Chapter 3
Frege, Russell, Ramsey and the Notion of an Arbitrary Function

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3.1 The Background

In *Frege's Philosophy of Language*, Dummett claims that Frege's notion of a function coincides with the notion of an arbitrary correspondence ([68], pp. 223 and 177):

> [...] Frege had not the slightest qualm about the legitimacy or intelligibility of higher-order quantification: he used it from the first, in *Begriffsschrift*, freely and without apology, and did not even see first-order logic as constituting a fragment having any special significance.
>
> [...] it is true enough, in a sense, that, once we know what objects there are, then we also know what functions there are, at least, so long as we are prepared, as Frege was, to admit all "arbitrary" functions defined over all objects.

Against this background, I claimed with Hintikka, in [129], that Frege's notions of a function and a class cannot be that of an arbitrary correspondence or arbitrary collection of objects and that Frege favoured, instead, some variety of non-standard interpretation, for which the domain of the function variables is something less than the characteristic functions of all subsets of the domain over which the individual variables range. When we wrote our paper, we were unaware of Dummett's argument in *Frege's Philosophy of Mathematics* which shows that the author changed his mind *vis à vis* his earlier position emerging from the above quote. Here is what he write there ([74], pp. 219–220):

> [...] Frege fails to pay due attention to the fact that the introduction of the [class] abstraction operator brings with it, not only new singular terms, but an extension of the domain. [...] [I]t may be seen as making an inconsistent demand on the size of the domain D, namely that, where D comprises n objects, we should have $n^n \leq n$, which holds only when $n = 1$, whereas we must have $n \geq 2$, since the two truth-values are distinct: for there must be n^n extensionally non-equivalent functions of one argument and hence n^n distinct value-ranges. But this assumes that the function-variables range over the entire classical totality of functions from D into D, and there is meagre evidence for attributing such a conception to Frege. His formulations make it more likely that he thought of his function-variables as

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> ranging over only those functions that could be referred to by functional expressions of his symbolism (and thus over a denumerable totality of functions), and of the domain D of objects as comprising value-ranges only of such functions.

The last two sentences, which show Dummett attributing to Frege a non-standard interpretation of his function-variables, are alike in spirit to some of the considerations we put forward in [129]. For example this one ([129], p. 292):

> Sometimes, a non-standard interpretation is guided by the idea that only such properties, relations, and functions can be assumed to exist as can be defined or otherwise captured by a suitable expression of one's language. In the case of theories with infinite models, this leads inevitably to a non-standard interpretation, for there can be only a countable number of such definitions or characterisations available for this purpose. Hence they cannot capture all the subsets of $do(M)$, for there is an uncountable number of them.

In a rejoinder to our paper, Bell and Demopoulos [62] took side with Dummett's standard interpretation of Frege's function variables in [68], and argued that Frege's concept of a function coincides with the set-theoretic notion of an arbitrary correspondence. The main idea behind that paper is summarised in [61], p. 5:

> Our thought was that whatever covert role the neglect of Cantor's theorem might have played in the inconsistency of [...][*Grundgesetze*], it is unlikely that Frege sought to ignore the theorem by assuming that the totality of functions, like the totality of expressions, is countably infinite. But we sided with Dummett in [...] [74] and supposed that Frege might very well have been misled into assuming that what holds for certain countable interpretations of the function variables holds in general; hence we agreed with Dummett's evaluation of the sense in which Frege missed the significance of the possibility of different interpretations for his program.

For me and Hintikka the definability of functions in one's suitable symbolism is just one possible manifestation of the basic idea underlying the non-standard interpretation: the connection between an argument and the corresponding value of a function is determined by a formal law, norm or property. This idea stands in contrast to the conception underlying the standard interpretation according to which the correlation between values and arguments is purely arbitrary and not determined by such a law. For this reason, the main argument of our paper was intended to focus on the distinction between, on one side, the idea of arbitrary variation between values and arguments, and the idea of a correlation as determined by a formal law, on the other. We argued that Frege could not have had a standard interpretation of his function-variables given that the notion of a law was important for him when characterising functions. Part of our argument was Frege's discussion of the inadequacies of the definition of a function proposed by Czuber. We contrasted Czuber's notion of correlation which involves no assertion as to the law of correlation, and which can be set up in the most various ways, with Frege's conception of correlation which focuses on the idea of a law ([100], p. 662; [104], p. 112):

> Correlation, then, takes place according to a law, and different laws of this sort can be thought of. In that case, the expression y is a function of x' has no sense, unless it is completed by the law of correlation.

To the question of how such a law is specified, Frege answers (*ibid.*):

> Our general way of expressing such a law of correlation is an equation in which the letter 'y' stands on the left side whereas on the right there appears a mathematical expression consisting of numerals, mathematical signs, and the letter 'x', e.g. '$y = x^2 + 3x$'.

Frege also remarks that with the introduction of the notion of a law, "variability has dropped out of sight, and instead generality comes into view, for that is what the word 'law' indicates" (*ibid.*). One of Frege's conclusions is that the notion of a function has nothing to do with variation, that 'x' does not denote an "indefinite" or "variable" number, but serves to express generality.

In another rejoinder to our paper, Heck and Stanley [124] claimed that we placed too much emphasis on Frege's remarks. They admit that Frege manifests a tendency to explain the notion of a function in terms of the nature of functional expressions, but that this should not obscure the fact that functions, for Frege, are the kind of unsaturated entities which only need to have arguments and values.

In [173] I considered the notion of an arbitrary correlation in the context of Ramsey's criticism of *Principia*'s notion of classes and his moving away from a predicative notion of a function towards the notion of a function-in-extension, which is an arbitrary correlation between arguments and propositions. The idea was to bring another, indirect evidence to my earlier claim with Hintikka to the effect that Frege could not have defended the idea of arbitrary correlation, for that would have placed him in the same camp with Ramsey, against Russell. In fact, I thought that Russell's notion of a propositional function and Frege's notion of a concept stand in deep contrast to Ramsey's notion of a function in extension in his "Foundations of Mathematics" [165]. Some of the arguments in my paper determined Demopoulos to reconsider, in [61], his earlier position with Bell which had attributed to Frege a standard interpretation. The present paper contains some reflections on these matters. The main focus will be on the notion of an arbitrary correlation, but let me start by saying few things on the connection between this notion and Dedekind theorem.

3.2 The Standard versus Non-standard Distinction and Dedekind Theorem

In [129] I and Hintikka claimed that it is the standard interpretation which is the most important for foundations of mathematics, for it is the only one which allows one to formulate descriptively complete categorical axiomatisations of mathematical theories such as number theory and the theory of real numbers (*ibid.*, p. 295). The only concrete example we gave was Dedekind's characterisation of real numbers by means of the cut principle, which says that every bounded set of reals has a least upper. This characterisation is a categorical one only if the sets involved are arbitrary and not restricted, as in Frege's system, to courses of values of concepts expressible in the language of arithmetic. We concluded that "there is a deep sense in which Frege's system is not adequate for interpreting results in contemporary set theory and mathematical theorising, for instance in real analysis" (*ibid.*, p. 314). It is this connection between Frege's non-standard interpretation and the failure of his system to formulate categoricity results that irritated some of our critics, including Demopoulos ([61], pp. 4–5):

> But although Dummett shares Hintikka and Sandu's conclusion that Frege tended toward a non-standard interpretation, his analysis does not support Hintikka and Sandu's evaluation of Frege's foundational contributions. If we follow Dummett, Frege missed the fact that the consistency of [...][*Grundgesetze*], relative to a non-standard interpretation, does not necessarily extend to its consistency when the logic is given a full interpretation. This is certainly an oversight, but it is not the oversight that is appealed to in those of Hintikka and Sandu's criticisms of Frege that so offended some of their critics, as for example, whether, without having isolated the notion of a standard interpretation, Frege could have even conceptualised results like Dedekind's categoricity theorem.

In [61], Demopoulos refers to [120], who showed that Frege proved an analogy of Dedekind's theorem using an axiomatisation of arithmetic that is only a slight variant of the Peano-Dedekind axiomatisation. He also refers to [62] for an argument which questions the systematic dependence of categoricity results on the standard interpretation. The argument shows that categoricity proofs can also be given in a suitably rich first-order theory such as Zermelo-Fraenkel set theory, and these proofs have pretty much the same form as categoricity proofs in second-order logic. His conclusion is this ([61], p. 5):

> Hintikka and Sandu's claim that Frege could not even have formulated (let alone appreciated) these results because of their dependence on the standard interpretation is therefore incorrect both historically and methodologically. It is incorrect historically because Frege successfully proved a categoricity theorem like Dedekind's. And it is incorrect systematically because essentially the same argument establishes the categoricity of second-order arithmetic in any of the usual systems of set theory. And surely it is implausible that only someone familiar with the categoricity of the Peano-Dedekind Axioms as a theorem of second-order logic has really grasped the theorem or its proof. At most, Frege might be charged with having missed a subtlety concerning the distinction between formal and semi-formal systems; but this is hardly surprising for the period in which he wrote.

3.3 The Isomorphism Theorem

In [129] we did not explicitly claim that Frege was unable to formulate, let alone to appreciate, results like Dedekind's theorem. But it is true that our paper suggested it. In the light of [120], that was certainly an oversight. Few things should be said, however. As pointed out by Heck, the statement of this theorem, that is, that two structures which satisfy the Dedekind-Peano axioms are isomorphic, does not appear in *Grundgesetze*. Heck shows how it can be extracted from the proof of the famous theorem 263, which states the conditions under which the number of objects falling under a concept G is *Endlos* (cf. the introduction to the present volume, p. ix, above). The conditions state that there exists a relation Q which is functional, and thus determines a sequence, no object follows after itself in this sequence, each G stands in the relation Q to some object in the series, and the G's are the members of the Q-sequence beginning with some object.

In the proof of this theorem, Frege builds up by induction a binary relation which maps the natural numbers into the members of the Q-sequence, and vice versa. That is, the members of this relation are the pairs $(0, x_0)$, $(1, x_1)$, ..., where $x_0, x_1, \ldots$ are the G's in the order determined by Q. This relation is functional and it preserves both the orderings of the natural numbers and the Q-ordering. For this reason, Heck proposes to call theorem 263 (or rather the theorem 254 which proves the general result that all simple and endless series are isomorphic) 'the Isomorphism Theorem'. It can be proved in second-order logic augmented by the ordered pair axiom. Heck shows the ordered pairs to be dispensable and also suggests that Frege knew his use of ordered pairs to be dispensable so that finally this is a "theorem of second-order arithmetics and logic simpliciter" ([120], p. 322). And when the conditions of the Isomorphism Theorem are rewritten so that one can easily derive from them the more familiar Dedekind-Peano axioms, then the proof of theorem 263 shows that "any two structures satisfying *Frege*'s axioms for arithmetic are isomorphic" (*ibid.*, pp. 324–325).[1]

But although we are told that a modern reader should take the Isomorphism Theorem to show that any two structures satisfying certain conditions are isomorphic, this theorem is not put to much use in *Grundgesetze*. I take Frege's proof and Heck's reconstruction of it to give us a derivation in second-order logic. In the remaining of this section I will look at a more recent argument about categoricity proofs in second-order logic and set theory that seems to support Demopoulos' conclusion.

According to Väänänen ([198], p. 378):

> [...] the situation is entirely *similar* in second-order logic and in set theory. [...] All the usual mathematical structures can be characterised up to isomorphism in set theory by appeal to their second-order characterisation but letting the second-order variables range over sets that are subsets of the structure to be characterised. The only difference to the approach of second-order logic is that in set theory these structures are indeed explicitly defined while in second-order logic they are merely described. In this respect second-order logic is closer to the standard mathematical practice of not paying attention to what the "objects" e.g. complex numbers really are, as long as they obey the right rules.

In the perspective of second-order logic, in mathematics one studies statements of the form

$$\mathbb{M} \vDash \varphi \tag{3.1}$$

where $\mathbb{M}$ is a mathematical structure and φ is a mathematical statement written in second-order logic. Väänänen remarks, that if $\mathbb{N}$ is the structure $\mathbb{N} = (N, +, \times, <)$ and $\varphi_{\mathbb{N}}$ is a second-order axiomatisation of arithmetic, so that we have

$$\forall \mathbb{M}(\mathbb{M} \vDash \varphi_{\mathbb{N}} \Leftrightarrow \mathbb{M} \cong \mathbb{N})$$

[1] What Heck calls 'Frege's axioms for arithmetic' are just the four conditions stated above for the number of objects falling under G to be *Endlos*, where Q is instantiated by the successor relation and G by the concept of a natural number: cf. [119], Sect. 6.

the statement (3.1) can be expressed as a second-order logical truth

$$\models \varphi_{\mathbb{N}} \to \varphi \tag{3.2}$$

The problem knowingly is that the second-order logical truth is not recursively axiomatisable. But, he continues, there are two stronger versions of (3.2), one in set theory and the other one in second-order logic:

$$ZFC \vdash \forall \mathbb{M}(\mathbb{M} \models \varphi_{\mathbb{N}} \to \mathbb{M} \models \varphi) \tag{3.3}$$

and

$$CA \vdash \varphi_{\mathbb{N}} \to \varphi \tag{3.4}$$

where *CA* is a second-order axiomatisation of second-order logic including a comprehension axiom and the axiom of choice. He thinks, then, that it is reasonable to give this later statement as a justification of (3.2). And he immediately adds this (*ibid*., p. 375):

> I have called (3.3) and (3.4) *stronger* forms of (3.2) because I take it for granted that *ZFC* and *CA* are *true* axioms. It is not the main topic of this paper to investigate how much *ZFC* and *CA* can be weakened in this or that special instance of (3.3) and (3.4), as such considerations do not differentiate second-order logic and set theory from each other in any essential way.

Väänänen (*ibid*., p. 376) notices, in fact, an apparent difference between (3.4) and (3.1): (3.1) is about the material truth of the statement φ in a (standard) model $\mathbb{N}$, whereas (3.4) seems to assert something that holds in all the models of *CA*, standard and non-standard. But he immediately points out this is only an appearance, as it can be seen by considering two versions of the sentence $\varphi_{\mathbb{N}}$ for the structure $\mathbb{N} = (N, +, \times)$: $\varphi^1_{\mathbb{N}}$ in the vocabulary $\{+_1, \times_1\}$, and $\varphi^2_{\mathbb{N}}$ in the vocabulary $\{+_2, \times_2\}$. If '*CA*' now denotes the axiomatisation of second-order logic in a vocabulary that includes both $\{+_1, \times_1\}$ and $\{+_2, \times_2\}$, then we have

$$CA \vdash \varphi^1_N \wedge \varphi^2_N \to Isom_{1,2}$$

where '$Isom_{1,2}$' denotes the statement of second-order logic stating that there is a bijection f such that

$$\forall x \forall y [f(x +_1 y) = f(x +_2 y)]$$
$$\forall x \forall y [f(x \times_1 y) = f(x \times_2 y)]$$

The conclusion is as follows (*ibid.*):

> [...] in this subtle sense, (3.4) really asserts the truth of φ in one and only one model, namely the standard model. [...] Naturally, *CA* itself has non-standard models but they should not be the concern in connection with (3.4) because we are not studying *CA* but the structure [...][$\mathbb{N}$]. In fact the whole concept of a model of *CA* is out of place here as *CA* is used as a medium of evidence for (3.2). We can convince ourselves of the correctness of the evidence by simply looking at the proof given in *CA* very carefully. There is no infinitistic element in this.

The situation is similar in set theory. In this perspective, in mathematics one studies statements of the form

$$\Phi(a) \tag{3.5}$$

where "$\Phi(x)$ is a first-order formula with variables ranging over the universe of sets, and a is a set" (*ibid.*, p. 377). Now we are told that (*ibid.*):

> If we compare (3.1) and (3.5), we observe that the former is restricted to one presumably rather limited structure [...][$\mathbb{M}$] while (3.5) refers to the entire universe. This is one often quoted difference between second order-logic and set theory. Second-order logic takes one structure at a time and asserts second-order properties about that structure, while set theory tries to govern the whole universe at a time.

Two qualifications are added. The first is this, "while it is true that (3.5) refers to the entire universe, typical mathematical propositions are really statements about some V_α such that $a \in V_\alpha$ (*ibid.*). The second qualification concerns the justification of (3.5), which raises the same worries as the justification of (3.1) in second-order logic. The justification is given by the stronger statement

$$ZFC \vdash \Phi(a) \tag{3.6}$$

where a is assumed to be a definable set. Väänänen concludes that there is no fundamental difference between set theory and second-order logic.

But, then, Väänänen wonders, "which is the right way to do mathematics: second-order logic or set theory?" (*ibid.*, p. 379). And here is, finally, his answer (*ibid.*):

> Let us leave aside the question whether the higher ordinals that exist in set theory are really needed. The point is that set theory is just a "taller" version of second-order logic, and if one does not need (or like) the tallness, then one can replace set theory by second - (or higher-) order logic. However, this does not yield more categoricity, for both second-order logic and set theory are equally "internally categorical". If we look at second-order logic and set theory from the outside we enter meta-mathematics. Then we can build formalisations of the semantics of either second-order logic or set theory and prove their categoricity in "full" models as well as their non-categoricity in "Henkin" models.

This ends my exposition of Väänänen's arguments. The point I want to underline, against their author, which is the same as the point I raised earlier in connection with Demopoulos's and Heck's discussion of Frege's proof of Dedekinf theorem, is that (3.4), like its set-theoretical counterpart (3.6), is just a formal derivation. None of them stands by itself in the context of justification. One cannot "take for granted that

ZFC and *CA* are true axioms" and assert in the same time that "the whole concept of a model of *CA* is out of place here as *CA* is used as a medium of evidence for (3.2)". The point is not so much that of looking at second-order logic or set theory from the inside or outside, but rather that of a derivation having a content or not. If it does not, then it cannot serve as a "medium of evidence", and this for the simple reason that it does it not refer to the concepts it purports to refer.

3.4 Ramsey's Notion of a Predicative Function in "Foundations of Mathematics"

In [173], I suggested to look at the question of the notion of arbitrary correspondence from a different angle: Ramsey's criticisms of the notion of propositional function in Whitehead and Russell's *Principia* [203]. My punching line was that Frege's functions (concepts) are as predicative as *Principia*'s propositional functions and thereby Ramsey's criticisms of the logic of *Principia* and his conclusion that this logic is inadequate for the logicism programme (the reduction of mathematics to logic) apply *mutatis mutandis* to Frege's logic.

Ramsey criticises *Principia*'s notion of a propositional function arguing for the need of its extension, in the context of the logicist reduction of mathematics to logic. Ramsey anticipates Carnap's distinction between two kinds of logicist reductions, respectively depending on:

1. the definition of all concepts of a mathematical theory in terms of logical notions.
2. (1) plus the derivation of the axioms of the resulting theory from purely logical axioms.

Carnap [41] points out that Russell operated a reduction of type (1). In his anticipation of this distinction, Ramsey observes that a reduction of type (1) would show the generality of mathematics, while a reduction of type (2) would illustrate the necessity of mathematics. The sense of necessity Ramsey is concerned with in his remarks is that according to which tautologies, in Wittgenstein' sense (namely sentences true in every universe of discourse), are necessary. Ramsey observes that in order to perform a reduction of type (2), one would have to give up the notion of propositional function to be found in *Principia*.

Let me comment first on Ramsey's notion of a predicative function in [165]. I will rely heavily on Trueman's reconstruction of it in [194].

We start with a class of propositions built from a stock of atomic propositions of the form 'John is tall', through (possibly an infinite application of) truth-functional connectives. Ramsey ([165], p. 35) defines a propositional function of individuals, as "a symbol of the form '$f(\hat{x}, \hat{y}, \hat{z}, \ldots)$'" such that every replacement in it of '$\hat{x}$','$\hat{y}$','$\hat{z}$', …with names of individuals yields a proposition of the initial class. Ramsey takes, moreover, a propositional function '$f(\hat{x}, \hat{y}, \hat{z}, \ldots)$' to be identical with '$g(\hat{x}, \hat{y}, \hat{z}, \ldots)$' if the substitution of the same set of names in one and the other yields the same proposition, that is, if '$f(\hat{x}, \hat{y}, \hat{z}, \ldots)$' and '$g(\hat{x}, \hat{y}, \hat{z}, \ldots)$' have the same truth-table. The definition extends to higher-order propositional functions.

To specify the subclass of propositional functions that Ramsey calls 'predicative functions' we need first to specify atomic predicative function of individuals. They are "the result of replacing by variables any of the names of individuals in an atomic proposition expressed by using names alone" (*ibid.*, p. 38). Thus '$\hat{x}$ is tall' is an atomic predicative function.

This notion is then extended to cover truth-functions of propositional functions and propositions. Ramsey's definition is as follows (*ibid.*):

> Suppose we have functions $\phi_1(\hat{x}, \hat{y})$, $\phi_2(\hat{x}, \hat{y})$, etc., then saying that a function $\psi(\hat{x}, \hat{y})$ is a certain truth-function [...] of the functions $\phi_1(\hat{x}, \hat{y})$, $\phi_2(\hat{x}, \hat{y})$, etc. and the proposition p, q, etc., we mean that any value of $\psi(\hat{x}, \hat{y})$, say $\psi(a, b)$, is that truth-function of the corresponding values of $\phi_1(\hat{x}, \hat{y})$, $\phi_2(\hat{x}, \hat{y})$, etc., i.e. $\phi_1(a, b)$, $\phi_2(a, b)$, etc. and the propositions p, q, etc.

Hence, '$F(\hat{x_0}, \hat{x_1}, \ldots, \hat{x_n})$' is a truth-function of some propositional functions and propositions if and only if any of its values for some appropriate arguments is the corresponding truth-function of the values of these propositional functions for these arguments and of these propositions. To take an example '$[G(\hat{x_1}, \hat{x_2}) \vee p] \wedge [H(\hat{x_1}, \hat{x_2}) \vee q]$' is a certain truth-function of '$G(\hat{x_1}, \hat{x_2})$', '$H(\hat{x_1}, \hat{x_2})$', p, and q, since for whatever names 'a', 'b', 'c, 'd', '$[G(a, b) \vee p] \wedge [H(c, d) \vee q]$' is this same truth-function of '$G(a, b)$', '$H(c, d)$', p, q.

Finally Ramsey defines predicative functions of individuals as follows (*ibid.*, p. 39):

> A *predicative function* of individuals is one which is any truth-function of arguments which, whether finite or infinite in number, are all either atomic functions of individuals or propositions.

Hence, a predicative function of individuals is "a (perhaps infinite) truth-function of atomic predicati[...][ve] functions of individuals and propositions", and, vice versa such a truth-function is a predicative function of individuals ([194], p. 294).

What needs to be emphasized is that, according to this definition, the predicativity of a propositional function '$F(\hat{x})$' consists, as generally agreed, in the fact that the proposition '$F(a)$', which '$F(\hat{x})$' assigns to 'a', says or predicates the same thing of a as the proposition '$F(b)$', which '$F(\hat{x})$' assigns to 'b', does of b.

I regard Ramsey's definition of a predicative function as a manifestation of the phenomenon we discussed in connection with Frege: the specification of a function by a formal law. In this case the "glue" which keeps the arguments and values together is a propositional function. A good example is our earlier propositional function '$\hat{x}$ is tall', which maps 'Socrates' to 'Socrates is tall' and 'Plato' to 'Plato is tall', i.e.,

'F(Socrates)' is 'Socrates is tall'
'F(Plato)' is 'Plato is tall'.

Let me finally mention that Ramsey uses his notion of propositional function to give an account of quantification in the *Tractatus*. The proposition '$\forall x F(x)$' is conceived of as the conjunction of all the values of '$F(\hat{x})$', and the proposition '$\exists x F(x)$' as the disjunction of all these propositions. Similarly, the higher-order '$\forall\varphi f(\varphi(\hat{x}))$' is the conjunction of all the values of '$f(\varphi(\hat{x}))$' ([165], p. 40). In addition, the use of quantifiers is governed by what is known as the exclusive interpretation of quantifiers, e.g. '$\exists x R(x, a)$' is conceived of as a disjunction of all the values of '$R(\hat{x}, a)$' except for '$R(a, a)$', and similarly for '$\forall x R(x, a)$', which is conceived of as the conjunction of these values.

3.5 Ramsey's Reduction of Type (2)

Reduction (2) is achieved in two steps, following Whitehead and Russell. In the first step mathematics is reduced to the theory of classes, e.g., each natural number n is defined as the class of all n-membered classes of individuals. For instance, 1 is defined as the class of all singletons, 2 as the class of all doubletons, etc. In the second step, the theory of classes is reduced to logic. It is here that propositional functions are needed, as class-terms are partially eliminated in favour of propositional functions. As a result of this process, every class is presented as the extension of a propositional function. But as Trueman observes Ramsey realised that if the only admissible propositional functions are predicative functions, then there can be no reduction of mathematics to logic. As logical truths are tautologies, then the failure of this reduction would also be a failure to show that mathematical truths are tautologies in Wittgenstein's sense.

Trueman spells out nicely what is at stake here ([194], p. 296):

> If 1 is defined as the class of singletons of individuals and 2 as the class of doubletons of individuals, then the mathematical truth that $1 \neq 2$ requires that there be a singleton or a doubleton: otherwise, 1 and 2 would both be empty and hence identical. If, in turn, the existence of a singleton or doubleton of individuals is to be reduced to logic then, assuming […][that all logical truths are tautologies, and vice versa], it must be a tautology that some propositional function is true of exactly one or exactly two individuals. But, if every propositional function is predicati[…][ve] then this is not a tautology, and this is because predicati[…][ve] functions do not logically discriminate between individuals, meaning that it is not contradictory for every individual to satisfy exactly the same predicati[…][ve] functions as every other individual

These considerations illustrate the kind of challenge Ramsey faced. Assume there are only two individuals, a and b. If it is a tautology that some propositional function is true of only these two individuals, then it must be a contradiction that, say, every atomic predicative propositional function '$F(\hat{x})$' is satisfied by both 'a' and 'b'. But the fact that every atomic predicative propositional function '$F(\hat{x})$' may be satisfied by both 'a' and 'b' is something that follows from the logical independence of atomic propositions: atomic propositional functions do not discriminate between individuals.

The argument extends then to the general case. As every predicative function is a truth-function of atomic predicative functions and propositions, it is not a contradiction that every individual which satisfies one function, satisfies also another. So as Trueman points out it cannot be a tautology that some predicative function is true of exactly one individual, or exactly two individuals, etc.

3.5.1 Logical Necessity versus Analytical Necessity

It has been emphasised (*ibid.*) that the argument establishing that predicative functions do not logically discriminate between individuals at no point appeals to the Tractarian assumption that all necessity is logical necessity. We could, for instance, introduce a different notion of necessity, call it 'analytic necessity', which is necessity in virtue of meaning. This will not rule out the possibility that there are two individuals who satisfy the same predicative functions, provided the atomic propositions would remain logically independent in the above sense of logical necessity.

Such a notion of necessity has been considered, among others, by the Finnish logician Erik Stenius [186]. According to Stenius, a statement is analytic if it is true in virtue of the semantic conventions for certain of its symbols. Alternatively, a statement is analytic if, according to the semantic conventions for some of its expressions, no state of affairs is a truth restriction for it (that is, no state of affairs makes it false).

The statement 'If a is red, then a is not green' symbolised by

$$\text{`}R\,(a) \rightarrow \neg G\,(a)\text{'} \tag{3.7}$$

can be shown to be analytic in Stenius's sense. Its truth-table is

$R\,(a)$	$G\,(a)$	$\neg G\,(a)$	$R\,(a) \rightarrow \neg G\,(a)$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

This truth-table seems to possess what Stenius calls, a truth-restriction, that is, a state of affairs which renders the proposition '$R\,(a) \rightarrow \neg G\,(a)$' false. But for Stenius this truth restriction is not a state of affairs because the colours green and red are logically incompatible. Therefore the first line must be erased and the truth restriction vanishes.

We witness here a violation of the logical independence of the atomic propositions which is of a different kind than the one we have considered so far: the truth of '$R\,(a)$' is incompatible with that of '$G\,(a)$', that is, one and the same individual cannot simultaneously be the argument of both propositional functions '$R(\hat{x})$' and '$G(\hat{x})$'. Given that (3.7) is analytic for each a, then so is

'No red objects are green'

symbolised by

$$\text{‘}\forall x\,[R(x) \to \neg G\,(x)]\text{’}$$

Thus the failure of logical independence in this new sense leads to the new notion of analytical necessity. But allowing for this kind of non-logical, analytical necessity is perfectly compatible with the logical independence of atomic propositions from the previous section.

Ramsey wanted to show that mathematical statements are logically necessary in the Tractarian sense. For this he needed to give up the kind of logical independence of atomic propositions we considered in the previous section and find a notion of propositional function which would discriminate between individuals and would not be grounded in the notion of analytical necessity illustrated in this section. He did that by introducing the notion of a propositional function in extension. Before discussing it, let me point out that Frege went a different way: although he wanted to show that mathematical (arithmetical) statements are reducible to logic, he did not conceive of logical statements as necessary in the Tractarian sense. For Frege logical statements are the most general statements about a universe of discourse. No wonder that the modern discussion around Frege's logicism ended up in debating whether the ultimate logical principles to which arithmetics is reduced are analytic or not.

3.5.2 Ramsey's Propositional Functions in Extension

Ramsey needed, then, a new notion of a propositional function which would allow him to distinguish between individuals. In order to do this, he needed to extend the notion of a function to cover also non-predicative propositional functions, where predicativity is understood as above. Here is how he expresses himself ([165], p. 52):

> The only practicable way to do it as radically and drastically as possible; to drop altogether the notion that $\varphi\,(a)$ says about a what $\varphi\,(b)$ says about b, to treat propositional functions like mathematical functions, that is, to extensionalise them completely. Indeed it is clear that, mathematical functions being derived from propositional, we shall get an adequate extensional account of the former only by taking a completely extensional view of the latter.

Ramsey is well aware that he cannot give an explicit definition of a function in extension and for this reason he contents himself to explain this notion rather than define it. His explanation is given in terms of the notion of correlation, that is, a relation in extension between propositions and individuals, which associates to each individual a unique proposition. In specifying the nature of this correlation, he remarks that it may be "practicable or impracticable" (*ibid.*). I take this to be just another way for Ramsey to say that the correlation is not determined by a formal law, but is an arbitrary association between individuals and propositions.

Ramsey uses propositional functions in extension to define identity:

$$x = y \quad =_{df} \quad \forall \varphi_e\,[\varphi_e\,(x) \equiv \varphi_e\,(y)]$$

where φ_e takes propositional functions in extension as values. We notice that when a and b are the same individual, then '$a = b$' is the conjunction of '$p \equiv p$', '$q \equiv q$', ..., which is a tautology. On the other side, when a is distinct from b, then there is a propositional function '$\varphi_e \left(\hat{x}\right)$' such that '$\varphi_e (a)$' is '$p$' while '$\varphi_e (b)$' is '$\neg p$'. In this case '$a = b$' is a conjunction of propositions which includes '$p \equiv \neg p$', that is, a contradiction.

After having defined identity, Ramsey can introduce set-theoretical notions. In order to introduce singletons, he considers the propositional function '$\hat{x} = a$', where a is an arbitrary individual. When identity is defined as above, then '$a = a$' is a tautology, and for any other b, '$b = a$' is a contradiction. Hence it is a tautology that some propositional function is true of exactly one individual. By a similar reasoning one can introduce doubletons. The propositional function '$\hat{x} = a \vee \hat{x} = b$' is true of exactly two individuals. (I am indebted to Trueman for this argument.)

It is perhaps worth comparing Ramsey's definition of identity to the modern model-theoretic definition of identity in second-order logic with the standard interpretation:

$$x = y \Leftrightarrow \forall X \left[X(x) \Leftrightarrow X(y)\right]$$

Here X is a second-order variable ranging over sets. By the standard interpretation we mean that every model for the second-order language is such that the range of the second-order variables is the full power set of the set which is the range of the first-order variables. In this setting, instead of showing that '$a = a$' is a tautology, and for any individual b distinct from a, '$b = a$' is a contradiction, we can show that in every model in which a and b are the same individual, then '$\forall X \left[X (x) \Leftrightarrow X (y)\right]$' is (trivially) true. And in every model in which a is distinct from b, '$\forall X \left[X (x) \Leftrightarrow X (y)\right]$' is false. Indeed, the first claim is true: it follows from the principle of extensionality of sets. As for the second claim, the set $\{a\}$ falsifies the formula '$\forall X \left[X (x) \Leftrightarrow X (y)\right]$'. Given the standard interpretation, this set exists. Notice that the only principle we need to rely on is the extensionality of sets.

We can achieve the same result by using functions. In this case the definition of identity would be

$$x = y \Leftrightarrow \forall f \left[f(x) \Leftrightarrow f(y)\right]$$

where we may take f to be a function from individuals to truth-values. Then '$a = a$' is a tautology and '$b = a$' is false in every model in which a and b are distinct individuals: take a function f which maps a to T and b to F. Here we need the standard interpretation of function variables and the notion of function in extension. Then, we can go on and reconstruct singletons and doubletons as Ramsey did.

In the remaining of mys paper let me consider two objections against the notion of propositional functions in extension discussed in [194]. One of them is due to Sullivan, the other to Wittgenstein.

3.5.3 Sullivan's Objection to the Notion of Propositional Function in Extension: Containment

According to Sullivan [187], the main difference between propositional functions and propositional functions in extension lies in the fact that the former are contained in their values in a way in which the latter are not. In other words, a propositional function in extension needs all its values to be individuated, whereas one single value suffices for the individuation of a (predicative) propositional function. It is not difficult, intuitively, to see why this is so. Take any argument and consider the proposition which is the value of the propositional function for that argument. By deleting the argument, you can recover the propositional function. To take an example, if '$F(\hat{x})$' is a propositional function and you know that 'F (John)' is 'John is tall', then you also know that 'F (Peter)' is 'Peter is tall', etc. On the other side, if you know that 'φ_e (John)' is 'Paris is beautiful' then you cannot infer anything about 'φ_e (Peter)', when φ_e is a function in extension.

Trueman ([194], Sect. 4) gives an example of a propositional function which shows Sullivan's argument to be invalid. Here it is take the function

$$\text{`}P(\hat{x}) \vee \exists y \left[T\ (\text{Plato}) \wedge \neg P\ (\hat{y})\right]\text{'} \tag{3.8}$$

where '$T(\hat{x})$' is

$$\text{`}P(\hat{x}) \vee \neg P(\hat{x})\text{'}$$

Consider first the value of this function for an argument 'a' other than 'Plato', i.e.

$$\text{`}P\ (a) \vee \exists y \left[T\ (\text{Plato}) \wedge \neg P\ (\hat{y})\right]\text{'} \tag{3.9}$$

By the convention governing the use of quantifiers, '$\exists y \left[T\ (\text{Plato}) \wedge \neg P\ (\hat{y})\right]$' is a disjunction of the values of 'T (Plato) $\wedge \neg P\ (\hat{y})$' for every argument other than 'Plato'. But given that 'T (Plato)' is a tautology, and the conjunction of a proposition with a tautology is that proposition itself, this conjunction is equivalent to the conjunction of the values of '$\neg P\ (\hat{y})$', for every argument other than 'Plato', one conjunct of which will be '$\neg P\ (a)$'. Hence (3.9) will be a disjunction including both '$P\ (a)$' and '$\neg P\ (a)$' as disjuncts, and will, then, be a tautology. On the other side, the value of (3.8) for 'Plato' is the the disjunction of 'P (Plato)' and the values of '$\neg P\ (\hat{y})$' for every argument other than 'Plato'. It will, then, be the disjunction of an atomic proposition with the negation of another atomic proposition, and will not be a tautology.

It follows that (3.8) maps every name other than 'Plato' to a tautology, and 'Plato' to a non-tautology. Trueman concludes that we need all the values of this propositional functions in order to establish its identity, and thereby Sullivan's claim should be restricted to atomic predicative propositional functions: only in this case the propositional function may be recovered by whatever value of the function one considers.

As nice as this example is, one should not overestimate its importance, though (I also take this to be Trueman's position). Its particularity is due to the convention governing the use of quantifiers that we discussed above. Even if the property of containment held only for atomic predicative propositional functions, it would still explain why these functions are more accessible than their extensional relatives and how our conceptual system can somehow integrate and manipulate potentially infinite correlations of arguments and values. For in the absence of properties like containment or other mechanisms which perform a similar function, the question still remains: How are we to understand the notion of mapping? Moreover, is there any way for us to grasp potentially infinite correlations?

3.5.4 Substitution

I take the notion of containment to provide an answer to the second question. One possible answer to the first question that Trueman considers is to understand mapping in terms of substitution. This is an expected move: after all, we needed substitution when we explained the notion of atomic predicative propositional function in the first place. We took such a function to be the result of replacing by variables any of the names of individuals in an atomic proposition. An example may help. For '$F(\hat{x})$' standing for the atomic propositional function '$\hat{x}$ is wise', when we substitute '$\hat{x}$' with 'Socrates' we thereby generate 'Socrates is wise' in which 'Socrates' occurs as a name of Socrates. The sense in which non-atomic predicative functions "map" names to propositions is explained analogously. It is quite clear that substitution, as a mechanical operation on expressions in an underlying language, explains the property of containment and thus also answers the second question considered above.

The operation of substitution cannot obviously ground the notion of mapping that underlies propositional functions in extension. One has to try something else. Returning to our last example, we notice that the operation of substitution generates a table:

$F(\hat{x})$	
'Socrates'	'Socrates is wise'
'Plato'	'Plato is wise'

Following the same idea, we could also introduce atomic predicative propositional functions in extension by tables, e.g.:

$F_e(\hat{x})$	
'Socrates'	'Queen Anne is dead'
'Plato'	'Einstein is a great man'

As expected, this suggestion is not shared by those who oppose arbitrary correlations. For the whole matter of dispute is the nature of the relation between the name on the left side, and the corresponding proposition on the right side. Trueman endorses an argument by Wittgenstein ([204], part II, Chap. 16) who points out that the name 'Socrates' appears here only to direct us to a line of the table. We could have marked instead the lines of this table with any signs we liked: numerals, letters or squares of colour, etc. The fact that we chose to mark each line of this table with strings which look like the names of Socrates and Plato should not mislead us into thinking that they are those names. Consequently, if '$F_e\left(\hat{x}\right)$' is a predicative function defined as in the table above, then the first and second occurrences of the string 'Socrates' in 'F(Socrates)$\wedge F_e$ (Socrates)' have different significances, the first is an occurrence of the name of Socrates and the latter is not.

We are back to square one. Wittgenstein's criticism is nothing else but a milder expression of the requirement that we have seen at work in the case of predicative propositional functions. According to that requirement, the proposition '$F\ (a)$' that the propositional function '$F\left(\hat{x}\right)$' assigns to 'a', must say or predicate the same thing of a as the proposition '$F\ (b)$', which '$F\left(\hat{x}\right)$' assigns to b, predicates of b. The present version is milder because it only asks for the value that the function assigned to 'a' to say something about a. Still it is obvious that in both cases we witness the refusal to accept the idea that what is important for individuating a function is an arbitrary correlation of values and arguments.

3.5.5 Arbitrary Functions

This is, then, Wittgenstein's criticism of Ramsey's notion of a propositional function in extension: the argument of such a function is there "only to direct us to a line of the table". In other words, when a function in extension is introduced, one abstracts from the nature of the connection between arguments and values and makes sure only that, to each argument, there is a line in the table.

Wittgenstein's criticism sounds surprisingly similar to Frege's criticism of the extensional notion of a set and of the individuation of sets through their members. In [129], p. 301, we pointed out two different reasons for Frege to reject the individuation of classes through their members. The first one concerns the definition of the empty class. Here is what Frege writes in an undated letter to Peano ([106], vol. II, p. 177; [108], p. 109):

> Of course, one must not then regard a class as made out by the objects (individual, entities), that belong to it; for removing the objects one would then also be removing the class constituted by them. Instead, one must regard the class as made out by the characteristic marks, i.e., the properties which an object must have if it is to belong to it. It can then happen that these properties contradict one another, or that there occurs no object that combines them in itself. The class is then empty but without being logically objectionable for that reason.

The second reason concerns the individuation of infinite classes. According to Frege, from the finiteness of the human intellect it follows that an infinite class cannot be given solely by its members. The only way it can be given is by deriving it from a concept, that is, by takings it as "yielded by thought". Here is what Frege writes in "Booles rechnende Logik und die Begriffsschrift" ([106], vol. I, p. 38; [107], p. 34; notice that this passage illustrates the first reason, too). This is also made clear in the following passage:

> But it is surely a highly arbitrary procedure to form concepts merely by assembling individuals, and one devoid of significance for actual thinking unless the objects are held together by having characteristics in common. It is precisely these which constitute the essence of the concept. Indeed one can form concepts under which no object form, where it might perhaps require lengthy investigation to discover that this was so. Moreover, a concept, such as that of number, can apply to infinitely many individuals. Such a concept would never be attained by logical addition. Nor finally may we presuppose that the individuals are given *in toto*, since some, such as e.g. the numbers, are only yielded by thought.

As we observed in [129], p. 305, "what made possible the conception of an arbitrary set was the gradual disentanglement of the notion of set from intensional ingredients such as concepts, properties, etc., and the definition of sethood in an alternative way". In a parallel development, the modern notion of arbitrary function emerged through the gradual disentanglement of the notion of correlation from Fregean concepts, equations and other formal rules, or from requirements like predicativity. Ramsey's notion of a propositional function in extension is one step in this process of emancipation. Modern logic has developed Ramsey's idea and taken functional dependencies as arbitrary correlations between values and arguments. Here is one example which illustrates this trend taken from [116].

The idea, made possible by the development of model-theoretical semantics, is not to define arbitrary functional correlations, but to introduce a new logical constant in the object language and then give its meaning through a semantical clause. More specifically, the syntax of first-order logic is extended with atomic formulas of the form

$$=(\overrightarrow{x}, y)$$

intended to express arbitrary functional dependence: the (values of the) variables $\overrightarrow{x}$ totally (functionally) determine (the value of) y. Such an atom is interpreted in a model by a set X of (partial) assignments in the universe of the model. The semantical clause that we need is:

1. X makes the formula '$=(\overrightarrow{x}, y)$' true if and only if for any two distinct assignments s and s' in X, whenever s and s' agree on the values of the variables in $\overrightarrow{x}$, they also agree on the values of y.

The right-hand side of this double implication defines a functional correlation in purely extensional terms, without appealing to any particular relation between an argument and its value. Here is an example ([197], p. 11), which also illustrates the kind of extensional correlation Wittgenstein objected to:

	x_0	x_1	x_2
s_0	1.5	4	0.51
s_1	2.1	4	0.55
s_2	2.1	4	0.53
s_3	5.1	4	0.54
s_4	8.9	4	0.53
s_5	21	4	0.54
s_6	100	4	0.54

The set X consisting of the six assignments $s_0, \ldots, s_6$ makes both $=(x_0, x_1)$ and $=(x_0, x_1)$ true.

Grädel and Väänänen define also independence. To this purpose, the syntax of first-order logic is extended with atomic formulas of the form

$$x \perp y$$

with the intended interpretation: the (values of the) variable y is (are) independent of the (values of the) variable x. Such a formula is interpreted by a set X of assignments, as in the previous case, but now the interpretative semantical clause is:

2. X makes '$x \perp y$' true if and only if for any two assignments s and s' in X there is a third assignment s'' such that s'' agrees with s on the value of x and it agrees with s' on the value of y.

This definition tells us that the value $s(x)$ of x alone does not determine the value $s(y)$ of y, for there may be another assignment s' in X which assigns to y a distinct value, i.e. $s'(y) \neq s(y)$. But then according to the proposed definition, there is a third assignment s'' such that $s''(y) = s'(y)$ and $s''(x) = s(x)$. That is, just when we thought that on the basis of $s(x)$ we can conclude that the value of y is $s(y)$, we discover s'' which gives the same value for x but a different value for y. In other words, borrowing Wittgenstein's jargon, there is an argument which "points to two lines" in the table. In our example, X makes both '$x_1 \perp x_2$' and '$x_0 \perp x_2$' true.

3.6 Conclusion

In [62], Bell and Demopoulos accept Dummett's initial view to the effect that Frege's interpretation of the function variables is the standard one and that Frege's concept of a function coincides with the set-theoretic notion of an arbitrary correspondence, in which case the domain of the function variables is in one-one correspondence with the power-set of the domain of the individual variables. In [61], reconsiders this matter (*ibid.*, pp. 5–6):

> More recently, reflection occasioned by reading [173] has convinced me that the equation of Frege's concept of a function with the notion of an arbitrary correspondence should be reconsidered, and that it might be fruitful to reconsider it from the perspective of Ramsey's interpretation of *Principia*s's propositional functions.

Demopoulos's conclusion is that Frege's assimilation of concepts to functions which map into truth values is as predicative as Russell. The correspondence is not arbitrary, but is constrained by the principle that if a function maps two objects to the True, they must fall under a common concept. But he also observes that Fregean functions and concepts lack the explicit association with propositions that is characteristic of Russellian propositional functions. *Principia*'s propositional functions "map to the truth values only by 'passing through' a proposition" whereas "Frege's concepts map directly to the truth values" (*ibid.*, p. 16). Despite his acknowledgement that Fregean concepts are constrained in the way mentioned above, Demopoulos is reluctant to explicitly admit that Frege's notions of a function is not extensionalist in nature. He prefers to close his paper in a rather ambiguous way, as follows (*ibid.*, pp. 16–17):

> Fregean functions and concepts [...] lack the explicit association with propositions that is characteristic of propositional functions; an extensionalist interpretation of a Fregean concept as an arbitrary mapping of objects to truth values is arguably still a Fregean concept. However its utility for Frege's theory of classes is unclear. According to a theory like Frege's, concepts provide the principle which gives classes their 'unity', and they also serve the epistemological function of providing the principle under which a collection of objects can be regarded as a separate object of thought. A class that is generated by an arbitrary pairing of individuals with truth values might be one that is 'determined by a concept', but the concept which determines it seems no more epistemically accessible than the collection itself. Even if it can be convincingly argued that such concepts sustain the unity of the classes they determine, it can hardly be maintained that they are capable of playing the epistemological role which the predicative interpretation can claim for its functions and concepts.

The overall conception that dominates the present paper as well as the ideas developed in [129] and [173], is that Russell's notion of a propositional function and Ramsey's notion of predicative function are one more manifestation, albeit a special one, of the same phenomenon which governs Fregean concepts: their determination by a norm (rule, equation, concept). If that were not the case, then they would not be able to perform, the epistemological function that Demopoulos attributes to them. Now, in the last quote Demopoulos speculates with the idea that a class that is generated by an arbitrary function might still be generated by a concept which is epistemically inaccessible. I take the point of this remark and of those that follow it to be that of emphasizing that there is still a considerable gap between Fregean concepts (functions) on one side, and Russell's and Ramsey's predicative functions, on the other.

A detailed comparison between predicative functions and Fregean concepts is outside the purpose of this paper. The point I tried to defend here and elsewhere is that both Frege's and Russell's conceptions of a function stand in clear contrast to Ramsey's notion of a propositional function in extension and to the extensionalist notion of a function illustrated by clause **1** in Sect. 3.5.5, above. There is no doubt that Frege could not have such a conception, for he tells us ([97], Sect. I.10; [110], p. 161):

> We have only a way always to recognise a value-range as the same if it is designated by a name such as $\dot{\varepsilon}\Phi(\varepsilon)$, whereby it is already recognisable as a value-range. However, we cannot decide yet whether an object that is not given to us as a value-range is a value-range or which function it may belong to; nor can we decide in general whether a given value-range has a given property if we do not know that this property is connected with a property of the corresponding function.

As this passage illustrates, For Frege, "an object that is not given to us as a value-range", i.e. that is not introduced as the extension of a law or concept, does not tell us what function that value-range corresponds to. True, Frege was possibly thinking here of any object whatsoever, and not necessarily of one that is easily identifiable as a value-range of some indeterminate function; his point seems to be that taking the value-range of a function $\Phi\ (\xi)$ to be the same as the value-range of a function $\Psi\ (\xi)$ if and only if the values of these functions are the same for any argument does not allow us to decide whether a certain table, the Mount Blanc, or Julius Caesar are value-ranges. But, it is a matter of fact that his claim is general, and it also applies, then, to objects that are easily identifiable as value-ranges, namely to classes. In this case, the point becomes that, when a class is given to us as such and not as a value-range of a determinate function, there is no vantage point from which we can say what function it is the value-range of. Ramsey's notion of propositional function in extension and the notion of functional dependence illustrated by clause **1** in Sect. 3.5.5, above may be seen as the perfect target of Frege's critical remark: the set of assignments, or, as we may call it, the value-range X in that clause, may be the extension, as we all know, of many functional laws.

What Frege and Russell ignored and Ramsey realized, is that one can and needs to talk about a function even when one is not able to individuate it through the law that generates is, like for instance when one talks about the properties *all* functions have. In that case one abstracts from the nature of the formal law that generates the corresponding extension. The framework outlined in the previous section allows one to do just that. According to clause **1**, a functional dependence is, indeed, just a set satisfying appropriate conditions (namely Armstrong axioms in data base theory, as showed by Väänänen in [197], Sects. 8.1 and 8.2). This provides an extensionalist notion of a function, akin in spirit to Ramsey's notion of a function in extension, which anticipates its treatment in contemporary mathematics: a notion that stands opposite both to Frege's conception, according to which a function is constrained by a law, and to Russell's idea of a propositional function.