The Exact Discrete Time Representation of Continuous Time Models with Unequally Spaced Data

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Abstract

This thesis presents the exact discrete time representations of first order continuous time models with unequally spaced stocks, flows and mixed data. With unequally spaced data, given that the underlying continuous time models have constant coefficients and homeskedastic disturbances, the exact discrete time representations exhibit more complicated properties such as time-varying coefficients and heteroskedastic moving average disturbances, which arise due to the irregularity in sampling intervals. When data are purely stock variables, the exact discrete time representation follows a VAR(1) process with time-varying coefficients and serially uncorrelated heteroskedastic disturbances. When data are purely flow variables or a mixture of stocks and flows, the exact discrete time representation follows a VARMA(1, 1) process with time-varying coefficients and moving average heteroskedastic disturbances. Based on unequally spaced real life data, the empirical results show that the parameter estimates are different when accounting for the unequal sampling intervals compared to the approach that assumes data are equally spaced. In addition, the Monte Carlo evidences indicate that there are gains to be made in the estimation, such as smaller estimation bias, when the irregular sampling intervals are correctly accounted for.

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1 Introduction

1.1 Why Modelling in Continuous Time

The last few decades have seen growing applications of continuous time models in macroeconomics and finance. The economy and financial markets are operating continuously, where agents are making decisions continuously and economic variables are adjusting gradually in response to deviations from equilibrium. Such adjustment process may be difficult to specify using traditional discrete time models, while continuous time models could provide more realistic description since they account for the interaction among variables during the observation interval.

One major advantage of continuous time models is the flexibility in modelling the underlying interactions. A continuous time model does not restrict the model structures to match the sample interval, therefore the specification of such models is independent of the available data frequency. When the underlying phenomenon is continuous, modelling in continuous time could avoid misspecification due to the discrepancies in intervals between the model and the observations. Moreover, a continuous time model would allow variables to adjust continuously in analysing the dynamic adjustment mechanisms, which permit more realistic specifications of the partial adjustment process. Since the specification of a continuous time model does not depend on the data frequency, continuous time models allow one to forecast at different (shorter) intervals to the data used in the estimation. However, a discrete time model would generate forecasts at the same frequency as the data.

Another advantage of modelling in continuous time is that such models reduces temporal aggregation bias. In discrete time models, there are no distinction between stock variables (observed at discrete points) and flow variables (observed over discrete intervals), which generates temporal aggregation bias associated with the mistreatment of flow variables (which are observed at a slower rate than at which it operates). Ignoring temporal aggregation could result in biased estimates. On the other hand, a continuous time model provides the correct treatment of both stock variables and flow variables, which helps to reduce the temporal aggregation bias in estimating discrete time models with flow variables (see Bergstrom, 1996 and Bergstrom and Nowman, 2007 for the advantages of continuous time models over discrete time models).

1.2 The Econometric Issues

Despite the fact that continuous time models have several advantages and broad applications in different fields such as finance and macroeconomics, there are costs of modelling in continuous time. Since most observed data are available only at discrete intervals while the model is formulated in continuous time, the major econometric challenge is to relate the parameters of the continuous time model to the discretely observed data. To estimate parameters of a continuous time model, it is possible to obtain a discrete time representation of the continuous time model and then estimate those parameters from discrete data. Early approaches involve using the approximation method to obtain an approximate discrete time model (see, for example, in Phillips, 1959, 1974; Bergstrom, 1976; and Phillips and Yu, 2009). Other approaches include the frequency domain approach (see Hannan, 1970; Robinson, 1976a, 1976b, 1993); the Kalman filter method with state space equations (see Harvey and Stock, 1985; and Zadrozny 1988); and the exact discrete time representation (see Bergstrom, 1983, 1986).

The fundamental issues of modelling in continuous time relate to the aliasing problem as well as the difficulty of deriving statistical distribution theory that underpins estimators of the continuous time parameters based on discrete time data. One issue the practitioner could face in estimating the continuous time models of stochastic processes would be the identification of the structural parameters such that the continuous time spectral density function cannot be inferred from the (equispaced) discrete time data, which is referred to as the aliasing problem (Hansen and Sargent, 1983; and Bergstrom and Nowman, 2007). With multivariate continuous time Markov process, there are many aliases of the continuous time (CT) coefficient matrix such that they could generate the same (equally spaced) discrete data through taking the place of the CT matrix (McCrorie, 2009). The aliasing identification problem might be solved by imposing Cowles Commission type restrictions on the structural parameters (Phillips, 1973), or by making the sampling interval (unit observation period) sufficiently small (Hansen and Sargent, 1983). An alternative approach, as suggested by Bergstrom, Nowman and Wymer (1992), involves the use of bounds and the speed of adjustment parameters.

The other issue of estimation continuous time models with discrete time data relates to deriving the asymptotic distribution of the estimator of continuous time parameters. The Maximum Likelihood (ML) method is commonly used for parameter estimation and statistical inference as the ML estimates have good asymptotic properties (Yu, 2014). Common approaches to deriving the asymptotic distribution include the long-span asymptotic theory and the in-fill asymptotic theory. The long-span asymptotic distribution is obtained when the sample size $T \to \infty$ and the sampling interval h is fixed; while the in-fill asymptotic distribution is obtained when the sampling interval $h \to 0$ and the sample size T is fixed. However, ML estimates of continuous time parameters may be biased (even with large sample size and small sampling intervals) due to the finite sample issues, particularly when the continuous time process is nearly a unit root. Although the finite sample problems are applicable to other estimation methods as well, including GMM, nonlinear least squares and Quasi ML, this issue could potentially limit the applications to some financial time series which contain a root near unity, such as interest rates and volatility. The finite sample performances of ML estimates might be improved via the Jackknife estimation (see, for example, in Phillips and Yu, 2005 and Yu, 2014) or via indirect inference estimation (see in Phillips and Yu, 2009b; and Fasen-Hartmann and Kimmig, 2018). Addressing these issues is beyond the scope of this thesis, nonetheless, these are the fundamental issues (particularly for practitioners) in continuous time modelling, which are worth exploring in future work.

1.3 Approaches to Estimate Continuous Time Models

1.3.1 Approximation

Early literature has shown great interest as well as development in estimating continuous time models based on approximate discrete models. As argued by Strotz and Wold (1960), systems of stochastic differential equations can be approximated using a simultaneous equation model. Concerned with the specification errors associated with the approximation using nonrecursive models, Bergstrom (1966) provided a numerical example using a three-equation model with restriction on certain elements of the continuous time coefficient matrix to be zero. By applying the three-stage least squares method, one can compute the exact asymptotic bias of the continuous time parameter estimates. Based on Bergstrom's framework, Sargan (1974, 1976) worked on approximating discrete models by applying the methods of two-stage least squares, three-stage least squares, and full information maximum likelihood. Later, Wymer extends the work on estimating approximate discrete time models by developing the method for obtaining full information maximum likelihood estimates (Wymer, 1972) and providing applications of this method to modelling UK financial market (Wymer, 1973). Such method could see wide applications in the empirical work (see discussions in Bergstrom, 1988), nevertheless, Bergstrom and Wymer (1976) indicate that the temporal aggregation bias arises with the full information maximum likelihood estimator. Another important work on the approximation scheme was done by Phillips (1972) where he applied the minimum-distance procedure to estimate a three-equation trade cycle model. The results show larger root mean square errors of estimates in the approximate discrete model compared to those in the exact discrete model, and using the exact discrete model rather than the simultaneous equations approximation helps to eliminate the asymptotic bias.

Another common approach of the approximation method is the Euler approximation. The Euler scheme involves approximating the transition densities to obtain an approximation to the exact discrete time model, which takes the first order term in some Taylor expansions (see details in Phillips and Yu, 2009a). Under the Euler approximation scheme, it is easy to obtain the likelihood function, which associates with low computational cost and can be applied to a wide range of models. However, the accuracy of the approximation depends on the observation interval. With relatively high frequency data (such as daily or higher), the Euler approximation provides a good approximation discrete time model. With lower frequency data (when the observation interval is large), such method would generate the discretization bias whose magnitude is determined by the observation interval. Specifically, the estimator is inconsistent when the observation interval is fixed (see Lo, 1988; Phillips, 1974; and Sargan, 1974). Even though a number of improvements to the Euler method have been suggested (see in Sargan, 1974; Phillips, 1974; and Lo, 1988), estimators obtained from the approximated models remain inconsistent with a fixed observation interval. Other approximation methods include the closed-form approximation and infill approximations (see Phillips and Yu, 2009a for review of these methods).

1.3.2 Kalman Filter

Apart from the approximation methods, the Kalman filter approach can also be used in the estimation of continuous time systems with discrete data. Jones (1981) proposed a Kalman filter state space representation to calculate the exact likelihood for Gaussian ARMA processes, which can be used to calculate the likelihood for equally or unequally spaced data. Built on Jones' work, Harvey and Stock (1985) extended the work to estimate higher-order continuous time autoregressive models using the Kalman filter method, considering three cases - observations are stocks, flows and a mixture of stocks and flows (while Jones' work did not consider flows). In their further works, Harvey and Stock (1988, 1989) applied the Kalman filter method for estimating the parameters of continuous time autoregressive models with cointegrated and integrated variables. Besides, in a benchmark work, Zadrozny (1988) extended Harvey and Stock's (1985) treatment of mixed samples at the same frequency to mixed frequencies, where he provided the Kalman filtering based algorithm for computing the Gaussian likelihood function in continuous time ARMA systems, which can also include exogenous variables. In a more recent work, Singer (1995) presented a Kalman filtering based method for estimating the continuous state space model with irregular samples, where the likelihood function is computed using analytic derivatives.

As an alternative approach to Maximum likelihood estimation, the Kalman filter method can be modified to deal with mixed samples as well as irregularly sampled data (including missing observations), be extended to allow for measurement error, and be extended to nonstationary models and models including exogenous variables. Such method seems more general compared to exact discrete methods and computationally attractive. In particular, it is computationally efficient in equally spaced observation cases (Harvey and Stock, 1985). Rather than deriving the full exact discrete time model, with the Kalman filter approach, one needs to derive just the first-order difference equation with the state vector which includes both unobservable components and the observed variables (Chambers, McCrorie and Thornton, 2018). In addition, the Kalman filter produces the optimal estimate of the unobservable components of the state vector; however, it is “less readily comparable” (Chambers, McCrorie and Thornton, 2018) to exact discrete time representations.

1.3.3 Frequency Domain

When the autocorrelation structure of unobservable disturbances is not parameterized, as pointed by Robinson (1991), it is possible to estimate their spectra based on initial and consistent estimates of the parametric part of the model, which leads to the frequency domain methods (of estimating spectra nonparametrically). In an important work, Hannan (1970) used the spectral estimates to obtain parameter estimates of GLS type for static and dynamic models. Later Robinson (1976a, 1976b) developed Fourier methods for the estimation of open continuous time dynamic models. Then he extended the work to estimate closed stationary continuous time dynamic models by maximizing a frequency-domain Gaussian likelihood approximation (Robinson, 1977a) and to estimate closed and open systems with mixed samples (Robinson, 1993). The frequency domain methods offer relatively wide applications, such as closed and open systems, flow variables, mixed samples, and exogenous variables. Such methods can be powerful and computationally efficient for estimating stationary continuous time models, but less attractive than the Kalman filter or exact discrete methods when dealing with unequally spaced data (see Bergstrom, 1988; and Chambers, McCrorie and Thornton, 2018 for detailed reviews).

1.3.4 Exact Discrete Time Representation

More recent work on fitting discrete data in continuous time systems has been concerned with methods that take account of the exact restrictions on the distribution of the discrete data implied by continuous time system. Work on the exact method of estimating higher-order continuous time models commenced with Bergstrom's seminal paper concerned with the efficient estimation of higher order continuous time dynamic models (Bergstrom, 1983). The article presented the derivation of exact discrete time representations of continuous time systems with stock and flow variables (mixed sample case was also outlined) and proved the existence and uniqueness of the solution to a higher order system. Under appropriate assumptions and subject to the boundary conditions, the exact Gaussian method yields the exact maximum likelihood estimates, which are asymptotically efficient and convenient to compute.

Bergstrom's paper (1983) proposed a different algorithm for obtaining the exact Gaussian estimates from the Kalman filter methods, which provides the possibility of extended research, in several ways, on the exact Gaussian or quasi-maximum likelihood estimation of continuous time models. In a subsequent paper, Bergstrom (1985) derived an efficient algorithm for computing the exact Gaussian likelihood for nonstationary second order continuous time models, which can be applied to a system of any order with mixed data. In another subsequent paper, he extended the work on the exact Gaussian likelihood estimation of closed higher order continuous time models by introducing exogenous variables (Bergstrom, 1986). Further extension, based on this paper, to investigate exogenous variables can be found in McCrorie (2001). Later, Bergstrom (1989) derived a model for forecasting stock and flow data generated by a higher order continuous time system. This paper proposed an algorithm that provides the optimal forecasts of the post-sample discrete observations, which has potential practical applications in economic forecasting. A further extended work developed a computational algorithm for mixed order models with stochastic trends and mixed stock and flow data (Bergstrom, 1997). The development in the algorithm for the exact Gaussian estimates and its computing technology makes it feasible to apply the exact discrete methods to estimating higher order continuous time macroeconometric model (see Bergstrom and Nowman, 2007; and Bergstrom, Nowman and Wymer, 1992).

1.4 Developments in the Exact Discrete Estimation

Bergstrom's works on the exact Gaussian estimation mostly consider continuous time models of higher order (second order), while Agbeyegbe (1984, 1987) extended the work by considering some special cases. Agbeyegbe (1984) pointed out that in Bergstrom and Wymer's (1976) study of a neo-classical growth model of the UK economy, the authors treated all thirteen equations as first order equations, which would generate misspecification bias. To avoid such misspecification bias, he proposed the derivation of the exact discrete analog to a closed linear mixed order system. This paper focused on cases of stock variables, while cases of flows and mixed data can be considered in further research. In a further research, Agbeyegbe (1987) derived the exact discrete model of a first order continuous time system with mixed data. This paper not only is a special case of Bergstrom's (1986) work but also provides a general approach to estimate first order systems.

Bergstrom's work (1983) requires the continuous time coefficient matrix to be non-singular, which restricts its applications to stationary process and hence rules out important cases such as unit roots and cointegration. Although he extended the work to nonstationary systems (for example, in Bergstrom, 1985), there have been significant development in the work on estimation of nonstationary continuous time models. For instance, Phillips (1991) derived an exact discrete time Error Correction Model (ECM) of a triangular cointegrated system, where the continuous time coefficients are always identified in the discrete time reduced form. Further, Chambers (1999) extended the (original) model to non-stationary higher order systems (of order greater than two) with mixed sample data. The exact discrete time representations derived in the paper can be applied in stationary, nonstationary and explosive systems. For estimating the exact discrete time representations of cointegrated systems, for example, Chambers (2003) used the representation in a theoretical analysis of the asymptotic efficiency of optimal estimators; Chambers and McCrorie (2007) used the frequency domain methods; while Chambers (2009) used the time domain methods.

When the data are observed at different frequencies, the approach to aggregate all variables to the lowest frequency would potentially discard information contained in the high frequency data (Chambers, McCrorie and Thornton, 2018). Therefore it is important to incorporate mixed frequency data in the context of continuous time models. For example, Chambers (2011) analysed the effects of sampling frequency on estimators of cointegrating parameters. In a more recent paper he derived the exact discrete model of a CAR(1) process with mixed sample data that are observed at mixed frequencies (Chambers, 2016). The discrete time representations are applicable to both stationary and nonstationary (including cointegrated) series, while the approach might be extended to deal with continuous time ARMA processes.

Besides mixed frequency data, another recent development in the work on continuous time modelling is to extend the continuous time models to continuous time ARMA systems by including moving average disturbances. In Chambers and Thornton's (2012) paper, the authors derived exact discrete time representations of a continuous time ARMA system (or CARMA models) with mixed stocks and flows. The approach does not incorporate mixed processes, therefore Thornton and Chambers (2017) proposed the representation and estimation of mixed CARMA systems. In another joint work, the authors presented estimation of CARMA models based on the exact discrete approach with examples of applications in finance (Thornton and Chambers, 2016). In particular, this paper outlined the process of calculating the matrix exponential based on results by Van Loan (1978), which is used in the estimation of cointegrated systems in this thesis. More reviews on continuous time modelling can be found in Chambers, McCrorie and Thornton (2018).

1.5 Unequally Spaced Data

In most of publish works on continuous time modelling, it is common (and possibly easier) to treat observations as equally spaced, where the sampling interval can be normalised as unity for simplicity reasons. In many time series, however, the observation interval is not constant across sample. In real life data, some monthly data would have unequally spaced intervals which vary with the variation in the length of calendar month, leading to the observation intervals vary from 28 days to 31 days (with approximately 10% difference). Moreover, for instance, data on infrequent trade could also exhibit irregularity in observation intervals. Irregularity in observation intervals may also associate with missing observations or occur in "jittered" sampling when there are small random deviations in the sample intervals, which is more common in financial data especially when data frequency is high.

The existence of unequally spaced data has drawn attention in the continuous time literature. In earlier works by Jones (1962, 1971) and Parzen (1963), the authors used spectral analysis to deal with missing observations. Robinson (1977b) argued that structure analysis may be less preferable in the estimation of finite parameter models from unequally spaced data, and presented the exact discrete representation of a first order univariate continuous time model with an unequally spaced stock variable. Based on Gaussianity requirement, Dunsmuir (1983) derived a central limit theorem (CLT) for estimates of parameters of a stationary Gaussian time series with zero mean. Alternative Kalman filter methods can also be used in the estimation of continuous time models from unequally spaced data. As discussed above, Harvey and Stock (1985) and Jones (1981) proposed the method for computing the exact maximum likelihood of continuous time models based on the Kalman filter approach. The method can be extended easily to incorporate unequally spaced data including missing observations.

Until now, the Kalman filter methods appears to be a common approach to deal with unequally spaced data in the context of continuous time modelling. In a more recent paper on estimating continuous time models with time-varying coefficients, Robinson (2009) pointed out the possibility to extend the work by investigating irregular intervals in the sample. Chambers (2016) has also indicated the potential to allow for the sample intervals to vary when combining monthly and daily data as the numbers of days differ in calendar months.

Although Robinson (1977b) has presented an example of estimating continuous time models from unequally spaced data based on exact discrete time representations, unfortunately there has not been sufficient (extended) work on applying the exact discrete methods to handle unequally spaced data in the literature.

One important advantage of the Kalman filter algorithm compared to the exact discrete time representation is its broader applications, including cases of missing observations, measurement errors as well as data observed at unequal intervals (see discussions in Bergstrom, 1985). On the other hand, the exact discrete representation approach has a number of advantages over the Kalman filter approach. Firstly, it is computationally less costly than the Kalman filter methods when the sample size is sufficiently large. Additionally, when the exact discrete time model is required for other reasons, the exact discrete time representations would provide comparative advantage over the Kalman filter method even when the samples are small. Moreover, the exact discrete representations method does not require the system to have distinct eigenvalues. Besides, the Kalman filter method is arguably computationally efficient when observations are equally spaced and none are missing as the sampling interval can then be normalised as unity and the Kalman filter converges to a steady state (Harvey and Stock, 1985). Given unequally spaced data, advantages in the Kalman filter algorithm may be less obvious. It is therefore worth exploring the method for estimating continuous time models from unequally spaced data based on the exact discrete time representations, which is the aim of this thesis. While comparisons with other methods might be interesting, these are beyond the scope of this thesis but can be explored in future work.

1.6 Discussion on Specification and Applications of Continuous Time Models

One specification concern in modelling in continuous time is whether it is more appropriate to model the dynamics of a single variable (univariate case) or the dynamics of multiple variables (multivariate case). The univariate continuous time models are simpler (to identify and compute) and have relatively wide applications, for example, in finance. In the univariate case, the diffusion is reducible, through some one-on-one transformation into some diffusion, whose diffusion matrix is the identity matrix (Ait-Sahalia, 2007). The density of the transformed diffusion can be approximated around a standard normal distribution in the form of an expansion in Hermite polynomials; while the coefficients of the expansion can be computed in closed form

(Ait-Sshalia, 2002). The closed-form likelihood expressions make maximum likelihood a feasible method for estimating parameters in continuous-time diffusion models with discrete samples.

In econometric literature, many models of interest require multivariate specification, especially in macroeconomic analysis. Unlike in the univariate case where every diffusion is reducible, the reducibility of multivariate diffusion depends on the specification of its variance matrix (Ait-Sahalia, 2007). Therefore, the multivariate models are not obtained as extending the univariate models by replacing the scalars by matrices. With a reducible continuous-time diffusion, an expansion can be computed for the transition density of such diffusion by computing the density of the transformed diffusion and then transforming it back to the continuous-time diffusion, while irreducible diffusions would involve a more complex process (see Ait-Sahalia, 2007 for details).

In the estimation of multivariate models, the identification of the parameter vector could be problematic when the continuous time Markov process is not reversible, which would cause the aliasing problem (discussed in 1.2). One potential solution would be restricting the continuous time parameter matrix to have no complex eigenvalues and no confluence in the eigenvalues (McCrorie, 2009). This restriction, however, rules out time series with cyclical behaviour or trend behaviour. Another issue associated with multivariate case is the controllability problem, where the covariance matrix estimator is not positive semi-definite. For example, Barndorff-Nielsen, Lunde and Shephard (2011) extended the univariate model to a multivariate model for modelling time-varying financial volatility, which requires removing the effects of non-synchronous trading as well as the covariance matrix estimator to be positive semi-definite. The authors show that the univariate realised kernel estimator converges at a faster rate than the multivariate realised kernel estimator. However, the former relies on the assumption that the noise is white noise, which rules out tick-by-tick data, while the latter allows for a general form of noise, which can be applied to tick-by-tick data. Moreover, the convergence rate of the multivariate estimator enables one to construct a positive semi-definite estimator, which is not the case in the univariate case.

One popular development in continuous time modelling is in linear-in-the-variables models, where the benchmark work is by Bergstrom. In particular, linear-in-the-variables models are common (and perhaps popular) in estimating continuous time models based on the exact discrete time methods, which is also the focus of this thesis. The linear models make it possible to estimate (continuous time) parameters via a likelihood function derived based on the transition probability density of the discrete data (McCrorie, 2009). In the univariate case (see in 2.6 as an example), the analytical solution of the maximum likelihood estimator (MLE) is equivalent to the OLS estimator; while in the multivariate case, the analytical solution of the MLE is obtained by evaluating (maximizing) the log-likelihood function derived from the transition probability density of the discrete data.

From the practitioners' prospective, linear models may have certain limitations in capturing more extreme events such as jumps and asymmetric adjustments across cycles (Martin, Hurn and Harris, 2013). The nonlinear behaviour might be captured by models with nonlinearities such as threshold time series models, bilinear models and Markov switching models. In continuous time econometrics literature, many financial (and macroeconomic) time series are modelled in the nonlinear form, while in many nonlinear models, the transition density and log-likelihood function cannot be written in closed form (McCrorie, 2009). However, on the other hand, the alias problem may be less of an issue with small system nonlinear (in the parameters) continuous time models. Although the focus of this thesis is on the theoretical framework of estimating linear continuous time models with unequally-sampled discrete data based on the exact method, these specification issues may be considered in extended work, particularly in empirical work.

As discussed above (in 1.5), the irregularity in the sampling intervals may impose some challenges on formulating and estimating continuous time models via the exact discrete time model, such as time-varying coefficients, which occur due to the variation in the sampling intervals, and the covariance matrix does not converge (which may cause additional computational cost), unlike in the equally sampled case. In the continuous time literature, especially in the high-frequency econometrics, there are several studies addressing the issue of unequally spaced data. One important empirical application of high-frequency unequally spaced data is in the realized measures, which use the intra-day price observations to measure and forecast the unobservable asset volatility (Corsi, Peluso and Audrino, 2015). When the price behaves like a continuous Brownian semimartingale, the integrated variance of a Brownian motion can be approximated by the sum of a large number of intra-day squared returns. Corsi, Peluso and Audrino (2015) pointed out that the standard realized covariance measures may generate attenuation bias when the irregularity in the sampling intervals is not taken into account; and the bias increases with sampling frequency. The authors proposed a Kalman Filter state space approach to address this (asynchronicity) problem by treating the the irregularly spaced data as synchronous ultra-high-frequency data with missing observations. The Kalman smoother and expectation maximization (KEM) approach estimator is robust to both asynchronicity and microstructure noise, which is feasible on large dimensions as well. In a related work, Phillips and Yu (2009b) applied the realized measures to estimate integrated volatility based on a flat trading model, which incorporates flat trading features into an efficient price process. To deal with samples that are intermittently observed (observed at random times) at high frequency, the authors extended the Stopping Time model based on the work of Mykland and Zhang (2006) and Jacos (1993) by allowing for the stopping time scheme (sampling points) to be random and depend on past prices. Such models may perform well in removing the effect of noise when high-frequency data with random sampling intervals are employed. The irregular sampling intervals in high-frequency data could appear in non-synchronous trading, where any two assets rarely trade at the same instant. Barndorff-f-Nielsen, Lunde and Shephard (2011) noted that the irregularly spaced and non-synchronous trading is a distinctive feature of multivariate financial data. The authors proposed a multivariate kernel for estimating time-varying financial volatility and the estimator, which can be applied to deal with non-synchronous trading, presents good properties, including consistency and asymptotic mixed Guassainity, the guarantee of positive semi-definite and being robust to measurement errors. More recently, Dias, Fernandes and Scherrer (2021) investigated how the standard continuous time price discovery measures from discretely sampled (high-frequency) prices vary with the sampling interval via the Information share (IS) and Component share (CS) measures. Under the exact discretization, the CS measure is invariant to sampling intervals, which indicates the continuous time price discovery model fits the discrete data well. In addition, the authors show that continuous time models may be more appropriate for estimating based on very high frequency data. The method in this chapter offers the possibility to address the issue of high-frequency unequally spaced data in continuous time modelling based on the exact discrete time representation as an alternative approach. The analysis of high-frequency unequally spaced data is outside the scope of this thesis, but it would be an interesting (and potentially important) extended work in the future.

1.7 About this Thesis

The aim of this thesis is to derive the exact discrete time representations of multivariate continuous time models with unequally spaced data. The underlying continuous time models are of first order and the involved data are either all stocks, or all flows, or a mixture of stocks and flows. These methods, alternative to the Kalman filter methods, are applicable to both stationary processes and nonstationary processes under a certain weak assumption. The theoretical framework indicates that with unequally spaced data, given that the underlying continuous time models have constant coefficients and homeskedastic disturbances, the exact discrete time representations exhibit more complicated properties such as time-varying coefficients and heteroskedastic moving average disturbances (which arise due to the irregularity in sampling intervals). In addition, the Monte Carlo evidences indicate that there are gains to be made in the estimation, such as smaller estimation bias, when the irregular sampling intervals are correctly accounted for. The thesis is organised as follows:

Chapter 2 derives the exact discrete time representation of continuous time models with stocks, flows and mixed data. The continuous time model is in the form of a first-order multivariate system with deterministic time trends. Exact discrete time representations are provided for the three main cases of interest, where the observed vector is comprised purely of stock variables, purely of flow variables, or of a mixture of both stocks and flows. For stocks, the exact discrete time representation follows a VAR(1) process with serially uncorrelated heteroskedastic disturbances; while for flows or mixed data, the exact discrete time representation follows a VARMA(1, 1) process with moving average heteroskedastic disturbances. In all cases, the exact discrete time representation has time-varying coefficients and heteroskedastic disturbances although the underlying continuous time system is time invariant and has homeskedastic disturbances. These characteristics arise mainly due to the irregularity of the observation interval. Results from a simple Monte Carlo simulation indicates improvement in estimate properties when the unequal sampling intervals are correctly measured.

Results in Chapter 2 require the underlying continuous time system to be stationary with flow variables or mixed data. Such restrictions would limit the applications to non-stationary systems such as unit roots and cointegration. Chapter 3, therefore, provides an extended work on deriving the exact discrete time representation of non-stationary continuous time models with unequally spaced data. This chapter presents an approach to derive the exact discrete time model with flows, which does not require the continuous time coefficient matrix to be non-singular hence the results can be applied to both stationary and non-stationary processes. For mixed data, the result relies on a weak assumption that a sub-matrix of the continuous time coefficient matrix to be non-singular, which unfortunately rules out cointegration in stock variables. Still, this approach provides broader applications compared to that in the previous chapter. Following the theoretical work, a Monte Carlo simulation is conducted, which estimates a cointegrated continuous time system with unequally spaced flows. Simulation evidence suggests gains (reduced estimation bias etc.) from correctly accounting for the unequal sampling intervals.

Chapter 4 presents some empirical applications of the theoretical models provided in the previous two chapters. Based on the work of the previous two chapters, this chapter presents the Gaussian estimation of continuous time models with unequally spaced macroeconomic data, aiming to illustrate gains from accounting for the unequal sampling intervals in reducing estimation bias. This chapter considers two cases - a univariate model with a stock variable (vacancy stock) and a multivariate (bivariate) model with flow variables (vacancy inflow and outflow). The data are reported labour market vacancies, whose count dates are not made at regular frequency. The empirical results show that the parameter estimates are different when accounting for the unequal sampling intervals compared to the approach that assumes data are equally spaced. In addition, the Monte Carlo simulation evidence suggests that estimation bias is smaller when accounting for the unequally spaced intervals, indicating potential gains in estimation when the appropriate approach is applied. Especially for the model with flow variables, the bias in parameter estimates is obviously smaller under the appropriate approach. Even with relatively small variation in sampling intervals, there are gains to be made by incorporating the correct discrete time representation of the continuous time models. These evidences support the argument that the unequal spacing should be taken into account in the estimation procedure. Finally, Chapter 5 briefly summerizes the main results and contributions in this thesis, in addition to discussing the limitations and suggestions for further research.

2 Discrete Time Representation of Continuous Time Models with Unequally Spaced Stocks, Flows and Mixed Data

This chapter deals with estimation of continuous time systems with data that are observed at unequally spaced intervals. Exact discrete time representations are provided for the three main cases of interest, where the observed vector is comprised purely stock variables, purely flow variables, or of mixed stocks and flows. In all cases, the exact discrete time representations exhibit time-varying coefficient and heteroskedastic disturbances. With stock variables, the exact discrete time representation follow a VAR(1) process, while the discrete time representation follow a VARMA(1, 1) process with flow variables or mixed stocks and flows. Results of some Monte Carlo simulation are reported that attempt to examine the estimate properties when the unequal sampling intervals are correctly measured. Evidence from simulation indicates that estimates have better properties from the exact discrete time model where the irregularity of sampling intervals are correctly accounted for.

2.1 Introduction

The major technical issue for modelling in continuous time models, perhaps, is the discrepancy between the models, which are defined in continuous time, and the available data, which are observed in discrete time. This requires for econometric estimation methods for fitting continuous time models with discrete time data. The general solution is to obtain the discrete form of the continuous time models to fit the data.

One commonly used approach is to approximate the likelihood of the continuous time models (see Phillips and Yu, 2009 for an overview of the approximation scheme). This approach provides accurate approximation only when data frequency is high (when observation interval is small) since ignoring the higher-order term in the discrete model does not cause much information loss. Whereas the exact discrete time representation, as a good alternative to approximation methods, allows one to obtain estimate (of parameters) of a continuous time model with good properties regardless of data frequency (Chambers, 1999). Recent literature in modelling in continuous time shows a favour of the latter approach.

A benchmark research on this problem is done by Bergstrom (1983) in which he provided derivation of exact discrete time representation of a closed high-order continuous time system with stocks, flows and mixed sample, respectively. The approach to deriving the exact discrete time models was later applied to estimate non-stationary higher-order continuous systems (see Bergstrom, 1985; and Chambers, 1999 as representative research); and to open higher-order continuous time dynamic models where exogenous variables were incorporated (see Bergstrom, 1986; and Chambers, 1991). Further, Chambers (2009) presented the exact discrete representation of cointegrated continuous time systems with mixed sample. More recently, Thornton and Chambers (2013) extended the work on continuous time dynamic models to estimating continuous time autoregressive moving average (CARMA) process with discrete data (see also in Chambers and Thornton, 2012).

Existing research has shown great interests in estimation of continuous time models, as well as applications of continuous time models. In financial markets, for instance, continuous time models are used in modelling asset and option prices, measuring stochastic volatilities and also in forecasting post-sample observations (see, for example; Bergstrom, 1989; and Robinson, 1977b). In addition to analysis on financial data, continuous time models have found various applications in macroeconomics. For instance, Bergstrom, Nowman and Wymer (1992) provided the first application of the exact Gaussian estimation methods for higher-order continuous time models to a macroeconometric model with mixed sample.

Current literature on continuous time models has focused greatly on estimating models with data observed at equally spaced intervals or even at the same frequency, while in many time series, the observation interval is not constant over time as pointed out in Robinson (1977b). For example, observation intervals may vary with the variation in the length of calendar months which is common in monthly data. Additionally, irregularity in observation intervals may occur in "jittered" sampling when there are small random deviations in the sample intervals. This phenomenon is more common in financial data especially when data frequency is high.

In Robinson's (1977b) paper he briefly presented a discrete time AR(1) model with stock variables observed at unequally spaced intervals and argued the potential for modelling irregularly sampled time series. This paper only considered the univariate case for a stock variable, however, he did not continue to work on the unequally spaced data in the context of continuous time modelling, although he pointed out the possible extensive work on investigating irregular intervals in the sample in a more recent paper (Robinson, 2009). Nevertheless, little work has been done in addressing this problem properly. Deriving the exact discrete time representation of the continuous time models with unequally spaced data, given the length of observation intervals are known, looks a feasible research to fill this gap in literature. Such models may potentially provide estimate that has better properties; and may be applied to address a variety of problems, for instance, in prediction (forecasting), which is commonly used in financial analysis.

Until now the sample intervals in discrete time models are treated as equal, while observations in some data are not equally spaced hence the sample intervals are mis-measured. Giving that the discrete time parameter is a function of the observation interval, if the sample interval is mis-measured, estimation using such models could be (more) biased. One feasible solution to improve estimation accuracy is to derive the discrete time model in which the irregular sampling intervals are correctly measured. The key to deriving suitable discrete time representations is to write the model in the form such that observation intervals are allowed to vary across observations. Inspired by Robinson's (1977b) work, this chapter extends his work on estimating unequally spaced time series based on the exact discrete method to multivariate cases as well as flow data and a mixture of stocks and flows. In particular, the discretization of the continuous time models with mixed sample is an extended work based on an earlier work by Agbeyegbe (1987), which is also a benchmark to this chapter. Agbeyegbe (1987) provided the exact discrete time model of a first-order continuous time system with equally spaced mixed sample, while this chapter modified the model by allowing for the sampling intervals to vary across observation, where the equally spaced data is merely a special case. Results suggest that the discrete time model follows a heteroskedastic VAR(1) process with time-varying coefficients with stocks only, the discrete time disturbance vector is heteroskedastic but serially uncorrelated. While with flows or mixed samples the discrete time model follows a VARMA(1, 1) process of order one and the discrete time disturbance vector is a heteroskedastic moving average process.

A simple Monte Carlo study is conducted to examine whether the model with unequally spaced data outperforms the model with equally spaced data, giving that the data are unequally spaced whose intervals are known and only stocks. The results indicate that when observation intervals are measured correctly, the estimates of continuous time parameters from the model with unequally spaced data seem to have better properties than the estimates of continuous time parameters from the model with equally spaced data.

Although the main focus of the chapter is to provide a theoretical framework of derivation of the exact discrete time representation of a first-order multivariate continuous time vector autoregressive (CVAR(1)) model with unequally spaced data, the results may be broadly applied in macroeconomics and finance, where many time series are not observed at a regular basis. In particular, the CVAR specification might be more directed towards macroeconomic analysis where low-frequency data modelling is still dominating (McCrorie, 2009). However, this typical specification may have limited applications in finance since the small-order nonlinear continuous time models are preferred in measuring high-frequency financial time series. As discussed in 1.6, the exact discrete approach could face certain limitations, including the aliasing problem with multivariate models and the lack of guarantee that the discrete time covariance matrix is positive semi-definite, which may cause the controllability problem. In the exact discrete model, coefficients are time varying, which occur entirely due to the variation in the sampling intervals; and the discrete time covariance is not converging, which may impose some computational complexities. The aliasing problem may be solved by modelling in discrete time models, which, however, suffer from a lack of time invariance (McCrorie, 2009; Chambers, McCrorie & Thornton, 2017). Alternatively, as discussed in McCrorie (2009), the aliasing problem may be solved by imposing the restriction on the continuous-time coefficient matrix such that the matrix has no complex eigenvalues and no confluence in the eigenvalues. This unfortunately limits the applications to time series with cyclical or trend behaviour.

Despite some limitations of the exact discrete approach in the continuous time estimation, this chapter offers a potentially important alternative to the Kalman filter method for estimation based on unequally spaced data. The Kalman filter method imposes less restriction with transition, while the discretization is an approximation. The exact discrete method requires replacing the unobservable terms, which imposes more restrictions, while the discretization is exact rather than being an approximation. The Kalman filter approach seem to provide a good method for estimating continuous time models with unequally spaced data, which has been regarded as an advantage over the exact discrete approach (see discussions in 1.3.2). Nevertheless, the results of this chapter indicate that with unequally spaced data it is also possible to estimate continuous time models using the exact discrete time model. This may be particularly attractive to mixed frequency data since the discretization based on the exact discrete method is not an approximation, and one does not need to worry about different frequency.

The chapter is organised as follows: Section 2.2 briefly presents the continuous time model and its solution. Section 2.3 derives the exact discrete time representation of the continuous time model with stock variables and the result is presented in Theorem 2.1. Section 2.4 presents the derivation of discrete time form of the continuous time model with flow variables, while Section 2.5 derives the discrete time model from a continuous time model where data are mixed of stocks and flows. Results of Section 2.4 and 2.5 are presented in Theorem 2.2 and 2.3, respectively. Then in Section 2.6, a Monte Carlo simulation is conducted on testing the estimation performance of models with unequally spaced data and with equally spaced data, respectively. Section 2.7 concludes the main results and briefly discusses some possible further research questions.

2.2 The Model and its Solution

This section briefly reviews the solution of the continuous time model in Bergstrom (1983, 1984) on which the methods of this chapter are based. Consider the system of stochastic differential equations where an intercept and a deterministic time trend are included:

$$dx(t) = [\mu + \gamma t + Ax(t)]dt + \zeta(dt), \tag{2.1}$$

where $x(t)$ is an $(n \times 1)$ vector of random processes, μ is an $(n \times 1)$vector of unknown constants, γt is an $(n \times 1)$ vector of deterministic time trend with γ being the unknown slope and A is an $(n \times n)$ matrix of unknown coefficients. The disturbance vector, $\zeta(dt)$, is assumed to be a vector stochastic process which has the following properties:

Assumption 2.1.

$$E[\zeta(dt)] = 0$$

$$E[\zeta(dt)\zeta(dt)'] = \Sigma dt,$$

where Σ is an unknown symmetric positive definite matrix and

$$E[\zeta_i(\Delta_1)\zeta_j(\Delta_2)'] = 0,$$

for $i, j = 1, 2, ..., n$; $i \neq j$; and $\Delta_1 \cap \Delta_2 = \emptyset$.

The system (2.1) is loosely described as a closed-form linear system of first order stochastic differential equations. Since the derivative $(d/dt)X(t)$ is not well defined, system (2.1) is not mean square differentiable. An interpretation of system (2.1) is to take integration of system (2.1) over the interval from t_1 to t with $t_1 < t$, from which we obtain:

$$x(t) - x(t_1) = A\int_{t_1}^{t} x(r)\,dr + \int_{t_1}^{t}[\mu + \gamma r]\,dr + \int_{t_1}^{t}\zeta(dr), \tag{2.2}$$

where $\int_{t_1}^{t}\zeta(dr) = \zeta(t_1, t]$.

Conditional on $x(0)$, a predetermined boundary condition, the solution to (2.1) is given by:

$$x(t) = \int_0^t e^{(t-r)A}\,\zeta(dr) + e^{tA}x(0) + \int_0^t e^{(t-r)A}[\mu + \gamma r]\,dr, \tag{2.3}$$

where $e^{tA} = \sum_{j=0}^{\infty}(j!)^{-1}(tA)^j$ for any square matrix A.

2.3 Discrete Time Representation of Continuous Time Model with Stocks

In this section, a first order system with a sample of observations at discrete points of time is concerned. We shall consider a system that only includes stock variables $x(t_i) = [x_1(t_i), x_2(t_i), ..., x_n(t_i)]'$, that are observed at each discrete point of time t_i, with $i = 1, 2, ..., T$. The sample interval is defined as $\delta_i = t_i - t_{i-1}$ for $i = 1, 2, ..., T$, which might not be equal to unity.

The system of stock variables, based on solution to system (2.1), can be written as

$$x(t_i) = \int_0^{t_i} e^{(t_i - r)A}\,\zeta(dr) + e^{t_i A}x(0) + \int_0^{t_i} e^{(t_i - r)A}[\mu + \gamma r]\,dr, \tag{2.4}$$

with boundary conditions $x(0) = \alpha$ for $t_0 = 0$ and α is any constant vector such that at time $t = 0$, the observation $x(0)$ is pre-determined.

Given that Assumption 2.1 is satisfied, the exact discrete time representation of system (2.4) is given by Theorem 2.1.

Theorem 2.1. Let $x(t)$ be generated by (2.1). Then, under Assumption 2.1, subject to the boundary condition, the discrete time data satisfy

$$x(t_i) = e^{\delta_i A}x(t_{i-1}) + \mu_i + \gamma_i t_i + \eta(t_i), \;\; i = 2, ..., T. \tag{2.5}$$

$$E[\eta(t_i)] = 0,$$

$$E[\eta(t_i)\eta(t_i)'] \;\; = \;\; \Omega_i = \int_0^{\delta_i} [e^{rA}\Sigma e^{rA'}]dr,$$

$$E[\eta(t_i)\eta(t_j)'] = 0 \;\; for \;\; i \neq j,$$

where

$$\mu_i = G_i\mu - H_i\gamma,$$

$$\gamma_i = G_i\gamma,$$

$$G_i = \int_0^{\delta_i} e^{sA}\, ds,$$

$$H_i = \int_0^{\delta_i} e^{sA} s\, ds,$$

Proof. The derivation of the exact discrete model of (2.4) is straightforward. By partitioning (2.4)

$$\begin{aligned}
x(t_i) &= \int_0^{t_{i-1}} e^{(t_i-r)A}\,\zeta(dr) + \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr) + e^{t_iA}x(0) \\
&+ \int_0^{t_{i-1}} e^{(t_i-r)A}[\mu+\gamma r]\,dr + \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}[\mu+\gamma r]\,dr \\
&= e^{\delta_iA}\left\{\int_0^{t_{i-1}} e^{(t_{i-1}-r)A}\,\zeta(dr) + e^{t_{i-1}A}x(0) + \int_0^{t_{i-1}} e^{(t_{i-1}-r)A}[\mu+\gamma r]\,dr\right\} \\
&+ \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}[\mu+\gamma r]\,dr + \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr) \\
&= e^{\delta_iA}x(t_{i-1}) + \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}[\mu+\gamma r]\,dr + \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr),
\end{aligned} \tag{2.6}$$

we obtain (2.5) with

$$\begin{aligned}
c_i &= \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}[\mu+\gamma r]\,dr, \\
&= \int_0^{\delta_i} e^{sA}\,ds\mu + \int_0^{\delta_i} e^{sA}(t_i-s)\,ds\gamma \\
&= \int_0^{\delta_i} e^{sA}\,ds\mu - \int_0^{\delta_i} e^{sA}s\,ds\gamma + \int_0^{\delta_i} e^{sA}\,ds\gamma t_i \\
&= \mu_i + \gamma_i t_i,
\end{aligned}$$

and

$$\eta(t_i) = \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr).$$

The properties of the discrete time disturbance vector $\eta(t_i)$ are derived as follows:

The mean of vector $\eta(t_i)$ is obtained as

$$\begin{aligned} E[\eta(t_i)] &= E\left[\int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr)\right] \\ &= \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,E[\zeta(dr)] \\ &= 0. \end{aligned}$$

The variance of $\eta(t_i)$ is obtained as

$$\begin{aligned} E[\eta(t_i)\eta(t_i)'] &= E\left[\int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr)\right]' \\ &= \int_{t_{i-1}}^{t_i} [e^{(t_i-r)A}\Sigma e^{(t_i-r)A'}]\,dr \\ &= \int_{0}^{\delta_i} [e^{rA}\Sigma e^{rA'}]dr. \end{aligned}$$

The autocovariances of $\eta(t_i)$ is obtained as

$$\begin{aligned} E[\eta(t_i)\eta(t_j)'] &= E\left[\int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr)\right]\left[\int_{t_{j-1}}^{t_j} e^{(t_j-r)A}\,\zeta(dr)\right]' \\ &= 0, \end{aligned}$$

for $i \neq j$. Since $i \neq j$ implies $\delta_i \neq \delta_j$, and hence $[t_{i-1}, t_i] \cap [t_{j-1}, t_j] = \emptyset$. End of proof.

As shown in Theorem 2.1, the discrete time disturbances are functions of vector $\zeta(dt)$. The properties of the discrete time disturbance vector $\eta(t_i)$ depend on the properties of continuous time disturbance vector $\zeta(dt)$ in system (2.1). Note that the original form of the discrete time representation of a continuous time process was defined in Robinson (1977b), in which he only presented a univariate first-order process of stock variable without any intercept or time trend (i.e. $n = 1$ and $\mu = \gamma = 0$). Theorem 2.1 extends his result to a more general case where a multivariate process with an intercept term and a deterministic time trend is involved.

The discrete time disturbance vector $\eta(t_i)$ in the model with stock variables is a heteroskedastic serially uncorrelated process. It has zero mean, heteroskedastic variance as the variances Ω_i changes across observations with the sample interval δ_i, and zero covariance. The discrete time model with stocks exhibits a heteroskedastic VAR(1) process with time-varying coefficients. Further more, when $n = 1$, (2.1) becomes a univariate continuous time series where μ, γ and a are scalars and the system contains only one variable $x(t)$. When $\mu = \gamma = 0$, (2.1) becomes a system of purely stochastic differential process without any drift or trend. The derivation of the exact discrete time model with stock variables does not require the assumption that the matrix A is non-singular, which suggest that Theorem 2.1 is valid also for non-stationary process (i.e. systems that contain unit roots and cointegration).

2.4 Discrete Time Representation of Model with Flows

This section presents the derivation of the exact discrete time representation of a first order system with a sample of unequally spaced integral observations [1], generated by (2.1). We shall consider a system that includes only flow variables - a sample of random vectors $[x_{t_1}, x_{t_2}, ..., x_{t_T}]$ observed as aggregations over discrete intervals $[t_{i-1}, t_i]$

$$x_{t_i} = \int_{t_{i-1}}^{t_i} x(r)\, dr, \tag{2.7}$$

for $i = 1, 2, ..T$, and $x(t_i)$ is the solution of the model (as presented in section 2.2).

In order to derive the exact discrete model, it is convenient to make an assumption on the coefficient matrix A such that

Assumption 2.2. The matrix A is non-singular.

Note that Assumption 2.2 rules out any systems that contain unit roots, for example $I(1)$ systems or cointegrated systems. The result suggested in Theorem 2.2 is valid for only stationary processes.

Given that Assumption 2.1 and 2.2 are satisfied and the boundary condition holds, the exact discrete model is given by Theorem 2.2.

Theorem 2.2. Let $x(t_i)$ be generated by (2.1). Then, under Assumption 2.1 and 2.2, subject to the boundary condition, the discrete time observations on the flow variables satisfy

$$x_{t_i} = e^{\delta_i A} x_{t_{i-1}} + m_{0i} + m_{1i} t_i + \xi_{t_i}, \quad i = 2, \ldots, T, \tag{2.8}$$

$$E[\xi_{t_i}] = 0,$$

$$E[\xi_{t_i}\xi_{t_i}'] \;=\; V_i = \int_0^{\delta_i} \Phi_i(r)\Sigma\Phi_i'(r)\, dr + \int_0^{\delta_{i-1}} \Psi_i(r)\Sigma\Psi_i'(r)\, dr,$$

$$\mathrm{E}[\xi_{t_i}\xi_{t_j}'] = \begin{cases} P_i = \int_0^{\delta_{i-1}} \Psi_i(r)\Sigma\Phi_i'(r)\, dr & \text{if } j = i-1 \\ L_i = \int_0^{\delta_i} \Phi_i(r)\Sigma\Psi_i'(r)\, dr & \text{if } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

[1] I am grateful to my supervisor for suggesting the method for deriving the exact discrete time model with flows.

where

$$m_{0i} = \alpha_i \mu - \beta_i \gamma,$$

$$\alpha_i = \int_0^{\delta_i} \Phi_i(s)\, ds + \int_0^{\delta_{i-1}} \Psi_i(s)\, ds,$$

$$\beta_i = \int_0^{\delta_i} \Phi_i(s)\, sds + \int_0^{\delta_{i-1}} \Psi_i(s)\, sds + \int_0^{\delta_{i-1}} \Psi_i(s)\, ds\delta_i,$$

$$m_{1i} = \int_0^{\delta_i} \Phi_i(s)\, ds\gamma,$$

$$\Phi_i(r) = A^{-1}[e^{rA} - I],$$

$$\Psi_i(r) = A^{-1}[e^{\delta_i A} - e^{rA}].$$

Proof. Integrating (2.1) over the interval $(t_{i-1}, t_i]$ yields

$$x(t_i) - x(t_{i-1}) = \int_{t_{i-1}}^{t_i} (\mu + \gamma r)dr + Ax_{t_i} + \int_{t_{i-1}}^{t_i} \zeta(dr). \tag{2.9}$$

But $t_{i-1} = t_i - \delta_i$ and so $x(t_i) - x(t_{i-1}) = (1 - L^{\delta_i})x(t_i) = \Delta_i x(t_i)$ where L denotes the lag operator and $\Delta_i = 1 - L^{\delta_i}$. Applying the operator Δ_i to (2.5) we obtain

$$\Delta_i x(t_i) = e^{\delta_i A}\Delta_i x(t_{i-1}) + \Delta_i c_{t_i} + \Delta_i \eta(t_i). \tag{2.10}$$

But $\Delta_i x(t_{i-1}) = \Delta_i L^{\delta_i} x(t_i) = L^{\delta_i}\Delta_i x(t_i)$, and so (2.10) can be written

$$(I - e^{\delta_i A}L^{\delta_i})\Delta_i x(t_i) = \Delta_i c_{t_i} + \Delta_i \eta(t_i). \tag{2.11}$$

Applying the operator $(I - e^{\delta_i A}L^{\delta_i})$ to (2.9) and using (2.11) to substitute for the term on the left-hand-side we obtain

$$\Delta_i c_{t_i} + \Delta_i \eta(t_i) = (I - e^{\delta_i A}L^{\delta_i})\int_{t_{i-1}}^{t_i} (\mu + \gamma r)dr + (I - e^{\delta_i A}L^{\delta_i})Ax_{t_i} + (I - e^{\delta_i A}L^{\delta_i})\int_{t_{i-1}}^{t_i} \zeta(dr),$$

which can be re-written in the form

$$\begin{aligned} Ax_{t_i} &= e^{\delta_i A} A x_{t_{i-1}} + \Delta_i c_{t_i} - (I - e^{\delta_i A} L^{\delta_i}) \int_{t_{i-1}}^{t_i} (\mu + \gamma r) dr \\ &+ \Delta_i \eta(t_i) - (I - e^{\delta_i A} L^{\delta_i}) \int_{t_{i-1}}^{t_i} \zeta(dr). \end{aligned} \tag{2.12}$$

But the matrices A and $e^{\delta_i A}$ commute i.e. $Ae^{\delta_i A} = e^{\delta_i A} A$ and so

$$\begin{aligned} x_{t_i} &= e^{\delta_i A} x_{t_{i-1}} + A^{-1} \left(\Delta_i c_{t_i} - (I - e^{\delta_i A} L^{\delta_i}) \int_{t_{i-1}}^{t_i} (\mu + \gamma r) dr \right) \\ &+ A^{-1} \left(\Delta_i \eta(t_i) - (I - e^{\delta_i A} L^{\delta_i}) \int_{t_{i-1}}^{t_i} \zeta(dr) \right) \end{aligned} \tag{2.13}$$

which, upon collection of terms, is the required equation (2.8).

In (2.13), uing $t_{i-1} = t_i - \delta_i$ we have

$$\begin{aligned} g_i &= A^{-1} \int_{t_{i-1}}^{t_i} (e^{(t_i - r)A} - I)(\mu + \gamma r) dr + A^{-1} \int_{t_{i-2}}^{t_{i-1}} (e^{\delta_i A} - e^{(t_{i-1} - r)A})(\mu + \gamma r) dr, \\ &= \int_0^{\delta_i} \Phi_i(s)\, ds\mu + \int_0^{\delta_{i-1}} \Psi_i(s)\, ds\mu - \left[\int_0^{\delta_i} \Phi_i(s)\, sds\gamma + \int_0^{\delta_{i-1}} \Psi_i(s)\, sds\gamma \right] \\ &+ \int_0^{\delta_i} \Phi_i(s)\, ds\gamma t_i + \int_0^{\delta_{i-1}} \Psi_i(s)\, ds\gamma(t_i - \delta_i) \\ &= m_{0i} + m_{1i} t_i, \end{aligned}$$

and

$$\xi_{t_i} = \int_{t_{i-1}}^{t_i} \Phi_i(t_i - r)\, \zeta(dr) + \int_{t_{i-2}}^{t_{i-1}} \Psi_i(t_{i-1} - r)\, \zeta(dr).$$

Properties of the discrete time disturbances ξ_{t_i} are characterized as follows.

The mean of the disturbance vector is obtained as

$$\begin{aligned} E[\xi_{t_i}] &= E\left[\int_{t_{i-1}}^{t_i} \Phi_i(t_i - r)\, \zeta(dr) + \int_{t_{i-2}}^{t_{i-1}} \Psi_i(t_{i-1} - r) \right] \\ &= \int_{t_{i-1}}^{t_i} \Phi_i(t_i - r)\, E[\zeta(dr)] + \int_{t_{i-2}}^{t_{i-1}} \Psi_i(t_{i-1} - r)\, E[\zeta(dr)] \\ &= 0. \end{aligned}$$

The variance of the disturbance vector is obtained as

$$\begin{aligned}
E[\xi_{t_i}\xi'_{t_i}] &= E\left[\int_{t_{i-1}}^{t_i} \Phi_i(t_i - r)\,\zeta(dr) + \int_{t_{i-2}}^{t_{i-1}} \Psi_i(t_{i-1} - r)\right] \\
&\times \left[\int_{t_{i-1}}^{t_i} \Phi_i(t_i - r)\,\zeta(dr) + \int_{t_{i-2}}^{t_{i-1}} \Psi_i(t_{i-1} - r)\right]' \\
&= \left[\int_{t_{i-1}}^{t_i} \Phi_i(t_i - r)\Sigma\Phi'_i(t_i - r)\,dr\right] \\
&+ \left[\int_{t_{i-2}}^{t_{i-1}} \Psi_i(t_{i-1} - r)\Sigma\Psi'_i(t_{i-1} - r)\,dr\right] \\
&= \left[\int_0^{\delta_i} \Phi_i(r)\Sigma\Phi'_i(r)\,dr\right] + \left[\int_0^{\delta_{i-1}} \Psi_i(r)\Sigma\Psi'_i(r)\,dr\right].
\end{aligned}$$

The autocovariance of the disturbance vector is expressed as

$$\begin{aligned}
E[\xi_{t_i}\xi'_{t_j}] &= E\left[\int_{t_{i-1}}^{t_i} \Phi_i(t_i - r)\,\zeta(dr) + \int_{t_{i-2}}^{t_{i-1}} \Psi_i(t_{i-1} - r)\right] \\
&\times \left[\int_{t_{j-1}}^{t_j} \Phi_j(t_j - r)\,\zeta(dr) + \int_{t_{j-2}}^{t_{j-1}} \Psi_j(t_{j-1} - r)\right]'.
\end{aligned}$$

If $j = i - 1$, then the covariance becomes

$$\begin{aligned}
E[\xi_{t_i}\xi'_{t_{i-1}}] &= \left[\int_{t_{i-2}}^{t_{i-1}} \Psi_i(t_{i-1} - r)\Sigma\Phi'_i(t_{i-1} - r)\,dr\right] \\
&= \left[\int_0^{\delta_{i-1}} \Psi_i(r)\Sigma\Phi'_i(r)\,dr\right].
\end{aligned}$$

If $j = i + 1$, then the covariance becomes

$$\begin{aligned}
E[\xi_{t_i}\xi'_{t_{i+1}}] &= \left[\int_{t_{i-1}}^{t_i} \Phi_i(t_i - r)\Sigma\Psi'_i(t_i - r)\,dr\right] \\
&= \left[\int_0^{\delta_i} \Phi_i(r)\Sigma\Psi'_i(r)\,dr\right].
\end{aligned}$$

End of proof.

The properties of the discrete time disturbance vector ξ_{t_i} depend on the properties of continuous time disturbance vector $\zeta(dt)$ in system (2.1). Obviously, the discrete disturbance vector ξ_{t_i} has moving average

(MA) properties since it has zero mean, heteroskedastic variance and are correlated with one lead and one lag respectively.

The discrete time model with flow variables follows a heteroskedastic VARMA(1, 1) process with time-varying coefficients. When allowing for irregularity in observation intervals, the discrete time model appears to exhibit more complicated properties such as the coefficients are no longer constant but change with observations over time (time-varying), the variance of the discrete time disturbances is heteroskedastic in both stock-variable model and flow-variable model, while the auto-covariance of the disturbances in the model with flow variables is asymmetric. The derivation of the estimation method requires taking account of time--varying coefficients, heteroskedastic variance and asymmetric covariance simultaneously.

2.5 Discrete Time Representation of Models with Mixed Sample

In this section, we shall consider a system that includes both stock variables and flow variables. The following of this section derives the discrete model from a mixed sample, which follows Agbeyegbe's (1987) procedure. Consider a system includes a sample of observations that contain a mixture of stocks and flows. The observation can be partitioned into stocks and flows as

$$x(t_i) = \begin{bmatrix} x^s(t_i) \\ x^f(t_i) \end{bmatrix}, i = 1, 2, ..., T.$$

Note $x^s(t_i)$ is a vector of $(n^s \times 1)$ stock variables and $x^f(t_i)$ is a vector of $(n^f \times 1)$ flow variables, with $n^s + n^f = n$. Since both types of variables are observed at the same frequency, the numbers of observations for stocks and flows are equal. Stock variables are observed at discrete points of time: $t = t_1, t_2, ..., t_T$, while flow variables are observed as intervals over $[t_{i-1}, t_i]$, with $t = t_1, t_2, ..., t_T$.

The system of stock and flow variables, generated by (2.1), is partitioned as

$$d(x^s(t)) = [A^{ss}x^s(t) + A^{sf}x^f(t) + \mu^s + \gamma^s t]dt + \zeta^s(dt), \tag{2.14}$$

$$d(x^f(t)) = [A^{fs}x^s(t) + A^{ff}x^f(t) + \mu^f + \gamma^f t]dt + \zeta^f(dt), \tag{2.15}$$

where $A = \begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix}, \mu = \begin{bmatrix} \mu^s \\ \mu^f \end{bmatrix}, \gamma = \begin{bmatrix} \gamma^s \\ \gamma^f \end{bmatrix}$,and $\zeta(dt) = \begin{bmatrix} \zeta^s(dt) \\ \zeta^f(dt) \end{bmatrix}$.

In order to derive the exact discrete time model, given that assumption 2.1 and 2.2 are satisfied and the Boundary Condition holds, the result of the exact discrete time representation is valid if the following assumptions on the sub-matrix of matrix A is also valid.

Assumption 2.3. The matrix $e^{\delta_i A} - I$ is non-singular for all $i = 1, 2, ..., T$;

Assumption 2.4. The sub-matrix A^{ss} is non-singular.

To derive the discrete time model with a mixture of stocks and flows, it is necessary to define an $(n \times 1)$ random vector $z_{t_1}, z_{t_2}, ..., z_{t_n}$ in the form

$$z_{t_i} = \begin{bmatrix} x^s(t_i) - x^s(t_{i-1}) \\ \int_{t_{i-1}}^{t_i} x^f(r)\, dr \end{bmatrix}, i = 1, 2, ..., T. \tag{2.16}$$

The vector z_{t_i} defined above represents a mixture of stock variables and flow variables. Given that Assumption 2.1 to 2.4 are satisfied and the boundary condition holds, the exact discrete time model is given by Theorem 2.3.

Theorem 2.3. Under Assumption 2.1 to 2.4, subject to the boundary condition, the random vectors $z_{t_1}, z_{t_2}, ..., z_{t_n}$ defined by (2.17) satisfy the system

$$z_{t_i} = \Pi_i z_{t_{i-1}} + k_{t_i} + \epsilon_{t_i}, \tag{2.17}$$

$$E[\epsilon_{t_i}] = 0,$$

$$E[\epsilon_{t_i}\epsilon'_{t_i}] = W_i = \int_0^{\delta_i} [M(r)\Sigma M(r)']\, dr + \int_0^{\delta_{i-1}} [N(r)\Sigma N(r)']\, dr,$$

$$\mathrm{E}[\epsilon_{t_i}\epsilon'_{t_j}] = \begin{cases} Q_i = \int_0^{\delta_{i-1}} [N(r)\Sigma M'(r)]\, dr & \text{if } j = i-1 \\ R_i = \int_0^{\delta_i} [M(r)\Sigma N'(r)]\, dr & \text{if } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Pi_i = \begin{bmatrix} \Pi_i^{11} & \Pi_i^{12} \\ \Pi_i^{21} & \Pi_i^{22} \end{bmatrix},$$

$$k_{t_i} = \begin{bmatrix} k_{t_i}^s \\ k_{t_i}^f \end{bmatrix},$$

$$\begin{aligned} \epsilon_{t_i} &= \int_{t_{i-1}}^{t_i} M(t_i - r) \begin{bmatrix} \zeta^s(dr) \\ \zeta^f(dr) \end{bmatrix} + \int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r) \begin{bmatrix} \zeta^s(dr) \\ \zeta^f(dr) \end{bmatrix} \\ &= \begin{bmatrix} \epsilon_{t_i}^s \\ \epsilon_{t_i}^f \end{bmatrix}, \end{aligned}$$

$$\Pi_i^{11} = [A^{ss}\theta_i^{ss} + A^{sf}\theta_i^{fs}][A^{ss}]^{-1},$$

$$\Pi_i^{12} = [A^{ss}\theta_i^{sf} + A^{sf}\theta_i^{ff}] - \Pi_i^{11}A^{sf},$$

$$\Pi_i^{21} = \theta_i^{fs}[A^{ss}]^{-1},$$

$$\Pi_i^{22} = \theta_i^{ff} - \Pi_i^{21}A^{sf},$$

$$k_{t_i}^s = A^{ss}q_i^s + A^{sf}q_i^f + \int_{t_{i-1}}^{t_i}[\mu^s + \gamma^s r]\,dr - \Pi_i^{11}\int_{t_{i-2}}^{t_{i-1}}[\mu^s + \gamma^s r]\,dr$$

$$k_{t_i}^f = q_i^f - \Pi_i^{21}\int_{t_{i-2}}^{t_{i-1}}[\mu^s + \gamma^s r]\,dr,$$

$$\begin{aligned} q_i &= A^{-1}\int_{t_{i-1}}^{t_i}[e^{(t_i-r)A} - I][\mu + \gamma r]\,dr \\ &+ A^{-1}[e^{\delta_i A} - I]\int_{t_{i-2}}^{t_{i-1}}\left\{e^{(t_{i-1}-r)A} - e^{\delta_{i-1}A}[e^{\delta_{i-1}A} - I]^{-1}[e^{(t_{i-1}-r)A} - I]\right\}[\mu + \gamma r]\,dr \\ &= \begin{bmatrix} q_i^s \\ q_i^f \end{bmatrix}, \end{aligned}$$

$$\epsilon_{t_i}^s = \int_{t_{i-1}}^{t_i}\zeta^s(dr) + A^{ss}\zeta_{t_i}^s + A^{sf}\zeta_{t_i}^f - \Pi_i^{11}\int_{t_{i-2}}^{t_{i-1}}\zeta^s(dr),$$

$$\epsilon_{t_i}^f = \zeta_{t_i}^f - \Pi_i^{21}\int_{t_{i-2}}^{t_{i-1}}\zeta^s(dr),$$

$$M(t_i - r) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{ss} & A^{sf} \\ 0 & I \end{bmatrix}\begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix}^{-1}\begin{bmatrix} [e^{(t_i-r)A}]^{ss} - I & [e^{(t_i-r)A}]^{sf} \\ [e^{(t_i-r)A}]^{fs} & [e^{(t_i-r)A}]^{ff} - I \end{bmatrix},$$

$$\begin{aligned} N(t_{i-1} - r) &= \begin{bmatrix} A^{ss} & A^{sf} \\ 0 & I \end{bmatrix}\begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix}^{-1}\begin{bmatrix} [e^{\delta_i A}]^{ss} - I & [e^{\delta_i A}]^{sf} \\ [e^{\delta_i A}]^{fs} & [e^{\delta_i A}]^{ff} - I \end{bmatrix} \\ &\times \left\{\begin{bmatrix} [e^{(t_{i-1}-r)A}]^{ss} & [e^{(t_{i-1}-r)A}]^{sf} \\ [e^{(t_{i-1}-r)A}]^{fs} & [e^{(t_{i-1}-r)A}]^{ff} \end{bmatrix} - \begin{bmatrix} [e^{\delta_{i-1}A}]^{ss} & [e^{\delta_{i-1}A}]^{sf} \\ [e^{\delta_{i-1}A}]^{fs} & [e^{\delta_{i-1}A}]^{ff} \end{bmatrix}\right. \\ &\times \left.\begin{bmatrix} [e^{\delta_{i-1}A}]^{ss} - I & [e^{\delta_{i-1}A}]^{sf} \\ [e^{\delta_{i-1}A}]^{fs} & [e^{\delta_{i-1}A}]^{ff} - I \end{bmatrix}^{-1}\begin{bmatrix} [e^{(t_{i-1}-r)A}]^{ss} - I & [e^{(t_{i-1}-r)A}]^{sf} \\ [e^{(t_{i-1}-r)A}]^{fs} & [e^{(t_{i-1}-r)A}]^{ff} - I \end{bmatrix}\right\} \\ &- \begin{bmatrix} \Pi_i^{11} & 0 \\ \Pi_i^{21} & 0 \end{bmatrix}. \end{aligned}$$

Proof: Integrating (2.1) over the interval $[t_{i-1}, t_i]$ obtains

$$x(t_i) - x(t_{i-1}) = A\int_{t_{i-1}}^{t_i} x(r)\,dr + \int_{t_{i-1}}^{t_i} [\mu + \gamma r]\,dr + \int_{t_{i-1}}^{t_i} \zeta(dr), \tag{2.18}$$

while the first row of equation (2.18) is

$$x^s(t_i) - x^s(t_{i-1}) = A^{ss}\int_{t_{i-1}}^{t_i} x^s(r)\,dr + A^{sf}\int_{t_{i-1}}^{t_i} x^f(r)\,dr + \int_{t_{i-1}}^{t_i} [\mu^s + \gamma^s r]\,dr + \int_{t_{i-1}}^{t_i} \zeta^s(dr). \tag{2.19}$$

Re-arranging (2.18) as

$$\int_{t_{i-1}}^{t_i} x(r)\,dr \quad = \quad A^{-1}[x(t_i) - x(t_{i-1})] - A^{-1}\int_{t_{i-1}}^{t_i} [\mu + \gamma r]\,dr - A^{-1}\int_{t_{i-1}}^{t_i} \zeta(dr). \tag{2.20}$$

Since (2.5) is valid

$$x(t_i) = e^{\delta_i A}x(t_{i-1}) + \int_{t_{i-1}}^{t_i} e^{(t_i - r)A}[\mu + \gamma r]\,dr + \int_{t_{i-1}}^{t_i} e^{(t_i - r)A}\,\zeta(dr), \tag{2.5}$$

where

$$\begin{aligned} e^{\delta_i A} &= \begin{bmatrix} [e^{\delta_i A}]^{ss} & [e^{\delta_i A}]^{sf} \\ [e^{\delta_i A}]^{fs} & [e^{\delta_i A}]^{ff} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \delta_i \begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix} + {\delta_i}^2/2! \begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix}^2 + ..., \end{aligned}$$

and

$$e^{(t_i - r)A} = \begin{bmatrix} [e^{(t_i - r)A}]^{ss} & [e^{(t_i - r)A}]^{sf} \\ [e^{(t_i - r)A}]^{fs} & [e^{(t_i - r)A}]^{ff} \end{bmatrix}.$$

Subtracting one lag from (2.5) yields

$$\begin{aligned} x(t_i) - x(t_{i-1}) \quad &= \quad e^{\delta_i A}x(t_{i-1}) - e^{\delta_{i-1} A}x(t_{i-2}) + \int_{t_{i-1}}^{t_i} e^{(t_i - r)A}[\mu + \gamma r]\,dr \\ &- \quad \int_{t_{i-2}}^{t_{i-1}} e^{(t_{i-1} - r)A}[\mu + \gamma r]\,dr + \int_{t_{i-1}}^{t_i} e^{(t_i - r)A}\,\zeta(dr) \\ &- \quad \int_{t_{i-2}}^{t_{i-1}} e^{(t_{i-1} - r)A}\,\zeta(dr). \end{aligned} \tag{2.21}$$

Lagging (2.5) for one period yields

$$x(t_{i-1}) = e^{\delta_{i-1}A}x(t_{i-2}) + \int_{t_{i-2}}^{t_{i-1}} e^{(t_{i-1}-r)A}[\mu + \gamma r]\,dr + \int_{t_{i-2}}^{t_{i-1}} e^{(t_{i-1}-r)A}\,\zeta(dr). \tag{2.22}$$

Substituting out $x(t_{i-1})$ in (2.21) on the right hand side by (2.22) yields

$$\begin{aligned} x(t_i) - x(t_{i-1}) &= [e^{\delta_i A} - I]e^{\delta_{i-1}A}x(t_{i-2}) + \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}[\mu + \gamma r]\,dr + \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\,\zeta(dr) \\ &+ \int_{t_{i-2}}^{t_{i-1}} [e^{\delta_i A} - I]e^{(t_{i-1}-r)A}[\mu + \gamma r]\,dr + \int_{t_{i-2}}^{t_{i-1}} [e^{\delta_i A} - I]e^{(t_{i-1}-r)A}\,\zeta(dr), \end{aligned} \tag{2.23}$$

where

$$e^{\delta_i A} - I = \begin{bmatrix} [e^{\delta_i A}]^{ss} - I & [e^{\delta_i A}]^{sf} \\ [e^{\delta_i A}]^{fs} & [e^{\delta_i A}]^{ff} - I \end{bmatrix}.$$

Lagging (2.18) for one period

$$x(t_{i-1}) - x(t_{i-2}) = A\int_{t_{i-2}}^{t_{i-1}} x(r)\,dr + \int_{t_{i-2}}^{t_{i-1}} [\mu + \gamma r]\,dr + \int_{t_{i-2}}^{t_{i-1}} \zeta(dr). \tag{2.24}$$

Substituting out $x(t_{i-1})$ in (2.24) by (2.22) yields:

$$\begin{aligned} [e^{\delta_{i-1}A} - I]x(t_{i-2}) &= A\int_{t_{i-2}}^{t_{i-1}} x(r)\,dr - \int_{t_{i-2}}^{t_{i-1}} [e^{(t_{i-1}-r)A} - I][\mu + \gamma r]\,dr \\ &- \int_{t_{i-2}}^{t_{i-1}} [e^{(t_{i-1}-r)A} - I]\zeta(dr), \end{aligned} \tag{2.25}$$

where

$$e^{(t_{i-1}-r)A} - I = \begin{bmatrix} [e^{(t_{i-1}-r)A}]^{ss} - I & [e^{(t_{i-1}-r)A}]^{sf} \\ [e^{(t_{i-1}-r)A}]^{fs} & [e^{(t_{i-1}-r)A}]^{ff} - I \end{bmatrix}.$$

Re-arranging (2.25) as

$$
\begin{aligned}
x(t_{i-2}) &= [e^{\delta_{i-1}A} - I]^{-1} A \int_{t_{i-2}}^{t_{i-1}} x(r)\, dr - [e^{\delta_{i-1}A} - I]^{-1} \int_{t_{i-2}}^{t_{i-1}} [e^{(t_{i-1}-r)A} - I][\mu + \gamma r]\, dr \\
&- [e^{\delta_{i-1}A} - I]^{-1} \int_{t_{i-2}}^{t_{i-1}} [e^{(t_{i-1}-r)A} - I]\zeta(dr). \qquad (2.26)
\end{aligned}
$$

Substituting out $x(t_{i-2})$ in (2.23) by (2.26) yields

$$
\begin{aligned}
x(t_i) - x(t_{i-1}) &= [e^{\delta_i A} - I]e^{\delta_{i-1}A}[e^{\delta_{i-1}A} - I]^{-1} A \int_{t_{i-2}}^{t_{i-1}} x(r)\, dr \\
&- [e^{\delta_i A} - I]e^{\delta_{i-1}A}[e^{\delta_{i-1}A} - I]^{-1} \int_{t_{i-2}}^{t_{i-1}} [e^{(t_{i-1}-r)A} - I][\mu + \gamma r]\, dr \\
&- [e^{\delta_i A} - I]e^{\delta_{i-1}A}[e^{\delta_{i-1}A} - I]^{-1} \int_{t_{i-2}}^{t_{i-1}} [e^{(t_{i-1}-r)A} - I]\zeta(dr) \\
&+ \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}[\mu + \gamma r]\, dr + \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\, \zeta(dr) \\
&+ \int_{t_{i-2}}^{t_{i-1}} [e^{\delta_i A} - I]e^{(t_{i-1}-r)A}[\mu + \gamma r]\, dr + \int_{t_{i-2}}^{t_{i-1}} [e^{\delta_i A} - I]e^{(t_{i-1}-r)A}\, \zeta(dr) \qquad (2.27)
\end{aligned}
$$

Combining (2.27) with (2.20), obtains

$$
\begin{aligned}
\int_{t_{i-1}}^{t_i} x(r)dr &= A^{-1}[e^{\delta_i A} - I]e^{\delta_{i-1}A}[e^{\delta_{i-1}A} - I]^{-1} A \int_{t_{i-2}}^{t_{i-1}} x(r)\, dr \\
&- A^{-1}[e^{\delta_i A} - I]e^{\delta_{i-1}A}[e^{\delta_{i-1}A} - I]^{-1} \int_{t_{i-2}}^{t_{i-1}} [e^{(t_{i-1}-r)A} - I][\mu + \gamma r]\, dr \\
&- A^{-1}[e^{\delta_i A} - I]e^{\delta_{i-1}A}[e^{\delta_{i-1}A} - I]^{-1} \int_{t_{i-2}}^{t_{i-1}} [e^{(t_{i-1}-r)A} - I]\zeta(dr) \\
&+ A^{-1} \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}[\mu + \gamma r]\, dr + A^{-1} \int_{t_{i-1}}^{t_i} e^{(t_i-r)A}\, \zeta(dr) \\
&+ A^{-1} \int_{t_{i-2}}^{t_{i-1}} [e^{\delta_i A} - I]e^{(t_{i-1}-r)A}[\mu + \gamma r]\, dr + A^{-1} \int_{t_{i-2}}^{t_{i-1}} [e^{\delta_i A} - I]e^{(t_{i-1}-r)A}\, \zeta(dr) \\
&- A^{-1} \int_{t_{i-1}}^{t_i} [\mu + \gamma r]\, dr - A^{-1} \int_{t_{i-1}}^{t_i} \zeta(dr) \\
&= \Theta_i \int_{t_{i-2}}^{t_{i-1}} x(r)\, dr + q_i + \zeta_{t_i}, \qquad (2.28)
\end{aligned}
$$

where

$$\begin{aligned}\Theta_i &= A^{-1}[e^{\delta_i A} - I]e^{\delta_{i-1}A}[e^{\delta_{i-1}A} - I]^{-1}A \\ &= \begin{bmatrix} \Theta_i^{ss} & \Theta_i^{sf} \\ \Theta_i^{fs} & \Theta_i^{ff} \end{bmatrix},\end{aligned}$$

$$q_i = \begin{bmatrix} q_i^s \\ q_i^f \end{bmatrix}$$

is define by (2.17),

$$\begin{aligned}\zeta_{t_i} &= A^{-1}\int_{t_{i-1}}^{t_i} [e^{(t_i - r)A} - I]\,\zeta(dr) \\ &+ A^{-1}[e^{\delta_i A} - I]\int_{t_{i-2}}^{t_{i-1}} \left\{e^{(t_{i-1}-r)A} - e^{\delta_{i-1}A}[e^{\delta_{i-1}A} - I]^{-1}[e^{(t_{i-1}-r)A} - I]\right\}\,\zeta(dr) \\ &= \begin{bmatrix} \zeta_{t_i}^s \\ \zeta_{t_i}^f \end{bmatrix}.\end{aligned}$$

Partitioning (2.28) as

$$\int_{t_{i-1}}^{t_i} x^s(r)dr = \int_{t_{i-2}}^{t_{i-1}} \theta_i^{ss} x^s(r)\,dr + \int_{t_{i-2}}^{t_{i-1}} \theta_i^{sf} x^f(r)\,dr + q_i^s + \zeta_{t_i}^s, \tag{2.29}$$

$$\int_{t_{i-1}}^{t_i} x^f(r)dr = \int_{t_{i-2}}^{t_{i-1}} \theta_i^{fs} x^s(r)\,dr + \int_{t_{i-2}}^{t_{i-1}} \theta_i^{ff} x^f(r)\,dr + q_i^f + \zeta_{t_i}^f. \tag{2.30}$$

Lagging (2.19) for one period obtains

$$\begin{aligned}x^s(t_{i-1}) - x^s(t_{i-2}) &= A^{ss}\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr + A^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \\ &+ \int_{t_{i-2}}^{t_{i-1}} [\mu^s + \gamma^s r]\,dr + \int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr).\end{aligned} \tag{2.31}$$

Re-arranging (2.31) as

$$\begin{aligned}\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr &= [A^{ss}]^{-1}[x^s(t_{i-1}) - x^s(t_{i-2})] - [A^{ss}]^{-1}A^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \\ &- [A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}[\mu^s + \gamma^s r]\,dr - [A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}\zeta^s(dr). \qquad (2.32)\end{aligned}$$

Substituting out $\int_{t_{i-1}}^{t_i} x^s(r)dr$ and $\int_{t_{i-1}}^{t_i} x^f(r)dr$ in (2.19) by (2.29) and (2.30), respectively, yields

$$\begin{aligned}x^s(t_i) - x^s(t_{i-1}) &= [A^{ss}\theta_i^{ss} + A^{sf}\theta_i^{fs}]\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr + [A^{ss}\theta_i^{sf} + A^{sf}\theta_i^{ff}]\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \\ &+ A^{ss}q_i^s + A^{ss}\zeta_{t_i}^s + A^{sf}q_i^f + A^{sf}\zeta_{t_i}^f \\ &+ \int_{t_{i-1}}^{t_i}[\mu^s + \gamma^s r]\,dr + \int_{t_{i-1}}^{t_i}\zeta^s(dr). \qquad (2.33)\end{aligned}$$

Then the object is to eliminate the unobservable term $\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr$ in (2.33) and (2.30), respectively. Substituting out $\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr$ in (2.33) using (2.32)

$$\begin{aligned}x^s(t_i) - x^s(t_{i-1}) &= [A^{ss}\theta_i^{ss} + A^{sf}\theta_i^{fs}][A^{ss}]^{-1}[[x^s(t_{i-1}) - x^s(t_{i-2})] \\ &- A^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr - \int_{t_{i-2}}^{t_{i-1}}[\mu^s + \gamma^s r]\,dr - \int_{t_{i-2}}^{t_{i-1}}\zeta^s(dr)] \\ &+ [A^{ss}\theta_i^{ss} + A^{sf}\theta_i^{ff}]\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + A^{ss}q_i^s + A^{ss}\zeta_{t_i}^s + A^{sf}q_i^f + A^{sf}\zeta_{t_i}^f \\ &+ \int_{t_{i-1}}^{t_i}[\mu^s + \gamma^s r]\,dr + \int_{t_{i-1}}^{t_i}\zeta^s(dr) \\ &= \Pi_i^{11}[x^s(t_{i-1}) - x^s(t_{i-2})] + \Pi_i^{12}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + g_i^s + \epsilon_{t_i}^s. \qquad (2.34)\end{aligned}$$

Substituting out $\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr$ in (2.30) using (2.32) yields

$$\begin{aligned}\int_{t_{i-1}}^{t_i} x^f(r)dr &= \theta_i^{fs}[A^{ss}]^{-1}[[x^s(t_{i-1}) - x^s(t_{i-2})] \\ &- A^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr - \int_{t_{i-2}}^{t_{i-1}}[\mu^s + \gamma^s r]\,dr - \int_{t_{i-2}}^{t_{i-1}}\zeta^s(dr)] \\ &+ \theta_i^{ff}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + q_i^f + \zeta_{t_i}^f \\ &= \Pi_i^{21}[x^s(t_{i-1}) - x^s(t_{i-2})] + \Pi_i^{22}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + g_i^f + \epsilon_{t_i}^f. \qquad (2.35)\end{aligned}$$

Combining (2.34) and (2.35) obtain

$$\begin{bmatrix} x^s(t_i) - x^s(t_{i-1}) \\ \int_{t_{i-1}}^{t_i} x^f(r)\, dr \end{bmatrix} = \begin{bmatrix} \Pi_i^{11} & \Pi_i^{12} \\ \Pi_i^{21} & \Pi_i^{22} \end{bmatrix} \begin{bmatrix} x^s(t_{i-1}) - x^s(t_{i-2}) \\ \int_{t_{i-2}}^{t_{i-1}} x^f(r)\, dr \end{bmatrix} + \begin{bmatrix} g_i^s \\ g_i^f \end{bmatrix} + \begin{bmatrix} \epsilon_{t_i}^s \\ \epsilon_{t_i}^f \end{bmatrix}. \tag{2.36}$$

Properties of vector ϵ_{t_i} depend on properties of the continuous time disturbance vector $\zeta(dt)$ in system (2.1).

The mean of the disturbance vector is obtained as

$$E[\epsilon_{t_i}] = 0.$$

The variance of the disturbance vector is obtained as

$$\begin{aligned} E[\epsilon_{t_i}\epsilon_{t_i}'] &= \int_{t_{i-1}}^{t_i} [M(t_i - r)\Sigma M(t_i - r)']\, dr + \int_{t_{i-2}}^{t_{i-1}} [N(t_{i-1} - r)\Sigma N(t_{i-1} - r)']\, dr \\ &= \int_0^{\delta_i} [M(r)\Sigma M(r)']\, dr + \int_0^{\delta_{i-1}} [N(r)\Sigma N(r)']\, dr. \end{aligned}$$

The autocovariance of the disturbance vector is presented as

$$\begin{aligned} E[\epsilon_{t_i}\epsilon_{t_j}'] &= E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_{j-1}}^{t_j} M(t_j - r)\zeta(dr)\right]' \\ &+ E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_{j-2}}^{t_{j-1}} N(t_{j-1} - r)\zeta(dr)\right]' \\ &+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{j-1}}^{t_j} M(t_j - r)\zeta(dr)\right]' \\ &+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{j-2}}^{t_{j-1}} N(t_{j-1} - r)\zeta(dr)\right]'. \end{aligned}$$

If $j = i - 1$, the autocovariance becomes

$$\begin{aligned} E[\epsilon_{t_i} \epsilon'_{t_{i-1}}] &= \int_{t_{i-2}}^{t_{i-1}} [N(t_{i-1} - r)] \Sigma M'(t_{i-1} - r)] \, dr \\ &= \int_{0}^{\delta_{i-1}} [N(r) \Sigma M'(r)] \, dr. \end{aligned}$$

If $j = i + 1$, the autocovariance becomes

$$\begin{aligned} E[\epsilon_{t_i} \epsilon'_{t_{i+1}}] &= \int_{t_{i-1}}^{t_i} [M(t_i - r)] \Sigma N'(t_i - r)] \, dr \\ &= \int_{0}^{\delta_i} [M(r) \Sigma N'(r)] \, dr. \end{aligned}$$

End of proof.

The exact discrete time model presented in Theorem 2.3 follows a VARMA(1, 1) process, which holds for stationary process of first order. Similar to the discrete disturbance vector in the flow variable case, the discrete time vector ϵ_{t_i} forms a MA(1) process of order one with zero mean, heteroskedastic variance and are correlated with one lead and one lag respectively. Interestingly, the covariance of vector ϵ_{t_i} is time-varying, whose value depends the length of the observation interval.

2.6 A Monte Carlo Simulation Study

To test whether the discrete time model with unequally spaced sample improves estimation performance (such as smaller estimation bias), a Monte Carlo study is conducted. That is, given an unequally spaced sample whose intervals are known, would the estimates exhibit better properties when the irregular sampling intervals are correctly meadured?

This study only concerns the simplest case in which a first-order scalar continuous time model with stock variable is considered. The main procedure of this study is to simulate a sample of monthly stock data for a 20-year span with the sample interval varies with the length of calendar month. Then, using the same simulated data, the study compares estimation results by using the "unequally-sampled" model where observations are unequally spaced to the estimation results using the "equally-sampled" model where observations are treated as equally spaced. As has been expected, the simulation results show that the "unequally-sampled" model provides estimate that has better properties, compared to the "equally-sampled" model. The study also explores how the parameter estimation results vary with the value of the continuous time parameter a. Both maximum likelihood estimate ("unequally-sampled" model) and OLS estimate ("equally-sampled" model) perform well, while the former provides less biased estimations. With non-stationary time series, when the value of a increases the data explodes at a faster rate, the OLS estimate turn to have poor properties.

2.6.1 A Brief Introduction

This subsection briefly describes the procedure of the Monte Carlo study. We shall consider a simple case: a first-order scalar continuous time model

$$dx(t) = ax(t)dt + \zeta(dt), \tag{2.37}$$

with

$$\zeta(dt) \sim NID(0, \sigma^2 dt).$$

Only stock variable is included. Let $x(t)$ be observed at a sequence of known unequal time points as $x(t_i)$ with $i = 1, 2, ..., T$.

Based on the result suggested in Theorem 2.1, the exact discrete time model of (2.37) is obtained as

$$x(t_i) = e^{a\delta_i} x(t_{i-1}) + \eta(t_i), \tag{2.38}$$

with

$$\eta(t_i) = \int_{t_{i-1}}^{t_i} e^{a(t_i - r)} \zeta(dr).$$

Next we characterize the properties of the discrete time disturbance $\eta(t_i)$. The mean of $\eta(t_i)$ is obtained as

$$E[\eta(t_i)] \quad = \quad 0,$$

while the variance of $\eta(t_i)$ is obtained as

$$\begin{aligned} E[\eta^2(t_i)] &= \sigma^2 \int_{t_{i-1}}^{t_i} e^{2a(t_i - r)} \zeta(dr) \\ &= \sigma^2 (e^{2a\delta_i} - 1)/2a \\ &= {\sigma_i}^2. \end{aligned}$$

We shall then allocate some values to the continuous time parameters a and σ. To explore how the estimation results vary with the value of a, we allocate a with a range of values from -0.95 to 0.03 (presented in table 1), while $\sigma = 1$ in all cases. Note that the results in Theorem 2.1 are valid for also non-stationary process, so we allow a to take small positive values as well.

In the next step, we shall specify a small sample of stock variable, which includes 240 monthly stocks for 20 year starts from January. The first observation is observed in January, thus its interval is normalised as $31/30$, which equals to 1.03. Whereas the second observation is made in February, whose interval is thus 0.93 $(28/90)$. Hence the vector of observation intervals for the first 12 observations (first year) is $[1.03, 0.93, 1.03, 1.00, 1.03, 1.00, 1.03, 1.03, 1.00, 1.03, 1.00, 1.03]'$, and is repeated 20 times. Note that, to simplify the estimation procedure, all Februaries are assumed to have 28 days.

We have the sample size $n = 240$, and discrete time parameters $e^{a\delta_i}$ and ${\sigma_i}^2 = \sigma^2(e^{2a\delta_i} - 1)/2a$, which is used to generate data for $x(t_i)$. Given the values of a and σ, we are able to calculate the values of the discrete time parameters. Then data for $x(t_i)$, whose form is given in equation (2.38), is simulated by using a random generation process. Each sample includes 240 observations and the simulation is repeated for 10,000 times (namely, 10,000 replications).

The last step is to estimate the continuous time parameters using the data simulated in the previous steps. The Gaussian estimates of parameters $\hat{a}$ and $\hat{\sigma}^2$ is obtained when the Gaussian Log-likelihood function

$$L(a, \sigma^2) = -(n/2)\ln(2\pi) - 1/2\sum_{i=1}^{n}\ln\sigma_i^2 - 1/2\sum_{i=1}^{n}[x(t_i) - e^{a\delta_i}x(t_{i-1})]^2/\sigma_i^2$$

is maximized. Note that the Gaussian likelihood function is a function of the continuous time parameters. After the 10,000 replications we obtain the distribution of the maximum likelihood estimates. The closer the mean of the estimates $E[\hat{a}]$ and $E[\hat{\sigma}]$ to their "true values" ($a = -0.95, ..., 0.03$ and $\sigma = 1$), the better the estimate properties are.

The next object is to estimate the "equally-sampled" model in which observations are treated as equally spaced by using the same data simulated before. If the "unequally-sampled" model improves estimation accuracy, we would expect the values of the estimates from the "unequally-sampled" model to be closer to their "true values", compared to the values of the estimates from the "equally-sampled" model.

The exact discrete model assuming equally spaced sample is obtained as

$$x(t) = \phi x(t-1) + \epsilon(t), \tag{2.39}$$

where

$$\phi = e^a,$$

and

$$\epsilon(t) = \int_{t-1}^{t} e^{a(t-r)}\zeta(dr).$$

The discrete observation interval $h = t - (t-1) = 1$ for all observations. The mean of the discrete time disturbance $\epsilon(t)$ is

$$\begin{aligned} E[\epsilon(t)] &= \int_{t-1}^{t} e^{a(t-r)} E[\zeta(dr)] \\ &= 0, \end{aligned}$$

and the variance of $\epsilon(t)$ is

$$\begin{aligned} E[\epsilon^2(t)] &= \sigma^2 \int_{t-1}^{t} e^{2a(t-r)} dr \\ &= \sigma^2 \int_{1}^{0} e^{2ar} dr \\ &= \sigma^2 (e^{2a} - 1)/2a \\ &= \sigma_\epsilon^2. \end{aligned}$$

Note $\epsilon(t)$ is white noise and $x(t)$ is an AR(1) process. The variance of $\epsilon(t)$ depends on the continuous time autoregressive parameter a. With stock variables, (2.48) could be estimated consistently by OLS, which gives the same estimates as maximum likelihood estimate. Note this gives estimates of the discrete parameters $\hat{\phi}$ and $\hat{\sigma}_\epsilon$. Then the estimates of the continuous time parameters a and σ are obtained as

$$\hat{a} = \ln \hat{\phi},$$

and

$$\hat{\sigma}^2 = 2\hat{a}\hat{\sigma}_\epsilon^2/(e^{2\hat{a}} - 1).$$

The next subsection discusses the Monte Carlo simulation results.

2.6.2 Simulation Results

As expected, by considering irregular observation intervals, the model provides more accurate estimates. The Maximum likelihood estimate (based on "unequally-sampled" model) has better properties compared to OLS estimate (based on "equally-sampled" model). Table 2.1 presents bias and standard errors (in parenthesis) of each estimator under different values of a while σ always equals to 1. $\hat{a}_1$ and $\hat{\sigma}_1^2$ represents maximum likelihood estimators of a and σ^2, respectively; while $\hat{a}_2$ and $\hat{\sigma}_2^2$ are the OLS estimators of a and σ^2, respectively.

With stationary time series (when a takes negative values), maximum likelihood estimate is obviously less biased. Both maximum likelihood and OLS estimates underestimate a (as suggested by the negative sign) while overestimate σ^2. In particular, estimation bias of OLS estimator $\hat{\sigma}_2^2$, is significantly larger than that of maximum likelihood estimator $\hat{\sigma}_1^2$. With non-stationary process, both estimates present similar results despite that OLS estimator of σ^2 is more biased. When a equals to 0 the process becomes a pure random walk, maximum likelihood estimate provides less biased estimation of a while OLS estimate provides less biased estimation of σ^2, but the two estimates provide similar results. With the increase in value of a, the process explode at a faster rate, while maximum likelihood still perform well, OLS estimate turns to over estimate the continuous time variance σ^2. When a takes value of 0.03, as shown in column 8, the bias of OLS estimate of σ^2 is 1.427 with standard error of 2.07, which is far from its "true" value.

Overall, the Monte Carlo simulation results suggest that estimation from a model where observation intervals are measured correctly rather than being assumed as equal, turn to have better properties.

Table 2.1: Monte Carlo Estimation Results

Estimator	Value of a						
	-0.95	-0.50	-0.05	0	0.01	0.02	0.03
Bias & standard error							
$\hat{a}_1$	-0.020	-0.012	-0.008	-0.007	-0.005	-0.001	0.000
	(0.164)	(0.088)	(0.025)	(0.013)	(0.010)	(0.006)	(0.004)
$\hat{a}_2$	-0.033	-0.019	-0.009	-0.008	-0.005	-0.001	-0.000
	(0.166)	(0.089)	(0.025)	(0.014)	(0.011)	(0.006)	(0.004)
$\hat{\sigma}_1^2$	0.008	0.003	-0.001	0.019	-0.006	-0.010	-0.009
	(0.147)	(0.059)	(0.094)	(0.093)	(0.091)	(0.090)	(0.091)
$\hat{\sigma}_2^2$	0.026	0.021	0.017	0.009	0.016	0.021	1.427
	(0.150)	(0.120)	(0.096)	(0.093)	(0.093)	(0.096)	(2.076)

2.7 Conclusion and Discussion

This chapter has provided exact discrete time representations of continuous time models with stocks, flows and mixed sample that are irregularly spaced. The discrete time model with stock variables follows a VAR(1) process, which has heteroskedastic disturbances which are serially uncorrelated. While the discrete time models with flow variables or with mixed sample follow a VARMA$(1, 1)$ process, which have heteroskedastic disturbances with asymmetric covariance.

Given that some data are not regularly spaced and the length of observation intervals are known, the chapter then provides a simple Monte Carlo simulation aiming at examining whether our model improves estimation accuracy. In the study, a sample of monthly stock observations is simulated. The Monte Carlo study is completed by estimating the continuous time parameters from the "unequally soaced" model and the "equally spaced" model using the same data simulated.

The simulation results suggest that by introducing irregularity in observation intervals to the discrete time models, one would be able to obtain estimates that have better properties. The estimates (maximum likelihood) of the parameters from the "unequally-sampled" model are more accurate than estimates (OLS) from the "equally-sampled" model. The simulation is repeated several times with different values of the continuous time coefficient parameter a (from -0.95 to 0.03). The maximum likelihood estimate performs well in all cases and is less biased. The OLS estimate performs well when a is smaller than 0 (with stationary process), however, it turns to over estimate the continuous time variance parameter σ^2 significantly when a exceeds 0 and gets larger (with non-stationary process).

This chapter has brought up a few issues which can be addressed in future research. Firstly, results presented in Theorem 2.2 and 2.3 are valid for only stationary systems, which could limit the applications to non-stationary systems such as unite root or cointegrated systems. It might be possible to derive the exact discrete time model without relying on Assumptions 2.2 to 2.4 using other technique. One potential alternative assumption is the existence of the mean-square derivative of the stock variables since the integral of the mean square derivative of the stocks is the first difference of the stock variables as defined in (2.16). Given such assumption, McCrorie (2000) proposed an alternative method for deriving the exact discrete time model by integrating the solution of the continuous time model in state space form; and then the covariance matrix is derived via a nonstandard change in the order of three type of integration without further restrictions on the data. By using Cholesky factorization of the covariance matrix, the exact discrete time representation is obtained as a VARMA process. Although this approach provides the possibility to derive the discrete time representation without ruling out nonstationary processes, it comes at the cost of additional complexity in the form of the covariance matrix.

Other issues related to the exact discrete method include the aliasing problem, the controllability problem and finite sample estimation bias. These issues may be considered in further extended research. In addition, the model might be extended to higher order multivariate system, which might be used in macroeconomic studies. However, estimating higher-order models could be much more complicated, which involves problems of defining observable vectors with mixed data as well as requiring the rank condition to eliminate unobservable variables (see detailed discussions in McCrorie, 2009).

2.8 Appendix A

Characterise Properties of Vector ϵ_{t_i}

$$
\begin{aligned}
\epsilon_{t_i} &= \begin{bmatrix} \int_{t_{i-1}}^{t_i} \zeta^s(dr) + A^{ss}\zeta_{t_i}^s + A^{sf}\zeta_{t_i}^f - \Pi_i^{11}\int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr) \\ \zeta_{t_i}^f - \Pi_i^{21}\int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr) \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \int_{t_{i-1}}^{t_i} \zeta^s(dr) \\ \int_{t_{i-1}}^{t_i} \zeta^f(dr) \end{bmatrix} + \begin{bmatrix} A^{ss} & A^{sf} \\ 0 & I \end{bmatrix} \begin{bmatrix} \zeta_{t_i}^s \\ \zeta_{t_i}^f \end{bmatrix} \\
&- \begin{bmatrix} \Pi_i^{11} & 0 \\ \Pi_i^{21} & 0 \end{bmatrix} \begin{bmatrix} \int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr) \\ \int_{t_{i-2}}^{t_{i-1}} \zeta^f(dr) \end{bmatrix} \\
&= \int_{t_{i-1}}^{t_i} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta^s(dr) \\ \zeta^f(dr) \end{bmatrix} + \int_{t_{i-1}}^{t_i} \begin{bmatrix} A^{ss} & A^{sf} \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix}^{-1} \\
&\times \begin{bmatrix} [e^{(t_i-r)A}]^{ss} - I & [e^{(t_i-r)A}]^{sf} \\ [e^{(t_i-r)A}]^{fs} & [e^{(t_i-r)A}]^{ff} - I \end{bmatrix} \begin{bmatrix} \zeta^s(dr) \\ \zeta^f(dr) \end{bmatrix} \\
&+ \int_{t_{i-2}}^{t_{i-1}} \begin{bmatrix} A^{ss} & A^{sf} \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix}^{-1} \begin{bmatrix} [e^{\delta_i A}]^{ss} - I & [e^{\delta_i A}]^{sf} \\ [e^{\delta_i A}]^{fs} & [e^{\delta_i A}]^{ff} - I \end{bmatrix} \\
&\times \left\{ \begin{bmatrix} [e^{(t_{i-1}-r)A}]^{ss} & [e^{(t_{i-1}-r)A}]^{sf} \\ [e^{(t_{i-1}-r)A}]^{fs} & [e^{(t_{i-1}-r)A}]^{ff} \end{bmatrix} - \begin{bmatrix} [e^{\delta_{i-1} A}]^{ss} & [e^{\delta_{i-1} A}]^{sf} \\ [e^{\delta_{i-1} A}]^{fs} & [e^{\delta_{i-1} A}]^{ff} \end{bmatrix} \right. \\
&\times \left. \begin{bmatrix} [e^{\delta_{i-1} A}]^{ss} - I & [e^{\delta_{i-1} A}]^{sf} \\ [e^{\delta_{i-1} A}]^{fs} & [e^{\delta_{i-1} A}]^{ff} - I \end{bmatrix}^{-1} \begin{bmatrix} [e^{(t_{i-1}-r)A}]^{ss} - I & [e^{(t_{i-1}-r)A}]^{sf} \\ [e^{(t_{i-1}-r)A}]^{fs} & [e^{(t_{i-1}-r)A}]^{ff} - I \end{bmatrix} \right\} \\
&\times \begin{bmatrix} \zeta^s(dr) \\ \zeta^f(dr) \end{bmatrix} - \int_{t_{i-2}}^{t_{i-1}} \begin{bmatrix} \Pi_i^{11} & 0 \\ \Pi_i^{21} & 0 \end{bmatrix} \begin{bmatrix} \zeta^s(dr) \\ \zeta^f(dr) \end{bmatrix} \\
&= \int_{t_{i-1}}^{t_i} M(t_i - r) \begin{bmatrix} \zeta^s(dr) \\ \zeta^f(dr) \end{bmatrix} + \int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r) \begin{bmatrix} \zeta^s(dr) \\ \zeta^f(dr) \end{bmatrix}
\end{aligned}
$$

The variance is obtained as

$$
\begin{aligned}
E[\epsilon_{t_i}\epsilon_{t_i}'] &= E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]' \\
&= \int_{t_{i-1}}^{t_i} [M(t_i - r)\Sigma M(t_i - r)']\,dr + \int_{t_{i-2}}^{t_{i-1}} [N(t_{i-1} - r)\Sigma N(t_{i-1} - r)']\,dr \\
&= \int_0^{\delta_i} [M(r)\Sigma M(r)']\,dr + \int_0^{\delta_{i-1}} [N(r)\Sigma N(r)']\,dr \\
&= W_i
\end{aligned}
$$

The covariance

$$
\begin{aligned}
E[\epsilon_{t_i}\epsilon_{t_j}'] &= E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_{j-1}}^{t_j} M(t_j - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_{j-2}}^{t_{j-1}} N(t_{j-1} - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{j-1}}^{t_j} M(t_j - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{j-2}}^{t_{j-1}} N(t_{j-1} - r)\zeta(dr)\right]'
\end{aligned}
$$

If $j = i - 1$, the covariance becomes:

$$\begin{aligned}
E[\epsilon_{t_i}\epsilon'_{t_{i-1}}] &= E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}} M(t_{i-1} - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_{i-3}}^{t_{i-2}} N(t_{i-2} - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}} M(t_{i-1} - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{j-3}}^{t_{i-2}} N(t_{i-2} - r)\zeta(dr)\right]' \\
&= \int_{t_{i-2}}^{t_{i-1}} [N(t_{i-1} - r)]\Sigma M'(t_{i-1} - r)]\, dr \\
&= \int_{0}^{\delta_{i-1}} [N(r)\Sigma M'(r)]\, dr
\end{aligned}$$

If $j = i + 1$, the covariance becomes:

$$\begin{aligned}
E[\epsilon_{t_i}\epsilon'_{t_{i+1}}] &= E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_i}^{t_{i+1}} M(t_{i+1} - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-1}}^{t_i} M(t_i - r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} N(t_i - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_i}^{t_{i+1}} M(t_{i+1} - r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}} N(t_{i-1} - r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} N(t_i - r)\zeta(dr)\right]' \\
&= \int_{t_{i-1}}^{t_i} [M(t_i - r)]\Sigma N'(t_i - r)]\, dr \\
&= \int_{0}^{\delta_i} [M(r)\Sigma N'(r)]\, dr
\end{aligned}$$

3 Discrete Time Representation of Non-stationary Continuous Time Models with Unequally Spaced Data

This chapter presents the exact discrete time representation of non-stationary continuous time systems with unequally spaced flows or a mixture of stocks and flows. The approach to obtain the exact discrete time representation with flow variables does not depend on the continuous time parameter matrix being non-singular, namely the underlying continuous time system may be non-stationary. In both cases the exact discrete time representations follow a VARMA(1, 1) process with time-varying parameters and heteroskedasticity, despite that the underlying continuous time model has constant parameters and homoskedasticity. The time-varying parameters and the heteroskedastic variance arise due to the variations in the sampling intervals, whereas the moving average disturbances arise due to the flow nature of the observations. A Monte Carlo simulation on estimation of a cointegrated continuous time system with unequally spaced flows is conducted, aiming at assessing estimate properties when unequal sampling intervals are correctly accounted for. Simulation evidence indicates the favour of exact discrete time models accounting for the irregularity of sampling intervals.

3.1 Introduction

Estimating continuous time models based on the exact discrete time analogue has been a popular topic in time series analysis for decades. Most research in estimations of continuous time models assume data are observed over the same interval, which is often time normalised as unity. The fact that some data are not observed on a regular basis has drawn some attention, for instance, Robinson (1977b) pointed out the possibility for modelling irregularly sampled time series.

Unequally spaced data can be found in a number of fields including economics and finance. A leading example can be found in monthly data, in which observation intervals may vary with the variation in the length of calendar months ranging from 28 days to 31 days with roughly 10 percent difference. Unequally spaced data could also appear in financial data, such as data on trades that take place infrequently. In addition, for daily closing price of stock exchange, weekends and public holidays would lead to the irregularity in the sampling intervals. Such type of data could also be obtained in other fields such as the timing of elections, which does not happen on regular basis, in political science.

Several work have addressed the issue in estimating continuous time models with unequally spaced data. One approach to estimate such models would be adopting state space representations. For example, Harvey and Stock (1985) estimated continuous time autoregressive systems using Kalman filter recursions. Their study is further extended to allow for exogenous variables and mixed frequency in (unequally spaced) data by Zadrozny (1988). Harvey and Stock (1993) later provide estimation of continuous time structural time series model where data are stocks, flows or a mixture of both that are unequally spaced. In Koopman et al. (2018) paper, the authors estimated continuous time structural models via the state space approach with high frequency traffic data observed at unequally spaced points in time.

In previous work I provide the derivation of exact discrete time representations of continuous time systems when data are unequally spaced. Exact discrete time representations are provided in three cases: when data are purely stock variables, purely flow variables, or mixed of both stocks and flows. In all cases the exact discrete time representations exhibit time-varying parameters and heteroskedasticity. When data are purely stock variables or a mixture of stocks and flows, the exact discrete time representations require the underlying continuous time system to be stationary. Such restriction would limit the applications to non-stationary systems such as unit root or cointegrated systems.

The focus of this chapter is on providing an approach to derive exact discrete time representation of non-stationary continuous time systems with unequally spaced flows and mixed data. The approach does not impose restrictions on the continuous time coefficient matrix. The discrete time representation is exact and is applicable to non-stationary systems as well. Despite that the underlying continuous time system has constant parameters and is homoskedastic, the exact discrete time representations, in both cases, follow a VARMA(1, 1) process with time-varying parameters and heteroskedasticity. Such a scenario arises when the continuous time system is observed at unequally spaced intervals. Both time-varying parameters and heteroskedastic variances arise due to the variations in the sampling intervals, whereas the moving average disturbances arise due to the flow nature of the observations. The time-varying parameters and variances arise systematically, which are entirely due to the unequally spaced intervals, indicating that such time variation in the discrete time models may merely be a manifestation of the unequally spaced data rather than any inherent time variation in the model itself. In contrast with the discrete-time literature, where time-varying parameters may be employed in discrete time series models (such as AR or ARMA models) in order to better fit for data that are generated in finer intervals, the method proposed in this chapter solves the issue of incompatibility in discrete time models (where parameter estimates are tied only to a specific sampling frequency) since the specification of the exact discrete time model is independent of the sampling interval (see further discussions in Chambers, McCrorie and Thornton, 2018; and in McCrorie, 2009).

The Monte Carlos simulation shows an example of estimation of a continuous time cointegrate system of two flow variables whose intervals vary with the variation of the calendar month. The example in 3.4 only considers a simple case (the Monte Carlo simulation is presented for illustrating the potential gain from correctly accounting for the unequal sampling intervals), whereas estimating cointegrated continuous time system has been an important development in the continuous time literature. Existing research has shown a number of techniques for estimating the exact discrete time models of the cointegrated continuous time systems, including frequency domain (Phillips, 1991; Chambers and McCrorie, 2007), the representation in a theoretical analysis of the asymptotic efficiency fo optimal estimator (Chambers, 2003) and the time domain method (Chambers, 2009). In addition, Kessler and Rahbek (2001) derived the asymptotic behaviour of the maximum likelihood estimators of the (multivariate) cointegrated systems and showed that the limiting distributions derived are the same as in discrete time AR model. Later the authors provided a method to re-address the aliasing issue for both general ergodic and cointegrated models, which relaxes the assumption that the original coefficient matrix is diagonalizable with distinct eigenvalues (Kessler and Rahbek, 2004). It is therefore possible to extend the result not only to deal with unequally spaced data, but also to solve the aliasing problem in multivariate cointegrated systems in future research.

In the following, section 3.2 provides the derivation of the exact discrete time representation of a continuous time system where the variables are observed over unequally spaced discrete intervals. The model considered is multivariate and includes a deterministic time trend. The discrete time representation has time-varying parameters and heteroskedsticity. In particular, the disturbance vector is a time-varying moving average, where the covariance matrix is time dependent.

Section 3.3 considers the case where the variables of interest are a mixture of stocks and flows, where the discrete time representation relies on the assumption that a sub-matrix of the continuous time parameter is non-singular (hence is invertible). This assumption, although limiting the potential applications, for example, to systems involving zero roots, is weaker than many that have appeared in the literature to date. The discrete time representation also has time-varying parameters and heteroskedastic moving average disturbances.

Results of a Monte Carlo simulation study are reported in Section 3.4. The study considers a cointegrated system of flow variables whose sampling intervals coincide with the variation of calendar months. Simulation results indicate that estimation bias is reduced when the unequal sampling intervals are correctly accounted for (rather than assuming all intervals are the same). Section 3.5 contains some concluding comments and detailed Monte Carlo simulation procedures are provided in the Appendix B.

3.2 An Exact Discrete Time Model with Flows

This section provides derivations of discrete time representation of a continuous model. [2] The continuous time model is a system of first-order stochastic differential equations with flow variables and stochastic trends.

Let $x(t)$ be an $n \times 1$ stochastic process generated by

$$dx(t) = [\mu + \gamma t + Ax(t)]dt + \zeta(dt), \ \ t > 0, \tag{3.1}$$

where μ and γ are $n \times 1$ parameter vectors, A is an $n \times n$ matrix, and $\zeta(dt)$ is an $n \times 1$ vector of random measures satisfying:

Assumption 3.1.

$$E[\zeta(dt)] = 0$$

$$E[\zeta(dt)\zeta(dt)'] = \Sigma dt,$$

where Σ is an unknown symmetric positive definite matrix and

$$E[\zeta_i(\Delta_1)\zeta_j(\Delta_2)'] = 0,$$

for $i, j = 1, 2, \cdots, n$; $i \neq j$; and $\Delta_1 \cap \Delta_2 = \emptyset$.

In what follows, it is assumed that samples are observed at the points t_i $(i = 1, \ldots, T)$ such that $0 < t_1 < \ldots < t_T$ and $t_i = t_{i-1} + \delta_i$ for some $\delta_i > 0$ $(i = 1, \ldots, T)$. In the case of a stock variable the sequence of observations is of the form

$$x(t_1), \ x(t_2), \ \cdots, \ x(t_T). \tag{3.2}$$

Extensive use is made of the matrix exponential and various functions thereof.The matrix exponential is defined as

$$e^A = \sum_{j=0}^{\infty} \frac{1}{j!} A^j,$$

[2]The method for deriving the exact discrete time model with flows follows the joint paper with my supervisor- Time-Varying Parameters and Heteroskedasticity: Continuous Time Systems with Unequally-Spaced Data.

and it is convenient to define the matrix functions

$$\begin{aligned} F(z) &= e^{Az}, \\ G(z) &= \int_0^z e^{As} ds, \\ H(z) &= \int_0^z s e^{As} ds, \\ J(z) = \int_0^z G(s) ds &= \int_0^z \left(\int_0^s e^{Ar} dr \right) ds, \\ K(z) = \int_0^z H(s) ds &= \int_0^z \left(\int_0^s r e^{Ar} dr \right) ds, \\ M(z) = \int_0^z s G(s) ds &= \int_0^z s \left(\int_0^s e^{Ar} dr \right) ds, \end{aligned}$$

in all cases z is a known constant. In particular, when $z = \delta_i$ the particular matrices are defined as

$$F_i = F(\delta_i), G_i = G(\delta_i), H_i = H(\delta_i), J_i = J(\delta_i), K_i = K(\delta_i), M_i = M(\delta_i), (i = 1, \ldots, T).$$

In the case of unequally spaced stock variables (when $x(t)$ is a stock variable), based on results of Theorem 2.1 in Chapter 2, the discrete time representation of (3.1) is obtained as

$$x(t_i) = c_{0i} + c_{1i} t_i + F_i x(t_{i-1}) + \eta(t_i), \;\; i = 1, \ldots, T, \tag{3.3}$$

where $c_{0i} = G_i \mu - H_i \gamma$, $c_{1i} = G_i \gamma$, $\eta(t_i) = \int_{t_{i-1}}^{t_i} e^{A(t_i - r)} \zeta(dr)$ and $\eta(t_i)$ satisfies $E\left(\eta(t_i)\right) = 0_{n\times 1}, E\left(\eta(t_i)\eta(t_j)'\right) = 0_{n\times n}$ for $i \neq j$ and $E\left(\eta(t_i)\eta(t_i)'\right) = \Omega_i = \int_0^{\delta_i} e^{Ar} \Sigma e^{A'r} dr, \;\; i = 1, \ldots, N.$

The discrete time model with unequally spaced stock data generated by (3.1) follows a VAR(1) process. In the discrete time model, the coefficients are time-varying and the disturbances are heteroskedastic while the parameters in the continuous time model (equation (3.1)) are constant and the variance is homoskedastic. These discrepancies are generated by the variations of the sampling intervals.

In the case of unequally spaced flow variables, the observations constitute a sequence of flow vectors of the form

$$x_{t_i} = \int_{t_{i-1}}^{t_i} x(r) dr = \int_0^{\delta_i} x(t_i - r) dr = \int_0^{\delta_i} x(t_{i-1} + r) dr, \;\; i = 1, \ldots, T. \tag{3.4}$$

With equally spaced observations a discrete time representation can be obtained by integrating (3.3) over the common observation interval. This procedure, however, is inappropriate when the observations are unequally spaced due to the following reason. Integration over $(t_{i-1}, t_i]$ will yield x_{t_i} on the left-hand-side but, on the right-hand-side,

$$\int_{t_{i-1}}^{t_i} x(r-\delta_i)dr = \int_{t_{i-1}-\delta_i}^{t_i-\delta_i} x(s)ds = \int_{t_{i-1}-\delta_i}^{t_{i-1}} x(s)ds \neq x_{t_{i-1}} = \int_{t_{i-2}}^{t_{i-1}} x(s)ds.$$

The problem concerns the lower limit where $t_{i-1} - \delta_i \neq t_{i-2} = t_{i-1} - \delta_{i-1}$. The approach to derive the discrete time representation, which is presented in the previous chapter imposes restrictions on the matrix A to be nonsingular. This rules out applications to systems involving unit roots and cointegration. This section provides the discrete time representation which has the advantage of not requiring any additional conditions beyond Assumption 3.1. The derivation relies on the following lemma.

Lemma 3.1. G_i is nonsingular for all $i = 1, \ldots, T$.

Proof. From the series expansion of exp$\{As\}$ we find that

$$\begin{aligned} G_i = \int_0^{\delta_i} e^{As} ds &= \int_0^{\delta_i} \sum_{j=0}^{\infty} \frac{A^j s^j}{j!} ds \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \left(\int_0^{\delta_i} s^j ds \right) A^j \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\delta_i^{j+1}}{j+1} \right) A^j \\ &= \sum_{j=0}^{\infty} c_j A^j \end{aligned}$$

where $c_j = \delta_i^{j+1}/(j+1)!$. It is shown by Abadir and Magnus (2005, p.262) that, if $\Phi(A) = \sum_{j=0}^{\infty} c_j A^j$, then $|\Phi(A)| = \prod_{i=1}^{n} \phi(\lambda_i)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (not necessarily distinct) and $\phi(\lambda) = \sum_{j=0}^{\infty} c_j \lambda^j$. The matrix G_i is clearly of the form $\Phi(A)$ and we shall demonstrate that $|G_i| \neq 0$, using the above result, and, hence, that G_i is nonsingular. Note that, if an eigenvalue of A is zero, then $\phi(0) = c_0 = \delta_i$ whereas, for real or complex $\lambda \neq 0$,

$$\phi(\lambda) = \sum_{j=0}^{\infty} c_j \lambda^j = \sum_{j=0}^{\infty} \frac{\delta_i^{j+1} \lambda^j}{(j+1)!} = \frac{1}{\lambda} \sum_{j=0}^{\infty} \frac{\delta_i^{j+1} \lambda^{j+1}}{(j+1)!} = \frac{1}{\lambda} \int_0^{\delta_i \lambda} e^s ds$$

i.e. $\phi(\lambda) = (e^{\delta_i \lambda} - 1)/\lambda$. Let there be n_1 zero eigenvalues and n_2 non-zero eigenvalues, where $n_1 + n_2 = n$, ordered so that $\lambda_j = 0$ $(j = 1, \ldots, n_1)$ and $\lambda_j \neq 0$ $(j = n_1 + 1, \ldots, n)$. Then

$$|G_i| = \prod_{j=1}^{n_1} \phi(0) \prod_{j=n_1+1}^{n} \phi(\lambda_j) = \delta_i^{n_1} \prod_{j=n_1+1}^{n} \frac{(e^{\delta_i \lambda_j} - 1)}{\lambda_j}$$

because $\phi(0) = \delta_i$. This expression can be zero only if $\delta_i = 0$ or if $e^{\delta_i \lambda_j} - 1 = 0$. The first possibility is ruled out because $\delta_i > 0$ and the second because $\delta_i \lambda_j \neq 0$ owing to $\delta_i > 0$ and $\lambda_j \neq 0$ for $j = n_1 + 1, \ldots, n$. Hence $|G_i| \neq 0$ and G_i is nonsingular as claimed. End of proof.

The invertibility of G_i is used in the derivation of the exact discrete time model; Lemma 3.1 shows that no further conditions need to be imposed on the matrix A for this property to hold. The discrete time representation is given by Theorem 3.1.

Theorem 1. Let $x(t)$ be a flow variable generated by (3.1) which is observed as the sequence in (3.4). Under Assumption 3.1, the observations satisfy

$$x_{t_1} = m_{01} + G_1 x(0) + \xi_{t_1},$$

$$x_{t_i} = m_{0i} + m_{1i} t_i + \Phi_i x_{t_{i-1}} + \xi_{t_i}, \quad i = 2, \ldots, T,$$

where $m_{01} = \rho_{01} + \rho_{11}\delta_1$ and, for $i = 2, \ldots, N$, $\Phi_i = G_i F_{i-1} G_{i-1}^{-1}$,

$$m_{0i} = \rho_{0i} + G_i(c_{0,i-1} - c_{1,i-1}\delta_i) - \Phi_i(\rho_{0,i-1} - \rho_{1,i-1}\delta_i),$$

$$m_{1i} = \rho_{1i} + G_i c_{1,i-1} - \Phi_i \rho_{1,i-1},$$

$$\rho_{0i} = J_i \mu + (M_i - K_i - J_i \delta_i)\gamma$$

$$\rho_{1i} = J_i \gamma.$$

Furthermore, ξ_{t_i} is a heteroskedastic MA(1) process with autocovariance matrices given by

$$\Omega_{0,i} = E[\xi_{t_i}\xi_{t_i}'] = \begin{cases} \int_0^{\delta_1} G(s)\Sigma G(s)' ds, & i = 1, \\ \int_0^{\delta_i} G(s)\Sigma G(s)' ds + \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma\Gamma_i(s)' ds, & i = 2, \ldots, T, \end{cases}$$

$$\Omega_{-1,i} = E[\xi_{t_i}\xi_{t_{i-1}}'] = \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma G(s)' ds, \quad i = 2, \ldots, T,$$

$$\Omega_{1,i} = E[\xi_{t_i}\xi_{t_{i+1}}'] = \int_0^{\delta_i} G(s)\Sigma\Gamma_{i+1}(s)' ds, \quad i = 1, \ldots, T-1,$$

where $\Gamma_i(x) = G_i F(x) - \Phi_i G(x)$.

Proof. We first derive the equations for $i = 2, \ldots, N$ and then for $i = 1$. (3.4) implies that

$$x(t_{i-1} + s) = c_s + e^{As} x(t_{i-1}) + \int_{t_{i-1}}^{t_{i-1}+s} e^{A(t_{i-1}+s-r)} \zeta(dr), 0 < s < \delta_i \tag{3.5}$$

where

$$c_s = \int_{t_{i-1}}^{t_{i-1}+s} e^{A(t_{i-1}+s-r)} (\mu + \gamma r)\, dr.$$

Evaluating this deterministic integral enables us to show that

$$c_s = G(s)\mu - H(s)\gamma + G(s)\gamma(t_{i-1} + s).$$

Hence integrating (3.5) over $s \in (0, \delta_i]$ results in

$$\begin{aligned} \int_0^{\delta_i} x(t_{i-1} + s) ds \quad &= \quad \int_0^{\delta_i} G(s) ds \mu - \int_0^{\delta_i} H(s) ds \gamma + \int_0^{\delta_i} G(s)(t_{i-1} + s) ds \gamma \\ &+ \left(\int_0^{\delta_i} e^{As} ds \right) x(t_{i-1}) + \int_0^{\delta_i} \int_{t_{i-1}}^{t_{i-1}+s} e^{A(t_{i-1}+s-r)} \zeta(dr) ds. \end{aligned}$$

Given that $t_{i-1} = t_i - \delta_i$, the above equation can be written as

$$x_{t_i} = \rho_{0i} + \rho_{1i} t_i + G_i x(t_{i-1}) + e_{t_i}, \;\; i = 1, \ldots, T, \tag{3.6}$$

where $\rho_{0i} = J_i \mu + (M_i - K_i - J_i \delta_i)\gamma$, $\rho_{1i} = J_i \gamma$, and $e_{t_i} = \int_0^{\delta_i} \int_{t_{i-1}}^{t_{i-1}+s} e^{A(t_{i-1}+s-r)} \zeta(dr) ds$.

Using Lemma 1 we can solve (3.6) for $x(t_{i-1})$:

$$x(t_{i-1}) = G_i^{-1} \left(x_{t_i} - \rho_{0i} - \rho_{1i} t_i - e_{t_i} \right). \tag{3.7}$$

But, from (3.3), we know that

$$x(t_{i-1}) = c_{0,i-1} + c_{1,i-1} t_{i-1} + F_{i-1} x(t_{i-2}) + \eta_{t_{i-1}}. \tag{3.8}$$

Using (3.7) and its lag to substitute for $x(t_{i-1})$ and $x(t_{i-2})$ in (3.8) results in

$$\begin{aligned} &G_i^{-1} \left(x_{t_i} - \rho_{0i} - \rho_{1i} t_i - e_{t_i} \right) = c_{0,i-1} + c_{1,i-1} t_{i-1} \\ &\quad + F_{i-1} G_{i-1}^{-1} \left(x_{t_{i-1}} - \rho_{0,i-1} - \rho_{1,i-1} t_{i-1} - e_{t_{i-1}} \right) + \eta_{t_{i-1}}. \end{aligned} \tag{3.9}$$

Multiplying (3.9) by G_i, using $t_{i-1} = t_i - \delta_i$, we obtain

$$x_{t_i} = m_{0i} + m_{1i}t_i + \Phi_i x_{t_{i-1}} + \xi_{t_i}, \ \ i = 1, \dots, T,$$

$$where \Phi_i = G_i F_{i-1} G_{i-1}^{-1}, m_{0i} = \rho_{0i} + G_i(c_{0,i-1} - c_{1,i-1}\delta_i) - \Phi_i(\rho_{0,i-1} - \rho_{1,i-1}\delta_i),$$

$$m_{1i} = \rho_{1i} + G_i c_{1,i-1} - \Phi_i \rho_{1,i-1} \ and \ \xi_{t_i} = e_{t_i} - \Phi_i e_{t_{i-1}} + G_i \eta_{t_{i-1}}. \tag{3.10}$$

The equation for $i = 1$ is obtained in a similar manner; setting $t_{i-1} = 0$ in (3.6), and noting that $t_1 = \delta_1$, we obtain

$$x_{t_1} = m_{01} + G_1 x(0) + \xi_{t_1},$$

where $m_{01} = \rho_{01} + \rho_{11}\delta_1$ and $\xi_{t_1} = e_{t_1}$.

To derive properties of the disturbances, it is necessary to reduce the double integral defining e_{t_i} to a more convenient form:

$$\begin{aligned} e_{t_i} &= \int_{t_{i-1}}^{t_i} \left(\int_{r-t_{i-1}}^{\delta_i} e^{A(t_{i-1}+s-r)} ds \right) \zeta(dr) \\ &= \int_{t_{i-1}}^{t_i} \left(\int_0^{t_i - r} e^{Aw} dw \right) \zeta(dr) \\ &= \int_{t_{i-1}}^{t_i} G(t_i - r)\zeta(dr), \ \ i = 1, \dots, T. \end{aligned}$$

Hence, using (3.10), for $i = 2, \dots, T$, ξ_{t_i} can be written as

$$\begin{aligned} \xi_{t_i} &= \int_{t_{i-1}}^{t_i} G(t_i - r)\zeta(dr) - \Phi_i \int_{t_{i-2}}^{t_{i-1}} G(t_{i-1} - r)\zeta(dr) + G_i \int_{t_{i-2}}^{t_{i-1}} F(t_{i-1} - r)\zeta(dr) \\ &= \int_{t_{i-1}}^{t_i} G(t_i - r)\zeta(dr) + \int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1} - r)\zeta(dr), \end{aligned}$$

where $\Gamma_i(x) = G_i F(x) - \Phi_i G(x)$, while for $i = 1$ we have

$$\xi_{t_1} = \int_0^{t_1} G(t_1 - r)\zeta(dr).$$

The autocovariances follow from these expressions. Properties of ξ are obtained as

$$E[\xi_{t_i}] = 0, i = 1, \dots, T,$$

$$
\begin{aligned}
E[\xi_{t_1}\xi_{t_1}{}'] &= E\left[\int_0^{t_1} G(t_1-r)\zeta(dr)\right]\left[\int_0^{t_1} G(t_1-r)\zeta(dr)\right]' \\
&= \int_0^{t_1} G(t_1-r)\Sigma G(t_1-r)'dr \\
&= \int_0^{\delta_1} G(s)\Sigma G(s)'ds, i=1,
\end{aligned}
$$

$$
\begin{aligned}
E[\xi_{t_i}\xi_{t_i}{}'] &= E\left[\int_{t_{i-1}}^{t_i} G(t_i-r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} G(t_i-r)\zeta(dr)\right]' \\
&+ E\left[\int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1}-r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1}-r)\zeta(dr)\right]' \\
&= \int_{t_{i-1}}^{t_i} G(t_i-r)G(t_i-r)'dr + \int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1}-r)\Sigma\Gamma_i(t_{i-1}-r)'dr \\
&= \int_0^{\delta_i} G(s)\Sigma G(s)'ds + \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma\Gamma_i(s)'ds, i=2,\ldots,T,
\end{aligned}
$$

$$
\begin{aligned}
E[\xi_{t_i}\xi_{t_{i-1}}'] &= E\left[\int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1}-r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}} G(t_{i-1}-r)\zeta(dr)\right]' \\
&= \int_{t_{i-2}}^{t_{i-1}} \Gamma_i(t_{i-1}-r)\Sigma G(t_{i-1}-r)'dr \\
&= \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma G(s)'ds, i=2,\ldots,T,
\end{aligned}
$$

$$
\begin{aligned}
E[\xi_{t_i}\xi_{i+1}{}'] &= E\left[\int_{t_{i-1}}^{t_i} G(t_i-r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i} \Gamma_i(t_i-r)\zeta(dr)\right] \\
&= \int_{t_{i-1}}^{t_i} G(t_i-r)\Sigma\Gamma_i(t_i-r)'dr \\
&= \int_0^{\delta_i} G(s)\Sigma\Gamma_i(s)'ds, i=1,\ldots,T-1.
\end{aligned}
$$

End of Proof.

Theorem 3.1 shows that the discrete time model with flow variables follows a VARMA(1, 1) process with time-varying coefficients and heteroskedasticity. The heteroskedastic variances arises due to the variations in the sampling intervals while the heteroskedastic MA(1) disturbances arise due to the flow nature of the observations. Furthermore, Theorem 3.1 does not require restrictions on the matrix A (i.e. requiring matrix A to be nonsingular), which indicates that the results of the theorem are applicable in nonstationary and cointegrated models as well as stationary systems. In addition, Theorem 3.1 can be used when data are equally spaced, namely, when $\delta_i = 1$ for all i. The advantage of this approach is that matrix A is not required to be nonsingular, which hence does not rule out applications to nonstationary systems.

3.3 An Exact Discrete Time Model with Mixed Samples

In this section, a system that includes both stock and flow variables is considered. The derivation of the exact discrete time representation follows Agbeyegbe's (1987) procedure. In the case of mixed samples, both the stocks and flows are assumed to be observed at the same frequency. The observations are of the form

$$x(t_i) = \begin{bmatrix} x^s(t_i) \\ x^f(t_i) \end{bmatrix} = \begin{bmatrix} x^s(t_i) \\ \int_{t_{i-1}}^{t_i} x^f(r)dr \end{bmatrix}, i = 1, 2, \cdots, T. \tag{3.11}$$

Note $x^s(t_i)$ is a vector of $(n^s \times 1)$ stock variables and $x^f(t_i)$ is a vector of $(n^f \times 1)$ flow variables, with $n^s + n^f = n$. The system of stock and flow variables, generated by (3.1), is partitioned as

$$d(x^s(t)) = [A^{ss}x^s(t) + A^{sf}x^f(t) + \mu^s + \gamma^s t]dt + \zeta^s(dt), \tag{3.12}$$

$$d(x^f(t)) = [A^{fs}x^s(t) + A^{ff}x^f(t) + \mu^f + \gamma^f t]dt + \zeta^f(dt), \tag{3.13}$$

where $A = \begin{bmatrix} A^{ss} & A^{sf} \\ A^{fs} & A^{ff} \end{bmatrix}$, $\mu = \begin{bmatrix} \mu^s \\ \mu^f \end{bmatrix}$, $\gamma = \begin{bmatrix} \gamma^s \\ \gamma^f \end{bmatrix}$,and $\zeta(dt) = \begin{bmatrix} \zeta^s(dt) \\ \zeta^f(dt) \end{bmatrix}$.

In order for Theorem 3.2 to be valid, we shall need the following assumption on the sub-matrix of A.

Assumption 3.2. The sub-matrix A^{ss} is non-singular.

The main challenge with mixed data is eliminating unobservable terms from the system: integrals of stock variables, $\int_{t_{i-1}}^{t_i} x^s(r)dr$, and the levels of flow variables, $x^f(t_i)$. To derive the exact discrete time model, it is necessary to define an $(n \times 1)$ random vector $z_{t_1}, z_{t_2}, \cdots, z_{t_n}$ in the form

$$z_{t_i} = \begin{bmatrix} x^s(t_i) - x^s(t_{i-1}) \\ \int_{t_{i-1}}^{t_i} x^f(r)\, dr \end{bmatrix}, i = 1, 2, \cdots, T. \tag{3.14}$$

The vector z_{t_i} defined above represents a mixture of stock variables and flow variables. The exact discrete time model for mixed data is given by Theorem 3.2.

Theorem 3.2. Let $x(t)$ be generated by (3.1) which is observed as the mixed-sample sequence in (3.11). Under Assumption 3.1 and 3.2, the random vectors $z_{t_1}, z_{t_2}, \cdots, z_{t_n}$ defined by (3.14) satisfy the system

$$z_{t_i} = \Pi_i z_{t_{i-1}} + g_i + \epsilon_{t_i}, \tag{3.15}$$

$$E[\epsilon_{t_i}] = 0,$$

$$\begin{aligned} V_i &= E[\epsilon_{t_i}\epsilon'_{t_i}] \\ &= \begin{cases} \int_0^{\delta_1} \Psi(s)\Sigma\Psi(s)'\,ds & i=1, \\ \int_0^{\delta_i} \Psi(s)\Sigma\Psi(s)'\,ds + \int_0^{\delta_{i-1}} S(s)\Sigma S(s)'\,ds, & i=2,\cdots,T, \end{cases} \end{aligned}$$

$$\begin{aligned} &W_{-1,i} = \mathrm{E}[\epsilon_{t_i}\epsilon'_{t_1}] = \int_0^{\delta_{i-1}} S(s)\Sigma\Psi(s)'\,ds \qquad && i=2,\cdots,T, \\ &W_i = \int_0^{\delta_i} \Psi(s)\Sigma S(s)'\,ds && i=1,\cdots,T-1, \end{aligned}$$

where

$$\Pi_i = \begin{bmatrix} \Pi_i^{ss} & \Pi_i^{sf} \\ \Pi_i^{fs} & \Pi_i^{ff} \end{bmatrix},$$

$$g_i = \begin{bmatrix} g_i^s \\ g_i^f \end{bmatrix},$$

$$\begin{aligned} \epsilon_{t_i} &= \int_{t_{i-1}}^{t_i} \Psi(t_i - r)\zeta(dr) + \int_{t_{i-2}}^{t_{i-1}} S(t_{i-1} - r)\zeta(dr) \\ &= \begin{bmatrix} \epsilon_{t_i}^s \\ \epsilon_{t_i}^f \end{bmatrix}, \end{aligned}$$

$$\Pi_i^{ss} = [A^{ss}\Phi_i^{ss} + A^{sf}\Phi_i^{fs}][A^{ss}]^{-1},$$

$$\Pi_i^{sf} = [A^{ss}\Phi_i^{sf} + A^{sf}\Phi_i^{ff}] - \Pi_i^{11}A^{sf},$$

$$\Pi_i^{fs} = \Phi_i^{fs}[A^{ss}]^{-1},$$

$$\Pi_i^{ff} = \Phi_i^{ff} - \Pi_i^{21}A^{sf},$$

$$g_i^s = A^{ss} m_{0i}^s + A^{sf} m_{0i}^f + (A^{ss} m_{1i}^s + A^{sf} m_{1i}^f) t_i + \int_{t_{i-1}}^{t_i} [\mu^s + \gamma^s r] \, dr - \Pi_i^{ss} \int_{t_{i-2}}^{t_{i-1}} [\mu^s + \gamma^s r] \, dr,$$

$$g_i^f = m_{0i}^f + m_{1i}^f t_i - \Pi_i^{fs} \int_{t_{i-2}}^{t_{i-1}} [\mu^s + \gamma^s r] \, dr,$$

$$\epsilon_{t_i}^s = \int_{t_{i-1}}^{t_i} \zeta^s(dr) + A^{ss} \xi_{t_i}^s + A^{sf} \xi_{t_i}^f - \Pi_i^{ss} \int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr),$$

$$\epsilon_{t_i}^f = \xi_{t_i}^f - \Pi_i^{fs} \int_{t_{i-2}}^{t_{i-1}} \zeta^s(dr),$$

$$\Psi(t_i - r) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{ss} & A^{sf} \\ 0 & I \end{bmatrix} \begin{bmatrix} [G(t_i - r)]^{ss} & [G(t_i - r)]^{sf} \\ [G(t_i - r)]^{fs} & [G(t_i - r)]^{ff} \end{bmatrix},$$

$$S(t_{i-1} - r) = \begin{bmatrix} A^{ss} & A^{sf} \\ 0 & I \end{bmatrix} \begin{bmatrix} [\Gamma_i(t_{i-1} - r)]^{ss} & [\Gamma_i(t_{i-1} - r)]^{sf} \\ [\Gamma_i(t_{i-1} - r)]^{fs} & [\Gamma_i(t_{i-1} - r)]^{ff} \end{bmatrix} - \begin{bmatrix} \Pi_i^{ss} & 0 \\ \Pi_i^{fs} & 0 \end{bmatrix},$$

$$G(t_i - r) = \begin{bmatrix} [G(t_i - r)]^{ss} & [G(t_i - r)]^{sf} \\ [G(t_i - r)]^{fs} & [G(t_i - r)]^{ff} \end{bmatrix},$$

$$\Gamma_i(t_{i-1} - r) = \begin{bmatrix} [\Gamma_i(t_{i-1} - r)]^{ss} & [\Gamma_i(t_{i-1} - r)]^{sf} \\ [\Gamma_i(t_{i-1} - r)]^{fs} & [\Gamma_i(t_{i-1} - r)]^{ff} \end{bmatrix}.$$

Proof. Integrating (3.1) over the interval $[t_{i-1}, t_i]$ obtains

$$x(t_i) - x(t_{i-1}) = A \int_{t_{i-1}}^{t_i} x(r) \, dr + \int_{t_{i-1}}^{t_i} [\mu + \gamma r] \, dr + \int_{t_{i-1}}^{t_i} \zeta(dr), \tag{3.16}$$

while the first row of equation (3.16) is

$$x^s(t_i) - x^s(t_{i-1}) = A^{ss} \int_{t_{i-1}}^{t_i} x^s(r) \, dr + A^{sf} \int_{t_{i-1}}^{t_i} x^f(r) \, dr + \int_{t_{i-1}}^{t_i} [\mu^s + \gamma^s r] \, dr + \int_{t_{i-1}}^{t_i} \zeta^s(dr). \tag{3.17}$$

Partitioning (3.10) as

$$\int_{t_{i-1}}^{t_i} x^s(r) \, dr = {\Phi_i}^{ss} \int_{t_{i-2}}^{t_{i-1}} x^s(r) \, dr + {\Phi_i}^{sf} \int_{t_{i-2}}^{t_{i-1}} x^f(r) \, dr + {m_{0i}}^s + {m_{1i}}^s t_i + {\xi_{ti}}^s, \tag{3.18}$$

$$\int_{t_{i-1}}^{t_i} x^f(r)\,dr = \Phi_i{}^{fs}\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr + \Phi_i{}^{ff}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + m_{0i}{}^f + m_{1i}{}^f t_i + \xi_{ti}{}^f, \tag{3.19}$$

where

$$\Phi_i = \begin{bmatrix} \Phi_i{}^{ss} & \Phi_i{}^{sf} \\ \Phi_i{}^{fs} & \Phi_i{}^{ff} \end{bmatrix},$$

$$m_{0i} = \begin{bmatrix} m_{0i}{}^s \\ m_{0i}{}^f \end{bmatrix},$$

$$m_{1i} = \begin{bmatrix} m_{1i}{}^s \\ m_{1i}{}^f \end{bmatrix},$$

and

$$\xi_{ti} = \begin{bmatrix} \xi_{ti}{}^s \\ \xi_{ti}{}^f \end{bmatrix}.$$

Substituting out $\int_{t_{i-1}}^{t_i} x^s(r)\,dr$ and $\int_{t_{i-1}}^{t_i} x^f(r)\,dr$ in (3.17) by (3.18) and (3.19), respectively

$$\begin{aligned} x^s(t_i) - x^s(t_{i-1}) &= [A^{ss}\Phi_i{}^{ss} + A^{sf}\Phi_i{}^{fs}]\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr + [A^{ss}\Phi_i{}^{sf} + A^{sf}\Phi_i{}^{ff}]\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \\ &+ A^{ss}m_{0i}{}^s + A^{sf}m_{0i}{}^f + [A^{ss}m_{1i}{}^s + A^{sf}m_{1i}{}^f]t_i + \int_{t_{i-1}}^{t_i}[\mu^s + \gamma^s r]\,dr \\ &+ A^{ss}\xi_{ti}{}^s + A^{sf}\xi_{ti}{}^f + \int_{t_{i-1}}^{t_i}\zeta^s(dr). \end{aligned} \tag{3.20}$$

From (3.17) we obtain

$$\begin{aligned} \int_{t_{i-1}}^{t_i} x^s(r)\,dr &= [A^{ss}]^{-1}[x^s(t_i) - x^s(t_{i-1})] - [A^{ss}]^{-1}A^{sf}\int_{t_{i-1}}^{t_i} x^f(r)\,dr \\ &- [A^{ss}]^{-1}\int_{t_{i-1}}^{t_i}[\mu^s + \gamma^s r]\,dr - [A^{ss}]^{-1}\int_{t_{i-1}}^{t_i}\zeta^s(dr). \end{aligned} \tag{3.21}$$

Lagging (3.21) for one period

$$\begin{aligned} \int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr &= [A^{ss}]^{-1}[x^s(t_{i-1}) - x^s(t_{i-2})] - [A^{ss}]^{-1}A^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \\ &- [A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}[\mu^s + \gamma^s r]\,dr - [A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}\zeta^s(dr). \end{aligned} \tag{3.22}$$

The object now is to eliminate the unobservale term, $\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr$ in (3.20) and (3.19). Substituting out $\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr$ in (3.20) using (3.22)

$$\begin{aligned}
x^s(t_i) - x^s(t_{i-1}) &= [A^{ss}\Phi_i{}^{ss} + A^{sf}\Phi_i{}^{fs}][A^{ss}]^{-1}\left\{[x^s(t_{i-1}) - x^s(t_{i-2})] - A^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr\right\} \\
&+ [A^{ss}\Phi_i{}^{sf} + A^{sf}\Phi_i{}^{ff}]\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + A^{ss}m_{0i}{}^s + A^{sf}m_{0i}{}^f \\
&+ [A^{ss}m_{1i}{}^s + A^{sf}m_{1i}{}^f]t_i + \int_{t_{i-1}}^{t_i}[\mu^s + \gamma^s r]\,dr \\
&- [A^{ss}\Phi_i{}^{ss} + A^{sf}\Phi_i{}^{fs}][A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}[\mu^s + \gamma^s r]\,dr \\
&+ A^{ss}\xi_{ti}{}^s + A^{sf}\xi_{ti}{}^f + \int_{t_{i-1}}^{t_i}\zeta^s(dr) \\
&- [A^{ss}\Phi_i{}^{ss} + A^{sf}\Phi_i{}^{fs}][A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}\zeta^s(dr) \\
&= \Pi_i{}^{ss}[x^s(t_{i-1}) - x^s(t_{i-2})] + \Pi_i{}^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + g_i{}^s + \epsilon_{ti}{}^s. \qquad (3.23)
\end{aligned}$$

Substituting out $\int_{t_{i-2}}^{t_{i-1}} x^s(r)\,dr$ in (3.19) using (3.22)

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} x^f(r)\,dr &= \Phi_i{}^{fs}[A^{ss}]^{-1}\left\{[x^s(t_{i-1}) - x^s(t_{i-2})] - A^{sf}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr\right\} \\
&+ \Phi_i{}^{ff}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + m_{0i}{}^f + m_{1i}{}^f t_i \\
&- \Phi_i{}^{fs}[A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}[\mu^s + \gamma^s r]\,dr + \xi_{ti}{}^f - \Phi_i{}^{fs}[A^{ss}]^{-1}\int_{t_{i-2}}^{t_{i-1}}\zeta^s(dr) \\
&= \Pi_i{}^{fs}[x^s(t_{i-1}) - x^s(t_{i-2})] + \Pi_i{}^{ff}\int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr + g_i{}^f + \epsilon_{ti}{}^f. \qquad (3.24)
\end{aligned}$$

Combining (3.23) and (3.24) we obtain (3.15)

$$\begin{bmatrix} x^s(t_i) - x^s(t_{i-1}) \\ \int_{t_{i-1}}^{t_i} x^f(r)\,dr \end{bmatrix} = \begin{bmatrix} \Pi_i^{ss} & \Pi_i^{sf} \\ \Pi_i^{fs} & \Pi_i^{ff} \end{bmatrix}\begin{bmatrix} x^s(t_{i-1}) - x^s(t_{i-2}) \\ \int_{t_{i-2}}^{t_{i-1}} x^f(r)\,dr \end{bmatrix} + \begin{bmatrix} g_i^s \\ g_i^f \end{bmatrix} + \begin{bmatrix} \epsilon_{t_i}^s \\ \epsilon_{t_i}^f \end{bmatrix}.$$

Properties of vector ϵ_{t_i} depend on properties of the continuous time disturbance vector $\zeta(dt)$.

The mean of ϵ_{t_i}:

$$E[\epsilon_{t_i}] = 0, i = 1, \ldots, T.$$

The variance of ϵ_{t_i}:

$$\begin{aligned} E[\epsilon_{t_1}\epsilon_{t_1}'] &= E\left[\int_0^{t_1}\Psi(t_1-r)\zeta(dr)\right]\left[\int_0^{t_1}\Psi(t_1-r)\zeta(dr)\right]' \\ &= \int_0^{t_1}\Psi(t_1-r)\Sigma\Psi(t_1-r)'dr \\ &= \int_0^{\delta_1}\Psi(s)\Sigma\Psi(s)'ds, i=1, \end{aligned}$$

$$\begin{aligned} E[\epsilon_{t_i}\epsilon_{t_i}'] &= E\left[\int_{t_{i-1}}^{t_i}\Psi(t_i-r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i}\Psi(t_i-r)\zeta(dr)\right]' \\ &+ E\left[\int_{t_{i-2}}^{t_{i-1}}S(t_{i-1}-r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}}S(t_{i-1}-r)\zeta(dr)\right]' \\ &= \int_{t_{i-1}}^{t_i}\Psi(t_i-r)\Sigma\Psi(t_i-r)'\,dr + \int_{t_{i-2}}^{t_{i-1}}S(t_{i-1}-r)\Sigma S'(t_{i-1}-r)\,dr \\ &= \int_0^{\delta_i}\Psi(s)\Sigma\Psi(s)'\,ds + \int_0^{\delta_{i-1}}S(s)\Sigma S(s)'\,ds, i=2,\ldots,T. \end{aligned}$$

The autocorariance of ϵ_{t_i}:

$$\begin{aligned} E[\epsilon_{t_i}\epsilon_{t_{i-1}}'] &= E\left[\int_{t_{i-2}}^{t_{i-1}}S(t_{i-1}-r)\zeta(dr)\right]\left[\int_{t_{i-2}}^{t_{i-1}}\Psi(t_{i-1}-r)\zeta(dr)\right]' \\ &= \int_{t_{i-2}}^{t_{i-1}}S(t_{i-1}-r)\Sigma\Psi(t_{i-1}-r)'dr \\ &= \int_0^{\delta_{i-1}}S(s)\Sigma\Psi(s)'\,ds, i=2,\ldots,T, \end{aligned}$$

$$\begin{aligned} E[\epsilon_{t_i}\epsilon_{t_{i+1}}'] &= E\left[\int_{t_{i-1}}^{t_i}\Psi(t_i-r)\zeta(dr)\right]\left[\int_{t_{i-1}}^{t_i}S(t_i-r)\zeta(dr)\right]' \\ &= \int_{t_{i-1}}^{t_i}\Psi(t_i-r)\Sigma S(t_i-r)'dr \\ &= \int_0^{\delta_i}\Psi(s)\Sigma S(s)'\,ds, i=1,\ldots,T-1. \end{aligned}$$

End of proof.

Theorem 3.2 shows that, with mixed data, the exact discrete time model follows a VARMA(1, 1) process with time-varying coefficient, and the disturbance vector ϵ_{ti} is a heteroskedastic MA(1). The underlying continuous time model, instead, has constant parameters and homoskedastic disturbance. The derivation procedure in 2.5 requires assumption 2.3 to be valid since (2.26) involves the inverse of coefficient matrix exponential; while the procedure in this section uses a different technique for discretization, which imposes, at least, a weaker assumption.

3.4 Simulation Evidence

A Monte Carlo simulation is conducted to examine the performance of estimation of continuous time models where unequal sampling intervals are correctly measured. This study considered a cointegrated system of flow variables whose sampling intervals coincide with the variation of calendar months. The lengths of monthly sampling intervals vary from 28 days to 31 days, which are normalised by dividing each interval by 30. Note that we ignore leap years, assuming each February has 28 days for reducing computation cost. The resulting sampling intervals are $\delta_{\text{min}} = 0.9\dot{3}$, 1.00 and $\delta_{\text{max}} = 1.0\dot{3}$. The model of interest is

$$dx(t) = Ax(t)dt + \zeta(dt), \;\; t > 0,$$

where $A = \begin{bmatrix} \alpha_1 & -\alpha_1\beta \\ \alpha_2 & -\alpha_2\beta \end{bmatrix}$ is an $n \times n$ coefficient matrix with $\beta = 1$ and $\alpha_1 - \alpha_2\beta < 0$ and $\zeta(dt)$ satisfy Assumption 3.1. In particular, the variance of $\zeta(dt)$ is $\Sigma = \sigma^2 I_n$ with σ^2 being some random variable.

The observations are made at points t_i with $t_i = t_{i-1} - \delta_i, i = 2, \cdots, T$ and δ_i denotes sample intervals. T is the sample size and, specifically, we assume $t_0 = 0$ and $x(0) = 0$ as the Boundary Condition. In this case we have 2 variables hence $n = 2$.

To explore the impact of values of the parameters and sample size on the estimation results, we compare the simulation results with α_1 and α_2 are -1.25 and 0.75 respectively to the results with α_1 and α_2 are -0.95 and -0.05 respectively; while the sample size change from 120 (10 year span) to 240 (20 year span). $\beta = 1$ and $\sigma^2 = 0.25$ in all cases.

Using results from Theorem 3.1, the discrete time model is obtained as

$$x_{t_1} = G_1 x(0) + \xi_{t_1},$$

$$x_{t_i} = \Phi_i x_{t_{i-1}} + \xi_{t_i}, \;\; i = 2, \ldots, T,$$

where properties of ξ_{t_i} satisfy Theorem 3.1. The parameters to be estimated are $\theta = [\alpha_1, \alpha_2, \beta, \sigma^2]'$. Estimates of θ are obtained when the Gaussian log-likelihood function is maximised.

$$L(\theta) = -\frac{T}{2}ln2\pi - \frac{1}{2}ln|\Omega| - \frac{1}{2}\xi'\Omega^{-1}\xi,$$

where $\xi = [\xi_1{}', \cdots, \xi_T{}']'$ is an $nT \times 1$ vector of disturbances and the $nT \times nT$ covariance matrix of ξ is

$$\begin{aligned}\Omega &= E\left[\xi\xi'\right] \\ &= \begin{bmatrix} \Omega_{0,1} & \Omega_{1,1} & 0 & 0 & \cdots & \cdots & 0 \\ \Omega_{-1,2} & \Omega_{0,2} & \Omega_{1,2} & 0 & \cdots & \cdots & 0 \\ 0 & \Omega_{-1,3} & \Omega_{0,3} & \Omega_{1,3} & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & \Omega_{-1,T-1} & \Omega_{0,T-1} & \Omega_{1,T-1} \\ 0 & 0 & \cdots & \cdots & 0 & \Omega_{-1,T} & \Omega_{0,T} \end{bmatrix}.\end{aligned}$$

Note that estimating the parameters, θ, by maximising the above log likelihood function may not be convenient since inverting the matrix Ω is (computationally) costly. An alternative method is to find the Choleskey factorization of Ω, then follow a recursive procedure that avoids directly inverting Ω (see Bergstrom, 1985, 1990).

Let M be the real $nT \times nT$ lower triangular matrix with positive elements along the diagonal such that $MM' = \Omega$. Thus $|\Omega| = |MM'| = |M||M| = |M|^2$ and $\Omega^{-1} = (M')^{-1}(M)^{-1}$. The sub-matrices of M, $M_{11}, \cdots, M_{t,t-1}, M_{tt}(t = 2, \cdots, T)$ can be computed as

$$M_{11}M_{11}' = \Omega_{00},$$

$$M_{i,i-1} = \Omega_{1,i-1}'(M_{i-1,i-1}')^{-1},$$

$$M_{i,i}M_{i,i}' = \Omega_{0,i} - M_{i,i-1}M_{i,i-1}', i = 2, \cdots, T.$$

Then define a normalised $nT \times 1$ vector ϵ, satisfying $E\left[\epsilon\right] = 0$ and $E\left[\epsilon\epsilon'\right] = I$, such that $M\epsilon = \xi$. Hence we have $\xi'\Omega^{-1}\xi = \xi'(M')^{-1}(M)^{-1}\xi = \epsilon'\epsilon$. Then log-likelihood function can thus be evaluated as

$$L = \sum_{i=1}^{nT}({\epsilon_i}^2 + 2ln(m_{ii})),$$

where m_{ii} is the i-th diagonal element of M.

Then ξ can be computed recursively as

$$\xi_1 = M_{11}\epsilon_1,$$

$$\xi_i = M_{i,i-1}\epsilon_{i-1} + M_{i,i}\epsilon_i, i = 2, \cdots, T.$$

The Gaussian estimates of θ are obtained when L is minimised. See Appendix B for derivation details.

With the simulated unequally-spaced data, we re-estimated the parameters using the model, which sampling intervals are treated as equal ("equally-spaced" model) and are normalised as unity. Namely, $\delta = t-(t-1) = 1$ for all observations. The estimation procedure is very similar to the model of interest ("unequally-spaced" model). See Appendix B for derivations. We then compared the estimations results from using the two models, expecting the estimates of "unequally-spaced" model to have smaller estimation bias.

The results from $10,000$ replications in each case are presented in Table 3.1. The table contains the simulation bias (calculated as estimated value minus fixed value) and standard error for each estimator (in the parenthesis under). The estimates of $\alpha_1, \alpha_2, \beta$ and σ^2 are denoted by $\hat{\alpha_1}, \hat{\alpha_2}, \hat{\beta}$ and $\hat{\sigma^2}$, respectively. "Model I" indicates the "unequally-spaced" model while "Model II" indicates the "equally-spaced" model. The estimation bias (in absolute terms) in Model I are smaller than that in Model II, except for estimates of σ^2. The bias in estimates of σ^2 is smaller in Model II, though the standard errors of these estimates are slightly larger in Model II. The standard errors are smaller in Model I than for Model II in all cases. Moreover, estimation bias get smaller with the increase in sample size in both models. Interestingly, the bias of estimates of β is of different signs in the two different parameter configurations. Overall, the results are broadly favouring Model I, which correctly accounts for the unequal sampling intervals, suggesting that there are improvements to be made in the estimation procedures when the sampling intervals are correctly measured.

Table 3.1: Monte Carlo Simulation Results

	Parameter	α_1	α_2	β	σ^2
	Fixed Value	-1.25	0.75	1	0.25
$T = 120$	Estimate	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\beta}$	$\hat{\sigma^2}$
	Bias (Model I)	-0.022	0.035	0.033	-0.127
		(0.0665)	(0.0762)	(0.0749)	(0.0006)
	Bias (Model II)	0.040	0.046	0.040	-0.122
		(0.0716)	(0.0828)	(0.0862)	(0.0007)
$T = 240$	Bias (Model I)	-0.014	0.018	0.012	-0.125
		(0.0291)	(0.0348)	(0.0276)	(0.0003)
	Bias (Model II)	-0.031	0.030	0.016	-0.120
		(0.0316)	(0.0393)	(0.0339)	(0.0003)
	Parameter	α_1	α_2	β	σ^2
	Fixed Value	-0.95	-0.05	1	0.25
$T = 120$	Estimate	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\beta}$	$\hat{\sigma^2}$
	Bias (Model I)	-0.042	0.043	-0.016	-0.122
		(0.026)	(0.0181)	(0.0283)	(0.0007)
	Bias (Model II)	-0.061	0.052	-0.016	-0.115
		(0.0277)	(0.0202)	(0.0332)	(0.0009)
$T = 240$	Bias (Model I)	-0.019	0.021	-0.011	-0.123
		(0.0104)	(0.007)	(0.0117)	(0.0003)
	Bias (Model II)	-0.040	0.027	-0.013	-0.114
		(0.0113)	(0.0087)	(0.0156)	(0.0005)

3.5 Conclusion

For discretizing continuous time models with unequally-spaced date, the previous chapter provides a method, which imposes restrictions on the parameter matrix A to be nonsingular. This, however, rules out applications to nonstationary systems such as unit root and cointegrated systems. This chapter presents an alternative method to derive the exact discrete time representation of continuous time models with unequally-spaced flows and mixed data. In all cases the discrete time representations follow a VARMA(1, 1) process with time-varying parameters and heteroskedasticity, despite that the underlying continuous time model has constant parameters and homoskedasticity. The time-varying parameters and the heteroskedastic variance arise due to the variations in the sampling intervals, whereas the moving average disturbances arise due to the flow nature of the observations.

The exact discrete time representation for flow variables can be applied to nonstationary systems such as unit root and cointegrated systems since it imposed no restrictions on the matrix A; while the exact discrete time model for mixed samples requires the sub-matrix A^{ss} to be nonsingular. This restriction limits the potential applications to systems involving zero roots and cointegration between the stocks.

A Monte Carlo simulation study is conducted, aiming at examining estimates properties for the model which correctly measures the unequal sampling intervals. The main procedure of the study is to simulate unequally-spaced data (monthly data) and then estimate the continuous time parameters using the exact discrete time model, which accounts for the unequal sampling intervals. Comparing to estimations results, based on the simulated data, from using the model, which treats sampling intervals as equal, the simulation results suggest that estimation bias is reduced when the unequal sampling intervals are measured correctly.

In the Monte Carlo study, we only simulate monthly data, which presents relatively small variation in sampling intervals. Though the simulation evidence indicates the favour of exact discrete time models accounting for the irregularity of sampling intervals, the estimation results are close when using different discrete time models. These relatively small estimation bias discrepancies may be explained by the small variations in the sampling intervals. With more irregularly spaced data, the advantage of accounting for the unequal sampling intervals could get bigger. Another potential extended work could be deriving the exact discrete time representation for mixed data, which does not impose restriction on the sub-matrix A^{ss}, such that the results could have broader applications of interest. This possibly request a different method which does not require inverting A^{ss}.

The exact discrete time representation provided in this chapter have wide applicability since restrictions on the underlying continuous time model are relatively weak, hence it can be applied to both stationary and non-stationary processes. This chapter focuses on deriving the exact discrete time representation, the approach may be extended in several directions in further research. One potential extension of the model is to deal with mixed frequency (unequally spaced) data, which is suggested in Chambers' (2016) paper where the derivation of exact representations of multivariate continuous time model with mixed frequency data is presented. The methods in Chambers (2016) may be applied to derive the exact discrete time model with data that are observed at different frequencies as well as unequal intervals. Although the process could be complex, such approach could provide a more realistic setting as well as wider applicability.

3.6 Appendix B

3.6.1 Cholesky factorization of the covariance matrix Ω

Let M be the real $nT \times nT$ lower triangular matrix with positive elements along the diagonal:

$$M = \begin{bmatrix} M_{11} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ M_{21} & M_{22} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & M_{32} & M_{33} & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & M_{T-1,T-2} & M_{T-1,T-1} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & M_{T,T-1} & M_{T,T} \end{bmatrix}.$$

The sub-matrices, $M_{11}, \cdots, M_{t,t-1}, M_{tt} (t = 2, \cdots, T)$ can be computed as

$$\begin{aligned} M_{11}M'_{11} &= \Omega_0, \\ M_{21} &= \Omega_1 (M'_{11})^{-1}, \\ M_{22}M'_{22} &= \Omega_0 - M_{21}M'_{21}, \\ &\vdots \\ M_{t,t-1} &= \Omega_1 (M'_{t-1,t-1})^{-1}, \\ M_{tt}M'_{tt} &= \Omega_0 - M_{t,t-1}M'_{t,t-1}, t = 2, \cdots, T. \end{aligned}$$

To compute M, we need to compute (elements of) Ω; then we need to compute ξ. It is necessary to define a normally distributed $nT \times 1$ vector satisfying $M\epsilon = \xi$: Define an $nT \times 1$ vector $\epsilon = [\epsilon'_1, \cdots, \epsilon'_T]'$ such that $M\epsilon = \xi$, where $E[\epsilon] = 0, E[\epsilon\epsilon'] = I_{nT}$ and $E[\epsilon_t] = 0, E[\epsilon_t\epsilon'_t] = I_n, E[\epsilon_t\epsilon'_s] = 0$ for $s \neq t$ and $s, t = 1, \cdots, T$. Therefore, $\xi'\Omega^{-1}\xi = \xi'(M')^{-1}M^{-1}\xi = \epsilon'\epsilon$. Then ξ is computed as $\xi = M\epsilon$, whose procedure is given in section 4.

3.6.2 Computing elements of Ω

Elements of the matrix Ω include

$$\begin{aligned}\Omega_{01} &= \int_0^{\delta_1} G(s)\Sigma G(s)'ds \\ &= \int_0^{\delta_1}\int_0^s\int_0^s e^{Ar}\Sigma e^{A'w}dwdrds \\ &= \Psi(\delta_1),\end{aligned}$$

$$\begin{aligned}\Omega_{1,i} &= \int_0^{\delta_i} G(s)\Sigma\Gamma(s)'ds \\ &= \left(\int_0^{\delta_i}\int_0^s e^{Ar}\Sigma e^{A's}drds\right)G'_{i+1} - \left(\int_0^{\delta_i}\int_0^s\int_0^s e^{Ar}\Sigma e^{A'w}dwdrds\right)\Phi'_{i+1}, \\ &= \Lambda(\delta_i)G'_{i+1} - \Psi(\delta_i)\Phi'_{i+1},\end{aligned}$$

where $\Lambda(\delta_i) = \int_0^{\delta_i}\int_0^s e^{Ar}\Sigma e^{A's}drds = \int_0^{\delta_i} G(s)\Sigma F(s)'ds$, and $\Psi(\delta_i) = \int_0^{\delta_i}\int_0^s\int_0^s e^{Ar}\Sigma e^{A'w}dwdrds = \int_0^{\delta_i} G(s)\Sigma G(s)'ds, i = 2, \cdots, T$ in the following,

$$\begin{aligned}\Omega_{-1,i} &= \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma G(s)'ds \\ &= G_i\Lambda(\delta_{i-1})' - \Phi_i\Psi(\delta_{i-1})',\end{aligned}$$

$$\begin{aligned}\Omega_{0,i} &= \int_0^{\delta_i} G(s)\Sigma G(s)'ds + \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma\Gamma_i(s)'ds \\ &= \Psi(\delta_i) + G_iL(\delta_{i-1})G'_i - G_i\Lambda(\delta_{i-1})'\Phi'_i - \Phi_i\Lambda(\delta_{i-1})G'_i + \Phi_i\Psi(\delta_{i-1})\Phi'_i,\end{aligned}$$

where $A(\delta_i) = \int_0^{\delta_i} e^{As}\Sigma e^{A's}ds = \int_0^{\delta_i} F(s)\Sigma F(s)'ds$.

In order to compute elements of Ω, we need to compute following matrix exponential and its integrals:

$$F_i = e^{\delta_i A}, G_i = \int_0^{\delta_i} e^{As}ds, L(\delta_i) = \int_0^{\delta_i} e^{As}\Sigma e^{A's}dsds,$$
$$\Lambda(\delta_i) = \int_0^{\delta_i}\int_0^s e^{Ar}\Sigma e^{A's}drds, \Psi(\delta_i) = \int_0^{\delta_i}\int_0^s\int_0^s e^{Ar}\Sigma e^{A'w}dwdrds.$$

Since the matrix A is singular, we cannot directly compute $\int_0^{\delta_i} e^{As}ds = A^{-1}(e^{\delta_i A} - I)$. The matrix exponential, $e^{A\delta_i}$, and the integrals of the matrix exponential can be obtained from the computation of a $4n \times 4n$

matrix exponential (see Van Loan, 1978; and Thornton and Chambers, 2016).

Let C be the $4n \times 4n$ upper triangular matrix, defined by

$$C = \begin{bmatrix} -A & I & 0 & 0 \\ 0 & -A & \Sigma & 0 \\ 0 & 0 & A' & I \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then for $\delta_i \geqslant 0$ for all i,

$$e^{c\delta_i} = exp\left\{ \delta_i \begin{bmatrix} -A & I & 0 & 0 \\ 0 & -A & \Sigma & 0 \\ 0 & 0 & A' & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} = \begin{bmatrix} F_1(\delta_i) & G_1(\delta_i) & H_1(\delta_i) & K_1(\delta_i) \\ 0 & F_2(\delta_i) & G_2(\delta_i) & H_2(\delta_i) \\ 0 & 0 & F_3(\delta_i) & G_3(\delta_i) \\ 0 & 0 & 0 & F_3(\delta_i) \end{bmatrix},$$

where

$$F_3(\delta_i) = e^{A'\delta_i},$$

$$\begin{aligned} G_2(\delta_i) &= \int_0^{\delta_i} e^{-A(\delta_i - s)} \Sigma e^{A's} ds \\ &= e^{-A\delta_i} \int_0^{\delta_i} e^{As} \Sigma e^{A's} ds, \end{aligned}$$

$$G_3(\delta_i) = \int_0^{\delta_i} e^{A(\delta_i - s)} ds,$$

$$H_2(\delta_i) = e^{-A\delta_i} \int_0^{\delta_i} \int_0^{\delta_s} e^{As} \Sigma e^{A'r} dr ds,$$

Therefore,

$$F_i = F_3(\delta_i)',$$

$$G_i = G_3(\delta_i),$$

$$L(\delta_i) = F_3(\delta_i)' G_2(\delta_i),$$

$$\Lambda(\delta_i) = F_3(\delta_i)' H_2(\delta_i),$$

$$\Psi(\delta_i) = F_3(\delta_i)' K_1(\delta_i) + K_1(\delta_i)' F_3(\delta_i).$$

3.6.3 Derivation of "equally-spaced" model

In the case of model with equally spaced flows, observations, x_t, are made over equally spaced discrete integrals, $(t-1,t)$, such that $x_t = \int_{t-1}^{t} x(r)dr, t = 1, \cdots, T$.

Let $x(t)$ be an $n \times 1$ stochastic process generated by

$$dx(t) = Ax(t)dt + \zeta(dt), \ \ t > 0,$$

where $A = \begin{bmatrix} \alpha_1 & -\alpha_1\beta \\ \alpha_2 & -\alpha_2\beta \end{bmatrix}$.

If $x(t)$ is a stock variable, then the discrete time form of (1) obtained as

$$x(t) = Fx(t-1) + \eta(t), t = 1, \cdots, T,$$

where $F = e^{A\delta} = e^{A}$, given that $\delta = 1$, $\eta(t) = \int_{t-1}^{t} e^{A(t-r)}\zeta(dr)$.

If the observations are flow variables then

$$x_t = \int_{t-1}^{t} x(r)dr = \int_{0}^{1} x(t-r)dr = \int_{0}^{1} x(t-1+r)dr.$$

From the discrete time model for stock variables we obtain

$$x(t-1+s) = e^{As}x(t-r) + \int_{t-1}^{t-1+s} \zeta(dr).$$

Integrating the above equation over the interval $s \in (0,h]$ obtains

$$\int_{0}^{1} x(t-1+s)ds = (\int_{0}^{1} e^{As})x(t-1) + \int_{0}^{1}\int_{t-1}^{t-1+s} e^{A(t-1+s-r)}\zeta(dr)ds,$$

which can be represented as

$$x_t = Gx(t-1) + e_t,$$

where $G = \int_{0}^{1} e^{As}ds, e_t = \int_{0}^{1}\int_{t-1}^{t-1+s} e^{A(t-1+s-r)}\zeta(dr)ds = \int_{t-1}^{t} G(t-r)\zeta(dr), t = 1, \cdots, T.$

Re-arranging the above equation yields

$$x(t-1) = G^{-1}(x_t - e_t).$$

Lagging the discrete time model for stocks for one period and substituting out $x(t-1)$ using the above equation obtains

$$G^{-1}(x_t - e_t) = FG^{-1}(x_{t-1} - e_{t-1}) + \eta(t-1).$$

Re-arranging the above equation obtains the reduced-form discrete time model

$$x_1 = \Phi x(0) + \epsilon_1,$$

$$x_t = \Phi x_{t-1} + \epsilon_t, \;\; t = 2, \ldots, T,$$

where $\epsilon_1 = e_1 = \int_0^1 G(1-r)\zeta(dr)$ with $t(0) = 0$ and $t_1 = \delta = 1$, $\epsilon_t = \int_{t-1}^t G(t-r)\zeta(dr) + \int_{t-2}^{t-1} \Gamma(t-1-r)\zeta(dr), t = 2, \ldots, T.$

Properties of the disturbances are given by

$$\Omega_0 = E[\epsilon_t \epsilon_t'] = \begin{cases} \int_0^1 G(s)\Sigma G(s)' ds, & t = 1, \\ \int_0^1 G(s)\Sigma G(s)' ds + \int_0^1 \Gamma(s)\Sigma\Gamma(s)' ds, & t = 2, \ldots, T, \end{cases}$$

$$\Omega_{-1} = E[\epsilon_t \epsilon_{t-1}'] = \int_0^1 \Gamma(s)\Sigma G(s)' ds, \;\; t = 2, \ldots, T,$$

$$\Omega_1 = E[\epsilon_t \epsilon_{t+1}'] = \int_0^1 G(s)\Sigma\Gamma(s)' ds, \;\; t = 1, \ldots, T-1,$$

Furthermore, let Ω_{00} denote the variance when $t = 1$ and the we have

$$\Omega_{00} = \int_0^1 G(s)\Sigma G(s)' ds,$$

and the covariance matrix is

$$\begin{aligned} \Omega &= E\left[\epsilon\epsilon'\right] \\ &= \begin{bmatrix} \Omega_{00} & \Omega_1 & 0 & 0 & \cdots & \cdots & 0 \\ \Omega_{-1} & \Omega_0 & \Omega_1 & 0 & \cdots & \cdots & 0 \\ 0 & \Omega_{-1} & \Omega_0 & \Omega_1 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & \Omega_{-1} & \Omega_0 & \Omega_1 \\ 0 & 0 & \cdots & \cdots & 0 & \Omega_{-1} & \Omega_0 \end{bmatrix}. \end{aligned}$$

The simulation procedure is the same as in the “unequally-spaced” model.

4 Exact Gaussian Estimation of Continuous Time Models with Unequally Spaced Data

This chapter presents empirical applications of the exact discrete time representation method to estimate continuous time models with data observed at unequally spaced intervals. Two cases are considered - a univariate model with a stock variable (vacancy stock), and a bivariate model with two flow variables (vacancy inflow and outflow). The data are reported labour market vacancies, whose count dates are not released at a regular basis, which leads to irregular sampling intervals. The empirical results show that the parameter estimates are different when accounting for the unequal sampling intervals compared to the approach that assumes data are equally spaced. In addition, the Monte Carlo simulation evidence suggests that estimation bias is smaller when accounting for the unequally spaced intervals, indicating potential gains in estimation when the appropriate approach is applied. Especially for the model with flow variables, the bias in parameter estimates is obviously smaller under the appropriate approach. Even with relatively small variation in sampling intervals, there are gains to be made by incorporating the correct discrete time representation of the continuous time models. These evidences support the argument that the unequal spacing should be taken into account in the estimation procedure.

4.1 Introduction

Since the seminal work by Bergstrom (1983) on the Gaussian estimation of continuous time dynamic models based on the exact discrete time representation, continuous time modelling has been applied in a wide range of areas. Advantages of modelling in continuous time over discrete time have been broadly discussed in Bergstrom et al. (1996) and Bergstrom and Nowman (2007). The development and application of continuous time models has been an important contribution and extension of the original work to the existing literature. One major ongoing development has been the estimation of continuous time models in finance. In a benchmark work by Black and Scholes (1973) the continuous time model was used for option pricing. Further applications include modelling interest rates (for example, in Vasicek, 1977; Chan et al., 1992) and asset pricing (see Huang, 1987). Continuous time models have also been extensively applied in macroeconomic modelling since Bergstrom and Wymer (1976) first derived a continuous time macroeconometric model. For example, Bergstrom and Chambers (1990) and Chambers (1992) presented a continuous time model of consumer's demand in the UK. Further, Campbell et al. (2004) used continuous time models to model optimal intertemporal portfolio and consumption choice. More recently, Diez de los Rios and Sentana (2011) used a continuous time approach to derive a new test of uncovered interest parity.

In much of the work on estimation of continuous time models, the data are treated as equally spaced, while in much time series data the observation interval is not constant over time. For example, irregular observation intervals may arise due to missing observations, irregular observing behaviour, small random deviations in the interval (in “jittered” samples) or simply the variation in the length of calendar months. Early approaches to estimate with irregular sampled data include spectral analysis. For instance, Jones (1962, 1971) and Parzen (1963) have employed spectral estimation with missing observations. However, with relatively small samples, the spectral analysis method may be not preferred in estimating finite-parameter models from unequally spaced data as argued by Robinson (1977b). In his paper he presented the discretisation of a univariate continuous time model with an unequally spaced stock variable as an alternative method to spectral analysis.

Robinson's paper (1977b) has provided the possibility to extend the original model of Bergstrom (1983) to estimate continuous time models with unequally spaced data based on the exact discrete time method. However, he did not continue to work on deriving the discrete time model with irregularly sampled time series. In my previous chapters I have derived the theoretical framework of the exact discrete representation of continuous time systems with unequally spaced samples that are observed at a sequence of known real time points (rather than missing observations). The exact discrete time representations exhibit more complicated characteristics such as heteroskedastic errors and time-varying coefficients compared to regularly sampled case. When the irregularly sampled data are mistreated as equally spaced, estimation results are likely to be more biased, which is consistent with simulation evidences. Given the available unequally spaced data, this chapter aims to provide some examples of empirical applications of the theoretical model, with an illustration on gains (such as reduced estimation bias) in the estimation procedure when the unequal sampling intervals are correctly measured.

The data employed in the empirical studies are the monthly job vacancies in UK posted through Jobcentre Plus, which include vacancy stocks, inflows and outflows from 2004 to 2012 (when the administrative data collection work discontinued). Firstly, this dataset provides data on vacancies, which is an important topic in the macroeconomic and labour economic literature. The number of vacancies in a labour market can be treated as a direct measure of the demand for labour since firms usually recruit via posting vacancies. Significance of vacancies in the context of theoretical macro-labour literature is emphasised in search and matching models following the benchmark work by Diamond (1982), Mortensen and Pissarides (1994), and Pissarides (2000). The Diamond-Mortensen-Pissarides (DMP) model has become the canonical model of labour macroeconomics. Therotically, in the DMP model, the (endogenous) stock of vacancies is a component of labour market tightness (measured as the ratio of vacancy to unemployment), and workers job-finding rates (and therefore unemployment duration, and aggregate unemployment rates) are determined by tightness. On the other hand, empirically, Shimer (2005) indicated that there is a very strong relationship between tightness and job finding rate. Therefore, understanding the behaviour of vacancy stocks and flows is crucial for understanding workers' labour market outcomes and the aggregate dynamics of the labour market. Secondly, the data are observed at irregular intervals that vary from 28 days to 35 days according to the count dates, which fit the purpose of this chapter well. Although there are available up-to-date vacancy data, which can be found, for example, in the ONS website, these data do not contain vacancy flows as well as the precise count dates, making it difficult to measure any irregular sampling intervals. Rather than addressing a specific issue in macroeconomics (or labour economics), the focus of this chapter is to provide evidence on gains in the estimation when the unequal sampling intervals are correctly measured via empirical applications.

This chapter provides two applications of the methods in the earlier chapters. The first application presents the exact Gaussian estimation of a univariate continuous time model of job vacancy stock variable. Results show small differences in estimations when comparing to estimation assuming equally spaced data, but the standard deviations are significantly smaller when using the approach that correctly accounts for the unequal sampling intervals. In practitioners' view, perhaps a more interesting related empirical exercise would be to estimate a model of the joint dynamics of vacancies and unemployment stocks (e.g. a VAR model). However, one limitation of the NOMIS data for this purpose is that the count dates of vacancy and unemployment are different and the method developed in this thesis requires that both sticks are observed at the same basis. Therefore, this exercise is outside the scope of this thesis, but could be interesting to investigate in future research.

The second application is the estimation of a bivariate continuous time models with vacancy inflow and outflow that are observed at the same basis. Comparing to the approach that treats the data as equally spaced, the differences in the estimated drift parameters are still small, but the differences in variance estimates are large. The results still indicate the possibility of gains from correctly accounting for the unequal sampling intervals. The approach presented in this chapter may be extended/modified to labour economic/macroeconomic analysis such as mixed-sample case (e.g. vacancy flows and stocks), although such work could involve complex process in model specification as well as estimation. In addition, given that the model is multivariate, the aliasing problem could impose some additional challenge in solving such problem. Again, solving this problem is beyond the scope of this thesis, but it may offer a new insight into labour market as a future research.

The rest of the chapter is organised as follows. Section 4.2 considers an application to estimating a univariate continuous time model with unequally spaced vacancy stocks based on the exact discrete time method. The empirical study involves two sets of estimations based on the same (unequally spaced) data: one approach takes the irregular sampling intervals into account while the other assumes data are equally spaced. The estimation results indicate that there are differences in parameter estimates and smaller standard errors for estimates from the approach which accounted for the unequal sampling intervals. Then a Monte Carlo simulation is conducted in order to examine the impact of correctly measuring sampling intervals on estimate's properties. Simulation evidence suggests that there are gains (such as smaller estimation bias and standard errors) to be made in the estimation of continuous time models when the unequal sampling intervals are correctly accounted for, even with relatively small variation in the sampling interval.

Section 4.3 estimates a bivariate continuous time system with vacancy inflows and outflows from the same dataset. The underlying continuous time model is estimated based on two different approaches, one considers the unequal sampling intervals while the other assumes intervals are equal. Estimation results are different for the two approaches, particularly in the estimate of the variance. Following a Monte Carlo simulation study, the simulation results suggest that with flow variables, the Gaussian estimation bias is obviously larger when the sampling intervals are mistreated as equally spaced. The bias is reduced when the unequal sampling intervals are correctly measured, particularly in estimates of the variance. Section 4.4 concludes the main results.

4.2 Estimation of Continuous Time Models with an Unequally Spaced Stock Variable

This section presents an application of estimation of continuous time models with stock variables based on the exact discrete time representation method. A simple case is considered: the continuous time model is in the form of a first corder univariate autoregressive model (AR(1) process) while the data are vacancy stocks in UK.

The data were obtained from the Nomis (ONS) website, which are vacancies posted through Jobcentre Plus, including monthly vacancy stocks, inflows and outflows from April 2004 to April 2012 with irregular count date. In the data, sampling intervals vary from 28 days to 35 days. Note that in the original data set, what followed August 2010 observation was October 2010 observation, leading to the corresponding sampling interval become 63 days and the sample size to be 96, which equals to 8-year monthly sample size. This administrative data collection work started in 2004 and was discontinued in November 2012. More recent (up-to-date) vacancy data could be found in the ONS website, however, these dataset firstly do not contain inflows and outflows, and secondly do not provide precise count dates, which is inconvenient for correctly measuring the irregular sampling intervals. These data from the Jobcentre Plus, even though have been discontinued unfortunately, provide a set of vacancy stocks and flows with precise count dates, which is appropriate for the models involved in this paper since they provide both stock and flow data and allow us to observe the irregular intervals.

Given the available unequally spaced data, this section compares estimation results from the model that accounts for the irregular sampling intervals to estimations results from the model that assumes data are equally spaced (using the same data). Then the estimation was repeated based on simulated data in order to examine the impact of correctly accounting for unequal sampling intervals.

4.2.1 The Exact Discrete Time Representation

Model I - Unequal Sampling Intervals

Let $x(t)$ be an stochastic process generated by

$$dx(t) = ax(t) + \zeta(dt), \;\; t > 0, \tag{4.1}$$

where a is a scalar and $\zeta(dt)$ is a random measure satisfying Assumption 4.1, such that:

$$E[\zeta(dt)] = 0$$

$$E[\zeta(dt)^2] = \sigma^2 dt,$$

where σ is an unknown positive constant and

$$E[\zeta_i(\Delta_1)\zeta_j(\Delta_2)] = 0,$$

for $i, j = 1, 2, \cdots, n$; $i \neq j$; and $\Delta_1 \cap \Delta_2 = \emptyset$.

In what follows, it is assumed that samples are observed at the discrete points t_i $(i = 0, \ldots, T)$ such that $0 < t_1 < \ldots < t_T$ and $t_i = t_{i-1} + \delta_i$ for some $\delta_i > 0$ $(i = 0, \ldots, T)$. In the case of a stock variable the sequence of observations is of the form

$$x(t_0),\ x(t_1),\ \cdots,\ x(t_T), \tag{4.2}$$

and henceforth denote $x(t_0)$ by $x(0)$ for convenience.

The system of a stock variable can be written as

$$x(t_i) = \int_0^{t_i} e^{(t_i - r)a}\, \zeta(dr) + e^{t_i a} x(0), \tag{4.3}$$

for $i = 1, \ldots, T$, under the Boundary Condition that $x(0)$ is observed.

Then under Assumption 4.1, subject to the above boundary condition, the exact discrete time representation of (4.1) is obtained as

$$x(t_i) = e^{a\delta_i} x(t_{i-1}) + \eta(t_i), \ \ i = 1, \ldots, T, \tag{4.4}$$

where

$$\eta(t_i) = \int_{t_{i-1}}^{t_i} e^{a(t_i - r)} \zeta(dr)$$

$$\begin{aligned} E[\eta(t_i)] &= 0, \\ E[\eta(t_i)^2] &= \sigma^2 \int_{t_{i-1}}^{t_i} e^{2a(t_i - r)} \zeta(dr) \\ &= \frac{\sigma^2(e^{2a\delta_i} - 1)}{2a} \\ &= {\sigma_i}^2, \end{aligned}$$

$$E[\eta(t_i)\eta(t_j)] = 0, \ \ for \ \ i \neq j.$$

The exact discrete model has time-varying coefficient and serially uncorrelated heteroskedastic disturbances. The Gaussian estimates of parameters of the continuous time model, $\hat{a}$ and $\hat{\sigma^2}$, can be obtained when the Gaussian log-likelihood function of (4.4)

$$L(a, \sigma^2) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{T}\ln\sigma_i^2 - \frac{1}{2}\sum_{i=1}^{T}\frac{[x(t_i) - e^{a\delta_i}x(t_{i-1})]^2}{\sigma_i^2}, \ \ i = 1, \ldots, T. \tag{4.5}$$

is maximized. Note that (4.5) is a function of the continuous time parameters a and σ^2, (4.5) can be further written as

$$L(a, \sigma^2) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{T}\ln\left[\frac{\sigma^2(e^{2a\delta_i} - 1)}{2a}\right] - \frac{1}{2}\sum_{i=1}^{T}\frac{2a[x(t_i) - e^{a\delta_i}x(t_{i-1})]^2}{\sigma^2(e^{2a\delta_i} - 1)}, \ \ i = 1, \ldots, T. \tag{4.6}$$

The maximum likelihood estimate (MLE) of a, denoted by $\hat{a}$, is directly obtained when (4.6) is maximised. We first obtained the MLE, $\hat{\sigma}$, and the MLE, $\hat{\sigma}^2$, is obtained by squaring $\hat{\sigma}$. This is to avoid getting any negative values in $\hat{\sigma}^2$. Then its standard deviation is obtained based on the delta method.

Model II - Equal Sampling Intervals

Given that data are generated by (4.1), and based on the assumption that the sampling intervals are equally spaced. Namely the samples are observed at discrete points $t = 0, 1, \ldots, T$ and the sampling interval $\delta = t - (t-1) = 1$. A stock variable consists a sequence of observations of the form

$$x(0), x(1), \ldots, x(T), \tag{4.7}$$

and the exact discrete time representation of such observations generated by (4.1) is obtained as

$$x(t) = \phi x(t-1) + u(t), \ \ t = 1, \ldots, T, \tag{4.8}$$

where

$$\phi = e^a,$$

$$u(t) \sim NID(0, \ \ {\sigma_u}^2),$$

$${\sigma_u}^2 = \frac{\sigma^2(e^{2a} - 1)}{2a},$$

under Assumption 4.1 and subject to the Boundary Condition that $x(0)$ is observed.

The Gaussian maximum likelihood estimates of parameters a and σ^2 are obtained when the log-likelihood function of (4.8)

$$L(a,\sigma^2) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln\sigma_u^2 - \frac{1}{2}\sum_{t=1}^{T}\frac{[x(t)-e^a x(t-1)]^2}{\sigma_u^2}, \;\; i = 1,\ldots,T. \tag{4.9}$$

is maximised. These are equivalent to the OLS estimates. The OLS estimate of ϕ is obtained as $\hat{\phi} = \frac{\sum_{t=1}^{T} x(t-1)^2}{\sum_{t=1}^{T} x(t-1)x(t)}$. Then estimate of a can be obtained taking the logarithm of $\hat{\phi}$ as $\hat{a} = ln\hat{\phi}$. Given the estimate value of $\hat{\phi}$, the OLS estimate of ${\sigma_u}^2$ is obtained as $\hat{\sigma_u}^2 = \frac{1}{T-1}\sum_{t=2}^{T}[x(t) - \hat{\phi}x(t-1)]^2$ and the estimate of σ is then obtained as $\hat{\sigma^2} = \frac{2\hat{a}\hat{\sigma_u}^2}{e^{2\hat{a}}-1}$. The standard errors are calculated based on the delta method. Estimation results from both models are presented in the next subsection.

4.2.2 Estimation Results

Parameter estimates and standard errors (in parenthesis) from both models are presented in Table 4.1. In the data, sampling intervals are normalised by dividing by 30, leading to the intervals, δ, vary from $28/30 = 0.933$ to $35/30 = 1.167$, except for the observation in August 2010 whose interval is 63 days, which is the maximum $\delta_{max} = 63/30 = 2.1$ in the sample. The sample size $N = T + 1 = 96$. Estimation results from Model I (unequal sampling intervals) are presented as “MLE” while estimation results from Model II (equal sampling intervals) are presented as “OLS” in the table.

Overall the estimated values of the continuous time parameters from both models are similar, while the differences in $\hat{\sigma^2}$ are slightly larger than the differences in $\hat{a}$. Specifically, the standard errors of the estimates in Model I are significantly smaller than the standard errors in Model II. For instance, the standard error of the MLE $\hat{\sigma^2}$ is 0.0377 while the standard error of the OLS estimate $\hat{\sigma^2}$ is 1.588, which is approximately five times (more than one standard deviation) larger than the former. Accounting for sampling irregularities leads to only very small change in the estimated drift parameter and a somewhat larger (yet also small) increase in the estimated volatility parameter. However, the biggest gain from correctly accounting for the unequal sampling intervals seems to be the significant reduction in standard errors, suggesting that all parameters may be more precisely estimated.

As discussed in Shimer (2005), one aspect in which the volatility of vacancies is important is that standard versions of the DMP model imply much higher volatility of vacancies than observed in the data, which has led several studies (e.g. Fujita, 2003) to introduce features to reduce the volatility of vacancies in the model. Even though the estimated differences in volatility here are too small to account for the large differences between model and data discussed by Shimer (2005), it is still clear that precise estimation of this volatility parameter could be of significance for studies calibrating the DMP model.

The estimation results indicate small differences in parameter estimates and potentially better properties of estimates from Model I, where the unequal sampling intervals are correctly accounted for, part of the reason for such small differences could be the small sample size as well as relatively small variation in sampling intervals. Still, there could be potential gains in the estimation, such as smaller estimation bias, which is explored in the next section. In 4.2.3, a Monte Carlo simulation on repeating the above estimation process is presented in order to examine the impact of correctly measuring the irregular sampling intervals.

Table 4.1: Estimation Results - Univariate Model

$\delta_{min} = 0.933$,	$\delta_{max} = 2.1$,	$N = 96$
	$\hat{a}$	$\hat{\sigma^2}$
MLE (Model I)	-0.0118	1.1426
	(0.0778)	(0.0377)
OLS (Model II)	-0.0110	1.1002
	(0.1741)	(0.1588)

4.2.3 Monte Carlo Simulation

This section presents some simulation evidence on improvement in estimate properties when correctly accounting for the unequal sampling intervals. Based on the estimated parameters, the simulated data are monthly data with the sample size N range from 96 to 240. The lengths of the intervals range from 0.933 to 1.167, which is the same as the real data in the empirical estimation. The observations are generated by (4.1)

$$dx(t) = ax(t) + \zeta(dt), \ \ \zeta(dt) \sim NID(o, \sigma^2 dt),$$

with $\sigma^2 = 1.1$ and $a = -0.011$. Subject to the Boundary Condition the initial value taken is $x_0 = x(0) = 0$ for t_0 or $t(0)$, estimate of a and σ^2 are obtained by maximising the Gaussian log-likelihood function

$$L(a, \sigma^2) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{T}\ln\sigma_i^2 - \frac{1}{2}\sum_{i=1}^{T}\frac{[x(t_i) - e^{a\delta_i}x(t_{i-1})]^2}{\sigma_i^2}, \ \ {\sigma_i}^2 = \frac{\sigma^2(e^{2a\delta_i} - 1)}{2a},$$

and the resulting estimates denoted as MLE from Model I (accounting for unequal sampling intervals) and OLS from Model II (assuming equally spaced sampling intervals).

The results from 10,000 replications for each sample size (namely 96, 120 and 240) are presented in Table 4.2. Table 4.2 contains the estimated values, estimation bias and standard error (in parenthesis) for each estimator. The estimation bias for a is negative for both MLE and OLS in all cases and is with a smaller magnitude for MLE. The differences become smaller as the sample size gets smaller. The estimation bias for σ^2 is positive and is smaller for MLE, the differences are larger than the differences in estimates of a. The standard errors are very similar but slightly smaller for MLE in all cases. As the sample size decreases, both estimation bias and the standard errors increase for both estimators. Overall, these results indicate some gains in the estimation of continuous time models, such as smaller estimation bias and standard errors, when the variation in sampling intervals are correctly accounted for, even though the variation in observation intervals is relatively small. With data observed at more irregular intervals (larger variation in intervals), the advantages of using the correct discrete time representation could get larger, particularly for large sample size. Even when such variation is limited, it is still beneficial to correctly account for the unequal sampling intervals in estimation procedure.

Table 4.2: Simulation Results

Simulation Results - Univariate Model				
$\delta_{min} = 0.933$,	$\delta_{max} = 1.167,$	$N = 240,$	$a = -0.011$	$\hat{\sigma}^2 = 1.1$
	$\hat{a}$	$\hat{\sigma^2}$		
MLE (Model I)	-0.0189	1.1069		
bias	-0.0079	0.0069		
	(0.0163)	(0.1020)		
OLS (Model II)	-0.0191	1.1240		
bias	-0.0081	0.0240		
	(0.0164)	(0.1042)		
$\delta_{min} = 0.933$,	$\delta_{max} = 1.167,$	$N = 120,$	$a = -0.011$	$\hat{\sigma}^2 = 1.1$
	$\hat{a}$	$\hat{\sigma^2}$		
MLE (Model I)	-0.0268	1.1100		
bias	-0.0158	0.0100		
	(0.0298)	(0.1445)		
OLS (Model II)	-0.0271	1.1315		
bias	-0.0161	0.0315		
	(0.0300)	(0.1480)		
$\delta_{min} = 0.933$,	$\delta_{max} = 1.167,$	$N = 96,$	$a = -0.011$	$\hat{\sigma}^2 = 1.1$
	$\hat{a}$	$\hat{\sigma^2}$		
MLE (Model I)	-0.0310	1.1148		
bias	-0.020	0.0148		
	(0.0314)	(0.1646)		
OLS (Model II)	-0.0314	1.1388		
bias	-0.0204	0.0388		
	(0.0377)	(0.1692)		

4.3 Estimation of Continuous Time Models with Unequally Spaced Flows

This section presents the empirical estimation which concerns a bivariate continuous time system with two flow variables - vacancy inflows and outflows. The sample is from the same dataset as in the previous section, which starts in April 2004 and finishes in April 2012, yielding 96 unequally spaced observations for each variable. The normalised observation intervals vary from 0.933 to 1.167 except for the observation in August 2010, which has the maximum interval $\delta_{max} = 2.1$.

4.3.1 The Exact Discrete Time Representation

Let $x(t)$ be an $n \times 1$ stochastic process generated by

$$dx(t) = Ax(t)dt + \zeta(dt), \;\; t > 0, \tag{4.10}$$

where A is an $n \times n$ matrix, and $\zeta(dt)$ is an $n \times 1$ vector of random measures satisfying Assumption 4.2, such that

$$E[\zeta(dt)] = 0,$$
$$E[\zeta(dt)\zeta(dt)'] = \Sigma dt,$$
$$E[\zeta_i(\Delta_1)\zeta_j(\Delta_2)'] = 0,$$

where Σ is an unknown symmetric positive definite matrix; $i, j = 1, 2, ..., n$; $i \neq j$; and $\Delta_1 \cap \Delta_2 = \emptyset$

When $n = 2$, (4.10) becomes a bivariate model given by

$$\begin{bmatrix} dx^I(t) \\ dx^O(t) \end{bmatrix} = \begin{bmatrix} a_{II} & a_{IO} \\ a_{OI} & a_{OO} \end{bmatrix} \begin{bmatrix} x^I(t) \\ x^O(t) \end{bmatrix} dt + \begin{bmatrix} \zeta^I(t) \\ \zeta^O(t) \end{bmatrix}, \tag{4.11}$$

where $x^I(t)$ denotes vacancy inflows and $x^O(t)$ denotes vacancy outflows. $[\zeta^I(t), \zeta^O(t)]'$ satisfies Assumption 4.2 with variance matrix Σdt such that

$$\Sigma = \begin{bmatrix} \sigma_I^2 & \sigma_{IO}^2 \\ \sigma_{IO}^2 & \sigma_O^2 \end{bmatrix}.$$

Model I: Unequal Sampling Intervals

In the case of unequally spaced flow variables, the observations constitute a sequence of flow vectors of the form

$$x_{t_i} = \int_{t_{i-1}}^{t_i} x(r)dr = \int_0^{\delta_i} x(t_i - r)dr = \int_0^{\delta_i} x(t_{i-1} + r)dr, \;\; i = 1, \ldots, T. \tag{4.12}$$

According to Theorem 3.1 in chapter 3, the exact discrete time representation of (4.1) observed as the sequence in (4.12) is obtained as

$$\begin{aligned} x_{t_1} &= G_1 x(0) + \xi_{t_1}, \\ x_{t_i} &= \Phi_i x_{t_{i-1}} + \xi_{t_i}, \;\; i = 2, \ldots, T, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} e^A &= \sum_{j=0}^{\infty} \frac{1}{j!} A^j, \\ F_i = F(\delta_i) &= e^{A\delta_i}, \\ G_i = G(\delta_i) &= \int_0^{\delta_i} e^{As} ds, \\ G_1 &= \int_0^{\delta_1} e^{As} ds, \\ \Phi_i &= G_i F_{i-1} G_{i-1}^{-1}. \end{aligned}$$

The disturbance vector in the exact discrete time model (4.13), ξ_{t_i}, is a heteroskedastic MA(1) process with autocovariance matrices given by

$$\Omega_{0,i} = E[\xi_{t_i}\xi_{t_i}'] = \begin{cases} \int_0^{\delta_1} G(s)\Sigma G(s)' ds, & i = 1, \\ \int_0^{\delta_i} G(s)\Sigma G(s)' ds + \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma\Gamma_i(s)' ds, & i = 2, \ldots, T, \end{cases}$$

$$\Omega_{-1,i} = E[\xi_{t_i}\xi_{t_{i-1}}'] = \int_0^{\delta_{i-1}} \Gamma_i(s)\Sigma G(s)' ds, \;\; i = 2, \ldots, T,$$

$$\Omega_{1,i} = E[\xi_{t_i}\xi_{t_{i+1}}'] = \int_0^{\delta_i} G(s)\Sigma\Gamma_{i+1}(s)' ds, \;\; i = 1, \ldots, T-1,$$

where

$$\Gamma_i(x) = G_i F(x) - \Phi_i G(x).$$

The parameters to be estimated are $\theta = [A, \;\Sigma] = [a_{II}, \; a_{IO}, \; a_{OI}, \; a_{OO}, \; \sigma_I^2, \; \sigma_{IO}, \; \sigma_O^2]'$. The maximum likelihood estimates of these parameters are obtained when the Gaussian likelihood function

$$L(\theta) = -\frac{T}{2} - \frac{1}{2} ln|\Omega_U| - \frac{1}{2}\xi'\Omega_U\xi, \;\; \xi = [\xi_1', \ldots, \xi_T']' \tag{4.14}$$

is maximised. The covariance matrix

$$
\begin{aligned}
\Omega_U &= E[\xi\xi'] \\
&= \begin{bmatrix}
\Omega_{01} & \Omega_{1,1} & 0 & 0 & \cdots & \cdots & 0 \\
\Omega_{-1,2} & \Omega_{0,2} & \Omega_{1,2} & 0 & \cdots & \cdots & 0 \\
0 & \Omega_{-1,3} & \Omega_{0,3} & \Omega_{1,3} & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \ddots & & \vdots \\
0 & 0 & \cdots & \cdots & \Omega_{-1,T-1} & \Omega_{0,T-1} & \Omega_{1,T-1} \\
0 & 0 & \cdots & \cdots & 0 & \Omega_{-1,T} & \Omega_{0,T}
\end{bmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega_{01} &= \Psi(\delta_1), \\
\Omega_{1,i} &= \Lambda(\delta_i)G'_{i+1} - \Psi(\delta_i)\Phi'_{i+1}, \\
\Omega_{-1,i} &= G_i\Lambda(\delta_{i-1})' - \Phi_i\Psi(\delta_{i-1})', \\
\Omega_{0,i} &= \Psi(\delta_i) + G_iL(\delta_{i-1})G'_i - G_i\Lambda(\delta_{i-1})'\Phi'_i - \Phi_i\Lambda(\delta_{i-1})G'_i + \Phi_i\Psi(\delta_{i-1})\Phi'_i, \\
\Lambda(\delta_i) &= \int_0^{\delta_i} G(s)\Sigma F(s)'ds, \\
\Psi(\delta_i) &= \int_0^{\delta_i} G(s)\Sigma G(s)'ds, \\
A(\delta_i) &= \int_0^{\delta_i} F(s)\Sigma F(s)'ds.
\end{aligned}
$$

In order to compute Ω_U it is necessary to compute the matrix exponential F_i and its integrals, $G_i,\ L(\delta_i),\ \Lambda(\delta_i)$, and $\Psi(\delta_i)$. If the matrix A is non-singular, we can simply compute $\int_0^{\delta_i} e^{As}ds = A^{-1}(e^{\delta_i A} - I)$. Since we do not restrict A to be non-singular, a more appropriate method to obtain the matrix exponential and its integrals would be computing the exponential of an appropriate $4n \times 4n$ matrix (8×8 in this case).

Let C be the 8×8 upper triangular matrix such that

$$
C = \begin{bmatrix}
-A & I & O & O \\
O & -A & \Sigma & O \\
O & O & A' & I \\
O & O & O & O
\end{bmatrix},
$$

then the exponential of C is computed as

$$P = e^{C\delta_i} = exp\left\{ \delta_i \begin{bmatrix} -A & I & O & O \\ O & -A & \Sigma & O \\ O & O & A' & I \\ O & O & O & O \end{bmatrix} \right\} = \begin{bmatrix} F_1(\delta_i) & G_1(\delta_i) & H_1(\delta_i) & K_1(\delta_i) \\ O & F_2(\delta_i) & G_2(\delta_i) & H_2(\delta_i) \\ O & O & F_3(\delta_i) & G_3(\delta_i) \\ O & O & O & F_4(\delta_i) \end{bmatrix}.$$

It can be shown that $F_i = F_3(\delta_i)'$, $G_i = G_3(\delta_i)$, $L(\delta_i) = F_3(\delta_i)'G_2(\delta_i)$, $\Lambda(\delta_i) = F_3(\delta_i)'H_2(\delta_i)$ and $\Psi(\delta_i) = F_3(\delta_i)'K_1(\delta_i) + K_1(\delta_i)'F_3(\delta_i)$. Maximising the log-likelihood function (4.14) involves inverting the covariance matrix Ω_U, which is computationally costly. It is convenient to find the Cholesky factorization of Ω_U then follow a recursive procedure as follows:

Denote M by the real $nT \times nT$ lower triangular matrix with positive elements along the diagonal such that $MM' = \Omega_U$. Therefore $|\Omega_U| = |MM'| = |M|^2$ and $\Omega_U^{-1} = (M')^{-1}M^{-1}$. Elements of M are computed as

$$M_{11}M'_{11} = \Omega_{01},$$

$$M_{i,i-1} = \Omega'_{1,i-1}(M'_{i-1,i-1})^{-1},$$

$$M_{i,i}M'_{i,i} = \Omega_{0,i} - M_{i,i-1}M'_{i,i-1}, \;\; i = 2, \ldots, T.$$

To compute ξ, it is necessary to define a $nT \times 1$ vector ϵ such that $M\epsilon = \xi$, where $E[\epsilon] = 0$ and $E[\epsilon\epsilon'] = I_{nT}$. Vector ξ can be computed recursively as

$$\xi_1 = M_{11}\epsilon_1,$$

$$\xi_i = M_{i,i-1}\epsilon_{i-1} + M_{i,i}\epsilon_i, \;\; i = 2, \ldots, T.$$

Since $\xi = M\epsilon$, we have $\xi'\Omega_U^{-1}\xi = \xi'(M')^{-1}M^{-1}\xi = \epsilon'M'(M')^{-1}M^{-1}M\epsilon = \epsilon'\epsilon$. Then the log-likelihood function (4.14) can be written as

$$L(\theta) = \sum_{i=1}^{nT}(\epsilon_i^2 + 2lnm_{ii}), \tag{4.15}$$

where m_{ii} is the i-th diagonal element of M. Then the Gaussian maximum likelihood estimate, θ is obtained by maximising (4.15).

Model II: Equal Sampling Intervals

In the case of equally spaced flow variables, the observations generated by (4.10) are in the form of

$$x_t = \int_{t-1}^{t} x(r)dr, \tag{4.16}$$

and the observation intervals $\delta_i = \delta = t - (t-1) = 1$.

The exact discrete time representation is obtained as

$$\begin{aligned} x_1 &= \Phi x(0) + v_1, \\ x_t &= \Phi x_{t-1} + v_t, \;\; i = 2, \ldots, T, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} \Phi &= GFG^{-1}, \\ v_1 &= \int_0^1 G(1-r)\zeta(dr), \\ v_t &= \int_{t-1}^{t} G(t-r)\zeta(dr) + \int_{t-2}^{t-1} \Gamma(t-1-r)\zeta(dr), \\ \Gamma(z) &= GF(z) - \Phi G(z). \end{aligned}$$

The covariance matrix

$$\begin{aligned} \Omega &= E[v'v] \\ &= \begin{bmatrix} \Omega_{00} & \Omega_1 & 0 & \cdots & \cdots & 0 & 0 \\ \Omega_{-1} & \Omega_0 & \Omega_1 & \cdots & \cdots & 0 & 0 \\ 0 & \Omega_{-1} & \Omega_0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \Omega_{-1} & \Omega_0 & \Omega_{-1} \\ 0 & 0 & \cdots & \cdots & 0 & \Omega_{-1} & \Omega_0 \end{bmatrix}, \end{aligned}$$

where

$$\Omega_{00} = \int_0^1 G(s)\Sigma G(s)' ds,$$

$$\Omega_1 = \Lambda G' - \Omega_{00}\Phi',$$

$$\Omega_0 = \Omega_{00} + G\Psi G' - G\Lambda'\Phi' - \Phi\Lambda G' + \Phi\Omega_{00}\Phi',$$

$$\Omega_{-1} = \Omega_1',$$

$$\Psi = \int_0^1 e^{As}\Sigma e^{A's} ds,$$

$$\Lambda = \int_0^1 \int_0^s e^{Ar}\Sigma e^{A's} drds.$$

The maximum likelihood estimates of parameters θ are obtained by maximising the log-likelihood function

$$L(\theta) = -\frac{T}{2} - \frac{1}{2}ln|\Omega| - \frac{1}{2}v'\Omega v, \quad v = [v_1', \ldots, v']'. \tag{4.18}$$

The estimation process is the same as in Model I. Section 4.3.2 presents estimation results from both models.

4.3.2 Estimation Results

Table 4.3 presents parameter estimates and standard errors (in parenthesis) from both models. All estimates are statistically significant. Observation intervals of flow variables are at the same frequency as the stock variable, namely the intervals vary from 0.933 to 1.167, while the observation in August 2010 has an interval of 2.1, which is the maximum. Both inflows and outflows contain 96 observations, $n_I = n_O = 96$.

The results show some differences between the two approaches, particularly in estimates of σ. The estimated values of parameters are slightly larger in Model II, while the standard errors in Model II are slightly smaller except for $\hat{\sigma_I}^2$ and $\hat{\sigma_{IO}}$. In particular, the estimate of $\hat{\sigma_I}^2$ in Model I is 1.2115 with standard deviation of 0.1679, while the The estimate in Model II is 1.8724 with standard deviation of 0.2265, suggesting that the volatility in vacancy inflows is much smaller than estimated without accounting for the unequal sampling interval. The coefficient between inflows and outflows $\hat{\sigma_{IO}}$ is positive in both models, 0.3623 and 0.5520 respectively, indicating that inflows and outflows are positively correlated. The differences in estimation results of the two approaches are relatively small, which is possibly due to the relatively small variation in observation intervals and small sample size. But still, these results suggest potential gains in the estimation procedure when the unequal sampling intervals are correctly accounted for.

Table 4.3: Estimation Results - Bivariate Model

$\delta_{min} = 0.933$,	$\delta_{max} = 2.1$, $n_I = n_O = 96$						
	$\hat{a_{II}}$	$\hat{a_{IO}}$	$\hat{a_{OI}}$	$\hat{a_{OO}}$	$\hat{\sigma_I}^2$	$\hat{\sigma_O}^2$	$\hat{\sigma_{IO}}$
Unequal (Model I)	-0.8728	0.8814	0.8647	-0.9376	1.2115	1.0025	0.3623
	(0.1711)	(0.2088)	(0.1667)	(0.1905)	(0.1679)	(0.1366)	(0.1364)
Equal (Model II)	-0.8791	0.9253	0.8075	-0.9370	1.8724	1.0246	0.5520
	(0.1561)	(0.1969)	(0.1680)	(0.1829)	(0.2265)	(0.1295)	(0.1433)

4.3.3 Monte Carlo Simulation

Given the estimation results, it is interesting to examine improvements in the performance of estimation when the unequally spaced intervals are correctly measured. In the following Monte Carlo simulation, the two models are (re-)estimated using simulated data based on the estimated parameters. The main purpose of the simulation is to test if the estimation bias is reduced under the appropriate approach (Model I), and possibly at what costs (eg. larger standard errors).

The data generated are two flow variables which consist of three sets of data with different size - 96 (same as in the real data), 120 and 240. The lengths of intervals are also the same as in the real data, which vary from 0.933 to 1.167. The observations are generated by (11)

$$\begin{bmatrix} dx^I(t) \\ dx^O(t) \end{bmatrix} = \begin{bmatrix} a_{II} & a_{IO} \\ a_{OI} & a_{OO} \end{bmatrix} \begin{bmatrix} x^I(t) \\ x^O(t) \end{bmatrix} dt + \begin{bmatrix} \zeta^I(t) \\ \zeta^O(t) \end{bmatrix},$$

with $a_{II} = -0.87$, $a_{IO} = 0.88$, $a_{OI} = 0.86$, $a_{OO} = -0.94$, ${\sigma_I}^2 = 1.21$, ${\sigma_O}^2 = 1$ and $\sigma_{IO} = 0.36$. Subjected to the Boundary Condition on initial values, Estimates of these parameters are obtained when the Gaussian likelihood function

$$L(A, \Sigma) = -\frac{T}{2} - \frac{1}{2} ln|\Omega_i| - \frac{1}{2}\xi'\Omega_i\xi$$

is maximised.

The estimation procedure is replicated 10,000 times for each sample size, T. Simulation results are presented in Table 4.4, which contains the estimated values, simulated bias and standard errors. Results from Model I are presented as "Unequally Spaced" while results from Model II are presented as "Equally Spaced". In all cases, the estimation bias in Model I is smaller than that in Model II, which is consistent with histograms and kernel graphs (see Apendix C); while the standard errors of coefficient estimates are very similar in both models. In particular, the biases and standard errors for the estimates of variance, σ_I^2, σ_O^2 and σ_{IO}, are significantly larger in Model II. The estimates of all σs are approximately twice as much as the true values in Model II. The estimation biases for a_{II} and a_{OO} are negative in both models; while the bias for σ_{IO} is negative in Model I but positive in Model II. It seems both approaches turn to overestimate parameters with positive estimate and underestimate those with negative estimate. As the sample size decreases, estimation biases and standard errors increase in both models, except that the bias in σ's decreases in Model II. Interestingly, with smaller sample size, estimates of σ turn to get closer to the true value. However, the advantage of using Model I is still dominating.

Since there are relatively large differences in estimates of variances, it might be interesting to check how close is the estimate to the true value. Figures 1 to 6 (in Appendix C) present comparisons of histograms (Figures 1 to 3) and kernel graphs (Figures 4 to 6) of estimates in both models to the true values when the sample size equals to 240, 120 and 96, respectively. In all cases, Model I provides better fit than Model II, particularly for σ, where Model II seems to perform obviously worse than Model I. Overall, these results are in favour of Model I, suggesting that the unequal spacing should be taken into account in the estimation procedure.

Table 4.4: Simulation Results - Bivariate Model

$\delta_{min} = 0.933,$	$\delta_{max} = 1.167,$ $T = 240$						
	a_{II}	a_{IO}	a_{OI}	a_{OO}	${\sigma_I}^2$	${\sigma_O}^2$	σ_{IO}
True Value	-0.87	0.88	0.86	-0.94	1.21	1	0.36
Unequally Spaced (Model I)	-0.8962	0.9008	0.8810	-0.9677	1.2252	1.0107	0.3471
Bias	-0.0262	0.0208	0.0210	-0.0277	0.0152	0.0107	-0.0129
	(0.1668)	(0.1880)	(0.1854)	(0.1830)	(0.1552)	(0.1293)	(0.1201)
Equally Spaced (Model II)	-0.9386	0.9053	0.8873	-1.0114	2.0590	1.7926	1.1083
Bias	-0.0686	0.0253	0.0273	-0.0714	0.8492	0.7926	0.7483
	(0.1709)	(0.1864)	(0.1828)	(0.1873)	(0.4188)	(0.3902)	(0.3868)
$\delta_{min} = 0.933$,	$\delta_{max} = 1.167,$ $T = 120$						
	a_{II}	a_{IO}	a_{OI}	a_{OO}	${\sigma_I}^2$	${\sigma_O}^2$	σ_{IO}
True Value	-0.87	0.88	0.86	-0.94	1.21	1	0.36
Unequally Spaced (Model I)	-0.9312	0.9297	0.9120	-1.0045	1.2503	1.0336	0.3269
Bias	-0.0612	0.0497	0.0520	-0.0645	0.0403	0.0336	-0.0331
	(0.2882)	(0.3195)	(0.3132)	(0.3149)	(0.2719)	(0.2419)	(0.2312)
Equally Spaced (Model II)	-0.9733	0.9332	0.9152	-1.0478	2.0280	1.7646	1.0356
Bias	-0.1033	0.0532	0.0552	-0.1078	0.8180	0.7646	0.6756
	(0.2878)	(0.3113)	(0.3060)	(0.3150)	(0.5667)	(0.5293)	(0.5246)
$\delta_{min} = 0.933$,	$\delta_{max} = 1.167,$ $T = 96$						
	a_{II}	a_{IO}	a_{OI}	a_{OO}	${\sigma_I}^2$	${\sigma_O}^2$	σ_{IO}
True Value	-0.87	0.88	0.86	-0.94	1.21	1	0.36
Unequally Spaced (Model I)	-0.9492	0.9470	0.9272	-1.0245	1.2651	1.0466	0.3147
Bias	-0.0792	0.0670	0.0672	-0.0845	0.0551	0.0466	-0.0453
	(0.3407)	(0.3811)	(0.3686)	(0.3762)	(0.3340)	(0.2965)	(0.2881)
Equally Spaced (Model II)	-0.9902	0.9470	0.9286	-1.0657	2.0128	1.7516	0.9983
Bias	-0.1202	0.0670	0.0686	-0.1257	0.8028	0.7516	0.6383

4.4 Conclusion

This chapter has presented two applications on estimating continuous time models with unequally spaced data based on the exact discrete time representation method. The first case considers the estimation of a univariate continuous time model with a labour market vacancy stock variable; while the second case presents estimation of a bivariate continuous time model with vacancy inflows and outflows. Both the stock variable and flow variables are observed at irregular intervals at the same frequency.

When taking the unequally spaced intervals into account, the exact discrete time representation exhibit different characteristics such as time-varying parameters and heteroskedasticity compared to the approach which assumes data are equally spaced, as shown in the theoretical model. By applying real-life data that are observed at irregular intervals, estimation from the two different approaches (one accounts for the unequally spaced intervals and the other treats them as equally spaced) present different results in both cases. Even with relatively small variation in sampling intervals and a small sample size (of 96 observations), there are differences in parameter estimates, which indicate possible potential gains in estimation when the unequal sampling intervals are correctly accounted for.

In order to examine the impact (possibly improved estimation results) of correctly measuring unequal sampling intervals on the estimated parameters of continuous time models, the estimation procedures were replicated using simulated data (based on estimated parameters). Simulation evidences show that the estimation bias is reduced when the unequal sampling intervals are accounted for in both cases. With flow variables, estimation biases are significantly larger (especially for variances and covariance) when the unequal sampling intervals are treated as equally spaced, which is consistent with the histograms and kernel graphs. With larger sized samples, the advantage of using the correct approach (Model I) for estimating the continuous time models become more obvious. In conclusion, simulation evidences support the argument that the unequal spacing intervals should be taken into account in estimating continuous time models, even with relatively small variation in the intervals.

This chapter provides some simple examples of empirical applications of estimating continuous time models with unequally spaced data based on the exact discrete time representation method. The work can be extended in a number of directions. For example, the multivariate continuous time model adopted in 4.3 is a bivariate model with two flow variables, which would face the identification (aliasing) problem, therefore one obvious extension would be providing an approach that solves (or at least limits) the aliasing problem. The methods presented in McCrorie's (2009) Appendix for illustrating the identification problem can be used to address this issue, which would potentially increase the applicability of the results of this thesis. Another extension can be made on exploring the comparisons between the estimation bias and discretization bias in estimation in multivariate continuous time models with discrete data. For instance, Wang, Phillips and Yu (2011) compared the performance of Euler and trapezoidal approximations relative to the exact maximum likelihood in terms of estimation bias and discretization bias. Their results suggest that the discretization bias and estimation bias have the opposite signs, and the bias in the two approximation method is smaller than the exact maximum likelihood estimator, suggesting the possibility that the estimator based on approximation may outperform the exact maximum likelihood estimator. Wang, Phillips and Yu's (2011) study assumes equally spaced sampling intervals, it might be interesting to exploring this problem by allowing for unequally spaced sampling intervals in future research.

4.5 Appendix C - Histograms and Kernel Graphs

Figure 1. Histograms of Estimates T=240

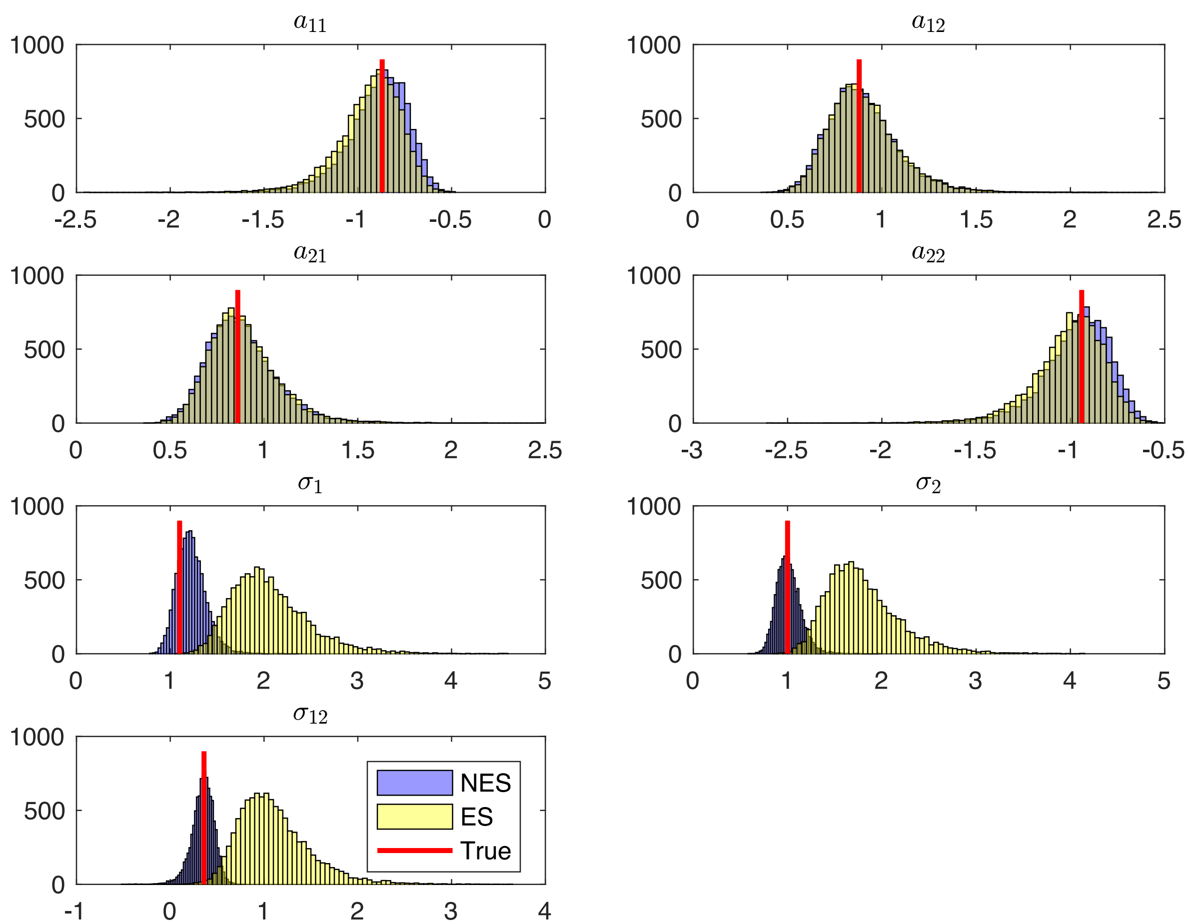


Figure 2. Histograms of Estimates T=120

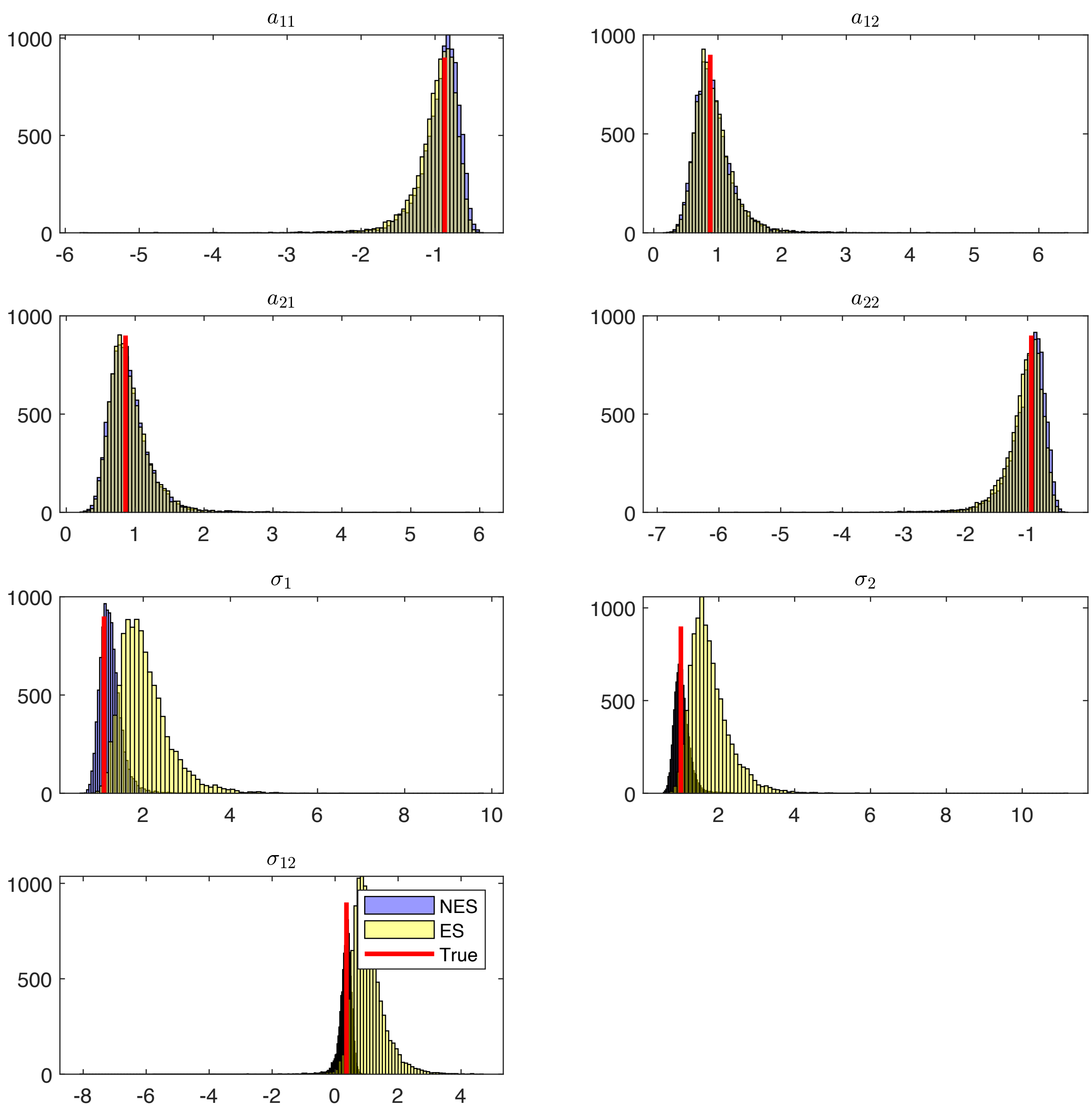


Figure 3. Histograms of Estimates T=96

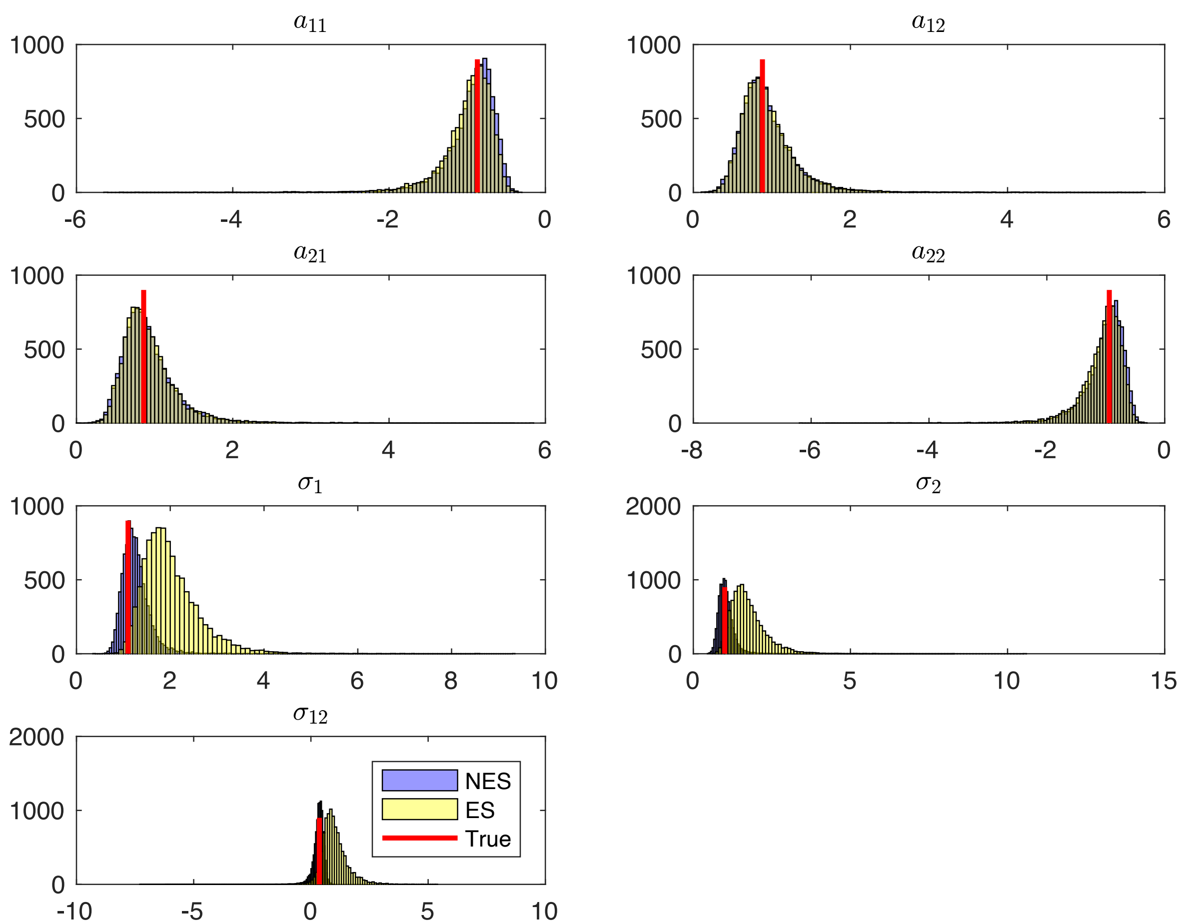


Figure 4. Kernel Graphs of Estimates T=240

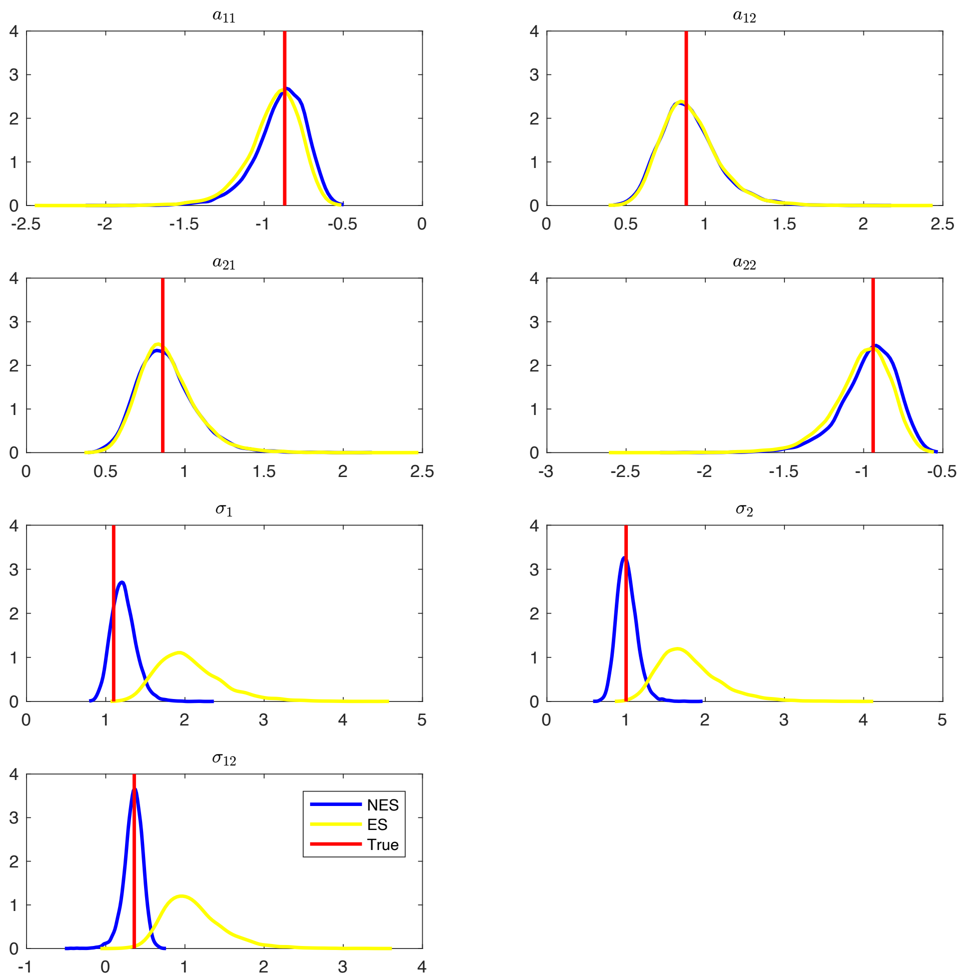


Figure 5. Kernel Graphs of Estimates T=120

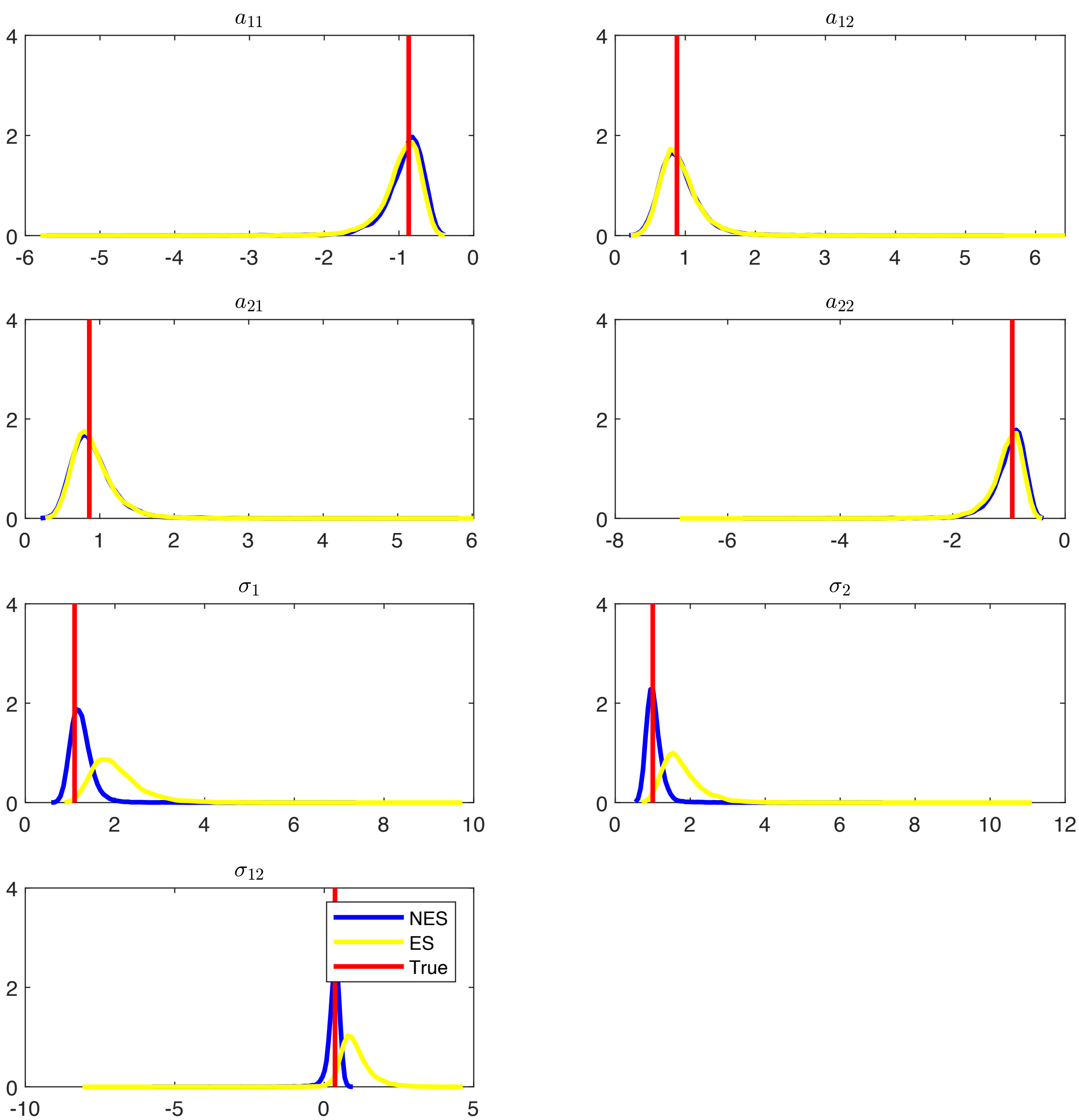
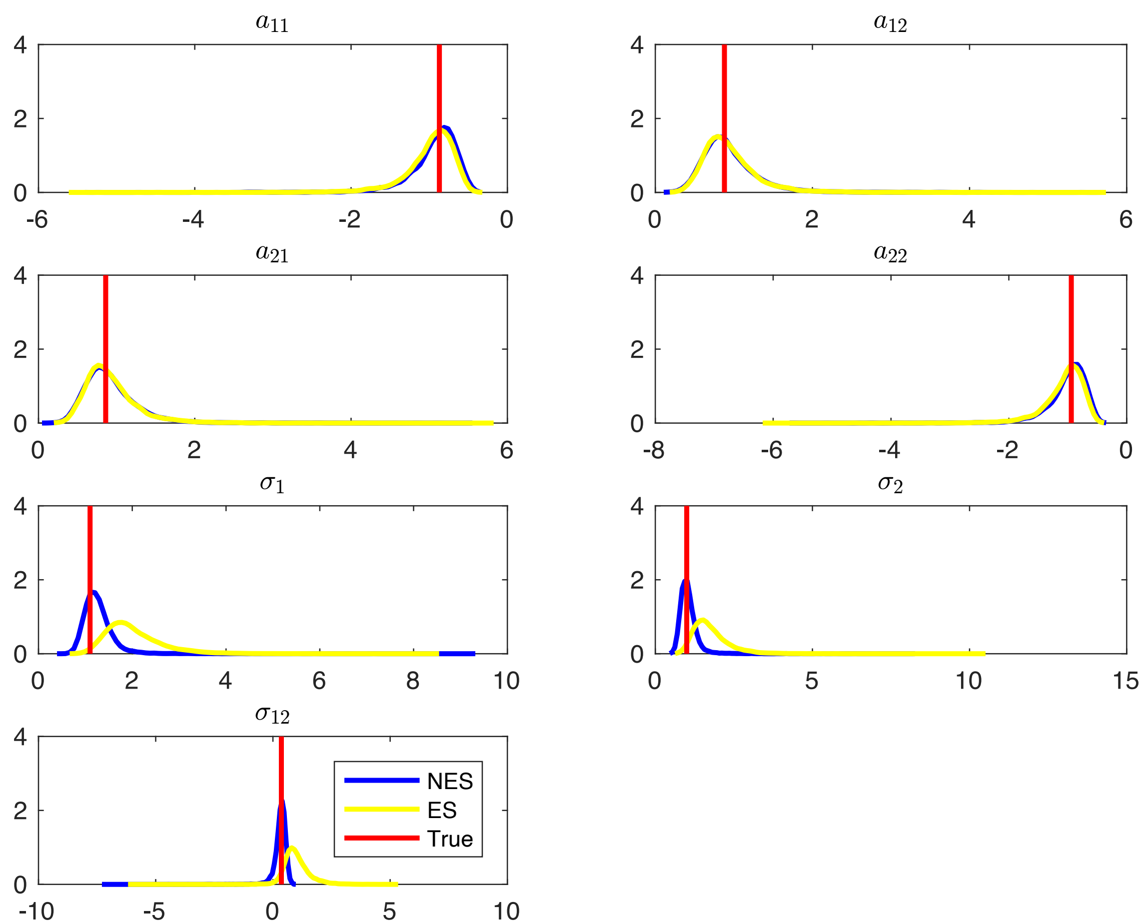


Figure 6. Kernel Graphs of Estimates T=96



5 Concluding Comments

This thesis presents a theoretical exact discrete approach for estimating parameters of continuous time models based on unequally spaced discrete data. Exact discrete time representations are provided for the three main cases of interest - where the variables are purely stocks (observed at discrete points), purely flows (observed over discrete intervals), or a mixture of stocks and flows. In all cases the exact discrete time models exhibit time-varying parameters and heteroskedasticity, while the underlying continuous time model is time invariant. The time variation in the discrete time parameters and variances arise systematically, which are entirely due to the unequal sampling intervals, indicating that such time variation may merely be a manifestation of the unequally spaced data rather than any inherent time variation in the discrete time model itself. The results of some Monte Carlo simulation studies are reported for assessing the extent that correctly accounting for the irregular intervals can have on the parameter estimation outcome (measured by estimation bias). Overall, the results are broadly favouring the approach that correctly accounts for the unequal sampling intervals, suggesting that there are improvements (such as smaller estimation bias) to be made in the estimation procedures when the sampling intervals are properly measured.

Chapter 2 provides an approach to obtain the exact discretization subject to the assumption that the original coefficient matrix to be nonsingular, which is unattractive as it rules out many important cases involving unit roots and cointegration. Chapter 3 extends the previous work to the non-stationary case by providing an alternative approach that does not depend on the original coefficient matrix being non-singular. With mixed data, the result imposes a weak assumption that a sub-matrix to be non-singular, which rules out conitegration in stock variables, but still represents a significant advance on the work. Following the theoretical work, Chapter 4 provides two applications of the methods in the previous chapters, where a univariate case with labour market job vacancy stock variable and a bivariate model with vacancy inflow and outflow variables.

This thesis provides an important alternative to the Kalman filter method for estimation based on unequally spaced data. In the continuous time econometrics literature, the Kalman filter is usually used to deal with unequally spaced data in the context of continuous time modelling, which has been regarded as an advantage over the exact discrete approach. However, in this thesis, we show that the exact discrete approach can be used to deal with unequally spaced data as well. This could be particularly attractive when the exact discrete time model is required for other reasons. In addition to this, this thesis provides a theoretical frame work of discretization of first-order multivariate continuous time systems with unequally spaced data based on the exact discrete method. The results offers the possibility of wide applications in both finance and economics.

The approach (based on the exact discretization) provided in this thesis imposes some limitations. First and foremost, the aliasing problem could arise when the original coefficient matrix is not well identified and many different matrices could share the same exponential of the original matrix, which is fundamental to (multivariate) continuous time systems. Another problem for the exact discrete approach is the controllability problem, where the discrete time covariance matrix may not be guaranteed to be positive definite. Moreover, it is technically more difficult to derive the asymptotic sampling properties of estimators from continuous time models than from discrete time models (see McCrorie, 2009 for discussions on probelms with the exact discrete approach). As discussed in 4.4, there is possibly a trade-off between the estimation bias and discretization bias in the estimation of continuous time models, which indicate that the estimator based on approximation may outperform the exact maximum likelihood estimator. Furthermore, in the mixed sample case, the approach can be applied to non-stationary processes, but still imposes a weak assumption, which rules out some interesting applications such as cointegration in stock variables.

Results of this thesis can be extended in a number of directions in future work. For example, the model can be extended to deal with high-frequency data, which has wide applications in financial econometric analysis. It is also possible to extend the model to estimate mixed frequency data, which would be complex but interesting. The model can be also applied to estimate multivariate cointegrated systems as discussed in Chapter 3. Another possible extension would be providing an approach that addresses the aliasing problem (which is a main limitation of the thesis) in estimation based on unequally spaced discrete data. Considering another limitation of this thesis, it might be interesting to explore the comparisons between the estimation bias and discretization bias in future work. In addition, it would be interesting to explore any alternative method that does not rely on assumption 3.2 in the future research. Last but not the least, the approach can be applied in macroeconomic or labour economic analysis. One example would be estimating a (VAR) model of joint dynamics of vacancies and unemployment stocks; and another example would be estimating a model of a mixture of (job) vacancy inflow, outflow and stock variables.

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