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Mereology in Aristotle's Assertoric Syllogistic

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Abstract

How does Aristotle think about sentences like 'Every x is y' in the *Prior Analytics*? A recently popular answer conceives of these sentences as expressing a mereological relationship between x and y: the sentence is true just in case x is, in some sense, a part of y. I argue that the motivations for this interpretation have so far not been compelling. I provide a new justification for the mereological interpretation. First, I prove a very general algebraic soundness and completeness result that unifies the most important soundness and completeness results to date. Then I argue that this result vindicates the mereological interpretation. In contrast to previous interpretations, this argument shows how Aristotle's conception of predication in mereological terms can do important logical work.

1. Introduction

At the heart of Aristotle's syllogistic are four kinds of predication:

- *b**A**a* Universal Affirmative: b belongs to (or is predicated of) every a. (Equivalent to saying 'Every a is b').
- *b**E**a* Universal Negative: b belongs to no a. ('No a is b').
- *b**I**a* Particular Affirmative: b belongs to some a. ('Some a is b').
- *b**O**a* Particular Negative: b does not belong to some a. ('Not every a is b').

In the *Prior Analytics*, Aristotle's investigations of logical relations between these kinds of propositions are primarily proof-theoretic. However, we get a hint of the semantics of these propositions in the famous *dictum de omni et nullo*:

> We use the expression 'predicated of every' when nothing can be taken of which the other term cannot be said, and we use 'predicated of none' likewise.[1]

The *dictum* gives the meaning of Aristotle's technical phrases 'predicated of every' and 'predicated of none'. The basic idea is that 'a is predicated of every b' means that for every z, if b is predicated of z, then a is also predicated of z. Similarly, 'a is predicated of no b' means that for every z, if b is predicated of z, then a is *not* predicated of z.[2] Because the *dictum* is used to justify the syllogisms of the first figure to which all the others are reduced, it carries enormous weight in understanding Aristotle's semantics. It is, as *Morison 2015* says, 'a governing principle in Aristotle's logic' (p. 112). Moreover, while it directly explains the meanings of universal affir-

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[1]λέγομεν δὲ τὸ κατὰ παντὸς κατηγορεῖσθαι ὅταν μηδὲν ᾖ λαβεῖν καθ' οὗ θάτερον οὐ λεχθήσεται· καὶ τὸ κατὰ μηδενὸς ὡσαύτως (*APr* 24b28-30). With Ross in excising 'τοῦ ὑποκειμένου'. Although it is present in all the manuscripts, it is absent in Alexander's citation and an otherwise unparalleled use of the term in the *Analytics*.

[2]It is disputed whether the phrases 'a is predicated of every b' and 'a is predicated of no b' are defined in terms of a notion of predication *different from* universal affirmative. See *Corcoran 1972, 1973, Barnes 2007*, for the view that they are different, but *Malink 2013, Morison 2008*, following Michael Frede for the view that they are the same. While I will stay neutral with respect to the dispute between these two readings, the former reading is usually associated with a view of predication akin to Frege's, which takes predication to be a relation between two different syntactic types. This view has been discredited by *Mignucci 2000, Malink 2009, Corkum 2015*. See *Crager 2015* for a version of the Corcoran/Barnes reading that does not use the Fregean conception of predication. In this paper, I will not take a stand on whether the two kinds of predication are different. Everything in the semantics will be neutral between those two interpretations.

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mative and universal negative propositions, it indirectly explains the meanings of particular propositions, which are each the *contradictory* of a universal proposition.[3]

How are we supposed to understand this notion of predication?

My aim is to argue that the relation of predication defined or elucidated by the *dictum de omni et nullo* is a mereological relation between universals.

This position has been gaining in popularity in recent years. In the early days of mathematical reconstructions of Aristotle's logic, the schematic letters (which are placeholders for terms) were taken to denote non-empty sets of individuals and syllogistic propositions were understood to be about the extensional relations that hold between them.[4] Despite its initial promise, the inadequacy of this account has been made clear, by Malink and others.[5]

In its place, the mereological interpretation of predication has been proposed as a viable alternative.[6] While this conception does not suffer from the same problems as the set-theoretic interpretation, I will argue that the reasons so far given to support it are not compelling. Instead, I will give a new argument in support of the mereological interpretation. The core of this argument is a very general algebraic soundness and completeness result for Corcoran's deductive system RD, the standard natural deduction system used to study Aristotle's logic. In this proof, I show how RD is sound for the class of Preorders $\mathbb{P}$ and complete for the class of Finite Boolean Algebras $\mathbb{FBA}$. This result has the corollary that any class of models $\mathbb{M}$ such that $\mathbb{FBA} \subseteq \mathbb{M} \subseteq \mathbb{P}$ is also sound and complete for RD. The systems of *Corcoran 1972* and *Martin 1997* thus emerge as special cases of the much more general soundness and completeness result proven here. In the final section, I argue 1) that we should interpret the predication relation as a preorder, since no further structure is needed to capture Aristotle's validities and invalidities, 2) that preorders generally capture the formal structure of part-whole relations, and 3) that the mereological interpretation is thereby vindicated.

2. The Mereological Interpretation of Predication

In recent literature on Aristotle's notion of predication, there have been two arguments for a mereological interpretation.[7]

The first reason to think that Aristotle has a mereological conception of predication is because of the language he uses. Aristotle's terminology for predication is steeped in mereological language. He marks his phrase for universal predication ('belongs to all') as equivalent to 'is in as a *whole*' and his terms for different kinds of propositions—'universal' (*katholou*) and 'particular' (*kata meros/en merei*)—are themselves derived from the language of whole (*holon*) and part (*meros*). If universal predication is a mereological relation, we would also expect it to be defined in terms of notions related to mereology.

While the language is highly suggestive, it does not settle the case in favor of the mereological interpretation. In particular, the linguistic information, on its own, does not tell us that Aristotle isn't being metaphorical. Aristotle could be reaching for mereological language just because he feels it is the best approximation of what he has in mind. Nevertheless, we are not entitled to conclude that the mereology talk should be taken literally, for the same reason that we should

[3]The exact manner of the explanation is is a matter of dispute. Some take the *dictum* to be a biconditional. For *Barnes 2007*, it is an explicit definition along Tarskian lines. For *Malink 2013*, take it to give an implicit definition of a-predication. *Morison 2015* thinks Aristotle is not giving a definition at all but specifying a rule of inference that 'characterizes' the meaning of the universal propositions. In what follows, I will follow the Barnes/Malink biconditional reading, but the account I propose could be translated into a rule-based account along the lines Morison endorses without any serious difficulty. Instead of one rule, we would have two corresponding to the biconditional. This account, unlike Morison's, would provide both an introduction and an elimination rule for the universal propositions.

[4]See especially the pioneering completeness results of *Corcoran 1972, 1973, Smiley 1973*, although see *Stekeler-Weithofer 1986* for an early completeness result that does not make this assumption.

[5]Two areas of particular difficulty are some claims about conversion and Aristotle's modal syllogistic. See *Malink 2009, 2013*.

[6]See especially *Mignucci 2000, Malink 2009, Corkum 2015* as well as *Malink 2013* for an extension of this basic account to Aristotle's modal syllogistic.

[7]See *Mignucci 2000, Malink 2009, Corkum 2015*. This was also the view of the ancient commentators: Alexander *in APr* 25.2–4 *Wallies 1883*, Philoponus *in APr*. 47.23–48.2, 73.22–3, 104.11–16, 164.4–7 *Wallies 1905*.

not take Aristotle's use of the word 'ὕλη' to mean that all matter is timber. What we want, and what the linguistic information does not yet give us, is a reason *why* Aristotle would want to conceive of predication mereologically. What *philosophical work* does it do? Without a good sense of its purpose, the metaphorical objection seems hard to answer.

One suggestion as to why Aristotle uses the language of part and whole comes from thinking about a specific kind of predication. When discussing the relation between genus and species, Aristotle explicitly says that they are related as whole to part (*Metaphysics* V.25, 26).[8] On this account, Aristotle inherits a mereological conception of the genus-species relation, presumably from Plato's method of division.[9] In the method of division, a genus is *divided* into the species that are its parts. Because of its connection to the method, it seems like we can make good sense of the mereological language.[10] Then, on this account, Aristotle *extends* the mereological conception from this special case of predication to predication in general. However, what is the evidence for this extension? It isn't clear that there is the same motivation for predication generally being a part-whole relation. To see this, we should turn to the passages in the *Metaphysics* in which the species/genus, part/whole connection is made.

> The elements into which the kind might be divided apart from the quantity, are also called parts of it; for which reason we say the species are parts of the genus.[11]
>
> We call a whole (1) that from which is absent none of the parts of which it is said to be naturally a whole, and (2) that which so contains the things it contains that they form a unity; and this in two senses?either as each and all one, or as making up the unity between them. For (a) that which is true of a whole class and is said to hold good as a whole (which implies that it is a kind of whole) is true of a whole in the sense that it contains many things by being predicated of each, and that each and all of them, e.g. man, horse, god, are one, because all are living things.[12]

In the first passage, Aristotle says that species are part of a genus *because* it is the product of a division. If this is the explanation, then why should we think that the broader, non-essential predication, can also be explained in the same way? For instance, does it motivate the claim that *Human* is a part of *Colored Thing*? In these cases, there simply does not seem to be a division that would do the job. The second passage is less clear. The language of predication could suggest that anything that a kind is predicated of is its part, in which case Aristotle's example at the end of the passage does not mean that only species are parts of genera. On the other hand, when Aristotle discusses a genus 'containing' things elsewhere in his corpus, it is inevitably species, differences, or individual substances.[13] Nowhere is it so much as hinted that containment includes accidental relations, as it must on this account. Therefore, the fact that Aristotle explicitly and literally claims the genus-species relation to be a relation of whole to part does not support by itself the broader view that predication *in general* is a mereological relation. For Aristotle's *reasons* for making the claim in the genus-species case do not carry over to non-essential predication. So, even if Aristotle *did* get the mereological language in this way, we have come no closer to understanding the work that it is supposed to do, since that does not obviously apply to the extended uses of the mereological relation.

[8] *Malink 2013* p. 84 also cites Aristotle's claim that medicine is a part of science (*APr* 2.15 64a17, 64b12–13), which is just a particular case of the general principle.

[9] In the *Statesman*, Plato is most explicit about this point: 'That whenever there is a kind of something, it is necessarily also a part of whatever thing it is said to be a kind of, but it is not at all necessary that a part is a kind. You must always assert, Socrates, that this is what I say rather than the other way around' (263b7-10). For the connection to Aristotle's syllogistic, see *Solmsen 1929* pp. 87, 177 ff. and *Mignucci 2000* pp. 4-5.

[10] Could this too be metaphorical? I find that less plausible, since Plato is insistent and explicit about the parthood language in a way that would be strange if we were to take him metaphorically. See n. 9.

[11] ἔτι εἰς ἃ τὸ εἶδος διαιρεθείη ἂν ἄνευ τοῦ ποσοῦ, καὶ ταῦτα μόρια λέγεται τούτου· διὸ τὰ εἴδη τοῦ γένους φασὶν εἶναι μόρια (*Met.* 1023b17-19), trans. Ross in *Barnes 1984*.

[12] Ὅλον λέγεται οὗ τε μηθὲν ἄπεστι μέρος ἐξ ὧν λέγεται ὅλον φύσει, καὶ τὸ περιέχον τὰ περιεχόμενα ὥστε ἕν τι εἶναι ἐκεῖνα· τοῦτο δὲ διχῶς· ἢ γὰρ ὡς ἕκαστον ἓν ἢ ὡς ἐκ τούτων τὸ ἕν. τὸ μὲν γὰρ καθόλου, καὶ τὸ ὅλως λεγόμενον ὡς ὅλον τι ὄν, οὕτως ἐστὶ καθόλου ὡς πολλὰ περιέχον τῷ κατηγορεῖσθαι καθ' ἑκάστου καὶ ἓν ἅπαντα εἶναι ὡς ἕκαστον, οἷον ἄνθρωπον ἵππον θεόν, διότι ἅπαντα ζῷα (*Met.* 1023b26-32).

[13] *HA* 490b16, 504b13, 534b13, *PA* 644a13, *Pol* 1285a2, *Topics* 121b24-9, 1139b38, 144b12. This use of 'containment' is also present in Plato *Ti.* 31a2–8, 33b2–7, and possibly *Soph.* 250b8. See *Gill 2012*, pp. 206 ff for a different reading of the *Sophist* passage.

In what follows, I will give a new piece of evidence, which also has a clear philosophical payoff. The minimal formal structure of the part-whole relation is sufficient for giving semantic clauses for all the syllogistic propositions and defining a consequence relation that is sound and complete for the standard proof-system for studying Aristotle's logic.

This reason shows the *adequacy* of the mereological conception of predication in Aristotle's logical theory. Even if Aristotle is using the terminology metaphorically, or extending spatial and physical language to apply to logic, this reason allows us to see what logical work such an association can do. I show that we do not need anything beyond what is given in the *dictum* to understand the meaning of the syllogistic propositions, provided that the predication relation is preorder, the general formal structure of any mereological relation. If conceiving of the meanings of the syllogistic propositions in mereological terms can do this kind of work, Aristotle's use of mereological language should be taken seriously.[14]

3. Algebraic Semantics

Here we introduce the assertoric syllogistic, the fragment of Aristotle's syllogistic that only deals with non-modal propositions. Soundness will be shown for the class $\mathbb{P}$ of Preorder models (domains with a reflexive and transitive relation) and completeness for the class $\mathbb{FBA}$ of Finite Boolean Algebras.[15]

The language of assertoric syllogistic is defined by a finite set of terms: $T = \{a, b, ...\}$ and copulae $C = \{\boldsymbol{A}, \boldsymbol{E}, \boldsymbol{I}, \boldsymbol{O}\}$ such that, $\mathcal{L} = \{x\boldsymbol{Z}y \mid x, y \in T \wedge x \neq y \wedge \boldsymbol{Z} \in C\}$.[16]

A model $\mathcal{M}$ is a tuple, where $\langle D, \sqsubseteq \rangle$ where $D = \{A, B, \ldots\}$, and $\sqsubseteq \subseteq D \times D$. We write $B \sqsubseteq A$ instead of $\sqsubseteq (B, A)$. The denotation function $[\![t]\!]^{\mathcal{M}} : T \to D$. The semantic clauses for the formulae:

- $\mathcal{M} \models x\boldsymbol{A}y$ iff $[\![y]\!]^{\mathcal{M}} \sqsubseteq [\![x]\!]^{\mathcal{M}}$.
- $\mathcal{M} \models x\boldsymbol{E}y$ iff $\neg\exists Z \in D\left(Z \sqsubseteq [\![x]\!]^{\mathcal{M}} \wedge Z \sqsubseteq [\![y]\!]^{\mathcal{M}}\right)$.
- $\mathcal{M} \models x\boldsymbol{I}y$ iff $\exists Z \in D\left(Z \sqsubseteq [\![x]\!]^{\mathcal{M}} \wedge Z \sqsubseteq [\![y]\!]^{\mathcal{M}}\right)$.
- $\mathcal{M} \models x\boldsymbol{O}y$ iff $\neg[\![y]\!]^{\mathcal{M}} \sqsubseteq [\![x]\!]^{\mathcal{M}}$.

We say that $\Gamma \models \psi$ if and only if for every $\mathcal{M}$ such that for all $\varphi \in \Gamma$, $\mathcal{M} \models \varphi$, $\mathcal{M} \models \psi$.

All of the accounts of the semantics of Aristotelian syllogistic propositions (Barnes, Corcoran, Malink, Martin, Smiley, etc) agree about the semantic clauses at this level of abstraction. What distinguishes them are particular choices about what the set D and the relation $\sqsubseteq$ are. For example, Corcoran, Smiley, and Barnes take D to be a set of non-empty sets and $\sqsubseteq$ to be the subset relation. For Martin, $\langle D, 0, \sqsubseteq \rangle$ is a meet semi-lattice.[17] For Malink, D is a set of terms and $\sqsubseteq$ is the a-predication relation itself, which is assumed to be a preorder. As we will see from the results below, using a single semantic clause while varying the classes of models will reveal how these accounts differ precisely in terms of how much structure they attribute to the $\sqsubseteq$ relation. And knowing how these accounts relate in this way will allow us to better assess

[14] I do not want to suggest that this is the only way that Aristotle thinks of predication. Singular propositions stand out as the most likely exception. Aristotle never claims that an individual is a part of a universal, but see *Mignucci2000*. He also sometimes uses form and matter to understand predication (e.g., *Metaphysics* Z.17, H.6) and this is not mereological either.

[15] As will become clear, the algebraic semantics presented here is a generalization of nearly every semantics present in the literature, including *Corcoran 1973* and *Martin 1997*. The only exception is the incomparable approach of *Andrade-Lotero and Dutilh Novaes 2010*, who examine the first order models given by a translation from Aristotle's language.

[16] The choice to exclude reflexive sentences is motivated by the fact that Aristotle's treatment of such sentences is controversial. On the one hand, he allows them in *APr* II.15, but on the other, allowing reflexive sentences seems to require that the conclusion of a syllogism can be among its premises. For instance, if one of the premises of Barbara were reflexive, the conclusion would be the same as the other premise. For the purposes of the results proven here, the choice of whether or not to include reflexive sentences is inessential. Soundness and completeness would go through in exactly the same way with the additional 0-premise inference rule that concluded $x\boldsymbol{A}x$. Things are more complicated for Smiley's system, since $a\boldsymbol{A}b, a\boldsymbol{O}b \vdash a\boldsymbol{O}a$ is now derivable in the deductive system, but $\{a\boldsymbol{A}b, a\boldsymbol{O}b, a\boldsymbol{A}a\}$ is not an "antilogism", that is, a set of at least two unsatisfiable sentences all of whose subsets are satisfiable and therefore it is not the case that $a\boldsymbol{A}b, a\boldsymbol{O}b \models a\boldsymbol{O}a$. See n. 22.

[17] A preorder $\leq$ is a meet semi-lattice if it is anti-symmetric ($\forall x, y\left((x \leq y \wedge y \leq x) \to x = y\right)$) and has the greatest-lower bound property ($\forall x, y \exists z\left(z \leq x \wedge z \leq y \wedge \forall w\left((w \leq x \wedge w \leq y) \to w \leq z\right)\right)$).

them as interpretations of Aristotle's syllogistic.

It might be objected that the semantic clauses for A and O propositions misconstrue the *dictum de omni et nullo*, which should instead use a clause like: $\forall Z \in D\left(Z \sqsubseteq [\![x]\!]^{\mathcal{M}} \rightarrow Z \sqsubseteq [\![y]\!]^{\mathcal{M}}\right)$. However, such semantic clauses have a significant drawback: with logic alone, these clauses validate all and only the valid syllogistic moods.[18] But then $x \sqsubseteq y$ could be any relation at all, for instance x *is more beautiful than* y or x *is a donkey and* y *is a sheep*. It is hard to believe that Aristotle wouldn't have wanted the character of the predication relation used in the *dictum* to play no role whatsoever in the explanation of why the valid moods are valid. Therefore, we should not use such a semantic clause in our explanation.

But how is the above semantic clause at all connected to the *dictum*? My answer begins with two observations familiar from the recent literature. First, the *dictum* does not need to provide an explicit definition of a-predication in order for it to explicate the meaning. Nor does it need to provide truth-conditions. Rather, the *dictum characterizes* the features of the predication relation that are operative in the semantics.[19] Each of the different accounts presented above offers a different account of exactly what that relevant characteristic is and thus what the *dictum* says. Second, Aristotle reasons back and forth between $x\boldsymbol{A}y$ being true and the predication relation described in the *dictum* holding between two terms.[20] Putting these two observations together, we can see that the semantic clauses offered here are both consistent with what Aristotle says about predication and, unlike the other proposed semantic clause, actually gives some work for the predication relation to do. As we will see, the various interpretations attribute different structure to the relation $\sqsubseteq$ and these semantic clauses will allow us to see what structure does the philosophical heavy-lifting in Aristotle's syllogistic.

The deductive system (Corcoran's RD) that we will be using is a natural deduction system consisting of the following rules (order does not matter):[21]

$$\text{(I)}\ \frac{x\boldsymbol{E}y}{y\boldsymbol{E}x}\quad \text{(II)}\ \frac{x\boldsymbol{A}y}{x\boldsymbol{I}y}\quad \text{(III)}\ \frac{\begin{array}{c}x\boldsymbol{A}y\\ y\boldsymbol{A}z\end{array}}{x\boldsymbol{A}z}\quad \text{(IV)}\ \frac{\begin{array}{c}x\boldsymbol{E}y\\ y\boldsymbol{A}z\end{array}}{x\boldsymbol{E}z}$$

A sequence of sentences $\langle p_1, \ldots, p_n\rangle$ is a *direct deduction* of φ from Γ if and only if $\varphi = p_n$ and for all $i \leq n$ one of the following holds:

(1) $p_i \in \Gamma$.
(2) $\exists j < i : p_i$ is obtained from p_j by (I) or (II).
(3) $\exists j, k < i : p_i$ is obtained from p_j and p_k by (III) or (IV).

A sequence of sentences $\langle p_1, \ldots, p_n\rangle$ is an *indirect deduction* of φ from Γ if and only if $\exists j < n\left(p_j = c\left(p_n\right)\right)$ (where $c : \mathcal{L} \rightarrow \mathcal{L}$ such that $c\left(x\boldsymbol{A}y\right) = x\boldsymbol{O}y$, $c\left(x\boldsymbol{E}y\right) = x\boldsymbol{I}y$, $c\left(x\boldsymbol{I}y\right) = x\boldsymbol{E}y$, $c\left(x\boldsymbol{O}y\right) = x\boldsymbol{A}y$) and for all $i \leq n$ one of the following holds:

(1) $p_i \in \Gamma \cup \{c\left(\varphi\right)\}$.
(2) $\exists j < i : p_i$ is obtained from p_j by (I) or (II).
(3) $\exists j, k < i : p_i$ is obtained from p_j and p_k by (III) or (IV).

We say that $\Gamma \vdash \varphi$ if and only if there is a direct or an indirect deduction of φ from Γ.[22]

[18]See *Malink 2013*, p. 39.

[19]*Malink 2013*, p. 66; *Morison 2008*, p. 214; *Morison 2015*, p. 132.

[20]*Malink 2013*, pp. 52-53, 63 ff. Note that this equivalence can be hold regardless of whether Malink is right to identify these two relations.

[21]This system lacks what is known as i-conversion (the inference from $x\boldsymbol{I}y$ to $y\boldsymbol{I}x$), which Aristotle uses to reduce some moods of the second and third figures, as well as iterated uses of reductio ad absurdum. However, as *Corcoran 1972* shows, these can be eliminated without any loss of deductive power. Disamis, for instance, can be proven with indirect deduction and Celarent, without any conversions at all. Datisi can be proven by reductio, e-conversion, and Celarent. While these proofs differ from Aristotle's own, they use principles that Aristotle himself accepts. For metalogical purposes, the more minimal proof theory is more convenient.

[22]This deductive system does not perfectly capture Aristotle's notion of the syllogism: it validates explosion (which Aristotle denies in *APr* II.15), allows for conclusions to be in the premises, and superfluous premises (these last two being ruled out by the definition of the syllogism). In these respects, *Smiley 1973* seems to be an improvement. His deductive system has none of these properties. However, he still has difficulties with the doctrine of *APr* II.15, since he cannot make sense of the possibility that there

3.1. *Soundness*

Theorem. *Let $\mathbb{P}$ be the class of Preorders, where $\sqsubseteq$ is reflexive and transitive. With the semantic clauses provided above, if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

Proof We proceed by first showing that (I)-(IV) above are satisfied in all models. We then show that, if a contradiction is proved in RD, then no model satisfies the premises so that $\Gamma \models \varphi$ is trivial, thus showing that indirect deductions are sound.

***E*-conversion**
Suppose $\mathcal{M} \models x\boldsymbol{E}y$. Then $\neg\exists Z\left(Z \sqsubseteq [\![x]\!]^{\mathcal{M}} \wedge Z \sqsubseteq [\![y]\!]^{\mathcal{M}}\right)$. So, $\mathcal{M} \models y\boldsymbol{E}x$.

***A*-subalternation**
Suppose $\mathcal{M} \models x\boldsymbol{A}y$. Then $[\![y]\!]^{\mathcal{M}} \sqsubseteq [\![x]\!]^{\mathcal{M}}$, but since $\sqsubseteq$ is reflexive, $[\![y]\!]^{\mathcal{M}} \sqsubseteq [\![y]\!]^{\mathcal{M}}$ and hence $\mathcal{M} \models x\boldsymbol{I}y$.

Barbara
Suppose $\mathcal{M} \models x\boldsymbol{A}y$ and $\mathcal{M} \models y\boldsymbol{A}z$. Then $[\![y]\!]^{\mathcal{M}} \sqsubseteq [\![x]\!]^{\mathcal{M}}$ and $[\![z]\!]^{\mathcal{M}} \sqsubseteq [\![y]\!]^{\mathcal{M}}$, but since $\sqsubseteq$ is transitive, $[\![z]\!]^{\mathcal{M}} \sqsubseteq [\![x]\!]^{\mathcal{M}}$ and hence $\mathcal{M} \models x\boldsymbol{A}z$.

Celarent
Suppose $\mathcal{M} \models x\boldsymbol{E}y$ and $\mathcal{M} \models y\boldsymbol{A}z$. Then $\neg\exists W\left(W \sqsubseteq [\![x]\!]^{\mathcal{M}} \wedge W \sqsubseteq [\![y]\!]^{\mathcal{M}}\right)$ and $[\![z]\!]^{\mathcal{M}} \sqsubseteq [\![y]\!]^{\mathcal{M}}$. If there is a $W \sqsubseteq [\![x]\!]^{\mathcal{M}}$ and $W \sqsubseteq [\![z]\!]^{\mathcal{M}}$, then by transitivity $W \sqsubseteq [\![y]\!]^{\mathcal{M}}$, contradicting the assumption. Hence $\mathcal{M} \models x\boldsymbol{E}z$.

Reductio
Suppose there is an indirect proof of φ from Γ where the contradicting sentences in the proof are ψ and $c(\psi)$. By the semantic clauses, no model satisfies both ψ and $c(\psi)$. Since all models that satisfy $\Gamma \cup \{c(\varphi)\}$ satisfy both ψ and $c(\psi)$ because of the validity of rules (I)-(IV), no models satisfy $\Gamma \cup \{c(\varphi)\}$. Either there is a model for Γ or not. If not, then vacuously every model that satisfies Γ also satisfies φ. If there is a model, it cannot also satisfy $c(\varphi)$. But by the semantic clauses, for any $\mathcal{M}$, $\mathcal{M} \vDash \varphi \Leftrightarrow \mathcal{M} \nvDash c(\varphi)$, so that model will satisfy φ. Therefore, if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

3.2. *Completeness*

Now we show that if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$. In fact we will show something significantly stronger, by constructing a counter-model that is a Finite Boolean Algebra $\mathcal{M} = \langle D, 0, \sqsubseteq, - \rangle$, where $0 \notin D$.[23] The semantic clauses remain the same.

Theorem. *Suppose that $\Gamma \nvdash \varphi$. There is a Finite Boolean Algebra model $\mathcal{M}$ such that, for all $\gamma \in \Gamma \cup \{c(\varphi)\}$, $\mathcal{M} \vDash \gamma$ and thus, that $\Gamma \nvDash \varphi$.*

Proof. First, extend $\Gamma \cup \{c(\varphi)\}$ by introducing witnesses for the existential statements. We do this by adjoining a new set of constants T' to the set of terms, giving rise to a new set of terms T^* and a new language $\mathcal{L}^*$. For each statement of the form $x\boldsymbol{I}y \in \Gamma \cup \{c(\varphi)\}$, add the statements $x\boldsymbol{A}c$ and $y\boldsymbol{A}c$, with a different $c \in T'$ resulting in a new theory Γ^* that contains $\Gamma \cup \{c(\varphi)\}$ and all the witnesses.

Proposition. *For any $\psi \in \mathcal{L}$, $\Gamma \cup \{c(\varphi)\} \vdash \psi \Leftrightarrow \Gamma^* \vdash \psi$.*

are *any* deductions that come from contradictory premises. I use Corcoran's system primarily because it has a more straightforward connection to the Tarskian definition of logical consequence and this simplifies various parts of the proof. However, Smiley uses the same semantic clauses as Corcoran and defines his notion of logical consequence in terms of Tarskian satisfaction. He defines $\Gamma \vDash \varphi$ as holding if and only if $\Gamma \cup \{c(\varphi)\}$ is an antilogism, which is formally defined as $\Gamma \cup \{c(\varphi)\}$ being unsatisfiable in the Tarskian sense, while every proper subset of $\Gamma \cup \{c(\varphi)\}$ is satisfiable, and for the cardinality of $\Gamma \cup \{c(\varphi)\}$ to be greater than one. Since he has proven his system to be complete for finite set-theoretic semantics, all that remains is to show that his deductive system is sound on the preorder semantics. The proof of this is essentially the same as that provided below, since his basic rules are the same and his structural rules are weaker than Corcoran's, so that if Corcoran's system is sound, so are Smiley's.

[23] This unusual notation for Boolean Algebras is equivalent to the usual $\langle D^*, 0, 1, -, +, \times \rangle$, where both 0 and 1 are elements of D^*. Just define $D^* = D \cup \{0\}$, define 1 as the top element of D, and define + and $\times$ as the least upper bound and greatest lower bound of $\sqsubseteq$ respectively. This presentation clarifies how for Corcoran, Smiley, et al the semantic value of a term (the range of the interpretation function) is a non-empty set, or equivalently, a non-bottom member of a Boolean Algebra. Thus we can use the general semantic clauses defined above without having to exclude the bottom element.

Proof The left to right direction is trivial, as the deductive system is monotonic. For the right to left direction, consider some $\chi \in \Gamma^* \setminus (\Gamma \cup \{c(\varphi)\})$ is used in a derivation to prove some $\gamma \in \mathcal{L}$. This is only possible if it is used in conjunction with another proposition $\chi' \in \Gamma^* \setminus (\Gamma \cup \{c(\varphi)\})$ where χ and χ' share a term in T^*. For each χ, there is exactly one such formula by construction, so we have $a\boldsymbol{A}c$ and $b\boldsymbol{A}c$, for $a, b \in T$ and $c \in T^*$. The only formulae in $\mathcal{L}$ that these can prove are: $a\boldsymbol{I}b$ and $b\boldsymbol{I}a$. However, these are both derivable from $\Gamma \cup \{c(\varphi)\}$, since they are elements of $\Gamma \cup \{c(\varphi)\}$.

Corollary. *$\Gamma \cup \{c(\varphi)\}$ and Γ^* are consistent.*

Proof A set of sentences $\Delta \subseteq \mathcal{L}$ is inconsistent iff for every $\psi \in \mathcal{L}$, $\Delta \vdash \psi$. By the initial hypothesis, $\Gamma \cup \{c(\varphi)\}$ is consistent (otherwise, φ could be derived by indirect deduction). Since, according to the Proposition, they prove all the same sentences, Γ^* is also consistent.

The counter model to be constructed will be the power set algebra of T^*, a Finite Boolean Algebra (a Boolean Algebra by construction and finite because T^* is finite) and hence a Preorder model. The denotation function of a term $t \in T$ will be defined as:

$$[\![t]\!]^{\mathcal{M}} = \{s \in T^* : \Gamma^* \vdash t\boldsymbol{A}s \vee s = t\}.$$

(Note that the denotation function always sends t to a non-empty subset of T^*.) Now we will show that every $\gamma \in \Gamma \cup \{c(\varphi)\}$ is satisfied in $\mathcal{M}$, proving it for each type of proposition:

- By the semantic clauses and definition of the interpretation function, $\mathcal{M} \vDash b\boldsymbol{A}a$ iff $[\![a]\!]^{\mathcal{M}} \subseteq [\![b]\!]^{\mathcal{M}}$ iff $\forall s \in T^* ((\Gamma^* \vdash a\boldsymbol{A}s \vee a = s) \to (\Gamma^* \vdash b\boldsymbol{A}s \vee b = s))$. Suppose that $b\boldsymbol{A}a \in \Gamma \cup \{c(\varphi)\}$. If, for an arbitrary $s \in T^*$, $\Gamma^* \vdash a\boldsymbol{A}s \vee a = s$, it will be the case that $\Gamma^* \vdash b\boldsymbol{A}s$ either because of substitution or Barbara. Hence $\mathcal{M} \vDash b\boldsymbol{A}a$.
- $\mathcal{M} \vDash b\boldsymbol{I}a$ iff $\exists s \in T^* ((\Gamma^* \vdash a\boldsymbol{A}s \vee a = s) \wedge (\Gamma^* \vdash b\boldsymbol{A}s \vee b = s))$. Suppose $b\boldsymbol{I}a \in \Gamma \cup \{c(\varphi)\}$. By the construction of Γ^*, for some $s \in T^*$, Γ^* contains $b\boldsymbol{A}s$ and $a\boldsymbol{A}s$. Hence $\mathcal{M} \vDash b\boldsymbol{I}a$.
- $\mathcal{M} \vDash b\boldsymbol{E}a$ iff $\neg\exists s \in T^* ((\Gamma^* \vdash a\boldsymbol{A}s \vee a = s) \wedge (\Gamma^* \vdash b\boldsymbol{A}s \vee b = s))$. Suppose $b\boldsymbol{E}a \in \Gamma \cup \{c(\varphi)\}$. Suppose for reductio that $\exists s \in T^* ((\Gamma^* \vdash a\boldsymbol{A}s \vee a = s) \wedge (\Gamma^* \vdash b\boldsymbol{A}s \vee b = s))$. Four cases are possible and in all of them a contradiction can be derived:
  - $\Gamma^* \vdash a\boldsymbol{A}s$ and $\Gamma^* \vdash b\boldsymbol{A}s$. Then $\Gamma^* \vdash a\boldsymbol{I}b$. But in that case Γ^* is inconsistent, which is impossible given *Corollary*.
  - $\Gamma^* \vdash a\boldsymbol{A}s$ and $s = b$. Then $\Gamma^* \vdash a\boldsymbol{A}b$, so Γ^* is inconsistent.
  - $\Gamma^* \vdash b\boldsymbol{A}s$ and $s = a$. Same as 2.
  - $s = a$ and $s = b$. But this is impossible because $a\boldsymbol{E}a \notin \mathcal{L}$.
- $\mathcal{M} \vDash b\boldsymbol{O}a$ iff $\exists s \in T^* ((\Gamma^* \vdash a\boldsymbol{A}s \vee s = a) \wedge \Gamma^* \nvdash b\boldsymbol{A}s \wedge b \neq s)$. Suppose $b\boldsymbol{O}a \in \Gamma \cup \{c(\varphi)\}$. Γ^* is consistent, so $\Gamma \nvdash b\boldsymbol{A}a$. Because $a \neq b$, $a = a$, and $\Gamma^* \nvdash b\boldsymbol{A}a$, $\mathcal{M} \vDash b\boldsymbol{O}a$.

This completes the proof.

3.3. *A Semantic Hierarchy*

Recall that we showed that every Preorder model is sound for Aristotle's deductive system and that Aristotle's deductive system is complete with respect to Finite Boolean Algebras. Consider any class of models intermediate between Preorders and Boolean Algebras. The above result immediately gives soundness and completeness results to this class as well. We see then that Aristotle's language and deductive system cannot distinguish between these classes of models.[24]

[24] Extending Aristotle's language, by including, say, identity or complex terms would be able to distinguish these classes of models, although this would not be particularly helpful in trying to interpret Aristotle's assertoric syllogistic, as he operates there with this relatively impoverished language.

4. The Mereological Interpretation Vindicated

In this section, I will argue that the technical result has bearing on how we ought to interpret Aristotle's notion of predication. The argument proceeds in two steps. First, I will argue that the above results motivate interpreting the predication relation as a preorder. Then, I will argue that the preorder captures the formal structure of the part-whole relations. Putting these two together, we have a good reason to interpret Aristotelian predication as a part-whole relation.

Recall that the result above immediately gives soundness and completeness results for a whole range of classes of models. Two of these classes deserve special mention: meet semi-lattices and Boolean Algebras. *Corcoran 1972* used a set-theoretic semantics equivalent to Boolean Algebras (by the Stone Representation Theorem) in proving his seminal completeness result. *Martin 1997* argued for the use of meet semi-lattices. We have seen above, however, that it is possible to capture everything in Aristotle's assertoric syllogistic captured by Boolean Algebras and meet semi-lattices with just preorders, *with the very same semantic clauses.* There is nothing to be gained by positing all the additional structure in previous interpretations, since the result above shows that it can all be done with preorders. These other interpretations, therefore, go beyond Aristotle's text. Since they all make the same predictions about the validities and invalidities, we should start with the weakest coherent reading of the text and only add structure when it is explicitly motivated.[25]

So we should understand Aristotle's predication relation in the assertoric syllogistic as a preorder. But why should that be understood as a *mereological* relation? It is because the preorder captures the crucial formal structure of the part-whole relation. Recall that Preorders have two essential properties: transitivity and reflexivity. While it is easy to see how transitivity is part of the formal structure of the part-whole relation, the case is significantly harder with reflexivity. Aristotle's preferred use of the language of part and whole is generally proper: nothing is a part of itself. By contrast, any mereological relation used to model quantification must make use of improper parthood, since 'A belongs to all A' is evidently true. Why would Aristotle have used the language of parthood when it seems so ill-suited to the task of modeling this reflexive aspect of quantification?

Indeed, Aristotle's terminology frequently has this problem. Another way he refers to universal affirmative predications is by saying that the predicate is 'above' the subject. He is willing to use this terminology even in cases where the subject and predicate convert. For instance, he argues in *APr* I.31 that the method of division can produce some, albeit trivial, syllogisms of the following form:

(1) Every animal is mortal or immortal.
(2) Every human is an animal.
(3) So, every human is mortal or immortal.

At the outset of the passage, he claims that division 'always deduces something higher up' (συλλογίζεται δ' ἀεί τι τῶν ἄνωθεν, *APr* 46a34). Later in the passage, Aristotle says more generally:

> In demonstrations, on the one hand, whenever it should be necessary to deduce that something belongs, it is necessary that the middle term through which the syllogism comes about is always less than and not a universal of the first of the extremes.[26]

In these passages, Aristotle is saying that animal is 'less than' the 'higher up' disjunctive term mortal or immortal, despite the fact that they clearly convert. So, although he uses language that suggests an irreflexive relation, it clearly is meant to indicate a reflexive one. How are we

[25]Indeed, *Malink 2009* argues that, outside of the sections on the assertoric syllogistic, Aristotle makes claims inconsistent with both Corcoran's and Martin's interpretations. It might be objected that *Corcoran 1972* and *Martin 1997* are only giving formal models, not interpretations of Aristotle's text. While this is explicitly true of *Andrade-Lotero and Dutilh Novaes 2010*, Corcoran and Martin both use the terms 'model' and 'interpretation' interchangeably, except, of course, when talking about semantic models.

[26]ἐν μὲν οὖν ταῖς ἀποδείξεσιν, ὅταν δέῃ τι συλλογίσασθαι ὑπάρχειν, δεῖ τὸ μέσον, δι' οὗ γίνεται ὁ συλλογισμός, καὶ ἧττον ἀεὶ εἶναι καὶ μὴ καθόλου τοῦ πρώτου τῶν ἄκρων (*APr* 46a39-b2).

to reconcile this tension?

The answer comes in two stages. First, reflexive predications are not at the forefront of Aristotle's thinking in his account of predication. The only point where it gets explicitly discussed is *APr* II.15, where he is clearly assuming that 'All A is A' is always true. But this text is the exception that proves the rule: while the syllogistic does in principle allow for such reflexive sentences, it is not designed to account for them. So the fact that the language of parthood does not fit well with it does not seem such a huge cost. Rather, reflexivity represents a sort of limit case of universal quantification.

Second, there do not seem to be any better relations at hand for expressing what Aristotle does want. It needs to generally be a kind of ordering, that is transitive, but not an equivalence relation. Perhaps he could have used the relation 'part of or identical to' to model predication, but that is cumbersome. 'Prior to' faces the same problems as 'part'. It seems to me that the language of parthood is the closest that Aristotle could have come describing preorders without modern technical terminology.

Corkum2015, p. 804 argues that any genuine mereology must satisfy the principle of Weak Supplementation (where *PP* stands for Proper Part and *O* for Overlap):

$$\forall x, y\, (PPxy \rightarrow \exists z(PPzy \wedge \neg Ozx))$$

This principle states that whenever x is a proper part of y, there is a part of y 'left over'. I take no stand on our contemporary conception of parthood, but there is significant evidence from ancient mathematics that shows that the Greeks did not require Weak Supplementation to hold for the *meros-holon* relation. In Greek mathematics as early as Aristotle,[27] there are two technical uses of the term *meros*:

> We call a part (1) that into which a quantity can in any way be divided; for that which is taken from a quantity qua quantity is always called a part of it, e.g. two is called in a sense a part of three.—(2) It means, of the parts in the first sense, only those which measure the whole; this is why two, though in one sense it is, in another is not, a part of three.[28]

From the passage, we see the semantic range of the term is quite wide—it can apply to any sorts of *quantities*. According to either definition in Aristotle's text, the *meros-holon* relation need not obey Weak Supplementation. For simplicity, just consider numbers. On the broad definition, 2 is a proper part of 3. However, there is no number less than 3 that does not overlap with 2. On the narrow definition, 2 is a proper part of 4, but again, there are no numbers that 4 factors into that do not overlap with 2. This shows that, just because Aristotle conceptualized the predication relation in mereological terms, we do not thereby need to think that he was committed to Weak Supplementation for that relation.

So why did Aristotle use the mereological language in his account of predication? I submit that he was looking for a way to talk about preorders, but, without the resources of contemporary logic, was unable to formally describe the conditions of transitivity and reflexivity that he wanted to impose on the predication relation. So he reached for the most natural, general relation with the structural features of a preorder. This was the part-whole relation. He then used the language of parthood to describe the predication relation. Unlike previous accounts, this reason for the mereological interpretation makes the part-whole relation do important philosophical work for Aristotle. The soundness and completeness result makes the work explicit,

[27] For its use in mathematics, see Euclid, *Elements* V Def 1 and *Heath 1926*, p. 115. The argument here does not depend on any special features of numbers and could be given for any quantities: lines, shapes, harmonic intervals, etc. Weak Supplementation fails for all of them.

[28] Μέρος λέγεται ἕνα μὲν τρόπον εἰς ὃ διαιρεθείη ἂν τὸ ποσὸν ὁπωσοῦν (ἀεὶ γὰρ τὸ ἀφαιρούμενον τοῦ ποσοῦ ᾗ ποσὸν μέρος λέγεται ἐκείνου, οἷον τῶν τριῶν τὰ δύο μέρος λέγεταί πως), ἄλλον δὲ τρόπον τὰ καταμετροῦντα τῶν τοιούτων μόνον· διὸ τὰ δύο τῶν τριῶν ἔστι μὲν ὥς λέγεται μέρος, ἔστι δ' ὥς οὔ (*Metaphysics* 1023b12-17).

showing how nothing more is necessary to account for the validities and invalidities of the deductive system.

In Aristotle's own practice, the work of the part-whole relation comes out differently and somewhat more indirectly. Central to his practice, as Morison has recently emphasized, is the proofs of first figure syllogisms by means of the *dictum de omni et nullo* and the other figures through these. Thus, at bottom, all that Aristotle has is the *dictum*, and hence the conception of predication that I have argued is grounded in a mereological conception of predication. But he does not have the tools to prove soundness and completeness, instead only supplying the semantic machinery to a small segment of his deductive system and using proof-theoretical tools to complete the other syllogisms.

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