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Complete List of Authors:	Duan, Huoyuan; Wuhan University, School of Mathematics and Statistics Cao, Jiwei; Henan University of Economics and Law, College of Mathematics and Information Science Lin, Ping; University of Dundee, Division of Mathematics Tan, Roger; National University of Singapore, Department of Mathematics
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Approximation of singular solutions and singular data for Maxwell's equations by Lagrange elements

Huoyuan Duan*, Jiwei Cao†, Ping Lin‡ and Roger C. E. Tan§

Abstract

A Lagrange finite element method is proposed for Maxwell's equations in Lipschitz domains. The method is suitable for the approximation of singular solutions lying outside $(H^1(\Omega))^3$ and nonhomogeneous singular boundary data in the tangential trace space of $H(\mathbf{curl}\,;\Omega)$ and a singular right-hand side source term in $(H_0(\mathbf{curl}\,;\Omega))'$ (the dual space of $H_0(\mathbf{curl}\,;\Omega)$). Numerical results are presented to illustrate the performance and the theoretical results.

Keywords Maxwell equations, Lagrange element, finite element method.

Mathematics Subject Classification(2000) 65N30.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a simply-connected bounded domain, with a connected Lipschitz-continuous boundary Γ. Let $\mathbf{n}$ denote the outward unit normal to Γ. Introduce the curl operator $\mathbf{curl}\,\mathbf{v} = \nabla \times \mathbf{v}$ and the div operator $\operatorname{div}\mathbf{v} = \nabla \cdot \mathbf{v}$, with ∇ denoting the gradient operator. Let λ be a given real number, $\mathbf{f}$ the source term and $\boldsymbol{\chi}$ the boundary data. We consider the three-dimensional Maxwell equations as follows:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \lambda\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

along with the Dirichlet boundary condition:

$$\mathbf{n} \times \mathbf{u} = \boldsymbol{\chi} \quad \text{on } \Gamma. \tag{1.2}$$

There are many difficulties in the finite element discretizations of the Maxwell equations (1.1)-(1.2). In this paper, we are concerned with singular solution and singular data. By singular solution we mean that

$$\mathbf{u} \notin (H^1(\Omega))^3.$$

In fact, in the variational setting, the Sobolev space for the solution $\mathbf{u}$ is commonly chosen as

$$H(\mathbf{curl}\,;\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \mathbf{curl}\,\mathbf{v} \in (L^2(\Omega))^3\},$$

as opposed to the usual $(H^1(\Omega))^3$ for Poisson equations. Corresponding to the variational space $H(\mathbf{curl}\,;\Omega)$ of the solution, we assume that

$$\mathbf{f} \in (H_0(\mathbf{curl}\,;\Omega))', \tag{1.3}$$

*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China. E-mail: hyduan.math@whu.edu.cn.
†College of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450002, China. E-mail: caojiwei@huel.edu.cn.
‡Division of Mathematics, University of Dundee, Dundee, DD1 4HN, Scotland, UK. E-mail: plin@maths.dundee.ac.uk.
§Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543. E-mail: uprtance@nus.edu.sg.

where $(H_0(\mathbf{curl}\,;\Omega))'$ denotes the dual of $H_0(\mathbf{curl}\,;\Omega) = \{\mathbf{v} \in H(\mathbf{curl}\,;\Omega) : \mathbf{n} \times \mathbf{v}|_\Gamma = \mathbf{0}\}$, and that the boundary data $\boldsymbol{\chi}$ lies in the tangential trace space of $H(\mathbf{curl}\,;\Omega)$, denoted as $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$, i.e.,

$$\boldsymbol{\chi} \in H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma). \tag{1.4}$$

These data $\mathbf{f}$ and $\boldsymbol{\chi}$ which may belong to negative-order Sobolev spaces are also referred to as singular.

Even if the domain Ω is of $C^{1,1}$ class (cf., [2]), these singular data $\mathbf{f}$ and $\boldsymbol{\chi}$ would still bring about a singular solution. On the other hand, even if $\mathbf{f}$ and $\boldsymbol{\chi}$ are smooth enough, say $\boldsymbol{\chi} = \mathbf{0}$ and $\mathbf{f} \in (L^2(\Omega))^3$ with $\mathrm{div}\,\mathbf{f} = 0$, the solution $\mathbf{u}$ may still be singular whenever the domain Ω is nonsmooth with reentrant corners and edges (e.g., see [3], [2], [27]). So, it is commonplace that the solution of (1.1)-(1.2) is singular, namely, not belonging to $(H^1(\Omega))^3$.

In this paper, we consider a least-squares approach, with the use of the Lagrange elements which are nodal-continuous and $(H^1(\Omega))^3$-conforming, dealing with the singular data and the singular solution which does not belong to $(H^1(\Omega))^3$. For this purpose, we first note that from (1.3) and (1.4), (1.1) holds in the dual space $(H_0(\mathbf{curl}\,;\Omega))'$ and (1.2) in the trace space $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$, and the appropriate space for the solution is $H(\mathbf{curl}\,;\Omega)$. Keeping this fact in mind, we next identify the dual space $(H_0(\mathbf{curl}\,;\Omega))'$ with a subspace of the product of $(H^{-1}(\Omega))^3 \times H^{-1}(\Omega)$ in the sense that $(H_0(\mathbf{curl}\,;\Omega))' = \{\mathbf{f} \in (H^{-1}(\Omega))^3 : \mathrm{div}\,\mathbf{f} \in H^{-1}(\Omega)\}$, see Lemma 3.1. We then find that (1.1) and (1.3) can be stated as follows: $g := -\lambda^{-1}\mathrm{div}\,\mathbf{f}$(if $\lambda = 0$, then $g := 0$),

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \lambda\mathbf{u} = \mathbf{f} \quad \text{in } (H^{-1}(\Omega))^3, \tag{1.5}$$

$$\mathrm{div}\,\mathbf{u} = g \quad \text{in } H^{-1}(\Omega). \tag{1.6}$$

or in variational setting, for all $\mathbf{z} \in (H_0^1(\Omega))^3$ and $q \in H_0^1(\Omega)$,

$$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{z}) - \lambda(\mathbf{u}, \mathbf{z}) = \langle \mathbf{f}, \mathbf{z} \rangle, \tag{1.5}$$

$$-(\mathbf{u}, \nabla q) = \langle g, q \rangle, \tag{1.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ or between $(H^{-1}(\Omega))^3$ and $(H_0^1(\Omega))^3$. Further, the trace space $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ is identified as a subspace of the product of $(H^{-\frac{1}{2}}(\Gamma))^3 \times H^{-\frac{1}{2}}(\Gamma)$(see Remark 3.1), in the following sense that

$$\mathbf{n} \times \mathbf{u} \in (H^{-\frac{1}{2}}(\Gamma))^3, \tag{1.7}$$

$$\mathbf{curl}\,\mathbf{u} \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\Gamma). \tag{1.8}$$

Such characterizations follow from the results in [28] which are not trivial. Thus, in variational setting, (1.2) and (1.4) can be stated as follows: for all $\psi \in H^1(\Omega)$ and $\mathbf{y} \in (H^1(\Omega))^3$,

$$(\mathbf{curl}\,\mathbf{u}, \nabla\psi) = \langle \boldsymbol{\chi}, \mathbf{n} \times (\nabla\psi \times \mathbf{n}) \rangle_{\Gamma,*}, \tag{1.7}$$

$$(\mathbf{curl}\,\mathbf{u}, \mathbf{y}) - (\mathbf{u}, \mathbf{curl}\,\mathbf{y}) = \langle \boldsymbol{\chi}, \mathbf{n} \times (\mathbf{y} \times \mathbf{n}) \rangle_\Gamma, \tag{1.8}$$

where $\langle \cdot, \cdot \rangle_{\Gamma,*}$ is the duality between the dual $(H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$and $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ and $\langle \cdot, \cdot \rangle_\Gamma$ the duality between $(H^{-\frac{1}{2}}(\Gamma))^3$ and $(H^{\frac{1}{2}}(\Gamma))^3$. The statements in (1.5)-(1.8) more explicitly reveal the regularity of the data $\mathbf{f}$ and $\boldsymbol{\chi}$ in the usual negative-order Hilbert spaces $(H^{-1}(\Omega))^3 \times H^{-1}(\Omega)$ and $(H^{-\frac{1}{2}}(\Gamma))^3 \times H^{-\frac{1}{2}}(\Gamma)$, respectively. Naturally, we propose a least-squares(LS) method by measuring the residuals in these usual negative-order Sobolev spaces, and a minimization problem for solving $\mathbf{u}$ of problem (1.1)-(1.2) can be stated as follows:

$$\mathscr{F}_1(\mathbf{u}; \mathbf{f}, g, \boldsymbol{\chi}) = \min_{\mathbf{v} \in H(\mathbf{curl}\,;\Omega)} \mathscr{F}_1(\mathbf{v}; \mathbf{f}, g, \boldsymbol{\chi}), \tag{1.9}$$

where the LS functional is $\mathscr{F}_1(\mathbf{v};\mathbf{f},g,\boldsymbol{\chi}) := ||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}-\lambda\mathbf{v}-\mathbf{f}||_{-1}^2 + ||\mathrm{div}\,\mathbf{v}-g||_{-1}^2 + ||\mathbf{n}\times\mathbf{v}-\boldsymbol{\chi}||_{-\frac{1}{2},\Gamma}^2 + ||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}-L(\boldsymbol{\chi})||_{-\frac{1}{2},\Gamma}^2$, where $L(\boldsymbol{\chi})$, which is a scalar linear functional of the trace $\boldsymbol{\chi}$ based on (1.8), will be defined later. Also, naturally, the Lagrange elements which are $(H^1(\Omega))^3$-conforming and nodal-continuous are suitable for discretizing the minimization problem (1.9).

A big advantage of this functional (1.9) is that any conforming finite element space of $H(\mathbf{curl}\,;\Omega)$ can be employed for the solutions, particularly, the classical nodal-continuous Lagrange elements which are $(H^1(\Omega))^3$-conforming can be used. Of course, the Nédélec elements([18, 20]) which are $H(\mathbf{curl}\,;\Omega)$-conforming but $(H^1(\Omega))^3$-nonconforming can be used as well. In the discretization of (1.9), by the Riesz-representation, we lift $H^{-1}(\Omega)$ and $H^{-\frac{1}{2}}(\Gamma)$ onto $H_0^1(\Omega)$ and $H^1(\Omega)$, respectively, and then we deal with it in $H_0^1(\Omega)$ and $H^1(\Omega)$ instead. The approximations of $H_0^1(\Omega)$ and $H^1(\Omega)$ are classical Lagrange elements, and we only use the linear element to compute the Sobolev negative-order norms in the functional $\mathscr{F}_1(\cdot;\mathbf{f},g,\boldsymbol{\chi})$. Of course, a drawback of the method here is that it involves higher computational cost than the existing methods by additionally numerically solving a number of Poisson equations of Dirichlet and Neumann boundary conditions while these are done only with *linear elements*(See Remark 4.1).

Featuring scalar degrees of freedom and nodal continuity, the Lagrange elements are still very useful, although they have not been used so widely in computational electromagnetism as the Nédélec elements have. As a matter of fact, there have been a number of methods available with the use of Lagrange elements, e.g., see [31], [17], [10], [6], [4], [5], [15], [19], [22], [25], [33], [34], [11], and references therein, etc. In these methods, error estimates of the finite element approximations may not be satisfactory. For example, some obtained the error estimates in weighted $H(\mathbf{curl}\,;\Omega)$ norm with a geometrical weight function(e.g., [25],[4], [5]); some obtained the error estimates in L^2 norm and no error estimates in $||\mathbf{curl}\,\cdot||_0$-semi-norm are possible for singular solutions(e.g.,[31], [10], [6], [11], [22]); some obtained error estimates of low order in $H(\mathbf{curl}\,;\Omega)$ norm(e.g., [15], [17], [19]). Some of these methods involve special Lagrange elements on composite meshes(e.g., [33], [34], [25]). All these methods do not deal with singular data.

One goal here is to develop a finite element method, with the standard Lagrange elements and with no special meshes, to obtain the best convergence rate that can be attained in $H(\mathbf{curl}\,;\Omega)$ norm. In fact, to the authors' knowledge, in these existing methods, none could reach such goal. The other goal here is to assume minimal regularity on the solution and on the data of the right-hand side and the boundary data. In realistic world, due to the complexity of the domain, the issue of the minimal regularity is important and meaningful. But, we are not aware of these methods which dealt with such issue, even in the literature of edge element methods, and the discontinuous methods, e.g., see [9], [16], [21], [38], [37], [30], and references therein. Precisely, we shall deal with only $H(\mathbf{curl}\,;\Omega)$ solution and the data (1.3) and (1.4), and prove the quasi-optimal error estimates and the convergence in (1.12) and (1.13), which are not known for other methods(Lagrange element method, edge element method, discontinuous method, etc), to the authors' knowledge, and we prove the convergence rate (1.15), which is the best that can be attained, not known for other Lagrange element methods, in general. In addition, all the techniques and arguments are new, to the authors' knowledge, only the proof of Lemma 5.1 follows the technique in [10].

The finite element discretization of (1.9) we propose, although Lagrange elements are used for the solution, is capable of approximating a singular solution, with $\mathbf{u}\in(H^r(\Omega))^3, \mathbf{curl}\,\mathbf{u}\in(H^r(\Omega))^3$, $0\le r\le 1$. Moreover, the method is suitable for singular data in $(H_0(\mathbf{curl}\,;\Omega))'\times H^{-\frac{1}{2}}(\mathrm{div}\,_\Gamma;\Gamma)$. We prove that the Lagrange finite element method is coercive, and the convergence and the error bounds are obtained in the norm of $H(\mathbf{curl}\,;\Omega)$. More precisely, we first establish the $H(\mathbf{curl}\,;\Omega)$-norm equivalence

$$c||\mathbf{v}||_{0,\mathbf{curl}}^2 \le \mathscr{F}_1(\mathbf{v};\mathbf{0},0,\mathbf{0}) \le c^{-1}||\mathbf{v}||_{0,\mathbf{curl}}^2 \quad \forall\mathbf{v}\in H(\mathbf{curl}\,;\Omega), \tag{1.10}$$

and let $\mathscr{F}_h$ be the finite element discretization of $\mathscr{F}_1$(see section 4), there also holds an $H(\mathbf{curl}\,;\Omega)$-norm equivalence on $\mathbf{U}_h$(a Lagrange finite element space $\mathbf{U}_h\subset(H^1(\Omega))^3$ of the solution), i.e.,

$$c||\mathbf{v}_h||_{0,\mathbf{curl}}^2 \le \mathscr{F}_h(\mathbf{v}_h;\mathbf{0},0,\mathbf{0}) \le c^{-1}||\mathbf{v}_h||_{0,\mathbf{curl}}^2 \quad \forall\mathbf{v}_h\in\mathbf{U}_h. \tag{1.11}$$

Then, an $H(\mathbf{curl}\,;\Omega)$-quasi-optimal error estimate is established, namely, for the exact solution $\mathbf{u}\in H(\mathbf{curl}\,;\Omega)$ and the finite element solution $\mathbf{u}_h\in\mathbf{U}_h$,

$$||\mathbf{u}-\mathbf{u}_h||_{0,\mathbf{curl}} \le c\inf_{\mathbf{v}_h\in\mathbf{U}_h}||\mathbf{u}-\mathbf{v}_h||_{0,\mathbf{curl}}\,. \tag{1.12}$$

As a result, for $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$ with $\mathbf{f} \in (H_0(\mathbf{curl}\,;\Omega))'$ and $\boldsymbol{\chi} \in H^{-\frac{1}{2}}(\text{div}_\Gamma;\Gamma)$, the convergence holds:

$$\lim_{h\to 0} ||\mathbf{u}-\mathbf{u}_h||_{0,\mathbf{curl}} = 0. \tag{1.13}$$

For a more regular solution but still possibly singular (not in $(H^1(\Omega))^3$), i.e.,

$$\mathbf{u}, \mathbf{curl}\,\mathbf{u} \in (H^r(\Omega))^3, \quad 0 \le r \le 1, \tag{1.14}$$

we obtain the convergence rate as follows:

$$||\mathbf{u}-\mathbf{u}_h||_{0,\mathbf{curl}} \le ch^{\frac{r\ell}{\ell+1}}(||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r), \quad \ell \ge 1, \tag{1.15}$$

where ℓ stands for the order of approximation of the Lagrange finite element space $\mathbf{U}_h$ (i.e., the standard Lagrange element of polynomial $\mathcal{P}_\ell$). If the solution possesses a higher-order regularity, say $\mathbf{u} \in (H^{1+\ell}(\Omega))^3$, the convergence rate in $H(\mathbf{curl}\,;\Omega)$-norm is optimal ℓ the same as the order of the approximation, i.e.,

$$||\mathbf{u}-\mathbf{u}_h||_{0,\mathbf{curl}} \le ch^{\ell}||\mathbf{u}||_{\ell+1}, \quad \ell \ge 1. \tag{1.16}$$

Although the rate of convergence in (1.15) is not the desired optimal value r, several observations are well worth being noted.

(i) The error bound in (1.15) reveals an interesting fact how the order of approximation affects the convergence rate. As ℓ becomes greater, the convergence rate will become better, asymptotically tending to r(the optimal order for singular solution (1.14)).

(ii) If the finite element space $\mathbf{U}_h$ could contain the gradient fields of a scalar $H^1(\Omega)$-conforming finite element space, the optimal r order can be recovered[1].

When the meshes are composite meshes such as Clough-Tocher/Alfeld macro meshes, then $\mathbf{U}_h$ does satisfy such property(cf. [34]), and as a consequence, (1.15) restores to the optimal r order.

Nevertheless, we note that the optimal error bound (1.16) holds with no need of the gradients of a scalar $H^1(\Omega)$-conforming finite element space.

(iii) As far as the Lagrange elements are concerned, the error bound in $H(\mathbf{curl}\,;\Omega)$-norm that can be obtained is at most (1.15) for singular solution, in general.

To the authors' knowledge, in the regime of Lagrange finite element methods, the proposed method is the only one that can achieve the error bound (1.15) in the $H(\mathbf{curl}\,;\Omega)$ norm.

Moreover, to the authors' knowledge, both the quasi-optimal error estimates (1.12) and the convergence (1.13) are not known for other methods in the literature.

The rest of this paper is outlined as follows. In section 2, we give notations and review of L^2-orthogonal and regular-singular decompositions of L^2-vector fields. In section 3, a norm equivalence is proven, and an LS method of Maxwell equations is formulated. In section 4, the finite element method (FEM) is defined. In section 5, coercivity is established. In section 6, convergence and error bound are obtained. In section 7, the implementation issue is addressed, and numerical results are given in the last section.

2 Preliminaries

For an open set $D \subseteq \mathbb{R}^3$, we shall use the Sobolev spaces $H^s(D)$, for $s \in \mathbb{R}$, equipped with norm $||\cdot||_{s,D}$. Reader may refer to [1] for details. If $D = \Omega$, the subscript $D = \Omega$ in $||\cdot||_{s,D}$ is omitted, i.e., $||\cdot||_s = ||\cdot||_{s,\Omega}$. Particularly, for $s = 1$, we use $|q|_1 = ||\nabla q||_0$ to denote the H^1-semi-norm of $H^1(\Omega)$. When $s = 0$, we use the notation $L^2(D) = H^0(D)$, with the L^2-inner product $(p,q)_{0,D} = \int_D pq$, and

[1] We note that the reason for why the Nédélec elements can have the optimal r order is because they contain the gradient fields of a scalar $H^1(\Omega)$-conforming finite element space.

in the case $D = \Omega$, both subscripts $0, D$ are omitted in the L^2-inner product, i.e., $(p,q) = (p,q)_{0,\Omega}$. The trace space of $H^1(\Omega)$ is denoted by $H^{\frac{1}{2}}(\Gamma)$, while the dual space is $H^{-\frac{1}{2}}(\Gamma)$, where the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$ is denoted by $\langle\cdot,\cdot\rangle_\Gamma$. We use $H_0^s(D)$ as the completion of $\mathcal{C}_0^\infty(D)$ with respect to the norm $||\cdot||_{s,D}$, where $\mathcal{C}_0^\infty(D)$ denotes the linear space of infinitely differentiable functions, with compact support in D. We also use the duality pairing $\langle\cdot,\cdot\rangle$ between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, where $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_\Gamma = 0\}$. If $\eta \in L^2(\Gamma)$ then the duality paring $\langle\eta,\xi\rangle_\Gamma$ is identified as L^2-inner product $(\eta,\xi)_\Gamma$ for $\xi \in H^{\frac{1}{2}}(\Gamma)$; similarly, if $f \in L^2(\Omega)$ then the duality paring $\langle f, v\rangle$ is identified with L^2-inner product (f,v) for $v \in H_0^1(\Omega)$.

In what follows, referring to [2, 27, 12, 3], we review and collect the Helmholtz-Hodge L^2-orthogonal decomposition and regular-singular decomposition of vector fields on Lipschitz domains. In addition to $H(\mathbf{curl}\,;\Omega)$ and $H_0(\mathbf{curl}\,;\Omega)$ that have been introduced earlier, we introduce $H(\mathrm{div}\,;\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}$ which is equipped with the norm $||\mathbf{v}||^2_{0,\mathrm{div}} = ||\mathbf{v}||_0^2 + ||\mathrm{div}\,\mathbf{v}||_0^2$. Let $H(\mathrm{div}\,^0;\Omega) = \{\mathbf{v} \in H(\mathrm{div}\,;\Omega) : \mathrm{div}\,\mathbf{v} = 0\}$, $H_0(\mathrm{div}\,;\Omega) = \{\mathbf{v} \in H(\mathrm{div}\,;\Omega) : \mathbf{v}\cdot\mathbf{n}|_\Gamma = 0\}$, and $H_0(\mathrm{div}\,^0;\Omega) = H_0(\mathrm{div}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$. Denote by $||\mathbf{v}||^2_{0,\mathbf{curl}\,,\mathrm{div}} = ||\mathbf{v}||_0^2 + ||\mathbf{curl}\,\mathbf{v}||_0^2 + ||\mathrm{div}\,\mathbf{v}||_0^2$ the norm of $H(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\Omega)$. The dual of $H_0(\mathbf{curl}\,;\Omega)$ is denoted by $(H_0(\mathbf{curl}\,;\Omega))'$, with the duality pairing being denoted by the notation $\langle\cdot,\cdot\rangle_*$. The tangential trace space of $H(\mathbf{curl}\,;\Omega)$ is denoted by $H^{-\frac{1}{2}}(\mathrm{div}\,_\Gamma;\Gamma)$(see [28]), and the dual of $H^{-\frac{1}{2}}(\mathrm{div}\,_\Gamma;\Gamma)$ is denoted as $(H^{-\frac{1}{2}}(\mathrm{div}\,_\Gamma;\Gamma))'$, where the notation $\langle\cdot,\cdot\rangle_{\Gamma,*}$ will be used as the duality pairing. About the definition of the trace space $H^{-\frac{1}{2}}(\mathrm{div}\,_\Gamma;\Gamma)$, since it is quite complicated involving a number of nonstandard Sobolev spaces on Γ, we would rather refer the readers to [28] for details.

For $\mathbf{v} \in H_0(\mathbf{curl}\,;\Omega)\cap H(\mathrm{div}\,;\Omega)$ or $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)\cap H_0(\mathrm{div}\,;\Omega)$, we have the following Poincaré inequality (e.g., see [2], Lemma 3.4 on page 52, Lemma 3.6 on page 53):

$$c||\mathbf{v}||_0 \le ||\mathbf{v}||_{\mathbf{curl}\,;\mathrm{div}} := \sqrt{||\mathbf{curl}\,\mathbf{v}||_0^2 + ||\mathrm{div}\,\mathbf{v}||_0^2}. \tag{2.1}$$

For a given $\mathbf{v} \in (L^2(\Omega))^3$, a Helmholtz-Hodge L^2-orthogonal decompositions of $\mathbf{v}$ are as follows (e.g., see [2], Theorem 2.7 on page 30, and [27], (3.35)-(3.37) on pages 847-848):

$$\mathbf{v} = \mathbf{v}^1 + \nabla q^1, \tag{2.2}$$

where

$$\mathbf{v}^1 \in H_0(\mathrm{div}\,^0;\Omega), \quad q^1 \in H^1(\Omega)/\mathbb{R}, \quad ||\mathbf{v}^1||_0^2 + ||\nabla q^1||_0^2 = ||\mathbf{v}||_0^2, \tag{2.3}$$

and the following vector potential exists (see [2], Theorem 3.6 on page 48, and [27], Theorem 3.17 on page 844):

$$\mathbf{v}^1 \in H_0(\mathrm{div}\,^0;\Omega), \quad \mathbf{v}^1 = \mathbf{curl}\,\mathbf{v}^{11}, \quad \mathbf{v}^{11} \in H_0(\mathbf{curl}\,;\Omega)\cap H(\mathrm{div}\,^0;\Omega), \tag{2.4}$$

$$||\mathbf{v}^{11}||_{0,\mathbf{curl}\,,\mathrm{div}} = ||\mathbf{v}^{11}||_{0,\mathbf{curl}} \le c||\mathbf{curl}\,\mathbf{v}^{11}||_0 = c||\mathbf{v}^1||_0, \tag{2.5}$$

where we have used (2.2) in deriving (2.5). For a more regular $\mathbf{v}$, there are some regular-singular decompositions in the following. Let$\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$. Since $\mathbf{curl}\,\mathbf{v} \in H(\mathrm{div}\,^0;\Omega)$, combining Theorem 3.4 on page 45 and Theorem 2.9 on page 31 in [2], we have the following regular-singular decomposition of $H(\mathbf{curl}\,;\Omega)$ functions:

$$\mathbf{v} \in H(\mathbf{curl}\,;\Omega), \quad \mathbf{v} = \mathbf{v}^* + \nabla q^*, \tag{2.6}$$

$$\mathbf{v}^* \in H(\mathrm{div}\,^0;\Omega)\cap (H^1(\Omega))^3, \quad q^* \in H^1(\Omega)/\mathbb{R},$$

$$||\mathbf{v}^*||_1 + ||q^*||_1 \le c||\mathbf{v}||_{0,\mathbf{curl}}\,. \tag{2.7}$$

It is noted that no boundary condition must be imposed on the vector potential $\mathbf{v}^*$. Otherwise, $\mathbf{v}^*$ would not belong to $(H^1(\Omega))^3$, unless the domain boundary Γ is smoother than Lipschitizian (e.g., Γ is of $C^{1,1}$ class) or Ω is a convex polyhedron, see [2], Theorem 3.5 on page 47 and Theorem 3.6 on page 48. A second regular-singular decomposition which plays a fundamental role in our argument for characterizing $(H_0(\mathbf{curl}\,;\Omega))'$ in the next section is that, see the proof in proving Proposition 5.1 on page 2034 in [12],

$$\mathbf{v} \in H_0(\mathbf{curl}\,;\Omega), \quad \mathbf{v} = \mathbf{v}^\diamondsuit + \nabla q^\diamondsuit, \tag{2.8}$$

$$\mathbf{v}^{\diamondsuit} \in (H_0^1(\Omega))^3, \quad q^{\diamondsuit} \in H_0^1(\Omega),$$
$$||\mathbf{v}^{\diamondsuit}||_1 + ||q^{\diamondsuit}||_1 \le c||\mathbf{v}||_{0,\mathbf{curl}} . \tag{2.9}$$

Associated with the curl operator, the Green formula of integration by parts is recalled as follows:

$$(\mathbf{curl}\,\mathbf{v}, \boldsymbol{\phi}) - (\mathbf{v}, \mathbf{curl}\,\boldsymbol{\phi}) = \langle \mathbf{n} \times \mathbf{v}, \boldsymbol{\phi} \rangle_\Gamma \tag{2.10}$$

for all $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$, $\boldsymbol{\phi} \in (H^1(\Omega))^3$, where if $\boldsymbol{\phi} \in H(\mathbf{curl}\,;\Omega)$ only, then the above right-hand side should be replaced by $\langle \mathbf{n} \times \mathbf{v}, \mathbf{n} \times (\boldsymbol{\phi} \times \mathbf{n}) \rangle_{\Gamma,*}$ (see Theorem 3.31 on page 59 in [23] and see also [28]). From [28] we have

$$||\mathbf{v} \times \mathbf{n}||_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)} \le c||\mathbf{v}||_{0,\mathbf{curl}} \quad \forall \mathbf{v} \in H(\mathbf{curl}\,;\Omega), \tag{2.11}$$

$$||\mathbf{n} \times (\boldsymbol{\phi} \times \mathbf{n})||_{(H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'} \le c||\boldsymbol{\phi}||_{0,\mathbf{curl}} \quad \forall \boldsymbol{\phi} \in H(\mathbf{curl}\,;\Omega). \tag{2.12}$$

Associated with the div operator and the gradient operator, the Green formula of integration by parts is also recalled: for all $\mathbf{v} \in H(\mathrm{div}\,;\Omega)$, $q \in H^1(\Omega)$,

$$(\mathbf{v}, \nabla q) + (\mathrm{div}\,\mathbf{v}, q) = \langle \mathbf{v} \cdot \mathbf{n}, q \rangle_\Gamma. \tag{2.13}$$

Throughout this paper, all the constants which are denoted by a generic notation c may depend on λ and may also be different at different occurrences. We assume that if $\lambda > 0$, it is not a Maxwell eigenvalue. All the analysis and theoretical results hold for any such λ.

3 Norm equivalence

In this section, we formulate a norm equivalence, relating the norm of $H(\mathbf{curl}\,;\Omega)$ to the norms of both the dual $(H_0(\mathbf{curl}\,;\Omega))'$ and the tangential trace space of $H(\mathbf{curl}\,;\Omega)$. The norm equivalence provides an guidance to design the FEM.

We first characterize $(H_0(\mathbf{curl}\,;\Omega))'$ (the dual of $H_0(\mathbf{curl}\,;\Omega)$).

Lemma 3.1.

$$(H_0(\mathbf{curl}\,;\Omega))' = \{\mathbf{f} \in (H^{-1}(\Omega))^3 : \mathrm{div}\,\mathbf{f} \in H^{-1}(\Omega)\},$$
$$c_1(||\mathbf{f}||_{-1} + ||\mathrm{div}\,\mathbf{f}||_{-1}) \le ||\mathbf{f}||_{(H_0(\mathbf{curl}\,;\Omega))'} \le c_2(||\mathbf{f}||_{-1} + |||\mathrm{div}\,\mathbf{f}||_{-1}),$$

where $c_i, i = 1, 2$, are two positive constants which depend on Ω but not on $\mathbf{f}$.

Proof. Let $\mathbf{f} \in (H_0(\mathbf{curl}\,;\Omega))'$, it is obvious that $\mathbf{f} \in \{\mathbf{f} \in (H^{-1}(\Omega))^3 : \mathrm{div}\,\mathbf{f} \in H^{-1}(\Omega)\}$, since $(H_0^1(\Omega))^3 \subset H_0(\mathbf{curl}\,;\Omega)$ and $\nabla H_0^1(\Omega) \subset H_0(\mathbf{curl}\,;\Omega)$. Below, we show the converse. Let $\mathbf{v} \in H_0(\mathbf{curl}\,;\Omega)$. From (2.8) and (2.9), for all $\mathbf{v} \in H_0(\mathbf{curl}\,;\Omega)$, we have

$$\mathbf{v} = \mathbf{w} + \nabla q, \quad \mathbf{w} \in (H_0^1(\Omega))^3, \quad q \in H_0^1(\Omega),$$
$$||\mathbf{w}||_1 + ||q||_1 \le c||\mathbf{v}||_{0,\mathbf{curl}} .$$

If $\mathbf{f} \in \{\mathbf{f} \in (H^{-1}(\Omega))^3 : \mathrm{div}\,\mathbf{f} \in H^{-1}(\Omega)\}$, letting

$$\langle \mathbf{f}, \mathbf{v} \rangle_* := \langle \mathbf{f}, \mathbf{w} \rangle - \langle \mathrm{div}\,\mathbf{f}, q \rangle,$$

we have

$$|\langle \mathbf{f}, \mathbf{v} \rangle_*| \le ||\mathbf{f}||_{-1}||\mathbf{w}||_1 + ||\mathrm{div}\,\mathbf{f}||_{-1}||q||_1 \le c(||\mathbf{f}||_{-1} + ||\mathrm{div}\,\mathbf{f}||_{-1})||\mathbf{v}||_{0,\mathbf{curl}} ,$$

that is to say, $\mathbf{f} \in (H_0(\mathbf{curl}\,;\Omega))'$. □

Just put

$$||\mathbf{f}||^2_{(H_0(\mathbf{curl}\,;\Omega))'} := ||\mathbf{f}||^2_{-1} + ||\mathrm{div}\,\mathbf{f}||^2_{-1},$$

which is equivalent to the *canonical* norm of $(H_0(\mathbf{curl}\,;\Omega))'$ defined by $\sup\limits_{\mathbf{0}\neq\mathbf{v}\in H_0(\mathbf{curl}\,;\Omega)} \frac{\langle \mathbf{f},\mathbf{v}\rangle_*}{||\mathbf{v}||_{0,\mathbf{curl}}}$. For a $\mathbf{v} \in H(\mathrm{div}\,;\Omega)$,

$$||\mathbf{v}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma} = \sup_{0\neq\psi\in H^{\frac{1}{2}}(\Gamma)} \frac{\langle \mathbf{v}\cdot\mathbf{n},\psi\rangle_\Gamma}{||\psi||_{\frac{1}{2},\Gamma}},$$

and, for a $\boldsymbol{\chi} \in (H^{-\frac{1}{2}}(\Gamma))^3$,

$$||\boldsymbol{\chi}||_{-\frac{1}{2},\Gamma} = \sup_{\mathbf{0}\neq\mathbf{y}\in(H^{\frac{1}{2}}(\Gamma))^3} \frac{\langle \boldsymbol{\chi},\mathbf{y}\rangle_\Gamma}{||\mathbf{y}||_{\frac{1}{2},\Gamma}}.$$

We now study the norm equivalence. This consists of the following two lemmas and one theorem.

Lemma 3.2. *Assume that λ is not a Maxwell eigenvalue. For any $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$,*

$$||\mathbf{u}||_0 \leq c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \lambda\mathbf{u}||_{-1} + ||\mathrm{div}\,\mathbf{u}||_{-1} + ||\mathbf{n}\times\mathbf{u}||_{-\frac{1}{2},\Gamma} + ||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma}),$$

where c depends on λ.

Proof. Let $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$ be given. Consider the following auxiliary problem: to find $\mathbf{v} \in H_0(\mathbf{curl}\,;\Omega)$ such that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v} = \mathbf{u} \quad \text{in } \Omega, \qquad \mathbf{v}\times\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

From [23] (Theorem 4.17, page 95) we have

$$||\mathbf{v}||_{0,\mathbf{curl}} + ||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}||_0 \leq c||\mathbf{u}||_0.$$

It is a trivial fact that

$$\mathbf{v} \in H_0(\mathbf{curl}\,;\Omega), \quad \mathbf{curl}\,\mathbf{v} \in H(\mathbf{curl}\,;\Omega) \cap H_0(\mathrm{div}\,^0;\Omega).$$

Then

$$||\mathbf{u}||_0^2 = (\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v},\mathbf{u}).$$

For $\mathbf{curl}\,\mathbf{v}$, from (2.6) and (2.7) we have a regular-singular decomposition of $\mathbf{curl}\,\mathbf{v}$ as follows:

$$\mathbf{curl}\,\mathbf{v} = \mathbf{A} + \nabla\phi, \quad \mathbf{A} \in (H^1(\Omega))^3, \quad \phi \in H^1(\Omega),$$

$$||\mathbf{A}||_1 + ||\phi||_1 \leq c(||\mathbf{curl}\,\mathbf{v}||_0 + ||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}||_0) \leq c||\mathbf{u}||_0.$$

We have

$$(\mathbf{curl}\,\mathbf{curl}\,\mathbf{v},\mathbf{u}) = (\mathbf{curl}\,\mathbf{A},\mathbf{u}) = (\mathbf{A},\mathbf{curl}\,\mathbf{u}) - \langle \mathbf{n}\times\mathbf{u},\mathbf{n}\times(\mathbf{A}\times\mathbf{n})\rangle_\Gamma.$$

From (2.2), (2.3), (2.4), (2.5), (2.8), (2.9), and (2.1), we further decompose $\mathbf{A}$ in the following:

$$\mathbf{A} = \mathbf{curl}\,\mathbf{B} + \nabla\psi, \quad \mathbf{B} \in (H_0^1(\Omega))^3, \quad \psi \in H^1(\Omega)/\mathbb{R},$$

where

$$||\mathbf{curl}\,\mathbf{B}||_0^2 + ||\nabla\psi||_0^2 = ||\mathbf{A}||_0^2,$$
$$||\mathbf{B}||_1 \leq c||\mathbf{A}||_0 \leq c||\mathbf{u}||_0.$$

Thus,

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{B}, \quad \mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,\mathbf{B} + \nabla\theta, \quad \theta := \phi + \psi \in H^1(\Omega), \quad \frac{\partial\theta}{\partial n} = 0.$$

Obviously,

$$\theta = \text{constant}, \quad \mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,\mathbf{B},$$

where, since $\mathbf{curl}\,(\mathbf{v}-\mathbf{B})=\mathbf{0}$, from [2, 27], there exists a scalar potential p such that

$$\mathbf{v}=\mathbf{B}+\nabla p, \quad p\in H_0^1(\Omega),$$

$$||p||_1\le c(||\mathbf{v}||_0+||\mathbf{B}||_0)\le c||\mathbf{u}||_0.$$

Hence,

$$(\mathbf{A},\mathbf{curl}\,\mathbf{u})=(\mathbf{curl}\,\mathbf{B}+\nabla\psi,\mathbf{curl}\,\mathbf{u})=\langle\mathbf{curl}\,\mathbf{curl}\,\mathbf{u},\mathbf{B}\rangle+\langle\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n},\psi\rangle_\Gamma,$$

$$(-\lambda\mathbf{v},\mathbf{u})=\langle-\lambda\mathbf{u},\mathbf{B}\rangle+(\nabla p,-\lambda\mathbf{u}),$$

$$(\nabla p,-\lambda\mathbf{u})=\lambda\langle\mathrm{div}\,\mathbf{u},p\rangle.$$

Combining the above, we find that

$$\begin{aligned}||\mathbf{u}||_0^2 &\le c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}-\lambda\mathbf{u}||_{-1}||\mathbf{B}||_1+||\lambda\mathrm{div}\,\mathbf{u}||_{-1}||p||_1\\ &+||\mathbf{n}\times\mathbf{u}||_{-\frac{1}{2},\Gamma}||\mathbf{A}||_1+||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma}||\psi||_1)\\ &\le c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}-\lambda\mathbf{u}||_{-1}+||\mathrm{div}\,\mathbf{u}||_{-1}+||\mathbf{u}\times\mathbf{n}||_{-\frac{1}{2},\Gamma}+||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma})||\mathbf{u}||_0.\end{aligned}$$

The proof is complete. □

Lemma 3.3. *Under the same assumption in Lemma 3.2, for any* $\mathbf{u}\in H(\mathbf{curl}\,;\Omega)$,

$$||\mathbf{curl}\,\mathbf{u}||_0\le c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}-\lambda\mathbf{u}||_{-1}+||\mathrm{div}\,\mathbf{u}||_{-1}+||\mathbf{n}\times\mathbf{u}||_{-\frac{1}{2},\Gamma}+||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma}).$$

Proof. From (2.2), (2.3), (2.4) and (2.5), we have

$$\mathbf{curl}\,\mathbf{u}=\mathbf{curl}\,\mathbf{w}+\nabla p, \quad \mathbf{w}\in H_0(\mathbf{curl}\,;\Omega)\cap H(\mathrm{div}\,^0;\Omega), \quad p\in H^1(\Omega)/\mathbb{R},$$

$$||\mathbf{curl}\,\mathbf{w}||_0+||p||_1\le c||\mathbf{curl}\,\mathbf{u}||_0.$$

From (2.8), (2.9) and (2.1), we have

$$\mathbf{curl}\,\mathbf{w}=\mathbf{curl}\,\mathbf{z}, \quad \mathbf{z}\in(H_0^1(\Omega))^3,$$

$$||\mathbf{z}||_1\le c||\mathbf{w}||_{0,\mathbf{curl}}\le c||\mathbf{curl}\,\mathbf{w}||_0\le c||\mathbf{curl}\,\mathbf{u}||_0.$$

Thus, we have

$$||\mathbf{curl}\,\mathbf{u}||_0^2=(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{u})=(\mathbf{curl}\,\mathbf{z}+\nabla p,\mathbf{curl}\,\mathbf{u}),$$

where

$$(\mathbf{curl}\,\mathbf{z},\mathbf{curl}\,\mathbf{u})=\langle\mathbf{curl}\,\mathbf{curl}\,\mathbf{u},\mathbf{z}\rangle=\langle\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}-\lambda\mathbf{u},\mathbf{z}\rangle+\lambda\langle\mathbf{u},\mathbf{z}\rangle,$$

$$\langle\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}-\lambda\mathbf{u},\mathbf{z}\rangle\le||\mathbf{z}||_1||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}-\lambda\mathbf{u}||_{-1}\le c||\mathbf{curl}\,\mathbf{u}||_0||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}-\lambda\mathbf{u}||_{-1},$$

$$\lambda\langle\mathbf{u},\mathbf{z}\rangle=\lambda(\mathbf{u},\mathbf{z})\le c||\mathbf{z}||_0||\mathbf{u}||_0\le c||\mathbf{curl}\,\mathbf{u}||_0||\mathbf{u}||_0,$$

$$\begin{aligned}(\nabla p,\mathbf{curl}\,\mathbf{u})= \quad &\langle\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n},p\rangle_\Gamma\le c||p||_{\frac{1}{2},\Gamma}||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma}\\ &\le c||p||_1||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma}\le c||\mathbf{curl}\,\mathbf{u}||_0||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma},\end{aligned}$$

and we have

$$||\mathbf{curl}\,\mathbf{u}||_0\le c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}-\lambda\mathbf{u}||_{-1}+||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma})+c||\mathbf{u}||_0,$$

Hence, combining Lemma 3.2 for $||\mathbf{u}||_0$ and the above, we obtain the desired. □

Combining Lemmas 3.2 and 3.3, we obtain the following theorem on the norm equivalence.

Theorem 3.1. *Assume that λ is not a Maxwell eigenvalue. For all* $\mathbf{u}\in H(\mathbf{curl}\,;\Omega)$, *we have the following norm equivalence: there are two positive constants* c_3,c_4 *such that*

$$c_3||\mathbf{u}||_{0,\mathbf{curl}}\le||\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}-\lambda\mathbf{u}||_{-1}+||\mathrm{div}\,\mathbf{u}||_{-1}+||\mathbf{n}\times\mathbf{u}||_{-\frac{1}{2},\Gamma}+||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma}\le c_4||\mathbf{u}||_{0,\mathbf{curl}}\,.$$

Proof. From Lemmas 3.2 and 3.3, we have the left-hand side inequality in the stated norm equivalence, while the right-hand side inequality is obvious, since

$$\langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \lambda\mathbf{u}, \mathbf{z}\rangle = (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{z}) - \lambda(\mathbf{u}, \mathbf{z}) \quad \forall \mathbf{z} \in (H_0^1(\Omega))^3,$$

$$\langle \mathrm{div}\,\mathbf{u}, q\rangle = -(\mathbf{u}, \nabla q) \quad \forall q \in H_0^1(\Omega),$$

$$\langle \mathbf{n}\times\mathbf{u}, \mathbf{y}\rangle_\Gamma = (\mathbf{curl}\,\mathbf{u}, \mathbf{y}) - (\mathbf{u}, \mathbf{curl}\,\mathbf{y}) \quad \forall \mathbf{y} \in (H^1(\Omega))^3,$$

$$\langle \mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}, \psi\rangle_\Gamma = (\mathbf{curl}\,\mathbf{u}, \nabla\psi) \quad \forall \psi \in H^1(\Omega).$$

□

Remark 3.1. *According to [28], for* $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$*, the tangential trace* $\mathbf{u}\times\mathbf{n} \in H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$*, and equivalently,* $\mathbf{u}\times\mathbf{n} \in (H^{-\frac{1}{2}}(\Gamma))^3$ *and* $\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$.

Corollary 3.1. *Assume that* λ *is not a Maxwell eigenvalue. Let* $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$ *be the solution to problem* (1.1) *and* (1.2)*, with* $\mathbf{f} \in (H_0(\mathbf{curl}\,;\Omega))'$ *and* $\boldsymbol{\chi} \in H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$*. Then, the following stability holds:*

$$c||\mathbf{u}||_{0,\mathbf{curl}} \leq ||\mathbf{f}||_{(H_0(\mathbf{curl}\,;\Omega))'} + ||\boldsymbol{\chi}||_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)}.$$

Proof. Since, for $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$ with $\mathbf{n}\times\mathbf{u} = \boldsymbol{\chi}$ on Γ, from (2.13),

$$||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma} \leq c \sup_{0\neq\psi\in H^1(\Omega)} \frac{\langle \mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}, \psi\rangle_\Gamma}{||\psi||_1} = c \sup_{0\neq\psi\in H^1(\Omega)} \frac{(\mathbf{curl}\,\mathbf{u}, \nabla\psi)}{||\psi||_1}$$

where, from (2.10)-(2.12),

$$(\mathbf{curl}\,\mathbf{u}, \nabla\psi) = \langle \mathbf{n}\times\mathbf{u}, \mathbf{n}\times(\nabla\psi\times\mathbf{n})\rangle_{\Gamma,*} = \langle \boldsymbol{\chi}, \mathbf{n}\times(\nabla\psi\times\mathbf{n})\rangle_{\Gamma,*},$$

$$\langle \boldsymbol{\chi}, \mathbf{n}\times(\nabla\psi\times\mathbf{n})\rangle_{\Gamma,*} \leq ||\boldsymbol{\chi}||_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)} ||\mathbf{n}\times(\nabla\psi\times\mathbf{n})||_{(H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'},$$

$$||\mathbf{n}\times(\nabla\psi\times\mathbf{n})||_{(H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'} \leq ||\nabla\psi||_{0,\mathbf{curl}} = ||\nabla\psi||_0 \leq ||\psi||_1,$$

we have

$$||\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma} \leq c||\boldsymbol{\chi}||_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)}.$$

Therefore, combining Theorem 3.1 and Lemma 3.1, we arrive at the conclusion. □

Recalling the minimization problem (1.9), we find the solution $\mathbf{u}$ of (1.1)-(1.2) to satisfy

$$\mathscr{F}_1(\mathbf{u};\mathbf{f},g,\boldsymbol{\chi}) = \min_{\mathbf{v}\in H(\mathbf{curl}\,;\Omega)} \mathscr{F}_1(\mathbf{v};\mathbf{f},g,\boldsymbol{\chi}),$$

where $\mathscr{F}_1(\mathbf{u};\mathbf{f},g,\boldsymbol{\chi})$ is introduced by (1.9), while $L(\boldsymbol{\chi}) \in H^{-\frac{1}{2}}(\Gamma)$ is defined by

$$\langle L(\boldsymbol{\chi}), \vartheta\rangle_\Gamma := \langle \boldsymbol{\chi}, \mathbf{n}\times(\nabla\vartheta\times\mathbf{n})\rangle_{\Gamma,*} \quad \forall\vartheta \in H^1(\Omega).$$

To formulate the variational problem which is equivalent to the LS minimization problem as stated in the above and to motivate the design of the FEM, we introduce some Riesz-representation lifting operators from the negative-order Hilbert spaces onto the usual Hilbert spaces. All these operators are associated with the $H^1(\Omega)$ space. Denote the inner product of $H^1(\Omega)$ by $(\cdot,\cdot)_{0,\nabla} = (\cdot,\cdot) + (\nabla\cdot,\nabla\cdot)$, where $(\cdot,\cdot)$ is the L^2-inner product, and the norm is $||\cdot||_1$(i.e., $||q||_1 = \sqrt{(q,q)_{0,\nabla}}$).

Let $\mathbf{f} \in (H^{-1}(\Omega))^3$. Find $\mathbf{R}(\mathbf{f}) \in (H_0^1(\Omega))^3$ such that

$$(\mathbf{R}(\mathbf{f}), \mathbf{z})_{0,\nabla} = \langle \mathbf{f}, \mathbf{z}\rangle \quad \forall \mathbf{z} \in (H_0^1(\Omega))^3. \tag{3.1}$$

Let $\boldsymbol{\chi} \in (H^{-\frac{1}{2}}(\Gamma))^3$. Find $\boldsymbol{\Upsilon}^\Gamma(\boldsymbol{\chi}) \in (H^1(\Omega))^3$ such that

$$(\boldsymbol{\Upsilon}^\Gamma(\boldsymbol{\chi}), \mathbf{y})_{0,\nabla} = \langle \boldsymbol{\chi}, \mathbf{n}\times(\mathbf{y}\times\mathbf{n})\rangle_\Gamma \quad \forall \mathbf{y} \in (H^1(\Omega))^3. \tag{3.2}$$

Let $g \in H^{-1}(\Omega)$. Find $S(g) \in H_0^1(\Omega)$ such that

$$(S(g), q)_{0,\nabla} = \langle g, q \rangle \quad \forall q \in H_0^1(\Omega). \tag{3.3}$$

Let $\kappa \in H^{-\frac{1}{2}}(\Gamma)$. Find $\Lambda^\Gamma(\kappa) \in H^1(\Omega)$ such that

$$(\Lambda^\Gamma(\kappa), \vartheta)_{0,\nabla} = \langle \kappa, \vartheta \rangle_\Gamma \quad \forall \vartheta \in H^1(\Omega). \tag{3.4}$$

With the above Riesz-representation liftings, instead of the LS functional $\mathscr{F}_1$ which is introduced earlier in (1.9), we define

$$\begin{aligned} \mathscr{F}(\mathbf{v}; \mathbf{f}, g, \boldsymbol{\chi}) := \quad & ||\mathbf{R}(\mathbf{curl\,curl\,v} - \lambda \mathbf{v} - \mathbf{f})||_1^2 + ||S(\mathrm{div}\,\mathbf{v} - g)||_1^2 \\ & + ||\boldsymbol{\Upsilon}^\Gamma(\mathbf{n} \times \mathbf{v} - \boldsymbol{\chi})||_1^2 + ||\Lambda^\Gamma(\mathbf{curl\,v} \cdot \mathbf{n} - L(\boldsymbol{\chi}))||_1^2. \end{aligned} \tag{3.5}$$

The advantage of $\mathscr{F}$ over $\mathscr{F}_1$ is that the norm $||\cdot||_1$ is easier discretizable and computable than the negative-order norms, particularly much more convenient than computing the norm $||\cdot||_{-\frac{1}{2},\Gamma}$. By the definitions (3.1)-(3.4), it is not difficult to show that both functionals $\mathscr{F}(\mathbf{v}; \mathbf{f}, g, \boldsymbol{\chi})$ and $\mathscr{F}_1(\mathbf{v}; \mathbf{f}, g, \boldsymbol{\chi})$ are equivalent, and now, minimizing the functional $\mathscr{F}$, we find the solution $\mathbf{u}$ of problem (1.1)-(1.2) to satisfy

$$\mathscr{F}(\mathbf{u}; \mathbf{f}, g, \boldsymbol{\chi}) = \min_{\mathbf{v} \in H(\mathbf{curl}\,;\Omega)} \mathscr{F}(\mathbf{v}; \mathbf{f}, g, \boldsymbol{\chi}).$$

Defining

$$\begin{aligned} \mathcal{L}(\mathbf{u}, \mathbf{v}) := \quad & (\mathbf{R}(\mathbf{curl\,curl\,u} - \lambda \mathbf{u}), \mathbf{R}(\mathbf{curl\,curl\,v} - \lambda \mathbf{v}))_{0,\nabla} + (S(\mathrm{div}\,\mathbf{u}), S(\mathrm{div}\,\mathbf{v}))_{0,\nabla} \\ & + (\boldsymbol{\Upsilon}^\Gamma(\mathbf{n} \times \mathbf{u}), \boldsymbol{\Upsilon}^\Gamma(\mathbf{n} \times \mathbf{v}))_{0,\nabla} + (\Lambda^\Gamma(\mathbf{curl\,u} \cdot \mathbf{n}), \Lambda^\Gamma(\mathbf{curl\,v} \cdot \mathbf{n}))_{0,\nabla}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \mathcal{G}(\mathbf{v}) := \quad & (\mathbf{R}(\mathbf{f}), \mathbf{R}(\mathbf{curl\,curl\,v} - \lambda \mathbf{v}))_{0,\nabla} + (S(g), S(\mathrm{div}\,\mathbf{v}))_{0,\nabla} \\ & + (\boldsymbol{\Upsilon}^\Gamma(\boldsymbol{\chi}), \boldsymbol{\Upsilon}^\Gamma(\mathbf{n} \times \mathbf{v}))_{0,\nabla} + \langle \boldsymbol{\chi}, \mathbf{n} \times (\nabla \Lambda^\Gamma(\mathbf{curl\,v} \cdot \mathbf{n}) \times \mathbf{n}) \rangle_{\Gamma,*}, \end{aligned} \tag{3.7}$$

we find that the variational problem which is equivalent to the above LS minimization problem is to find $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$ such that

$$\mathcal{L}(\mathbf{u}, \mathbf{v}) = \mathcal{G}(\mathbf{v}) \quad \forall \mathbf{v} \in H(\mathbf{curl}\,;\Omega). \tag{3.8}$$

From the definitions (3.3)-(3.4) and Theorem 3.1, we have the coercivity

$$\begin{aligned} \mathcal{L}(\mathbf{v}, \mathbf{v}) \quad & = ||\mathbf{R}(\mathbf{curl\,curl\,v} - \lambda \mathbf{v})||_1^2 + ||S(\mathrm{div}\,\mathbf{v})||_1^2 + ||\boldsymbol{\Upsilon}^\Gamma(\mathbf{n} \times \mathbf{v})||_1^2 + ||\Lambda^\Gamma(\mathbf{curl\,v} \cdot \mathbf{n})||_1^2 \\ & \geq c(||\mathbf{curl\,curl\,v} - \lambda \mathbf{v}||_{-1}^2 + ||\mathrm{div}\,\mathbf{v}||_{-1}^2 + ||\mathbf{n} \times \mathbf{v}||_{-\frac{1}{2},\Gamma}^2 + ||\mathbf{curl\,v} \cdot \mathbf{n}||_{-\frac{1}{2},\Gamma}^2) \\ & \geq c||\mathbf{v}||_{0,\mathbf{curl}}^2 \end{aligned}$$

and the boundedness

$$|\mathcal{L}(\mathbf{u}, \mathbf{v})| \leq c||\mathbf{u}||_{0,\mathbf{curl}}\, ||\mathbf{v}||_{0,\mathbf{curl}}\,,$$

and we also have, from Corollary 3.1,

$$\begin{aligned} \mathcal{G}(\mathbf{v}) \quad & = (\mathbf{R}(\mathbf{f}), \mathbf{R}(\mathbf{curl\,curl\,v} - \lambda \mathbf{v}))_{0,\nabla} + (S(g), S(\mathrm{div}\,\mathbf{v}))_{0,\nabla} \\ & + (\boldsymbol{\Upsilon}^\Gamma(\boldsymbol{\chi}), \boldsymbol{\Upsilon}^\Gamma(\mathbf{n} \times \mathbf{v}))_{0,\nabla} + \langle \boldsymbol{\chi}, \mathbf{n} \times (\nabla \Lambda^\Gamma(\mathbf{curl\,v} \cdot \mathbf{n}) \times \mathbf{n}) \rangle_{\Gamma,*} \\ & \leq c(||\mathbf{f}||_{(H_0(\mathbf{curl}\,;\Omega))'} + ||\boldsymbol{\chi}||_{H^{-\frac{1}{2}}(\mathrm{div}_{\,\Gamma};\Gamma)})||\mathbf{v}||_{0,\mathbf{curl}}\,. \end{aligned}$$

Thus, the well-posedness of the variational problem (3.8) directly follows from the classical Lax-Milgram lemma. The above coercivity and boundedness say that $\forall \mathbf{v} \in H(\mathbf{curl}\,;\Omega)$,

$$c||\mathbf{v}||_{0,\mathbf{curl}}^2 \leq \mathcal{L}(\mathbf{v}, \mathbf{v}) = \mathscr{F}_1(\mathbf{v}; \mathbf{0}, 0, \mathbf{0}) = \mathscr{F}(\mathbf{v}; \mathbf{0}, 0, \mathbf{0}) \leq c^{-1}||\mathbf{v}||_{0,\mathbf{curl}}^2\,, \tag{3.9}$$

which is (1.10) in the Introduction section 1. In the next section, we shall design the finite element method(FEM) to discretize the variational problem (3.8) to preserve the above coercivity or the norm equivalence in the finite element space.

4 FEM

In this section, we state the finite element problem. The guide for designing the finite element method is to preserve the norm equivalence in Theorem 3.1 or in (3.9). We need to discretize the Riesz-representation liftings by (3.3)-(3.4). The replacement by the discrete Riesz-representation liftings brings about the loss of the coercivity. To remedy such loss we shall use some stabilizations.

We shall assume that Ω is a polyhedral domain, with piecewise planar boundary Γ. This assumption is only for simplifying the meshes of the finite element method so that the technicalities involved in treating the curved elements are avoided. Let $\mathcal{T}_h$ denote the usual conforming and shape-regular triangulation of Ω into tetrahedra (cf. [7]) , where the mesh size $h = \max_{K\in\mathcal{T}_h} h_K$, and h_K is the diameter of the tetrahedron K. Let $\mathcal{F}_h$ denote the set of all element faces in $\mathcal{T}_h$, while $\mathcal{F}_h^\Omega \subset \mathcal{F}_h$ denotes the set of all interior element faces in Ω and $\mathcal{F}_h^\Gamma \subset \mathcal{F}_h$ the set of element faces on Γ. It is assumed that $\mathcal{F}_h^\Gamma$ which is composed of straight-sided triangles constitutes a conforming and shape-regular triangulation of Γ. Let h_F denote the diameter of $F \in \mathcal{F}_h$. Over $K \in \mathcal{T}_h$, let $\mathcal{P}_\ell(K)$ stand for the space of polynomials of total degree not greater than the integer $\ell \geq 1$.

Let $\lambda_j^K \in \mathcal{P}_1(K)$, $j = 1,2,3,4$, be the jth local basis of linear polynomial corresponding to the ith vertex of the tetrahedron element $K \in \mathcal{T}_h$. For a face $F \subset \partial K$, letting a_i, $i = 1,2,3$, be the three vertices locating on F, we define the face bubble $b_F^K \in H^1(K)$:

$$b_F^K = \lambda_1^K\lambda_2^K\lambda_3^K.$$

This face bubble satisfies

$$b_F^K|_F \in H_0^1(F), \quad b_F^K|_{F'} = 0 \quad \forall F' \neq F.$$

We also introduce the element bubble:

$$b_K = \lambda_1^K\lambda_2^K\lambda_3^K\lambda_4^K \in H_0^1(K).$$

Define Lagrange finite element spaces as follows:

$$\mathbf{U}_h = \{\mathbf{v} \in (H^1(\Omega))^3 : \mathbf{v}|_K \in (\mathcal{P}_\ell(K))^3, \forall K \in \mathcal{T}_h\}, \tag{4.1}$$

$$\mathbf{Z}_h = \{\mathbf{z} \in (H_0^1(\Omega))^3 : \mathbf{z}|_K \in (\mathcal{P}_1(K))^3, \forall K \in \mathcal{T}_h\}, \tag{4.2}$$

$$\mathbf{Y}_h = \{\mathbf{y} \in (H^1(\Omega))^3 : \mathbf{y}|_K \in (\mathcal{P}_1(K))^3, \forall K \in \mathcal{T}_h\}, \tag{4.3}$$

$$Q_h = \{q \in H_0^1(\Omega) : q|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_h\}, \tag{4.4}$$

$$\Psi_h = \{\psi \in H^1(\Omega) : \psi|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_h\}, \tag{4.5}$$

where $\mathbf{U}_h$ is used for the solution, while the rest which are always chosen as *linear* elements are auxiliary spaces. These two pairs $(\mathbf{Z}_h, Q_h)$ and $(\mathbf{Y}_h, \Psi_h)$ will be used respectively for *lifting* the right-hand side $\mathbf{f}$ and $\boldsymbol{\chi}$ to smooth finite element functions, i.e., they are used for discretizing the Riesz-representation lifting operators in (3.1)-(3.4).

Corresponding to (3.1)-(3.4), define the so-called discrete Riesz-representation liftings as follows:

Let $\mathbf{f} \in (H^{-1}(\Omega))^3$. Find $\mathbf{R}_h(\mathbf{f}) \in \mathbf{Z}_h$ such that

$$(\mathbf{R}_h(\mathbf{f}), \mathbf{z}_h)_{0,\nabla} = \langle \mathbf{f}, \mathbf{z}_h\rangle \quad \forall \mathbf{z}_h \in \mathbf{Z}_h. \tag{4.6}$$

Let $\boldsymbol{\chi} \in (H^{-\frac{1}{2}}(\Gamma))^3$. Find $\boldsymbol{\Upsilon}_h^\Gamma(\boldsymbol{\chi}) \in \mathbf{Y}_h$ such that

$$(\boldsymbol{\Upsilon}_h^\Gamma(\boldsymbol{\chi}), \mathbf{y}_h)_{0,\nabla} = \langle \boldsymbol{\chi}, \mathbf{n}\times(\mathbf{y}_h\times\mathbf{n})\rangle_\Gamma \quad \forall \mathbf{y}_h \in \mathbf{Y}_h. \tag{4.7}$$

Let $g \in H^{-1}(\Omega)$. Find $S_h(g) \in Q_h$ such that

$$(S_h(g), q_h)_{0,\nabla} = \langle g, q_h \rangle \quad \forall q_h \in Q_h. \tag{4.8}$$

Let $\kappa \in H^{-\frac{1}{2}}(\Gamma)$. Find $\Lambda_h^\Gamma(\kappa) \in \Psi_h$ such that

$$(\Lambda_h^\Gamma(\kappa), \vartheta_h)_{0,\nabla} = \langle \kappa, \vartheta_h \rangle_\Gamma \quad \forall \vartheta_h \in \Psi_h. \tag{4.9}$$

Remark 4.1. *All these discrete Riesz-representation liftings which are in fact the linear finite element solutions of the Poisson equations with different right-hand sides are also called H^1-projections. We remark that the role of these projections is essentially to lift the singular data $\mathbf{f}$ and $\boldsymbol{\chi}$ to smooth functions (finite element functions). In addition, all the computations of (4.6)-(4.9) involve the same coefficient matrix for linear finite elements, and they can be realized by the same codes up to right-hand sides.*

Alternatively, instead of exactly solving these Riesz-representation liftings, we can replace all the linear element liftings $\mathbf{R}_h, \boldsymbol{\Upsilon}_h^\Gamma, S_h, \Lambda_h^\Gamma$ by the associated spectrally equivalent linear element preconditioners. Such replacement means that we do not need to really solve (4.6)-(4.9); also such replacement will not affect all the theoretical analysis and results in the sequel. For ease of the presentation, we shall not dwell on such replacement, since it is a well-known technique in the context of LS methods, e.g., see [36] and references therein.

Since there occurs a loss of the coercivity due to the discrete Riesz-representation liftings in place of those continuous ones, below, we shall define some mesh-dependent stabilizations. All of them are evaluated locally. For that purpose, we need to define some sets of element-bubble and face-bubble polynomials. Firstly, according to the nodes which are chosen as the principal lattice of K (See the definition of the principal lattice by (A.19) on page 99 in [2]), we introduce

$$\mathcal{P}_\ell(K) = \mathrm{span}\{\varphi_i^{\ell,K}, 1 \le i \le m_\ell\}, \tag{4.10}$$

where $m_\ell = \frac{(\ell+1)(\ell+2)(\ell+3)}{6}$ and $\varphi_i^{\ell,K}$ is the 'canonical' local basis corresponding to the ith node, satisfying $\varphi_i^{\ell,K}(a_j) = \delta_{ij}$. Here a_j denotes the jth node, $1 \le j \le m_\ell$, and the notation δ_{ij} is the Kronecker delta (i.e., $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$). For example, $\mathcal{P}_1(K) = \mathrm{span}\{\varphi_i^{1,K} := \lambda_i^K, 1 \le i \le m_1 = 4\}$. In this example, the set of nodes is just the set of four vertices of the tetrahedron K. Secondly, for any given $F \in \mathcal{F}_h$, we define a so-called macro-element M_F, where M_F is the union of the two tetrahedra sharing F as the common face if $F \in \mathcal{F}_h^\Omega$, and M_F is K_F if $F \in \mathcal{F}_h^\Gamma$. Here K_F denotes the tetrahedron with its face F in Γ. For any given $F \in \mathcal{F}_h^\Omega$, corresponding to the macro-element $M_F = K_1 \cup K_2$ and $\partial K_1 \cap \partial K_2 = F$, we introduce the macro-element bubble in the following way:

$$b_F^{M_F} = \begin{cases} b_F^{K_1} & \text{in } K_1, \\ b_F^{K_2} & \text{in } K_2. \end{cases}$$

Obviously,

$$b_F^{M_F} \in H_0^1(M_F), \quad b_F^{M_F}|_F \in H_0^1(F).$$

Thirdly, we define

$$\Delta^{\ell,K} = \mathrm{span}\{\phi_i^{\ell,K} = \varphi_i^{\ell,K} b_K, 1 \le i \le m_\ell\} \subset H_0^1(K)$$

where $\varphi_i^{0,K} :\equiv 1$ and $m_0 := 1$. For macro-element $M_F = K_1 \cup K_2$ with the common face F, we define

$$\Delta^{\ell,M_F} = \mathrm{span}\{\phi_i^{\ell,M_F}, 1 \le i \le m_\ell^F, \phi_i^{\ell,M_F}|_{K_j} = \varphi_i^{\ell,K_j} b_F^{M_F}, j = 1, 2, \} \subset H_0^1(M_F),$$

where φ_i^{ℓ,K_j} are basis corresponding to the nodes on F and $m_\ell^F = \frac{(\ell+1)(\ell+2)}{2}$ and $m_0^F := 1$. Put $\boldsymbol{\varphi}_i^{\ell,K} := (\varphi_i^{\ell,K}, \varphi_i^{\ell,K}, \varphi_i^{\ell,K})$ and $\boldsymbol{\phi}_i^{\ell-1,M_F} := (\phi_i^{\ell-1,M_F}, \phi_i^{\ell-1,M_F}, \phi_i^{\ell-1,M_F})$, then define

$$Q^{\ell-1,K} = \Delta^{\ell-1,K} = \mathrm{span}\{\phi_i^{\ell-1,K} = \varphi_i^{\ell-1,K} b_K, 1 \le i \le m_{\ell-1}\} \subset H_0^1(K), \tag{4.11}$$

$$\mathbf{Z}^{\ell,K} = (\Delta^{\ell,K})^3 = \mathrm{span}\{\boldsymbol{\phi}_i^{\ell,K} = \boldsymbol{\varphi}_i^{\ell,K} b_K, 1 \le i \le m_\ell\} \subset (H_0^1(K))^3, \tag{4.12}$$

$$\mathbf{Z}^{\ell-1,M_F} = (\Delta^{\ell-1,M_F})^3 = \text{span}\{\boldsymbol{\phi}_i^{\ell-1,M_F}, 1 \le i \le m_{\ell-1}^F\} \subset (H_0^1(M_F))^3, \tag{4.13}$$

$$\mathbf{Y}^{\ell,K_F} = \text{span}\{\boldsymbol{\rho}_i^{\ell,K_F} = \boldsymbol{\varphi}_i^{\ell,K_F} b_F^{K_F}, 1 \le i \le m_\ell^F\} \subset (H^1(K_F))^3, \tag{4.14}$$

$$\Psi^{\ell-1,K_F} = \text{span}\{o_i^{\ell-1,K_F} = \varphi_i^{\ell-1,K_F} b_F^{K_F}, 1 \le i \le m_{\ell-1}^F\} \subset H^1(K_F), \tag{4.15}$$

where K_F is any of the two elements with the given face $F \subset \partial K_F$, and we have

$$\boldsymbol{\rho}_i^{\ell,K_F}|_F \in (H_0^1(F))^3, \quad \boldsymbol{\rho}_i^{\ell,K_F}|_{F'} = \mathbf{0} \quad \forall F' \ne F, \quad F' \subset \partial K_F,$$

$$o_i^{\ell-1,K_F}|_F \in H_0^1(F), \quad o_i^{\ell-1,K_F}|_{F'} = 0 \quad \forall F' \ne F, \quad F' \subset \partial K_F.$$

With the above element-bubbles and face-bubbles, we define the following mesh-dependent stabilizations:

$$\mathscr{S}_{\mathbf{curl\ curl}}(\mathbf{u},\mathbf{v}) = \sum_{K\in\mathcal{T}_h} \frac{\sum_{i=1}^{m_\ell}((\mathbf{curl\ u},\mathbf{curl}\ \boldsymbol{\phi}_i^{\ell,K})_{0,K} - \lambda(\mathbf{u},\boldsymbol{\phi}_i^{\ell,K})_{0,K})((\mathbf{curl\ v},\mathbf{curl}\ \boldsymbol{\phi}_i^{\ell,K})_{0,K} - \lambda(\mathbf{v},\boldsymbol{\phi}_i^{\ell,K})_{0,K})}{\sum_{i=1}^{m_\ell}||\boldsymbol{\phi}_i^{\ell,K}||_{0,\nabla,K}^2}, \tag{4.16}$$

where $||\mathbf{v}||_{0,\nabla,K}^2 = ||\mathbf{v}||_{0,K}^2 + ||\nabla\mathbf{v}||_{0,K}^2$,

$$\mathscr{R}_{\mathbf{curl\ curl}}(\mathbf{f},\mathbf{v}) = \sum_{K\in\mathcal{T}_h} \frac{\sum_{i=1}^{m_\ell}\langle \mathbf{f},\boldsymbol{\phi}_i^{\ell,K}\rangle_K((\mathbf{curl\ v},\mathbf{curl}\ \boldsymbol{\phi}_i^{\ell,K})_{0,K} - \lambda(\mathbf{v},\boldsymbol{\phi}_i^{\ell,K})_{0,K})}{\sum_{i=1}^{m_\ell}||\boldsymbol{\phi}_i^{\ell,K}||_{0,\nabla,K}^2}, \tag{4.17}$$

where $\langle \mathbf{f},\boldsymbol{\phi}_i^{\ell,K}\rangle_K$ is understood as $\langle \mathbf{f},\boldsymbol{\phi}_i^{\ell,K}\rangle$ after the zero extension of $\boldsymbol{\phi}_i^{\ell,K} \in (H_0^1(K))^3$ outside K to the whole domain Ω,

$$\mathscr{S}_{\mathbf{curl\ curl},\times\mathbf{n}}(\mathbf{u},\mathbf{v}) = \sum_{F\in\mathcal{F}_h^\Omega} \frac{\sum_{i=1}^{m_{\ell-1}^F}((\mathbf{curl\ u},\mathbf{curl}\ \boldsymbol{\phi}_i^{\ell-1,M_F})_{0,M_F} - \lambda(\mathbf{u},\boldsymbol{\phi}_i^{\ell-1,M_F})_{0,M_F})((\mathbf{curl\ v},\mathbf{curl}\ \boldsymbol{\phi}_i^{\ell-1,M_F})_{0,M_F} - \lambda(\mathbf{v},\boldsymbol{\phi}_i^{\ell-1,M_F})_{0,M_F})}{\sum_{i=1}^{m_{\ell-1}^F}||\boldsymbol{\phi}_i^{\ell-1,K_F}||_{0,\nabla,M_F}^2}, \tag{4.18}$$

where $||\mathbf{v}||_{0,\nabla,M_F}^2 = ||\mathbf{v}||_{0,M_F}^2 + ||\nabla\mathbf{v}||_{0,M_F}^2$,

$$\mathscr{R}_{\mathbf{curl\ curl},\times\mathbf{n}}(\mathbf{f},\mathbf{v}) = \sum_{F\in\mathcal{F}_h^\Omega} \frac{\sum_{i=1}^{m_{\ell-1}^F}\langle \mathbf{f},\boldsymbol{\phi}_i^{\ell-1,M_F}\rangle_{M_F}((\mathbf{curl\ v},\mathbf{curl}\ \boldsymbol{\phi}_i^{\ell-1,M_F})_{0,M_F} - \lambda(\mathbf{v},\boldsymbol{\phi}_i^{\ell-1,M_F})_{0,M_F})}{\sum_{i=1}^{m_{\ell-1}^F}||\boldsymbol{\phi}_i^{\ell-1,M_F}||_{0,\nabla,M_F}^2}, \tag{4.19}$$

where $\langle \mathbf{f},\boldsymbol{\phi}_i^{\ell-1,M_F}\rangle_{M_F}$ is understood as $\langle \mathbf{f},\boldsymbol{\phi}_i^{\ell-1,M_F}\rangle$ after the zero extension of $\boldsymbol{\phi}_i^{\ell-1,M_F} \in (H_0^1(M_F))^3$ outside M_F to the whole domain Ω,

$$\mathscr{S}_{\rm div}\,(\mathbf{u},\mathbf{v}) = \sum_{K\in\mathcal{T}_h} \frac{\sum_{i=1}^{m_{\ell-1}} (\mathbf{u},\nabla\phi_i^{\ell-1,K})_{0,K}(\mathbf{v},\nabla\phi_i^{\ell-1,K})_{0,K}}{\sum_{i=1}^{m_{\ell-1}} ||\nabla\phi_i^{\ell-1,K}||_{0,K}^2}, \tag{4.20}$$

$$\mathscr{R}_{\rm div}\,(g,\mathbf{v}) = -\sum_{K\in\mathcal{T}_h} \frac{\sum_{i=1}^{m_{\ell-1}} \langle g,\phi_i^{\ell-1,K}\rangle_K(\mathbf{v},\nabla\phi_i^{\ell-1,K})_{0,K}}{\sum_{i=1}^{m_{\ell-1}} ||\nabla\phi_i^{\ell-1,K}||_{0,K}^2}, \tag{4.21}$$

where $\langle g,\phi_i^{\ell-1,K}\rangle_K$ is understood as $\langle g,\phi_i^{\ell-1,K}\rangle$ after the zero extension of $\phi_i^{\ell-1,K}\in H_0^1(K)$ outside K to the whole domain Ω,

$$\mathscr{S}_{\times\mathbf{n}}(\mathbf{u},\mathbf{v}) = \sum_{F\in\mathcal{F}_h^\Gamma} \frac{\sum_{i=1}^{m_\ell^F}((\mathbf{curl}\,\mathbf{u},\boldsymbol{\rho}_i^{\ell,K_F})_{0,K_F} - (\mathbf{u},\mathbf{curl}\,\boldsymbol{\rho}_i^{\ell,K_F})_{0,K_F})((\mathbf{curl}\,\mathbf{v},\boldsymbol{\rho}_i^{\ell,K_F})_{0,K_F} - (\mathbf{v},\mathbf{curl}\,\boldsymbol{\rho}_i^{\ell,K_F})_{0,K_F})}{\sum_{i=1}^{m_\ell^F} ||\boldsymbol{\rho}_i^{\ell,K_F}||_{0,\nabla,K_F}^2}, \tag{4.22}$$

where $K_F\in\mathcal{T}_h^\Gamma$ denotes the tetrahedron element which shares the face F with Γ,

$$\mathscr{R}_{\times\mathbf{n}}(\boldsymbol{\chi},\mathbf{v}) = \sum_{F\in\mathcal{F}_h^\Gamma} \frac{\sum_{i=1}^{m_\ell^F}\langle\boldsymbol{\chi},\mathbf{n}\times(\boldsymbol{\rho}_i^{\ell,K_F}\times\mathbf{n})\rangle_F((\mathbf{curl}\,\mathbf{v},\boldsymbol{\rho}_i^{\ell,K_F})_{0,K_F} - (\mathbf{v},\mathbf{curl}\,\boldsymbol{\rho}_i^{\ell,K_F})_{0,K_F})}{\sum_{i=1}^{m_\ell^F} ||\boldsymbol{\rho}_i^{\ell,K_F}||_{0,\nabla,K_F}^2}, \tag{4.23}$$

where $\langle\boldsymbol{\chi},\mathbf{n}\times(\boldsymbol{\rho}_i^{\ell,K_F}\times\mathbf{n})\rangle_F$ is understood as $\langle\boldsymbol{\chi},\mathbf{n}\times(\boldsymbol{\rho}_i^{\ell,K_F}\times\mathbf{n})\rangle_\Gamma = \langle\boldsymbol{\chi},\boldsymbol{\rho}_i^{\ell,K_F}\rangle_\Gamma$ after the zero extension of $\boldsymbol{\rho}_i^{\ell,K_F}$ outside K_F to the whole domain Ω,

$$\mathscr{S}_{\cdot\mathbf{n}}(\mathbf{u},\mathbf{v}) = \sum_{F\in\mathcal{F}_h^\Gamma} \frac{\sum_{i=1}^{m_{\ell-1}^F}(\mathbf{curl}\,\mathbf{u},\nabla o_i^{\ell-1,K_F})_{0,K_F}(\mathbf{curl}\,\mathbf{v},\nabla o_i^{\ell-1,K_F})_{0,K_F}}{\sum_{i=1}^{m_{\ell-1}^F} ||\nabla o_i^{\ell-1,K_F}||_{0,K_F}^2}, \tag{4.24}$$

$$\mathscr{R}_{\cdot\mathbf{n}}(\boldsymbol{\chi},\mathbf{v}) = \sum_{F\in\mathcal{F}_h^\Gamma} \frac{\sum_{i=1}^{m_{\ell-1}^F}\langle\boldsymbol{\chi},\mathbf{n}\times(\nabla o_i^{\ell-1,K_F}\times\mathbf{n})\rangle_F(\mathbf{curl}\,\mathbf{v},\nabla o_i^{\ell-1,K_F})_{0,K_F}}{\sum_{i=1}^{m_{\ell-1}^F} ||\nabla o_i^{\ell-1,K_F}||_{0,K_F}^2}, \tag{4.25}$$

where $\langle\boldsymbol{\chi},\mathbf{n}\times(\nabla o_i^{\ell-1,K_F}\times\mathbf{n})\rangle_F$ is understood as $\langle\boldsymbol{\chi},\mathbf{n}\times(\nabla o_i^{\ell-1,K_F}\times\mathbf{n})\rangle_{\Gamma,*}$ after the zero extension of $o_i^{\ell-1,K_F}$outside K_F to the whole domain Ω.

Remark 4.2. *The rationale for designing the above stabilizations will be seen respectively in Lemma 5.1, and Lemma 6.1, Remark 5.1 and Remark 6.1. From the proof of the coercivity of the finite element problem in the next section, one can see the role of the stabilizations. Also, these stabilizations can accommodate singular solution* $\mathbf{u}$ *and singular data* $\mathbf{f}$ *and* $\boldsymbol{\chi}$*.*

We are now in a position to state the finite element problem.
Let β be a constant to be determined later. Put

$$\mathscr{S}_h(\mathbf{u},\mathbf{v}) := \mathscr{S}_{\mathbf{curl\,curl}}(\mathbf{u},\mathbf{v}) + \beta\mathscr{S}_{\mathbf{curl\,curl},\times\mathbf{n}}(\mathbf{u},\mathbf{v}) + \mathscr{S}_{\mathrm{div}}(\mathbf{u},\mathbf{v}) + \mathscr{S}_{\times\mathbf{n}}(\mathbf{u},\mathbf{v}) + \mathscr{S}_{\cdot\mathbf{n}}(\mathbf{u},\mathbf{v}), \quad (4.26)$$

$$\mathscr{R}_h(\mathbf{f},g,\boldsymbol{\chi};\mathbf{v}) := \mathscr{R}_{\mathbf{curl\,curl}}(\mathbf{f},\mathbf{v}) + \beta\mathscr{R}_{\mathbf{curl\,curl},\times\mathbf{n}}(\mathbf{f},\mathbf{v}) + \mathscr{R}_{\mathrm{div}}(g,\mathbf{v}) + \mathscr{R}_{\times\mathbf{n}}(\boldsymbol{\chi},\mathbf{v}) + \mathscr{R}_{\cdot\mathbf{n}}(\boldsymbol{\chi},\mathbf{v}). \quad (4.27)$$

The finite element problem reads as follows: Find $\mathbf{u}_h \in \mathbf{U}_h$ such that, for all $\mathbf{v}_h \in \mathbf{U}_h$,

$$\mathcal{L}_h(\mathbf{u}_h,\mathbf{v}_h) = \mathcal{G}_h(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{U}_h, \quad (4.28)$$

where

$$\begin{aligned}\mathcal{L}_h(\mathbf{u},\mathbf{v}) &= (\mathbf{R}_h(\mathbf{curl\,curl\,u} - \lambda\mathbf{u}), \mathbf{R}_h(\mathbf{curl\,curl\,v} - \lambda\mathbf{v}))_{0,\nabla} + (S_h(\mathrm{div}\,\mathbf{u}), S_h(\mathrm{div}\,\mathbf{v}))_{0,\nabla}\\ &+(\boldsymbol{\Upsilon}_h^\Gamma(\mathbf{n}\times\mathbf{u}), \boldsymbol{\Upsilon}_h^\Gamma(\mathbf{n}\times\mathbf{v}))_{0,\nabla} + (\Lambda_h^\Gamma(\mathbf{curl\,u}\cdot\mathbf{n}), \Lambda_h^\Gamma(\mathbf{curl\,v}\cdot\mathbf{n}))_{0,\nabla} + \mathscr{S}_h(\mathbf{u},\mathbf{v}),\end{aligned} \quad (4.29)$$

$$\begin{aligned}\mathcal{G}_h(\mathbf{v}) &= (\mathbf{R}_h(\mathbf{f}), \mathbf{R}_h(\mathbf{curl\,curl\,v} - \lambda\mathbf{v}))_{0,\nabla} + (S_h(g), S_h(\mathrm{div}\,\mathbf{v}))_{0,\nabla}\\ &+(\boldsymbol{\Upsilon}_h^\Gamma(\boldsymbol{\chi}), \boldsymbol{\Upsilon}_h^\Gamma(\mathbf{n}\times\mathbf{v}))_{0,\nabla} + \langle\boldsymbol{\chi}, \mathbf{n}\times(\nabla\Lambda_h^\Gamma(\mathbf{curl\,v}\cdot\mathbf{n})\times\mathbf{n})\rangle_{\Gamma,*} + \mathscr{R}_h(\mathbf{f},g,\boldsymbol{\chi};\mathbf{v}).\end{aligned} \quad (4.30)$$

It can be verified from (4.29) and (4.30) that the finite element formulation (4.28) is consistent, or the error orthogonality property holds, i.e., for the exact solution $\mathbf{u}$ and the finite element solution $\mathbf{u}_h$,

$$\mathcal{L}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{U}_h. \quad (4.31)$$

5 Coercivity

In this section, we shall establish the coercivity of the proposed finite element method (4.28). As a consequence, the resulting algebraic linear system is symmetric, positive definite. For that goal, we first give two lemmas below on the stabilizations defined in the previous section.

Lemma 5.1. *For all $\mathbf{v} \in \mathbf{U}_h$, we have*

$$\mathscr{S}_{\mathbf{curl\,curl}}(\mathbf{v},\mathbf{v}) \geq c\sum_{K\in\mathcal{T}_h} h_K^2||\mathbf{curl\,curl\,v} - \lambda\mathbf{v}||_{0,K}^2,$$

$$\mathscr{S}_{\mathbf{curl\,curl},\times\mathbf{n}}(\mathbf{v},\mathbf{v}) \geq c_5\sum_{F\in\mathcal{F}_h^\Omega} h_F||[\mathbf{n}\times(\mathbf{curl\,v}\times\mathbf{n})]||_{0,F}^2 - c_6\sum_{K\in\mathcal{T}_h} h_K^2||\mathbf{curl\,curl\,v} - \lambda\mathbf{v}||_{0,K}^2,$$

where $[\cdot]$ denotes the jump across F,

$$\mathscr{S}_{\mathrm{div}}(\mathbf{v},\mathbf{v}) \geq c\sum_{K\in\mathcal{T}_h} h_K^2||\mathrm{div}\,\mathbf{v}||_{0,K}^2,$$

$$\mathscr{S}_{\times\mathbf{n}}(\mathbf{v},\mathbf{v}) \geq c\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{v}\times\mathbf{n}||_{0,F}^2,$$

$$\mathscr{S}_{\cdot\mathbf{n}}(\mathbf{v},\mathbf{v}) \geq c\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{curl\,v}\cdot\mathbf{n}||_{0,F}^2.$$

Proof. For $\mathbf{v} \in \mathbf{U}_h$, from the construction of the element-bubble and face-bubble spaces in (4.11)-(4.15), we know that

$$(\mathbf{curl\,curl\,v} - \lambda\mathbf{v})b_K|_K \in \mathbf{Z}^{\ell,K} \quad \forall K \in \mathcal{T}_h,$$

$$[\mathbf{n}\times(\mathbf{curl\,v}\times\mathbf{n})]b_F^{M_F}|_F \in \mathbf{Z}^{\ell-1,M_F}|_F \quad \forall F \in \mathcal{F}_h^\Omega,$$

$$(\text{div } \mathbf{v})b_K|_K \in Q^{\ell-1,K} \quad \forall K \in \mathcal{T}_h,$$
$$(\mathbf{v} \times \mathbf{n})b_F^K|_F \in \mathbf{Y}^{\ell,K_F}|_F \quad \forall F \in \mathcal{F}_h^\Gamma,$$
$$(\mathbf{curl}\ \mathbf{v} \cdot \mathbf{n})b_F^K|_F \in \Psi^{\ell-1,K_F}|_F \quad \forall F \in \mathcal{F}_h^\Gamma.$$

Thus, it is not difficult to adapt the argument in [10] (Lemma 4.3 on page 1285) to have the conclusion. For the sake of completeness and for illustrating the idea and the argument therein, below we give a demonstration of the third inequality in the above Lemma 5.1, i.e.,

$$\mathscr{S}_{\text{div}}(\mathbf{v}, \mathbf{v}) \geq c \sum_{K \in \mathcal{T}_h} h_K^2 ||\text{div } \mathbf{v}||_{0,K}^2 \quad \forall \mathbf{v} \in \mathbf{U}_h, \tag{5.1}$$

using the fact that

$$(\text{div } \mathbf{v})b_K|_K \in Q^{\ell-1,K} \quad \forall K \in \mathcal{T}_h, \quad \forall \mathbf{v} \in \mathbf{U}_h, \tag{5.2}$$

where $\mathscr{S}_{\text{div}}(\mathbf{v}, \mathbf{v})$ is defined by (4.20) while $Q^{\ell-1,K}$ is defined by (4.11). From (4.20) we have

$$\mathscr{S}_{\text{div}}(\mathbf{v}, \mathbf{v}) = \sum_{K \in \mathcal{T}_h} \frac{\sum_{i=1}^{m_{\ell-1}} \left((\mathbf{v}, \nabla \phi_i^{\ell-1,K})_{0,K}\right)^2}{\sum_{i=1}^{m_{\ell-1}} ||\nabla \phi_i^{\ell-1,K}||_{0,K}^2}, \tag{5.3}$$

and since $\phi_i^{\ell-1,K} = \varphi_i^{\ell-1,K} b_K \in H_0^1(K)$, by the Green formula of integration by parts we have

$$\mathscr{S}_{\text{div}}(\mathbf{v}, \mathbf{v}) = \sum_{K \in \mathcal{T}_h} \frac{\sum_{i=1}^{m_{\ell-1}} \left((\text{div } \mathbf{v}, \phi_i^{\ell-1,K})_{0,K}\right)^2}{\sum_{i=1}^{m_{\ell-1}} ||\nabla \phi_i^{\ell-1,K}||_{0,K}^2}. \tag{5.4}$$

But, from the construction of $Q^{\ell-1,K}$ by (4.11)

$$Q^{\ell-1,K} = \Delta^{\ell-1,K} = \text{span}\{\phi_i^{\ell-1,K} = \varphi_i^{\ell-1,K} b_K, 1 \leq i \leq m_{\ell-1}\} \subset H_0^1(K)$$

and (5.2) which equivalently says that

$$\text{div } \mathbf{v}|_K \in \text{span}\{\varphi_i^{\ell-1,K}, 1 \leq i \leq m_{\ell-1}\}, \tag{5.5}$$

we can write

$$\text{div } \mathbf{v}|_K = \sum_{j=1}^{m_{\ell-1}} d_j \varphi_j^{\ell-1,K},$$

where $d_j, 1 \leq j \leq m_{\ell-1}$, are coefficients. Now, since $\phi_i^{\ell-1,K} = \varphi_i^{\ell-1,K} b_K$, we have

$$(\text{div } \mathbf{v}, \phi_i^{\ell-1,K})_{0,K} = \sum_{j=1}^{m_{\ell-1}} d_j (\varphi_j^{\ell-1,K}, \varphi_i^{\ell-1,K} b_K)_{0,K},$$

and we find that

$$\sum_{i=1}^{m_{\ell-1}} \left((\text{div } \mathbf{v}, \phi_i^{\ell-1,K})_{0,K}\right)^2 = \mathbf{d}^t B \mathbf{d}, \tag{5.6}$$

where $\mathbf{d}^t = (d_1, d_2, \cdots, d_{m_{\ell-1}}) \in \mathbb{R}^{m_{\ell-1}}$ and

$$B = (B_{ij}) \in \mathbb{R}^{m_{\ell-1} \times m_{\ell-1}}, \quad B_{ij} = \sum_{k=1}^{m_{\ell-1}} (\varphi_i^{\ell-1,K}, \varphi_k^{\ell-1,K} b_K)_{0,K} (\varphi_k^{\ell-1,K}, \varphi_j^{\ell-1,K} b_K)_{0,K}.$$

Since $\{\varphi_i^{\ell-1,K}, 1 \le i \le m_{\ell-1}\}$ is a group of linearly independent basis, the corresponding *Gram* matrix

$$G = (G_{ij}) \in \mathbb{R}^{m_{\ell-1}\times m_{\ell-1}}, \quad G_{ij} = (\varphi_i^{\ell-1,K}, \varphi_j^{\ell-1,K} b_K)_{0,K},$$

is symmetric, positive definite, and we have

$$B = (G)^2.$$

Further, by the well-known scaling argument(cf. [2, page 95]) through an affine mapping $\mathbf{x} = \mathcal{F}_K(\hat{\mathbf{x}})$ from the reference element $\hat{K}$ (a unit tetrahedron in the reference $\hat{\mathbf{x}}$-coordinates system) to the physical element K(a tetrahedron in the $\mathbf{x}$-coordinates system) and by the fact that $\{\varphi_i^{\ell-1,K}, 1 \le i \le m_{\ell-1}\}$ are concretely given basis functions, it is not difficult in finding that under the shape-regularity condition of the meshes, the least eigenvalue of G satisfies

$$\mu_{\min} \ge c|K|, \tag{5.7}$$

where c depends on $\{\hat{\varphi}_i^{\ell-1} := \varphi_i^{\ell-1,K} \circ \mathcal{F}_K, 1 \le i \le m_{\ell-1}\}$ and on $\hat{b} := b_K \circ \mathcal{F}_K$ all of which live on the reference element $\hat{K}$, but it is independent of K, h and $\mathbf{v}$. On the other hand, we find that

$$(\text{div }\mathbf{v}, (\text{div }\mathbf{v})b_K)_{0,K} = \mathbf{d}^t G\mathbf{d}, \tag{5.8}$$

and from a similar argument in [2, page 142] that

$$(\text{div }\mathbf{v}, (\text{div }\mathbf{v})b_K)_{0,K} \ge c||\text{div }\mathbf{v}||_{0,K}^2, \tag{5.9}$$

where c depends on $\hat{b}$ but it is independent of K, h and $\mathbf{v}$. From any standard textbook of linear algebra, it is easy to show that

$$\mathbf{d}^t B\mathbf{d} = \mathbf{d}^t (G)^2\mathbf{d} \ge \mu_{\min}\mathbf{d}^t G\mathbf{d}. \tag{5.10}$$

Also by the scaling argument, under the shape-regularity condition of the meshes, we find that

$$\sum_{i=1}^{m_{\ell-1}} ||\nabla\phi_i^{\ell-1,K}||_{0,K}^2 \le ch_K^{-2}|K|, \tag{5.11}$$

where c depends on $\{\hat{\varphi}_i^{\ell-1}, 1 \le i \le m_{\ell-1}\}$ and on $\hat{b}$, but it is independent of K, h and $\mathbf{v}$. Summarizing (5.3), (5.4), (5.5), (5.6), (5.7), (5.8), (5.9), (5.10) and (5.11), we obtain

$$\frac{\sum_{i=1}^{m_{\ell-1}} \left((\text{div }\mathbf{v}, \phi_i^{\ell-1,K})_{0,K}\right)^2}{\sum_{i=1}^{m_{\ell-1}} ||\nabla\phi_i^{\ell-1,K}||_{0,K}^2} \ge ch_K^2||\text{div }\mathbf{v}||_{0,K}^2, \tag{5.12}$$

where c is independent of K, h and $\mathbf{v}$. It follows from (5.12) and (5.3) that (5.1) holds. Other coercivity results in Lemma 5.1 can be similarly shown. $\square$

As a consequence of Lemma 5.1, for a suitable value for β, for all $\mathbf{v} \in \mathbf{U}_h$ we have

$$\begin{aligned} &\mathscr{S}_{\mathbf{curl\ curl}}(\mathbf{v},\mathbf{v}) + \beta\mathscr{S}_{\mathbf{curl\ curl}, \times\mathbf{n}}(\mathbf{v},\mathbf{v}) \ge \\ &c\left(\sum_{F\in\mathcal{F}_h^\Omega} h_F||[\mathbf{n}\times(\mathbf{curl\ v}\times\mathbf{n})]||_{0,F}^2 + \sum_{K\in\mathcal{T}_h} h_K^2||\mathbf{curl\ curl\ v} - \lambda\mathbf{v}||_{0,K}^2\right). \end{aligned} \tag{5.13}$$

Below, we can prove the coercivity.

Theorem 5.1. *For all* $\mathbf{v} \in \mathbf{U}_h$*, for a suitable value of* β*, we have*

$$\mathcal{L}_h(\mathbf{v},\mathbf{v}) \ge c||\mathbf{v}||_{0,\mathbf{curl}}^2.$$

Proof. For $\mathbf{v} \in \mathbf{U}_h$, From Theorem 3.1, we have

$$c||\mathbf{v}||_{0,\mathbf{curl}} \le ||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda \mathbf{v}||_{-1} + ||\mathrm{div}\,\mathbf{v}||_{-1} + ||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma} + ||\mathbf{v}\times\mathbf{n}||_{-\frac{1}{2},\Gamma}. \tag{5.14}$$

In what follows, we show that

$$||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda \mathbf{v}||_{-1} + ||\mathrm{div}\,\mathbf{v}||_{-1} + ||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma} + ||\mathbf{v}\times\mathbf{n}||_{-\frac{1}{2},\Gamma} \le c\Big(\mathcal{L}_h(\mathbf{v},\mathbf{v})\Big)^{\frac{1}{2}}. \tag{5.15}$$

We first consider the first term in the above left-hand side. Notice that

$$||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda \mathbf{v}||_{-1} = \sup_{\mathbf{0}\neq\mathbf{z}\in(H_0^1(\Omega))^3} \frac{\langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}, \mathbf{z}\rangle}{||\mathbf{z}||_1}.$$

Denote by $\mathbf{\Xi}_h : (H^1(\Omega))^3 \to \mathbf{Z}_h$ the classical interpolation operator such as the Clément interpolation operator or the Scott-Zhang interpolation operator(e.g., see [8, 2, 26]). Choosing $\mathbf{z}_h := \mathbf{\Xi}_h\mathbf{z} \in \mathbf{Z}_h$ so that

$$\left(\sum_{K\in\mathcal{T}_h} h_K^{-2}||\mathbf{z}-\mathbf{z}_h||_{0,h}^2\right)^{\frac{1}{2}} + \left(\sum_{F\in\mathcal{F}_h^\Omega} h_F^{-1}||\mathbf{z}-\mathbf{z}_h||_{0,F}^2\right)^{\frac{1}{2}} + ||\mathbf{z}_h||_1 \le c||\mathbf{z}||_1,$$

we have

$$\langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}, \mathbf{z}\rangle = \langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}, \mathbf{z}_h\rangle + \langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}, \mathbf{z}-\mathbf{z}_h\rangle,$$

where

$$\begin{aligned}
&\langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}, \mathbf{z}-\mathbf{z}_h\rangle = (\mathbf{curl}\,\mathbf{v}, \mathbf{curl}\,(\mathbf{z}-\mathbf{z}_h)) - \lambda(\mathbf{v}, \mathbf{z}-\mathbf{z}_h)\\
&= \sum_{K\in\mathcal{T}_h} (\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}, \mathbf{z}-\mathbf{z}_h)_{0,K} + \sum_{F\in\mathcal{F}_h^\Omega} \int_F (\mathbf{n}\times(\mathbf{z}-\mathbf{z}_h))\cdot[\mathbf{n}\times(\mathbf{curl}\,\mathbf{v}\times\mathbf{n})]\\
&\le \left(\sum_{K\in\mathcal{T}_h} h_K^2||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}||_{0,K}^2\right)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{T}_h} h_K^{-2}||\mathbf{z}-\mathbf{z}_h||_{0,K}^2\right)^{\frac{1}{2}}\\
&+c\left(\sum_{F\in\mathcal{F}_h^\Omega} h_F||[\mathbf{n}\times(\mathbf{curl}\,\mathbf{v}\times\mathbf{n})]||_{0,F}^2\right)^{\frac{1}{2}} \left(\sum_{F\in\mathcal{F}_h^\Omega} h_F^{-1}||\mathbf{z}-\mathbf{z}_h||_{0,F}^2\right)^{\frac{1}{2}}\\
&\le c\left(\sum_{K\in\mathcal{T}_h} h_K^2||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}||_{0,K}^2 + \sum_{F\in\mathcal{F}_h^\Omega} h_F||[\mathbf{n}\times(\mathbf{curl}\,\mathbf{v}\times\mathbf{n})]||_{0,F}^2\right)^{\frac{1}{2}} ||\mathbf{z}||_1,
\end{aligned}$$

and from (4.6), we have

$$\begin{aligned}
\langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}, \mathbf{z}_h\rangle &= (R_h(\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}), \mathbf{z}_h)_{0,\nabla}\\
&\le ||R_h(\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v})||_{0,\nabla}||\mathbf{z}_h||_{0,\nabla}\\
&\le ||R_h(\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v})||_{0,\nabla}||\mathbf{z}||_1.
\end{aligned}$$

Hence,

$$\begin{aligned}
&c||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}||_{-1} \le ||R_h(\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v})||_{0,\nabla}\\
&+\left(\sum_{K\in\mathcal{T}_h} h_K^2||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{v}||_{0,K}^2 + \sum_{F\in\mathcal{F}_h^\Omega} h_F||[\mathbf{n}\times(\mathbf{curl}\,\mathbf{v}\times\mathbf{n})]||_{0,F}^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.16}$$

Similarly, for the second-term, we have

$$c||\mathrm{div}\,\mathbf{v}||_{-1} \le ||S_h(\mathrm{div}\,\mathbf{v})||_{0,\nabla} + \left(\sum_{K\in\mathcal{T}_h} h_K^2||\mathrm{div}\,\mathbf{v}||_{0,K}^2\right)^{\frac{1}{2}}. \tag{5.17}$$

Below, we estimate the two terms $||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma}$ and $||\mathbf{v}\times\mathbf{n}||_{-\frac{1}{2},\Gamma}$. Note that

$$||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma} = \sup_{0\neq\psi\in H^{\frac{1}{2}}(\Gamma)} \frac{\langle \mathbf{curl}\,\mathbf{v}\cdot\mathbf{n},\psi\rangle_\Gamma}{||\psi||_{\frac{1}{2},\Gamma}}.$$

Let $\Psi_h(\mathcal{F}_h^\Gamma)$ denote the restriction of Ψ_h over $\mathcal{F}_h^\Gamma$ (the conforming and shape-regular triangulation of Γ). Denote by $\Pi_h : L^1(\Gamma)\to\Psi_h(\mathcal{F}_h^\Gamma)$ the Clément interpolation operator (cf., [8, 2]). Choose a $\psi_h := \Pi_h\psi \in \Psi_h(\mathcal{F}_h^\Gamma)$ so that

$$\left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F^{-1}||\psi_h-\psi||_{0,F}^2\right)^{\frac{1}{2}} + ||\psi_h||_{\frac{1}{2},\Gamma} \le c||\psi||_{\frac{1}{2},\Gamma}.$$

Denote by $E : H^{\frac{1}{2}}(\Gamma)\to H^1(\Omega)$ the lifting operator. Let $E\psi_h\in H^1(\Omega)$ be the lifting of the boundary trace $\psi_h\in H^{\frac{1}{2}}(\Gamma)$, satisfying

$$||E\psi_h||_1 \le c||\psi_h||_{\frac{1}{2},\Gamma} \le c||\psi||_{\frac{1}{2},\Gamma}.$$

Now, denote by $\pi_h : H^1(\Omega)\to\Psi_h$ the Scott-Zhang interpolation operator, and define $\pi_h E\psi_h\in\Psi_h$, which preserves the piecewise polynomials on the boundary Γ, satisfying (see [26])

$$\pi_h E\psi_h|_\Gamma = \psi_h,$$
$$||\pi_h E\psi_h||_1 \le c||E\psi_h||_1 \le c||\psi_h||_{\frac{1}{2},\Gamma} \le c||\psi||_{\frac{1}{2},\Gamma}.$$

Since

$$\langle \mathbf{curl}\,\mathbf{v}\cdot\mathbf{n},\psi\rangle_\Gamma = \langle \mathbf{curl}\,\mathbf{v}\cdot\mathbf{n},\psi_h\rangle_\Gamma + \langle \mathbf{curl}\,\mathbf{v}\cdot\mathbf{n},\psi-\psi_h\rangle_\Gamma,$$

where, from (4.9), we have

$$\begin{aligned}\langle \mathbf{curl}\,\mathbf{v}\cdot\mathbf{n},\psi_h\rangle_\Gamma &= \langle \mathbf{curl}\,\mathbf{v}\cdot\mathbf{n},\pi_h E\psi_h\rangle_\Gamma = (\Lambda_h^\Gamma(\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}),\pi_h E\psi_h)_{0,\nabla}\\ &\le ||\Lambda_h^\Gamma(\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n})||_{0,\nabla}||\pi_h E\psi_h||_1 \le c||\Lambda_h^\Gamma(\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n})||_{0,\nabla}||\psi||_{\frac{1}{2},\Gamma},\end{aligned}$$

$$\begin{aligned}\langle \mathbf{curl}\,\mathbf{v}\cdot\mathbf{n},\psi-\psi_h\rangle_\Gamma &= \textstyle\int_\Gamma(\psi-\psi_h)\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}\\ &\le \left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{0,F}^2\right)^{\frac{1}{2}}\left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F^{-1}||\psi_h-\psi||_{0,F}^2\right)^{\frac{1}{2}}\\ &\le c\left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{0,F}^2\right)^{\frac{1}{2}}||\psi||_{\frac{1}{2},\Gamma},\end{aligned}$$

and we thus have

$$c||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma} \le ||\Lambda_h^\Gamma(\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n})||_{0,\nabla} + \left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{0,F}^2\right)^{\frac{1}{2}}. \tag{5.18}$$

Similarly, we have

$$c||\mathbf{v}\times\mathbf{n}||_{-\frac{1}{2},\Gamma} \le ||\Upsilon_h^\Gamma(\mathbf{n}\times\mathbf{v})||_{0,\nabla} + \left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{v}\times\mathbf{n}||_{0,F}^2\right)^{\frac{1}{2}}. \tag{5.19}$$

Summarizing (5.16)-(5.19), we have

$$\begin{aligned}&c(||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}-\lambda\mathbf{v}||_{-1} + ||\mathrm{div}\,\mathbf{v}||_{-1} + ||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{-\frac{1}{2},\Gamma} + ||\mathbf{v}\times\mathbf{n}||_{-\frac{1}{2},\Gamma}) \le\\ &||R_h(\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}-\lambda\mathbf{v})||_{0,\nabla} + ||S_h(\mathrm{div}\,\mathbf{v})||_{0,\nabla} + ||\Upsilon_h^\Gamma(\mathbf{n}\times\mathbf{v})||_{0,\nabla} + ||\Lambda_h^\Gamma(\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n})||_{0,\nabla}\\ &+\left(\sum_{K\in\mathcal{T}_h} h_K^2||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}-\lambda\mathbf{v}||_{0,K}^2 + \sum_{F\in\mathcal{F}_h^\Omega} h_F||[\mathbf{n}\times(\mathbf{curl}\,\mathbf{v}\times\mathbf{n})]||_{0,F}^2\right)^{\frac{1}{2}}\\ &+\left(\sum_{K\in\mathcal{T}_h} h_K^2||\mathrm{div}\,\mathbf{v}||_{0,K}^2\right)^{\frac{1}{2}} + \left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{0,F}^2\right)^{\frac{1}{2}} + \left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{v}\times\mathbf{n}||_{0,F}^2\right)^{\frac{1}{2}},\end{aligned} \tag{5.20}$$

and, hence, combining Lemma 5.1, we conclude (5.15). □

Remark 5.1. *From Lemma 5.1, all the stabilizations in the finite element problem can be replaced by simpler forms as follows:*

$$\begin{aligned}&\left(\sum_{K\in\mathcal{T}_h} h_K^2||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}-\lambda\mathbf{v}||_{0,K}^2+\sum_{F\in\mathcal{F}_h^\Omega} h_F||[\mathbf{n}\times(\mathbf{curl}\,\mathbf{v}\times\mathbf{n})]||_{0,F}^2\right)^{\frac{1}{2}}\\&+\left(\sum_{K\in\mathcal{T}_h} h_K^2||\mathrm{div}\,\mathbf{v}||_{0,K}^2\right)^{\frac{1}{2}}+\left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{0,F}^2\right)^{\frac{1}{2}}+\left(\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{v}\times\mathbf{n}||_{0,F}^2\right)^{\frac{1}{2}}.\end{aligned}\tag{5.21}$$

The above is much easier to implement.

However, clearly, these stabilizations are only suitable for regular enough $\mathbf{f}$ *and* $\boldsymbol{\chi}$*, say* $\mathbf{f}\in(L^2(\Omega))^3$ *and* $\boldsymbol{\chi}\in(L^2(\Gamma))^3$*. Moreover, it is easy to define the corresponding right-hand sides, but not for the stabilization term* $(\sum_{F\in\mathcal{F}_h^\Gamma} h_F||\mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}||_{0,F}^2)^{1/2}$*. This term requires that the solution* $\mathbf{u}$ *is smooth enough so that* $\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}\in L^2(\Gamma)$*. For this term, if* $\boldsymbol{\chi}=\mathbf{0}$*, then no right-hand side is needed. If* $\boldsymbol{\chi}\neq\mathbf{0}$*, we cannot find a corresponding right-hand side, and the property of consistency or error orthogonality will be lost. Without this property, higher-order elements (i.e.,* $\ell>1$*) cannot result in higher-order convergence rate for smooth solution (say,* $\mathbf{u}\in(H^{1+\ell}(\Omega))^3$*). Here, why we design the stabilizations as in* (4.16)-(4.25) *is explained. See a further explanation in Remark 6.1.*

Remark 5.2. *Upon the interest of one of the referees, here we consider to use the element-bubbles and face-bubbles from* $\mathbf{Z}^{\ell,K},\mathbf{Z}^{\ell-1,M_F},\mathbf{Y}^{\ell,K_F}$*,* $Q^{\ell-1,K}$*,* $\Psi^{\ell-1,K_F}$ *to enrich all the auxiliary finite element spaces* $\mathbf{Z}_h$*,* $\mathbf{Y}_h$*,* Q_h *and* Ψ_h*, then, denoting the enriched finite element spaces by* $\mathbf{Z}_h^{\flat+}$*,* $\mathbf{Y}_h^{\flat+}$*,* $Q_h^{\flat+}$*, and* $\Psi_h^{\flat+}$*, all the discrete Riesz-representation liftings* $\mathbf{R}_h,\boldsymbol{\Upsilon}_h^\Gamma,S_h,\Lambda_h^\Gamma$ *are now defined on* $\mathbf{Z}_h^{\flat+}$*,* $\mathbf{Y}_h^{\flat+}$*,* $Q_h^{\flat+}$*, and* $\Psi_h^{\flat+}$*, respectively. Under such replacements, Theorem 5.1 still holds, while all the stabilizations are unnecessary and can be dropped. In fact, with* $\mathbf{Z}_h^{\flat+}$*, in reasoning* (5.16)*, but now choosing a* $\mathbf{z}_h\in\mathbf{Z}_h^{\flat+}$*, for* $\mathbf{v}\in\mathbf{U}_h$*, so that*

$$\langle\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}-\lambda\mathbf{v},\mathbf{z}-\mathbf{z}_h\rangle=0,$$

and as a result, the term

$$\left(\sum_{K\in\mathcal{T}_h} h_K^2||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}-\lambda\mathbf{v}||_{0,K}^2+\sum_{F\in\mathcal{F}_h^\Omega} h_F||[\mathbf{n}\times(\mathbf{curl}\,\mathbf{v}\times\mathbf{n})]||_{0,F}^2\right)^{\frac{1}{2}}$$

disappears in (5.16)*, i.e.,* (5.16) *now reads as follows:*

$$c||\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}-\lambda\mathbf{v}||_{-1}\leq||R_h(\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}-\lambda\mathbf{v})||_{0,\nabla}.$$

Thus, the related stabilizations $\mathscr{S}_{\mathbf{curl}\,\mathbf{curl}}(\mathbf{u},\mathbf{v}),\mathscr{S}_{\mathbf{curl}\,\mathbf{curl}\,,\times\mathbf{n}}(\mathbf{u},\mathbf{v})$*, together with their corresponding right-hand sides* $\mathscr{R}_{\mathbf{curl}\,\mathbf{curl}}(\mathbf{f},\mathbf{v}),\mathscr{R}_{\mathbf{curl}\,\mathbf{curl}\,,\times\mathbf{n}}(\mathbf{f},\mathbf{v})$*, are unnecessary and can be dropped. Similar arguments apply if* $(Q_h^{\flat+},\mathbf{Y}_h^{\flat+},\Psi_h^{\flat+})$ *are in place of* $(Q_h,\mathbf{Y}_h,\Psi_h)$*, and the related stabilizations and their right-hand sides are unnecessary and can be dropped, too.*

6 Convergence and error bound

In this section, we shall establish the $H(\mathbf{curl}\,;\Omega)$-quasi-optimal error estimate, convergence and error bound.

We first give the boundedness of the bilinear form $\mathcal{L}_h$ and the linear form $\mathcal{G}_h$, and of the stabilizations, where $\mathcal{L}_h$ and $\mathcal{G}_h$ are defined in (4.28), and the stabilizations are defined in (4.16)-(4.25).

Lemma 6.1. *Let* $\mathbf{f} \in (H_0(\mathbf{curl}\,;\Omega))' = \{\mathbf{f} \in (H^{-1}(\Omega))^3 : \mathrm{div}\,\mathbf{f} \in H^{-1}(\Omega)\}$ *and* $\boldsymbol{\chi} \in H^{-\frac{1}{2}}(\mathrm{div}_{\,\Gamma};\Gamma)$ *and* $g = -\lambda \mathrm{div}\,\mathbf{f} \in H^{-1}(\Omega)$. *For all* $\mathbf{u}, \mathbf{v} \in H(\mathbf{curl}\,;\Omega)$*, we have*

$$\mathscr{S}_{\mathbf{curl}\ \mathbf{curl}}\,(\mathbf{u},\mathbf{v}) \leq c||\mathbf{u}||_{0,\mathbf{curl}}\,||\mathbf{v}||_{0,\mathbf{curl}}\,,$$

$$\mathscr{S}_{\mathbf{curl}\ \mathbf{curl}\,,\times\mathbf{n}}(\mathbf{u},\mathbf{v}) \leq c||\mathbf{u}||_{0,\mathbf{curl}}\,||\mathbf{v}||_{0,\mathbf{curl}}\,,$$

$$\mathscr{S}_{\mathrm{div}}\,(\mathbf{u},\mathbf{v}) \leq c||\mathbf{u}||_0||\mathbf{v}||_0,$$

$$\mathscr{S}_{\times\mathbf{n}}(\mathbf{u},\mathbf{v}) \leq c||\mathbf{u}||_{0,\mathbf{curl}}\,||\mathbf{v}||_{0,\mathbf{curl}}\,,$$

$$\mathscr{S}_{\cdot\mathbf{n}}(\mathbf{u},\mathbf{v}) \leq c||\mathbf{curl}\ \mathbf{u}||_0||\mathbf{curl}\ \mathbf{v}||_0,$$

$$\mathscr{R}_{\mathbf{curl}\ \mathbf{curl}}\,(\mathbf{f},\mathbf{v}) \leq c||\mathbf{f}||_{(H_0(\mathbf{curl}\,;\Omega))'}||\mathbf{v}||_{0,\mathbf{curl}}\,,$$

$$\mathscr{R}_{\mathbf{curl}\ \mathbf{curl}\,,\times\mathbf{n}}(\mathbf{f},\mathbf{v}) \leq c||\mathbf{f}||_{(H_0(\mathbf{curl}\,;\Omega))'}||\mathbf{v}||_{0,\mathbf{curl}}\,,$$

$$\mathscr{R}_{\mathrm{div}}\,(g,\mathbf{v}) \leq c||g||_{-1}||\mathbf{v}||_0 \leq c||\mathbf{f}||_{(H_0(\mathbf{curl}\,;\Omega))'}||\mathbf{v}||_0,$$

$$\mathscr{R}_{\times\mathbf{n}}(\boldsymbol{\chi},\mathbf{v}) \leq c||\boldsymbol{\chi}||_{H^{-\frac{1}{2}}(\mathrm{div}_{\,\Gamma};\Gamma)}||\mathbf{v}||_{0,\mathbf{curl}}\,,$$

$$\mathscr{R}_{\cdot\mathbf{n}}(\boldsymbol{\chi},\mathbf{v}) \leq c||\boldsymbol{\chi}||_{H^{-\frac{1}{2}}(\mathrm{div}_{\,\Gamma};\Gamma)}||\mathbf{curl}\ \mathbf{v}||_0.$$

Proof. From the expressions (4.16)-(4.25) of the stabilizations, one can easily obtain the above boundedness. □

Remark 6.1. *If we adopt the stabilizations in* (5.21)*, we cannot obtain the* $H(\mathbf{curl}\,;\Omega)$*-norm boundedness in Lemma 6.1, and consequently, we cannot establish the* $H(\mathbf{curl}\,;\Omega)$*-quasi-optimal error estimate, and the convergence, the error bound in the sequel.*

Lemma 6.2. *For all* $\mathbf{u}, \mathbf{v} \in H(\mathbf{curl}\,;\Omega)$ *and for all* $\mathbf{f} \in (H_0(\mathbf{curl}\,;\Omega))'$ *and* $\boldsymbol{\chi} \in H^{-\frac{1}{2}}(\mathrm{div}_{\,\Gamma};\Gamma)$*, we have*

$$|\mathcal{L}_h(\mathbf{u},\mathbf{v})| \leq c||\mathbf{u}||_{0,\mathbf{curl}}\,||\mathbf{v}||_{0,\mathbf{curl}}\,,$$

$$|\mathcal{G}_h(\mathbf{v})| \leq c(||\mathbf{f}||_{(H_0(\mathbf{curl}\,;\Omega))'} + ||\boldsymbol{\chi}||_{H^{-\frac{1}{2}}(\mathrm{div}_{\,\Gamma};\Gamma)})||\mathbf{v}||_{0,\mathbf{curl}}\,.$$

Proof. First observe that both $\mathcal{L}_h$ and $\mathcal{G}_h$ are well-defined on $H(\mathbf{curl}\,;\Omega)$. From the definitions of those discrete Riesz-representation liftings or projections (4.6)-(4.9), for $\mathbf{f}' \in (H^{-1}(\Omega))^3, g' \in H^{-1}(\Omega), \boldsymbol{\chi}' \in (H^{-\frac{1}{2}}(\Gamma))^3, \kappa' \in H^{-\frac{1}{2}}(\Gamma)$, we have

$$||\mathbf{R}_h(\mathbf{f}')||_{0,\nabla} \leq c||\mathbf{f}'||_{-1},$$

$$||S_h(g')||_{0,\nabla} \leq c||g'||_{-1},$$

$$||\boldsymbol{\Upsilon}_h^\Gamma(\boldsymbol{\chi}')||_{0,\nabla} \leq c||\boldsymbol{\chi}'||_{-\frac{1}{2},\Gamma},$$

$$||\Lambda_h^\Gamma(\kappa')||_{0,\nabla} \leq c||\kappa'||_{-\frac{1}{2},\Gamma}.$$

For $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$, putting $\mathbf{f}' := \mathbf{curl}\,\mathbf{curl}\ \mathbf{u} - \lambda\mathbf{u}$, $g' := \mathrm{div}\ \mathbf{u}$, $\boldsymbol{\chi}' = \mathbf{n} \times \mathbf{u}$ and $\kappa' := \mathbf{curl}\ \mathbf{u}\cdot\mathbf{n}$, we have

$$||\mathbf{R}_h(\mathbf{f}')||_{0,\nabla} \leq c||\mathbf{f}'||_{-1} \leq c||\mathbf{u}||_{0,\mathbf{curl}}\,,$$

$$||S_h(g')||_{0,\nabla} \leq c||g'||_{-1} \leq c||\mathbf{u}||_0,$$

$$||\boldsymbol{\Upsilon}_h^\Gamma(\boldsymbol{\chi}')||_{0,\nabla} \leq c||\boldsymbol{\chi}'||_{-\frac{1}{2},\Gamma} \leq c||\mathbf{u}||_{0,\mathbf{curl}}\,,$$

$$||\Lambda_h^\Gamma(\kappa')||_{0,\nabla} \leq c||\kappa'||_{-\frac{1}{2},\Gamma} \leq c||\mathbf{curl}\ \mathbf{u}||_0.$$

For $\mathbf{v}$, we have the same. Combining Lemma 6.1, from (4.29) and (4.30), we conclude. □

Combining Theorem 5.1, Lemmas 5.1, 6.1 and 6.2, from the classical Lax-Milgram Lemma (cf., [7]), we conclude the following theorem on the stability of the finite element solution.

Theorem 6.1. *The finite element problem* (4.28) *is well-posed, i.e., there exists a unique solution* $\mathbf{u}_h \in \mathbf{U}_h$*, and for all* $\mathbf{f} \in (H_0(\mathbf{curl}\,;\Omega))'$ *and* $\boldsymbol{\chi} \in H^{-\frac{1}{2}}(\mathrm{div}_{\Gamma};\Gamma)$*, the following stability holds:*

$$||\mathbf{u}_h||_{0,\mathbf{curl}} \le c(||\mathbf{f}||_{(H_0(\mathbf{curl}\,;\Omega))'} + ||\boldsymbol{\chi}||_{H^{-\frac{1}{2}}(\mathrm{div}_{\Gamma};\Gamma)}).$$

In addition, under the assumption of quasi-uniform meshes, the condition number of the resulting algebraic linear system is $\mathcal{O}(h^{-2})$*, since*

$$c||\mathbf{v}||^2_{0,\mathbf{curl}} \le \mathcal{L}_h(\mathbf{v},\mathbf{v}) \le c^{-1}||\mathbf{v}||^2_{0,\mathbf{curl}} \quad \forall \mathbf{v} \in \mathbf{U}_h.$$

The following theorem is about the quasi-optimal error estimates and the convergence.

Theorem 6.2. *Let* $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$ *be the exact solution and* $\mathbf{u}_h \in \mathbf{U}_h$ *the finite element solution. We have the following quasi-optimal error estimates:*

$$||\mathbf{u}-\mathbf{u}_h||_{0,\mathbf{curl}} \le c \inf_{\mathbf{v}\in\mathbf{U}_h} ||\mathbf{u}-\mathbf{v}||_{0,\mathbf{curl}}.$$

Consequently, we have the convergence as follows:

$$\lim_{h\to 0} ||\mathbf{u}-\mathbf{u}_h||_{0,\mathbf{curl}} = 0.$$

Proof. From Theorem 5.1, the consistency or error orthogonality property (4.31), and Lemma 6.2, we have the quasi-optimal error estimates. Since, from Theorem 2.10 on page 34 in [2], the space $(\mathscr{D}(\bar{\Omega}))^3 \cap H(\mathbf{curl}\,;\Omega)$ is dense in $H(\mathbf{curl}\,;\Omega)$ with respect to the norm of $H(\mathbf{curl}\,;\Omega)$, where $\mathscr{D}(\bar{\Omega}) = \{\phi|_\Omega, \phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)\}$, we have a smooth enough function $\mathbf{v}$ which approximates $\mathbf{u}$ in the $H(\mathbf{curl}\,;\Omega)$ norm. Such $\mathbf{v}$ can be interpolated by the function in $\mathbf{U}_h$, following the argument in [7] (cf. Theorem 18.2 on page 139), it is not difficult to obtain the convergence. □

In order to establish the error bound, we first give an extension result.

Lemma 6.3. *For* $\mathbf{u}, \mathbf{curl}\,\mathbf{u} \in (H^r(\Omega))^3$ *with* $0 \le r \le 1$*, there exists an extension operator* $\mathbf{E}$ *such that* $\mathbf{Eu}, \mathbf{curl}\,\mathbf{Eu} \in (H^r(\mathbb{R}^3))^3$*, and that*

$$\mathbf{Eu} = \mathbf{u} \quad \text{in } \Omega,$$

$$||\mathbf{Eu}||_{r,\mathbb{R}^3} + ||\mathbf{curl}\,\mathbf{Eu}||_{r,\mathbb{R}^3} \le c(||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r).$$

Proof. Since $\mathbf{curl}\,\mathbf{u} \in H(\mathrm{div}^{\,0};\Omega)$ and $\mathbf{curl}\,\mathbf{u} \in (H^r(\Omega))^3$, from [2] (see Remark 3.12 on page 47) there exists a vector potential $\mathbf{A} \in (H^{1+r}(\Omega))^3$ such that

$$\mathbf{curl}\,\mathbf{u} = \mathbf{curl}\,\mathbf{A}, \quad ||\mathbf{A}||_{1+r} \le c||\mathbf{curl}\,\mathbf{u}||_r.$$

Thus, we have a scalar potential $p \in H^1(\Omega)$ such that

$$\mathbf{u} = \mathbf{A} + \nabla p.$$

We may require $p \in H^1(\Omega)/\mathbb{R}$ to have a unique p and

$$||p||_0 \le c|p|_1 \le c(||\mathbf{u}||_0 + ||\mathbf{A}||_0) \le c(||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r).$$

It also follows that $p \in H^{1+r}(\Omega)$, satisfying

$$|p|_{1+r} \le c(||\mathbf{u}||_r + ||\mathbf{A}||_r) \le c(||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r),$$

and we have

$$||p||_{1+r} \le c(||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r).$$

In other words, for $\mathbf{u}, \mathbf{curl}\,\mathbf{u} \in (H^r(\Omega))^3$, we have the following regular-singular decomposition (cf. (2.6)-(2.7) for the case $r = 0$):

$$\mathbf{u} = \mathbf{A} + \nabla p, \quad \mathbf{A} \in (H^{1+r}(\Omega))^3, \quad p \in H^{1+r}(\Omega),$$

where

$$||\mathbf{A}||_{1+r} + ||p||_{1+r} \le c(||\mathbf{u}||_r + ||\mathbf{curl}\ \mathbf{u}||_r).$$

Let $E_1 p \in H^{1+r}(\mathbb{R}^3)$ and $\mathbf{E}_1\mathbf{A} \in (H^{1+r}(\mathbb{R}^3))^3$ be the classical extensions of p and $\mathbf{A}$, satisfying (see [29, 1])

$$E_1 p = p \quad \text{in}\ \Omega, \quad ||E_1 p||_{1+r,\mathbb{R}^3} \le c||p||_{1+r},$$

$$\mathbf{E}_1\mathbf{A} = \mathbf{A} \quad \text{in}\ \Omega, \quad ||\mathbf{E}_1\mathbf{A}||_{1+r,\mathbb{R}^3} \le c||\mathbf{A}||_{1+r}.$$

Define

$$\mathbf{Eu} = \mathbf{E}_1\mathbf{A} + \nabla E_1 p.$$

Summarizing the above, we have the conclusion. □

Let $\delta > 0$ be a constant to be determined. Introduce the usual mollifier ρ_δ: $\rho_\delta(x) = \delta^{-3}\rho(x/\delta)$, where $\rho(x) = c_\xi^{-1}\xi(x)$, $\xi(x) = \exp(|x|^2 - 1)^{-1}$ if $|x| < 1$, otherwise, $\xi(x) = 0$, where $c_\xi = \int_{\mathbb{R}^3} \xi$. Let $\mathbf{v} \in (H^r(\mathbb{R}^3))^3$. Assume that $\mathbf{v}$ is compactly supported in $\mathbb{R}^3$. Define the mollification of $\mathbf{v}$ by setting $\mathbf{J}_\delta\mathbf{v} := \rho_\delta * \mathbf{v} = \int_{\mathbb{R}^3} \rho_\delta(x-y)\mathbf{v}(y)dy$. This $\mathbf{J}_\delta\mathbf{v} \in (\mathcal{C}_0^\infty(\mathbb{R}^3))^3$. It is not difficult to show the following commuting property, approximation property and inverse estimate for $\mathbf{v}, \mathbf{curl}\ \mathbf{v} \in (H^r(\mathbb{R}^3))^3$ (also, cf., [1], Theorem 5.33 on page 160):

$$\mathbf{curl}\ \mathbf{J}_\delta\mathbf{v} = \mathbf{J}_\delta\mathbf{curl}\ \mathbf{v},$$

$$||\mathbf{v} - \mathbf{J}_\delta\mathbf{v}||_{0,\mathbb{R}^3} \le c\delta^r||\mathbf{v}||_{r,\mathbb{R}^3},$$

$$||\mathbf{J}_\delta\mathbf{v}||_{s,\mathbb{R}^3} \le c\delta^{t-s}||\mathbf{v}||_{t,\mathbb{R}^3},$$

where $0 \le t \le s$ are any two real numbers, and $t \le r$.

Theorem 6.3. *Let $\mathbf{u} \in H(\mathbf{curl}\,;\Omega)$ be the exact solution and $\mathbf{u}_h \in \mathbf{U}_h$ the finite element solution. Assume that $\mathbf{u}, \mathbf{curl}\ \mathbf{u} \in (H^r(\Omega))^3$ for some $0 \le r \le 1$. Then,*

$$||\mathbf{u} - \mathbf{u}_h||_{0,\mathbf{curl}} \le ch^{\frac{\ell r}{\ell+1}}(||\mathbf{u}||_r + ||\mathbf{curl}\ \mathbf{u}||_r).$$

Proof. From Lemma 6.3, first extend $\mathbf{u} \in (H^r(\Omega))^3$ with $\mathbf{curl}\ \mathbf{u} \in (H^r(\Omega))^3$ from Ω to the whole space $\mathbb{R}^3$, denoted by $\mathbf{Eu} \in (H^r(\mathbb{R}^3))^3$ with $\mathbf{curl}\ \mathbf{Eu} \in (H^r(\mathbb{R}^3))^3$, satisfying

$$\mathbf{Eu} = \mathbf{u} \quad \text{in}\ \Omega,$$

$$||\mathbf{Eu}||_{r,\mathbb{R}^3} + ||\mathbf{curl}\ \mathbf{Eu}||_{r,\mathbb{R}^3} \le c(||\mathbf{u}||_r + ||\mathbf{curl}\ \mathbf{u}||_r).$$

Denote by $(\mathbf{Eu})_\delta = \mathbf{J}_\delta\mathbf{Eu} = \rho_\delta * \mathbf{Eu} \in (\mathcal{C}_0^\infty(\mathbb{R}^3))^3$ as the mollified counterpart of $\mathbf{Eu}$, satisfying

$$\mathbf{curl}\ (\mathbf{Eu})_\delta = (\mathbf{curl}\ \mathbf{Eu})_\delta,$$

and

$$\begin{aligned} ||\mathbf{u} - (\mathbf{Eu})_\delta||_0 &= ||\mathbf{Eu} - (\mathbf{Eu})_\delta||_0 \\ &\le ||\mathbf{Eu} - (\mathbf{Eu})_\delta||_{0,\mathbb{R}^3} \\ &\le c\delta^r||\mathbf{Eu}||_{r,\mathbb{R}^3} \\ &\le c\delta^r(||\mathbf{u}||_r + ||\mathbf{curl}\ \mathbf{u}||_r), \end{aligned}$$

$$\begin{aligned} ||\mathbf{curl}\ (\mathbf{u} - (\mathbf{Eu})_\delta)||_0 &= ||\mathbf{curl}\ \mathbf{Eu} - (\mathbf{curl}\ \mathbf{Eu})_\delta||_0 \\ &\le ||\mathbf{curl}\ \mathbf{Eu} - (\mathbf{curl}\ \mathbf{Eu})_\delta||_{0,\mathbb{R}^3} \\ &\le c\delta^r||\mathbf{curl}\ \mathbf{Eu}||_{r,\mathbb{R}^3} \\ &\le c\delta^r(||\mathbf{u}||_r + ||\mathbf{curl}\ \mathbf{u}||_r). \end{aligned}$$

Let $\mathbf{I}_h$ denote the classical finite element interpolation operator in $\mathbf{U}_h$, which is determined by nodal-values (e.g., see [7]). Define

$$\mathbf{v}_h = \mathbf{I}_h(\mathbf{Eu})_\delta \in \mathbf{U}_h,$$

satisfying

$$||(\mathbf{Eu})_\delta - \mathbf{I}_h(\mathbf{Eu})_\delta||_0 + h||(\mathbf{Eu})_\delta - \mathbf{I}_h(\mathbf{Eu})_\delta||_1 \le ch^{\ell+1}||(\mathbf{Eu})_\delta||_{\ell+1},$$

where

$$||(\mathbf{Eu})_\delta||_{\ell+1} \le ||(\mathbf{Eu})_\delta||_{\ell+1,\mathbb{R}^3} \le c\delta^{r-\ell-1}||\mathbf{Eu}||_{r,\mathbb{R}^3} \le c\delta^{r-\ell-1}(||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r).$$

Hence,

$$\begin{aligned} ||\mathbf{u} - \mathbf{v}_h||_{0,\mathbf{curl}} &\le ||\mathbf{u} - (\mathbf{Eu})_\delta||_{0,\mathbf{curl}} + ||(\mathbf{Eu})_\delta - \mathbf{v}_h||_{0,\mathbf{curl}} \\ &\le c\delta^r(||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r) + ch^\ell\delta^{r-\ell-1}(||\mathbf{u}||_r + ||\mathbf{curl}\,\mathbf{u}||_r). \end{aligned}$$

Choosing

$$\delta^r = h^\ell\delta^{r-\ell-1},$$

from Theorem 6.2, we obtain the error bound as claimed. □

Remark 6.2. *In the absence of gradient fields of a scalar H^1-conforming finite element space, the rate of convergence in Theorem 6.3 for the finite element solution $\mathbf{u}_h$ is the best that can be attained in $H(\mathbf{curl}\,;\Omega)$-norm for all $0 \le r \le 1$ and for all $\ell \ge 1$. Unless the finite element spaces contain the gradient fields of a scalar H^1-conforming finite element space, no existing methods in the literature can reach the optimal rate r in $H(\mathbf{curl}\,;\Omega)$-norm, to the authors' knowledge.*

To recover the optimal value r, one has to enrich $\mathbf{U}_h$ with the gradient fields of an H^1-conforming finite element space. There are four ways for achieving that goal.

The first way is to use the gradients of a scalar H^2-conforming C^1 element to enrich $\mathbf{U}_h$, and the enriched is still H^1-conforming. Thus, one can construct a finite element interpolation $\mathbf{v}_h = \mathbf{u}_h^* + \nabla p_h^*$ from the regular-singular decomposition $\mathbf{u} = \mathbf{u}^* + \nabla p^*$ for the solution $\mathbf{u}, \mathbf{curl}\,\mathbf{u} \in (H^r(\Omega))^3$, where $\mathbf{u}^* \in (H^{1+r}(\Omega))^3$ and $p^* \in H^{1+r}(\Omega)$. See Lemma 6.3 for an example of this type of regular-singular decomposition. From Theorem 6.2, then, it is not difficult to establish the optimal error bound $\mathcal{O}(h^r)$.

The second way is to artificially enrich $\mathbf{U}_h$ by the gradient fields of a scalar H^1-conforming C^0 element. The enriched space is no longer H^1-conforming, but, of course, it is still $H(\mathbf{curl}\,;\Omega)$-conforming. We use again the regular-singular decomposition $\mathbf{u} = \mathbf{u}^* + \nabla p^*$ to have the optimal error bound $\mathcal{O}(h^r)$.

The third way is simply to define $\mathbf{U}_h$ as the $H(\mathbf{curl}\,;\Omega)$-conforming only Nédélec element, which usually contains the gradient fields of a scalar H^1-conforming C^0 element. The optimal error bound $\mathcal{O}(h^r)$ follows, as is a classical result.

The fourth way is to use the composite meshes such as the Clough-Tocher/Alfeld macro meshes. Then, $\mathbf{U}_h$ can contain the gradients of a scalar H^2-conforming C^1 element, and the optimal r order can be restored, as done in [34], [33].

Remark 6.3. *For smooth $\mathbf{u} \in (H^{1+t}(\Omega))^3$ for $t \ge 0$, applying the classical finite element analysis (cf., [7]), from Theorem 6.2 one can have the error bound $||\mathbf{u} - \mathbf{u}_h||_{0,\mathbf{curl}} \le ch^{\min(t,\ell)}||\mathbf{u}||_{1+t}$, which is optimal in $H(\mathbf{curl}\,;\Omega)$-norm for $t = \ell$, the same as the order of approximation. No gradient fields are needed in the above remark.*

7 Implementation issue

In this section, we address the implementation of the finite element method proposed in section 4. There are two ways for solving the finite element problem:

- To realize (4.28)-(4.30) by a symmetric positive definite system.
- To realize (4.28)-(4.30) by a mixed system.

7.1 Implementation by a symmetric positive definite system

We first consider the first way.

We have seen that the finite element method involves the computations of four liftings. Each lifting needs to solve the *linear* finite element solution of the Poisson equation.

7.1.1 How to compute the duality and the liftings

Once the definitions of these functionals $\mathbf{f} \in (H^{-1}(\Omega))^3$, $\boldsymbol{\chi} \in (H^{-1/2}(\Gamma))^3$, $g \in H^{-1}(\Omega)$ and $\kappa \in H^{-1/2}(\Gamma)$ are given, we solve the corresponding linear finite element problems (4.6)-(4.9). When a functional is given, say $g \in H^{-1}(\Omega)$, it means that the action on the $H_0^1(\Omega)$ space is concretely prescribed, i.e., the formula of $\langle g, q\rangle$ for $q \in H_0^1(\Omega)$ is available. Generally, we have

$$\langle g, q\rangle = (g^0, q) + (\mathbf{g}^1, \nabla q),$$

where g^0 and $\mathbf{g}^1$ are known L^2-functions. For $q_h \in Q_h \subset H_0^1(\Omega)$, then we have the value of $\langle g, q_h\rangle$, and we have the right-hand side vector. Similarly, when we are given a $\kappa \in H^{-1/2}(\Gamma)$, it means that the formula of $\langle \kappa, \vartheta\rangle_\Gamma$ for $\vartheta \in H^{1/2}(\Gamma)$ or for $\vartheta \in H^1(\Omega)$ is known. Usually,

$$\langle \kappa, \vartheta\rangle_\Gamma = \langle \boldsymbol{\xi} \cdot \mathbf{n}, \vartheta\rangle_\Gamma = (\boldsymbol{\xi}, \nabla\vartheta) + (\operatorname{div} \boldsymbol{\xi}, \vartheta),$$

where $\boldsymbol{\xi}$ is a known vector function in $H(\operatorname{div}; \Omega)$. We then obtain the right-hand side vector from the formula of $\langle \kappa, \vartheta_h\rangle$ for $\vartheta_h \in \Psi_h \subset H^1(\Omega)$. For all these functionals $\mathbf{f} \in (H^{-1}(\Omega))^3$, $\boldsymbol{\chi} \in (H^{-1/2}(\Gamma))^3$, $g \in H^{-1}(\Omega)$ and $\kappa \in H^{-1/2}(\Gamma)$, the corresponding right-hand side vectors are obtained from the formula of the definitions of those functionals applying to the finite element spaces.

Solving (4.6)-(4.9), we will obtain four matrix operators of the linear element. The implementation is as follows. For any $\mathbf{v}_h \in \mathbf{U}_h$, put

$$\mathbf{f} := \mathbf{curl}\,\mathbf{curl}\,\mathbf{v}_h - \lambda\mathbf{v}_h \in (H^{-1}(\Omega))^3, \quad g := \operatorname{div} \mathbf{v}_h \in H^{-1}(\Omega),$$

$$\boldsymbol{\chi} := \mathbf{n} \times \mathbf{v}_h \in (H^{-\frac{1}{2}}(\Gamma))^3, \quad \kappa := \mathbf{curl}\,\mathbf{v}_h \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\Gamma).$$

For example, we obtain $\mathbf{R}_h(\mathbf{f}) \in \mathbf{Z}_h$ by solving the linear finite element problem

$$(\mathbf{R}_h(\mathbf{f}), \mathbf{z}_h)_{0,\nabla} = \langle \mathbf{f}, \mathbf{z}_h\rangle = (\mathbf{curl}\,\mathbf{v}_h, \mathbf{curl}\,\mathbf{z}_h) - \lambda(\mathbf{v}_h, \mathbf{z}_h) \quad \forall \mathbf{z}_h \in \mathbf{Z}_h.$$

To see the matrix form of $\mathbf{R}_h(\mathbf{f})$, we introduce the finite dimensional basis of $\mathbf{U}_h$ and $\mathbf{Z}_h$: $\mathbf{U}_h = (\operatorname{span}\{b_i^U, 1 \le i \le N^U\})^3$ and $\mathbf{Z}_h = (\operatorname{span}\{b_i^Z, 1 \le i \le N^Z\})^3$, where b_i^U, b_i^Z denote the global Lagrange basis functions. Denote by $\mathbf{v}_h = \sum_{i=1}^{N^U} \mathbf{c}_i b_i^U$ with $\mathbf{c}_i = (c_{1i}, c_{2i}, c_{3i})$ and $\mathbf{R}_h(\mathbf{f}) = \sum_{i=1}^{N^Z} \mathbf{d}_i b_i^Z$ with $\mathbf{d}_i = (d_{1i}, d_{2i}, d_{3i})$. Set $\mathbf{c} \in \mathbb{R}^{3N^U}$ as the coefficient column vector of $\mathbf{v}$, and similarly, set $\mathbf{d} \in \mathbb{R}^{3N^Z}$ as the coefficient column vector of $\mathbf{R}_h(\mathbf{f})$. We have, with $\mathbb{M}$ being the left-hand side matrix and $\mathbb{N}$ the right-hand side matrix,

$$\mathbb{M}\mathbf{d} = \mathbb{N}\mathbf{c}, \quad \mathbb{M} \in \mathbb{R}^{3N^Z \times 3N^Z}, \quad \mathbb{N} \in \mathbb{R}^{3N^Z \times 3N^U}.$$

Note that $\mathbb{M}$ is the resulting matrix from $\mathbf{Z}_h \subset (H_0^1(\Omega))^3$, the product of the linear elements. $\mathbb{M}$ is symmetric positive definite. Similarly, we obtain $\boldsymbol{\Upsilon}_h^\Gamma(\boldsymbol{\chi}) \in \mathbf{Y}_h$ by solving the linear finite element problem

$$(\boldsymbol{\Upsilon}_h^\Gamma(\boldsymbol{\chi}), \mathbf{y}_h)_{0,\nabla} = \langle \boldsymbol{\chi}, \mathbf{n} \times (\mathbf{y}_h \times \mathbf{n})\rangle_\Gamma = (\mathbf{curl}\,\mathbf{v}_h, \mathbf{y}_h) - (\mathbf{v}_h, \mathbf{curl}\,\mathbf{y}_h) \quad \forall \mathbf{y}_h \in \mathbf{Y}_h,$$

and obtain $S_h(g) \in Q_h$ by solving the linear finite element problem

$$(S_h(g), q_h)_{0,\nabla} = \langle g, q_h\rangle = (\operatorname{div} \mathbf{v}_h, q_h) \quad \forall q_h \in Q_h,$$

and obtain $\Lambda_h^\Gamma(\kappa) \in \Psi_h$ by solving the linear finite element problem

$$(\Lambda_h^\Gamma(\kappa), \vartheta_h)_{0,\nabla} = \langle \kappa, \vartheta_h\rangle_\Gamma = (\mathbf{curl}\,\mathbf{v}_h, \nabla\vartheta_h) \quad \forall \vartheta_h \in \Psi_h.$$

7.1.2 A symmetric positive definite system

Letting $\mathbf{u}_h = \sum_{i=1}^{N^U} \boldsymbol{\alpha}_i b_i^U$, with $\boldsymbol{\alpha} \in \mathbb{R}^{3N^U}$ being the coefficient column vector, from the bilinear form $\mathcal{L}_h$ in (4.29), with the computed liftings in the above, we can obtain

$$\mathcal{L}_h(\mathbf{u}_h, \mathbf{v}_h) = \mathbf{c}'\mathbb{A}\boldsymbol{\alpha},$$

where $\mathbb{A} \in \mathbb{R}^{3N^U \times 3N^U}$ is a symmetric positive definite matrix. Similarly, we can obtain the right-hand side vector from (4.30), i.e., we have

$$\mathcal{G}(\mathbf{v}_h) = \mathbf{c}'\boldsymbol{\eta}, \quad \boldsymbol{\eta} \in \mathbb{R}^{3N^U}.$$

From (4.28), the resultant system is $\mathbb{A}\boldsymbol{\alpha} = \boldsymbol{\eta}$. This is a symmetric positive definite system.

7.2 Implementation by a mixed system

Next, we consider the second way: an expanded mixed problem.

This way lies in that all the Riesz-lifting solutions in (4.6)-(4.9) and the solution $\mathbf{u}_h$ in (4.28) are solved simultaneously from a mixed problem.

Introduce $\mathbf{w}_h \in \mathbf{Z}_h, p_h \in Q_h, \mathbf{d}_h \in \mathbf{Y}_h, \gamma_h \in \Psi_h$, which are defined as follows:

$$\mathbf{w}_h := \mathbf{R}_h(\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}_h - \lambda\mathbf{u}_h), \quad p_h := S_h(\mathrm{div}\,\mathbf{u}_h), \quad \mathbf{d}_h := \boldsymbol{\Upsilon}_h^\Gamma(\mathbf{n}\times\mathbf{u}_h), \quad \gamma_h := \Lambda_h^\Gamma(\mathbf{curl}\,\mathbf{u}_h\cdot\mathbf{n}),$$

and introduce $\mathbf{f}_h \in \mathbf{Z}_h, g_h \in Q_h, \boldsymbol{\chi}_h \in \mathbf{Y}_h, \rho_h \in \Psi_h$, which are defined as follows:

$$\mathbf{f}_h := \mathbf{R}_h(\mathbf{f}), \quad g_h := S_h(g), \quad \boldsymbol{\chi}_h := \boldsymbol{\Upsilon}_h^\Gamma(\boldsymbol{\chi}), \quad \rho_h := \Lambda_h^\Gamma(\mathbf{curl}\,\mathbf{u}^*\cdot\mathbf{n}),$$

where $\mathbf{u}^* \in H(\mathbf{curl}\,;\Omega)$ is any function that satisfies $\mathbf{n}\times\mathbf{u}^* = \boldsymbol{\chi}$ on Γ. Note that we do not need to really know what $\mathbf{u}^*$ is. The introduction of $\mathbf{u}^*$ is only for giving the meaning of ρ_h. Then, from (4.28)-(4.30), we state the mixed problem: Find

$$\mathbf{u}_h \in \mathbf{U}_h, \quad \mathbf{w}_h \in \mathbf{Z}_h, \quad p_h \in Q_h, \quad \mathbf{d}_h \in \mathbf{Y}_h, \quad \gamma_h \in \Psi_h$$

and

$$\mathbf{f}_h \in \mathbf{Z}_h, \quad g_h \in Q_h, \quad \boldsymbol{\chi}_h \in \mathbf{Y}_h, \quad \rho_h \in \Psi_h,$$

such that

$$\begin{aligned}
&(\mathbf{curl}\,\mathbf{w}_h, \mathbf{curl}\,\mathbf{v}_h) - \lambda(\mathbf{w}_h,\mathbf{v}_h) - (\mathbf{v}_h,\nabla p_h) + (\mathbf{n}\times\mathbf{v}_h,\mathbf{d}_h)_\Gamma + (\mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{n},\gamma_h)_\Gamma + \mathscr{S}_h(\mathbf{u}_h,\mathbf{v}_h)\\
&-(\mathbf{curl}\,\mathbf{v}_h,\mathbf{curl}\,\mathbf{f}_h) + \lambda(\mathbf{v}_h,\mathbf{f}_h) + (\mathbf{v}_h,\nabla g_h) - (\mathbf{n}\times\mathbf{v}_h,\boldsymbol{\chi}_h)_\Gamma - (\mathbf{curl}\,\mathbf{v}_h\cdot\mathbf{n},\rho_h)_\Gamma = \mathscr{R}_h(\mathbf{f},g,\boldsymbol{\chi};\mathbf{v}_h),\\
&(\mathbf{w}_h,\mathbf{z}_h)_{0,\nabla} - (\mathbf{curl}\,\mathbf{u}_h,\mathbf{curl}\,\mathbf{z}_h) + \lambda(\mathbf{u}_h,\mathbf{z}_h) = 0,\\
&(p_h,q_h)_{0,\nabla} + (\mathbf{u}_h,\nabla q_h) = 0,\\
&(\mathbf{d}_h,\mathbf{y}_h)_{0,\nabla} - (\mathbf{n}\times\mathbf{u}_h,\mathbf{n}\times(\mathbf{y}_h\times\mathbf{n}))_\Gamma = 0,\\
&(\gamma_h,\vartheta_h)_{0,\nabla} - (\mathbf{curl}\,\mathbf{u}_h\cdot\mathbf{n},\vartheta_h)_\Gamma = 0,\\
&(\mathbf{f}_h,\widetilde{\mathbf{z}}_h)_{0,\nabla} = \langle\mathbf{f},\widetilde{\mathbf{z}}_h\rangle,\\
&(g_h,\widetilde{q}_h)_{0,\nabla} = \langle g,\widetilde{q}_h\rangle,\\
&(\boldsymbol{\chi}_h,\widetilde{y}_h)_{0,\nabla} = \langle\boldsymbol{\chi},\widetilde{y}_h\rangle_\Gamma,\\
&(\rho_h,\widetilde{\vartheta}_h)_{0,\nabla} = \langle\boldsymbol{\chi},\mathbf{n}\times(\nabla\widetilde{\vartheta}_h\times\mathbf{n})\rangle_{\Gamma,*}.
\end{aligned}$$

for all $\mathbf{v}_h \in \mathbf{U}_h, \ \mathbf{z}_h \in \mathbf{Z}_h, \ q_h \in Q_h, \ \mathbf{y}_h \in \mathbf{Y}_h, \ \vartheta_h \in \Psi_h$ and for all $\widetilde{\mathbf{z}}_h \in \mathbf{Z}_h, \ \widetilde{q}_h \in Q_h, \ \widetilde{\mathbf{y}}_h \in \mathbf{Y}_h, \ \widetilde{\vartheta}_h \in \Psi_h$.

In both ways, the total scalar unknowns are 19 (For two-dimensional problem, the total scalar unknowns are 10), and both ways involve the same computational cost. On the other hand, the coding for the mixed system is relatively easier.

8 Numerical experiments

In this section, we provide some numerical experiments for illustrating the FEM (4.28) and the theoretical results of error estimates. For ease of implementation we consider a two-dimensional problem of Maxwell equations over a square domain $\Omega = (0,1)^2$. We use the mixed formulation as formulated in the previous section to simultaneously solve all unknowns.

In all the numerical experiments, the exact solutions are known, either singular or smooth. If not indicated, $\lambda = 1$ is chosen. The singular solutions are in $(H^{1/2-\epsilon}(\Omega))^2$ and $(H^{2/3-\epsilon}(\Omega))^2$, respectively, for any small $\epsilon > 0$. The smooth solution is infinitely smooth. Since the exact solution $\mathbf{u}$ is known, all the data $\mathbf{f}$ and $\boldsymbol{\chi}$ are computed through (1.1) and (1.2), with $g := -\lambda^{-1}\mathrm{div}\,\mathbf{f}$. If the data are in negative-order Sobolev spaces, the data should be computed through their dual products in the following way (see also the previous section):

$$\langle\mathbf{f},\mathbf{z}\rangle = (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{z}) - \lambda(\mathbf{u},\mathbf{z}), \quad \langle g,q\rangle = -(\mathbf{u},\nabla q), \quad \langle\boldsymbol{\chi},\mathbf{y}\rangle_\Gamma = (\mathbf{curl}\,\mathbf{u},\mathbf{y}) - (\mathbf{u},\mathbf{curl}\,\mathbf{y}).$$

For the singular boundary data and the singular right-hand side data in these examples, we may also use the Gaussian quadrature rules with sufficiently many Gaussian points in the vicinity of the origin. With the Gaussian quadrature rules we used fourteen Gaussian points in the computation of the line integrals and nineteen Gaussian points in the computation of of the triangle element integrals[1] If the data are in L^2 spaces, the dual products are replaced by the L^2 inner products, e.g., $\langle \boldsymbol{\chi}, \mathbf{y}\rangle_\Gamma = (\boldsymbol{\chi}, \mathbf{y})_\Gamma$. Some of the singular solutions are the gradients of scalar functions while others are not. In fact, any $H(\mathbf{curl}\,;\Omega)$ function can always be decomposed into the sum of a regular part in the H^1 space and a singular part which is the gradient of a scalar function which is in the H^1 space(see section 2 for various decompositions).

We consider the linear element (denoted as $\mathcal{P}_1$ element) and the quadratic element (denoted as $\mathcal{P}_2$ element) in these experiments, in order to verify the theoretical results as stated in Theorem 6.3 for singular solution and Remark 6.3 for smooth solution. The mesh consists of uniform triangles.

8.1 Example 1

We take

$$\mathbf{u} = (-1/2\rho^{-1/2}\sin(\theta/2), 1/2\rho^{-1/2}\cos(\theta/2)),$$

where (ρ,θ) are the polar coordinates system in the plane, ρ the distance function and θ the opening angle from the origin. Such $\mathbf{u} \in (H^{1/2-\epsilon}(\Omega))^2$ where ϵ is any small positive number less than $1/2$, and $\mathbf{curl}\,\mathbf{u} = 0$.

The computed errors of $\mathcal{P}_1$ and $\mathcal{P}_2$ elements are reported in Table 1 and Table 2, respectively. The theoretical order in $||\cdot||_{0,\mathbf{curl}}$ -norm for $\mathcal{P}_1$ element is $r\ell/(\ell+1) = (1/2-\epsilon)/2 \approx 0.25$, while for $\mathcal{P}_2$ element $r\ell/(\ell+1) = (1/2-\epsilon)2/3 \approx 0.33$. From Tables 1 and 2, we see that the computed order is approximately consistent with the predicted. We also see that for the $\mathcal{P}_2$ element, the convergence rate in Table 2 seems to be close to optimal $1/2$. The reason is not clear so far.

Table 1: L^2, $H(\mathbf{curl}\,)$ errors of $\mathbf{u}_h$ for $\mathcal{P}_1$ element: $H^{1/2-\epsilon}$ solution

h	1/4	1/8	1/16	1/32	1/64	1/128
$\|\mathbf{u}-\mathbf{u}_h\|_0$	0.13678493	0.13218979	0.12194860	0.10641319	0.09035986	0.07622981
order	—	0.04929857	0.11633749	0.19659624	0.23592299	0.24532686
$\|\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\|\|_0$	0.04517498	0.04087683	0.03656654	0.02899549	0.02160393	0.01617408
order	—	0.14424071	0.16075919	0.33469559	0.42453508	0.41760948

Table 2: L^2, $H(\mathbf{curl}\,)$ errors of $\mathbf{u}_h$ for $\mathcal{P}_2$ element: $H^{1/2-\epsilon}$ solution

h	1/4	1/8	1/16	1/32	1/64
$\|\mathbf{u}-\mathbf{u}_h\|_0$	0.08945654	0.06765027	0.05022329	0.03702763	0.02719256
order	—	0.40309141	0.42973904	0.43975442	0.44539005
$\|\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\|\|_0$	0.04511309	0.03207375	0.02276016	0.01611810	0.01140330
order	—	0.49215306	0.49488262	0.49782858	0.49923029

8.2 Example 2

We take

$$\mathbf{u} = (-2/3\rho^{-1/3}\sin(\theta/3), 2/3\rho^{-1/3}\cos(\theta/3)).$$

Such $\mathbf{u} \in (H^{2/3-\epsilon}(\Omega))^2$, and $\mathbf{curl}\,\mathbf{u} = 0$. The computed errors of $\mathcal{P}_1$ and $\mathcal{P}_2$ elements are reported in Table 3 and Table 4, respectively. The theoretical order in $||\cdot||_{0,\mathbf{curl}}$ -norm for $\mathcal{P}_1$ and $\mathcal{P}_2$ elements is respectively $r\ell/(\ell+1) = (2/3-\epsilon)/2 \approx 0.33$ and approximately $r\ell/(\ell+1) = (2/3-\epsilon)2/3 \approx 0.44$. From

[1] D. A. Dunavant, High degree efficient symmetrical Gaussian quadrature rules for the triangle, Internat. J. Numer. Methods Engrg., 21(1985), pp. 1129-1148.

Tables 3 and 4, we see that the computed order is approximately consistent with the predicted. Likewise, we see again that the convergence rate in Table 4 for the $\mathcal{P}_2$ element seems to tend to optimal $2/3$.

Table 3: L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h$ for $\mathcal{P}_1$ element: $H^{2/3-\epsilon}$ solution

h	1/4	1/8	1/16	1/32	1/64	1/128
$\|\mathbf{u}-\mathbf{u}_h\|_0$	0.08695579	0.08165994	0.07210830	0.05921238	0.04724758	0.03760512
order	—	0.09065353	0.17946315	0.28426644	0.32565849	0.32931133
$\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\|_0$	0.02324863	0.02625699	0.02426743	0.01779159	0.01214099	0.00864408
order	—	-0.17555636	0.11368023	0.44782644	0.55130887	0.49010182

Table 4: L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h$ for $\mathcal{P}_2$ element: $H^{2/3-\epsilon}$ solution

h	1/4	1/8	1/16	1/32	1/64
$\|\mathbf{u}-\mathbf{u}_h\|_0$	0.04376512	0.02907927	0.01898241	0.01231933	0.00797417
order	—	0.58979051	0.61532817	0.62373931	0.62751718
$\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\|_0$	0.01773116	0.01124060	0.00712104	0.00449736	0.00283577
order	—	0.65756728	0.65856002	0.66300783	0.66534020

8.3 Example 3

We consider two exact solutions which are not **curl** -free, i.e., $\mathbf{curl}\,\mathbf{u} \neq 0$. Such solution is not gradient. To construct such solution, let $\mathbf{u} = (u_1, u_2)$ be the exact solution of subsection 8.1. The first non-gradient solution is

$$\mathbf{u}^* := \mathbf{u} + (\sin(y), \sin(x)).$$

In addition, we consider the second non-gradient singular solution:

$$\mathbf{u}^{**} := (u_2, -u_1).$$

The two functions $\mathbf{u}^*$ and $\mathbf{u}^{**}$ are no longer gradients of scalar functions. They are still singular; both $\mathbf{u}^*$ and $\mathbf{u}^{**}$ have the same regularity as $\mathbf{u}$. We use the $\mathcal{P}_1$ element. With the finite element solutions respectively denoted by $\mathbf{u}_h^*$ and $\mathbf{u}_h^{**}$, the computed errors are reported in Tables 5 and 6. From these tables, we see that the computed orders are still approximately consistent with the predicted.

Table 5: L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h^*$

h	1/4	1/8	1/16	1/32	1/64	1/128
$\|\mathbf{u}^*-\mathbf{u}_h^*\|_0$	0.15297168	0.13044549	0.12004545	0.10600884	0.09034701	0.07623370
order	—	0.22981758	0.11986630	0.17939613	0.23063581	0.24504795
$\|\mathbf{curl}\,(\mathbf{u}^*-\mathbf{u}_h^*)\|_0$	0.14092891	0.06301626	0.03577231	0.02831519	0.02156276	0.01616831
order	—	1.16117151	0.81688082	0.33726693	0.39303439	0.41537315

Table 6: L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h^{**}$

h	1/4	1/8	1/16	1/32	1/64	1/128
$\|\mathbf{u}^{**}-\mathbf{u}_h^{**}\|_0$	0.16383283	0.15626837	0.13598042	0.11231068	0.09415977	0.08000757
order	—	0.06819872	0.20062689	0.27590375	0.25431236	0.23497445
$\|\mathbf{curl}\,(\mathbf{u}^{**}-\mathbf{u}_h^{**})\|_0$	0.05148881	0.06424895	0.05297506	0.03206539	0.02022473	0.01519387
order	—	-0.31941398	0.27835953	0.72429652	0.66489654	0.41263111

8.4 Example 4

We take a smooth solution

$$\mathbf{u} = (-\sin(2\pi y)(\cos(2\pi x) - 1), \sin(2\pi x)(\cos(2\pi y) - 1))$$

to test the proposed method, and also compare the computational results between the proposed method and compared with the standard method $(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v}) - (\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - (\mathrm{div}\,\mathbf{f}, \mathrm{div}\,\mathbf{v})$, still with $\lambda = 1$. The computed errors of $\mathcal{P}_1$ and $\mathcal{P}_2$ elements of the standard formulation are reported in Table 7 and Table 8, respectively, while for the proposed method by Table 7′ and Table 8′. The theoretical order in $||\cdot||_{0,\mathbf{curl}}$ -norm for $\mathcal{P}_1$ and $\mathcal{P}_2$ elements is respectively $\ell = 1$ and $\ell = 2$. From Tables 7 and 8, the computed orders are optimal for the standard method in both L^2 norm and $H(\mathbf{curl}\,)$ norm. From Tables 7′ and 8′, the computed orders are about one and two, which have confirmed the predicted. In comparison with the standard method, there is a loss of one order in L^2 norm for the proposed method, but the $H(\mathbf{curl}\,)$ norm convergence orders are indeed optimal. The reason may be due to the fact that the standard method provides a much stronger stability in $||\cdot||_{0,\mathbf{curl}\,,\mathrm{div}}$ norm. It could thus be expected that the L^2 norm convergence order is higher by one order, just like the Poisson Dirichlet problem of Laplace operator. However, the standard method cannot correctly approximate singular solutions, as is well-known.

Table 7: (Standard method) L^2, $H(\mathbf{curl}\,)$ errors of $\mathbf{u}_h$ for $\mathcal{P}_1$ element: smooth solution

h	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$	$1/128$
$\\|\mathbf{u}-\mathbf{u}_h\\|_0$	0.43669559	0.13597740	0.03617140	0.00918962	0.00230677	0.00057728
order	—	1.68326108	1.91044569	1.97677290	1.99413171	1.99852865
$\\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\\|_0$	4.53862691	2.29056553	1.13284587	0.56400690	0.28166763	0.14079067
order	—	0.98655206	1.01575225	1.00616687	1.00171906	1.00044208

Table 7′: (Proposed method) L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h$ for $\mathcal{P}_1$ element: smooth solution

h	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$	$1/128$
$\\|\mathbf{u}-\mathbf{u}_h\\|_0$	1.08729924	0.65517320	0.37864667	0.28284629	0.19960536	0.11025138
order	—	0.73080081	0.79102409	0.42083400	0.50286781	0.85635372
$\\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\\|_0$	7.33457185	4.46214982	1.99958157	0.75975236	0.30417988	0.13947827
order	—	0.71697380	1.15804082	1.39609698	1.32060454	1.12488430

Table 8: (Standard method) L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h$ for $\mathcal{P}_2$ element: smooth solution

h	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$
$\\|\mathbf{u}-\mathbf{u}_h\\|_0$	0.05274715	0.00674109	0.00084982	0.00010652	0.00001332
order	—	2.96803942	2.98774487	2.99610309	2.99891081
$\\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\\|_0$	1.13338784	0.29795247	0.07477305	0.01866310	0.00465970
order	—	1.92748753	1.99449191	2.00232993	2.00187800

Table 8′: (Proposed method) L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h$ for $\mathcal{P}_2$ element: smooth solution

h	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$
$\\|\mathbf{u}-\mathbf{u}_h\\|_0$	0.30669052	0.04445678	0.00552012	0.00098352	0.00023148
order	—	2.78630831	3.00963054	2.48868018	2.08708815
$\\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\\|_0$	1.59586550	0.35138165	0.07536291	0.01808351	0.00449989
order	—	2.18322831	2.22111233	2.05917953	2.00671427

8.5 Example 5

In this subsection, we consider different values of λ to know about how λ affects the convergence rate. We take the exact solution of subsection 8.1 with the use of the $\mathcal{P}_1$ element and consider several values $\lambda = 1, 15, 30, 45$. As can be seen from the results in Table 9, the higher the wavenumbers λ are, the more the convergence orders deteriorate accordingly. Such issue has been well-known for Helmholtz-type equations with high wavenumber, and it deserves further studies, but it is beyond the scope of this paper.

Table 9: L^2, $H(\mathbf{curl}\,)$ errors of $\mathbf{u}_h$ for different λs

λ	h	1/4	1/8	1/16	1/32	1/64	1/128
1	$\|\mathbf{u}-\mathbf{u}_h\|_0$	0.13678493	0.13218979	0.12194860	0.10641319	0.09035986	0.07622981
	order	—	0.04929857	0.11633749	0.19659624	0.23592299	0.24532686
	$\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\|_0$	0.04517498	0.04087683	0.03656654	0.02899549	0.02160393	0.01617408
	order	—	0.14424071	0.16075919	0.33469559	0.42453508	0.41760948
15	$\|\mathbf{u}-\mathbf{u}_h\|_0$	0.09375371	0.10251241	0.10514734	0.10111516	0.09173078	0.07947803
	order	—	-0.12885093	-0.03661376	0.05641300	0.14052155	0.20684980
	$\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\|_0$	0.11340742	0.08103533	0.07691022	0.08884497	0.10040281	0.09554489
	order	—	0.48489215	0.07537557	-0.20811476	-0.17643757	0.07154905
30	$\|\mathbf{u}-\mathbf{u}_h\|_0$	0.07476354	0.08394423	0.08943865	0.09011749	0.08583046	0.07647305
	order	—	-0.16709626	-0.09146730	-0.01090870	0.07031744	0.16653836
	$\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\|_0$	0.21203683	0.15599929	0.11387433	0.10047165	0.09965822	0.08356459
	order	—	0.44277543	0.45409687	0.18065411	0.01172779	0.25409704
45	$\|\mathbf{u}-\mathbf{u}_h\|_0$	0.06553182	0.07359882	0.08010152	0.08241766	0.08061779	0.07448523
	order	—	-0.16748707	-0.12214695	-0.04112389	0.03185520	0.11414383
	$\|\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\|_0$	0.26848267	0.20101306	0.14180167	0.10771165	0.09697876	0.09547564
	order	—	0.41753971	0.50341473	0.39670019	0.15143361	0.02253612

8.6 Example 6

We consider a case in which the right-hand side $\mathbf{f}$ is singular. We also compute the residual of the divergence of the difference between the finite element solution and the exact solution. For this purpose, let $p = \rho^{1/2}\sin(\theta/2)$ which has the regularity of $H^{3/2-\epsilon}(\Omega)$ for any small positive number ϵ less than $1/2$, and let $p^\diamondsuit := 100\sin(x+1)\sin(y+1)p$. Consider

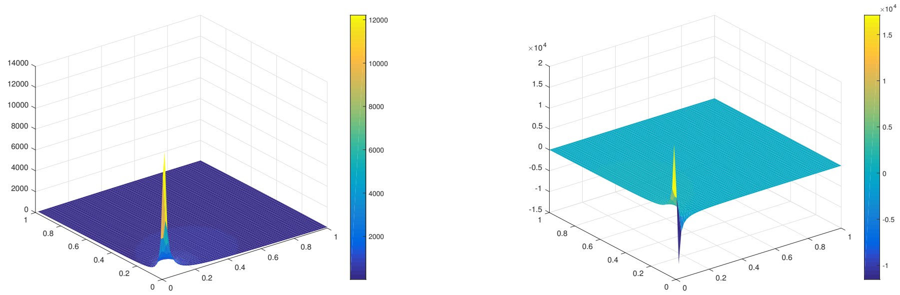
$$\mathbf{u}^\diamondsuit = \mathbf{curl}\, p^\diamondsuit = (\partial p^\diamondsuit/\partial y, -\partial p^\diamondsuit/\partial x).$$

Then, compute $\mathbf{f} = \mathbf{curl}\,\mathrm{curl}\,\mathbf{u}^\diamondsuit - \mathbf{u}^\diamondsuit$, still with $\lambda = 1$, and it follows that $\mathbf{f} \in (H^{-1/2-\epsilon}(\Omega))^2$. The singularity of $\mathbf{f}$ can be seen from the figures for the two components of $\mathbf{f} = (f_1, f_2)$ which are shown by Figures 1a and 1b in Figure 1.

We use the $\mathcal{P}_1$ element for the finite element solution denoted by $\mathbf{u}_h^\diamondsuit$ and report the numerical results in Table. Upon the interest of one of the referees of this paper, we in addition report the numerical results for the residuals of the divergence of the difference between the finite element solution and the exact solution in the norm $||\mathrm{div}\,\mathbf{u}_h^\diamondsuit||_{-1,h} := \sqrt{||S_h(\mathrm{div}\,\mathbf{u}_h^\diamondsuit)||^2_{0,\nabla} + \sum_{K\in\mathcal{T}_h} h_K^2 ||\mathrm{div}\,\mathbf{u}_h^\diamondsuit||^2_{0,K}}$. It can be shown that $||\mathrm{div}\,\cdot||_{-1,h}$ is equivalent to the H^{-1} norm $||\mathrm{div}\,\cdot||_{-1}$ over $\mathbf{U}_h$. Note that the exact solution satisfies $\mathrm{div}\,\mathbf{u}^\diamondsuit = 0$. The errors are reported in Table 10. The convergence order is about 0.25 as predicted. In addition, the $||\mathrm{div}\,\cdot||_{-1,h}$ norm of the finite element solution converges with about the same order, namely, $\mathrm{div}\,\mathbf{u}_h^\diamondsuit$ converges to $\mathrm{div}\,\mathbf{u}^\diamondsuit$ in the norm $||\cdot||_{-1}$ with about the same order.

8.7 Discussion on stabilization

In this subsection, we report some numerical results from the effects of the stabilizations. We take the exact solution from subsection 8.1. The regularity of the exact solution is $1/2-\epsilon$ for any small number



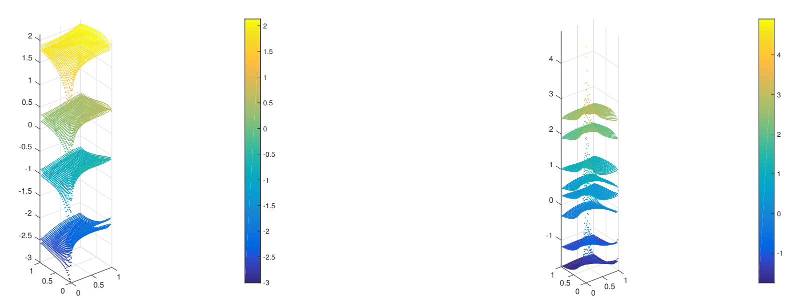
(a) Component f_1 (b) Component f_2

Figure 1: Singular right-hand side $\mathbf{f} = (f_1, f_2)$ of Example 6 in subsection 8.6

Table 10: L^2, $H(\mathbf{curl})$, $||\mathrm{div}\,||_{-1,h}$ errors of $\mathbf{u}_h^{\diamondsuit}$ with singular right-hand side in Example 6

h	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$
$\lvert\lvert\mathbf{u}^{\diamondsuit} - \mathbf{u}_h^{\diamondsuit}\rvert\rvert_0$	34.10050339	15.87895996	10.13679246	8.05924726	6.70825614
order	—	1.10267661	0.64751520	0.33088422	0.26470732
$\lvert\lvert\mathbf{curl}\,(\mathbf{u}^{\diamondsuit} - \mathbf{u}_h^{\diamondsuit})\rvert\rvert_0$	49.43313171	23.46731781	11.18508758	6.54473633	4.21271455
order	—	1.07482535	1.06907641	0.77316958	0.63558490
$\lvert\lvert\mathrm{div}\,\mathbf{u}_h^{\diamondsuit}\rvert\rvert_{-1,h}$	3.10607212	3.29645872	3.16358151	2.77816480	2.26491114
order	—	-0.08582569	0.05935825	0.18742658	0.29467773

$0 < \epsilon < 1/2$. We first consider the case without the stabilizations with the use of $\mathcal{P}_1$ element and $\mathcal{P}_2$ element. When using the $\mathcal{P}_2$ element, for $h = 1/2, 1/4, \cdots$, etc., all the resulting matrices are singular, and no finite element solutions can be obtained. When using the $\mathcal{P}_1$ element, the situation is a little bit strange. For $h = 1/2, 1/4$, the resulting matrices are still singular. For $h = 1/8, 1/16, 1/32, 1/64$, the resulting matrices are not singular, but the resulting finite element solutions look like clutters. For example, for $h = 1/64$, the finite element solution $\mathbf{u}_h = (u_{1h}, u_{2h})$ are shown in Figures 2a and 2b in Figure 2.



(a) Component u_{1h} (b) Component u_{2h}

Figure 2: Finite element solution of Example 1 in subsection 8.1 without stabilizations

We next report the numerical results with the stabilization in (5.21) in Tables 11 and 12. From these two tables, the computed convergence orders in $||\cdot||_{0,\mathbf{curl}}$ -norm are approximately consistent with the predicted $r\ell(\ell+1) = (1/2-\epsilon)/2 \approx 0.25$ for the $\mathcal{P}_1$ element and with the predicted $r\ell(\ell+1) = (1/2-\epsilon)2/3 \approx 0.33$ for the $\mathcal{P}_2$ element. For the $\mathcal{P}_2$ element, the computed order seems still to be close to the optimal order 0.5.

Corresponding to Remark 5.2, we report some additional numerical results, where all the discrete Riesz-representation lifting operators are defined in the enriched finite element spaces as described in Remark 5.2

so that all the stabilizations can be dropped. Here we still consider the exact solution in subsection 8.1 and use the $\mathcal{P}_1$ element. The numerical results are reported in Table 13, where the convergence order is comparable to that in Table 11. We also report the CPU times in Table 13, for example. The computations are performed in personal laptop using MATLAB codes.

Table 11: L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h$ with $\mathcal{P}_1$ element and (5.21)

h	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$
$\lVert\mathbf{u}-\mathbf{u}_h\rVert_0$	0.15213127	0.13832658	0.12352097	0.10613546	0.089657456
order	—	0.13723827	0.16332243	0.21884930	0.24474576
$\lVert\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\rVert_0$	0.02912832	0.03514196	0.03293941	0.02614453	0.01945651
order	—	-0.27077227	0.09337981	0.33330586	0.42625584

Table 12: L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h$ with $\mathcal{P}_2$ element and (5.21)

h	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$
$\lVert\mathbf{u}-\mathbf{u}_h\rVert_0$	0.10846542	0.07814245	0.05561342	0.03948855	0.02802773
order	—	0.47305672	0.49067358	0.49399864	0.49457967
$\lVert\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\rVert_0$	0.02069951	0.01268150	0.00817145	0.00552495	0.00383610
order	—	0.70687154	0.63406052	0.56463177	0.52632006

Table 13: L^2, $H(\mathbf{curl})$ errors of $\mathbf{u}_h$ with enriched FE spaces replacing stabilizations

h	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$
$\lVert\mathbf{u}-\mathbf{u}_h\rVert_0$	0.12084605	0.10684409	0.09061946	0.07620836	0.06404244
order	—	0.17766322	0.23761420	0.24987168	0.25092098
$\lVert\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)\rVert_0$	0.03427928	0.04506178	0.04520172	0.04094476	0.03550838
order	—	-0.39456743	-0.00447348	0.14269888	0.20551923
CPU time(s)	1.59	1.89	3.37	14.13	78.82

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Appendix

Upon the interest of one of the referees, in this Appendix section, we report some numerical results for the *weighted mixed method* in [32] in an L-shaped domain $\Omega := (-1,1)^2 \setminus ([0,1]\times[-1,0]) \subset \mathbb{R}^2$. Consider a two-dimensional problem of Maxwell's equations in Ω with boundary Γ and unit tangential vector $\boldsymbol{\tau}$, for given right-hand sides $\mathbf{f}, \kappa$ and boundary data χ, to find the electrical field $\mathbf{z}$ such that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{z} = \mathbf{f}, \quad \operatorname{div} \mathbf{z} = \kappa \quad \text{in } \Omega, \quad \mathbf{z}\cdot\boldsymbol{\tau} = \chi \quad \text{on } \Gamma. \tag{A.1}$$

With a multiplier introduced, *which identically equals zero and is called a dummy variable*, for the above Maxwell's equations, one obtains a mixed problem. Accordingly, from [32], the weighted mixed finite element method therein can be formulated as follows: Find $\mathbf{z}_h \in \mathcal{X}_h \subset H(\mathbf{curl}\,;\Omega), \theta_h \in M_h^* \subset H_0^1(\Omega)$, $\mathbf{z}_h\cdot\boldsymbol{\tau} = \chi_h$ where χ_h is an interpolation of χ, such that, $\forall \mathbf{v}_h \in \mathcal{X}_h \cap H_0(\mathbf{curl}\,,\Omega)$ and $\forall \vartheta_h \in M_h^*$,

$$\begin{cases} (\mathbf{curl}\,\mathbf{z}_h, \mathbf{curl}\,\mathbf{v}_h) + (\operatorname{div}\mathbf{z}_h, \operatorname{div}\mathbf{v}_h)_{0,\gamma} + (\theta_h, \operatorname{div}\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) + (\kappa, \operatorname{div}\mathbf{v}_h)_{0,\gamma}, \\ (\operatorname{div}\mathbf{z}_h, \vartheta_h) = (\kappa, \vartheta_h), \end{cases} \tag{A.2}$$

where for any two functions p and q, $(p,q)_{0,\gamma} := \int_\Omega d^{2\gamma} pq$, and d is the distance function to the reentrant corner (here is the origin) and $\gamma \in (\gamma_{\min}, 1]$ for $\gamma_{\min} \in (0, 1/2]$. In [32], γ is taken as 0.95. In [32, pages 507-508], the finite element space M_h^* for the multiplier θ_h is an *unusual* subspace of $H_0^1(\Omega)$, e.g.,

$$M_h^* := \{\vartheta_h \in H_0^1(\Omega) : \vartheta_h|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_h, \vartheta_h \equiv 0 \text{ in all those elements } K \text{ with } \partial K \cap \Gamma \neq \varnothing\}. \tag{A.3}$$

In other words, any function $\vartheta_h \in M_h^*$ not only is zero on the boundary Γ and but also is further identically zero on all the elements in the neighborhood of the boundary Γ.

From [32], for approximating the electrical field $\mathbf{z}$, it is known that $\mathcal{X}_h$ cannot be chosen as the linear element and is instead chosen as the quadratic element. Here, for the weighted mixed method (A.2), we report the numerical results for the linear element as well as the quadratic element for the $\mathcal{X}_h$, while M_h^* is the same(defined in (A.3)). For comparisons, we also report the numerical results of the proposed method with the linear element and with the quadratic element for approximating the electrical field $\mathbf{z}$ of the same problem (A.1). From Tables A1 and A2, indeed, the weighted mixed method (A.2) does not give a convergent solution for the linear element while it gives a convergent solution for the quadratic element, for both of which M_h^* in (A.3) is used. On the contrary, from Tables A3 and A4, for both linear element and quadratic element, the proposed method gives convergent solutions.

Table A1: L^2, $H(\mathbf{curl})$ errors of $\mathbf{z}_h$ with the method (A.2) of $\mathcal{P}_1$ element

h	1/4	1/8	1/16	1/32	1/64
$\|\|\mathbf{z}-\mathbf{z}_h\|\|_0$	0.22165551	0.18326176	0.19059676	0.23653279	0.30914111
order	—	0.27441350	-0.05661784	-0.31151659	-0.38622535
$\|\|\mathbf{curl}\,(\mathbf{z}-\mathbf{z}_h)\|\|_0$	0.36614843	0.42006414	0.49855785	0.58966605	0.68718886
order	—	-0.19818102	-0.24715130	-0.24213723	-0.22080850

Table A2: L^2, $H(\mathbf{curl})$ errors of $\mathbf{z}_h$ with the method (A.2) of $\mathcal{P}_2$ element

h	1/4	1/8	1/16	1/32	1/64
$\|\|\mathbf{z}-\mathbf{z}_h\|\|_0$	0.12818975	0.09567879	0.07396757	0.05709418	0.04255315
order	—	0.4220099	0.37130631	0.37354915	0.42407779
$\|\|\mathbf{curl}\,(\mathbf{z}-\mathbf{z}_h)\|\|_0$	0.21310929	0.17590991	0.13305225	0.09105296	0.05556513
order	—	0.27675668	0.40284385	0.54721506	0.71252623

Table A3: L^2, $H(\mathbf{curl})$ errors of $\mathbf{z}_h$ with the proposed method of $\mathcal{P}_1$ element

h	1/4	1/8	1/16	1/32	1/64
$\|\|\mathbf{z}-\mathbf{z}_h\|\|_0$	0.24437197	0.21751288	0.18764186	0.15537776	0.12514313
order	—	0.16797800	0.21311909	0.27220169	0.31220098
$\|\|\mathbf{curl}\,(\mathbf{z}-\mathbf{z}_h)\|\|_0$	0.03813998	0.04711603	0.04571838	0.03686610	0.02785113
order	—	-0.30491384	0.04344382	0.31047929	0.40455892

Table A4: L^2, $H(\mathbf{curl})$ errors of $\mathbf{z}_h$ with the proposed method of $\mathcal{P}_2$ element

h	1/4	1/8	1/16	1/32	1/64
$\|\|\mathbf{z}-\mathbf{z}_h\|\|_0$	0.12858677	0.08103176	0.05114443	0.03224953	0.02032916
order	—	0.66618284	0.66391026	0.66529913	0.66572721
$\|\|\mathbf{curl}\,(\mathbf{z}-\mathbf{z}_h)\|\|_0$	0.00847390	0.00438290	0.00242808	0.00141754	0.00085666
order	—	0.95114146	0.85206869	0.77642730	0.72660164

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