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Stochastic Equilibria: Noise in Actions or Beliefs?†

By Evan Friedman*

We introduce noisy belief equilibrium (NBE) for normal-form games in which players best respond to noisy belief realizations. Axioms restrict belief distributions to be unbiased with respect to and responsive to changes in the opponents' behavior. The axioms impose testable restrictions both within and across games, and we compare these restrictions to those of regular quantal response equilibrium (QRE) in which axioms are placed on the quantal response function as the primitive. NBE can generate similar predictions as QRE in several classes of games. Unlike QRE, NBE is a refinement of rationalizability and invariant to affine transformations of payoffs. (*JEL* C72, D83, D91)

Game theory rests on Nash equilibrium (NE) as its central concept, but despite its appeal and influence, it fails to capture the richness of experimental data. Systematic deviations from NE predictions have been documented, even in some of the simplest games.

NE rests on two assumptions. First, players form accurate beliefs over their opponents' actions. Second, players best respond to these beliefs. Efforts to reconcile theory with data typically amount to weakenings of these strict assumptions.

One leading example is quantal response equilibrium (QRE) (McKelvey and Palfrey 1995), which is very much like NE but relaxes the assumption of best response. That is, while each player forms correct beliefs over the distribution of opponents' actions, she fails to perfectly best respond, though she is more likely to take actions with higher expected payoffs. Simply put, QRE is an equilibrium model with "noise in actions."

In many contexts, however, the assumption of correct beliefs is unrealistic. Therefore, it is natural to consider equilibrium models that relax the other condition of NE by allowing for "noise in beliefs" while maintaining best response. In this

* Department of Economics, University of Essex (email: ef20396@essex.ac.uk). Michael Ostrovsky was coeditor for this article. I am indebted to Alessandra Casella, Mark Dean, and Navin Kartik for their guidance and encouragement. I thank Yeon-Koo Che, Ambuj Dewan, Duarte Gonçalves, Frank Heinemann, Judd Kessler, RC Lim, Claudio Mezzetti, Suanna Oh, Pietro Ortoleva, Tom Palfrey, Andrea Prat, Mike Woodford, and four anonymous referees for many helpful suggestions. I also thank participants at PSE (2020), NYU (2018), the Summer School of the Econometric Society (2017), the Stony Brook Game Theory Conference (2017), and Columbia (Experimental Lunch, Cognition and Decision Lab, and Microtheory Colloquium). This paper was previously circulated as "Noisy Belief Equilibrium." All errors are my own, due to both noise in actions and noise in beliefs.

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paper, we introduce such a model, and by comparing it to QRE, we ask: which of action or belief noise is more consistent with experimental data?

Since we do not want our conclusions to depend on specific functional forms, we begin by introducing a general class of equilibrium models with noisy beliefs. In a noisy belief equilibrium (NBE), players best respond to their beliefs, but their beliefs are drawn from distributions that depend on the opponents' equilibrium behavior. The belief distributions are restricted to satisfy several axioms. The important behavioral axioms are *belief responsiveness* and *unbiasedness*, which ensure that the belief distributions tend to track changes in opponents' behavior and are centered around the distribution of opponents' actions in equilibrium.

We study the testable restrictions of NBE, which we compare to those of regular QRE (Goeree, Holt, and Palfrey 2005) in which axioms embed a "sensitivity to payoffs" into the primitive quantal response function.[1] This is essentially the most flexible form of QRE that imposes testable restrictions on the data, and so we avoid altogether any concerns that QRE can "explain anything" (see, for example, Haile, Hortaçsu, and Kosenok 2008). Thus, we compare two families of stochastic equilibrium models, which inject noise into actions and beliefs, respectively.

While the idea of injecting noise into beliefs is not new (see *Related Literature* below), an approach that does not rely on parametric structure brings new insights. For example, some existing parametric models approximately satisfy our axioms and hence give predictions that can be approximated by NBE, and so our results have implications for understanding these models and their relationship to QRE.

In Section I, we introduce NBE for normal-form games and discuss the relationship of NBE to other concepts that relax the assumption of perfect beliefs. In particular, we show that NBE is a refinement of rationalizability (Bernheim 1984 and Pearce 1984) in the sense that only rationalizable actions are played with positive probability in equilibrium. This distinguishes NBE from QRE, and yet we show that the models make similar predictions in certain types of fully mixed games.

In Section II, we study two empirical regularities explained by QRE that lie at the heart of its success. Specifically, in fully mixed games, QRE predicts commonly observed deviations from NE within a game and the well-known "own payoff effect" across games.[2] The best evidence for these regularities comes from generalized matching pennies games, so we specialize results for this context. We begin by showing an equivalence result: NBE imposes the same testable restrictions as QRE within any one of these games. We then show that NBE also predicts the own payoff effect across games. In other words, by adding noise to beliefs, it is as if players are sensitive to expected payoffs—the mechanism behind QRE.

[1] For each player, the quantal response function maps her vector of expected payoffs (each element representing the expected payoff to some action) to a distribution over actions. The axioms impose that actions with higher payoffs are played more often (*monotonicity*) and that an all-else-equal increase in the payoff to some action increases the probability it is played (*responsiveness*).

[2] First, whereas NE predicts that players' choice probabilities keep their opponents indifferent, there are systematic deviations within a game: the empirical frequency of actions typically leads to a ranking of actions for each player by expected payoffs to which they noisily best respond. Second, whereas NE predicts that a change in a player's own payoffs does not affect her equilibrium behavior, subjects' behavior is systematically affected by certain transformations of payoffs. See, for example, Ochs (1995) and Goeree, Holt, and Palfrey (2003).

In Section III, we revisit a sticking point for QRE: that it overpredicts sensitivity to changes in payoff magnitude. The problem is well known for the parametric logit model: for fixed λ (rationality parameter), equilibrium predictions are sensitive to scaling one or more players' payoffs by positive constants. However, such scaling predictions are often rejected (e.g., McKelvey, Weber, and Palfrey 2000). We provide novel results to establish that this "scaling issue" is general to all regular QRE in the sense that if QRE is to explain the empirical regularities discussed in the previous paragraph, it must be nontrivially sensitive to scaling and/or translating payoffs. To be precise, while QRE can be invariant to scaling or translation, it cannot be invariant to both. If it is invariant to translation (scaling), it can be nearly invariant to scaling (translation), but the resulting model predicts nearly uniformly random behavior independent of payoffs. By contrast, NBE explains the empirical regularities while being invariant to both types of affine transformations.

In Section IV, we introduce a parametric NBE model that can be broadly applied. It is based on the logit transform and can be viewed as analogous to logit QRE.

In Section V, we consider an extension of NBE in which players make uniform trembles with some probability. While no longer a model of purely noisy beliefs, the noise in actions is not sensitive to payoffs, and so the model is fundamentally different from QRE. Without trembles, NBE predicts that only rationalizable actions are played with positive probability. With trembles, there are interesting deviations. We apply the model to games with dominated and iteratively deleted actions as well as to the agent normal form of extensive-form games after imposing a version of sequential rationality, similar to what has been done for QRE (McKelvey and Palfrey 1998). We find that like QRE, the model can explain experimentally observed behavior in the traveler's dilemma (Basu 1994) and centipede game (Binmore 1987).

In Section VI, we consider several datasets of fully mixed 2×2 and 3×3 games to test model predictions.[3] Revisiting the McKelvey, Weber, and Palfrey (2000) study on scale effects and using only the structure provided by the model's axioms, we show that NBE captures all qualitative features of the dataset. We then compare the performance of logit transform NBE to logit QRE using data from several existing studies. We find that the models perform similarly when fit to individual games or sets of games of similar scale, which is unsurprising due to our equivalence result. However, NBE outperforms QRE in fitting sets of games or in making out-of-sample prediction across games of different scale.

In Section VII, we conclude by discussing possible directions for future research, including a suggestion for testing the NBE axioms directly using elicited beliefs.

Related Literature.—Early QRE theory was developed in a series of papers (McKelvey and Palfrey 1995; McKelvey and Palfrey 1998; Chen, Friedman, and Thisse 1997; and others) and is surveyed in a recent monograph (Goeree, Holt, and Palfrey 2016). The logit specification was introduced in the original paper and has since found wide application in experimental studies where it is used to reconcile data with theoretical predictions.

[3] We focus on fully mixed games because the basic NBE model (without trembles) would be rejected trivially (i.e., have zero likelihood) if an iteratively deleted action is played even once.

In this paper, we compare equilibrium models with noise in actions to those with noise in beliefs. For each type of noise, we select a representative family of models.

For noisy actions, we choose the flexible, yet falsifiable, regular QRE (Goeree, Holt, and Palfrey 2005) in which axioms restrict the quantal response functions directly. Another alternative would have been the family of structural QRE in which quantal response is induced by players who choose actions that maximize the sum of expected utility and a random error. However, Haile, Hortaçsu, and Kosenok (2008) show that structural QRE can rationalize the data from any one game as long as the errors are not restricted to be i.i.d. across players' actions, though it should be noted that the practical relevance of this critique is limited since a majority of empirical applications are based on the logit model, which is derived from i.i.d. errors. In any case, regular QRE imposes testable restrictions and is strictly more general than the family of structural QRE with i.i.d. errors.

For noisy beliefs, we develop a new model that we call NBE. It is analogous to regular QRE in that the primitive belief distributions are restricted to satisfy several axioms. Like regular QRE, flexibility in its primitive typically leads to set predictions and, by excluding a measure of possible outcomes, is falsifiable.

For injecting noise into equilibrium beliefs, NBE adapts the basic framework of random belief equilibrium (RBE) of Friedman and Mezzetti (2005). In their model, players best respond to beliefs that depend stochastically on the opponents' behavior, but as they study the case in which belief noise "goes to zero" to develop a theory of equilibrium selection, their conditions on belief distributions do not impose any testable restrictions beyond ruling out weakly dominated actions. On the other hand, our paper is concerned with characterizing equilibria when belief noise is bounded away from zero, so we introduce a new model and provide nonoverlapping results. We compare NBE to RBE in greater detail in Section I.

Another related model is sampling equilibrium of Osborne and Rubinstein (2003), which was applied to experimental data in Selten and Chmura (2008). Sampling equilibrium is a parametric model of noisy beliefs, which approximately satisfies the NBE axioms and thus, up to technical conditions, is a special case of NBE (see Section I for details). Our results therefore suggest that it will behave similarly to QRE in certain datasets. NBE is also related to stochastic learning equilibrium (Goeree and Holt 2002), which is a generalization of sampling equilibrium in which noise in beliefs is driven by randomness in observed histories of actions. Similar in spirit to NBE, Rogers, Palfrey, and Camerer (2009) introduce a logit QRE model with belief heterogeneity, and Heller and Winter (2018) introduce an equilibrium model with biased but deterministic beliefs.

NBE, QRE, and the concepts mentioned in the previous paragraph predict systematic deviations from NE in many normal-form games of complete information. This is in contrast with the concepts of cursed equilibrium (Eyster and Rabin 2005) and analogy-based expectation equilibrium (Jehiel 2005), which collapse to NE in all such games. Hence, their mechanisms of coarse reasoning cannot explain the basic phenomena that occupy this paper.

We emphasize that NBE is invariant to affine transformations of payoffs. This is of interest because logit QRE is well known to overpredict sensitivity to changes in scale (see, for example, McKelvey, Weber, and Palfrey 2000). To address this "scaling issue," several parametric QRE models have been proposed. While there are variants of QRE that are invariant to scale such as the Luce model (Luce 1959), these can only be defined for strictly positive payoffs. Other approaches have included modifying logit QRE.[4] However, we show that for any translation invariant QRE model such as logit, its ability to explain deviations from NE requires significant sensitivity to scale.

NBE's prediction of perfect scale invariance is consistent with several, but not all, studies. McKelvey, Weber, and Palfrey (2000); Pulford, Colman, and Loomes (2018); Brocas et al. (2014); and Kocher, Martinsson, and Visser (2008) find no evidence for scaling effects in asymmetric matching pennies, 3×3 and 4×4 games without weakly or strongly dominant actions, two-player betting games, and public goods games, respectively. However, effects have been reported in centipede games (Rapoport et al. 2003), ultimatum games (e.g., Andersen et al. 2011), and 4×4 dominance solvable games (Esteban-Casanelles and Gonçalves 2020). Our sense is that reported effects are typically small unless both the scale factor is very large and the game has an iterative structure (e.g., dominance or backward induction solvability). In any case, we acknowledge that NBE's prediction of perfect scale invariance may be too extreme for some contexts. The broader point of this paper is that deviations from NE can be explained in part by noise in beliefs, a mechanism that does not contribute to scale sensitivity.

Our approach to mistaken beliefs can be contrasted with those that drop the equilibrium assumption altogether. Level k (Nagel 1995 and Stahl and Wilson 1995) and its successors (Camerer, Ho, and Chong 2004; Alaoui and Penta 2016; and others) assume that subjects' beliefs are determined by their "depths of reasoning" or how many iterations of best response they can calculate. Goeree and Holt (2004) models beliefs through a process of "noisy introspection," and Mauersberger (2019) models beliefs as random draws from a Bayesian posterior.

I. Stochastic Equilibria

We provide the notation for normal-form games, review QRE, introduce NBE, and discuss the relationship of NBE to other concepts.

A. *Normal-Form Games*

A finite, normal-form game $\Gamma = \{N, A, u\}$ is defined by a set of players $N = \{1, \ldots, n\}$, action space $A = A_1 \times \cdots \times A_n$ with $A_i = \{a_{i1}, \ldots, a_{iJ(i)}\}$ such that each player i has $J(i)$ possible pure actions, and a vector of payoff functions $u = (u_1, \ldots, u_n)$ with $u_i : A \to \mathbb{R}$.

Let Δ_i be the set of probability measures on A_i. Elements of Δ_i are of the form $\sigma_i : A_i \to \mathbb{R}$, where $\sum_{j=1}^{J(i)} \sigma_i(a_{ij}) = 1$ and $\sigma_i(a_{ij}) \geq 0$. For simplicity, set $\sigma_{ij} = \sigma_i(a_{ij})$. Define $\Delta = \Delta_1 \times \cdots \times \Delta_n$ and $\Delta_{-i} = \times_{k \neq i} \Delta_k$ with typical elements $\sigma \in \Delta$ and $\sigma_{-i} \in \Delta_{-i}$. As is standard, extend payoff functions $u = (u_1, \ldots, u_n)$

[4] Goeree, Holt, and Palfrey (2003) augment logit QRE with risk aversion; McKelvey, Weber, and Palfrey (2000) consider heterogenous λs; and McKelvey, Palfrey, and Weber (1997) and Friedman (2020) endogenize λ as a strategic decision.

to be defined over Δ via $u_i(\sigma) = \sum_{a \in A} \sigma(a) u_i(a)$. For convenience, we will call any element of Δ an "action" regardless of whether it is pure or mixed, and we use these terms only when the distinction is important.

B. *Quantal Response Equilibrium*

As is standard in the literature on QRE, we use additional notation for expected utilities. Given $\sigma_{-i} \in \Delta_{-i}$, player i's vector of expected utilities is given by $\bar{u}_i(\sigma_{-i}) = (\bar{u}_{i1}(\sigma_{-i}), \ldots, \bar{u}_{iJ(i)}(\sigma_{-i})) \in \mathbb{R}^{J(i)}$, where $\bar{u}_{ij}(\sigma_{-i}) = u_i(a_{ij}, \sigma_{-i})$ is the expected utility to action a_{ij} given behavior of the opponents. We use $v_i = (v_{i1}, \ldots, v_{iJ(i)}) \in \mathbb{R}^{J(i)}$ as shorthand for an arbitrary vector of expected utilities. That is, v_i is understood to satisfy $v_i = \bar{u}_i(\sigma'_{-i})$ for some σ'_{-i}.

Player i's behavior is modeled via the quantal response function $Q_i = (Q_{i1}, \ldots, Q_{iJ(i)}) : \mathbb{R}^{J(i)} \to \Delta_i$, which maps her vector of expected utilities to a distribution over actions. For any $v_i \in \mathbb{R}^{J(i)}$, component $Q_{ij}(v_i)$ gives the probability assigned to action j. Intuitively, Q_i allows for arbitrary probabilistic mistakes in taking actions given the expected utility to each action, resulting perhaps from unmodeled costs of information processing.

We follow Goeree, Holt, and Palfrey (2005) by imposing the regularity axioms on the quantal response functions. Regularity imposes testable restrictions while being flexible enough to include all structural QRE[5] with i.i.d. errors, such as logit. The class also includes many nonstructural QRE such as the Luce model (Luce 1959). We impose the axioms throughout.

ASSUMPTION 1: *Quantal response function Q_i satisfies (A1)–(A4):*

(A1) **Interiority**: $Q_{ij}(v_i) \in (0,1)$ *for all* $j \in 1, \ldots, J(i)$ *and for all* $v_i \in \mathbb{R}^{J(i)}$.

(A2) **Continuity**: $Q_{ij}(v_i)$ *is a continuous and differentiable function for all* $v_i \in \mathbb{R}^{J(i)}$.

(A3) **Responsiveness**: $\frac{\partial Q_{ij}(v_i)}{\partial v_{ij}} > 0$ *for all* $j \in 1, \ldots, J(i)$ *and for all* $v_i \in \mathbb{R}^{J(i)}$.

(A4) **Monotonicity**: $v_{ij} > v_{ik} \Rightarrow Q_{ij}(v_i) > Q_{ik}(v_i)$ *for all* $j,k \in 1, \ldots, J(i)$.

Responsiveness and *monotonicity* are the important behavioral axioms, imposing that stochastic choice is sensitive to payoffs. These require that an all-else-equal increase in the payoff to some action increases the probability that it is played and that actions with higher payoffs are played more often. The other axioms are technical in nature, ensuring existence and that all actions are played with positive probability.

A QRE is obtained when the distribution over all players' actions is consistent with their quantal response functions. Letting $Q = (Q_1, \ldots, Q_n)$ and $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$, QRE is any fixed point of the composite function $Q \circ \bar{u} : \Delta \to \Delta$.

[5] In a structural QRE, player i chooses the action that maximizes the sum of expected utility and a random error, and thus, $Q_{ij}(v_i) = \Pr(v_{ij} + \varepsilon_{ij} \geq v_{ik} + \varepsilon_{ik} \, \forall k)$.

DEFINITION 1: *Fix* $\{\Gamma, Q\}$. *A QRE is any* $\sigma \in \Delta$ *such that for all* $i \in 1, \ldots, n$ *and all* $j \in 1, \ldots, J(i)$, $\sigma_{ij} = Q_{ij}(\bar{u}_i(\sigma_{-i}))$.

C. *Noisy Belief Equilibrium*

In an NBE, players draw beliefs about their opponents' actions probabilistically, to which they best respond. This induces, for each player, an expected action. In equilibrium, the belief distributions are centered in some sense around the opponents' expected actions, which are similarly induced by best responding to realized beliefs.

Randomness in beliefs can be interpreted in several ways. It could result from mistakes in "solving" for an equilibrium or from noisy signals about opponents' behavior. It could also be that each player represents a population of individuals who form beliefs deterministically and the distribution of beliefs simply reflects heterogeneity in the population. To impose testable restrictions, the belief distributions are restricted to satisfy axioms. We argue that these capture the key restrictions imposed on beliefs by models of sampling (e.g., Osborne and Rubinstein 2003) but allow for very general sampling processes.

First, we introduce the primitive and axioms in the context of binary action games for which they take a simple form. Second, we define equilibrium for the example of generalized matching pennies, which helps to illustrate the main ideas. Third, we generalize the model to normal-form games.

The Belief Map with Two Actions.—We first consider a situation in which player k has two pure actions ($J(k) = 2$). To avoid using subscripts, we write player k's action as $r \in [0,1]$, which is the probability with which she takes one of her two actions.

We assume that player i's belief over k's action is drawn from a distribution that depends on r. In other words, for each value of r, player i's belief is a random variable that we denote $r^*(r)$ and that is supported on $[0,1]$. We call this family of random variables the *belief map* (following Friedman and Mezzetti 2005), and it is defined by a family of CDFs: for any potential belief $\bar{r} \in [0,1]$, $F_k^i(\bar{r}\,|\,r) = \Pr(r^*(r) \leq \bar{r})$ is the probability of realizing a belief less than or equal to $\bar{r}$ given that player k is playing r.

To impose testable restrictions on behavior, we assume that the belief map satisfies the following axioms whenever $J(k) = 2$. These capture the idea that while beliefs may be noisy, they are not systematically biased.

ASSUMPTION 2: *If* $J(k) = 2$, *the belief map* r^* *satisfies* (*B1*$'$)–(*B4*$'$):

(B1$'$) **Interior full support**: *For any* $r \in (0,1)$, $F_k^i(\bar{r}|r)$ *is strictly increasing and continuous in* $\bar{r} \in [0,1]$; $r^*(0) = 0$ *and* $r^*(1) = 1$ *with probability* 1.[6]

[6]This is equivalent to having CDFs that satisfy $F_k^i(\bar{r}|0) = 1$ and $F_k^i(\bar{r}|1) = \mathbf{1}_{\{\bar{r}=1\}}$ for $\bar{r} \in [0,1]$.

(B2′) **Continuity**: *For any* $\bar{r} \in (0,1)$, $F_k^i(\bar{r}|r)$ *is continuous in* $r \in [0,1]$.

(B3′) **Belief responsiveness**: *For all* $r < r' \in [0,1]$, $F_k^i(\bar{r}|r') < F_k^i(\bar{r}|r)$ *for* $\bar{r} \in (0,1)$.

(B4′) **Unbiasedness**: $F_k^i(r|r) = 1/2$ *for* $r \in (0,1)$.

Axioms (B1′) and (B2′) are technical in nature and will be shown to ensure existence of equilibria and that the other axioms are well defined. (B3′) restricts belief distributions to be responsive to changes in the opponent's behavior, (B4′) restricts belief distributions to be unbiased with respect to the opponent's action, and both axioms are required to meaningfully restrict the set of equilibrium outcomes. We explain each axiom in turn.

Interior full support (B1′) requires that belief distributions are atomless and have full support when the opponent's action is interior, i.e., for $r \in (0,1)$. This is a weak requirement, as the probability beliefs realize in any open subset of $[0,1]$ can be arbitrarily small. The axiom further imposes that beliefs are correct with probability 1 (and are therefore described by a single atom) when the opponent's action is on the boundary, i.e., for $r \in \{0,1\}$. Otherwise, beliefs would necessarily be biased.[7]

Continuity (B2′) imposes a particular sense in which belief distributions move continuously in r, which will be shown to ensure existence of equilibria. Importantly, while (B2′) ensures existence for any binary action game, (B1′) and (B2′) together imply that there are *discontinuities in beliefs*.

REMARK 1: *There exists a* (*Borel*) *subset of* $[0,1]$ *for which the probability that player i's beliefs realize in that set is discontinuous in k's action r.*

For example, consider what happens as the opponent's action approaches 0, i.e., as $r \to 0^+$. Axiom (B1′) requires that belief distributions are atomless (and have full support) for all interior r but are degenerate and correct with probability 1 when r is on the boundary. Hence, there is a discontinuity in the probability that belief $r' = 0$ realizes, which jumps from 0 to 1. Importantly, however, (B1′) and (B2′) together imply that for any $\epsilon \in (0,1)$, the probability beliefs realize in $[0,\epsilon)$ is continuous in r and approaches 1 as $r \to 0^+$. Hence, while the probability of realizing the boundary belief $r' = 0$ is discontinuous as $r \to 0^+$, beliefs concentrate continuously within a neighborhood of the boundary. More generally, there are other belief discontinuities, all of which are related to sets of realized beliefs nearby one of the boundaries and r approaching that same boundary. From (B1′) and (B2′), these are easy to characterize.[8]

[7] If beliefs are not correct with probability one when $r \in \{0,1\}$, beliefs would be biased on mean, and if they are correct with probability less than one-half, than they would be biased on median.

[8] *Characterizing belief discontinuities* when $J(k) = 2$: We discuss the case that r is nearby 0, with the case of r nearby 1 being symmetric. Let $\mu_k^i(\cdot|r)$ be the probability measure on $[0,1]$ derived from $F_k^i(\cdot|r)$. From (B1′) and (B2′), it is easy to check that (i) $\mu_k^i(\cdot|\cdot)$ satisfies $\mu_k^i(\{0\}|r) = 0$ for $r > 0$, (ii) $\mu_k^i(\{0\}|0) = 1$, (iii) $\mu_k^i([0,\epsilon)|r)$ is continuous in $r \in [0,1]$, and (iv) $\mu_k^i([0,\epsilon)|r) \to 1$ as $r \to 0^+$. Hence, there are discontinuities

Table 1—Generalized Matching Pennies

	L	R
U	b_U $a_L + c_L$	$b_U + d_U$ a_R
D	$b_D + d_D$ a_L	b_D $a_R + c_R$

U: up **D**: down
L: left **R**: right

Player 1's payoff in lower-left corner
Player 2's payoff in upper-right corner

$a_L, a_R, b_U, b_D \in \mathbb{R}$
$c_L, c_R, d_U, d_D > 0$

Belief responsiveness (B3′) requires that belief distributions shift in the same direction as changes in the opponent's action. To capture this idea, we use a notion of first-order stochastic dominance (FOSD). The notion is slightly stronger than the usual one in that we require a strict inequality for interior beliefs, which helps to reduce multiplicity of equilibria.

Unbiasedness (B4′) imposes that beliefs are unbiased on *median*. We favor median over mean unbiasedness for use in the applications we pursue in this paper, but we note that mean unbiasedness is consistent with (B4′) and the other axioms and therefore could be imposed in addition. In subsection ID, we discuss how both (B4′) and its mean-based counterpart can be microfounded via a model of sampling.

Example: The NBE of Generalized Matching Pennies.—Before defining NBE for normal-form games in the next section, we introduce it using our leading example: the family of generalized matching pennies games. Consisting of all 2×2 games with unique fully mixed NE, the NBE of these games take a simple form.

Generalized matching pennies is defined by the payoff matrix in Table 1, and we use Γ^m to refer to an arbitrary game in this family. The parameters a_L, a_R, b_U, and b_D give the base payoffs. The parameters c_L, c_R, d_U, and d_D are the payoff differences, which we assume are strictly positive to maintain the relevant features.[9] Since each player has only two pure actions in Γ^m, we identify Δ_i with $[0,1]$ and Δ with $[0,1]^2$. We also use σ_U and σ_L for actions: the probabilities of playing U and L, respectively. Note that the NE $\left(\sigma_U^{NE}, \sigma_L^{NE}\right) = \left(d_D/(d_U + d_D), c_R/(c_L + c_R)\right)$ depends only on the payoff differences.

We assume that players best respond to every belief realization. Player 1, who forms beliefs over player 2's action σ_L, must choose U when her realized belief is $\sigma_L' > \sigma_L^{NE}$ and D when her realized belief is $\sigma_L' < \sigma_L^{NE}$. In the event that her realized belief is $\sigma_L' = \sigma_L^{NE}$, she is indifferent and so may randomize arbitrarily between U and D. Similarly, player 2, who forms beliefs over player 1's action σ_U, must choose

in $\mu_k^i\big(\{0\}|r\big)$ and $\mu_k^i\big((0,\epsilon)|r\big)$ as $r \to 0^+$, which jump from 0 to 1 and 1 to 0, respectively. More generally, letting $A, B_k \subset [0,1]$ be Borel subsets, there is a discontinuity in $\mu_k^i(B_k|r)$ as $r \to 0^+$ if and only if $B_k = \{0\} \cup A$, where $\mathrm{cl}(A) \cap \{0\} = \emptyset$ or $B_k = (0,\epsilon) \cup A$ where $A \cap \{0\} = \emptyset$.

[9] Games in which the payoff differences are all strictly negative are equivalent up to the labeling of actions. We borrow this notation for generalized matching pennies from Selten and Chmura (2008).

L when her belief is $\sigma'_U < \sigma_U^{NE}$, R when her belief is $\sigma'_U > \sigma_U^{NE}$, and any mixture when her belief is $\sigma'_U = \sigma_U^{NE}$.

For any action of player k, player i draws her beliefs from a distribution according to the belief map and best responds to every belief realization. Player i's overall behavior is the *expected action* determined by integrating her best response correspondence with respect to the measure of beliefs. Thus, the belief map plus the behavioral rule of best response induces a mapping from player k's action to player i's action, which we call player i's *expected best response correspondence* or, simply, *reaction correspondence*.

Interior full support $(\mathrm{B}1')$ and the best response structure of Γ^m make the form of the reaction correspondences particularly simple. Since the belief that makes player i indifferent is interior, the probability of holding such a belief is zero from $(\mathrm{B}1')$. Hence, player 1's reaction correspondence Ψ_U is a single-valued function of σ_L indicating the probability with which U is a best response to realized beliefs, and player 2's reaction Ψ_L is defined similarly:

$$\Psi_U(\sigma_L) = 1 - F_2^1(\sigma_L^{NE}|\sigma_L),$$

$$\Psi_L(\sigma_U) = F_1^2(\sigma_U^{NE}|\sigma_U).$$

Note that $(\mathrm{B}1')$ also implies that each of these reactions is interior if and only if the opponent's action is interior, which will imply interior equilibria in Γ^m.

Continuity $(\mathrm{B}2')$ implies that $\Psi = (\Psi_U, \Psi_L) : [0,1]^2 \to [0,1]^2$ is continuous in (σ_U, σ_L). Notice that this is despite the existence of belief discontinuities, as described in Remark 1. This is because the reactions depend only on the probability beliefs realize in any of the "relevant" sets, the largest which induce unique best responses (e.g., $[0, \sigma_L^{NE})$ and $(\sigma_L^{NE}, 1]$ for player 1), and these probabilities are continuous in the opponent's behavior. This issue of belief discontinuities will require special care when generalizing to arbitrary games, but we will still find that the reactions are continuous in generic games. It is also easy to show that $(\mathrm{B}1')$ and $(\mathrm{B}2')$ together imply that Ψ is jointly continuous in (σ_U, σ_L) and the payoff parameters, which will ensure that equilibria do not jump for small changes in the game.

Belief responsiveness $(\mathrm{B}3')$ implies that $\Psi_U(\sigma_L)$ and $\Psi_L(\sigma_U)$ are strictly increasing and strictly decreasing, respectively. For player 1 (and similarly for player 2), as σ_L increases, the belief distribution over the probability that player 2 moves L shifts to the right. Since the expected payoff to U increases (and the payoff to D decreases) in this belief, player 1 must choose U with a higher probability. That the reactions are strictly monotonic implies a unique equilibrium in Γ^m.

Unbiasedness $(\mathrm{B}4')$ implies that belief realizations over- and underestimate the opponent's action equally often. Hence, if player k is playing the indifferent action that equalizes the (objective) expected utility to both of player i's actions, the probability of taking either action is exactly one-half. We note that replacing $(\mathrm{B}4')$ with *mean* unbiasedness would place no restriction on player i's reaction when k plays the indifferent action.

For a given belief map σ^*, NBE is defined as a fixed point of Ψ. This describes a situation in which each player best responds to belief realizations whose distribution are centered on the opponent's expected action, which is similarly induced by best responding to realized beliefs.

DEFINITION 2: *Fix* $\{\Gamma^m, \sigma^*\}$. *An NBE is any* $(\sigma_U, \sigma_L) \in [0,1]^2$ *such that* $\Psi_U(\sigma_L) = \sigma_U$ *and* $\Psi_L(\sigma_U) = \sigma_L$.

Fixing any belief map σ^*, an NBE of Γ^m exists since the reaction Ψ is continuous (*continuity*). The NBE is unique since Ψ_U and Ψ_L are strictly monotonic (*responsiveness*), and it is interior since Ψ_U and Ψ_L are interior if and only if the opponent's action is interior (*interior full support*), and there cannot be a fixed point on the boundary since this would be a pure strategy NE, which does not exist. Moreover, the game's payoff parameters only enter the expression for Ψ through the NE actions, so the NBE only depends on the NE.

THEOREM 1: *Fix* $\{\Gamma^m, \sigma^*\}$. (i) *An NBE exists and is unique and interior.* (ii) *The NBE only depends on the NE of* Γ^m $($*for fixed* $\sigma^*)$.

PROOF:

See Appendix A. ∎

NBE in Normal-Form Games.—We generalize NBE to normal-form games. To this end, we adapt the framework of RBE (Friedman and Mezzetti 2005) but restrict the belief distributions to satisfy axioms in order to impose testable restrictions on the data. The general axioms nest their binary action counterparts.

Given player k's action $\sigma_k \in \Delta_k$, player i's belief over k's action is given by random vector $\sigma_k^{i*}(\sigma_k) = \left(\sigma_{k1}^{i*}(\sigma_k), \ldots, \sigma_{kJ(k)}^{i*}(\sigma_k)\right)$ that is supported on Δ_k. We call this family of random vectors player i's *belief map* over player k's action. For convenience, refer to all players' belief maps by $\sigma^* = \left(\sigma_k^{i*}\right)_{i,k\neq i}$, and for all $\sigma_{-i} \in \Delta_{-i}$, define belief maps over i's opponents' actions by $\sigma_{-i}^*(\sigma_{-i}) = \left(\sigma_k^{i*}(\sigma_k)\right)_{k\neq i}$.

For each $\sigma_k \in \Delta_k$, $\sigma_k^{i*}(\sigma_k)$ is defined by probability measure $\mu_k^i(\cdot|\sigma_k)$ on $\mathcal{B}(\Delta_k)$, the Borel σ-algebra on Δ_k, which gives the probability that beliefs are realized in any $B_k \in \mathcal{B}(\Delta_k)$. Assume that all of k's opponents draw their beliefs about k independently conditional on σ_k and that player i's beliefs about any two of her opponents are drawn independently conditional on their actions. Thus, for all $\sigma_{-i} \in \Delta_{-i}$, $\sigma_{-i}^*(\sigma_{-i})$ is associated with the product measure: $\mu_{-i}(B|\sigma_{-i}) = \prod_{k\neq i} \mu_k^i(B_k|\sigma_k)$ for any $B = \times_{k\neq i} B_k \in \otimes_{k\neq i}\mathcal{B}(\Delta_k) = \mathcal{B}(\Delta_{-i})$.

Define the *ij-response set* $R_{ij} \subseteq \Delta_{-i}$ by

$$R_{ij} = \left\{\sigma'_{-i} : \bar{u}_{ij}(\sigma'_{-i}) \geq \bar{u}_{ik}(\sigma'_{-i}), \quad \forall k = 1, \ldots, J(i)\right\}. \tag{1}$$

This defines the set of beliefs about i's opponents for which action a_{ij} is a best response. A *strategy* for player i is a measurable function $s_i = \left(s_{i1}, \ldots, s_{iJ(i)}\right)$: $\Delta_{-i} \to \Delta_i$, where for all $\sigma'_{-i} \in \Delta_{-i}$, $s_{ij}(\sigma'_{-i}) \geq 0$ and $\sum_{j=1}^{J(i)} s_{ij}(\sigma'_{-i}) = 1$. This maps any realized belief to an action. Strategy s_i is *rational* if it only puts positive probability on best responses: $s_{ij}(\sigma'_{-i}) = 0$ if $\sigma'_{-i} \notin R_{ij}$.

Given any $\sigma_{-i} \in \Delta_{-i}$, player i's belief map σ^*_{-i} induces a distribution over her realized beliefs and thus also over her actions through her strategy s_i. Integrating i's strategy via the measure $\mu_{-i}(\cdot\,|\,\sigma_{-i})$ gives an expected action. Restricting attention to rational strategies in which player i best responds to realized beliefs, we define player i's *expected best response correspondence* or *reaction correspondence* as

$$\Psi_i(\sigma_{-i};\sigma^*_{-i}) = \left\{ \int_{\Delta_{-i}} s_i(\sigma'_{-i})\, d\mu_{-i}(\sigma'_{-i}\,|\,\sigma_{-i}) : s_i \text{ is rational} \right\}. \tag{2}$$

This maps the opponents' action profile σ_{-i} to the set of i's expected actions that can be induced by best responding to realized beliefs.

Correspondence (2) generalizes the best response correspondence of NE, and analogous to NE, NBE is defined as a fixed point of $(\Psi;\sigma^*) = ((\Psi_1;\sigma^*_{-1}),\ldots,(\Psi_n;\sigma^*_{-n})) : \Delta \rightrightarrows \Delta$. Note that while the belief distributions depend on the opponents' expected actions, the dependence is arbitrary without additional restrictions on σ^*.

We generalize the binary action axioms (B1′)–(B4′) to allow for arbitrary numbers of actions. Generalized *interior full support* requires that the support of every belief distribution is precisely the set of beliefs that assign positive probability to the opponent's pure actions that are played with positive probability, and only those actions. Generalized *continuity* requires that beliefs vary as continuously as possible given the restrictions imposed by *interior full support*. As in the binary action case, these technical axioms are necessary to accommodate our behavioral axioms but imply that the belief distributions involve belief discontinuities associated with opponents' actions nearby the boundary. However, as in the matching pennies example, the reactions of which NBE is a fixed point will be continuous in generic games (and upper hemicontinuous for all games).

To state the technical axioms, we use additional notation. For any $\sigma_k \in \Delta_k$, define $\Delta(\sigma_k) = \{\sigma'_k \in \Delta_k : \text{supp}(\sigma'_k) = \text{supp}(\sigma_k)\}$ as the subset of Δ_k in which each element is a probability vector with the same support as σ_k (i.e., has zeros in precisely the same components as σ_k). For example, if $\sigma_k = (0,1/2,1/2)$, then $\Delta(\sigma_k) = \{(0,p,1-p) : p \in (0,1)\}$. Let $\langle \Delta_k, \mathcal{B}(\Delta_k), \mathcal{L}_k\rangle$ be the Lebesgue probability space on Δ_k where $\mathcal{L}_k$ is the Lebesgue measure. For each σ_k, we also define the probability space $\langle \Delta(\sigma_k), \mathcal{B}(\Delta(\sigma_k)), \mathcal{L}_k^{\Delta(\sigma_k)}\rangle$, where $\mathcal{B}(\Delta(\sigma_k))$ is the Borel σ-algebra on $\Delta(\sigma_k)$ and $\mathcal{L}_k^{\Delta(\sigma_k)}$ is the Lebesgue measure on $\Delta(\sigma_k)$. Since $\sigma'_k \in \Delta(\sigma_k)$ implies $\Delta(\sigma'_k) = \Delta(\sigma_k)$, this defines only finite probability spaces. Note that if σ_k has 0 in some component and $A \in \mathcal{B}(\Delta(\sigma_k))$, then $\mathcal{L}_k(A) = 0$. For example, if $\sigma_k = (0,1/2,1/2)$, $\mathcal{L}_k(\Delta(\sigma_k)) = 0$ even though $\mathcal{L}_k^{\Delta(\sigma_k)}(\Delta(\sigma_k)) = 1$. Our technical axioms follow.

(B1) **Interior full support**: $\mu^i_k(B_k|\sigma_k) > 0$ if and only if $\mathcal{L}_k^{\Delta(\sigma_k)}(B_k \cap \Delta(\sigma_k)) > 0$.

(B2) **Continuity**: Let $\{\sigma^t_k\} \subset \Delta_k$ be a sequence with $\sigma^t_k \to \sigma^\infty_k$ as $t \to \infty$. $\lim_{t\to\infty} \mu^i_k(B_k|\sigma^t_k) = \mu^i_k(B_k|\sigma^\infty_k)$ if for sufficiently large t, either (*i*) $\{\sigma_k^t\} \subset \Delta(\sigma_k^\infty)$ or (*ii*) $\mathrm{cl}(B_k \cap \Delta(\sigma_k^t)) \cap \Delta(\sigma_k^\infty) = B_k \cap \Delta(\sigma_k^\infty)$ up to $\mathcal{L}_k^{\Delta(\sigma_k^\infty)}$-measure 0.[10]

Interior full support (B1) says that if the opponent's action is $\sigma_k \in \Delta_k$, the probability beliefs realize in B_k will be positive if and only if there is a nontrivial intersection of B_k with $\Delta(\sigma_k)$, which is the set of beliefs that assign positive probability to the opponent's pure actions that are played with positive probability and only those actions. The "if" direction requires that the support of the belief distribution is $\Delta(\sigma_k)$. The "only if" direction requires that the belief measure is absolutely continuous with respect to $\mathcal{L}_k^{\Delta(\sigma_k)}$, which is the Lebesgue measure defined on the support of the belief distribution. In particular, this rules out atoms unless the opponent plays an action with probability one, in which case beliefs are correct with probability one.

Continuity (B2) specifies pairs $(B_k, \{\sigma_k^t\})$, where B_k is a set of realized beliefs and $\{\sigma_k^t\}$ is a sequence of the opponent's action such that the probability beliefs realize in the set is continuous along the sequence. The axiom is best understood by contrast with a more standard notion. It is similar to requiring that for any sequence $\sigma_k^t \to \sigma_k^\infty$ and Borel set B_k, $\lim_{t\to\infty} \mu_k^i(B_k|\sigma_k^t) = \mu_k^i(B_k|\sigma_k^\infty)$, which is simply strong convergence of $\mu_k^i(\cdot|\sigma_k^t)$ to $\mu_k^i(\cdot|\sigma_k^\infty)$. This is the technical condition assumed in Friedman and Mezzetti (2005). However, this is incompatible with *interior full support*, which we require for the behavioral axioms. Hence, we weaken this condition by allowing for discontinuities associated with some $(B_k, \{\sigma_k^t\})$-pairs. Whereas in the one-dimensional (binary action) case, *interior full support* implies discontinuities when the opponent's action approaches the boundary (see Remark 1), the analogue for higher dimensions is when the opponent's action "gains zeros" in the limit, i.e., puts positive probability on fewer pure actions. Axiom (B2)-(*i*) states that if $\{\sigma_k^t\} \subset \Delta(\sigma_k^\infty)$ for sufficiently large t, meaning σ_k^t does not gain zeros in the limit, there are no discontinuities for any B_k. If σ_k^t does gain zeros in the limit, then there necessarily will be discontinuities for some B_k since the probability that beliefs realize in $\Delta(\sigma_k^\infty)$ goes from 0 to 1 by *interior full support*. However, (B2)-(*ii*) states that there is no discontinuity if $\mathrm{cl}(B_k \cap \Delta(\sigma_k^t)) \cap \Delta(\sigma_k^\infty) = B_k \cap \Delta(\sigma_k^\infty)$, meaning that the part of B_k in $\Delta(\sigma_k^t)$ "overlaps" with the part of B_k in $\Delta(\sigma_k^\infty)$.

By construction, belief discontinuities can only arise when the overlapping condition (B2)-(*ii*) fails.[11] In the binary action case, it is easy to characterize failures of the overlapping condition and thus rewrite (B2) for this special case.[12] To provide

[10] That is, set equality may only hold as $\{\mathrm{cl}(B_k \cap \Delta(\sigma_k^t)) \cap \Delta(\sigma_k^\infty)\} \cup c_1 = \{B_k \cap \Delta(\sigma_k^\infty)\} \cup c_2$ for some $c_1, c_2 \in \Delta(\sigma_k^\infty)$ with $\mathcal{L}_k^{\Delta(\sigma_k^\infty)}(c_1) = \mathcal{L}_k^{\Delta(\sigma_k^\infty)}(c_2) = 0$.

[11] (B2)-(*i*) is actually redundant since it implies (B2)-(*ii*), which is immediate after noting that $\{\sigma_k^t\} \subset \Delta(\sigma_k^\infty)$ implies $\Delta(\sigma_k^t) = \Delta(\sigma_k^\infty)$.

[12] *Continuity* (B2) *when* $J(k) = 2$: Consider a sequence with $r \to 0^+$. For $B_k \in \{\{0\}, (0,\epsilon)\}$, the overlapping condition fails: for $B_k = \{0\}$, $\mathrm{cl}(\{0\} \cap (0,1)) \cap \{0\} = \emptyset$ and $\{0\} \cap \{0\} = \{0\}$, and for $B_k = (0,\epsilon)$, $\mathrm{cl}((0,\epsilon) \cap (0,1)) \cap \{0\} = \{0\}$ and $(0,\epsilon) \cap \{0\} = \emptyset$. For $B_k = [0,\epsilon)$, the overlapping condition is satisfied: $\mathrm{cl}([0,\epsilon) \cap (0,1)) \cap \{0\} = \{0\}$ and $[0,\epsilon) \cap \{0\} = \{0\}$. Given these results, it is easy to show that (B2) becomes: (i) $\mu_k^i(B_k|r)$ is continuous for all $r \in (0,1)$, (ii) $\lim_{r\to 0^+} \mu_k^i(B_k|r) = \mu_k^i(B_k|0)$ for any $B_k = [0,\epsilon) \cup A$, (iii) $\lim_{r\to 0^+} \mu_k^i(B_k|r) = \mu_k^i(B_k|0)$ if $B_k \cap [0,\epsilon) = \emptyset$ for some $\epsilon > 0$, (iv) $\lim_{r\to 1^-} \mu_k^i(B_k|r) = \mu_k^i(B_k|1)$ for any $B_k = A \cup (\epsilon, 1]$, and (v) $\lim_{r\to 1^-} \mu_k^i(B_k|r) = \mu_k^i(B_k|1)$ if $B_k \cap (1-\epsilon, 1] = \emptyset$ for some $\epsilon > 0$.

intuition for (B1) and (B2) in higher dimensions, online Appendix A gives examples for the case of three pure actions.

To state the behavioral axioms, we define the *marginal belief distribution* (CDF) by $F^i_{kj}(\bar{\sigma}_{k0}|\sigma_k) = \mu^i_k(\{\sigma'_k \in \Delta_k : \sigma'_{kj} \in [0, \bar{\sigma}_{k0}]\}|\sigma_k)$. This gives the probability that player i believes player k plays action a_{kj} with probability weakly less than $\bar{\sigma}_{k0} \in [0,1]$ as a function of $\sigma_k \in \Delta_k$. *Belief responsiveness* requires that player i's jth marginal belief distribution shifts up as the probability that the opponent plays the corresponding action increases. *Unbiasedness* requires that the marginal belief distributions are unbiased on median.

(B3) **Belief responsiveness**: If for some j, σ_k and σ'_k satisfy $\sigma_{kj} < \sigma'_{kj}$ and $\sigma_{kl} \geq \sigma'_{kl}$ for all $l \neq j$, then $F^i_{kj}(\bar{\sigma}_{k0}|\sigma'_k) < F^i_{kj}(\bar{\sigma}_{k0}|\sigma_k)$ for $\bar{\sigma}_{k0} \in (0,1)$.

(B4) **Unbiasedness**: $F^i_{kj}(\sigma_{kj}|\sigma_k) = 1/2$ for σ_k with $\sigma_{kj} \in (0,1)$.

The general axioms nest their binary action counterparts. When $J(k) = 2$, it is immediate that (B1), (B3), and (B4) collapse to (B1′), (B3′), and (B4′), respectively. That (B2) collapses to (B2′) is less obvious but becomes clear once (B2) is rewritten for the binary action case (see footnote 12).

REMARK 2: *If* $J(k) = 2$, *(B1)–(B4) are equivalent to (B1′)–(B4′).*

Several other axioms come to mind as natural and, in fact, will be satisfied by our parametric model.[13] However, we only impose (B1)–(B4) to derive our theoretical results:

ASSUMPTION 3: *The belief map* σ^*_i *satisfies (B1)–(B4).*

DEFINITION 3: *Fix* $\{\Gamma, \sigma^*\}$. *An NBE is any* $\sigma \in \Delta$ *such that for all* $i \in 1, \ldots, n$, $\sigma_i \in \Psi_i(\sigma_{-i}; \sigma^*_{-i})$.

From Assumption 3 and the fact that the R_{ij} sets are closed in Δ, it can be shown that $\Psi : \Delta \rightrightarrows \Delta$ is upper hemicontinuous. Existence of NBE follows from standard arguments.

THEOREM 2: *Fix* $\{\Gamma, \sigma^*\}$. *An NBE exists.*

PROOF:

See Appendix A. ∎

[13] One is *belief monotonicity,* in which the distribution of $\sigma^{i*}_{kj}(\sigma_k)$ first-order stochastically dominates the distribution of $\sigma^{i*}_{kl}(\sigma_k)$ if $\sigma_{kj} > \sigma_{kl}$. Another is *label independence*, in which $\sigma^{i*}_{kl}(\sigma_k)$ and $\sigma^{i*}_{kj}(\sigma_k)$ have the same distribution if $\sigma_{kj} = \sigma_{kl}$, and if σ_k and σ'_k are the same up to permutation of components, then $\sigma^{i*}_{kj}(\sigma_k)$ has the same distribution as $\sigma^{i*}_{k\iota(j)}(\sigma'_k)$, where $\iota : \{1, \ldots, J(k)\} \to \{1, \ldots, J(k)\}$ is the permutation mapping.

In general, Ψ may not be single valued. In fact, Ψ is multivalued if and only if a player can be indifferent between two best responses with positive probability, i.e., if $\mu_{-i}(R_{ij} \cap R_{il} | \sigma_{-i}) > 0$ for some $\sigma_{-i} \in \Delta_{-i}$. Since *interior full support* requires that beliefs are correct with probability one when the opponents take pure actions, this necessarily occurs when a_{ij} and a_{il} are best responses to some pure action profile a_{-i}. In games without such actions, however, *interior full support* implies that Ψ is single valued. Intuitively, holding fixed beliefs over the actions of all opponents except one, indifference requires holding very particular mixed beliefs about the remaining opponent, and the probability of holding such mixed beliefs is zero no matter the opponent's action.

LEMMA 1: *Fix* $\{\Gamma, \sigma^*\}$. *If* $u_i(a_{ij}, a_{-i}) \neq u_i(a_{il}, a_{-i})$ *for all* i, $a_{ij} \neq a_{il}$, *and* a_{-i}, *then* Ψ *is single valued.*

PROOF:

See Appendix A. ∎

Since Ψ is upper hemicontinuous, the lemma implies that Ψ is a continuous function for generic games.

D. *Relationship to Other Concepts*

Rationalizability.—The theory of rationalizability (Bernheim 1984 and Pearce 1984) finds strategy profiles that cannot be ruled out on the basis of rationality and common knowledge of rationality alone, recognizing that these are not enough to form correct beliefs as required in NE. NBE is a compromise between NE and rationalizability in that it acknowledges the difficulty in forming correct beliefs and yet pins down distributions over beliefs and actions. What's more, NBE is a refinement of rationalizability in the following sense.

LEMMA 2: *If* $\sigma \in \Delta$ *is an NBE, then* $a_{ij} \in \operatorname{supp}(\sigma_i)$ *is rationalizable for all i and j.*

PROOF:

See Appendix A. ∎

The result follows from the fact that players best respond to all belief realizations, and all belief realizations assign positive probability only to pure actions that are played with positive probability by *interior full support* (B1). QRE, on the other hand, does not respect rationalizability, as *interiority* (A1) requires that all pure actions are played with positive probability. For example, in the prisoner's dilemma, NBE coincides with the unique NE, whereas the only restriction of QRE is that each player plays the dominant strategy with some probability strictly between one-half and one.

Random Belief Equilibrium.—NBE adopts the idea of the belief map and equilibrium condition from RBE (Friedman and Mezzetti 2005). The difference between the models lies in the restrictions imposed on the belief distributions, which are tailored for different purposes. Whereas we introduce NBE as a tool for understanding the testable restrictions of equilibria with belief noise that is "bounded away from zero," Friedman and Mezzetti (2005) use RBE for equilibrium selection and hence study the limiting case as belief noise "goes to zero." Specifically, they consider belief measures that converge weakly to the opponents' action profile. Along the sequence, the restrictions they impose on belief distributions are (i) full support and absolute continuity with respect to Lebesgue measure $(\mu_k^i(B_k|\sigma_k) > 0$ if and only if $\mathcal{L}_k(B_k) > 0)$ and (ii) a strong notion of continuity $(\mu_k^i(B_k|\sigma_k)$ is continuous in $\sigma_k \in \Delta_k)$.[14] The only restrictions imposed by these conditions are that weakly dominated actions are played with zero probability and undominated actions are played with positive probability. In particular, RBE does not respect rationalizability, as players must expect, incorrectly, that their opponents play never best responses. NBE's technical axioms (B1) and (B2) neither nest nor are nested in the RBE conditions. In particular, the RBE conditions imply that the belief map cannot be unbiased.[15]

Sampling Equilibrium.—Osborne and Rubinstein (2003) introduce sampling equilibrium in which players are frequentists who form beliefs by observing m-length samples of pure actions drawn from each opponent's equilibrium mixed action. Sampling from k's mixed action gives a multinomial with parameters m and $\sigma_k = (\sigma_{k1}, \ldots, \sigma_{kJ(k)})$; dividing the count data by m gives the corresponding belief distribution. Since no sample involves actions that are not played with positive probability and the variance of the sampling belief distribution goes to zero as the opponent puts increasing probability on a pure action, NBE's technical axioms capture belief formation that has a sampling flavor. Moreover, even though the sampling belief distribution is discrete, it is easy to show that it respects *belief responsiveness* and, in large samples, is approximately *unbiased* on both median and mean.[16] Hence, one can regard NBE as a generalized and "smoothed" sampling model that captures some of the key properties of sampling in reduced form. Our results therefore have implications for the empirical content of sampling models and their generalizations, such as stochastic learning equilibrium (Goeree and Holt 2002).[17]

[14] Friedman and Mezzetti (2005) also allow for belief distributions to have finite atoms, so these restrictions only apply to the absolutely continuous part of the belief measures.

[15] RBE's full support condition implies that the belief distribution must be biased if the opponent's action is on the boundary of the simplex, and thus, RBE's continuity condition implies that there must be bias for some interior actions as well. One implication is that belief distributions will necessarily be biased in equilibrium when the opponent has a dominated action.

[16] The jth marginal of the sampling distribution is binomial with parameters m and σ_{kj}; dividing the count data by m gives the corresponding marginal belief distribution. From results on the binomial distribution (e.g., Kaas and Buhrman 1980): (i) If $m\sigma_{kj}$ is an integer, then the unique (strong) median belief is $M = \sigma_{kj}$. (ii) If $m\sigma_{kj}$ is not an integer, then any (weak) median belief M satisfies $\lfloor m\sigma_{kj} \rfloor / m \leq M \leq \lceil m\sigma_{kj} \rceil / m$ due to discreteness; the bounds contain σ_{kj} and get arbitrarily tight as $m \to \infty$.

[17] SLE is a generalization of sampling equilibrium that allows for alternative "learning rules" that map histories of observed actions into beliefs as well as alternative "choice rules" that map these beliefs into actions. In equilibrium, the distribution of histories is consistent with the learning and choice rules jointly, and so randomness in beliefs is driven by randomness in observed histories. One can think of NBE as an SLE variant that maintains the choice rule of best response but allows for any learning rule that results in belief distributions with certain properties.

REMARK 3: *By* interior full support, *beliefs are correct with probability one when opponents play a pure action. Thus*, *any pure strategy NE is also an NBE*, *which implies the existence of games for which there are multiple NBE for any belief map. On the other hand*, *for any game*, *there exists a quantal response function for which the QRE is unique. This is true even for games with multiple pure strategy NE. For example*, *the logit QRE is always unique for sufficiently small* λ (*McKelvey and Palfrey 1995*). *In this sense*, *NBE is more like NE and could be paired with standard refinements such as trembling hand perfection.*

II. Within-Game Restrictions and the Own Payoff Effect

Contrary to the predictions of NE in fully mixed games, experimental studies report two regularities. First, whereas NE predicts that players' choice probabilities keep their opponents indifferent, there are systematic deviations within a game: the empirical frequency of actions typically leads to a ranking of actions for each player by expected payoffs to which they noisily best respond. Second, whereas NE predicts that a change in a player's own payoffs does not affect her equilibrium behavior, subjects exhibit the "own payoff effect."

The best evidence for these regularities comes from generalized matching pennies (see, for example, Ochs 1995; McKelvey, Weber, and Palfrey 2000; and Goeree and Holt 2001), and QRE is well known for capturing both effects in this context (Goeree, Holt, and Palfrey 2005). In this section, we show that NBE also captures both effects. Thus, these empirical patterns can be explained equally well by adding noise to actions or adding noise to beliefs without relying on any specific functional form.

We first show that NBE imposes the same testable restrictions as QRE for any individual matching pennies game and hence captures deviations from NE equally well.

THEOREM 3: *Fix* Γ^m. *The set of attainable NBE is equal to the set of attainable QRE.*

PROOF:

See Appendix A. ∎

The theorem states that any QRE that can be attained for some quantal response function satisfying (A1)–(A4) can be attained as an NBE for some belief map satisfying (B1′)–(B4′) and vice versa. The intuition is simple.

When player k takes the mixed action that equates the expected utilities to player i's actions, player i will take each action with one-half probability under both models. This follows from *monotonicity* (A4) in a QRE and *unbiasedness* (B4′) in an NBE (beliefs are equally likely to realize on either side of the indifferent belief). As player k's mixed action increases, then one of player i's actions increases in expected utility (while decreasing for the other). Player i will now play this action with probability greater than one-half in a QRE by *responsiveness* (A3) as well as in an NBE by *belief responsiveness* (B3′) (the belief distribution shifts up, so she is more likely to favor this action).

Table 2—Matching Pennies X

	L	**R**
U	$X, 0$	$0, 1$
D	$0, 1$	$1, 0$

In online Appendix B, we derive the set of attainable NBE for any matching pennies game, which corresponds to the set of QRE by Theorem 3. The following example, which illustrates such a set, was derived in Goeree, Holt, and Palfrey (2005) for QRE; we rederive the set using NBE.

EXAMPLE 1: *Let* $X > 0$. *In the game of Table* 2, (σ_U, σ_L) *is an NBE* (*QRE*) *if and only if*

$$\begin{cases} \sigma_U <(=) \ \frac{1}{2} & \text{if } \sigma_L <(=) \ \frac{1}{1+X} \\ \sigma_U >(=) \ \frac{1}{2} & \text{if } \sigma_L >(=) \ \frac{1}{1+X} \end{cases} \quad \text{and} \quad \begin{cases} \sigma_L >(=) \ \frac{1}{2} & \text{if } \sigma_U <(=) \ \frac{1}{2} \\ \sigma_L <(=) \ \frac{1}{2} & \text{if } \sigma_U >(=) \ \frac{1}{2}. \end{cases}$$

PROOF:

Suppose (σ_U, σ_L) is an NBE. By (B4′), the probability player 1 plays U when player 2 is playing $\sigma_L = 1/(1+X)$ (the action that makes player 1 indifferent) is exactly $\sigma_U = 1/2$. By (B3′), if $\sigma_L < 1/(1+X)$, the probability player 1 plays U is strictly less than $1/2$. The other inequalities are similar. Conversely, the flexibility in constructing the belief maps allows any (σ_U, σ_L) satisfying the inequalities to be attained as an NBE.[18] ∎

For any $X > 0$, the set of attainable NBE (QRE) is given by the inequalities in Example 1. The left panel of Figure 1 plots this set when $X = 4$ as a gray rectangle, in which case only 15 percent of outcomes are consistent with the model. A representative NBE is plotted as the intersection of reaction functions. Player 1's reaction must be strictly increasing in σ_L (*belief responsiveness*) and pass through the point $(\sigma_U, \sigma_L) = (1/2, 1/(1+X))$ (*unbiasedness*) as well as the corners of the square.[19] Similarly, player 2's reaction must be strictly decreasing in σ_U and pass through $(\sigma_U, \sigma_L) = (1/2, 1/2)$. These are the only restrictions on the reaction functions, and hence any (σ_U, σ_L) satisfying the inequalities can be attained in an NBE.

In the right panel of Figure 1, we illustrate the set of attainable NBE in which *unbiasedness* is modified so that beliefs are unbiased on mean instead of median (which we generalize to any matching pennies game in online Appendix B). The reaction function for player 1 must be increasing, fall between the upward-sloping lines, and include the corners of the square, with a similar condition for player 2.

[18] The inequalities define the set of outcomes that are *monotonic* in the sense of QRE axiom (A4), so any such outcome can be attained as a QRE and thus also as an NBE from Theorem 3.

[19] By *interior full support*, beliefs are correct with probability one when the opponent is playing a pure action to which a pure action is the unique best response. The QRE reactions would look qualitatively similar except they would be bounded away from the corners by *interiority*.

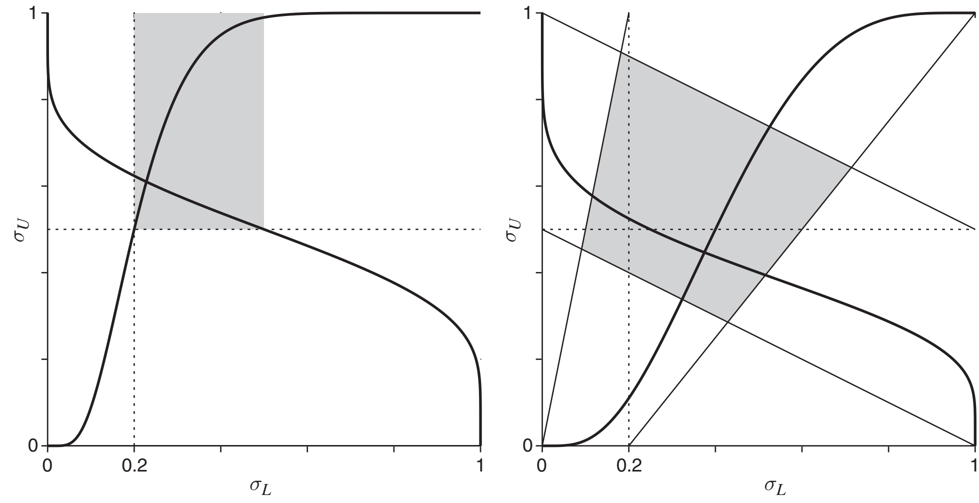

Figure 1. NBE (QRE) in Matching Pennies X

Notes: The left panel plots the set of attainable NBE (QRE) in the game of Table 2 ($X = 4$) as a gray region. The NE is given as the intersection of the best response correspondences (dotted lines), and a representative NBE is given as the intersection of reaction functions (black curves). The right panel plots the set of attainable NBE in which *unbiasedness* is modified so that beliefs are unbiased on mean instead of median.

Note that the reactions are unrestricted at the indifferent action but there are still testable restrictions on equilibria. If beliefs are unbiased on both median and mean, then the set of attainable NBE would be the intersection of the gray regions from the two panels and account for less than 10 percent of possible outcomes.

The next example illustrates the own payoff effect, which in this case is a simple comparative static in player 1's payoff parameter X. NE predicts that player 1's action does not change with X, as she must mix to keep her opponent indifferent, but empirical evidence suggests a different pattern that is well known to be explained by QRE (Goeree, Holt, and Palfrey 2005). We now show that NBE makes the same prediction. This is not a corollary of Theorem 3, which only concerns individual games.

EXAMPLE 2: *Let $X > 0$. In the NBE (QRE) of the game in Table 2, σ_U is strictly increasing in X, and σ_L is strictly decreasing in X.*

PROOF:

Fix σ^*. The NBE of this game is given as the unique fixed point

$$\sigma_U = \Psi_U(\underset{+}{\sigma_L}, \underset{+}{X}), \tag{3}$$

$$\sigma_L = \Psi_L(\underset{-}{\sigma_U}), \tag{4}$$

where $\Psi_U(\sigma_L, X) = 1 - F_2^1(1/(1+X)|\sigma_L)$ and $\Psi_L(\sigma_U) = F_1^2(1/2|\sigma_U)$. From (B1′) and (B3′), $\Psi_U(\sigma_L, X)$ is strictly increasing in both arguments, and $\Psi_L(\sigma_U)$ is strictly decreasing in σ_U. From (4), as X increases, it must be that either σ_U

Table 3—A 3 × 3 game with a Matching Pennies Core

	L	**R**	**R′**
U	4, 0	0, 1	0, −1
D	0, 1	1, 0	0, −1
D′	−1, 0	−1, 0	Z, −1

increases and σ_L decreases, σ_U decreases and σ_L increases, or both σ_U and σ_L remain constant. The latter two cases are impossible since (3) implies that as X increases, σ_U increases if σ_L is constant or increases. Thus, as X increases, σ_U must strictly increase and σ_L must strictly decrease. ∎

Example 2 generalizes to any matching pennies game.[20] For more general fully mixed games, however, NBE and QRE typically require more structure than is provided by the basic axioms to make unambiguous predictions in response to changing a single payoff parameter.

Our next example combines previous results to make the simple point that while NBE and QRE can make similar predictions, this depends crucially on the structure of the game. In particular, NBE's relationship to nonrationalizable actions is very different.

EXAMPLE 3: *The game of Table* 3 *is constructed from the game in Figure* 2 $(X = 4)$ *by giving each player an additional action. Here*, R' *is strictly dominated*, *and* D' *is either strictly dominated* $(for\ Z < 0)$ *or iteratively dominated after deleting* R' $(for\ Z > 0)$. *After removing* R' *and* D', *the reduced game is a standard matching pennies game. NBE respects rationalizability*, *so it is immediate that the set of attainable NBE for this game equals the set of NBE in the reduced game as depicted in the left panel of Figure 1*, *a result that holds for all values of* Z. *On the other hand*, *QRE predicts that both* D' *and* R' *are played with positive probability and that behavior is sensitive to changes in* Z.

The types of predictions from Examples 1 and 2 find strong support in data, and we have shown that they are explained equally well by noise in actions or noise in beliefs. By contrast, Example 3 suggests an experiment (varying Z in the game of Table 3) in which the two types of noise imply starkly different predictions.

III. The Effects of Payoff Magnitude

It is important to understand the effects of payoff magnitude in games. Indeed, games played in the lab are often meant to emulate their real-world counterparts but are typically played at much lower stakes.

[20] For any Γ^m, varying any one of player i's payoff parameters will shift her NBE or QRE reaction curve in some direction, say "up," which leads to an unambiguous comparative static.

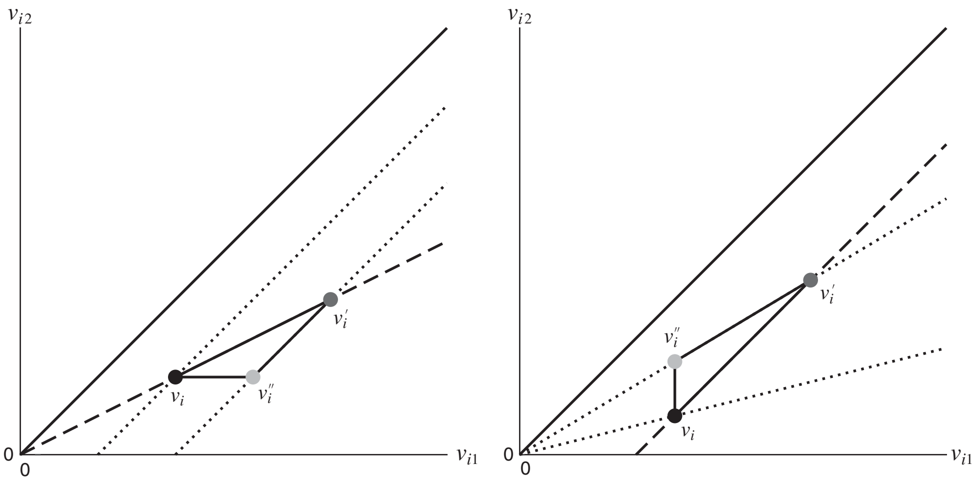

FIGURE 2. PROOF OF LEMMA 3

Notes: The left panel plots some isoquantal response curves (dotted lines) under translation invariance and illustrates the method of projection used in part (i). The right panel gives the analogous plot for scale invariant quantal response that is used in part (ii).

In applications of QRE, it is common to assume that the quantal response function takes the familiar logit form. When parameter λ is chosen to best explain data from individual games, the fit is often very good. However, for fixed λ, logit implies considerable sensitivity to scaling one or more players' payoffs by positive constants, and these predictions typically find little support. For instance, McKelvey, Weber, and Palfrey (2000) find no evidence for scale effects in generalized matching pennies games. Since equilibria vary continuously with payoffs, this "scaling issue" points to a more general difficulty in explaining behavior across games.

In this section, we establish that the scaling issue of logit is general to all QRE in the sense that if QRE is to explain the empirical regularities discussed in Section II, it must be nontrivially sensitive to scaling and/or translating payoffs. For the class of translation invariant QRE, which includes logit and, more generally, all structural QRE with i.i.d. errors, sensitivity to scale is inescapable. By contrast, we show that NBE, which can also explain the empirical regularities, is invariant to both scaling and translation.

To study QRE's properties, we begin by analyzing the quantal response functions directly before extending the results to games. Goeree, Holt, and Palfrey (2005) defined a notion of translation invariance for quantal response functions. We present their definition along with an analogous notion of scale invariance, which, for technical reasons, can only be defined for strictly positive utility vectors.[21] Quantal response is *translation invariant* if it is unaffected by shifting

[21] Consider the utility vector $v_i = (1,0,\ldots,0) \in \mathbb{R}^{J(i)}$. *Responsiveness* implies that $Q_{i1}(v_i) < Q_{i1}(\beta v_i)$ for $\beta > 1$, and hence, no quantal response function can be scale invariant over the entire domain $\mathbb{R}^{J(i)}$.

payoffs while keeping payoff differences fixed. Quantal response is *scale invariant* if it is unaffected by multiplying payoffs by positive constants while keeping payoff ratios fixed.

Translation Invariance: $Q_i(v_i) = Q_i(v_i + \gamma e_{J(i)})$ for all $v_i \in \mathbb{R}^{J(i)}$ and $\gamma \in \mathbb{R}$, where $e_{J(i)} = (1, \ldots, 1) \in \mathbb{R}^{J(i)}$ is a vector of ones.

Scale Invariance: $Q_i(v_i) = Q_i(\beta v_i)$ for all $v_i \in \mathbb{R}^{J(i)}_{++}$ and $\beta > 0$.

For some results, we introduce an additional condition called *weak substitutability*, which requires that an action is played more often when the payoffs to all other actions are weakly lowered. Though not implied by regularity alone, the condition is extremely weak: satisfied by all structural QRE with i.i.d. errors and implied by *responsiveness* when $J(i) = 2$.

Weak Substitutability: $Q_{ij}(v_i) > Q_{ij}(v_i')$ whenever $v_{ij} \geq v_{ij}'$ and $v_{ik} \leq v_{ik}'$ for all $k \neq j$ with strict inequality for some k.

An example of a quantal response function that is translation invariant but not scale invariant is logit: $Q_{ij}(v_i; \lambda) = e^{\lambda v_{ij}} / \sum_{k=1}^{J(i)} e^{\lambda v_{ik}}$, where parameter $\lambda \in [0, \infty)$ controls the sensitivity to payoff differences. More generally, Goeree, Holt, and Palfrey (2005) show that structural quantal response functions are translation invariant under weak conditions. An example of a quantal response function that is scale invariant but not translation invariant is the Luce model (Luce 1959) for strictly positive payoffs: $Q_{ij}(v_i; \mu) = v_{ij}^{\frac{1}{\mu}} / \sum_{k=1}^{J(i)} v_{ik}^{\frac{1}{\mu}}$, where parameter $\mu \in (0, \infty)$ controls the sensitivity to payoff ratios.

Hence, there exist quantal response functions that are translation invariant and those that are scale invariant. However, we show in Lemma 3 that no quantal response function satisfies both properties. In particular, for translation invariant Q_i, scale increases lead to increasing sensitivity: the high payoff actions are played with greater probability. For scale invariant Q_i, positive translations lead to diminishing sensitivity: the high payoff actions are played with smaller probability.

For simplicity, we give the result in the binary action case, whose proof has a simple geometry that we plot in Figure 2. In online Appendix C, we generalize the result to arbitrary numbers of actions with the additional assumption of weak substitutability.

LEMMA 3: *Fix* $J(i) = 2$, *and let* $v_i \in \mathbb{R}^2_{++}$ *be such that* $v_{i1} > v_{i2}$.

(*i*) *Let* Q_i *be translation invariant and* $\beta > 1$. *Then,* $Q_{i1}(\beta v_i) = Q_{i1}(v_{i1} + \delta(\beta), v_{i2}) > Q_{i1}(v_i)$, *where* $\delta(\beta) > 0$ *is strictly increasing in* β *and* $\lim_{\beta \to \infty} \delta(\beta) = \infty$.

(*ii*) *Let* Q_i *be scale invariant and* $\gamma > 0$. *Then,* $Q_{i1}(v_i + \gamma e_2) = Q_{i1}(v_{i1}, v_{i2} + \delta(\gamma)) < Q_{i1}(v_i)$, *where* $\delta(\gamma) > 0$ *is strictly increasing in* γ *and* $\lim_{\gamma \to \infty} \delta(\gamma) = v_{i1} - v_{i2} > 0$.

PROOF:

(i): Take any $v_i \in \mathbb{R}_{++}$ and translation invariant Q_i. Referring to the left panel of Figure 2, scaling by $\beta > 1$ causes a shift along the dashed line to $v_i' = \beta v_i$. By translation invariance of Q_i, v_i and v_i' are on different isoquantal response curves (dotted 45° lines). Define v_i'' as the projection of v_i' along its isoquantal response curve onto the horizontal line passing through v_i. This point is $v_i'' = \left(v_{i1} + \delta(\beta), v_{i2}\right)$, where $\delta(\beta) = (\beta - 1)(v_{i1} - v_{i2}) > 0$ is strictly increasing in β and $\lim_{\beta\to\infty}\delta(\beta) = \infty$. By construction, v_i'' is on the same isoquantal response curve as v_i' and directly to the right of v_i. Thus, $Q_{i1}\left(\underbrace{\beta v_i}_{=v_i'}\right) = Q_{i1}\left(\underbrace{v_{i1} + \delta(\beta), v_{i2}}_{=v_i''}\right) > Q_{i1}(v_i)$, where the inequality follows from *responsiveness* (A3).

(ii): Take any $v_i \in \mathbb{R}_{++}$ and scale invariant Q_i. Referring to the right panel of Figure 2, translating by $\gamma > 0$ causes a shift along the dashed line to $v_i' = v_i + \gamma e_2$. By scale invariance of Q_i, v_i and v_i' are on different isoquantal response curves (dotted rays through the origin). Define v_i'' as the projection of v_i' along its isoquantal response curve onto the vertical line passing through v_i. This point is $v_i'' = \left(v_{i1}, v_{i2} + \delta(\gamma)\right)$, where $\delta(\gamma) = \frac{v_{i1}}{v_{i1} + \gamma}(v_{i2} + \gamma) - v_{i2} > 0$ is strictly increasing in γ and $\lim_{\gamma\to\infty}\delta(\gamma) = v_{i1} - v_{i2} > 0$. By construction, v_i'' is on the same isoquantal response curve as v_i' and directly above v_i. Thus, $Q_{i1}\left(\underbrace{v_i + \gamma e_2}_{=v_i'}\right) = Q_{i1}\left(\underbrace{v_{i1}, v_{i2} + \delta(\gamma)}_{=v_i''}\right) < Q_{i1}(v_i)$, where the inequality follows from *responsiveness* (A3). ∎

Lemma 3 shows that quantal response cannot be invariant to both scale and translation, but it does not rule out translation invariant quantal response functions with very weak scale effects and vice versa. However, for translation (scale) invariant quantal response, the lemma shows that scaling (translating) payoffs has the same effect on quantal response as an increase in the payoff to some action. Intuitively, this implies that, if translation (scale) invariant quantal response is nearly insensitive to scale (translation), it must be nearly insensitive to differences in payoffs. This is formalized in the following corollary..

COROLLARY 1:

(*i*) *Let* $H(a,b,c) = \left\{(v_{i1}, v_{i2}) \in \mathbb{R}^2_{++} | v_{i1} \in [a,b], v_{i2} = c\right\}$ *for* $0 < c < a < b$ *be the set of payoff vectors on a horizontal line segment. Let* Q_i *be translation invariant and* $\beta > 1$. *There exists* $K^* > 0$ *such that for any* $\epsilon > 0$, $|Q_{i1}(\beta v_i) - Q_{i1}(v_i)| < \epsilon$ *for all* $v_i \in H(a,b,c)$ *implies* $|Q_{i1}(v_i') - Q_{i1}(v_i'')| < K^*\epsilon$ *for all* $v_i', v_i'' \in \left\{(v_i + \gamma e_2) | v_i \in H(a,b,c), \gamma \in \mathbb{R}\right\}$.

(*ii*) *Let* $V(a,b,c) = \left\{(v_{i1}, v_{i2}) \in \mathbb{R}^2_{++} | v_{i1} = c, v_{i2} \in [a,b]\right\}$ *for* $0 < a < b < c$ *be the set of payoff vectors on a vertical line segment. Let* Q_i *be scale invariant and* $\gamma > 0$. *There exists* $K^* > 0$ *such that for any*

$\epsilon > 0$, $|Q_{i1}(v_i + \gamma e_2) - Q_{i1}(v_i)| < \epsilon$ *for all* $v_i \in V(a,b,c)$ *implies* $|Q_{i1}(v_i') - Q_{i1}(v_i'')| < K^* \epsilon$ *for all* $v_i', v_i'' \in \{\beta v_i | v_i \in V(a,b,c), \beta > 0\}$.

PROOF:

See Appendix A. ∎

Part (i) states that if a translation invariant quantal response function is nearly insensitive to scaling all payoff vectors on a horizontal line segment, it assigns roughly the same quantal response to those payoff vectors and any translation of those vectors. Part (ii) states that if a scale invariant quantal response function is nearly insensitive to translating all payoff vectors on a vertical line segment, it assigns roughly the same quantal response to those payoff vectors and any scaling of those vectors. Hence, for translation (scale) invariant quantal response, insensitivity to scale (translation) on a small set of zero measure implies insensitivity to differences in payoffs over a large set of positive measure. If insensitivity to scale (translation) is extended to all payoff vectors,[22] quantal response must assign uniform probabilities to all actions, independent of payoffs. Hence, to explain the empirical regularities of Section II, QRE must be nontrivially sensitive to some type of affine transformation.

REMARK 4: *Corollary 1 generalizes what is known of logit QRE, for which λ controls both sensitivity to scale and sensitivity to payoff differences and, at one extreme* ($\lambda = 0$), *assigns uniform probabilities to all actions*, *independent of payoffs.*[23]

We now extend our results to games. To this end, we define families of games that differ only in affine transformations of payoffs.

DEFINITION 4: *Fix* $\Gamma = \{N, A, u\}$.

(*i*) *The* scale family $\mathcal{S}(\Gamma)$ *is the set of games* Γ' *such that* $N' = N$; $A' = A$; *and for all i*, *there exists* $\beta_i > 0$ *such that* $u_i' = \beta_i u_i$.

(*ii*) *The* translation family $\mathcal{T}(\Gamma)$ *is the set of games* Γ' *such that* $N' = N$; $A' = A$; *and for all i and* $a_{-i} \in A_{-i}$, *there exists* $\gamma_i(a_{-i}) \in \mathbb{R}$ *such that* $\bar{u}_{ij}'(a_{-i}) = \bar{u}_{ij}(a_{-i}) + \gamma_i(a_{-i})$ *for all j.*

(*iii*) *The* affine family $\mathcal{A}(\Gamma)$ *is the set of games* Γ' *such that* $N' = N$; $A' = A$; *and for all i and* $a_{-i} \in A_{-i}$, *there exists* $\beta_i > 0$ *and* $\gamma_i(a_{-i}) \in \mathbb{R}$ *such that* $\bar{u}_{ij}'(a_{-i}) = \beta_i \bar{u}_{ij}(a_{-i}) + \gamma_i(a_{-i})$ *for all j.*

[22] As $a \to c^+$ and $b \to \infty$, $\{(v_i + \gamma e_2) | v_i \in H(a,b,c), \gamma \in \mathbb{R}\}$ approaches $\{v_i \in \mathbb{R}^2 | v_{i1} > v_{i2}\}$. As $a \to 0^+$ and $b \to c^-$, $\{\beta v_i | v_i \in V(a,b,c), \beta > 0\}$ approaches $\{v_i \in \mathbb{R}^2_{++} | v_{i1} > v_{i2}\}$.

[23] Similarly, in the Luce model, μ controls both sensitivity to translation and sensitivity to payoff ratios, and at one extreme ($\mu = \infty$), it assigns uniform probabilities to all actions, independent of payoffs.

Theorem 4 extends the generalization of Lemma 3 (online Appendix C) to the QRE of games.

THEOREM 4: *Fix* $\{\Gamma, Q\}$.

(*i*) *If Q is translation* (*scale*) *invariant*, *the set of QRE is the same for all* $\Gamma' \in \mathcal{T}(\Gamma)$ $(\Gamma' \in \mathcal{S}(\Gamma))$.

(*ii*) *Let Q be weakly substitutable*, *and suppose* $\sigma \in \Delta$ *is a QRE in which no player is uniformly mixing* (*for all i*, $\sigma_{ij} \neq 1/J(i)$ *for some j*):

(*a*) *If Q is translation invariant*, σ *is not a QRE of* $\Gamma' \in \mathcal{S}(\Gamma)\backslash\Gamma$.
(*b*) *If Q is scale invariant*, σ *is not a QRE of* $\Gamma' \in \mathcal{T}(\Gamma)\backslash\Gamma$.

PROOF:
See Appendix A. ∎

The results of this section suggest that by augmenting QRE with translation or scale invariance as an additional axiom, we may sharpen comparative static predictions, i.e., predictions that hold across games for a given quantal response function. To this end, in online Appendix D, we provide an example set of games for which QRE makes an ambiguous comparative static prediction but the ambiguity is resolved by assuming translation or scale invariance. More generally, in online Appendix E, we derive necessary conditions for a dataset from sets of binary action games to be consistent with QRE for some quantal response function under the additional maintained assumptions of translation or scale invariance, respectively. This is done by extending the method of projection used in Lemma 3. Our result takes the form of inequalities that the empirical choice probabilities must satisfy. Melo, Pogorelskiy, and Shum (2018) derive a similar result for structural QRE in arbitrary games under additional maintained assumptions using results from convex analysis, and when the games have binary actions, our translation invariant inequalities coincide with theirs.

Unlike QRE, NBE is invariant to affine transformations of the game, which is no more than a simple observation. It follows from the fact that the best response structure of a game is unaffected by affine transformations.

THEOREM 5: *Fix* $\{\Gamma, \sigma^*\}$. *The set of NBE is the same for all* $\Gamma' \in \mathcal{A}(\Gamma)$.

PROOF:
See Appendix A. ∎

REMARK 5: *The key assumption implicit in Theorems 4 and 5 is that the model primitive*, *Q or* σ^*, *is held fixed across games. Importantly*, *however*, *if Q is generated via the structural approach with additive errors*, *fixing Q does not imply the "invariance" assumption—that the underlying error distribution is independent of* *payoffs* v_i (*see, for example, Haile, Hortaçsu, and Kosenok 2008*). *Theorem 4 therefore applies to a very broad class of QRE models.*[24]

IV. Logit Transform NBE

For applications, we introduce a parametric model based on the logit transform. In this section, we consider the case of binary actions, and we give its generalization to normal-form games in online Appendix F. The binary action model satisfies the axioms, while the general model introduces a small bias in belief distributions.

When actions are binary, player k's action is $r \in [0,1]$, and we derive player i's belief map through the following procedure:

(i) Map $r \in [0,1]$ to the extended real line via the *logit transform* $\mathcal{L}(r) = \ln(r/(1-r))$, using the convention that $\mathcal{L}(0) = -\infty$ and $\mathcal{L}(1) = \infty$.

(ii) Add $\tau\varepsilon_i$ to $\mathcal{L}(r)$, where $\varepsilon_i \sim_{iid} \mathcal{N}(0,1)$ and $\tau \in (0,\infty)$.

(iii) Map $\mathcal{L}(r) + \tau\varepsilon_i$ back to $[0,1]$ via the inverse logit transform

$$r^*(r;\tau) = \mathcal{L}^{-1}(\mathcal{L}(r) + \tau\varepsilon_i) = \frac{\exp\left(\ln\left(\frac{r}{1-r}\right) + \tau\varepsilon_i\right)}{1 + \exp\left(\ln\left(\frac{r}{1-r}\right) + \tau\varepsilon_i\right)}.$$

The parameter τ is the standard deviation of the noise added to the logit transformed action and thus can be interpreted as the "noisiness" of beliefs. This belief map admits a closed form CDF[25]

$$F_k^i(\bar{r}|r;\tau) = \Phi\left(\frac{1}{\tau}\left[\ln\left(\frac{\bar{r}}{1-\bar{r}}\right) - \ln\left(\frac{r}{1-r}\right)\right]\right), \tag{5}$$

which we derive in online Appendix G. We also plot the CDF and PDF for different values of τ and r and show that the model satisfies axioms (B1′)–(B4′).

V. Extensions: Trembles and Extensive-Form Games

We consider two extensions. First, since the basic NBE model makes extreme predictions in that nonrationalizable actions are played with probability zero, we consider a variant in which players make uniform trembles in taking actions. This makes the theory a statistical one in the sense that all datasets have positive likelihood, and it allows for a better qualitative match to data in certain classes of games. Second, to study extensive-form games, we introduce agent NBE, which applies

[24] Beyond applying to all structural QRE models with i.i.d. errors and many nonstructural models, the theorem also applies to structural models like that of Friedman (2020) for which Q is regular and translation invariant despite being generated via errors whose variance depends on v_i.

[25] To make the CDF well defined, we resolve indeterminacies as follows: $-\infty - (-\infty) = \infty$ and $\infty - \infty = \infty$. As is standard, we also need $\Phi(-\infty) = 0$ and $\Phi(\infty) = 1$.

NBE to the agent normal form of the game and imposes a version of sequential rationality.

A. *Trembles*

With probability $\epsilon \in [0,1]$, player i trembles and chooses a pure action uniformly randomly. Otherwise, player i best responds to her belief realization. The probability that player i takes action j is given by $\Psi^{\epsilon}_{ij}(\sigma_{-i};\sigma^*_{-i}) = \epsilon(1/J(i)) + (1-\epsilon)\Psi_{ij}(\sigma_{-i};\sigma^*_{-i})$, where Ψ_{ij} is as in (2), and equilibrium is similarly defined as a fixed point.

We make two observations. First, while this is no longer purely a model of noisy beliefs, the noise in actions is not sensitive to payoffs. For this reason, the model maintains invariance to affine transformations and is fundamentally different from QRE. Second, despite the additional degree of freedom from ϵ, the model is still falsifiable. In fact, for some games, such as for any matching pennies game Γ^m, the additional parameter has no effect on the set of attainable equilibria.[26]

More generally, we may expect effects, especially as Lemma 2 no longer holds for $\epsilon > 0$. For instance, in the game of Table 3 with $Z > 0$, player 2 takes the strictly dominated action R' with probability $\epsilon/3$, and player 1 takes iteratively deleted action D' with some probability strictly greater than $\epsilon/3$ (one-third of the time when trembling and with positive probability otherwise). However, holding fixed the belief map and Z, behavior converges to the $\epsilon = 0$ case as $\epsilon \to 0^+$. Hence, for sufficiently small ϵ, NBE with trembles predicts a complete ordering of all action frequencies for this game. In particular, there is a separation between rationalizable, iteratively deleted, and strictly dominated actions: $\sigma_U, \sigma_D, \sigma_L, \sigma_R > \sigma_{D'} > \sigma_{R'}$.

To take a starker example, we consider the traveler's dilemma (Basu 1994). In the experimental variant of Goeree and Holt (2001), each of two players simultaneously chooses integers between 180 and 300, inclusive. Both players are paid the lower of the two numbers, and in addition, an amount $R > 1$ is transferred from the player with the higher number to the player with the lower number. For all values of R, the unique rationalizable outcome, and thus the unique NBE, is for both players to choose 180. However, with trembles, the NBE can be very different. For simplicity, we numerically solve for the symmetric logit transform NBE for the values $R \in \{180,5\}$ considered by Goeree and Holt (2001). We plot the predicted distribution of choices in Figure 3 for $(\tau,\epsilon) = (8,0.25)$, which provides a close match both qualitatively and quantitatively to the data.[27] For $R = 180$, choices are skewed to the left of the interval, whereas for $R = 5$, it is just the opposite.

The intuition is simple. If player i believes that her opponent is going to choose some integer with probability one, then i's unique optimal action is to "undercut" her opponent by an increment of one, and this holds for any value of R. With

[26] This is because allowing for trembles only shrinks reactions toward $1/2$, and the reactions can already be made arbitrarily close to $1/2$ whenever the opponent's action is interior, as it must be in equilibrium.

[27] For $R = 180$ and $R = 5$, the mean choices predicted by NBE with trembles are 196 and 276, respectively, which closely match the values of 201 and 280 in the data. Unlike the NBE prediction, however, there is a sizable mass of subjects that choose exactly 300 when $R = 5$. This cannot occur in an NBE with trembles, since for no belief is 300 the optimal choice.

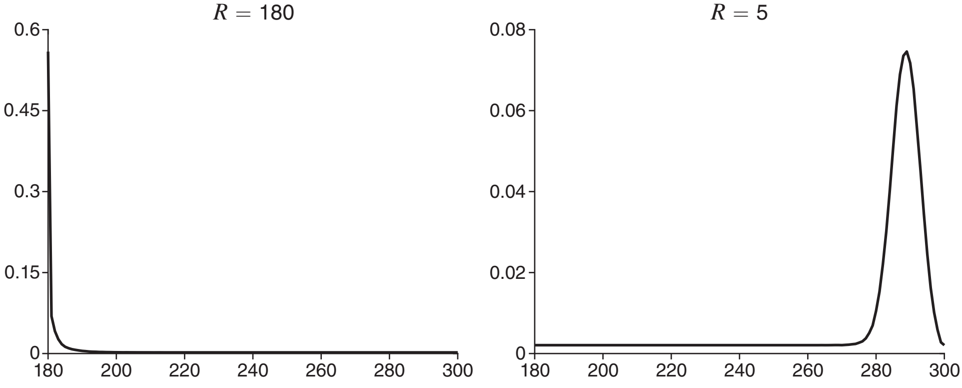

Figure 3. NBE with Trembles in the Traveler's Dilemma

Notes: This figure plots the predicted distribution of choices under logit transform NBE with trembles for $(\tau, \epsilon) = (8, 0.25)$. The left panel is for $R = 180$ and the right panel is for $R = 5$.

trembles, however, player i holds a nondegenerate belief with probability one by *interior full support*, in which case her optimal action balances the desire to choose a low number to undercut her opponent with the desire to choose a high number in case her opponent does too. The value of undercutting increases in R, which gives rise to the comparative static that we observe. Logit QRE is also capable of closely matching the data (Goeree and Holt 2001).

B. *Extensive-Form Games*

We apply NBE with trembles to extensive-form games. To do so, we introduce agent NBE, which applies NBE to the agent normal form of the game and imposes a version of sequential rationality. This closely mirrors the agent QRE of McKelvey and Palfrey (1998), which is defined analogously. A complete treatment is beyond the scope of this paper, so rather than defining the formal apparatus, we consider a few examples for which the concept is clear.

Consider the two-stage trust game of Goeree and Holt (2001) in Figure 4. Player 1 chooses S or R; S ends the game, and R gives player 2 the opportunity to choose P or N. Since each player has a single information set, the "agent" part of agent NBE is not relevant. The sequential rationality part requires that at each information set, players best respond to each of their belief realizations, and so the game can be solved by backward induction. First, we consider the case that $X = Y = 0$ without trembles. Player 2, conditional on reaching her information set, has no beliefs to form and so chooses N to maximize her payoff. Given that player 2 chooses N with probability one, player 1 knows this and so chooses R to maximize her payoff. Hence, agent NBE coincides with the subgame perfect NE.

With trembles, player 2 will choose N unless she trembles: $(\sigma_N, \sigma_P) = (1 - \epsilon/2, \epsilon/2)$. Now that player 2's action is nondegenerate, player 1 forms noisy beliefs. For some belief realizations, S is optimal, and for others, R is optimal. Thus, with trembles, it must be that $\min\{\sigma_S, \sigma_R\} > \epsilon/2$. For ϵ sufficiently

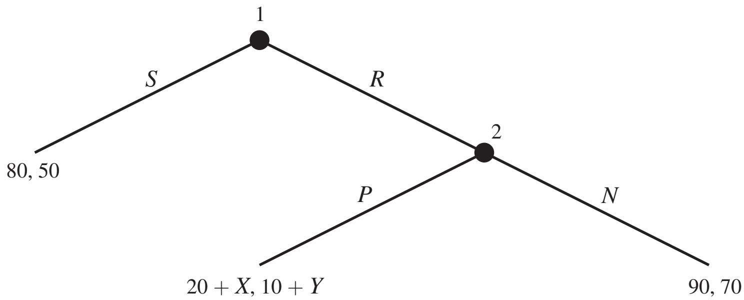

Figure 4. Two-Stage Trust Game

small ($\epsilon < 2/7$), most beliefs will induce an action of R, but for ϵ sufficiently large ($\epsilon > 2/7$), most beliefs will induce S. Thus, for small ϵ, we have $1 - \epsilon/2 = \sigma_N > \sigma_R > 1/2 > \sigma_S > \sigma_P = \epsilon/2$, and for large ϵ, we have $1 - \epsilon/2 = \sigma_N > \sigma_S > 1/2 > \sigma_R > \sigma_P = \epsilon/2$. The ordering for small ϵ is consistent with the Goeree and Holt (2001) data.

Goeree and Holt (2001) also consider the same game for $Y = 58$ so that player 2 still prefers N to P but the difference in payoffs is much smaller. They find that, consistent with the QRE prediction, player 2 is more likely to choose P than before and player 1 is more likely to choose S. Agent NBE with trembles cannot capture this effect since the increase in Y has no effect on player 2's behavior, and thus, player 1's beliefs and behavior remain unchanged as well. Though Goeree and Holt (2001) did not consider it, it is interesting to contrast this with the effect of an increase in X. Agent NBE with trembles predicts that while this has no effect on player 2's behavior, player 1 will take R more often, as this would expand the set of beliefs over player 2's action for which R yields a higher payoff than S.

Next, we consider the centipede game (Binmore 1987), focusing on the four-move, experimental variant of McKelvey and Palfrey (1992) shown in Figure 5. This game features two players who alternate in choosing to *take* (T) or *pass* (P). A prize is to be divided between the players. Taking ends the game and ensures that she who takes receives a larger share of the prize; passing increases the size of the prize but gives the opponent the next take/pass decision. The unique subgame perfect NE involves taking at every opportunity, with the outcome being *take* at the very first node. However, this game is often cited as a classic example of the failure of backward induction: experimental studies show considerable amounts of passing and that the rate of *take* increases monotonically with each node.

For convenience, we have labeled the nodes A, B, C, and D, where player 1 moves at A and C and player 2 moves at B and D. Agent NBE treats the agent at each node as a separate player. The agent at A forms noisy beliefs about the actions at B, C, and D, the agent at B forms noisy beliefs about the actions at C and D, the agent at C forms noisy beliefs about the action at D, and the agent at D has no belief to form. Beliefs drawn about actions at any two nodes are drawn independently. As with the game of Figure 4, without trembles, agent NBE coincides with the subgame perfect NE: the agent at D takes, and the game unravels from the end.

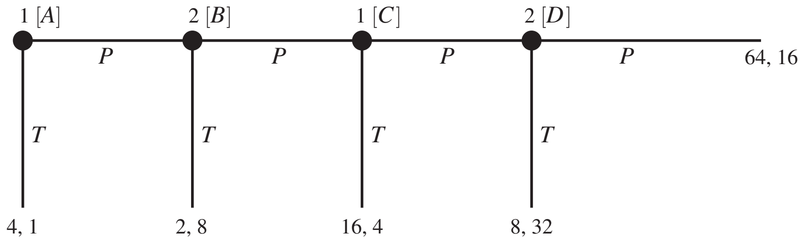

Figure 5. The Centipede Game

With trembles, however, agent NBE can generate the empirically observed pattern of increasing take probabilities.

EXAMPLE 4: *Consider the centipede game of Figure 5. Fix any* $\underline{\epsilon} < \overline{\epsilon} \in (0,1)$. *There exists a belief map (same for all agents) such that for all* $\epsilon \in [\underline{\epsilon}, \overline{\epsilon}]$, *the corresponding agent NBE with trembles satisfies* $\sigma_T^A < \sigma_T^B < \sigma_T^C < \sigma_T^D = 1 - \epsilon/2$.

PROOF:
See Appendix A. ■

We emphasize that while there is a considerable degree of freedom in constructing the belief map for this result, the belief map is restricted to be the same for all agents, and the construction does not depend on ϵ. Agent QRE is also capable of predicting increasing take probabilities (McKelvey and Palfrey 1998), which is unsurprising, as the mechanism is payoff sensitivity, and payoffs increase at later nodes. It would be interesting to explore how the NBE and QRE predictions may diverge in other versions of the centipede game, such as the constant sum variant (Fey, McKelvey, and Palfrey 1996).

VI. Analysis of Experimental Data

We consider data from several studies to test specific qualitative predictions as well as for quantitative measures of fit. We focus on three studies, whose inclusions we motivate on specific grounds.[28] We only consider fully mixed games, for which the basic NBE model assigns positive likelihood to all datasets.

A. *McKelvey, Weber, and Palfrey (2000)*

We include McKelvey, Weber, and Palfrey (2000) in our analysis because their study was designed to test the payoff magnitude predictions of QRE. First, we show that there is no evidence for scale effects, consistent with NBE but not translation

[28] The data sources are McKelvey, Palfrey, and Weber (accessed November 28, 2017); Selten and Chmura (accessed March 13, 2017); and Melo, Pogorelskiy, and Shum (accessed November 3, 2018). These were obtained through personal correspondence.

Table 4—Matching Pennies from McKelvey, Weber, and Palfrey (2000)

	A			*B*			*C*			*D*	
	L	**R**		**L**	**R**		**L**	**R**		**L**	**R**
U	9, 0	0, 1	**U**	9, 0	0, 4	**U**	36, 0	0, 4	**U**	4, 0	0, 1
D	0, 1	1, 0	**D**	0, 4	1, 0	**D**	0, 4	4, 0	**D**	0, 1	1, 0

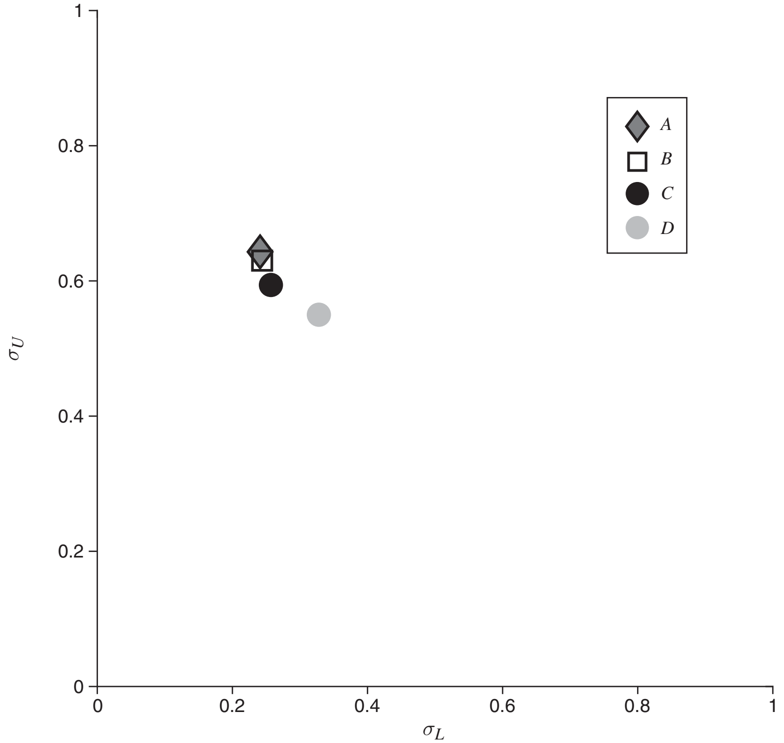

Figure 6. Data from McKelvey, Weber, and Palfrey (2000)

invariant QRE. Second, we show that both QRE and NBE predict the observed deviations from NE within a game and comparative statics unrelated to changes in scale. Third, to grasp the economic significance of these findings, we show that logit transform NBE and logit QRE perform similarly when fit to individual games but NBE outperforms QRE in fitting the pooled data and making out-of-sample predictions. Fourth, we consider risk aversion, which has been proposed to address QRE's oversensitivity to scale (Goeree, Holt, and Palfrey 2003); we find that, allowing for a risk aversion parameter, the gap in performance is much smaller but NBE still outperforms QRE.

Statistical Evidence for Scale Effects.—McKelvey, Weber, and Palfrey (2000) played the generalized matching pennies games in Table 4. Games *A*–*C* are part of the same scale family. Relative to *A*, player 2's payoffs are scaled by four in *B*, and both players' payoffs are scaled by four in *C*. Game *D* is not part of this family.

Table 5—Statistical Tests of Scale Effects

H_o	H_a	Actual	$\lvert t \rvert$	p-value
$\sigma_U^A = \sigma_U^B$	$\sigma_U^A \neq \sigma_U^B$	$0.643 > 0.630$	0.24	0.81
$\sigma_U^A = \sigma_U^C$	$\sigma_U^A \neq \sigma_U^C$	$0.643 > 0.594$	0.89	0.38
$\sigma_U^B = \sigma_U^C$	$\sigma_U^B \neq \sigma_U^C$	$0.630 > 0.594$	0.57	0.57
$\sigma_L^A = \sigma_L^B$	$\sigma_L^A \neq \sigma_L^B$	$0.241 < 0.244$	0.06	0.95
$\sigma_L^A = \sigma_L^C$	$\sigma_L^A \neq \sigma_L^C$	$0.241 < 0.257$	0.27	0.79
$\sigma_L^B = \sigma_L^C$	$\sigma_L^B \neq \sigma_L^C$	$0.244 < 0.257$	0.21	0.83

Note: This table reports the results of t-tests to determine if scale invariance can be rejected. Standard errors are clustered at the subject level.

Figure 6 plots the empirical action frequencies, which are also reported in online Appendix I along with sample sizes. The data from games A–C are very similar, with the data from game D standing out from the rest. This seems consistent with a hypothesis of scale invariance, which requires equilibria to be the same in A–C but allows for differences between D and the others.

Table 5 reports the results of t-tests to determine whether scale invariance can be rejected statistically. Separate tests are run for each pair of games in A–C. Since each subject in the study played a game 50 times, we cluster standard errors at the subject level to account for within-subject correlation between observed actions. In all cases, scale invariance cannot be rejected, with very large p-values ranging from 0.38 to 0.95. In the words of McKelvey, Weber, and Palfrey (2000, 534), there is an "apparent absence of payoff magnitude effects."

Qualitative Predictions.—We now statistically explore other qualitative predictions of NBE and QRE. Table 6 reports the results of standard t-tests of the models' predictions and is adapted from Table 6 of McKelvey, Weber, and Palfrey (2000). Some predictions are about the relative action frequencies across games, and some are predictions about action frequencies within a game relative to the NE benchmark. We label these two kinds of predictions as "*OOS*" for *out of sample* and "*IS*" for *in sample*. We mark the out-of-sample predictions across games A–C with an "*S*" since they are related to changes in scale. We also label in-sample predictions relative to the NE prediction with an "*NE*."

The NBE predictions in Table 6 hold for any belief map satisfying (B1)–(B4). For QRE to make unambiguous comparative static predictions across games A and B and across B and C, it is not enough to assume that the quantal response function satisfies (A1)–(A4), so we derive QRE predictions under the additional assumption of translation invariance following the discussion in Section III.[29] We note that the class of translation invariant QRE contains logit QRE as well as all structural QRE with i.i.d. errors. We have already derived several of the predictions in the table, as games A and D correspond to $X = 9$ and $X = 4$ in Table 2 and games A and B

[29] McKelvey, Weber, and Palfrey (2000) show that these predictions hold for logit QRE, but our results establish that they hold for any translation invariant QRE.

Table 6—Summary of Predictions versus Actual Behavior

Model	Prediction	Type	H_o	H_a	Actual	t	p-value
NBE	$\sigma_U^B > \sigma_U^D$	*OOS*	$\sigma_U^B = \sigma_U^D$	$\sigma_U^B > \sigma_U^D$	$0.630 > 0.550$	0.99	0.16
	$\sigma_U^C > \sigma_U^D$	*OOS*	$\sigma_U^C = \sigma_U^D$	$\sigma_U^C > \sigma_U^D$	$0.594 > 0.550$	0.54	0.30
QRE	$\sigma_U^A > \sigma_U^B$	*OOS* (*S*)	$\sigma_U^A = \sigma_U^B$	$\sigma_U^A > \sigma_U^B$	$0.643 > 0.630$	0.24	0.41
	$\sigma_U^B < \sigma_U^C$	*OOS* (*S*)	$\sigma_U^B = \sigma_U^C$	$\sigma_U^B < \sigma_U^C$	$0.630 > 0.594$	−0.57	0.72
	$\sigma_L^A > \sigma_L^B$	*OOS* (*S*)	$\sigma_L^A = \sigma_L^B$	$\sigma_L^A > \sigma_L^B$	$0.241 < 0.244$	−0.06	0.52
	$\sigma_L^A > \sigma_L^C$	*OOS* (*S*)	$\sigma_L^A = \sigma_L^C$	$\sigma_L^A > \sigma_L^C$	$0.241 < 0.257$	−0.27	0.61
	$\sigma_L^B > \sigma_L^C$	*OOS* (*S*)	$\sigma_L^B = \sigma_L^C$	$\sigma_L^B > \sigma_L^C$	$0.244 < 0.257$	−0.21	0.58
NBE, QRE	$\sigma_U^A > \sigma_U^D$	*OOS*	$\sigma_U^A = \sigma_U^D$	$\sigma_U^A > \sigma_U^D$	$0.643 > 0.550$	1.25	0.11
	$\sigma_L^A < \sigma_L^D$	*OOS*	$\sigma_L^A = \sigma_L^D$	$\sigma_L^A < \sigma_L^D$	$0.241 < 0.328$	1.03	0.15
	$\sigma_L^B < \sigma_L^D$	*OOS*	$\sigma_L^B = \sigma_L^D$	$\sigma_L^B < \sigma_L^D$	$0.244 < 0.328$	0.99	0.16
	$\sigma_L^C < \sigma_L^D$	*OOS*	$\sigma_L^C = \sigma_L^D$	$\sigma_L^C < \sigma_L^D$	$0.257 < 0.328$	0.84	0.20
	$\sigma_U^B > 0.5$	*IS* (*NE*)	$\sigma_U^B = 0.5$	$\sigma_U^B > 0.5$	$0.630 > 0.500$	2.96	0.00
	$\sigma_U^D > 0.5$	*IS* (*NE*)	$\sigma_U^D = 0.5$	$\sigma_U^D > 0.5$	$0.550 > 0.500$	0.72	0.24
	$\sigma_L^A < 0.5$	*IS* (*NE*)	$\sigma_L^A = 0.5$	$\sigma_L^A < 0.5$	$0.241 < 0.500$	6.17	0.00
	$\sigma_L^A > 0.1$	*IS* (*NE*)	$\sigma_L^A = 0.1$	$\sigma_L^A > 0.1$	$0.241 > 0.100$	3.34	0.00
	$\sigma_L^C > 0.1$	*IS* (*NE*)	$\sigma_L^C = 0.1$	$\sigma_L^C > 0.1$	$0.257 > 0.100$	3.65	0.00
	$\sigma_L^D < 0.5$	*IS* (*NE*)	$\sigma_L^D = 0.5$	$\sigma_L^D < 0.5$	$0.328 < 0.500$	2.25	0.02

Notes: This table reports the results of t-tests of model predictions. Standard errors are clustered at the subject level. Positive (negative) t-statistics indicate that the predicted direction of the effect is correct (incorrect).

correspond to $Y = 1$ and $Y = 4$ in Table 11 of online Appendix D. The remaining predictions can be similarly derived, and the set of predictions is nearly exhaustive.[30]

Since the predictions hold for all NBE and all translation invariant QRE, they can be visualized in Figure 7, which plots the sets of logit transform NBE and logit QRE, indexed by parameters τ and λ, respectively. The out-of-sample predictions in the table correspond to all those that can be made unambiguously from the figure, i.e., those that hold for any parameter value (held fixed across the pair of games).

The results are clear. All predictions shared by NBE and QRE are in the correct direction, with most of the in-sample predictions highly significant and the out-of-sample predictions marginally significant. All of the NBE-only predictions are in the correct direction, and only one out of five QRE-only predictions are in the correct direction. While none of the NBE-only or QRE-only predictions are significant at conventional levels, the p-values of the NBE-only predictions (0.16–0.30) are uniformly lower than those of the QRE-only predictions (0.41–0.72). In any case, the qualitative patterns in the data clearly favor NBE over QRE, especially in light of the absence of scale effects documented in Table 5.

Fitting the Data.—We have established the qualitative patterns in the McKelvey, Weber, and Palfrey (2000) data using statistical tests, which seem to favor NBE over translation invariant QRE. So far, we have only used the structure provided by the models' axioms. We now study their parametric forms for quantitative measures of fit.

[30] The out-of-sample predictions in the table constitute every such prediction that can be made unambiguously. The in-sample are the selection chosen by McKelvey, Weber, and Palfrey (2000). Additional such predictions are shared by both NBE and QRE, follow from transitivity of predictions already in the table, and are supported.

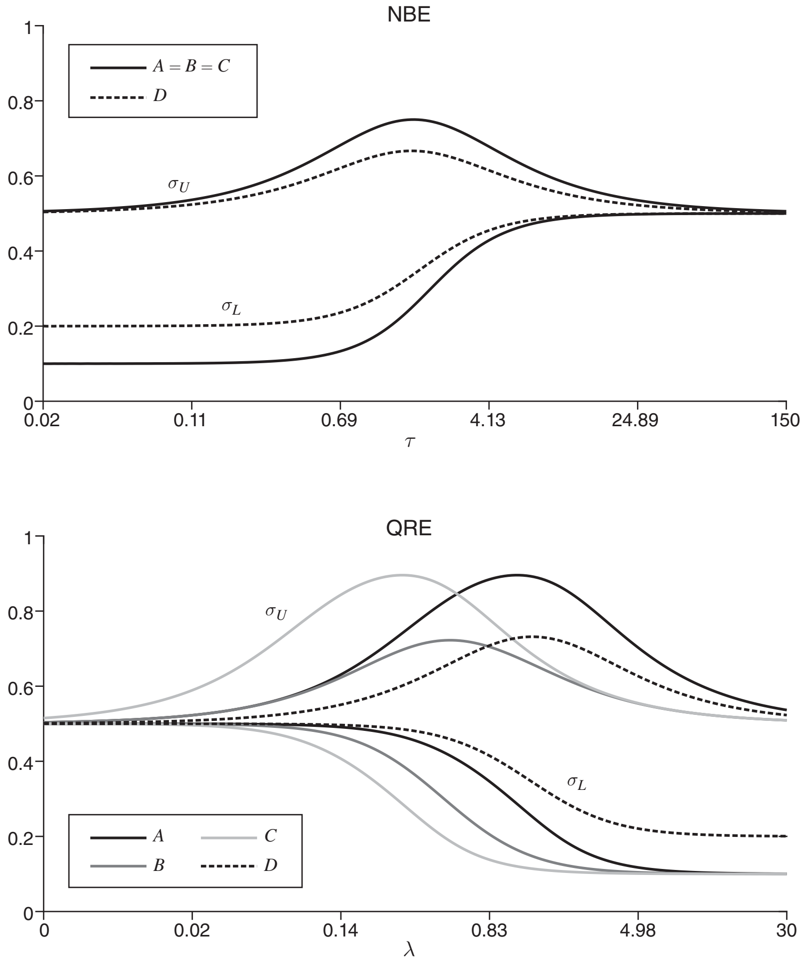

Figure 7. NBE and QRE Correspondences

Note: This figure plots the entire set of logit transform NBE and logit QRE as a function of their parameters for games *A*–*D*.

In Table 7, we compare the performance of the parametric models. Using maximum likelihood, we fit the models to each game individually as well as to the whole set of games pooled together (i.e., with a single parameter). We report the predicted values for each model, the best fit parameters, and the statistics of the Vuong test (Vuong 1989) of which (nonnested) model best matches the data. The log likelihoods, sample sizes, and empirical frequencies are reported in online Appendix I. In this application, the Vuong test somewhat overstates differences in model performance since the asymptotic distribution is calculated under the assumption of i.i.d. observations, which is violated in the data due to within-subject correlation. However, in online Appendix J, we show that our conclusions, based on Vuong and likelihood ratio tests, are robust to "throwing away" a large percentage of the data, which we argue proxies for within-subject correlation in the data-generating process.

Table 7—Summary of Estimates from McKelvey, Weber, and Palfrey (2000)

Game	NBE		QRE		NBE	QRE	
	σ_U	σ_L	σ_U	σ_L	$\hat{\tau}$	$\hat{\lambda}$	Vuong
A	0.75	0.22	0.69	0.11	1.43	5.41	4.4
B	0.75	0.23	0.71	0.22	1.47	0.76	−6.3
C	0.75	0.27	0.63	0.11	1.75	2.01	2.1
D	0.66	0.31	0.59	0.21	1.34	7.30	1.0
Pooled	—	—	—	—	1.47	4.50	5.9

Notes: The Vuong statistic is asymptotically distributed as a standard normal. If it is positive (negative) and exceeds the positive (falls below the negative) $(1-\alpha)$-tail of the standard normal distribution, NBE (QRE) performs best at significance level α.

When the models are fit to each game individually, the models perform very similarly. However, since axiomatic NBE and QRE cannot be distinguished by looking at any game in isolation (Theorem 3), these differences in performance must be related to model structures and should not be interpreted as fundamental. To distinguish the two models, we favor the fit of the games pooled together, and we find that NBE significantly outperforms QRE.

That NBE outperforms QRE in the pooled data is a finding we attribute to scale effects. Games *A*–*C* belong to the same scale family, and the data from these games are very similar. NBE with a single value of τ predicts the same behavior in these games, whereas QRE with a single value of λ predicts widely diverging behavior in *A*–*C*. Hence, while NBE and QRE perform similarly when fit to each game individually, QRE's performance suffers much more by restricting its parameter to be the same across games. This can be seen from likelihood ratio tests of the restriction that each model's parameter is fixed across games. One cannot reject the restriction for NBE ($p = 0.55$), but it is strongly rejected for QRE ($p < 0.001$). Inspecting the λ estimates gives intuition. For instance, game *C* is the same as *A* up to a scale factor of four, so QRE makes the same prediction in these games when $\lambda^A = 4\lambda^C$. Unsurprisingly, $\hat{\lambda}^A$ is much larger than $\hat{\lambda}^C$, and the pooled estimate satisfies $\hat{\lambda}^C < \hat{\lambda}^{pooled} < \hat{\lambda}^A$, implying oversensitivity to payoffs in *C* and undersensitivity in *A*.

That the estimated NBE parameters are more stable across games than the estimated QRE parameters suggests that NBE will outperform QRE in making out-of-sample predictions. In online Appendix H, we show that this is indeed the case.

Risk Aversion.—Risk aversion has been proposed to account for QRE's oversensitivity to scale, which is explored in Goeree, Holt, and Palfrey (2003), who fit logit QRE to games *A*–*D* by jointly estimating λ and a risk aversion parameter. With risk aversion, QRE predicts less sensitivity to scaling monetary payoffs since, holding fixed the opponent's action, scaling a game's monetary payoffs by a factor of four (say) increases expected utility differences by a factor less than four. It is also the case that for general risk-averse preferences, *A*–*C* need not be in the same scale family once expressed in utiles, and hence, NBE may give different predictions in these games for the same value of τ. In online Appendix K, we replicate the

Table 8—Summary of Estimates from Selten and Chmura (2008)

Game	NBE		QRE		NBE	QRE	
	σ_U	σ_L	σ_U	σ_L	$\hat{\tau}$	$\hat{\lambda}$	Vuong
1	0.06	0.68	0.04	0.67	0.98	1.23	11.5
2	0.18	0.50	0.19	0.49	1.06	0.57	−0.4
3	0.15	0.80	0.17	0.79	0.91	1.20	−0.9
4	0.27	0.75	0.28	0.74	0.69	1.10	−3.3
5	0.31	0.67	0.31	0.67	0.54	1.26	−3.6
6	0.42	0.62	0.42	0.61	0.44	1.34	−3.5
7	0.09	0.51	0.11	0.48	1.67	0.45	−1.2
8	0.17	0.56	0.17	0.53	0.75	0.73	3.9
9	0.20	0.88	0.20	0.86	0.36	2.34	−9.2
10	0.27	0.74	0.28	0.74	0.70	1.07	−14.5
11	0.31	0.66	0.31	0.66	0.63	1.09	−3.4
12	0.42	0.62	0.42	0.62	0.41	1.46	−2.3
Pooled	—	—	—	—	0.96	0.99	4.3

exercise from Goeree, Holt, and Palfrey (2003) by fitting both NBE and QRE with constant relative risk-averse (CRRA) utility to the data. We show that with CRRA utility, NBE *is* invariant to scaling monetary payoffs and continues to significantly outperforms QRE.

B. *Selten and Chmura* (*2008*)

For additional evidence, including across games that are not ordered by scale, we consider the study of Selten and Chmura (2008). They collect data from 12 generalized matching pennies games, whose payoffs are chosen systematically so that the NE span a wide region of the unit square and, in contrast to McKelvey, Weber, and Palfrey (2000), no player uniformly mixes.

Table 8 reports the fits of logit transform NBE and logit QRE (log likelihoods, sample sizes, and empirical frequencies are reported in online Appendix I). Unsurprisingly, the performance between the two models is very similar. QRE outperforms NBE in more games individually, but NBE outperforms QRE in the pooled data.

C. *Melo, Pogorelskiy, and Shum* (*2018*)

We wish to compare model performance in games for which players have more than two actions. To this end, we fit logit transform NBE and logit QRE to the three asymmetric 3×3 "joker" games from Melo, Pogorelskiy, and Shum (2018).[31] These are among the simplest fully mixed games with unique regular QRE in which each player has more than two actions.

Though we do not have strong theoretical results for 3×3 games, our hypothesis is that NBE will behave similarly to QRE, and this is indeed the case. The games are in Table 9, and the estimates are in Table 10 (log likelihoods, sample sizes, and

[31] Melo, Pogorelskiy, and Shum (2018) also play a symmetric game, but we omit it from our analysis since the parametric models (as well as NE) predict uniform play for all parameter values.

Table 9—Joker Games from Melo, Pogorelskiy, and Shum (2018)

	2				3				4		
	1	**2**	**J**		**1**	**2**	**J**		**1**	**2**	**J**
1	10, 30	30, 10	10, 30	**1**	25, 30	30, 10	10, 30	**1**	20, 30	30, 10	10, 30
2	30, 10	10, 30	10, 30	**2**	30, 10	25, 30	10, 30	**2**	30, 10	10, 30	10, 30
J	10, 30	10, 30	55, 10	**J**	10, 30	10, 30	30, 10	**J**	10, 30	10, 30	30, 10

Table 10—Summary of Estimates from Melo, Pogorelskiy, and Shum (2018)

		Player						
Game	Action	NBE		QRE		NBE	QRE	
		1	2	1	2	$\hat{\tau}$	$\hat{\lambda}$	Vuong
2	1	0.28	0.40	0.29	0.39			
	2	0.28	0.40	0.29	0.39	1.41	0.22	0.6
	J	0.44	0.20	0.42	0.21			
3	1	0.36	0.31	0.37	0.30			
	2	0.36	0.31	0.37	0.30	1.76	0.15	−3.9
	J	0.29	0.38	0.26	0.41			
4	1	0.39	0.38	0.38	0.39			
	2	0.30	0.23	0.31	0.22	1.00	0.43	−0.2
	J	0.30	0.38	0.31	0.39			
Pooled	—	—	—	—	—	1.30	0.21	−1.4

empirical frequencies are reported in online Appendix I). Not only is every logit transform NBE prediction also a regular QRE,[32] the NBE predictions are very similar to the logit QRE predictions. All three games are of similar scale, and so the pooled fits are also very similar, though QRE performs slightly better. These results suggest that it may be difficult to distinguish noise in actions from noise in beliefs in fully mixed games more generally.

VII. Conclusion

NE fails to explain the richness of experimental data, and many models have been proposed as a result. Whereas QRE relaxes the rationality requirement of NE by allowing for "noise in actions," we introduce NBE, which relaxes the other condition of NE by allowing for "noise in beliefs." In an NBE, axioms restrict belief distributions to be unbiased with respect to and responsive to changes in the opponents' behavior. We compare NBE to regular QRE in which axioms restrict the primitive quantal response function.

We find that NBE explains, just as QRE does, some commonly observed deviations from NE and the own payoff effect. The mechanism whereby QRE achieves

[32] As shown in Melo, Pogorelskiy, and Shum (2019), all regular QRE satisfy $\sigma_{11} = \sigma_{12} \in (0, 1/3)$ and $\sigma_{21} = \sigma_{22} \in (1/3, 9/22]$ in game 2, $\sigma_{11} = \sigma_{12} \in (1/3, 1)$ and $\sigma_{21} = \sigma_{22} \in [4/15, 1/3)$ in game 3, and $\sigma_{12} = \sigma_{1J} \in (0, 1/3)$ and $\sigma_{21} = \sigma_{2J} \in (1/3, 2/5]$ in game 4.

this is an explicit sensitivity to payoffs, which is linked inextricably to a sensitivity to affine transformations of payoffs. By contrast, NBE is invariant to affine transformations, which may be valuable in explaining experimental data. Unlike QRE, NBE respects rationalizability and hence has a fundamentally different relationship to dominated and iteratively deleted actions.

We see two directions for future work. First, to allow better discrimination between the models, one could fit NBE with trembles and compare it to QRE variants in a rich dataset of diverse games, including families of games that differ only in the payoffs to nonrationalizable actions. Second, it would be interesting to collect data to directly test the NBE axioms. One approach, currently being pursued in Friedman and Ward (2019), is to elicit point beliefs multiple times to build empirical belief distributions. Comparing these to the corresponding action frequencies for a set of games that differ only in payoffs allows for testing of both behavioral axioms.

Appendix

A. *Proofs*

PROOF OF THEOREM 1:

(i): $\Psi : [0,1]^2 \rightarrow [0,1]^2$ is a continuous function mapping from a compact and convex set to itself (from $(B2')$ as already shown). By Brouwer's fixed point theorem, there exists a fixed point of Ψ. To show interiority of any fixed points, suppose for purposes of contradiction that some player k is playing $r \in \{0,1\}$ in an NBE. But then, by $(B1')$, player i forms belief $r^*(r) = r$ w.p. 1, to which a pure action $s \in \{0,1\}$ is the only best response. Note that (r,s) cannot be an NBE, since if it were, it would also be an NE, and the game has no pure strategy NE. Thus, all fixed points of Ψ are interior, and we only need to check $(\sigma_U, \sigma_L) \in (0,1)^2$. That the fixed point is unique follows from the fact that $\Psi_U(\sigma_L)$ is strictly increasing in $\sigma_L \in (0,1)$ and $\Psi_L(\sigma_U)$ is strictly decreasing in $\sigma_U \in (0,1)$ by $(B3')$.

(ii): This follows immediately from the expressions for Ψ_U and Ψ_L, which depend on the game's payoff parameters only through the NE actions σ_L^{NE} and σ_U^{NE}. ∎

PROOF OF THEOREM 2:

An NBE is a fixed point of $\Psi : \Delta \rightrightarrows \Delta$. It is trivial to show that Ψ is nonempty and convex valued and that Δ is nonempty, compact, and convex. Existence of NBE follows from Kakutani's fixed point theorem after showing that Ψ_i (and thus Ψ) is upper hemicontinuous. To this end, let $z_i \subset \{1,2,\ldots,J(i)\}$ be an arbitrary, possibly empty, subset of action indices, and define

$$
\begin{aligned}
(6) \qquad r_i(z_i) &= \left\{\sigma'_{-i} \in \Delta_{-i} \,|\, \sigma'_{-i} \in \bigcap_{j \in z_i} R_{ij}, \sigma'_{-i} \notin R_{ik} \text{ for } k \notin z_i\right\}, \\
r_i(\emptyset) &= \emptyset
\end{aligned}
$$

as the set of beliefs for which actions indexed in z_i, and only those actions, are best responses. Friedman and Mezzetti (2005) previously defined this object, which is used in some of their results. Note that the collection

$\mathcal{R} = \{r_i(z_i)\}_{z_i \subset \{1,2,\ldots,J(i)\}}$ defines a partition of Δ_{-i}. Let $\{\sigma_{-i}^t\} \subset \Delta_{-i}$ be an arbitrary convergent sequence with $\sigma_{-i}^t \to \sigma_{-i}^{\infty}$ as $t \to \infty$. Note, Ψ_i is upper hemicontinuous if for all such sequences, there exists a rational strategy $s_i : \Delta_{-i} \to \Delta_i$ for which $\int_{\Delta_{-i}} s_i(\sigma'_{-i})\, d\mu_{-i}(\sigma'_{-i}|\sigma_{-i}^t) \to \int_{\Delta_{-i}} s_i(\sigma'_{-i})\, d\mu_{-i}(\sigma'_{-i}|\sigma_{-i}^{\infty})$. There are two cases to consider (any convergent sequence will fall into one of the cases for sufficiently high t), and for each, we construct such a strategy s_i.

Case 1: Let $\{\sigma_{-i}^t\} \subset \prod_{k \neq i} \Delta(\sigma_k^{\infty})$. Let $r_i \in \mathcal{R}$ be an arbitrary partition element. From (B2)-(i), $\mu_{-i}(r_i|\sigma_{-i}^t)$ is continuous for all t, and hence, we can set strategy s_i to be any that is rational and constant within each partition element of $\mathcal{R}$.

Case 2: Let $\{\sigma_{-i}^t\} \not\subset \prod_{k \neq i} \Delta(\sigma_k^{\infty})$, meaning σ_k^t "gains zeros" in the limit for some k. Further suppose without loss that t is greater than some $\underline{t}$ such that for all k, $\Delta(\sigma_k^{t'}) = \Delta(\sigma_k^{t''})$ for all $\underline{t} \leq t', t'' < \infty$. In other words, along the remaining sequence, σ_k^t does not gain or lose zeros except in the limit. We must modify the proof from that of case 1 because $\mu_{-i}(r_i|\sigma_{-i}^t)$ may be discontinuous for some $r_i \in \mathcal{R}$ as $t \to \infty$ by (B1).[33] Define $C_{ij} = \mathrm{cl}(R_{ij} \setminus \prod_{k \neq i} \Delta(\sigma_k^{\infty}))$ for all j. Intuitively, each C_{ij} is like the corresponding R_{ij}, except without the part of R_{ij} in $\prod_{k \neq i} \Delta(\sigma_k^{\infty})$ that does not "overlap" with the part of R_{ij} in $\Delta_{-i} \setminus \prod_{k \neq i} \Delta(\sigma_k^{\infty})$. In particular, $C_{ij} = \emptyset$ if $R_{ij} \subset \prod_{k \neq i} \Delta(\sigma_k^{\infty})$. The rest of the proof proceeds exactly as in case 1, with C_{ij} replacing R_{ij}. That is, define $c_i(\cdot)$ as in (6) except with C_{ij} replacing R_{ij}. Then, $\mathcal{C} = \{c_i(z_i)\}_{z_i \subset \{1,2,\ldots,J(i)\}}$ defines a partition of Δ_{-i} with $c_i \in \mathcal{C}$ an arbitrary element. By (B2)-(ii), $\mu_{-i}(c_i|\sigma_{-i}^t)$ is continuous for all t, and hence, we can set strategy s_i to be any that is rational and constant within each partition element of $\mathcal{C}$.[34] ∎

PROOF OF LEMMA 1:

Fix player i and any $\sigma_{-i} \in \Delta_{-i}$. If σ_{-i} is a pure action profile, then it is immediate from (B1) that player i will have correct beliefs with probability one, to which one of her pure actions is a strict best response by assumptions on u_i, making Ψ_i single valued. So assume not; i.e., at least one $k \neq i$ has an action j such that $\sigma_{kj} \in (0,1)$. By (B1), with probability one, player i's beliefs only put positive probability on pure actions in the support of σ_{-i}, and so we show that it is as if player i is playing a restricted game $\Gamma' = \{N', A', u'\}$ in which her opponents take a fully mixed profile. Specifically, define $N' = \{i, N'_{-i}(\sigma_{-i})\}$, where $N'_{-i}(\sigma_{-i}) = \{k \in N | k \neq i, \sigma_{kj} \in (0,1) \text{ for some } j\}$ is the set of i's opponents who are mixing (over at least two pure actions) under σ_{-i}. Define $A' = A_i \times (\times_{k \in N'_{-i}(\sigma_{-i})} A'_k(\sigma_k))$, where $A'_k(\sigma_k) = \{a_{kj} \in A_k | \sigma_k(a_{kj}) \in (0,1)\}$ is the set of pure actions played by k with interior

[33] Consider a 2×2 game in which player i (row) has two actions U and D in which U weakly dominates D. Only when player i's realized belief is $r' = 0$ is D a best response, and hence, $r_i(U) = (0,1]$, $r_i(D) = \emptyset$, and $r_i(U,D) = \{0\}$. (B1) implies that $\mu_k^i(r_i(U,D)|r) = 0$ for all $r \in (0,1)$ and $\mu_k^i(r_i(U,D)|0) = 1$ and hence $u_k^i(r_i(U,D)|r)$ is discontinuous as $r \to 0^+$.

[34] In the example from footnote 33, $c_i(U) = [0,1]$, $c_i(D) = \emptyset$, and $c_i(U,D) = \emptyset$, and so this construction implies the strategy $s_i(r') = 1$ (corresponding to U) for all $r' \in [0,1]$.

probability under σ_k. Define $u' = \left(u'_i, (u'_k)_{k\in N'_{-i}(\sigma_{-i})}\right) : A' \to \mathbb{R}^{|N'_{-i}(\sigma_{-i})|+1}$, where u'_i is defined by $u'_i\left(a_{ij}, a'_{-i}\right) = u_i\left(a_{ij}, b\left(a'_{-i}; \sigma_{-i}\right)\right)$, where b records a'_{-i} for $k \in N'_{-i}(\sigma_{-i})$ and the pure actions taken by i's opponents $k \notin N'_{-i}(\sigma_{-i})$. Each u'_k for $k \in N'_{-i}(\sigma_{-i})$ can be set arbitrarily. Finally, defining $\sigma'_{-i}(\sigma_{-i}) \in \Delta'_{-i} = \times_{k\in N'_{-i}(\sigma_{-i})} \Delta A'_k(\sigma_{-i})$ as the natural projection of $\sigma_{-i} \in \Delta_{-i}$ onto Δ'_{-i} after dropping players $k \notin N'_{-i}(\sigma_{-i})$ and zeros corresponding to $\sigma_{kj} = 0$ for $k \in N'_{-i}(\sigma_{-i})$, it is as if player i faces $\Gamma' = \{N', A', u'\}$ with opponents who are playing a fully mixed profile $\sigma'_{-i}(\sigma_{-i}) \in \Delta'_{-i}$. By (B1), player i's beliefs do not realize with positive probability in any subset of Δ'_{-i} with zero Lebesgue measure. By assumption, $u_i(a_{ij}, a_{-i}) \neq u_i(a_{il}, a_{-i})$ for all $a_{ij} \neq a_{il}$ and $a_{-i} \in A_{-i}$, and thus, for no two actions $a_{ij} \neq a_{il}$ is it the case that $u'_i(a_{ij}, a'_{-i}) = u'_i(a_{il}, a'_{-i})$ for all $a'_{-i} \in \times_{k\in N'_{-i}(\sigma_{-i})} A'_k(\sigma_k)$. By Lemma 8 of Friedman and Mezzetti (2005), the event that player i is indifferent between any two pure actions has zero Lebesgue measure, and hence, Ψ_i is single valued. ∎

PROOF OF LEMMA 2:

According to Definition 55.1 of Osborne and Rubinstein (1994), $a_{ij} \in A_i$ is rationalizable if for each player k, there is a set of pure actions $Z_k \subset A_k$ such that (i) $a_{ij} \in Z_i$ and (ii) every $a_{kj} \in Z_k$ is a best response to some belief $\sigma'_{-k} \in \Delta_{-k}$ whose support is a subset of $Z_{-k} = \times_{l\neq k} Z_l$. Osborne and Rubinstein (1994) show that this is equivalent to the more commonly known recursive definition (Lemma 56.1). Letting $\sigma \in \Delta$ be an NBE and $a_{ij} \in \text{supp}(\sigma_i)$, the result follows after setting $Z_k = \text{supp}(\sigma_k)$ for all k. That $a_{ij} \in Z_i$ follows from the hypothesis on a_{ij} and the definition of Z_i, which shows (i). Every $a_{kj} \in Z_k$ is played with positive probability in an NBE by definition of Z_k and thus is a best response to some belief realization σ'_{-k}. That σ'_{-k} is supported on a subset of Z_{-k} follows from (B1), which implies that $\text{supp}(\sigma'_{-k}) = Z_{-k}$ with probability one. This shows (ii). ∎

PROOF OF THEOREM 3:

To simplify the proof, we additionally assume that NBE axiom (B2′) contains a differentiability condition: for any $\bar{r} \in (0,1)$, $F^i_k(\bar{r}|r)$ is differentiable in $r \in (0,1)$. In particular, this implies that $\partial F^i_k(\bar{r}|r)/\partial r|_{\bar{r},r\in(0,1)} < 0$ by (B3′). Including differentiability has no effect on the result, as it does not effect the set of attainable NBE; it simplifies the proof because of an analogous differentiability condition assumed in QRE axiom (A2), which does not affect the set of QRE.

The proof proceeds by construction. For every NBE (satisfying (B1′)–(B4′)), we construct the corresponding QRE (satisfying (A1)–(A4)) and vice versa.

Step 1: *Every NBE is a QRE.*

Fix $\{\Gamma^m, \sigma^*\}$. Player i's belief map r^* induces NBE reaction function $\Psi_{ij} : [0,1] \to [0,1]$. By Theorem 1, all NBE are interior, so the unique NBE must be a fixed point of $\Psi = (\Psi_{1j}, \Psi_{2l}) : [\epsilon, 1-\epsilon]^2 \to [\epsilon, 1-\epsilon]^2$ for sufficiently small $\epsilon > 0$. For convenience, define $U_i(\epsilon) = \bar{u}_i([\epsilon, 1-\epsilon]) = (\bar{u}_{i1}(r), \bar{u}_{i2}(r))_{r\in[\epsilon,1-\epsilon]} \subset \mathbb{R}^2$ as the set of utility vectors associated with any belief $r \in [\epsilon, 1-\epsilon]$. Further, assume that $\partial\bar{u}_{i1}(r)/\partial r > 0$, $\partial\bar{u}_{i2}(r)/\partial r < 0$, which is without loss.

Step 1a: Construct a *prequantal response function* $\tilde{Q}_{ij} : U_i(\epsilon) \to [0,1]$ such that $\tilde{Q}_{ij} \circ \bar{u}_i|_{[\epsilon,1-\epsilon]} = \Psi_{ij}|_{[\epsilon,1-\epsilon]}$ and $\tilde{Q}_{ij}$ satisfies analogues of (A1)–(A4):

(A1°): $\tilde{Q}_{ij} \circ \bar{u}_i(r) \in (0,1)$ for all $r \in [\epsilon, 1-\epsilon]$.

(A2°): $\tilde{Q}_{ij} \circ \bar{u}_i(r)$ is a continuous and differentiable function for all $r \in [\epsilon, 1-\epsilon]$.

(A3°): $(\partial\tilde{Q}_{i1} \circ \bar{u}_i(r))/\partial r > 0$, $(\partial\tilde{Q}_{i2} \circ \bar{u}_i(r))/\partial r < 0$ for all $r \in [\epsilon, 1-\epsilon]$.

(A4°): For $r \in [\epsilon, 1-\epsilon]$ such that $\bar{u}_{ij}(r) > \bar{u}_{il}(r)$, $\tilde{Q}_{ij} \circ \bar{u}_i(r) > \tilde{Q}_{il} \circ \bar{u}_i(r)$.

From this, the result almost follows. Intuitively, $\tilde{Q}_{ij}$ is very much like a quantal response function but is restricted to the subset of $\mathbb{R}^2$ that is relevant for equilibrium in this game, $\tilde{Q}_{ij} \circ \bar{u}_i$ is a more convenient reparameterization, and (A1°)–(A4°) are just (A1)–(A4) restricted to the relevant space. Once $\tilde{Q}_{ij}$ is constructed for both players $i \in \{1,2\}$, the fixed point of Ψ representing the NBE is also the fixed point of $(\tilde{Q}_{1j} \circ \bar{u}_1, \tilde{Q}_{2l} \circ \bar{u}_2) : [\epsilon, 1-\epsilon]^2 \to (0,1)^2$ representing the corresponding QRE. All that remains is to extend $\tilde{Q}_{ij}$ to a proper quantal response function defined over $\mathbb{R}^2$ that satisfies (A1)–(A4), which we do in step 1b.

Take $\tilde{Q}_{ij} : U_i(\epsilon) \to [0,1]$ defined by $\tilde{Q}_{ij}(v_i) \equiv \Psi_{ij}(\bar{u}_i^{-1}(v_i))$ as the prequantal response function, which satisfies $\tilde{Q}_{ij} \circ \bar{u}_i|_{[\epsilon,1-\epsilon]} = \Psi_{ij}|_{[\epsilon,1-\epsilon]}$ by construction. We now show that $\tilde{Q}_{ij}$ satisfies (A1°)–(A4°). We make extensive use of the fact that (without loss) $\tilde{Q}_{i1} \circ \bar{u}_i(r) = \Psi_{i1}(r) = 1 - F_k^i(\bar{r}|r)$ and $\tilde{Q}_{i2} \circ \bar{u}_i(r) = \Psi_{i2}(r) = F_k^i(\bar{r}|r)$, where $\bar{r} \in (\epsilon, 1-\epsilon)$ is the unique value that satisfies $\bar{u}_{i1}(\bar{r}) = \bar{u}_{i2}(\bar{r})$.

(A1°): $\tilde{Q}_{ij}$ satisfies (A1°) because $\tilde{Q}_{i1} \circ \bar{u}_i(r) = 1 - F_k^i(\bar{r}|r) \in (0,1)$ for $r \in [\epsilon, 1-\epsilon]$ by (B1′).

(A2°): $\tilde{Q}_{ij}$ satisfies (A2°) because $\tilde{Q}_{i1} \circ \bar{u}_i(r) = 1 - F_k^i(\bar{r}|r)$ is continuous and differentiable for all $r \in [\epsilon, 1-\epsilon]$ by (B2′).

(A3°): Without loss of generality, for all $r \in [\epsilon, 1-\epsilon]$: $\partial\bar{u}_{i1}(r)/\partial r > 0$, $\partial\bar{u}_{i2}(r)/\partial r < 0$, $\tilde{Q}_{i1} \circ \bar{u}_i(r) = 1 - F_k^i(\bar{r}|r)$, and $\tilde{Q}_{i2} \circ \bar{u}_i(r) = F_k^i(\bar{r}|r)$. That $\tilde{Q}_{ij}$ satisfies (A3°) follows because $\partial F_k^i(\bar{r}|r)/\partial r|_{\bar{r},r\in[\epsilon,1-\epsilon]} < 0$ from (B3′).

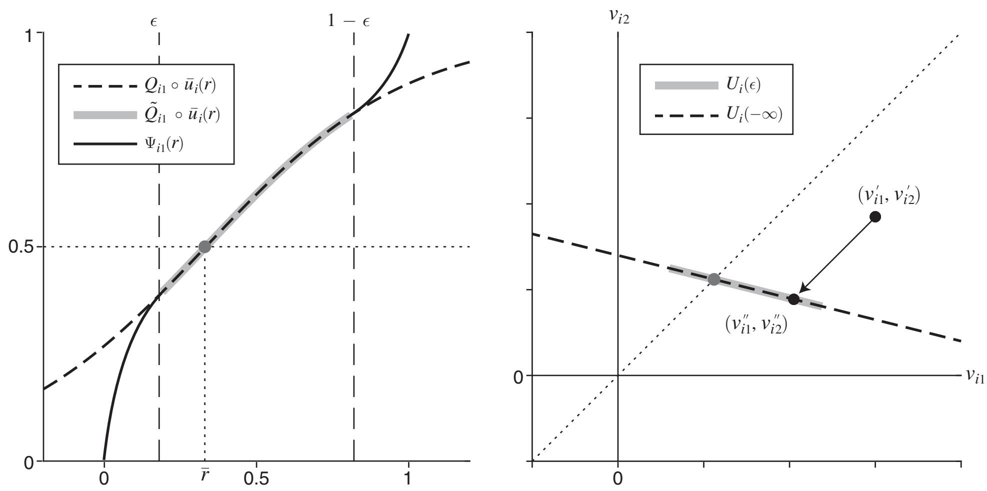

Figure 8. Construction of the Quantal Response Function

(A4°): Recall that $\bar{u}_{i1}(r) = \bar{u}_{i2}(r)$ if and only if $r = \bar{r}$. Notice that by (B4′), $\tilde{Q}_{i1} \circ \bar{u}_i(\bar{r}) = 1 - F_k^i(\bar{r}|\bar{r}) = 1 - 1/2 = 1/2$ and $\tilde{Q}_{i2} \circ \bar{u}_i(\bar{r}) = F_k^i(\bar{r}|\bar{r}) = 1/2$. Hence, by (B3′), $\tilde{Q}_{i1} \circ \bar{u}_i(r) = \tilde{Q}_{i2} \circ \bar{u}_i(r)$ if and only if $r = \bar{r}$. Axiom (A4°) then follows from (A3°).

Step 1b: Extend $\tilde{Q}_{ij} : U_i(\epsilon) \to [0,1]$ to a proper quantal response function $Q_{ij} : \mathbb{R}^2 \to [0,1]$ that satisfies (A1)–(A4).

We now construct the extension, which we illustrate in Figure 8. Define $U_i(-\infty) = (\bar{u}_{i1}(r), \bar{u}_{i2}(r))_{r \in (-\infty,\infty)}$ as the line that results from evaluating the expected utility vector for any r on the real line. Choose some function $Q_{i1} : U_i(-\infty) \to (0,1)$ such that $Q_{i1} \circ \bar{u}_i : (-\infty,\infty) \to (0,1)$ agrees with $\tilde{Q}_{i1} \circ \bar{u}_i(r)$ on $r \in [\epsilon, 1-\epsilon]$ and is strictly increasing and differentiable on $r \in (-\infty,\infty)$, which is possible because $\tilde{Q}_i$ satisfies (A1°)–(A3°) (see the left panel of Figure 8). Now extend Q_{i1} to $\mathbb{R}^2$ as follows. For any $(v'_{i1}, v'_{i2}) \in \mathbb{R}^2$, define $Q_{i1}(v'_{i1}, v'_{i2}) = Q_{i1}(v''_{i1}, v''_{i2})$, where (v''_{i1}, v''_{i2}) is the projection of (v'_{i1}, v'_{i2}) along the 45° line onto subspace $U_i(-\infty)$ (see right panel of Figure 8). It is easy to verify that Q_{ij} satisfies (A1)–(A4).

Step 2: *Every QRE is an NBE.*

We are now given quantal response function $Q_{ij} : \mathbb{R}^2 \to [0,1]$. First, we construct a family of CDFs $F_k^i(\cdot|r)$ representing belief map $r^*(r)$. We then show that $r^*(r)$ induces a reaction function $\Psi_{ij} : [0,1] \to [0,1]$ such that $\Psi_{ij}|_{[\epsilon,1-\epsilon]} = Q_{ij} \circ \bar{u}_i|_{[\epsilon,1-\epsilon]}$ and that $r^*(r)$ satisfies (B1′)–(B4′), from which the result follows.

We assume that $\bar{u}_{i1}(r)$ and $Q_{i1} \circ \bar{u}_i(r)$ are strictly decreasing in $r \in [0,1]$, which is without loss by (A3). For the unique $\bar{r} \in (\epsilon, 1-\epsilon)$ such that $\bar{u}_{i1}(\bar{r}) = \bar{u}_{i2}(\bar{r})$, define

$$F_k^i(\bar{r}|r) = \begin{cases} g(r) & \text{if } r \in [0,\epsilon) \\ Q_{i1} \circ \bar{u}_i(r) & \text{if } r \in [\epsilon, 1-\epsilon] \\ h(r) & \text{if } r \in (1-\epsilon, 1], \end{cases}$$

where $g(r)$ and $h(r)$ are any functions chosen so that the whole function is strictly decreasing, continuous, and differentiable and $F_k^i(\bar{r}|0) = 1$ and $F_k^i(\bar{r}|1) = 0$. That this is possible relies on (A1)–(A3). Notice that $F_k^i(\bar{r}|r)|_{[\epsilon,1-\epsilon]} = Q_{i1} \circ \bar{u}_i(r)|_{[\epsilon,1-\epsilon]}$ and $F_k^i(\bar{r}|r)$ goes from the top-left corner of the unit square to the bottom-right corner. Also, by (A4), we have that $F_k^i(\bar{r}|\bar{r}) = Q_{i1} \circ \bar{u}_i(\bar{r}) = 1/2$. Now choose positive number n_0 sufficiently large such that $(1-r)^{n_0} < F_k^i(\bar{r}|r) < 1 - r^{n_0}$ for all $r \in (0,1)$, which exists since as $n \to \infty$, $(1-r)^n \to 0$ and $(1-r^n) \to 1$ pointwise on $r \in (0,1)$. Figure 9 gives an illustration of the functions defined so far and is a useful reference for the whole construction.

Define $r_\circ = \{r : (1-r)^{n_0} = 1/2\}$ and $r^\circ = \{r : 1 - r^{n_0} = 1/2\}$, and notice that $r_\circ < \epsilon < \bar{r} < 1-\epsilon < r^\circ$. For all $\tilde{r} \in [r_\circ, \bar{r}]$, define $\alpha(\tilde{r}) = \{\alpha \in [0,1] : \alpha F_k^i(\bar{r}|\tilde{r}) + (1-\alpha)(1-\tilde{r})^{n_0} = 1/2\}$ and $F_k^i(\tilde{r}|r) = \alpha(\tilde{r}) F_k^i(\bar{r}|r) + (1-\alpha(\tilde{r}))(1-r)^{n_0}$. Similarly, for all $\tilde{r} \in [\bar{r}, r^\circ]$, define $\beta(\tilde{r}) = \{\beta \in [0,1] : \beta F_k^i(\bar{r}|\tilde{r}) + (1-\beta)(1-(\tilde{r})^{n_0}) = 1/2\}$ and $F_k^i(\tilde{r}|r) = \beta(\tilde{r}) F_k^i(\bar{r}|r) + (1-\beta(\tilde{r}))(1-r^{n_0})$. Now, for $\tilde{r} \in (0, r_\circ)$, define $F_k^i(\tilde{r}|r) = (1-r)^{m(\tilde{r})}$, where $m(\tilde{r}) = \{m \in [n_0, \infty) : (1-\tilde{r})^m = 1/2\}$. Similarly, for $\tilde{r} \in (r^\circ, 1)$, define $F_k^i(\tilde{r}|r) = 1 - r^{n(\tilde{r})}$, where $n(\tilde{r}) = \{n \in [n_0, \infty) : 1 - \tilde{r}^n = 1/2\}$. Finally, set $F_k^i(0|r) = 0$ and $F_k^i(1|r) = 1$ for $r \in (0,1)$. We have defined a family of CDFs $F_k^i(\tilde{r}|r)_{\tilde{r}\in[0,1], r\in(0,1)}$, pinning down belief map $r^*(r)$ for all $r \in (0,1)$. We may also impose that $r^*(0) = 0$ and $r^*(1) = 1$, which gives $F_k^i(\tilde{r}|1) = \mathbf{1}_{\{\tilde{r}=1\}}$ and $F_k^i(\tilde{r}|0) = 1$, and thus, we have constructed the entire family $F_k^i(\tilde{r}|r)_{\tilde{r}\in[0,1], r\in[0,1]}$. The NBE reaction is now given by $\Psi_{i1}(r) = F_k^i(\bar{r}|r)$ and $\Psi_{i2}(r) = 1 - F_k^i(\bar{r}|r)$, which by construction satisfies $\Psi_{ij}|_{[\epsilon,1-\epsilon]} = Q_{ij} \circ \bar{u}_i|_{[\epsilon,1-\epsilon]}$. Finally, that $r^*(r)$ satisfies (B1′)–(B4′) is immediate from construction of $F_k^i(\cdot|r)$. ∎

PROOF OF COROLLARY 1:

(i): Scale the payoff vector (a,c) by β, which yields $\beta(a,c)$, and then project it along the 45° line passing through it onto the horizontal line passing through (a,c) and (b,c), which is the point $(a + \delta^1(\beta), c)$ where $\delta^1(\beta) = (\beta - 1)(a - c)$ is, as in part (i) of Lemma 3. Since $|Q_{i1}(\beta(a,c)) - Q_{i1}(a,c)| < \epsilon$ and $Q_{i1}(\beta(a,c)) = Q_{i1}(a + \delta^1(\beta), c)$ by translation invariance, $|Q_{i1}(a + \delta^1(\beta), c) - Q_{i1}(a,c)| < \epsilon$ as well. Repeating this process of scaling and projecting $(a + \delta^1(\beta), c)$ gives the point $(a + \delta^1(\beta) + \delta^2(\beta), c)$ where $\delta^2(\beta) = (\beta - 1)(a + \delta^1(\beta) - c)$, and we have $|Q_{i1}(a + \delta^1(\beta) + \delta^2(\beta), c) - Q_{i1}(a + \delta^1(\beta), c)| < \epsilon$. Taken together, we

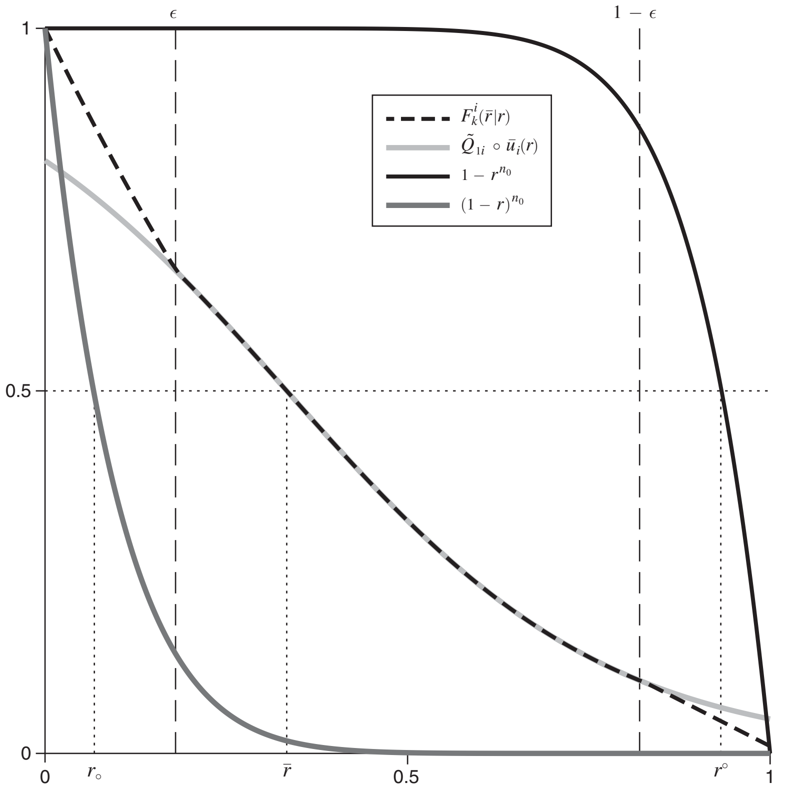

Figure 9. Construction of the Belief Map

have $|Q_{i1}\big(a+\delta^1(\beta)+\delta^2(\beta),c\big)-Q_{i1}(a,c)|<2\epsilon$. Having scaled and projected recursively in this fashion K times in total yields vector $\big(a+\sum_{k=1}^{K}\delta^k(\beta),c\big)$ for recursively defined $\delta^k(\beta)$, and $|Q_{i1}\big(a+\sum_{k=1}^{K}\delta^k(\beta),c\big)-Q_{i1}(a,c)|<K\epsilon$. Algebra reveals that $\sum_{k=1}^{K}\delta^k(\beta)=(\beta-1)(a-c)\sum_{k=1}^{K}\beta^{k-1}$, which approaches ∞ as $K\to\infty$, and so for sufficiently large K, we have $a+\sum_{k=1}^{K}\delta^k(\beta)>b$. Take K^* to be the smallest such K (note that this does not depend on ϵ). Thus, we have $Q_{i1}\big(a+\sum_{k=1}^{K^*}\delta^k(\beta),c\big)>Q_{i1}(b,c)$ by (A3). Since $Q_{i1}(b,c)\geq Q_{i1}(v_i)$ and $Q_{i1}(a,c)\leq Q_{i1}(v_i)$ for all $v_i\in H(a,b,c)$ by (A3), it is necessarily the case that $|Q_{i1}(v_i')-Q_{i1}(v_i'')|<K^*\epsilon$ for all $v_i',v_i''\in H(a,b,c)$. That $|Q_{i1}(v_i')-Q_{i1}(v_i'')|<K^*\epsilon$ for all $v_i',v_i''\in\big\{(v_i+\gamma e_2)|v_i\in H(a,b,c),\gamma\in\mathbb{R}\big\}$ follows from translation invariance.

(ii): Translate the payoff vector (c,a) by γ, which yields $(c,a)+\gamma e_2$, and then project it along the ray that passes through it and the origin onto the vertical line passing through (c,a) and (c,b), which is the point $\big(c,a+\delta^1(\gamma)\big)$ where $\delta^1(\gamma)=\big(c/(c+\gamma)\big)(a+\gamma)-a$ is, as in part (ii) of Lemma 3. Since $|Q_{i1}\big((c,a)+\gamma_2\big)-Q_{i1}(c,a)|<\epsilon$ and $Q_{i1}\big((c,a)+\gamma e_2\big)=Q_{i1}\big(c,a+\delta^1(\gamma)\big)$ by scale invariance, $|Q_{i1}\big(c,a+\delta^1(\gamma)\big)-Q_{i1}(a,c)|<\epsilon$ as well. Repeating this process of translating and projecting $(c, a + \delta^1(\gamma))$ gives the point $(c, a + \delta^1(\gamma) + \delta^2(\gamma))$ where $\delta^2(\gamma) = (c/(c+\gamma))(a + \delta^1(\gamma) + \gamma) - (a + \delta^1(\gamma))$, and we have $|Q_{i1}(c, a + \delta^1(\gamma) + \delta^2(\gamma)) - Q_{i1}(c, a + \delta^1(\gamma))| < \epsilon$. Taken together, we have $|Q_{i1}(c, a + \delta^1(\gamma) + \delta^2(\gamma)) - Q_{i1}(c, a)| < 2\epsilon$. Having translated and projected recursively in this fashion K times in total yields vector $(c, a + \sum_{k=1}^{K} \delta^k(\gamma))$ for recursively defined $\delta^k(\gamma)$, and $|Q_{i1}(c, a + \sum_{k=1}^{K} \delta^k(\gamma)) - Q_{i1}(a, c)| < K\epsilon$. Algebra reveals that $\sum_{k=1}^{K} \delta^k(\gamma) = \gamma \sum_{k=1}^{K} (c/(c+\gamma))^k + (c/(c+\gamma))^K a - a$, which approaches $c - a$ as $K \to \infty$, and so for sufficiently large K, we have $a + \sum_{k=1}^{K} \delta^k(\gamma) > b$. Take K^* to be the smallest such K (note that this does not depend on ϵ). Thus, we have $Q_{i1}(c, a + \sum_{k=1}^{K^*} \delta^k(\gamma)) < Q_{i1}(c, b)$ by (A3). Since $Q_{i1}(c, b) \leq Q_{i1}(v_i)$ and $Q_{i1}(c, a) \geq Q_{i1}(v_i)$ for all $v_i \in V(a, b, c)$ by (A3), it is necessarily the case that $|Q_{i1}(v_i') - Q_{i1}(v_i'')| < K^* \epsilon$ for all $v_i', v_i'' \in V(a, b, c)$. That $|Q_{i1}(v_i') - Q_{i1}(v_i'')| < K^* \epsilon$ for all $v_i', v_i'' \in \{\beta v_i | v_i \in V(a, b, c), \beta > 0\}$ follows from scale invariance. ∎

PROOF OF THEOREM 4:

(i): Fix $\{\Gamma, Q\}$, and let σ be a QRE: $Q_i(\bar{u}_i(\sigma_{-i})) = \sigma_i$ for all i. Let $\Gamma' \in \mathcal{T}(\Gamma)$ be associated with expected payoffs $\bar{u}_i'(\cdot)$ and translations $\gamma_i(a_{-i})$ for all i and a_{-i}. Then $\bar{u}_i'(\sigma_{-i}) = \bar{u}_i(\sigma_{-i}) + \bar{\gamma}_i(\sigma_{-i}) e_{J(i)}$, where $\bar{\gamma}_i(\sigma_{-i}) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}) \gamma_i(a_{-i})$. If Q is translation invariant, then $Q_i(\bar{u}_i'(\sigma_{-i})) = Q_i(\bar{u}_i(\sigma_{-i}) + \bar{\gamma}_i(\sigma_{-i}) e_{J(i)}) = Q_i(\bar{u}_i(\sigma_{-i})) = \sigma_i$ for all i, and thus, σ is also a QRE of Γ'. Similarly, any QRE of Γ' is also a QRE of Γ, and thus, the two sets of QRE are the same.

Fix $\{\Gamma, Q\}$, and let σ be a QRE: $Q_i(\bar{u}_i(\sigma_{-i})) = \sigma_i$ for all i. Let $\Gamma' \in \mathcal{S}(\Gamma)$ be associated with expected payoffs $\bar{u}_i'(\cdot)$ and scalings β_i for all i. Then $\bar{u}_i'(\sigma_{-i}) = \beta_i \bar{u}_i(\sigma_{-i})$. If Q is scale invariant, then $Q_i(\bar{u}_i'(\sigma_{-i})) = Q_i(\beta_i \bar{u}_i(\sigma_{-i})) = Q_i(\bar{u}_i(\sigma_{-i})) = \sigma_i$ for all i, and thus, σ is also a QRE of Γ'. Similarly, any QRE of Γ' is also a QRE of Γ, and thus, the two sets of QRE are the same.

(ii)-(a): Fix $\{\Gamma, Q\}$, and let σ be a QRE. First note that (A4) and the fact that no player is uniformly mixing implies that for all i, $\bar{u}_{ij}(\sigma_{-i}) \neq \bar{u}_{ik}(\sigma_{-i})$ for some j and k. If σ were also a QRE of $\Gamma' \in \mathcal{S}(\Gamma) \backslash \Gamma$, then it must be that for some player i, $Q_i(\bar{u}_i(\sigma_{-i})) = Q_i(\beta_i \bar{u}_i(\sigma_{-i}))$ for some $\beta_i \neq 1$. But this cannot be if Q is both translation invariant and weakly substitutable by Lemma 4 of online Appendix D.

(ii)-(b): Fix $\{\Gamma, Q\}$, and let σ be a QRE. First note that (A4) and the fact that no player is uniformly mixing implies that for all i, $\bar{u}_{ij}(\sigma_{-i}) \neq \bar{u}_{ik}(\sigma_{-i})$ for some j and k. If σ were also a QRE of $\Gamma' \in \mathcal{T}(\Gamma) \backslash \Gamma$, then it must be that for some player i, $Q_i(\bar{u}_i(\sigma_{-i})) = Q_i(\bar{u}_i(\sigma_{-i}) + \bar{\gamma}_i(\sigma_{-i}) e_{J(i)})$ for some $\bar{\gamma}_i(\sigma_{-i}) \neq 0$. But this cannot be if Q is both scale invariant and weakly substitutable by Lemma 4 of online Appendix C. ∎

PROOF OF THEOREM 5:

Fix $\{\Gamma, \sigma^*\}$. First, we show that response set R_{ij} is the same for all $\Gamma' \in \mathcal{A}(\Gamma)$. By definition of $\mathcal{A}$, for all i and a_{-i}, there exists β_i and $\gamma_i(a_{-i})$ such that $\bar{u}_{ij}'(a_{-i}) = \beta_i \bar{u}_{ij}(a_{-i}) + \gamma_i(a_{-i})$ for all j. By linearity of expected utility, for all i and $\tilde{\sigma}_{-i}$, β_i and

$\gamma_i(\tilde{\sigma}_{-i}) \equiv \sum_{a_{-i}} \tilde{\sigma}_{-i}(a_{-i})\gamma_i(a_{-i})$ satisfy $\bar{u}'_{ij}(\tilde{\sigma}_{-i}) = \beta_i \bar{u}_{ij}(\tilde{\sigma}_{-i}) + \gamma_i(\tilde{\sigma}_{-i})$ for all j. Thus, we have that

$$\begin{aligned} R'_{ij} &= \left\{\tilde{\sigma}_{-i} : \bar{u}'_{ij}(\tilde{\sigma}_{-i}) \geq \bar{u}'_{ik}(\tilde{\sigma}_{-i})\ \forall k = 1,\ldots,J(i)\right\} \\ &= \left\{\tilde{\sigma}_{-i} : \beta_i \bar{u}_{ij}(\tilde{\sigma}_{-i}) + \bar{\gamma}_i(\tilde{\sigma}_{-i}) \geq \beta_i \bar{u}_{ik}(\tilde{\sigma}_{-i}) + \bar{\gamma}_i(\tilde{\sigma}_{-i})\ \forall k = 1,\ldots,J(i)\right\} \\ &= \left\{\tilde{\sigma}_{-i} : \beta_i \bar{u}_{ij}(\tilde{\sigma}_{-i}) \geq \beta_i \bar{u}_{ik}(\tilde{\sigma}_{-i})\ \forall k = 1,\ldots,J(i)\right\} \\ &= \left\{\tilde{\sigma}_{-i} : \bar{u}_{ij}(\tilde{\sigma}_{-i}) \geq \bar{u}_{ik}(\tilde{\sigma}_{-i})\ \forall k = 1,\ldots,J(i)\right\} \\ &= R_{ij}. \end{aligned}$$

It is immediate that for any belief map σ^*, NBE reaction Ψ, and thus any NBE, is the same for all $\Gamma' \in \mathcal{A}(\Gamma)$. ■

PROOF OF EXAMPLE 4:

First, we construct a belief map $\sigma^*_T(\sigma_T)$ whose belief distributions have two mass points and that is defined over three disjoint intervals $[(2-\bar{\epsilon})/2, (2-\underline{\epsilon})/2]$, $[(3+\bar{\epsilon})/8, (5-\bar{\epsilon})/8]$, and $[(2+2\underline{\epsilon})/16, (5+3\bar{\epsilon})/16]$. Then, we show that the result holds for this belief map and argue that it can be modified into a proper belief map for which the result also holds. To this end, let

$$\sigma^*_T(\sigma_T) = \begin{cases} 0.2 \text{ w.p. } 1/2,\ 1 \text{ w.p. } 1/2, & \text{if } \sigma_T \in \left[\dfrac{2-\bar{\epsilon}}{2}, \dfrac{2-\underline{\epsilon}}{2}\right]; \\ 0.1 \text{ w.p. } 1/2,\ 0.8 \text{ w.p. } 1/2, & \text{if } \sigma_T \in \left[\dfrac{3+\bar{\epsilon}}{8}, \dfrac{5-\bar{\epsilon}}{8}\right]; \\ 0 \text{ w.p. } 1/2,\ 0.6 \text{ w.p. } 1/2, & \text{if } \sigma_T \in \left[\dfrac{2+2\underline{\epsilon}}{16}, \dfrac{5+3\bar{\epsilon}}{16}\right]. \end{cases}$$

The agent at D will choose T unless she trembles, and so her action will be $\sigma^D_T = 1 - \epsilon/2$, and thus, any $\sigma^D_T \in [(2-\bar{\epsilon})/2, (2-\underline{\epsilon})/2]$ is possible for some $\epsilon \in [\underline{\epsilon}, \bar{\epsilon}]$. According to σ^*_T, the agent at C will form belief 0.2 with probability $1/2$ and belief 1 with probability $1/2$ over D's action. Best responding to these belief realizations yields the action $\sigma^C_T = 1/2$. Since trembles are uniform, her action will be exactly $\sigma^C_T = 1/2$ for all $\epsilon \in [\underline{\epsilon}, \bar{\epsilon}]$, which is in the interior of $[(3+\bar{\epsilon})/8, (5-\bar{\epsilon})/8]$. According to σ^*_T, the agent at B will draw beliefs about D's action from the same belief distribution as C does and will form belief 0.1 with probability $1/2$ and belief 0.8 with probability $1/2$ over C's action. Best responding to these belief realizations yields the action $\sigma^B_T = 1/4$. With trembles, her action will be any $\sigma^B_T \in [(1+\underline{\epsilon})/4, (1+\bar{\epsilon})/4]$, and this interval is strictly nested in $[(2+2\underline{\epsilon})/16, (5+3\bar{\epsilon})/16]$. According to σ^*_T, the agent at A will draw beliefs about C and D's actions, respectively, from the same belief distributions as the other agents do and will form belief 0 with probability $1/2$ and belief 0.6 with probability $1/2$ over B's action. Best responding to these belief realizations yields the action $\sigma^A_T = 0$. With an ϵ-tremble, her action will be $\sigma^A_T = \epsilon/2$. In sum, the constructed belief map yields, for any $\epsilon \in [\underline{\epsilon}, \bar{\epsilon}]$, $\sigma^A_T = \epsilon/2 < \sigma^B_T = (1+\epsilon)/4 < \sigma^C_T = 1/2 < \sigma^D_T = 1 - \epsilon/2$. Further, notice that the belief distributions are unbiased on median for any $\epsilon \in [\underline{\epsilon}, \bar{\epsilon}]$

and the belief distributions about the agents at B, C, and D are ordered by FOSD.

It remains to show that this discrete belief map can be modified and extended to a proper belief map satisfying the axioms (B1′)–(B4′) in a way that preserves the ordering of equilibrium actions. This can be done since when another agent's action is in one of the nonintersecting intervals, say $\left[(2-\overline{\epsilon})/2, (2-\underline{\epsilon})/2\right]$, the discrete belief distribution can be approximated arbitrary well with continuous full support distributions that shift in the sense of strict FOSD ever so slightly as the agent's action varies within the interval. This can be done in such a way that each player's induced action from best responding to beliefs before trembles is arbitrarily close to that under the discrete belief distributions and, hence, after trembles remains within the necessary interval. Since the discrete belief distributions are unbiased on median, the smoothed ones can be made so as well. Furthermore, since the discrete belief distributions are ordered by FOSD and the intervals are disjoint (with an open interval between them) and separated from the boundary, the rest of the belief map can be constructed to satisfy all axioms. ∎

REFERENCES

Alaoui, Larbi, and Antonio Penta. 2016. "Endogenous Depth of Reasoning." *Review of Economic Studies* 83 (4): 1297–1333.

Andersen, Steffen, Seda Ertaç, Uri Gneezy, Moshe Hoffman, and John A. List. 2011. "Stakes Matter in Ultimatum Games." *American Economic Review* 101 (7): 3427–39.

Basu, Kuashik. 1994. "The Traveler's Dilemma: Paradoxes of Rationality in Game Theory." *American Economic Review* 84 (2): 391–95.

Bernheim, B. Douglas. 1984. "Rationalizable Strategic Behavior." *Econometrica* 52 (4): 1007–28.

Binmore, Ken. 1987. "Modeling Rational Players." *Economics and Philosophy* 3 (2): 179–214.

Brocas, Isabelle, Juan D. Carillo, Stephanie W. Wang, and Colin F. Camerer. 2014. "Imperfect Choice or Imperfect Attention? Understanding Strategic Thinking in Private Imperfect Choice or Imperfect Attention? Understanding Strategic Thinking in Private Information Games." *Review of Economic Studies* 81 (3): 944–70.

Camerer, Colin F., Teck-Hua Ho, and Juin-Kuan Chong. 2004. "A Cognitive Hierarchy Model of Games." *Quarterly Journal of Economics* 119 (3): 861–98.

Chen, Hsiao-Chi, James W. Friedman, and Jacques-François Thisse. 1997. "Boundedly Rational Nash Equilibrium: A Probabilistic Choice Approach." *Games and Economic Behavior* 18 (1): 32–54.

Esteban-Casanelles, Teresa, and Duarte Gonçalves. 2020. "The Effect of Incentives on Choices and Beliefs in Games an Experiment." https://duartegoncalves.com/papers/Esteban-Casanelles%20Goncalves%202020%2C%20The%20Effect%20of%20Incentives%20on%20Choices%20and%20Beliefs%20in%20Games.pdf.

Eyster, Erik, and Matthew Rabin. 2005. "Cursed Equilibrium." *Econometrica* 73 (5): 1623–72.

Fey, Mark, Richard D. McKelvey, and Thomas R. Palfrey. 1996. "An Experimental Study of Constant-Sum Centipede Games." *International Journal of Game Theory* 25: 269–87.

Friedman, Evan. 2020. "Endogenous Quantal Response Equilibrium." *Games and Economic Behavior* 124: 620–43.

Friedman, Evan. 2022. "Replication data for: Stochastic Equilibria: Noise in Actions or Beliefs?" American Economic Association [publisher], Inter-university Consortium for Political and Social Research [distributor]. https://doi.org/10.38886/E130401V1.

Friedman, James W., and Claudio Mezzetti. 2005. "Random Belief Equilibrium in Normal Form Games." *Games and Economic Behavior* 51 (2): 296–323.

Friedman, Evan, and Jeremy Ward. 2019. "Stochastic Choice and Noisy Beliefs in Games: An Experiment." https://www.dropbox.com/s/a0x3bnqwdeyuv68/SCNB.pdf?dl=0.

Goeree, Jacob K., and Charles A. Holt. 2001. "Ten Little Treasures of Game Theory and Ten Intuitive Contradictions." *American Economic Review* 91 (5): 1402–22.

Goeree, Jacob K., and Charles A. Holt. 2002. "Learning in Economics Experiments." In *Encyclopedia of Cognitive Science*, edited by Lynn Nadel. Basingstoke, UK: Palgrave MacMillan.
Goeree, Jacob K., and Charles A. Holt. 2004. "A Model of Noisy Introspection." *Games and Economic Behavior* 46 (2): 365–82.
Goeree, Jacob K., Charles A. Holt, and Thomas R. Palfrey. 2003. "Risk Averse Behavior in Generalized Matching Pennies Games." *Games and Economic Behavior* 45 (1): 97–113.
Goeree, Jacob K., Charles A. Holt, and Thomas R. Palfrey. 2005. "Regular Quantal Response Equilibrium." *Experimental Economics* 8: 347–67.
Goeree, Jacob K., Charles A. Holt, and Thomas R. Palfrey. 2016. *Quantal Response Equilibrium—A Stochastic Theory of Games*. Princeton, NJ: Princeton University Press.
Haile, Philip A., Ali Hortaçsu, and Grigory Kosenok. 2008. "On the Empirical Content of Quantal Response Equilibrium." *American Economic Review* 98 (1): 180–200.
Heller, Yuval, and Eyal Winter. 2018. "Biased-Belief Equilibrium." *American Economic Journal: Microeconomics*, 12 (2): 1-40.
Jehiel, Phillipe. 2005. "Analogy-Based Expectation Equilibrium." *Journal of Economic Theory* 123 (2): 81–104.
Kaas, R., and J.M. Buhrman. 1980. "Mean, Median, and Mode in Binomial Distributions." *Statistica Neerlandia* 34 (1): 13–18.
Kocher, Martin G., Peter Martinsson, and Martine Visser. 2008. "Does Stake Size Matter for Cooperation and Punishment?" *Economics Letters* 99 (3): 508–11.
Luce, R. Duncan. 1959. *Individual Choice Behavior: A Theoretical Analysis*. New York: Wiley.
Mauersberger, Felix. 2020. "Thompson Sampling: Endogenously Random Behavior in Games and Markets." https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3061481.
McKelvey, Richard D., and Thomas R. Palfrey. 1992. "An Experimental Study of the Centipede Game." *Econometrica* 60 (4): 803–36.
McKelvey, Richard D., and Thomas R. Palfrey. 1995. "Quantal Response Equilibria for Normal Form Games." *Games and Economic Behavior* 10 (1): 6–38.
McKelvey, Richard D., and Thomas R. Palfrey. 1998. "Quantal Response Equilibria for Extensive Form Games." *Experimental Economics* 1: 9–41.
McKelvey, Richard, Thomas Palfrey, and Roberto Weber. 1997. "Endogenous Rationality Equilibrium." Unpublished.
McKelvey, Richard, Thomas Palfrey, and Roberto Weber. n.d. "The Effects of Payoff Magnitude and Heterogeneity on Behavior in 2x2 Games with Unique Mixed Strategy Equilibria." Unpublished data (accessed November 28, 2017).
McKelvey, Richard D., Roberto Weber, and Thomas Palfrey. 2000. "The Effects of Payoff Magnitude and Heterogeneity on Behavior in 2x2 Games with Unique Mixed Strategy Equilibria." *Journal of Economic Behavior and Organization* 42 (4): 523–48.
Melo, Emerson, Kiril Pogorelskiy, and Matthew Shum. 2018. "Testing the Quantal Response Hypothesis." Unpublished data (accessed November 3, 2018).
Melo, Emerson, Kiril Pogorelskiy, and Matthew Shum. 2019. "Testing the Quantal Response Hypothesis." *International Economic Review* 60 (1): 53–74.
Nagel, Rosemarie. 1995. "Unraveling in Guessing Games: An Experimental Study." *American Economic Review* 85 (5): 1313–26.
Ochs, Jack. 1995. "Games with Unique, Mixed Strategy Equilibria: An Experimental Study." *Games and Economic Behavior* 10 (1): 202–17.
Osborne, Martin J., and Ariel Rubinstein. 1994. *A Course in Game Theory*. Cambridge, MA: MIT Press.
Osborne, Martin J., and Ariel Rubinstein. 2003. "Sampling Equilibrium, with an Application to Strategic Voting." *Games and Economic Behavior* 45 (2): 434–41.
Pearce, David G. 1984. "Rationalizable Strategic Behavior and the Problem of Perfection." *Econometrica* 52 (4): 1029–50.
Pulford, Briony D., Andrew M. Colman, and Graham Loomes. 2018. "Incentive Magnitude Effects in Experimental Games: Bigger Is Not Necessarily Better." *Games* 9 (1): 4.
Rapoport, Amnon, William E. Stein, James E. Parco, and Thomas E. Nicholas. 2003. "Equilibrium Play and Adaptive Learning in a Three-Person Centipede Game." *Games and Economic Behavior* 43 (2): 239–65.
Rogers, Brian W., Thomas R. Palfrey, and Colin F. Camerer. 2009. "Heterogeneous Quantal Response and Cognitive Hierarchies." *Journal of Economic Theory* 144 (4): 1440–67.

Selten, Reinhard, and Thorsten Chmura. 2008. "Stationary Concepts for Experimental 2x2-Games." *American Economic Review* 98 (3): 938–66.
Selten, Reinhard, and Thorsten Chmura. n.d. "Stationary Concepts for Experimental 2x2-Games." Unpublished data (accessed March 13, 2017).
Stahl, Dale, and Paul Wilson. 1995. "On Players' Models of Other Players: Theory and Experimental Evidence." *Games and Economic Behavior* 10 (1): 218–54.
Vuong, Quang H. 1989. "Likelihood Ratio Tests for Model Selection and Non-nested Hypotheses." *Econometrica* 57 (2): 307–33.