\vee \in \Delta_+^\vee} \left(1 + \frac{(\lambda, \alpha^\vee)}{(\rho, \alpha^\vee)}\right).$$

We can calculate $m_\lambda(\delta)$ using the notion of a pseudodivisor:

Definition 4.1. Let C be a smooth projective curve. A *pseudodivisor* μ on C is a formal linear combination $\mu = \sum_{p \in C} m_p \cdot p$ where $m_p \in \mathbb{R} \cup \{\pm\infty\}$ and all but finitely many m_p are 0. Let $H^0(C, \mu) = \{f \in \Bbbk(C) \mid \operatorname{div} f + \mu \geq 0\}$ where for all $x \in \mathbb{R}$, we set $x + (\pm\infty) = \pm\infty$.

If there is $p \in C$ with $m_p = -\infty$, then $H^0(C, \mu) = 0$. Otherwise, $H^0(C, \mu)$ is the space of global sections of the divisor $\lfloor \mu \rfloor = \sum_p \lfloor m_p \rfloor \cdot p$ on $C \setminus \{p \in C \mid m_p = +\infty\}$, where $\lfloor m_p \rfloor$ represents the floor of m_p.

Now let δ be as above. Note that $H^0(X, \mathcal{O}(\delta))^{(B)}$ is isomorphic to

$$\left\{ f_0 e_\lambda \mid f_0 \in K^B, \lambda \in \Lambda, \sum_{D \in \mathcal{B}(X)} [h_D \nu_{p_D}(f_0) + \langle \lambda, \ell_D \rangle + m_D] D \geq 0 \right\}.$$

Hence fix $\lambda \in \Lambda$ and consider the pseudodivisor

$$H_\lambda = H_\lambda(\delta) = \sum_{p \in \mathbb{P}^1} \left(\min_{p_D = p} \frac{\langle \lambda, \ell_D \rangle + m_D}{h_D} \right) p,$$

where we assume $\frac{x}{0} = +\infty$ for $x \geq 0$ and $\frac{x}{0} = -\infty$ for $x < 0$. It is clear from the above description of $H^0(X, \mathcal{O}(\delta))^{(B)}$ that $m_\lambda(\delta) = \dim H^0(\mathbb{P}^1, H_\lambda(\delta)) := h^0(\delta, \lambda)$.

We know that $h^0(\mathbb{P}^1, H_\lambda) = 0$ if any of its coefficients are $-\infty$. This is the case exactly when there is $p \in \mathbb{P}^1$ and $D \in \mathcal{B}(X)$ with $p_D = p$ satisfying $h_D = 0$ and $\langle \lambda, \ell_D \rangle < -m_D$. Hence we define a polyhedral domain

$$\mathcal{P}(\delta) = \{\lambda \in \Lambda \otimes \mathbb{R} \mid \langle \lambda, \ell_D \rangle \geq -m_D \text{ for all } D \text{ with } h_D = 0\}.$$

Then $H^0(\mathbb{P}^1, H_\lambda) = 0$ for all $\lambda \notin \mathcal{P}(\delta)$. Conversely, a coefficient of H_λ is $+\infty$ if and only if there is $p \in \mathbb{P}^1$ such that no divisor $D \in \mathcal{B}(X)$ with $p_D = p$ satisfies $h_D > 0$. This is the case e.g. if X is a B-chart of type I. Then $H^0(\mathbb{P}^1, H_\lambda)$ is the space of global sections of $\lfloor H_\lambda \rfloor$ on the affine curve $\mathbb{P}^1 \setminus \{p \mid m_p = +\infty\}$ and hence $h^0(\delta, \lambda) = \infty$ for all $\lambda \in \mathcal{P}(\delta)$.

Otherwise, H_λ is a 'standard' Weil divisor on $\mathbb{P}^1$, so by Riemann-Roch we have $h^0(\delta, \lambda) = \deg \lfloor H_\lambda \rfloor + 1 + h^1(\delta, \lambda)$, where $h^1(\delta, \lambda) := \dim H^1(\mathbb{P}^1, \lfloor H_\lambda \rfloor)$. If we define

$$A(\delta, \lambda) = \sum_{p \in \mathbb{P}^1} \min_{p_D = p} \frac{\langle \lambda, \ell_D \rangle + m_D}{h_D},$$

i.e. $A(\delta, \lambda) = \deg H_\lambda$, then $\deg \lfloor H_\lambda(\delta) \rfloor$ differs from $A(\delta, \lambda)$ by some bounded non-negative function $\sigma(\delta, \lambda)$ for all δ, λ. We then have $h^0(\delta, \lambda) = A(\delta, \lambda) - \sigma(\delta, \lambda) + h^1(\delta, \lambda) + 1$.

Proposition 4.3. *If $A(\delta, \lambda) < 0$, then $h^0(\delta, \lambda) = 0$. Otherwise, for large n, $h^0(n\delta, n\lambda) \sim nA(\delta, \lambda)$.*

Proof. [Tim11, §17.4] The first claim follows from the fact that $A(\delta,\lambda) < 0$ implies $\deg H_\lambda < 0$ and hence $\deg \lfloor H_\lambda \rfloor < 0$, so $\lfloor H_\lambda \rfloor$ has no global sections and $h^0(\delta,\lambda) = 0$.

Suppose $A(\delta,\lambda) \geq 0$. If $\deg \lfloor H_\lambda(\delta) \rfloor \geq 0$, i.e. $A(\delta,\lambda) \geq \sigma(\delta,\lambda)$, we have $h^0(\delta,\lambda) \leq \deg \lfloor H_\lambda \rfloor + 1$ (by [Har77, Ex. II.1.5]), so $h^1(\delta,\lambda) \leq 0$, i.e. $h^1(\delta,\lambda) = 0$. Since $\sigma(\delta,\lambda)$ is bounded while $A(n\delta, n\lambda) = nA(\delta,\lambda)$, the former condition will hold for all sufficiently large n. Then $h^0(n\delta, n\lambda) = nA(\delta,\lambda) - \sigma(n\delta, n\lambda) + 1 \sim nA(\delta,\lambda)$ as required. □

Inspired by the first part of the above Proposition, define the polyhedral domain

$$\mathcal{P}_+(\delta) = \{\lambda \in \mathcal{P}(\delta) \mid A(\delta,\lambda) \geq 0\}.$$

Then by the above and the definition of $\mathcal{P}(\delta)$, we have $h^0(\delta,\lambda) = 0$ for all $\lambda \notin \mathcal{P}_+(\delta)$.

4.1.4 Volume of Divisors

We discuss the notion of the volume of a divisor, as described in e.g. [Laz17a, §2.2.C].

Definition 4.2. Let δ be a Cartier divisor on a smooth projective variety X of dimension d. The *volume* of δ is

$$\operatorname{vol} \delta = \limsup_{n\to\infty} \frac{\dim H^0(X, \mathcal{O}(\delta)^{\otimes n})}{n^d/d!}.$$

In fact, by [Laz17b, Ex. 11.4.7] the lim sup is actually a limit.

Proposition 4.4. *If δ is ample,* $\operatorname{vol} \delta = \delta^d$.

Proof. By asymptotic Riemann-Roch, for sufficiently large n we have

$$\dim H^0(X, \mathcal{O}(\delta)^{\otimes n}) = \chi(X, \mathcal{O}(\delta)^{\otimes n}) = \frac{n^d \delta^d}{d!} + O(n^{d-1}).$$

Hence

$$\frac{\dim H^0(X, \mathcal{O}(\delta)^{\otimes n})}{n^d/d!} = \delta^d + O(n^{-1}).$$

Taking the limit $n \to \infty$ gives the result. □

Now assume that X is a smooth projective G-model. Let Δ be the root system of G, Δ_+ the positive roots determined by B, $\Delta_+^\vee$ the corresponding set of positive coroots and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha$. The following formula of Timashev [Tim00] allows us to compute the volume of a Cartier divisor on a complexity one variety.

Theorem 4.2. *Let δ be a B-stable Cartier divisor of weight λ_δ on a normal projective quasihomogeneous G-variety X of dimension d, complexity $c = 1$ and rank r. Then, in the notation of the previous subsection:*

$$d = c + r + |\Delta_+^\vee \setminus (\Lambda + \mathbb{Z}\lambda_\delta)^\perp|,$$

and

$$\operatorname{vol}\delta = d! \int\limits_{\lambda_\delta + \mathcal{P}_+(\delta)} A(\delta, \lambda - \lambda_\delta) \prod_{\alpha^\vee \in \Delta_+^\vee \setminus (\Lambda + \mathbb{Z}\lambda_\delta)^\perp} \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} \, \mathrm{d}\lambda$$

where the Lebesgue measure on $\Lambda \otimes \mathbb{R}$ is normalised such that a fundamental parallelepiped of Λ has volume 1.

Proof. [Tim11, Thm 18.8] By the dicussions in section 4.1.3, we have

$$\begin{aligned}
\dim H^0(X, \mathcal{O}(\delta)^{\otimes n}) &= \sum_{\lambda \in n\lambda_\delta + n\mathcal{P}(\delta) \cap \Lambda} m_{\lambda - n\lambda_\delta}(n\delta) \cdot \dim V_\lambda \\
&= \sum_{\lambda \in \lambda_\delta + \mathcal{P}(\delta) \cap \frac{1}{n}\Lambda} m_{n(\lambda - \lambda_\delta)}(n\delta) \cdot \dim V_{n\lambda} \\
&= \sum_{\lambda \in \lambda_\delta + \mathcal{P}_+(\delta) \cap \frac{1}{n}\Lambda} \prod_{\alpha^\vee \in \Delta_+^\vee} \left(1 + n\frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}\right) [nA(\delta, \lambda - \lambda_\delta) \\
&\quad - \sigma(n\delta, n(\lambda - \lambda_\delta)) + h^1(n\delta, n(\lambda - \lambda_\delta)) + 1]
\end{aligned}$$

Take $n \gg 0$: by Proposition 4.3 the term in square brackets tends to $nA(\delta, \lambda - \lambda_\delta)$; in the product, the $+1$ becomes negligible; and the scaling of the lattice Λ introduces a factor of n^r, r being $\operatorname{rk}\Lambda$. Finally, we replace the sum with an integral, giving:

$$\begin{aligned}
\dim H^0(X, \mathcal{O}(\delta)^{\otimes n}) &\sim n^r \int\limits_{\lambda_\delta + \mathcal{P}_+(\delta)} nA(\delta, \lambda - \lambda_\delta) \prod_{\alpha^\vee \in \Delta_+^\vee \setminus (\lambda_\delta + \mathcal{P}_+(\delta))^\perp} n\frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} d\lambda \\
&= n^{c + r + |\Delta_+^\vee \setminus (\lambda_\delta + \mathcal{P}_+(\delta))^\perp|} \int\limits_{\lambda_\delta + \mathcal{P}_+(\delta)} A(\delta, \lambda - \lambda_\delta) \prod_{\alpha^\vee \in \Delta_+^\vee \setminus (\lambda_\delta + \mathcal{P}_+(\delta))^\perp} \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} d\lambda
\end{aligned}$$

Hence it remains to prove that $(\lambda_\delta + \mathcal{P}_+(\delta))^\perp = (\Lambda + \mathbb{Z}\lambda_\delta)^\perp$ and the formula for d. For the former, it suffices to show that $\mathcal{P}_+(\delta)$ generates $\Lambda \otimes \mathbb{R}$. Suppose $\lambda \in \Lambda$ is the weight of a B-eigenfunction f on X. Write f as a quotient of B-eigensections of some $\mathcal{O}(\delta)^{\otimes n}$. Then λ is the difference of the weights of these eigensections. Since $h^0(\delta, \lambda) = 0$ outside $\mathcal{P}_+(\delta)$, these weights must lie in $\mathcal{P}_+(\delta)$ and we get the result.

For the formula for d, see [Tim11, Thm 18.8]. □

Example. Let $G = \mathrm{SL}_2$ so that $\Delta = \{\pm\alpha\}$, $\Delta_+ = \{\alpha\}$, $\Delta_+^\vee = \left\{\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}\right\}$ and $\rho = \frac{1}{2}\alpha$. Any quasihomogeneous complexity one SL_2-threefold X has $d = 3$, $c = 1$, $r = 1$, so $|\Delta_+^\vee \setminus (\Lambda + \mathbb{Z}\lambda_\delta)^\perp| = 1$, i.e. this set is just $\Delta_+^\vee = \{\alpha^\vee\}$. If we identify $\Lambda = \mathbb{Z}\alpha$ with $\mathbb{Z}$, and hence α with 1, we have

$$\prod_{\alpha^\vee \in \Delta_+^\vee \setminus (\Lambda + \mathbb{Z}\lambda_\delta)^\perp} \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} = \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \frac{\alpha}{2}, \alpha^\vee \rangle} = 2\lambda,$$

and the volume of a Cartier divisor δ on X is given by

$$\operatorname{vol} \delta = 6 \int_{\lambda_\delta + \mathcal{P}_+(\delta)} 2\lambda A(\delta, \lambda - \lambda_\delta) \, \mathrm{d}\lambda.$$

4.2 β-Invariant

Definition 4.3. Let X be a Fano variety. If $\sigma \colon Y \to X$ is any projective birational morphism with Y normal, we call a prime divisor $F \subseteq Y$ a *prime divisor over* X.

Proposition 4.5. *Let X be a smooth complex Fano projective variety. There is a bijective correspondence between prime divisors over X and test configurations on X (excluding the trivial test configuration).*

Proof. [Xu20, Lemma 3.7] Let $F \subseteq Y \to X$ be a prime divisor over X and let $-K_X$ be the anticanonical divisor of X. Consider the section ring $R = \bigoplus_{k \in \mathbb{Z}} H^0(X, -kK_X)$ of $-K_X$. The prime divisor F induces a filtration

$$\mathcal{F}^r R := \bigoplus_{k \in \mathbb{Z}} \{ f \in H^0(X, -kK_X) \mid \nu_F(f) \geq r \}$$

of R. Consider the Rees algebra

$$\mathcal{A} = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}^r R \cdot z^{-r}$$

of this filtration. Setting $r = -1$, we see that the $k = 0$ piece of $\mathcal{F}^r R$ is $\mathbb{C}$. It follows that $z \in \mathcal{A}_{(0,-1)}$, so there is an embedding $\mathbb{C}[z] \hookrightarrow \mathcal{A}$. This induces a morphism $\mathcal{X} = \operatorname{Proj} \mathcal{A} \to \mathbb{A}^1$, which is automatically compatible with the standard $\mathbb{C}^\times$ action on $\mathbb{A}^1$. Furthermore, since $R = \mathcal{A}/(z-1)$ we see that $X = \operatorname{Proj} R$ embeds into $\mathcal{X} = \operatorname{Proj} \mathcal{A}$ as a closed subscheme and is the preimage of $1 \in \mathbb{A}^1$ under the given morphism. Hence $(\mathcal{X}, \mathcal{O}(1)_{\mathcal{X}})$ is indeed a test configuration on $(X, -K_X)$.

Conversely, given a test configuration $(\mathcal{X}, \mathcal{L})$ on $(X, -K_X)$, the special fibre $X_0 \subseteq \mathcal{X}$ is a prime divisor, having a corresponding valuation ν_0 on $\Bbbk(\mathcal{X}) = \Bbbk(X \times \mathbb{A}^1)$. We can thus restrict ν_0 to $\Bbbk(X)$. Since the restriction of a geometric valuation is itself geometric, there exists some model Y of $\Bbbk(X)$ with a prime divisor $F \subseteq Y$ such that $\nu_F = \nu_0|_{\Bbbk(X)}$, i.e. a prime divisor over X. □

Definition 4.4. Let $\mathcal{F}^r$ be a filtration of an algebra R. The *associated graded ring* to the filtration $\mathcal{F}^r$ is the ring

$$\mathcal{B} = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}^r / \mathcal{F}^{r+1}.$$

Remark. In the notation of Proposition 4.5, the associated graded ring to the filtration $\mathcal{F}^r$ on R is $\mathcal{A}/(z)$. The ideal $(z) \subseteq \mathbb{C}[z]$ corresponds to $0 \in \mathbb{A}^1$, so $\mathcal{A}/(z)$ corresponds to the fibre over 0 of the morphism $\operatorname{Proj} \mathcal{A} \to \mathbb{A}^1$, i.e. the central fibre of the corresponding test configuration.

Definition 4.5. Let X be a Fano variety and let $F \subseteq Y \xrightarrow{\sigma} X$ be a prime divisor over X. The *log discrepancy of F over X* is $A_X(F) = \operatorname{ord}_F(K_{Y/X}) + 1$, where $K_{Y/X} = K_Y - \sigma^*(K_X)$ is the *relative canonical divisor.*

The usual definition of log discrepancy is more general, but we will only need the one above. For full details see e.g. [KM98, §2]. A useful consequence of this definition is that it is not hard to see that if σ is a sequence of n nested blow-ups, of which F is the final exceptional divisor, then $A_X(F) = n + 1$.

Definition 4.6. Let X be a smooth complex Fano variety of dimension n. Let $F \subseteq Y$ be a prime divisor over X. The *β-invariant* of F over X is

$$\beta_X(F) = A_X(F)(-K_X)^n - \int_0^\infty \operatorname{vol}(-K_X - xF)dx,$$

where $\operatorname{vol}(-K_X - xF)$ is shorthand for $\operatorname{vol}(\sigma^*(-K_X) - xF)$, where $\sigma\colon Y \to X$.

Theorem 4.3. *[Li17, Fuj16] A smooth complex Fano variety X is K-semistable if and only if $\beta_X(F) \geq 0$ for any prime divisor F over X, and K-stable if and only if the inequality is always strict.*

The proof of the above essentially amounts to the fact that, under the correspondence described in Proposition 4.5, the β-invariant of a prime divisor F over X is a positive multiple of the Donaldson-Futaki invariant of the corresponding test configuration. This gives a *K-polystability* criterion as well:

Corollary 4.2. *A smooth complex Fano variety X is K-polystable if and only if $\beta_F(X) \geq 0$ for any prime divisor F over X, and $\beta_F(X) = 0$ only when F corresponds to a product test configuration.*

In our perspective when X is equipped with a group action, we have:

Corollary 4.3. *If X comes equipped with the action of a reductive group G, then we need only check the β-invariant for prime divisors $F \subseteq Y \to X$ in the case where Y also has a G-action, the morphism $Y \to X$ is G-equivariant, and F is G-invariant in Y.*

Proof. This follows by combining the theorem of Fujita-Li with the theorem of Datar-Székelyhidi. □

4.3 K-Polystability of SL_2-Threefolds

The main results of this thesis are the following:

Theorem 4.4. *Let X be a smooth Fano SL_2-threefold. If any of the three conditions below holds, then X is K-polystable if $\beta_F(X) > 0$ for all* central *G-stable prime divisors over X:*

(i) *A finite subgroup $A \subseteq \operatorname{Aut} X$ acts on $\mathbb{P}^1$ with no fixed points, such that the rational B-quotient $X \dashrightarrow \mathbb{P}^1$ is A-equivariant*

(ii) *A finite subgroup $A \subseteq \operatorname{Aut} X$ acts on $\mathbb{P}^1$, interchanging two points in $\mathbb{P}^1$ corresponding to subregular colours of X, and the rational B-quotient $X \dashrightarrow \mathbb{P}^1$ is A-equivariant*

(iii) *X has subregular colours lying over three or more distinct points of $\mathbb{P}^1$.*

Remark. We expect but have not proved that Theorem 4.4 applies with only minor alterations to smooth Fano G-varieties of complexity one in general, rather than just to SL_2-threefolds.

Remark. We note the similarity of this result to [Süß13, Thm 1.1]

Theorem 4.5. *The smooth Fano threefolds, (1.16), (1.17), (2.27), (2.32), (3.17), (3.25) and (4.6) in the Mori-Mukai classification are K-polystable. The families (2.21) and (3.13) each contain a K-polystable variety.*

Remark. The same result were recently obtained independently by other authors using different methods, see [SC21, ACC$^+$21]. The K-polystability of the Mukai-Umemura threefold in the family (1.10) was already known by Donaldson [Don08], and the K-polystability of V_5 (1.15) was known by Cheltsov-Shramov [CS09].

We will prove the first of these theorems in the next section for each of the three cases. We then demonstrate that for each of the varieties listed in Theorem 4.5, one of these three conditions holds. The remainder of the proof of Theorem 4.5 will be given in the section after that, using an argument which shows that in all of the above examples, there can only be one central G-divisor over X, followed by an explicit calculation of the β-invariant of this divisor in each case.

4.4 Test Configurations From Non-Central Divisors

4.4.1 Action Without Fixed Points

Proof of Theorem 4.4(i). If A acts on $\mathbb{P}^1$ and the B-quotient $X \dashrightarrow \mathbb{P}^1$ is A-equivariant, we have an overall action on X of an extension G' of G by A which descends to the quotient. Suppose F is a non-central G-invariant prime divisor over X. Since F is non-central, it must lie over some point $P_F \in \mathbb{P}^1$. If A acts with no fixed points on $\mathbb{P}^1$, then in particular P_F is not fixed, so F cannot be G'-invariant. It follows that only central divisors can be G'-invariant. Since A is finite, G' is reductive, so by the theorems of Fujita-Li and Datar-Székelyhidi, X is K-polystable if $\beta(F) \geq 0$ for all G'-invariant prime divisors over X, i.e. for all central G-divisors over X, and is only 0 for divisors corresponding to product configurations. □

We now show that this case of Theorem 4.4 applies to $\mathbb{P}^3$ (1.17), the blow-up of $\mathbb{P}^3$ in two lines (3.25) and the blow-up of $\mathbb{P}^3$ in three lines (4.6).

Realise $\mathbb{P}^1$ as the unit sphere and consider the action of the symmetric group $A = S_3$ on $\mathbb{P}^1$ consisting of rotations permuting the vertices of an equilateral triangle inscribed in the equator. This action has no fixed points, one orbit of order 2 consisting of the north and south poles, and various other orbits of orders 3 and 6.

If $X = \mathbb{P}^3$, then $S_3 \subseteq \operatorname{Aut} X = \mathrm{PGL}_4$. Recall that the slices of the coloured hyperfan are all identical to each other except for that of the distinguished point, which lies two units to the left on the ℓ-axis relative to the other slices. Hence we can shift the slice of the distinguished point p_0 one unit to the right and shift the slice of some other point p_1 one unit to the left. Now $\mathcal{H}_{p_0}$ and $\mathcal{H}_{p_1}$ look the same, so we identify these two points with the north and south poles of the sphere. It follows that the A-action preserves the coloured hyperfan of X, in the sense that for $a \in A$ and $p \in \mathbb{P}^1$, we permute the slices of $\mathcal{H}$ via $\mathcal{H}_p \to \mathcal{H}_{a \cdot p}$ and the coloured data within each slice are invariant with respect to these permutations. This means that the action of A on X is such that the B-quotient is A-invariant and Theorem 4.4

If X is the blow-up of $\mathbb{P}^3$ along two lines, say Y_q and Y_r, then let q and r be the north and south poles of the sphere $\mathbb{P}^1$. Shift their slices of the hyperspace two units to the left each, then choose two other non-distinguished points p_1 and p_2 and shift their slices two units to the right each. Then the slices corresponding to p_1, p_2 and the distinguished point p_0 align, and we can choose these to be the vertices of the equilateral triangle acted on by S_3. Again, S_3 acts on X and preserves the coloured hyperfan up to balanced integral shifts, so Theorem 4.4 applies.

If X is the blow-up of $\mathbb{P}^3$ along three lines, Y_q, Y_r and Y_s, the method is the same as when $X = \mathbb{P}^3$, only making sure to choose q, r and s to be the vertices of the triangle. Again, Theorem 4.4 applies.

As an aside, it is worth mentioning that the blow-up of $\mathbb{P}^3$ along *one* line (2.33) has non-reductive automorphism group and is thus not K-polystable. It is easy to see that the method above does not work in this case, since the slice of hyperspace corresponding to the blown-up line is fundamentally distinct from all other slices for this variety, so no action on $\mathbb{P}^1$ which does not fix the corresponding point could ever preserve the coloured hyperfan.

4.4.2 Non-Normality

The proof of Theorem 4.4 parts (ii) and (iii) will be by showing that, under these conditions, the test configurations corresponding to non-central prime divisors over

X have non-normal central fibre, and hence are not *special* test configurations, and therefore we do not need to calculate their Donaldson-Futaki invariant (or, equivalently, the β invariant of F). We will use the correspondence hinted at in section 4.1.3 between B-semi-invariant sections of prime divisors on over X and sections of divisors on the B-quotient. We show that the filtrations defined by the resulting divisors on $\mathbb{P}^1$ give non-integrally closed rings which correspond to the central fibres of the given test configuration.

Divisors on $\mathbb{P}^1$

Theorem 4.6. *Let $H = \sum_{i=1}^m a_i Q_i = \sum_{i=1}^m \frac{b_i}{c_i} Q_i$ be a $\mathbb{Q}$-divisor of positive degree on $\mathbb{P}^1$, and let $P \in \mathbb{P}^1$. Let*

$$\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} H^0(kH)$$

be the section ring of H. Fix $q \in \mathbb{Z}$ and consider the filtration on $\mathcal{A}$ over $r \in \mathbb{Z}$ given by

$$\mathcal{F}_r^q = \bigoplus_{k \in \mathbb{Z}} \left\{ f \in H^0(kH) \mid \mathrm{ord}_P(f) \geq \frac{r}{q} \right\}.$$

Then take the associated graded ring

$$\mathcal{B}^q = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}_r^q / \mathcal{F}_{r+1}^q.$$

If at least two Q_i, both distinct from P, have non-integral coefficients $a_i \notin \mathbb{Z}$ in H, then for each $q \in \mathbb{Z}_{\geq 1}$, the ring $\mathcal{B}^q$ is not integrally closed.

We will prove this theorem in a number of steps, beginning with:

Proposition 4.6. *With all notation as in Theorem 4.6, for any $q \geq 1$ there exist integers k, r and n, with k and n positive, such that $\mathcal{B}^q_{(k,r)} = 0$ and $\mathcal{B}^q_{(nk,nr)} \neq 0$.*

Lemma 4.2. *In the above proposition and theorem, we can assume without loss of generality that $q = 1$.*

Proof. Let $q \geq 1$. We have $\mathcal{B}^1_{(k,r)} = \mathcal{B}^q_{(k,qr)}$, so if we find k, r and n with $\mathcal{B}^1_{(k,r)} = 0$ and $\mathcal{B}^1_{(nk,nr)} \neq 0$, then k, qr and n give the required result for $q > 1$. Hence we set $q = 1$ going forward, and we drop the corresponding superscript. □

Proof of Proposition 4.6. By the lemma, we assume that $q = 1$, and we want to find n, k and r with $\mathcal{B}_{(k,r)} = 0$ and $\mathcal{B}_{(nk,nr)} \neq 0$. We will rewrite $H = \sum_{i=1}^{m} a_i Q_i + a_P P$, with $a_P = \frac{b_P}{c_P}$, and assuming all $Q_i \neq P$. Sums and products indexed over i should be understood as running from $i = 1$ to $i = m$ and *excluding* P, a_P etc. unless otherwise specified.

Before we choose particular values of k, r and n, we will demonstrate an alternative description of $\mathcal{B}_{(k,r)}$. Denote by $\mathcal{F}_{(k,r)}$ the degree-k part of $\mathcal{F}_r$, i.e. $\mathcal{F}_{(k,r)} = \{f \in H^0(kH) \mid \mathrm{ord}_P(f) \geq r\}$. Then $\mathcal{B}_{(k,r)} = \mathcal{F}_{(k,r)}/\mathcal{F}_{(k,r+1)}$. Define

$$H_{(k,r)} = \begin{cases} kH & -r \geq \lfloor ka_P \rfloor \\ kH - ka_P P - rP & -r \leq \lfloor ka_P \rfloor. \end{cases}$$

We will show that $\mathcal{F}_{(k,r)} = H^0(H_{(k,r)})$, so $\mathcal{B}_{(k,r)} = H^0(H_{(k,r)})/H^0(H_{(k,r+1)})$.

Indeed, when $-r \geq \lfloor ka_P \rfloor$, it suffices to show that any $f \in H^0(kH)$ automatically has order at least r at P. Any such f must satisfy $\mathrm{ord}_P(f) + \lfloor ka_P \rfloor \geq 0$, and if $0 \geq r + \lfloor ka_P \rfloor$ then the result follows.

On the other hand, it is clear that $\mathcal{F}_{(k,r)} \subseteq H^0(kH - ka_P P - rP)$. Now suppose $f \in H^0(kH - ka_P P - rP)$ and $-r < \lfloor ka_P \rfloor$. We have $\mathrm{ord}_P(f) \geq r$ since the coefficient at P of $(f) + \lfloor kH - ka_P P - rP \rfloor$ is $\mathrm{ord}_P(f) - r$, so it remains to show that $f \in H^0(kH)$. Since kH only differs from $kH - ka_P P - rP$ at P, it is sufficient to note that $\mathrm{ord}_P(f) + \lfloor ka_P \rfloor \geq r + \lfloor ka_P \rfloor > 0$.

Hence to show $\mathcal{B}_{(k,r)} = 0$ it is sufficient either that $\deg \lfloor H_{(k,r)} \rfloor < 0$ or $H^0(H_{(k,r)}) = H^0(H_{(k,r+1)})$.

Likewise, for $\mathcal{B}_{(k,r)} \neq 0$ we must show that $\deg \lfloor H_{(k,r)} \rfloor \geq 0$ and $H^0(H_{(k,r)}) \neq H^0(H_{(k,r+1)})$. Note that when the first of these conditions holds and $-r \leq \lfloor ka_P \rfloor$, the second one also holds by definition of $H_{(k,r)}$. If $-r \geq \lfloor ka_P \rfloor$ and $\deg \lfloor kH \rfloor \geq 0$ we have

$$0 \leq \deg \lfloor kH \rfloor = \sum_i \lfloor ka_i \rfloor + \lfloor ka_i \rfloor \leq r + \lfloor ka_i \rfloor \leq 0,$$

so $-(r+1) < \lfloor ka_i \rfloor$ and $H^0(H_{(k,r)}) \neq H^0(H_{(k,r+1)})$ as well. Thus $\mathcal{B}_{(k,r)} \neq 0$ if and only if $\deg \lfloor H_{(k,r)} \rfloor \geq 0$.

<u>Choice of k</u>:

Our choice of k is motivated by two requirements, the reasons for which will be seen later, these being:

$$(1)\ \sum_i \{ka_i\} \geq 1$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of a real number x, and

$$(2)\ \deg \lfloor kH \rfloor \geq 0.$$

With that in mind, consider

$$k = \begin{cases} \prod_i c_i + 1 & \sum_i \{a_i\} \geq 1 \\ \prod_i c_i - 1 & \sum_i \{a_i\} < 1. \end{cases}$$

This choice satisfies requirement (1): in the first case we have $\{ka_i\} = \{a_i\}$ for each i (by the fact that $\{x+n\} = \{x\}$ for all integers n and real x), so $\sum_i \{ka_i\} = \sum_i \{a_i\} \geq 1$ by assumption. If $\sum_i \{a_i\} < 1$ we have

$$\begin{aligned} \sum_i \{ka_i\} &= \sum_i \{-a_i\} \\ &= |\{i \mid a_i \notin \mathbb{Z}\}| - \sum_i \{a_i\} \\ &\geq 2 - \sum_i \{a_i\} > 1 \end{aligned}$$

since $\{-x\}$ is $1 - \{x\}$ whenever $x \notin \mathbb{Z}$ and 0 when $x \in \mathbb{Z}$. Note that this is where we use our assumption that at least two a_i are non-integral, and it is essential.

However, this choice of k may not satisfy requirement (2). We have

$$\deg \lfloor kH \rfloor = \deg (kH) - \sum_i \{ka_i\} - \{ka_P\} > \deg (kH) - (m+1),$$

since $0 \leq \{x\} < 1$ for all x. Since $\deg H > 0$, for $k \gg 0$ we will have $\deg (kH) \geq m + 1$ and requirement (2) will be satisfied. Hence replace our initial choice of k with $k + \ell \prod_i c_i$ for ℓ large enough to give $\deg(kH) \geq m + 1$ - this choice will still satisfy requirement (1) since $\{(k + \ell \prod_i c_i)\, a_i\} = \{ka_i\}$ for each i in either case.

<u>Choice of r, $\mathcal{B}_{(k,r)} = 0$:</u>

Let $r = \lfloor \deg(kH - ka_P P) \rfloor = \lfloor \sum_i ka_i \rfloor$. Then

$$\begin{aligned} r + \lfloor ka_P \rfloor &= \left\lfloor \sum_i ka_i \right\rfloor + \lfloor ka_P \rfloor \\ &\geq \sum_i \lfloor ka_i \rfloor + \lfloor ka_P \rfloor \\ &= \deg \lfloor kH \rfloor \geq 0 \end{aligned}$$

by requirement (2) of our choice of k, so $-r \leq \lfloor ka_P \rfloor$ and $\mathcal{F}_{(k,r)} = H^0(H_{(k,r)}) = H^0(kH - ka_P P - rP)$.

We have

$$\begin{aligned} \deg \lfloor kH - ka_P P - rP \rfloor &= \sum_i \lfloor ka_i \rfloor + \lfloor -r \rfloor \\ &= \sum_i \lfloor ka_i \rfloor - \left\lfloor \sum_i ka_i \right\rfloor \\ &= \left\{ \sum_i ka_i \right\} - \sum_i \{ka_i\} \\ &< 1 - \sum_i \{ka_i\} \leq 0 \end{aligned}$$

since $\{x\} < 1$ for all x and $\sum_i \{ka_i\} \geq 1$ by requirement (1) of our choice of k. It follows that $\mathcal{B}_{(k,r)} = \mathcal{F}_{(k,r)}/\mathcal{F}_{(k,r+1)} = 0$.

Choice of n, $\mathcal{B}_{(nk,nr)} \neq 0$:

Next, choose $n = c_P \prod_i c_i$, so that nkH is an integral divisor. To show that $\mathcal{B}_{(nk,nr)} \neq 0$, recall that it suffices to show that $\deg \lfloor H_{(nk,nr)} \rfloor \geq 0$. We have

$$\begin{aligned} nr + \lfloor nka_P \rfloor &= n \left(\left\lfloor \sum_i ka_i \right\rfloor + ka_P \right) \\ &= n \left(\deg(kH) - \left\{ \sum_i ka_i \right\} \right) > 0, \end{aligned}$$

since we chose k satisfying $\deg(kH) \geq m + 1 > 1$. Hence $-nr \leq nka_P$, meaning

$H_{(nk,nr)} = nkH - nka_P - nrP$. This divisor has degree

$$\begin{aligned}\deg H_{(nk,nr)} &= \deg(nkH) - nka_P - nr \\ &= n\left(\deg(kH) - ka_P - \left\lfloor \sum_i ka_i \right\rfloor\right) \\ &= n\left\{\sum_i ka_i\right\} \geq 0\end{aligned}$$

since $\{x\} \geq 0$ for any x. Hence $\mathcal{B}_{(nk,nr)} \neq 0$, and we are done. □

Our goal is to show that for each H as above, the ring $\mathcal{B}$ is not integrally closed. We now know that there exist k, r and n with $\mathcal{B}_{(k,r)} = 0$ and $\mathcal{B}_{(nk,nr)} \neq 0$. Let K be the algebra consisting of fractions of homogeneous elements of $\mathcal{B}$. If there exists $f \in K_{(k,r)}$ with $f^n \in \mathcal{B}_{(nk,nr)}$ then the monic polynomial $x^n - f^n \in \mathcal{B}[x]$ has a root in K which does not lie in $\mathcal{B}$ (since such a root would lie in $\mathcal{B}_{(k,r)}$), which would prove non-normality. We now show the existence of such an element.

Proposition 4.7. *Let H, k, r and n be as above. There exist integers (k', r') such that $\mathcal{B}_{(k',r')}$ and $\mathcal{B}_{(k'+k,r'+r)}$ are both nonzero. Hence there exists a nonzero function in $K_{(k,r)}$.*

Proof. Recall from above that $\mathcal{B}_{(k,r)} \neq 0$ if and only if $\deg\lfloor H_{(k,r)}\rfloor \geq 0$.

We have $\deg\lfloor kH\rfloor = \deg(kH) - \sum_i\{ka_i\} - \{ka_P\} > \deg(kH) - t$, where t is the number of terms in kH with non-integral coefficients (possibly including ka_P). Hence choose k' such that $\deg(k'H) > t$ and choose $r' = -\lfloor k'a_P\rfloor$. Then $H_{(k',r')} = k'H = k'H - k'a_PP - r'P$ and $\deg\lfloor k'H\rfloor > 0$, so $\mathcal{B}_{(k',r')} \neq 0$.

Now let k and r be as in the proof of Proposition 4.6 and recall that for these values we have $-r \leq \lfloor ka_P\rfloor$. Hence

$$H_{(k+k',r+r')} = (k+k')H - (k+k')a_PP - (r+r')P = H_{(k,r)} + H_{(k',r')}.$$

Then $\deg\lfloor H_{(k+k',r+r')}\rfloor \geq \deg\lfloor H_{(k,r)}\rfloor + \deg\lfloor H_{(k',r')}\rfloor$. Since $\deg\lfloor H_{(k,r)}\rfloor$ is fixed, we may increase k' (and thus increase $\deg\lfloor H_{(k',r')}\rfloor$) to ensure that $\deg\lfloor H_{(k+k',r+r')}\rfloor \geq 0$, if necessary.

It follows that $\mathcal{B}_{(k+k',r+r')} \neq 0$ as required, and since $\mathcal{B}_{(k',r')} \neq 0$ as well, taking the quotient of a nonzero element of the former by a nonzero element of the latter gives a nonzero element in $K_{(k,r)}$. □

We can now prove the theorem:

Proof of Theorem 4.6. As always we are free to assume that $q = 1$ and drop the superscript.

By some previous remarks, it suffices to find a nonzero element $f \in K_{(k,r)}$ with $f^n \in \mathcal{B}_{(nk,nr)}$. The result above shows that $K_{(k,r)} \neq 0$, so we may choose $f \in K_{(k,r)}$ to be nonzero, so $f^n \in K_{(nk,nr)} \neq 0$. We will show that $K_{(nk,nr)}$ is a line, and since it contains $\mathcal{B}_{(nk,nr)} \neq 0$, it follows that the two are equal, giving the result.

Let k', r' be arbitrary integers, and suppose $g, h \in K_{(k',r')}$ are nonzero. Their quotient must lie in $K_{(0,0)}$, which consists of fractions of homogeneous elements of $\mathcal{B}$ of equal degree. Since $\mathcal{B}_{(k',r')} = H^0(H_{(k,r)})/H^0(H_{(k',r'+1)})$ by the previous proof, it is clear that $\dim \mathcal{B}_{(k',r')} \leq 1$. Therefore any fraction of elements of $\mathcal{B}$ of equal degree is constant, so $K_{(0,0)} = \Bbbk$. Then $\dim K_{(k',r')} \leq 1$ for any (k', r'), and since $K_{(nk,nr)} \neq 0$, it follows that $\dim K_{(nk,nr)} = 1$ as required. □

We will now work through an example to demonstrate this theorem.

Example. With all notation as in the rest of this section, let

$$H = \frac{1}{2}Q_1 + \frac{1}{3}Q_2 - \frac{1}{2}P.$$

We will show that the ring $\mathcal{B}$ arising from H as described in Theorem 4.6 is not integrally closed, since H has two non-integral coefficients at Q_1 and Q_2.

First, we find k, r and n. Since $\sum_i \{a_i\} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1$, we set $k = a_1 \cdot a_2 - 1 = 2 \cdot 3 - 1 = 5$. Now $r = \lfloor \sum_i k a_i \rfloor = \left\lfloor \frac{5}{2} + \frac{5}{3} \right\rfloor = \left\lfloor \frac{25}{6} \right\rfloor = 4$. Finally, $n = c_P \cdot c_1 \cdot c_2 = 2 \cdot 2 \cdot 3 = 12$.

This gives $H_{(k,r)} = \frac{5}{2}Q_1 + \frac{5}{3}Q_2 - 4P$. Then $\lfloor H_{(k,r)} \rfloor = 2Q_1 + Q_2 - 4P$. This divisor has no global sections because it has negative degree, so we see that $\mathcal{B}_{(k,r)} = H^0(H_{(k,r)})/H^0(H_{(k,r+1)}) = 0$ as required.

On the other hand, we have $H_{(nk,nr)} = 30Q_1 + 20Q_2 - 48P$, which has positive degree, so $\mathcal{B}_{(nk,nr)} \neq 0$. Indeed we have

$$\mathcal{B}_{(nk,nr)} = H^0(30Q_1 + 20Q_2 - 48P)/H^0(30Q_1 + 20Q_2 - 49P).$$

Let $\mathbb{P}^1$ have co-ordinates x and y and suppose $P = [0 : 1]$, $Q_1 = [1 : 0]$ and $Q_2 = [1 : 1]$. Then $\mathcal{B}_{(nk,nr)}$ is generated by the rational functions

$$\frac{x^{48}}{y^{30}(x-y)^{18}}, \frac{x^{48}}{y^{29}(x-y)^{19}}, \frac{x^{48}}{y^{28}(x-y)^{20}}.$$

However, note that because we quotient by $H^0(30Q_1 + 20Q_2 - 49P)$, we can show that:

$$\frac{x^{48}}{y^{30}(x-y)^{18}} + \frac{x^{48}}{y^{29}(x-y)^{19}} = \frac{x^{49}}{y^{30}(x-y)^{20}} = 0,$$

since the result is a section of that divisor. It follows that in $\mathcal{B}_{(nk,nr)}$ we have $\frac{y}{x-y} = -1$, and we will use this fact later.

Now we choose k' and r'. We can pick $k' = 4$ since that gives $k'H = 2Q_1 + \frac{4}{3}Q_2 - 2P$ which has degree $\frac{4}{3}$ and only one non-integral coefficient, so $\deg \lfloor k'H \rfloor > 0$. Then $r' = -\lfloor k' a_P \rfloor = 2$. We have

$$\mathcal{B}_{(k',r')} = H^0(2Q_1 + Q_2 - 2P)/H^0(2Q_1 + Q_2 - 3P)$$

and

$$\mathcal{B}_{(k+k',r+r')} = H^0(4Q_1 + 3Q_2 - 6P)/H^0(4Q_1 + 3Q_2 - 7P).$$

Hence take $f = \frac{x^2}{y(x-y)} \in \mathcal{B}_{(k',r')}$ and $g = \frac{x^6}{y^4(x-y)^2} \in \mathcal{B}_{(k+k',r+r')}$ and let

$$h = \frac{f}{g} = \frac{x^4}{y^3(x-y)} \in K_{(k,r)}.$$

We know that $h \notin \mathcal{B}$ since $\mathcal{B}_{(k,r)} = 0$. However, we have

$$h^n = \frac{x^{48}}{y^{36}(x-y)^{12}} = \frac{(x-y)^6}{y^6} \cdot \frac{x^{48}}{y^{30}(x-y)^{18}} = \frac{x^{48}}{y^{30}(x-y)^{18}} \in \mathcal{B}_{(nk,nr)},$$

since $\frac{(x-y)}{y} = -1$ in $\mathcal{B}$ as seen above.

It follows that h is a root in K, the field of fractions of $\mathcal{B}$, of the monic polynomial

$$z^n - \frac{x^{48}}{y^{30}(x-y)^{12}} \in \mathcal{B}[z].$$

Hence $\mathcal{B}$ is not integrally closed.

4.4.3 Divisor Correspondence

Now we show that prime divisors over X give rise to divisors on $\mathbb{P}^1$, and demonstrate how, subject to some conditions, Theorem 4.6 can be applied to show that the test configurations corresponding to these prime divisors have non-normal central fibres. We refer back to the notation of Proposition 4.5 and Definition 4.4.

Proposition 4.8. *Let X be a smooth G-variety of complexity one. The test configuration corresponding to a prime divisor F over X is special if and only if the graded ring $\mathcal{B}$ associated to the filtration on the section ring of $-K_X$ induced by F is integrally closed.*

Proof. The test configuration is special if and only if the central fibre X_0 is normal. In this case we have $X_0 = \pi^{-1}(0) = \operatorname{Proj} \mathcal{A}/(z)$, where $\mathcal{A}/(z) = \bigoplus_{r\in\mathbb{Z}} \mathcal{F}^r R/\mathcal{F}^{r+1}R = \mathcal{B}$. Since $X_0 = \operatorname{Proj} \mathcal{B}$, and in fact $\mathcal{B}$ is the section ring of (X_0, L_0), it follows that X_0 is normal, and the test configuration special, if and only if $\mathcal{B}$ is integrally closed. □

Now suppose X is a smooth Fano G-variety of complexity one. Let F be a non-central G-divisor over X. We know that the test configuration corresponding to X is special if and only if the bigraded ring $\mathcal{B}$ defined above is integrally closed. We will use the B-quotient to allow ourselves to check this using divisors on $\mathbb{P}^1$. Recall that $\mathcal{B} = \bigoplus_{r\in\mathbb{Z}} \mathcal{F}^r R/\mathcal{F}^{r+1}R$, where $R = \bigoplus_{k\in\mathbb{Z}} H^0(X, -kK_X)$ and $\mathcal{F}^r R = \bigoplus_{k\in\mathbb{Z}} \{f \in H^0(X, -kK_X) \mid \nu_F(f) \geq r\}$.

We know that we can find a B-invariant representative of the class $-K_X$, and given this, the section rings $H^0(X, -kK_X)$ gain a G-module structure. The B-semi-invariants of weight λ in this G-module are of the form $f_0 e_\lambda$ where $f_0 \in K^B = \Bbbk(\mathbb{P}^1)$. If $B(X)$ is the set of B-invariant divisors of X and we have $-K_X = \sum_{D\in B(X)} m_D D$, then

$$H^0(X, -K_X)^{(B)}_\lambda = \{f_0 e_\lambda \mid f_0 \in K^B, \sum_{D\in B(X)} [h_D \nu_{P_D}(f_0) + \langle \lambda, \ell_D\rangle + m_D] D \geq 0\}.$$

For a fixed weight $\lambda \in \Lambda$ we have $H^0(X, -K_X)^{(B)}_\lambda \cong H^0(\mathbb{P}^1, H_\lambda)$, where

$$H_\lambda = \sum_{P\in\mathbb{P}^1} \left(\min_{P_D = P} \frac{\langle \lambda, \ell_D\rangle + m_D}{h_D} \right) P.$$

This is a well defined divisor on $\mathbb{P}^1$ (i.e. has no coefficients $\pm\infty$) provided that (a): for any $P \in \mathbb{P}^1$ there exists a B-divisor D on X with $P_D = P$ and $h_D > 0$, and (b): λ lies in the polyhedral domain

$$\mathcal{P}(-K_X) = \{\lambda \in \Lambda \mid \langle \lambda, \ell_D\rangle \geq -m_D \text{ for all } D \text{ with } h_D = 0\}.$$

Condition (a) holds by completeness of X, but we need to be careful about condition (b).

Recall the function

$$A(-K_X, \lambda) = \sum_{P\in\mathbb{P}^1} \left(\min_{P_D = P} \frac{\langle \lambda, \ell_D\rangle + m_D}{h_D} \right)$$

which computes the degree of H_λ, and the polyhedral domain

$$\mathcal{P}_+(-K_X) = \{\lambda \in \mathcal{P} \mid A(-K_X, \lambda) \geq 0\}.$$

In this notation, H_λ is well-defined and has positive degree exactly when λ lies in the relative interior of $\mathcal{P}_+(-K_X)$.

Lemma 4.3. *We may assume that* 0 *is in the relative interior of* $\mathcal{P}_+(-K_X)$*.*

Proof. Since $-K_X$ is ample, we have seen that $-K_X^n = \operatorname{vol}(-K_X)$, and this quantity must be positive. We have also seen that $\operatorname{vol}(-K_X)$ can be expressed as an integral over $\lambda_{-K_X} + \mathcal{P}_+(-K_X)$. It follows that $\mathcal{P}_+(-K_X)$ has non-empty interior. Let λ lie in the interior of $\mathcal{P}_+(-K_X)$, and replace $-K_X$ with the equivalent divisor $-K_X' = -K_X + \operatorname{div} e_\lambda$.

Suppose $h_D = 0$ for some $D \in B(X)$. We know that $\langle \lambda, \ell_D \rangle \geq -m_D$. The former is defined to be $\nu_D(e_\lambda)$, so we have $m_D + \nu_D(e_\lambda) = m_D' \geq 0$, or equivalently $0 = \langle 0, \ell_D \rangle \geq -m_D'$, i.e. $0 \in \mathcal{P}(-K_X')$.

Likewise, since $\lambda \in \operatorname{relint} \mathcal{P}_+(-K_X)$, we have $A(-K_X, \lambda) > 0$. But

$$\begin{aligned} A(-K_X, \lambda) &= \sum_{P \in \mathbb{P}^1} \left(\min_{P_D = P} \frac{\langle \lambda, \ell_D \rangle + m_D}{h_D} \right) \\ &= \sum_{P \in \mathbb{P}^1} \left(\min_{P_D = P} \frac{m_D'}{h_D} \right) = A(-K_X', 0). \end{aligned}$$

So $A(-K_X', 0) > 0$ and the result follows. ∎

Now looking back to $\mathcal{B}$, we have

$$(\mathcal{F}^r)^{(B)} = \bigoplus_{k \in \mathbb{Z}} \left\{ f \in \bigoplus_{\lambda \in \Lambda} H^0(X, -kK_X)_\lambda^{(B)} \mid \nu_F(f) \geq r \right\}.$$

By discussions above, we can rewrite this as

$$(\mathcal{F}^r)^{(B)} = \bigoplus_{\lambda \in \Lambda} \bigoplus_{k \in \mathbb{Z}} \{ f \in H^0(\mathbb{P}^1, kH_\lambda) \mid \nu_F(f) \geq r \}.$$

Since $\nu_F(f) = h_F \operatorname{ord}_{P_F}(f) + \langle \lambda, \ell_F \rangle$, we have

$$(\mathcal{F}^r)^{(B)}_{(\lambda,k)} = \left\{ f \in H^0(kH_\lambda) \mid \operatorname{ord}_P(f) \geq \frac{r - \langle \lambda, \ell_F \rangle}{h_F} \right\}.$$

We have shown that if $\deg H_\lambda > 0$ and H_λ has non-integral coefficients at two points other than P_F, the ring $\mathcal{B}^{h_F}(H_\lambda)$ is not integrally closed. The latter is the sum over $(k, r) \in \mathbb{Z} \oplus \mathbb{Z}$ of

$$\mathcal{B}^{h_F}(H_\lambda)_{(k,r)} = \frac{\left\{ f \in H^0(kH_\lambda) \mid \operatorname{ord}_{P_F}(f) \geq \frac{r}{h_F} \right\}}{\left\{ f \in H^0(kH_\lambda) \mid \operatorname{ord}_{P_F}(f) \geq \frac{r+1}{h_F} \right\}}.$$

Note that $\mathcal{B}^{h_F}(H_\lambda)_{(k,r-\langle\lambda,\ell_F\rangle)} = (\mathcal{F}^r)^{(B)}_{(\lambda,k)}/(\mathcal{F}^{r+1})^{(B)}_{(\lambda,k)}$ as defined above. The shift of degrees by $\langle\lambda,\ell_F\rangle$ can be ignored as we sum over $\mathbb{Z}$ either way.

We can now write

$$\mathcal{B}^{(B)} = \bigoplus_{\lambda\in\Lambda}\bigoplus_{r\in\mathbb{Z}}\bigoplus_{k\in\mathbb{Z}}\mathcal{B}^{(B)}_{(\lambda,k,r)} = \bigoplus_{\lambda\in\Lambda}\bigoplus_{r\in\mathbb{Z}}\bigoplus_{k\in\mathbb{Z}}\mathcal{B}^{h_F}(H_\lambda)_{(k,r)} = \bigoplus_{\lambda\in\Lambda}\mathcal{B}^{h_F}(H_\lambda).$$

Lemma 4.4. *Let A be an integral domain with field of fractions K, and let K' be a subfield of K. Let $B = A\cap K'$. If B is not integrally closed in K', then A is not integrally closed in K.*

Proof. If B is not integrally closed in K', there exists a monic polynomial in $B[x]$ with a root $f\in K'$ which does not lie in B. Since $K'\subseteq K$ and $B[x]\subseteq A[x]$, we can also view f as a root in K of a monic polynomial in $A[x]$. If $f\in A$, then since $f\in K'$, we have $f\in A\cap K' = B$, a contradiction. Hence $f\notin A$ and A is therefore not integrally closed in K. □

Theorem 4.7. *If there exists $\lambda\in\mathcal{P}_+(-K_X)$ such that H_λ has two non-integral coefficients at points other than P_F, then $\mathcal{B}$ is not integrally closed.*

Proof. By Lemma 4.4 it suffices to show that $\mathcal{B}^{(B)}_0 = \mathcal{B}\cap K^B$ is not integrally closed. We have shown that $\mathcal{B}^{(B)} = \bigoplus_{\lambda\in\Lambda}\mathcal{B}^{h_F}(H_\lambda)$, so in particular $\mathcal{B}^{(B)}_0 = \mathcal{B}^{h_F}(H_0)$. Hence if $\mathcal{B}^{h_F}(H_0)$ is not integrally closed, then neither is $\mathcal{B}$. We have proved already that $\mathcal{B}^{h_F}(H_0)$ is not integrally closed when H_0 has positive degree and two non-integral points distinct from P_F. We may assume that H_0 has positive degree by Lemma 4.3.

If H_λ has non-integral coefficients at two points other than P_F, then replace $-K_X$ with $-K'_X = -K_X + \operatorname{div} e_\lambda$. Then $H'_0 = H_\lambda$ and the result follows using H'_0 instead of H_0. □

To summarise the results of this section, we have:

Corollary 4.4. *Let X be a smooth Fano G-variety of complexity one with anticanonical divisor $-K_X$. Let F be a non-central prime divisor over X corresponding to a point $P_F\in\mathbb{P}^1$ on the B-quotient. If there exists $\lambda\in\mathcal{P}_+(-K_X)$ such that H_λ has two non-integral coefficients at points other than P_F, then the test configuration corresponding to F has non-normal special fibre.*

4.4.4 Non-Integral Coefficients

We will now investigate exactly when the hypotheses of Corollary 4.4 actually hold.

Theorem 4.8. *Let X be one of the SL_2-threefolds listed in Theorem* 4.5*, other than $\mathbb{P}^3$ and the blow-up thereof at two or three lines. There exists $\lambda \in \mathcal{P}_+(-K_X)$ such that H_λ has non-integral coefficients at the points corresponding to subregular colours.*

We will demonstrate this result one variety at a time.

V_{22} (1.10)

It is known that for $X = V_{22}$, the Mukai-Umemura threefold, we can take $-K_X$ to be a hyperplane section. The B-invariant hyperplane section of X for our action is D_{-4}. This variety has subregular colours lying over $0, \infty, -4 \in \mathbb{P}^1$. With this representative of $-K_X$, we have

$$H_\lambda = -\frac{\lambda}{2}[0] + \frac{\lambda}{3}[\infty] + \min\left\{\frac{\lambda+1}{5}, 0\right\}[-4].$$

Choosing $\lambda = -5$ gives $\deg H_\lambda = \frac{1}{30}$ and non-integral coefficients at all three points.

V_5 (1.15)

This time $-K_X = 2D_{-4}$. We have

$$H_\lambda = -\frac{\lambda}{2}[0] + \frac{\lambda}{3}[\infty] + \min\left\{\frac{\lambda+2}{4}, 0\right\}[-4].$$

Again, $\lambda = -5$ works, giving $\deg H_\lambda = \frac{1}{12}$ and non-integral coefficients at all three points.

Q (1.16)

Q is a hypersurface of degree 2 in $\mathbb{P}^4$ so its anticanonical divisor is given by its intersection with a divisor of degree 3 in $\mathbb{P}^4$. The three subregular colours D_0, D_∞ and D_{-4} of Q are sections of prime divisors in $\mathbb{P}^4$ of degrees 3, 2 and 1, respectively, so we may take $-K_Q = 3D_{-4}$. Then

$$H_\lambda = \frac{-\lambda}{2}[0] + \frac{\lambda}{3}[\infty] + \min\left\{\frac{\lambda+3}{3}, 0\right\}[-4].$$

Choosing $\lambda = -5$ gives non-integral coefficients at each point and $\deg H_\lambda = \frac{1}{6}$.

Blow-up of Q (2.21)

With the same $-K_Q$ as before, blowing up in the twisted quartic, which is contained in D_0 and D_∞ but not in D_{-4}, gives $-K_X = 3\tilde{D}_{-4} - E$. Adding $\mathrm{div}(f_0 e_{2\alpha}^{-2})$, where $f_0 \in \mathbb{C}(\mathbb{P}^1)$ has divisor $[\infty] - [0]$ gives $-K_X = \tilde{D}_{-4} + \tilde{D}_\infty$. Then

$$H_\lambda = \frac{-\lambda+2}{2}[0] + \min\left\{\frac{\lambda+1}{3}, 0\right\}[\infty] + \min\left\{\frac{\lambda+1}{3}, 0\right\}[-4].$$

Taking $\lambda = -3$ gives non-integral coefficients and $\deg H_\lambda = \frac{1}{6}$.

Blow-up of $\mathbb{P}^3$ along a twisted cubic (2.27)

$\mathbb{P}^3$ with the cubic SL_2-action has subregular colours D_0, D_∞ and D_{-4} of respective degrees 3, 2 and 1. Hence we may choose $-K_{\mathbb{P}^3} = 4D_{-4}$. Then blowing up the twisted cubic gives $-K_X = 4\tilde{D}_{-4} - E$, where E is the exceptional divisor and lies over ∞. We add $\mathrm{div}(f_0 e_{2\alpha}^{-2})$ to $-K_X$, where $\mathrm{div}(f_0) = [0] - [\infty]$, giving $-K_X = 2\tilde{D}_{-4} + \tilde{D}_\infty$. We then have

$$H_\lambda = \frac{\lambda}{2}[0] + \min\left\{\frac{1-2\lambda}{3}, -\lambda\right\}[\infty] + \min\left\{\frac{\lambda+2}{2}, 0\right\}[-4].$$

Choosing $\lambda = -3$ gives non-integral coefficients at all points and $\deg H_\lambda = \frac{1}{3}$.

W (2.32)

The divisor W on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$ has anticanonical divisor class given by the intersection with W of a class of bidegree $(2,2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. The subregular colours D_0 and D_∞ have bidegrees $(1,0)$ and $(0,1)$, so we take $-K_W = 2D_0 + 2D_\infty$. This gives

$$H_\lambda = \min\left\{\frac{\lambda+2}{2}, 0\right\}[0] + \min\left\{\frac{\lambda+2}{2}, 0\right\}[\infty] - \frac{\lambda}{2}[-1].$$

Taking $\lambda = -3$ gives non-integral coefficients at all points and $\deg H_\lambda = \frac{1}{2}$.

$\mathbb{P}^1 \times \mathbb{P}^2$ (2.34)

The anticanonical divisor of $\mathbb{P}^1 \times \mathbb{P}^2$ has bidegree $(2,3)$ which we can obtain as $2D_\infty + 3D_{-1}$. This gives

$$H_\lambda = -\frac{\lambda}{2}[0] + \min\left\{\frac{2-\lambda}{2}, -\lambda\right\}[\infty] + \min\{2\lambda+3, \lambda\}[-1].$$

Taking $\lambda = -3$ gives non-integral coefficients at the points $0, \infty$ corresponding to subregular colours and $\deg H_\lambda = 1$.

Blow-up of W (3.13)

We blow up W in the curve of bidegree $(2,2)$ obtaining $-K_X = 2\tilde{D}_0 + 2\tilde{D}_\infty - E$. Adding $\mathrm{div}(e_{2\alpha}^{-1})$ gives $-K_X = \tilde{D}_0 + \tilde{D}_\infty + \tilde{D}_{-1}$. Then

$$H_\lambda = \min\left\{\frac{\lambda+1}{2}, 0\right\}[0] + \min\left\{\frac{\lambda+1}{2}, 0\right\}[\infty] + \min\left\{\frac{1-\lambda}{2}, -\lambda\right\}[-1].$$

Taking $\lambda = -2$ gives non-integral coefficients at all points and $\deg H_\lambda = \frac{1}{2}$.

Blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ (3.17)

Blowing up $\mathbb{P}^1 \times \mathbb{P}^2$ along the curve of bidegree $(1,1)$ gives $-K_X = 2\tilde{D}_\infty + 3\tilde{D}_{-1} - E$. We can add $\mathrm{div}(f_0 e_{2\alpha}^{-1})$, where $\mathrm{div}(f_0) = [-1] - [\infty]$ to get $-K_X = \tilde{D}_\infty + 2\tilde{D}_{-1} + \tilde{D}_0$. Then

$$H_\lambda = \min\left\{\frac{1-\lambda}{2}, -\lambda\right\}[0] + \min\left\{\frac{1-\lambda}{2}, -\lambda\right\}[\infty] + \min\{2\lambda+2, \lambda\}[-1].$$

Choosing $\lambda = -2$ gives non-integral coefficients at the points $0, \infty$ corresponding to subregular colours and $\deg H_\lambda = 1$.

Result

Putting together the results of the above subsections, we have:

Theorem 4.9. *Let X be one of the SL_2-threefolds mentioned in Theorem 4.8. Let F be a non-central prime divisor over X corresponding to a point $P_F \in \mathbb{P}^1$. If X has subregular colours lying over two points in $\mathbb{P}^1$ distinct from P_F, then the test configuration corresponding to F has non-normal central fibre.*

4.4.5 Action Interchanging Two Points

Proof of Theorem 4.4(ii). Suppose a finite subgroup $A \subseteq \mathrm{Aut}\, X$ acts on $\mathbb{P}^1$, interchanging two points P and Q corresponding to subregular colours of X and that the B-quotient is equivariant with respect to the A-action. We have an action on X of an extension G' of G by A. Any non-central G-invariant prime divisor F over X can only be G'-invariant if its corresponding point $P_F \in \mathbb{P}^1$ is fixed by A. Since P and Q are not fixed by A, they are distinct from P_F, and since each has a subregular colour lying over it, Theorem 4.9 applies, and the test configuration corresponding to F is not special. Therefore we may show that X is K-polystable by checking only central divisors. □

We now show that this case of Theorem 4.4 applies to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve of bidegree $(1, 1)$ (3.17).

Consider the $\mathbb{Z}_2$-symmetry of $\mathbb{P}^1$ given by $[\alpha : \beta] \mapsto [\beta : \alpha]$. This interchanges 0 and ∞, fixes 1 and -1, and puts every other point in an orbit of order 2. Recall that the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ has subregular colours lying over 0 and ∞, and has a distinguished point -1. Since the slices of the coloured hyperfan over 0 and ∞ are identical to each other, the interchange of these points by $\mathbb{Z}_2$ leaves the hyperfan invariant, hence the $\mathbb{Z}_2$ action on X respects the B-quotient map and Theorem 4.4 applies.

Oddly, this method seems not to apply to $\mathbb{P}^1 \times \mathbb{P}^2$ itself, even though this variety is known to be K-polystable, since in this case, one of the two slices with subregular colours contains a G-divisor while the other does not, and the $\mathbb{Z}_2$-symmetry therefore does not extend. This may require some further exploration.

4.4.6 Three or More Subregular Colours

Proof of Theorem 4.4(iii). If X has subregular colours lying over three or more distinct points of $\mathbb{P}^1$, then for any non-central prime divisor F over X corresponding to a point P_F in $\mathbb{P}^1$, there always exist at least two subregular colours lying over two points distinct from P_F and from each other. Then Theorem 4.9 applies and we need only check the β-invariant of central divisors. □

We have seen that any smooth Fano SL_2-threefold whose stabiliser subgroup H is one of $\{\tilde{D}_m, \tilde{T}, \tilde{C}, \tilde{I}\}$ has 3 subregular colours, all lying over distinct points in $\mathbb{P}^1$. Then in particular, Theorem 4.4 applies to V_{22} (1.10), V_5 (1.15), Q (1.16), the blow-up of Q along a twisted quartic (2.21), the blow-up of $\mathbb{P}^3$ along a twisted cubic (2.27), W (2.32) and the blow-up of W along a curve of bidegree $(2, 2)$ (3.13).

4.5 Central Divisors of SL_2-Threefolds

Here we prove Theorem 4.5 by calculating the β-invariant for the G-stable central prime divisors over each of our list of SL_2-threefolds. By Theorem 4.4, we need to show that (1): $\beta_X(F) \geq 0$ for central divisors F over each variety and (2): if $\beta_X(F) = 0$ then F corresponds to a product configuration. It will turn out that (2) is never the case. Throughout we will mostly use the notation and results of Section 3.3 for curves, divisors etc. lying in each variety, with any changes to this notation clearly signalled.

4.5.1 Existence and Uniqueness of Central Divisors

In this subsection we will prove the following:

Theorem 4.10. *Let X be a smooth Fano SL_2-threefold. There exists a unique central G-divisor over X. If X is of type I, then it contains this central divisor. If X is of type II, the central G-divisor over X lies on the type I variety over X whose existence is proved in Proposition* 3.14.

Definition 4.7. Let $\mathcal{H}$ be the hyperspace of a G-model X of complexity one. The *dimension* of $\mathcal{H}$ is the common dimension of each half-space $\mathcal{H}_p$ for $p \in \mathbb{P}^1$. If $\mathcal{C}_p$ is a coloured cone in $\mathcal{H}_p$, its dimension is the dimension of a minimal affine subspace of $\mathcal{H}_p$ containing $\mathcal{C}_p$. If $\mathcal{C}$ is a coloured hypercone of type II in $\mathcal{H}$, set $\dim \mathcal{C} = \max_{p\in\mathbb{P}^1} \dim(\mathcal{C} \cap \mathcal{H}_p)$.

Lemma 4.5. *Let X be a G-model of complexity one and rank r, and let $\mathcal{H}$ be the hyperspace of X. We have* $\dim \mathcal{H} = 1 + r$.

Proof. For each $p \in \mathbb{P}^1$, the slice $\mathcal{H}_p$ of hyperspace corresponding to p is isomorphic to $\Lambda^*_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0}$, where Λ is the weight lattice of X, which has dimension $1 + \dim \Lambda^*_{\mathbb{Q}} = 1 + \operatorname{rk} \Lambda = 1 + r$, by the definition of rank. □

Proposition 4.9. *Let X be a G-model and let $Y \subseteq X$ be a G-germ. The dimension of $\mathcal{C}_Y$ in $\mathcal{H}$ is equal to the codimension of Y in X.*

Proof. This follows from the fact that the coloured hypercone corresponding to X itself is $\{0\}$ and that inclusion of coloured hypercones as faces in one another corresponds to the reverse inclusion of the associated G-germs. □

Corollary 4.5. *Let X be a complete G-model of dimension d, rank r and complexity 1. Then minimal G-germs in X must have dimension $d - r - 1$.*

Proof. Since $\mathcal{V}$ is full dimensional in $\mathcal{H}$, it follows from completeness that the coloured (hyper)cone corresponding to a minimal G-germ Y in X must have the same dimension as $\mathcal{H}$, i.e. $1 + r$. Then $\operatorname{codim}_X Y = \dim \mathcal{C}_Y = 1 + r$, so $\dim Y = d - r - 1$. □

Now let $G = \mathrm{SL}_2(\Bbbk)$ and let X be a complete three dimensional G-model of complexity one, i.e. a normal projective threefold with a G-action having finite stabilisers. Then X contains an open orbit isomorphic to G/H for H a finite subgroup of G.

Let B be the Borel subgroup of G given by the upper triangular matrices. Then $\mathfrak{X}(B)$ is of rank 1, generated by the character α which picks out the upper-left entry. Hence in particular X is a rank 1 variety. The hyperspace $\mathcal{H}$ of X then has dimension 2, so minimal G-germs of X must have codimension 2, i.e. they are curves, and in particular X can contain no G-fixed points. The centre $\mathcal{Z}$ of $\mathcal{H}$ is a line. Most of the following results arise from this fact. In particular, note that for any finite $H \subseteq G$, the valuation cone $\mathcal{V}(G/H)$ intersects $\mathcal{Z}$ in a ray.

Proposition 4.10. *X contains at most one central G-divisor.*

Proof. Any central G-divisor must be mapped to the intersection $\mathcal{V} \cap \mathcal{Z}$ of the valuation cone and the central hyperplane. As noted, this intersection is a ray. If there were two distinct central G-divisors, their coloured (hyper)cones would then both be the same ray, a contradiction. □

Proposition 4.11. *X contains a central G-divisor if and only if X is a model of type I.*

Proof. Suppose X is of type I, i.e. every G-germ of X corresponds to a coloured cone in some slice of the hyperspace. By completeness, these coloured cones must cover $\mathcal{V}$. Since $\mathcal{V}$ intersects the central line $\mathcal{Z}$ in a ray ρ, there must be a G-germ of X whose coloured cone is ρ, i.e. a central G-divisor.

Now suppose X has a central G-divisor D and that $Y \subseteq X$ is a G-germ of type II with coloured hypercone $\mathcal{C}_Y$. We know that $\mathcal{C}_Y$ must intersect $\mathcal{Z}$ in $\mathcal{V}$, i.e. it contains the ray ρ corresponding to D. Then ν_D lies in the relative interior of $\mathcal{C}_Y$, hence in the support $\mathcal{S}_Y$ of Y. But $\nu_D \in \mathcal{S}_D$, so the supports of Y and D are not disjoint, contradicting separation of X. Hence all G-germs of X are of type I, as required. □

Proposition 4.12. *If X is of type II, there exists a central prime divisor over X.*

Proof. We know from Proposition 3.14 that there exists a projective birational morphism $\nu\colon \check{X} \to X$ where $\check{X}$ is of type I. Then by the above proposition there is a central prime divisor $D \subseteq \check{X}$. □

Proposition 4.13. *Let X be of type II. Then X has a unique G-germ of type II, a curve, which must be a closed G-orbit.*

Proof. We know that X contains at least one G-germ of type II. Suppose $Y, Y' \subseteq X$ are both G-germs of type II. Their corresponding coloured hypercones $\mathcal{C}_Y$, $\mathcal{C}_{Y'}$ must both intersect the central ray $\rho \subseteq \mathcal{V}$, hence their relative interiors must intersect in $\mathcal{V}$. It follows that $Y = Y'$, and X has exactly one G-germ of type II.

Then all other G-germs of X are of type I and so each defines a coloured cone in some $\mathcal{H}_p$. In particular, these coloured cones cannot contain $\mathcal{C}_Y$, so Y does not contain any other G-germ of X. Hence Y is a minimal G-germ, so in particular a curve and a closed G-orbit. □

Proposition 4.14. *Let X be of type II and suppose that every G-divisor of X maps to the boundary of the valuation cone. Then the unique minimal G-germ $Y \subseteq X$ of type II is contained in every G-divisor, and X has no minimal G-germs of type I. In particular, Y is the unique closed G-orbit of X.*

Proof. Let $F \subseteq X$ be a G-divisor mapping to the non-central boundary of $\mathcal{V}_p = \mathcal{V} \cap \mathcal{H}_p$. The ray $\mathcal{C}_F$ defined by F must then be a face of some coloured cone in $\mathcal{H}_p$ or coloured hypercone of type II in $\mathcal{H}$, i.e. F must contain some minimal G-germ.

We know that X contains a unique minimal G-germ Y of type II, and in particular the coloured hypercone $\mathcal{C}_Y$ must have $\mathcal{C}_F$ as a face, i.e. F contains Y. Now suppose F contains another G-germ Z. This must be of type I since Y is the only G-germ of type II, so $\mathcal{C}_Z$ must be a coloured cone in $\mathcal{H}_p$ with $\mathcal{C}_F$ as a face. However, $\mathcal{C}_Z$ must intersect $\mathcal{V}$ in its relative interior, and since $\mathcal{C}_F$ is the boundary of $\mathcal{V}$, it follows that $\mathcal{C}_Z$ and $\mathcal{C}_Y$ intersect in their relative interiors, a contradiction.

Hence for each $\mathcal{H}_p$ containing a G-divisor, the only minimal G-germ whose coloured (hyper)cone intersects $\mathcal{H}_p$ is Y. If $\mathcal{H}_p$ does not contain a G-divisor, it can also only support one coloured (hyper)cone since it only contains one B-divisor, the colour D_p.

Therefore Y is indeed the unique minimal G-germ of X and hence also the unique closed orbit. □

In all cases which we will consider here, we are given a smooth SL_2-threefold X, and either X is of type I and has a central divisor, which is unique, or X is of type II and we can obtain the unique central divisor over X by blowing up a finite sequence of G-invariant curves.

Combining Theorem 4.10 with Theorem 4.4, we see that to prove Theorem 4.5, we need only find the central divisor over each variety and show that its β-invariant is positive. In the following sections we will perform this calculation for each of the examples, thus proving their K-stability.

4.5.2 $\mathbb{P}^3$ and Blow-Ups Along Two or Three Lines

$\mathbb{P}^3$ (1.17)

The anticanonical divisor of $\mathbb{P}^3 = \mathbb{P}(M_2(\mathbb{C}))$ is the class of a divisor of degree 4, which we may take to be 2Δ, where Δ is the G-invariant divisor of singular matrices. This is also the unique central divisor on $\mathbb{P}^3$.

To calculate $\beta_X(\Delta)$ when $X = \mathbb{P}^3$, note that $(-K_X)^3 = 64$ and $A_X(\Delta) = 1$ since Δ lies on $\mathbb{P}^3$. It remains to calculate $\mathrm{vol}(\delta)$, where $\delta = -K_X - x\Delta = (2-x)\Delta$. Note that since δ is G-invariant we have $\lambda_\delta = 0$.

We have $\mathcal{P}(\delta) = \{\lambda \in \Lambda \mid \langle \lambda, \ell_\Delta \rangle \geq -(2-x)\}$. Since $\Lambda = \mathbb{Z}\alpha \cong \mathbb{Z}$ and $\ell_\Delta = -1$, we get $\mathcal{P}(\delta) = \{\lambda \in \mathbb{Z} \mid \lambda \leq 2-x\}$.

Consider

$$A(\delta, \lambda) = \sum_{p \in \mathbb{P}^1} \left(\min_{p_D = p} \frac{\langle \lambda, \ell_D \rangle + m_D}{h_D} \right).$$

We have $\ell_D = m_D = 0$ for all colours D other than the distinguished colour, which has $m_D = 0$, $\ell_D = 2$ and hence contributes a value of 2λ to $A(\delta, \lambda)$.

We therefore have $\mathcal{P}_+(\delta) = \{\lambda \leq 2-x \mid \lambda \geq 0\} = [0, 2-x]$. Hence

$$\mathrm{vol}(\delta) = 6 \int_0^{2-x} 2\lambda \cdot 2\lambda \, \mathrm{d}\lambda = 8(2-x)^3.$$

We therefore have

$$\beta(\Delta) = 64 - \int_0^2 8(2-x)^3 \, \mathrm{d}x = 64 - 32 = 32 > 0.$$

Hence by Theorem 4.4, $\mathbb{P}^3$ is K-polystable.

Blow-up of $\mathbb{P}^3$ Along Two Lines (3.25)

Starting with $-K_{\mathbb{P}^3} = 2\Delta$ as before, the anticanonical divisor after blowing up two lines Y_q and Y_r is $2\tilde{\Delta} + E_q + E_r$.

This time we have $(-K_X)^3 = 44$ and again $A_X(\Delta) = 1$. We must now compute $\beta(\tilde{\Delta})$ by calculating $\mathrm{vol}(\delta)$ where $\delta = (2-x)\tilde{\Delta} + E_q + E_r$. We still have $\lambda_\delta = 0$.

Likewise, $\mathcal{P}(\delta) = (-\infty, 2-x]$ as before. This time $A(\delta, \lambda)$ receives the same contribution of 2λ at the distinguished point, and there is no contribution other than from here and from q and r. At q, we have the exceptional divisor E_q with $\ell = -1$, $m = 1$ and $h = 1$, and the colour with $\ell = m = 0$. Thus there is a contribution to $A(\delta, \lambda)$ of $-\lambda + 1$ when this is less than or equal to 0, and a contribution of 0 otherwise. The same holds for r. Hence we have

$$A(\delta, \lambda) = \begin{cases} 2 & 1 \le \lambda \le 2-x \\ 2\lambda & \lambda < 1. \end{cases}$$

Therefore $\mathcal{P}_+(\delta) = [0, 2-x]$, and

$$\mathrm{vol}(\delta) = \begin{cases} 6\int_0^1 4\lambda^2 \,\mathrm{d}\lambda + 6\int_1^{2-x} 4\lambda \,\mathrm{d}\lambda & 0 \le x \le 1 \\ 6\int_0^{2-x} 4\lambda^2 \,\mathrm{d}\lambda & 1 < x \le 2. \end{cases}$$

We thus get

$$\beta_X(\tilde{\Delta}) = 44 - \int_0^2 \mathrm{vol}(\delta)\,\mathrm{d}x = 44 - 26 = 18 > 0.$$

Hence the blow-up of $\mathbb{P}^3$ at two lines is K-polystable.

Blow up of $\mathbb{P}^3$ Along Three Lines (4.6)

The anticanonical divisor of the blow-up of $\mathbb{P}^3$ along three lines Y_q, Y_r and Y_s is $2\tilde{\Delta} + E_q + E_r + E_s$. We have $(-K_X)^3 = 34$, $A_X(\tilde{\Delta}) = 1$, $\delta = (2-x)\tilde{\Delta} + E_q + E_r + E_s$, $\lambda_\delta = 0$ and $\mathcal{P}(\delta) = (-\infty, 2-x]$.

The calculation of $A(\delta, \lambda)$ goes much the same as in the previous case, only $A(\delta, \lambda)$ gains an extra contribution of $-\lambda + 1$ from E_s when $\lambda \ge 1$, giving

$$A(\delta, \lambda) = \begin{cases} 3-\lambda & 1 \le \lambda \le 2-x \\ 2\lambda & \lambda < 1. \end{cases}$$

Therefore $\mathcal{P}_+(\delta) = [0, 2-x]$, and

$$\mathrm{vol}(\delta) = \begin{cases} 6\int_0^1 4\lambda^2 \,\mathrm{d}\lambda + 6\int_1^{2-x} 6\lambda - 2\lambda^2 \,\mathrm{d}\lambda & 0 \le x \le 1 \\ 6\int_0^{2-x} 4\lambda^2 \,\mathrm{d}\lambda & 1 < x \le 2. \end{cases}$$

We thus get

$$\beta_X(\tilde{\Delta}) = 34 - \int_0^2 \mathrm{vol}(\delta)\ \mathrm{d}x = 34 - 23 = 11 > 0.$$

Hence the blow-up of $\mathbb{P}^3$ at three lines is K-polystable.

4.5.3 Blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$

Central Divisor

Let $X = \mathcal{Z}(x_0y_0z_2 + x_1y_1z_0 - x_0y_1z_1 - x_1y_0z_1) \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. This variety is the blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ along the $G = \mathrm{SL}_2$-stable curve $C = \mathcal{Z}(x_1z_0 - x_0z_1, x_0y_1 - x_1y_0)$. In X there are G-invariant divisors $\Delta = \mathcal{Z}(x_0y_1 - x_1y_0, x_0^2z_2 + x_1^2z_0 - 2x_0x_1z_1)$, $E = \mathcal{Z}(x_1z_0 - x_0z_1, x_0y_1 - x_1y_0)$ and $F = \mathcal{Z}(y_1z_0 - y_0z_1, y_0z_2 - y_1z_1)$. The curve $Z = F \cap \Delta \cap E$ is G-invariant and defined by $\mathcal{Z}(x_1z_0 - x_0z_1, x_1z_1 - x_0z_2, x_0y_1 - x_1y_0)$.

Taking $x_1 = y_1 = z_2 = 1$ gives a maximal B-chart U of Z, given by $U = \mathcal{Z}(x_0y_0 + z_0 - x_0z_1 - y_0z_1) \subseteq \mathbb{A}^4$, so eliminating z_0 gives $U = \operatorname{Spec} \Bbbk[x_0, y_0, z_1] \cong \mathbb{A}^3$. We have $\Delta \cap U = \mathcal{Z}(x_0 - y_0)$, $F \cap U = \mathcal{Z}(y_0 - z_1)$, $E \cap U = \mathcal{Z}(z_1 - x_0)$ and $Z \cap U = \mathcal{Z}(z_1 - x_0, x_0 - y_0)$. We blow up U in this curve to obtain the variety $\tilde{X} = \mathcal{Z}(u_1(z_1 - x_0) - u_0(x_0 - y_0)) \subseteq \mathbb{A}^3 \times \mathbb{P}^1$.

The B-invariant rational function

$$f = \frac{x_1^2(z_0z_2 - z_1^2)}{(x_0z_2 - x_1z_1)^2}$$

on X becomes, on $\tilde{X}$, $f = \frac{z_1 - y_0}{x_0 - z_1}$. From this one can see that the exceptional divisor $D = \mathcal{Z}(z_1 - y_0, x_0 - z_1)$ of the blow-up $\sigma\colon \tilde{X} \to X$ is central.

$\beta_X(F)$ (3.17)

We now want to calculate

$$\beta(D) = A_X(D)(-K_X)^3 - \int_0^\infty \mathrm{vol}_X\left(-K_X - xD\right)\ \mathrm{d}x.$$

We have $A_X(D) = 2$ since D is the exceptional divisor on a blow-up of X, and $(-K_X)^3 = 36$, so

$$\beta(F) = 72 - \int_0^\infty \mathrm{vol}_X\left(-K_X - xD\right)\ \mathrm{d}x = 72 - \int_0^\infty \mathrm{vol}_{\tilde{X}}\left(\sigma^*(-K_X) - xD\right)\ \mathrm{d}x.$$

To calculate $\sigma^*(-K_X)$, first let $\Delta = \mathcal{Z}(x_0^2z_2 + x_1^2z_0 - 2x_0x_1z_1)$ and $F = \mathcal{Z}(z_0z_2 - z_1^2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. The divisors Δ, F above are the strict transforms of these under the blow-up $\mu\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$. Since $-K_{\mathbb{P}^1\times\mathbb{P}^2}$ is the class of a divisor of bidegree $(2,3)$, we can represent it by $\Delta + F = (2,1) + (0,2)$.

Then $-K_X = \mu^*(\Delta + F) - E = (\Delta + E) + (F + E) - E = \Delta + F + E$. Hence $\sigma^*(-K_X) = \tilde{\Delta} + \tilde{F} + \tilde{E} + 3D$, and so we need to calculate the volume of $\delta = \tilde{\Delta} + \tilde{F} + \tilde{E} + (3 - x)D$. This is given by

$$\operatorname{vol}(\delta) = 6 \int\limits_{\lambda_\delta + \mathcal{P}_+(\delta)} 2\lambda A(\delta, \lambda - \lambda_\delta)\, \mathrm{d}\lambda.$$

Since δ is G-invariant, we have $\lambda_\delta = 0$. We also have

$$\mathcal{P}(\delta) = \{\lambda \in \Lambda \otimes \mathbb{R} \mid \langle \lambda, \ell_D \rangle \geq x - 3\} = \{\lambda \mid \lambda \leq 3 - x\}$$

and

$$\mathcal{P}_+(\delta) = \{\lambda \in \mathcal{P}(\delta) \mid A(\delta, \lambda) \geq 0\} = \{\lambda \leq 3 - x \mid A(\delta, \lambda) \geq 0\}$$

where

$$A(\delta, \lambda) = \sum_{p \in \mathbb{P}^1} \min_{p_D = p} \frac{\langle \lambda, \ell_D \rangle + m_D}{h_D}.$$

To calculate $A(\delta, \lambda)$, first note that for $p \neq 0, -1, \infty$, the only B-divisors with $p_D = p$ are the colours D_p with $\ell_{D_p} = m_{D_p} = 0$, $h_{D_p} = 1$, so there is no contribution in these cases.

For $p = 0$, the two divisors with $p_D = p$ are E, with $\ell = -1$, $m = 1$ and $h = 1$, and D_0 with $\ell = -1$, $m = 0$ and $h = 2$. Hence there is a contribution to $A(\delta, \lambda)$ of

$$\min\left\{1 - \lambda, -\frac{\lambda}{2}\right\} = \begin{cases} 1 - \lambda & \lambda \geq 2 \\ -\frac{\lambda}{2} & \lambda < 2 \end{cases}.$$

For $p = \infty$, the contribution is the same. The two divisors with $p_D = -1$ are Δ with $\ell = 1$, $m = 1$ and $h = 1$, and D_{-1} with $\ell = -2$, $m = 0$ and $h = 1$, so the contribution to $A(\delta, \lambda)$ is

$$\min\{1 + \lambda, 2\lambda\} = \begin{cases} 1 + \lambda & \lambda \geq 1 \\ 2\lambda & \lambda < 1 \end{cases}.$$

Hence we have

$$A(\delta, \lambda) = \begin{cases} 3 - \lambda & \lambda \geq 2 \\ 1 & 1 \leq \lambda \leq 2 \\ \lambda & \lambda < 1 \end{cases}.$$

It follows that $\mathcal{P}_+(\delta) = \{\lambda \leq 3 - x \mid 0 \leq \lambda \leq 3\}$, and since $x \geq 0$, $3 - x \leq 3$ and $\mathcal{P}_+(\delta)$ is empty if $x > 3$. Hence $\mathcal{P}_+(\delta) = [0 : 3 - x]$ where $0 \leq x \leq 3$.

Therefore

$$\operatorname{vol}\delta = 6 \begin{cases} \int_0^{3-x} 2\lambda^2 \,\mathrm{d}\lambda & 2 \leq x \leq 3 \\ \int_0^1 2\lambda^2 \,\mathrm{d}\lambda + \int_1^{3-x} 2\lambda \,\mathrm{d}\lambda & 1 \leq x \leq 2 \\ \int_0^1 2\lambda^2 \,\mathrm{d}\lambda + \int_1^2 2\lambda \,\mathrm{d}\lambda + \int_2^{3-x} 2\lambda(3-\lambda) \,\mathrm{d}\lambda & 0 \leq x \leq 1 \end{cases}$$

$$= \begin{cases} 4(3-x)^3 & 2 \leq x \leq 3 \\ 6x^2 - 36x + 52 & 1 \leq x \leq 2 \\ 4x^3 - 18x^2 + 36 & 0 \leq x \leq 1 \end{cases}$$

giving

$$\begin{aligned} \beta(F) &= 72 - \int_0^3 \operatorname{vol}\delta \,\mathrm{d}x \\ &= 72 - \int_0^1 4x^3 - 18x^2 + 36 \,\mathrm{d}x - \int_1^2 6x^2 - 36x + 52 \,\mathrm{d}x - \int_2^3 4(3-x)^3 \,\mathrm{d}x \\ &= 72 - 31 - 12 - 1 = 28. \end{aligned}$$

Hence X is K-polystable.

4.5.4 The Divisor W on $\mathbb{P}^2 \times \mathbb{P}^2$ and Its Blow-Up

Central Divisor

Let $W = \mathcal{Z}(x_0y_2 - 2x_1y_1 + x_2y_0) \subseteq \mathbb{P}^2 \times \mathbb{P}^2$. We know that W has G-invariant divisors $E_\infty = \mathcal{Z}(x_0x_2 - x_1^2) \cap W$ and $E_0 = \mathcal{Z}(y_0y_2 - y_1^2) \cap W$ whose intersection is a G-stable curve Z. We obtain the smooth Fano (3.13) by blowing up W along Z. The curve Z has a minimal B-chart $U = W \setminus (\mathcal{Z}(x_2) \cup \mathcal{Z}(y_2)) = \mathcal{Z}(x_0 - 2x_1y_1 + y_0) \subseteq \mathbb{A}^4$. We eliminate x_0 to obtain $U = \operatorname{Spec}\Bbbk[x_1, y_0, y_1] \cong \mathbb{A}^3$. Introducing new co-ordinates $x = x_1 - y_1$, $y = y_0$, $z = y_1$, the curve $Z \cap U$ is defined by $x = y - z^2 = 0$. The divisors $E_\infty \cap U$ and $E_0 \cap U$ are defined by $z^2 - y - x^2 = 0$ and $y - z^2 = 0$, respectively.

Hence blowing up U along $Z \cap U$ we obtain $X = \mathcal{Z}(vx - u(y - z^2)) \subseteq \mathbb{A}^3 \times \mathbb{P}^1$ with exceptional divisor $E = \mathcal{Z}(x, y - z^2)$ and strict transforms (abusing notation) $\tilde{E}_\infty = \mathcal{Z}(ux - v, z^2 - x^2 - y)$ and $\tilde{E}_0 = \mathcal{Z}(y - z^2, v)$. The intersection of these three G-invariant divisors (in fact any two of them) gives a G-invariant curve $Y = \mathcal{Z}(y - z^2, x, v)$, the unique minimal G-germ of X.

Take a chart $u = 1$ to obtain $X = \mathcal{Z}(vx - y + z^2)$, then eliminate y so that $X = \operatorname{Spec} \Bbbk[x, v, z] \cong \mathbb{A}^3$. Then we have $\tilde{E}_\infty = \mathcal{Z}(x - v)$, $\tilde{E}_0 = \mathcal{Z}(v)$, $E = \mathcal{Z}(x)$ and $Y = \mathcal{Z}(x, v)$.

Blowing up this chart along Y, we obtain $\tilde{X} = \mathcal{Z}(wx - sv) \subseteq \mathbb{A}^3 \times \mathbb{P}^1$. We now have $\tilde{E}_0 = \mathcal{Z}(v, w)$, $\tilde{E}_\infty = \mathcal{Z}(x - v, w - s)$, $\tilde{E} = \mathcal{Z}(x, s)$ and an exceptional divisor $F = \mathcal{Z}(x, v)$. The B-quotient map $W \dashrightarrow \mathbb{P}^1$ was originally given by $P \mapsto [y_2^2(x_0x_2 - x_1^2) : x_2^2(y_0y_2 - y_1^2)]$, which on $\tilde{X}$ reduces to $P \mapsto [v - x : v]$. From this one can see that the exceptional divisor F is central.

Since F is G-invariant we must have $\ell_F < 0$ (for it to lie in the valuation cone), and since it is a hyperplane we must then have $\ell_F = -1$. Since the coloured hyperfan of any model of type I must consist of strictly convex coloured cones, and the G-invariant valuations map injectively into the hyperspace, F is the unique central G-invariant prime divisor over W.

$\beta_W(F)$ **(2.32)**

To calculate $\beta_W(F)$, we first must calculate $-K_W$ and its pullback to the model containing F. Since W is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$, the adjunction formula gives $-K_W = (-K_{\mathbb{P}^2\times\mathbb{P}^2} - W)|_W$. The anticanonical class of $\mathbb{P}^2 \times \mathbb{P}^2$ is $(3, 3)$ where we identify the divisor class group with $\mathbb{Z} \oplus \mathbb{Z}$, and since W has bidegree $(1, 1)$ we get $-K_W = (2, 2)|_W$. Represent the divisor class $(2, 2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$ by $\mathcal{Z}(x_0x_2 - x_1^2) + \mathcal{Z}(y_0y_2 - y_1^2)$, so that the restriction to W of this class is represented by $E_\infty + E_0 = -K_W$.

After the two blow-ups, this class pulls back to $E_0 + E_\infty + 2E + 4F$, and we must calculate $\beta_W(F) = A_W(F)(-K_W)^3 - \int_0^\infty \operatorname{vol}(\delta)\, dx$ where $\delta = E_0 + E_\infty + 2E + (4 - x)F$. We have $(-K_W)^3 = 48$ and $A_W(F) = 3$ since F is the exceptional divisor of the second of two nested blow-ups of W.

We have $\lambda_\delta = 0$ and $\mathcal{P}(\delta) = (-\infty, 4 - x]$. To calculate $A(\delta, \lambda)$, first note that there is no contribution at points other than 0, ∞ and -1. At $p = 0, \infty$, we have divisors E_p with $\ell = 0$, $h = 1$ and $m = 1$, and D_p with $\ell = 1$, $h = 2$ and $m = 0$, so the contribution in each case is $\min\{1, \frac{\lambda}{2}\}$. At $p = -1$ we have E with $\ell = -1$, $h = 1$ and $m = 2$, and D_{-1} with $\ell = -1$, $h = 2$ and $m = 0$, so the contribution is $\min\{2 - \lambda, -\frac{\lambda}{2}\}$. Overall, we have

$$A(\delta,\lambda)=\begin{cases}4-\lambda & \lambda\geq 0\\ \frac{\lambda}{2} & \lambda<2.\end{cases}$$

Therefore we have $\mathcal{P}_+(\delta)=[0,4-x]$, so $0\leq x\leq 4$, and:

$$\begin{aligned}\mathrm{vol}(\delta)&=6\int_0^{2-x}2\lambda A(\delta,\lambda)\,\mathrm{d}\lambda\\ &=\begin{cases}6\int_0^{2-x}\lambda^2\,\mathrm{d}\lambda & 2\leq x\leq 4\\ 6\int_0^2\lambda^2\,\mathrm{d}\lambda+6\int_2^{2-x}8\lambda-2\lambda^2\,\mathrm{d}\lambda & 0\leq x\leq 2\end{cases}\\ &=\begin{cases}2(4-x)^3 & 2\leq x\leq 4\\ 4x^3-24x^2+80 & 0\leq x\leq 2.\end{cases}\end{aligned}$$

Hence

$$\beta_W(F)=3\cdot 48-\int_0^4\mathrm{vol}(\delta)\,\mathrm{d}x=144-120=24>0$$

so W is K-polystable.

$\beta_X(F)$ (3.13)

We now want to calculate

$$\beta_X(F)=A_X(F)(-K_X)^3-\int_0^\infty\mathrm{vol}_X\left(-K_X-xF\right)\,\mathrm{d}x.$$

We have $A_X(F)=2$ since F is a prime divisor on a blow-up of X, and $(-K_X)^3=30$. We have $-K_X=\mu^*(-K_W)-E$ where μ is the blow-up of W in $E_0\cap E_\infty$, which gives $-K_X=E_0+E_\infty+E$. Under the next blow-up to the model containing F, this pulls back to $E_0+E_\infty+E+3F$, so we set $\delta=E_0+E_\infty+E+(3-x)F$.

We have $\lambda_\delta=0$ and $\mathcal{P}(\delta)=(-\infty,3-x]$. To calculate $A(\delta,\lambda)$, first note that for $p\neq 0,-1,\infty$, there is no contribution. For $p=0,\infty$, the two divisors with $p_D=p$ are E_p, with $\ell=0$, $m=1$ and $h=1$, and D_p with $\ell=1$, $m=0$ and $h=2$. Hence in each case there is a contribution to $A(\delta,\lambda)$ of

$$\min\left\{1,\frac{\lambda}{2}\right\}=\begin{cases}1 & \lambda\geq 2\\ \frac{\lambda}{2} & \lambda<2\end{cases}.$$

For $p=-1$, the two divisors with $p_D=p$ are E with $\ell=-1$, $m=1$ and $h=1$, and D_p with $\ell=-1$, $m=0$ and $h=2$, so the contribution to $A(\delta,\lambda)$ is

$$\min\left\{1-\lambda,-\frac{\lambda}{2}\right\}=\begin{cases}1-\lambda & \lambda\geq 2\\ -\frac{\lambda}{2} & \lambda<2\end{cases}.$$

Hence we have

$$A(\delta,\lambda)=\begin{cases}3-\lambda & \lambda\geq 2\\ \frac{\lambda}{2} & \lambda<2\end{cases}.$$

It follows that $\mathcal{P}_+(\delta)=[0,3-x]$, so $0\leq x\leq 3$

Therefore

$$\begin{aligned}\operatorname{vol}\delta &= \begin{cases}6\int_0^{3-x}\lambda^2\,\mathrm{d}\lambda & 1\leq x\leq 3\\ 6\int_0^2\lambda^2\,\mathrm{d}\lambda+6\int_2^{3-x}6\lambda-2\lambda^2\,\mathrm{d}\lambda & 0\leq x\leq 1\end{cases}\\ &= \begin{cases}2(3-x)^3 & 1\leq x\leq 3\\ 4x^3-18x^2+30 & 0\leq x\leq 1.\end{cases}\end{aligned}$$

Hence

$$\beta(F)=60-\int_0^3\operatorname{vol}(\delta)\,\mathrm{d}x=60-33=27.$$

So X is K-polystable.

4.5.5 Blow up of $\mathbb{P}^3$ along the Twisted Cubic

Central Divisor

Let $G=\mathrm{SL}_2$ act on $\mathbb{P}^3=\mathbb{P}(S^3\mathbb{k}^2)$. The twisted cubic curve

$$C=\mathcal{Z}(x_0x_2-x_1^2,x_0x_3-x_1x_2,x_1x_3-x_2^2)$$

is G-invariant. The prime divisor

$$F=\mathcal{Z}(3x_1^2x_2^2-4x_1^3x_3-x_0^2x_3^2-4x_0x_2^3+6x_0x_1x_2x_3).$$

is G-invariant and contains C - indeed F is the secant variety to C and C is the singular locus of F, contained with multiplicity 2.

The smooth Fano (2.27) is obtained by blowing up $\mathbb{P}^3$ along C. We first take the open chart given by $x_3 = 1$, which is the minimal B-chart of C. In this chart,

$$C = \mathcal{Z}(x_0x_2 - x_1^2, x_0 - x_1x_2, x_1 - x_2^2) = \mathcal{Z}(x_0 - x_1x_2, x_1 - x_2^2),$$

and

$$F = \mathcal{Z}(3x_1^2x_2^2 - 4x_1^3 - x_0^2 - 4x_0x_2^3 + 6x_0x_1x_2)$$

Now consider the change of co-ordinates

$$(x_0, x_1, x_2) \mapsto (x_0 + 3x_1x_2 + x_2^3, x_1 + x_2^2, x_2).$$

It is easily checked to be an isomorphism, and it sends F to $\mathcal{Z}(x_0^2 - 4x_1^3)$ and C to $\mathcal{Z}(x_0, x_1)$. Hence we see that F is isomorphic to the product of a line (C) and a cuspidal cubic plane curve. Performing another transformation $x_1 \mapsto x_1/\sqrt[3]{4}$ gives $F = \mathcal{Z}(x_0^2 - x_1^3)$ and leaves C invariant.

Now we blow up C, giving $X = \mathcal{Z}(y_1x_0 - y_0x_1) \subseteq \mathbb{A}^3 \times \mathbb{P}^1$ with exceptional divisor $E = \mathcal{Z}(x_0, x_1)$. We have

$$\tilde{F} = \mathcal{Z}(x_0^2 - x_1^3, y_1x_0 - y_0x_1) \setminus \mathcal{Z}(x_0, x_1).$$

It is easy to check that this gives

$$\tilde{F} = \mathcal{Z}(x_0^2 - x_1^3, y_1x_0 - y_0x_1, y_0x_0 - x_1^2y_1, y_0^2 - y_1^2x_1).$$

The intersection $\tilde{F} \cap E$ is then given by $\mathcal{Z}(x_0, x_1, y_0)$. Since we don't yet have a central divisor, we will blow up this curve.

First, take the chart $y_1 = 1$. Then X becomes $\mathcal{Z}(x_0 - x_1y_0) \cong \operatorname{Spec} \Bbbk[x_1, x_2, y_0]$, F becomes $\mathcal{Z}(y_0^2 - x_1)$, and E becomes $\mathcal{Z}(x_1)$. Hence we obtain $\tilde{X} = \mathcal{Z}(z_0x_1 - z_1y_0) \subseteq \mathbb{A}^3 \times \mathbb{P}^1$, with exceptional divisor $D = \mathcal{Z}(x_1, y_0)$. The strict transforms of $\tilde{F}$ and E are $\mathcal{Z}(y_0^2 - x_1, z_0y_0 - z_1)$ and $\mathcal{Z}(x_1, z_1)$, respectively. It is straightforward to check that $\tilde{E}, \tilde{F}$ and D mutually intersect in the curve $\mathcal{Z}(x_1, y_0, z_1)$. In particular this shows that D is not central, so we must blow up again.

Take the chart $z_0 = 1$, giving $\tilde{X} = \mathcal{Z}(x_1 - z_1y_0) \cong \operatorname{Spec} \Bbbk[x_2, y_0, z_1]$, $\tilde{F} = \mathcal{Z}(y_0 - z_1)$, $\tilde{E} = \mathcal{Z}(z_1)$ and $D = \mathcal{Z}(y_0)$. Blowing up the intersection $\mathcal{Z}(y_0, z_1)$ of these divisors gives $\tilde{X}' = \mathcal{Z}(u_1y_0 - u_0z_1)$ with exceptional divisor $H = \mathcal{Z}(y_0, z_1)$. Now the strict transforms $\tilde{E}$, $\tilde{F}$ and $\tilde{D}$ all intersect H in different curves and are disjoint from each other: hence H is a central divisor.

$\beta(H)$ **(2.27)**

We now want to calculate $\beta(H)$. We have $(-K_X)^3 = 38$, and $A_X(H) = 3$ since H is a prime divisor on a variety obtained by two blow-ups of X. Hence

$$\beta(H) = 114 - \int_0^\infty \mathrm{vol}_X\left(-K_X - xH\right) \mathrm{d}x = 114 - \int_0^\infty \mathrm{vol}_{\tilde{X}'}\left(\sigma^*(-K_X) - xH\right) \mathrm{d}x$$

where $\sigma \colon \tilde{X}' \to X$ is the birational morphism given by composing the two blow-ups described above.

To calculate $\sigma^*(-K_X)$, first note that the anticanonical class of $\mathbb{P}^3$ is the class of a prime divisor of degree 4, so we can set $-K_{\mathbb{P}^3} = F$. Then, blowing up C, which is contained in F with multiplicity 2, gives $-K_X = (\tilde{F} + 2E) - E = \tilde{F} + E$. The pullback of this class under the blowing up of $\tilde{F} \cap E$ is then $(\tilde{F} + D) + (\tilde{E} + D) = \tilde{F} + \tilde{E} + 2D$. Finally, the second blow-up gives

$$\sigma^*(-K_X) = (\tilde{F} + H) + (\tilde{E} + H) + 2(\tilde{D} + H) = \tilde{F} + \tilde{E} + 2\tilde{D} + 4H.$$

Our next step is the calculate the volume of the divisor $\delta = \tilde{F} + \tilde{E} + 2\tilde{D} + (4 - x)H$. We have $\lambda_\delta = 0$ and $\mathcal{P}(\delta) = (-\infty, 4 - x]$ since we must have $\ell_H = -1$. Now we calculate $A(\delta, \lambda)$. The points $p \neq -4, 0, \infty$ contribute nothing as $p_D = p$ in this case only for colours D_p with $m_D = \ell_D = 0$.

At $p = -4$, we have two divisors: $\tilde{F}$, with $m = 1, \ell = 0$ and $h = 1$, and D_{-4}, with $m = 0$, $\ell = 1$ and $h = 2$. Hence there is a contribution of

$$\min\left\{1, \frac{\lambda}{2}\right\} = \begin{cases} 1 & \lambda \geq 2 \\ \frac{\lambda}{2} & \lambda < 2 \end{cases}.$$

At $p = 0$, we have two divisors: $\tilde{D}$, with $m = 2$, $\ell = 0$ and $h = 1$, and D_0, with $m = 0$, $\ell = 1$ and $h = 2$. Hence there is a contribution of

$$\min\left\{2, \frac{\lambda}{2}\right\} = \begin{cases} 2 & \lambda \geq 4 \\ \frac{\lambda}{2} & \lambda < 4 \end{cases}.$$

At $p = \infty$, we have two divisors: $\tilde{E}$, with $m = 1$, $\ell = -1$ and $h = 1$, and D_∞, with $m = 0$, $\ell = -2$ and $h = 3$. Hence there is a contribution of

$$\min\left\{1 - \lambda, -\frac{2\lambda}{3}\right\} = \begin{cases} 1 - \lambda & \lambda \geq 3 \\ -\frac{2\lambda}{3} & \lambda < 3 \end{cases}.$$

All in all, we have

$$A(\delta,\lambda)=\begin{cases}4-\lambda & \lambda\geq 4\\ 2-\frac{\lambda}{2} & 3\leq\lambda<4\\ 1-\frac{\lambda}{6} & 2\leq\lambda<3\\ \frac{\lambda}{3} & \lambda<2\end{cases}.$$

We can then read off $\mathcal{P}_+(\delta)=[0,4-x]$, so in particular $x\leq 4$. Hence

$$\mathrm{vol}(\delta)=6\int_0^{4-x}2\lambda A(\delta,\lambda)\ \mathrm{d}\lambda.$$

That is,

$$\begin{aligned}\mathrm{vol}(\delta)&=\begin{cases}\int_0^{4-x}4\lambda^2\ \mathrm{d}\lambda & 2\leq x\leq 4\\ \int_0^2 4\lambda^2\ \mathrm{d}\lambda+\int_2^{4-x}12\lambda-2\lambda^2\ \mathrm{d}\lambda & 1\leq x<2\\ \int_0^2 4\lambda^2\ \mathrm{d}\lambda+\int_2^3 12\lambda-2\lambda^2\ \mathrm{d}\lambda+\int_3^{4-x}24\lambda-6\lambda^2\ \mathrm{d}\lambda & 0\leq x<1\end{cases}\\ &=\begin{cases}-\frac{4}{3}(x-4)^3 & 2\leq x\leq 4\\ \frac{32}{3}+\frac{2}{3}(x^3-3x^2-24x+52) & 1\leq x<2\\ 28+2(x^3-6x^2+5) & 0\leq x<1.\end{cases}\end{aligned}$$

Hence we have

$$\beta(H)=114-\int_0^4\mathrm{vol}(\delta)=114-59=55.$$

So X is K-polystable.

4.5.6 The Quadric Threefold and Its Blow-Up

Central Divisor

Let $Q=\mathcal{Z}(3x_2^2-4x_1x_3+x_0x_4)\subseteq\mathbb{P}^4$. The twisted quartic curve

$$C=\mathcal{Z}(x_0x_2-x_1^2,x_0x_3-x_1x_2,x_1x_4-x_2x_3,x_2x_4-x_3^2)$$

is G-invariant. The prime divisor

$$F=\mathcal{Z}(4x_2^3+x_1^2x_4+x_0x_3^2-6x_1x_2x_3)\cap Q$$

is G-invariant and contains C.

The smooth Fano (2.21) is obtained by blowing up Q along C. The minimal B-chart of C is given by taking $x_4 = 1$. In this chart, the equation of Q allows us to eliminate x_0, so that $Q \cong \operatorname{Spec} \Bbbk[x_1, x_2, x_3]$. Then C is given by

$$C = \mathcal{Z}(x_1 - x_2x_3, x_2 - x_3^2)$$

and F by

$$F = \mathcal{Z}(4x_2^3 + x_1^2 + 4x_1x_3^3 - 3x_2^2x_3^2 - 6x_1x_2x_3)$$

One can read off immediately that there is an isomorphism from this B-chart in Q to the B-chart in $\mathbb{P}^3$ we took in the previous example and that this isomorphism preserves the G-invariant subvarieties F and C (again, F is the secant variety of C). Hence finding the central divisor is identical in this case to the previous one. Hence keeping the same notation as above, we must blow Q up three times, obtaining exceptional divisors E, D and H, with the latter being central.

$\beta_Q(H)$ **(1.16)**

To calculate $\beta_Q(H)$, we must first calculate $-K_Q$ and its pullback to the model containing H. Since Q is a hypersurface in $\mathbb{P}^4$, the adjunction formula gives $-K_Q = (-K_{\mathbb{P}^4} - Q)|_Q$. The anticanonical class of $\mathbb{P}^4$ is the class of a divisor of degree 5. If we represent this by $Q + \mathcal{Z}(4x_2^3 + x_1^2x_4 + x_0x_3^2 - 6x_1x_2x_3)$, we see that F is an anticanonical divisor of Q.

After the three blow-ups described above, this pulls back to $F + 2E + 3D + 6H$, so set $\delta = F + 2E + 3D + (6 - x)H$. We have $\lambda_\delta = 0$ and $\mathcal{P}(\delta) = (\infty, 6 - x]$. To calculate $A(\delta, \lambda)$, first note that there is no contribution at points $p \neq 0, \infty, -4$. At $p = 0$ we have two divisors: D with $\ell = -1$, $m = 3$ and $h = 1$, and D_0 with $\ell = -1$, $m = 0$ and $h = 2$. Hence at this point there is a contribution of

$$\min\left\{3 - \lambda, -\frac{\lambda}{2}\right\} = \begin{cases} 3 - \lambda & \lambda \geq 6 \\ -\frac{\lambda}{2} & \lambda < 6. \end{cases}$$

At $p = -4$ we have F with $\ell = 0$, $m = 1$, $h = 1$, and D_{-4} with $\ell = 1$, $m = 0$, $h = 3$. Hence the contribution is

$$\min\left\{1, \frac{\lambda}{3}\right\} = \begin{cases} 1 & \lambda \geq 3 \\ \frac{\lambda}{3} & \lambda < 3. \end{cases}$$

Finally, at $p = \infty$ we have E with $\ell = 0$, $m = 2$, $h = 1$, and D_∞ with $\ell = 1$, $m = 0$, $h = 3$. Hence the contribution is

$$\min\left\{2, \frac{\lambda}{3}\right\} = \begin{cases} 2 & \lambda \geq 6 \\ \frac{\lambda}{3} & \lambda < 6. \end{cases}$$

All in all, we have

$$A(\delta, \lambda) = \begin{cases} 6 - \lambda & \lambda \geq 6 \\ 1 - \frac{\lambda}{6} & 3 \leq \lambda \leq 6 \\ \frac{\lambda}{6} & \lambda \leq 3. \end{cases}$$

Hence $A(\delta, \lambda) \geq 0$ for $0 \leq \lambda \leq 6$, so we have $\mathcal{P}_+(\delta) = [0, 6 - x]$ and $0 \leq x \leq 6$.

Now we have

$$\begin{aligned} \mathrm{vol}(\delta) &= 6 \int_0^{6-x} 2\lambda A(\delta, \lambda)\ \mathrm{d}\lambda \\ &= \begin{cases} 6 \int_0^{6-x} \frac{\lambda^2}{3}\ \mathrm{d}\lambda & 3 \leq x \leq 6 \\ 6 \int_0^3 \frac{\lambda^2}{3}\ \mathrm{d}\lambda + 6 \int_3^{6-x} 2\lambda - \frac{\lambda^2}{3}\ \mathrm{d}\lambda & 0 \leq x \leq 3 \end{cases} \\ &= \begin{cases} \frac{2}{3}(6-x)^3 & 3 \leq x \leq 6 \\ \frac{2x^3}{3} - 6x^2 + 54 & 0 \leq x \leq 3. \end{cases} \end{aligned}$$

We have $(-K_Q)^3 = 54$ and $A_Q(H) = 4$, since we reached H as the final exceptional divisor after 3 nested blow-ups of Q. Therefore

$$\beta_Q(H) = 216 - \int_0^6 \mathrm{vol}(\delta)\ \mathrm{d}x = 216 - 135 = 81 > 0$$

and Q is K-polystable.

$\beta_X(H)$ **(2.21)**

We now calculate $\beta_X(H)$, where X is the blow-up of Q in the twisted quartic C, i.e. the smooth Fano (2.21). We have $(-K_X)^3 = 28$, and $A_X(H) = 3$.

Since $-K_Q = F$ and the curve C has multiplicity 2 in F, we have $-K_X = (F + 2E) - E = F + E$. Under thw two subsequent blow-ups to the model containing H, this pulls back to $F + E + 2D + 4H$, so we set $\delta = F + E + 2D + (4 - x)H$. We have $\lambda_\delta = 0$ and $\mathcal{P}(\delta) = (\infty, 4 - x]$. Moving on to calculating $A(\delta, \lambda)$: as before, points $p \neq -4, 0, \infty$ do not contribute.

At $p = -4$, we have two divisors: F, with $m = 1, \ell = 0$, $h = 1$, and D_{-4}, with $m = 0$, $\ell = 1$, $h = 3$. Hence there is a contribution of

$$\min\left\{1, \frac{\lambda}{3}\right\} = \begin{cases} 1 & \lambda \geq 3 \\ \frac{\lambda}{3} & \lambda < 3 \end{cases}.$$

At $p = \infty$, the situation is identical to that at $p = -4$, so we get the same contribution again.

At $p = 0$, we have two divisors: D, with $m = 2$, $\ell = -1$ and $h = 1$, and D_0, with $m = 0$, $\ell = -1$ and $h = 2$. Hence there is a contribution of

$$\min\left\{2 - \lambda, -\frac{\lambda}{2}\right\} = \begin{cases} 2 - \lambda & \lambda \geq 4 \\ -\frac{\lambda}{2} & \lambda < 4 \end{cases}.$$

Hence all things considered, we have

$$A(\delta, \lambda) = \begin{cases} 4 - \lambda & \lambda \geq 4 \\ 2 - \frac{\lambda}{2} & 3 \leq \lambda < 4 \\ \frac{\lambda}{6} & \lambda < 3 \end{cases} \cdot$$

Therefore $A(\delta, \lambda) \geq 0$ for $0 \leq \lambda \leq 4$. Hence $\mathcal{P}_+(\delta) = [0, 4 - x]$ with $0 \leq x \leq 4$. We thus have

$$\begin{aligned} \mathrm{vol}(\delta) &= 6 \int_0^{4-x} 2\lambda A(\delta, \lambda)\ \mathrm{d}\lambda \\ &= \begin{cases} \int_0^{4-x} 2\lambda^2\ \mathrm{d}\lambda & 1 \leq x \leq 4 \\ \int_0^3 2\lambda^2\ \mathrm{d}\lambda + \int_3^{4-x} 24\lambda - 6\lambda^2\ \mathrm{d}\lambda & 0 \leq x \leq 1 \end{cases} \\ &= \begin{cases} \frac{2}{3}(4 - x)^3 & 1 \leq x \leq 4 \\ 18 + 2(x^3 - 6x^2 + 5) & 0 \leq x \leq 1. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \beta(H) &= 84 - \int_0^4 \mathrm{vol}(\delta)\ \mathrm{d}x \\ &= 84 - \int_0^1 18 + 2(x^3 - 6x^2 + 5)\ \mathrm{d}x - \int_1^4 \frac{2}{3}(4 - x)^3\ \mathrm{d}x \\ &= 84 - \frac{49}{2} - \frac{27}{2} = 46 > 0, \end{aligned}$$

so X is K-polystable.

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