



Analytic integrability of certain resonant saddle

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ABSTRACT

We provide sufficient conditions of analytic integrability for a family of planar differential system with a $p : -q$ resonant saddle at the origin. The conditions are then shown to be necessary in several finite dimensional families where the calculations can be performed explicitly. It is conjectured that the conditions are in fact complete for all families.

Though the form of the equations is quite simple, their study merits further attention as they exhibit an interesting dichotomy between finite and arbitrary dimensional components of the center variety.

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1. Introduction and main results

The classical center problem at the origin for real planar differential systems

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

where P, Q are analytic functions without constant and linear terms, can be generalized to study the integrability problem of a $p : -q$ resonant singular point at the origin of systems

$$\dot{x} = px + F_1(x, y), \quad \dot{y} = -qy + F_2(x, y), \quad (1)$$

with $p, q \in \mathbb{N}$ and where $F_1(0, 0) = \partial_x F_1(0, 0) = \partial_y F_1(0, 0) = 0$ for $i = 1, 2$. In the case that F_1 and F_2 are specific polynomials and for some particular resonance ratios, the integrability problem of system (1) has been studied by several authors [1,2,8,10,11,13–15,17,21]. However few results are known for arbitrary resonance ratios $p : -q$. The order of the resonant saddle is recently studied in Dong et al. [3], Dong and Yang [4], Dong et al. [5] for polynomials of arbitrary degree.

In this paper we aim to give a characterization of the analytic integrable resonant saddles at the origin of the complex differential system in \mathbb{C}^2 ,

$$\dot{x} = px, \quad \dot{y} = -qy + f(y)x, \quad 0 \neq p, q \in \mathbb{N}, \quad (2)$$

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where $f(y) = \sum_{i \geq 1} a_i y^i$ is an analytic function without constant term. The resonance $1 : -1$ of system (2) was studied in Giné and Valls [12]. Some specific polynomial systems (2) with $p : -1$ resonance have been studied in Ferčec and Giné [7]. If system (1) has a analytic integrable saddle at the origin we say that the origin is a generalized center.

The main theorem is the following obtained studying the necessary conditions for low values of p, q and the degree of f .

Theorem 1. System (2) has an analytic integrable saddle at the origin if one of the following conditions holds:

- (1) $a_i = 0$ for $i \leq n$ with $n = 1 + \lfloor p/q \rfloor$;
- (2) $a_i = 0$ for $i \geq n$ with $n = 1 + \lceil p/q \rceil$;
- (3_k) $a_i = 0$ for $i \neq 1$ and $i \neq k + 1$ and p does not divide k .

In order to investigate the universality of these conditions, we have taken a system of reasonably large degree (f of degree 6) and for a range of p and q have obtained the necessary and sufficient conditions to have an analytic integrable saddle for system (2). The calculations grow rapidly in complexity for higher degrees of f or for large values of p and q . However, we have been able to verify the following after explicit computations.

Theorem 2. If f is a polynomial of degree at most 6, then conditions (1–3) of Theorem 1 are the necessary and sufficient conditions for a generalized center for all $p, q \geq 1$ with $p + q \leq 7$.

Proof. Theorem 1 establishes the sufficiency of the conditions. A number of necessary conditions were then computed using two approaches. The first is detailed in the next section. The second was a more traditional approach seeking a first integral of the form $x^q y^p + \dots$. Finally, a Groebner Basis computation was carried out to simplify the computed expressions to the ones given in Theorem 1. \square

These results strongly suggest the following conjecture.

Conjecture 3. *If f is a polynomial, conditions (1–3) of Theorem 1 are the necessary and sufficient conditions for a generalized center for all $p, q \geq 1$.*

Although the system (2) is of a very simple form, the completeness of the conditions in Theorem 1 is not at all obvious. Furthermore, the nature of the conditions is quite unusual—exhibiting an interesting dichotomy between finite codimension components (cases 1 and 3_k) of the center variety and arbitrary (resp. infinite) dimensional components (case 2) in the polynomial (resp. analytic) case.

Remark. A further investigation of this system would be of much interest—especially to understand to what extent the conjecture holds when f is an analytic function. Furthermore, if the analytic case is indeed more general, is it possible to write the additional center conditions in an explicit form?

2. Method to find the necessary conditions

The necessary conditions are usually obtained proposing a power series as a formal first integral of system (2). However we will see that this is not the case in this work. Any $p : -q$ resonant saddle can be written, doing a rotation and a change of time if necessary, as system (1). If we propose a formal first integral for system (1), the leading term $h_0(x, y)$ of this formal first integral must a first integral of the linear part of the system (1). Then we look for a formal first integral of the form $H(x, y) = x^q y^p + \dots$ where the dots denotes the higher order terms. Hence the derivative of H along the trajectories of system (1) give us

$$\dot{H}(x, y) = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y} = v_1 h_0^2(x, y) + v_3 h_0^3(x, y) + \dots, \tag{3}$$

where v_i are polynomials in the coefficients of system (1) called *resonant saddle quantities* of system (1) at the origin. We have formal integrability at the origin when all of the v_i are zero and in this case the system has a formal integrable resonant saddle at the origin, see [9,20,22]. Moreover the formal integrability around an isolated singular point implies local analytic integrability around it, see for instance [16]. Other methods to study integrability of a resonant saddle and the recursive formulas to compute resonant saddle quantities were also investigated in Wang and Huang [17], Wang et al. [18], Wang and Liu [19].

However we are not going to construct the formal first integral around the origin of system (2). We are going to use the blow-up method described in Ferčec and Giné [6,7]. The method is based on applying the blow-up $(x, y) \rightarrow (x, z) = (x, y/x)$ at the origin of system (1). Doing this blow-up the origin is replaced by the line $x = 0$, which contains two singular points that correspond to the separatrices at the origin of system (1). We call these new singular points p_1 and p_2 which are $(p + q) : -p$ and $(p + q) : -q$ resonant saddles, respectively. In [9] the following results are proved that we will use to derive the method for computing the necessary integrability conditions of system (1) around the origin.

Theorem 4. *The $p : -q$ resonant singular point at the origin of system (1) is analytically integrable if and only if one of the points p_1 or p_2 are analytically integrable.*

Proposition 5. *The first nonzero necessary condition for integrability of the $p : -q$ resonant singular point at the origin of system (1) is the same that the first nonzero necessary integrability condition of the points p_1 or p_2 .*

Hence, we apply the blow-up $(x, y) \rightarrow (x, z) = (x, y/x)$, to system (1) and in the variables (x, z) we get a system of the form

$$\dot{z} = -(p + q)z + x\mathcal{F}(x, z), \quad \dot{x} = px + x^2\mathcal{G}(x, z), \tag{4}$$

where $\mathcal{F}(0, 0) = 0$. The most powerful of the method is that now we can propose the power series

$$\tilde{H} = \sum_{i \geq 1}^{\infty} f_i(z)x^i, \tag{5}$$

as a formal first integral of system (4), where $f_i(z)$ are functions of z . However we will see that in the case that we have an integrable resonant saddle at the origin of system (4) these $f_i(z)$ will be polynomials of degree at most i . Now we compute the derivative of \tilde{H} along the trajectories of system (4), i.e.,

$$\dot{\tilde{H}} = \frac{\partial \tilde{H}}{\partial z} (-(p + q)z + x\mathcal{F}(x, z)) + \frac{\partial \tilde{H}}{\partial x} (px + x^2\mathcal{G}(x, z)),$$

and we get a system of equations defined by the coefficient of each power of x . The constant term gives the differential equation $p f_1(z) - (p + q)z f_1'(z) = 0$, whose solution is $f_1(z) = c_1 z^{p/(p+q)}$. Taking into account that to have formal integrability $f_1(z)$ must be a polynomial then we have that $c_1 = 0$ which implies that $f_1(z) = 0$.

The power x^1 has as a coefficient the differential equation is $2p f_2(z) - (p + q)z f_2'(z) = 0$, whose solution is $f_2(z) = c_2 z^{(2p)/(p+q)}$, which implies, either $(2p)/(p + q) \in \mathbb{N}$ or c_2 is zero and $f_2(z)$ too. From the fact that $p, q \in \mathbb{N}$ then exist $f_{k_0}(z)$ such that $(k_0 p)/(p + q) \in \mathbb{N}$ or $(k_0 q)/(p + q) \in \mathbb{N}$ (taking the other saddle point p_2). From here, for the next powers of x we have the following ordinary recursive differential equation

$$k p f_k(z) - (p + q)z f_k'(z) + g_k(z) = 0, \tag{6}$$

where $g_k(z)$ is a polynomial in z which depends on the previous polynomials f_i for $i = k_0, \dots, k - 1$. The solution of the differential Eq. (6) takes the form

$$f_k(z) = c_k z^{\frac{kp}{p+q}} + z^{\frac{kp}{p+q}} \int^z \frac{s^{-1-\frac{kp}{p+q}} g_k(s) ds}{p + q}, \tag{7}$$

where c_k is an arbitrary constant. In order to have formal integrability, $f_k(z)$ must be a polynomial. The integral that appears in expression (7) can give a logarithmic term and this term is the term that prevents formal integrability of system (4) and consequently by Theorem (4) of the original system (1). The logarithmic term appears, in fact, by the power s^{-1} with nonzero coefficient in the integrand of the integral of (7). The first nonzero coefficient of the power s^{-1} coincides with the first nonzero resonant saddle quantity defined in (3). However if the resonant saddle quantity is zero then coefficient of the power s^{-1} is also zero and the method always gives a polynomial attending to the computation of f_k in (7). Therefore the blow-up method always gives polynomials and sometimes also logarithmic terms. In fact this is true if we do not use the blow-up method presented here. More specifically, system (1) can be formal integrable around the origin but if we propose a formal first integral in the original variables (x, y) the recursive procedure can do not give always polynomials for the functions f_i , see for instance [6] and references therein.

Usually, induction method works to prove that, for any arbitrary i , if certain $f_i(z)$ does not have the logarithmic term then next function $f_{i+1}(z)$ does not have them either. However, sometimes this is not the case. Nevertheless, in Ferčec and Giné [6], it is

shown that if we add certain property \mathcal{P} that must satisfy any previous f_i , for $i = k_0, \dots, k-1$, then f_k does not have a logarithmic term. This method was applied to system (2) to find the necessary conditions that give the proof of Theorem 2.

3. Proof of Theorem 1

Since $\dot{x} = px$, the integrability and linearizability conditions of (2) are the same.

3.1. Case (1)

Under the assumptions of statement (1) of Theorem 1, system (2) takes the form

$$\dot{x} = px, \quad \dot{y} = -qy + f(y)x = -qy + \left(\sum_{i=n+1}^{\infty} a_i y^i \right) x, \quad (8)$$

where $n = 1 + \lfloor p/q \rfloor$. Hence the condition implies that $f(y) = y^{n+1}h(y)$ for some analytic function $h(y)$. Choosing the new variable $Y = y^n x$, the system (2) takes the form

$$\begin{aligned} \dot{Y} &= (p - nq)Y + x^2 y^{n-1} f(y) = Y(p - nq + Yh(y)), \\ \dot{y} &= (-q)y + Yh(y) = y(-q + Yh(y)). \end{aligned} \quad (9)$$

Since $n = 1 + \lfloor p/q \rfloor$, we have $n - p/q > 0$. Hence the system (9) has a node and can have no resonant terms because as it has two analytic separatrices. Consequently there exist a change of variables which linearizes the node. In particular, there is a function $Z = Y\phi(y, Y)$, with $\phi(0, 0) = 1$ such that $\dot{Z} = (p - nq)Z$. If we substitute $Z = Y\phi(y, Y)$ into $\dot{Z} = (p - nq)Z$ we find that $\dot{\phi} = -nh(y)Y\phi$.

Hence, from $Y = y^n x$, we find that $W = y\phi^{1/n}(y, y^n x)$ satisfies $\dot{W} = -qW$ and (x, W) becomes a linearizing change of coordinates.

3.2. Case (2)

Under the assumptions of statement (2) of Theorem 1, system (2) takes the form

$$\dot{x} = px, \quad \dot{y} = -qy + (a_1 y + a_2 y^2 + \dots + a_n y^{m+1})x. \quad (10)$$

where $m = n - 2 = \lceil p/q \rceil - 1 > 0$. If $m = 0$ then there is a first integral $H = x^q y^p e^{-a_1 x}$ and a linearizing change of coordinates $(x, ye^{-a_1 x/p})$.

If $m > 0$ then we perform the change of coordinates $X = x^{1/m}$ and $Y = yx^{1/m}$, bringing the system (10) to the form

$$\dot{X} = \frac{p}{m} X \quad \dot{Y} = Y \left(\frac{p}{m} - q + F(X, Y) \right), \quad (11)$$

where $YF(X, Y) = x^{(1/m)+1}(a_1 y + \dots + a_n y^{m+1}) = xYf(y)/y$. System (11) has a linearizable node at the origin because there are two analytic separatrices and the ratio of eigenvalues is

$$\frac{p/m - q}{p/m} = \frac{q}{p}(-m + \frac{p}{q}) > 0, \quad \text{for } m = \lceil p/q \rceil - 1.$$

Hence, there is an analytic function $\phi(X, Y)$ with $\phi(0, 0) = 1$ so that, taking $Z = Y\phi(X, Y)$, we have $\dot{Z} = (p/m - q)Z$. On substituting for $Z = Y\phi$ into $\dot{Z} = (p/m - q)Z$ we obtain $\dot{\phi} = -F(X, Y)\phi$. Thus

$$\begin{aligned} \dot{\phi}(x^{1/m}, yx^{1/m}) &= -F(x^{1/m}, yx^{1/m}) \\ \dot{\phi}(x^{1/m}, yx^{1/m}) &= -x(f(y)/y)\phi(x^{1/m}, yx^{1/m}). \end{aligned}$$

Clearly $\phi(w^k x^{1/m}, w^k yx^{1/m})$ will also have the same property for w a primitive m th root of unity. And so, replacing ϕ by

$$\tilde{\phi} = \left(\prod_{i=0}^{m-1} \phi(w^i x^{1/m}, w^i yx^{1/m}) \right)^{1/m},$$

if necessary, we can assume that ϕ is analytic in x and y .

Finally, if we take $W = y\phi(x^{1/m}, yx^{1/m})$ we see that $\dot{W} = -qW$ and (x, W) is a linearizing change of coordinates.

3.3. Case (3_k)

Under the assumptions of statement (3_k) of Theorem 1, system (2) takes the form

$$\dot{x} = px, \quad \dot{y} = -qy + x(ay + by^{k+1}). \quad (12)$$

System (10) has associated a Bernoulli equation. In this case we seek a Darboux factor of the form $\ell = 1 + y^k g(x)$, for some analytic function $g(x)$. Then ℓ has cofactor $k_\ell = kbxy^k$ if and only if

$$pxg'(x) + k(ax - q)g(x) = kbx.$$

The solution to this linear differential equation is

$$g = e^{-ak/p} x^{aqk/p} \int \frac{kb}{p} e^{ak/p} x^{-qk/p} dx. \quad (13)$$

If p does not divide qk then the exponential can be expanded as a power series and the integral can be evaluated term by term to express g as a power series in x . It is straight forward to show that this series must be convergent. On the other hand, if p divides qk , then there will be terms in $\log(x)$ in the series for g arising from terms of the form x^{-1} in the integrand of (13). Thus, assuming p and q are coprime, a sufficient condition for the existence of g , and hence ℓ , is that p does not divide k .

It is easy to construct a Darboux first integral from the Darboux factors x , y , e^x and ℓ . In fact the cofactors are $k_x = p$, $k_y = -q + x(a + by^k)$, $k_{e^x} = px$ and $k_\ell = kbxy^k$ and a Darboux first integral will be given by

$$H = x^q y^p e^{ax} (1 + y^k g(x))^{p/k}.$$

Furthermore, we can use these factors to construct an explicit linearizing change of coordinates. Indeed, if $H = x^q y^p \psi(x, y)$ is the first integral then $X = x$ and $Y = y\psi^{1/p}$ linearizes the system.

4. Conclusions

For system (2) we have computed the necessary conditions to have formal integrability using the new method developed in Ferčec and Giné [6,7] for certain degrees of the function $f(x)$. The main result (Theorem 1) gives the sufficiency of such conditions using different methods for each case, as the blow-up method to a node, as well as the recursive method used in the last case.

There are several methods to find the necessary conditions but there is no global method to find the sufficiency and in this work we propose some new techniques to find the sufficiency which are given in the proof of Theorem 1.

A conjecture is given for the analytic case of $f(x)$. Furthermore, if the analytic case is indeed more general, it is open if such additional center conditions can be written in an explicit form.

Declaration of Competing Interest

Authors declare that they have no conflict of interest.

CRediT authorship contribution statement

Colin Christopher: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing - original draft, Writing - review & editing, Visualization. **Jaume Giné:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing - original draft, Writing - review & editing, Visualization.

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