

ALGEBRAIC SYMBOLISM IN MEDIEVAL ARABIC ALGEBRA

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ABSTRACT

Medieval Arabic books read more or less like transcriptions of lectures, a fact that stems from the fundamentally oral nature of medieval Islamic civilization. Books covering algebra are no exception. There we find algebraic solutions to problems written all out in words, with no notation even for numbers. But problems were not solved rhetorically. Algebraists would work out a problem in some kind of notation on a dust board or other ephemeral surface, and to record it in a book a rhetorical version of the calculations would be composed. From early texts it appears that only the coefficients of polynomials were written in Hindi notation. But in the Maghreb in the twelfth century a notation specific to algebra developed in which the power was indicated also. This symbolic notation is not presented as a scientific development in the manuscripts, but is instead shown in textbooks to instruct students in its use. In this article we explain the notation, its context, and how it differs from modern notation.

1. Introduction

There are over eighty different extant medieval Arabic mathematics books with “algebra” in the title.¹ Add to these the many arithmetic texts which treat algebra in one or more chapters, and it may seem that we are in a good position to know how people worked out problems by algebra in those times. But because medieval Arabic books played a part in a predominantly oral culture, the algebra we encounter in the manuscripts is rhetorical, with all operations, equations, and even numbers themselves written out in words. The actual calculations were worked out in notation on a dust-board or other ephemeral surface, and authors of books then verbalized the results.

The few samples of notation we do find in medieval texts are provided as figures or illustrations. There are just enough of these snapshots to show that throughout most of this history algebraists appear to have jotted down and manipulated only the numbers (coefficients) of polynomials, with no notation for the powers themselves. It was only in the Maghreb, in the twelfth century CE, that a truly algebraic notation emerged. A dozen textbooks composed over the course of the next three centuries show students how to write

¹ Counted in (Rosenfeld & Ihsanoglu 2003). “Algebra” in Arabic is *al-jabr wa'l-muqābala*, or sometimes *al-jabr* for short. In this article the word “book” refers to any treatise, whether it exists in manuscript, in print, or even in memory.

polynomials and equations in notation, and later, in the mid-eighteenth century, the notation was briefly revived in Constantinople.

This notation was not the creation of the medieval mathematicians well-known today. The algebraic works of al-Khwārizmī, Abū Kāmil, al-Karajī, al-Samaw'al, al-Khayyām (Omar Khayyam), and Sharaf al-Dīn al-Ṭūsī show no hint of notation, and none of these authors worked in the region and time where it later flourished. Arabic algebraic notation instead developed from the textbook tradition in arithmetic, in conjunction with Hindu-Arabic (Hindi) numerals. The people responsible for it were interested in finding a practical way to work through problems, either for instruction or for practical use.

In this article I first describe the Maghrebian notations for arithmetic, taking into account the ways books and other writing media were used. Because medieval algebraists conceived of polynomials and equations differently than we do today, I introduce the algebraic notation in the course of outlining these differences. This is followed by an analysis of the notation to understand how it functioned and how it differs from modern notation.

2. Arabic arithmetic²

Three main systems of calculation coexisted in medieval Islam: finger reckoning, Hindi numerals, and the *jummal* or *abjad* system. In finger reckoning, the system preferred by merchants, calculations were performed mentally and intermediate results were stored by positioning the fingers in particular ways. Hindi numerals, which we call “Arabic numerals”, use the symbols 1, 2, ..., 9, and 0 in connection with the various algorithms for calculating with them. In the *jummal* system the 28 letters of the Arabic alphabet are assigned the values 1, 2, 3, ..., 10, 20, 30, ..., 100, 200, ..., 900, and 1000.³ For example, the number 36 was written as ٣٦, where the *lām* (ل) represents 30 and the *wāw* (و) represents 6. The hexadecimal numbers of the astronomers were written using the *jummal* numbers for 1 through 59.

The three systems are often covered in different chapters of the same arithmetic book, and some attempts were made to unify different aspects of them. For example, the system of fractions from finger-reckoning was often incorporated into calculation with Hindi numerals. Algebra appears to have emerged in Arabic as a problem-solving method associated with finger-reckoning. But the later Maghrebian

² See Saidan’s introduction in (al-Uqlīdisī 1978) for a good overview of Arabic arithmetic.

³ The designation of the letters was slightly different between west (Maghreb and al-Andalus) and east. See (al-Uqlīdisī 1978, 9).

notation for algebra builds from calculation with Hindi numerals, so I focus on this system.

Calculation with the nine symbols (1, 2, 3, ..., 9) and zero originated in India, and thus was called “Hindi reckoning” (*al-hisāb al-Hindī*) in Islamic countries. The numerals had already arrived in the Middle East by the seventh century CE, and the earliest known Arabic text on the subject, al-Khwārizmī’s *Book on Hindi Reckoning*,⁴ was composed in the early ninth century. This book explains algorithms for performing arithmetical operations using a dust board (*takht*), which is a flat surface covered with fine sand or dust in which the figures are drawn with a stylus. Because erasing is easy, one can work out the answer to a problem by erasing, writing over digits, and by shifting entire numbers.

Al-Uqlidīsī describes the disadvantages of the dust-board in his *Chapters on Hindi Arithmetic* (952/3 CE).⁵ The medium was associated with street astrologers, and because of erasing one cannot check one’s work without reproducing it. In his book al-Uqlidīsī shows algorithms for pen and paper, which are “simpler and quicker than the arithmetic

⁴ *Kitāb al-ḥisāb al-Hindī*. A critical edition and translation of the manuscripts known in the early 1990’s, all incomplete, is given in (al-Khwārizmī 1992). A complete Latin MS discovered later is edited and translated in (al-Khwārizmī 1997).

⁵ (al-Uqlidīsī 1978; al-Uqlidīsī 1984).

of the *takht*.”⁶ But despite al-Uqlidisī’s recommendation the dust-board remained popular.

Another surface for working out problems is the *lawḥa* (board; tablet). Here one writes with a cane stick dipped in ink on a board covered in soft clay. When finished, the work is wiped off. Al-Ḥaṣṣār, the twelfth century mathematician whose work will be quoted below, writes of calculations on both the dust-board and the *lawḥa*.

3. The role of books in medieval Islam⁷

All of our evidence for the practice of algebra in medieval Arabic comes from manuscripts. Because people related to books differently in medieval times than we do today, some brief remarks on the role of books are in order. Where today learning revolves around schools and the printed word, in Medieval Islam it was centered on the personal relationship between student and *shaykh* (teacher) and the recitation and memorization of texts. Books were considered to properly reside in memory, and were manifested through recitation. They were

⁶ (al-Uqlidisī 1978, 247).

⁷ For more on education and books in Islamic societies see (Makdisi, 1981; Pedersen, 1984; Berkey, 1992; Chamberlain, 1994).

“published” orally, and to learn a text a student would listen to a *shaykh* recite it, memorize it, and then recite it back.

Written books played a valuable role within this predominantly oral learning environment. Many authors composed memorization-friendly poems on arithmetic and algebra, and manuscripts were seen as aids to memorization. For instance, in his *Introduction to Geometry* Qusṭā ibn Lūqā (9th c.) wrote “I have composed this book in the form of question and answer, because this is easy to understand, simplifies the notions, and facilitates memorization.”⁸ When studying from a written book after class, the student would read it aloud, “for what the ear hears becomes firmly established in the heart.”⁹

Even if longer and more advanced works were certainly not routinely memorized, they still retain characteristics of oral learning. Because notation serves no purpose when speaking aloud, manuscripts read like transcriptions of lectures in which everything, even numbers, is written out in words. The comprehensive books on algebra by Abū Kāmil, al-Karajī, al-Khayyām, Ibn al-Bannā³, and Ibn al-Hā³im, for example, are devoid of notation.¹⁰

⁸ Translated in (Hogendijk 2008, 176).

⁹ Abū 'l-Hilāl al-Ḥassan al-^ᶜAskarī, quoted in (Berkey 1992, 27).

¹⁰ For al-Karajī I have in mind his *al-Fakhrī*.

4. Hindi notation: figures vs. text

The rules for performing calculations with Hindi numerals are presented rhetorically in the manuscripts. The numerals themselves are not part of the running text, but appear as illustrations, in much the same way as geometric figures.¹¹ Ibn al-Yāsamin, for example, poses this problem in his late twelfth century *Grafting of Opinions of the Work on Dust Figures*: “If [someone] said to you, divide one hundred fifty three thousand three hundred sixty by one hundred forty four. Its figure is

$$\begin{array}{r} 1\ 5\ 3\ 3\ 6\ 0 \\ 1\ 4\ 4 \end{array}$$

.” He then gives verbal instructions on how to manipulate the numbers to do the division. The one intermediate

figure shown in the course of these instructions is $\begin{array}{r} 1\ 0 \\ 9\ 3\ 6\ 0 \\ 1\ 4\ 4 \end{array}$, and at the



end the answer is given as “one thousands sixty-five. Its figure is 1065.”¹²

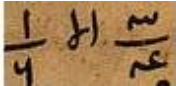
Arabic reads right to left, and in the notation the numerals are written starting with the units, then the tens, hundreds, etc. When Hindi notation found its way into Latin the direction of the digits was not reversed. Because the representations of numbers in this system were regarded as figures, they retained their original orientation.

¹¹ A few texts, like al-Uqlīdisī’s *Chapters on Hindu Arithmetic*, do show the numerals as part of the text.

¹² (Ibn al-Yāsamin 1993, 122).

Charles Burnett writes “The retention of the *visual* order of the Indian numerals, I think, adds to the evidence for their being conceived as *different* from written text...Numerals were *hors de texte*, and like pictures, or geometrical diagrams, they kept the same directionality as they passed from one language context to another.”¹³

In the Maghreb, beginning in the twelfth century, other notation besides the numerals make their appearance in arithmetic books. Sometimes the words *illā*  (“less”) and *wa*  (“and”) occur in figures with the same meaning they holds in ordinary text. *illā* indicates that what follows has been taken away from the preceding number, and *wa* is sometimes placed between numbers that are appended, or added together. An example of *illā* from Al-Ḥaṣṣār is “three fourths less a sixth”,¹⁴ which is shown in notation as


 . This is equivalent to our $\frac{3}{4} - \frac{1}{6}$, but I will transcribe it as $\frac{3}{4} \ell \frac{1}{6}$, where the “ ℓ ” is an abbreviation for “less”.¹⁵ Most later texts omit the first letter of *illā* in figures, as will be seen below. Al-Ḥaṣṣār’s book is not only the earliest we have seen which uses *illā* in figures, but it is also the earliest known text in which fractions are written with numerator over denominator, separated by a horizontal bar. This is a



¹³ (Burnett 2006, 29), his emphases.

¹⁴ (al-Ḥaṣṣār MS, f. 64v).

¹⁵ I explain below in §6 why this is not identical to $\frac{3}{4} - \frac{1}{6}$.

characteristically North African notation, which was later transmitted to Europe.¹⁶

Another notation is the letter *jīm* , the first letter in the word *jidhr* (“root”). This was placed over a number to indicate its square root. We first encounter it in the works of Ibn al-Yāsamīn at the end of the twelfth century.¹⁷ Al-Qatrawānī explains the notation in his fourteenth century *Sucking the Nectar from the Mouths of the Operations of Arithmetic*:

In some computations, one needs to specify the value of the [square] root of a number; but some numbers do not have a root.¹⁸ It follows that if we take an approximate value of the root and if we operate on the squares of these non-rational numbers, the resulting calculation is then faulty. It was therefore agreed to put the elongated letter *jīm*, as in this figure , above the number whose root is to be calculated, and to put two *jīms*, as in this figure: ,

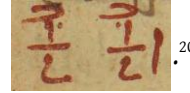
¹⁶ The few common fractions reproduced in this article do not hint of the fact that there were many competing, sometimes ambiguous, variations on this notation. See (Abdeljaouad 2005; Djebbar 1992).

¹⁷ Ibn al-Yāsamīn shows the sign for roots in his *Grafting of Opinions of the Work on Dust Figures*, and he explains the use of the symbol in his *Urjūza fī'l-judhūr* (*Poem on Roots*).

¹⁸ i.e., irrational square roots cannot be expressed precisely with integers and fractions.

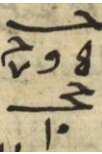
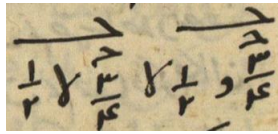
for the root of the root of this number, and to put above them a new *jīm* as many times as the word *jidhr* (“root”) is repeated, since the root of the root of a number is never the root of the root of any other number.¹⁹

Here is an example of the use of the *jīm* from al-Mawāhidī’s fourteenth century *Obtaining the Wish in Commenting on the Condensed [Book] of Ibn al-Bannā*². The number expressed as “the root of one and a half and the root of a half” is shown with this figure:



Note that there is no notation indicating “+”, since the number is merely the aggregation of two roots. In the next two examples, from Abī al-Faṭḥ al-Ṣūfī’s book on roots (1491-2), the *jīm* is used recursively.

The first,





is equivalent to our $\sqrt{\sqrt{\frac{3}{4} + \frac{1}{2}} - \sqrt{\sqrt{\frac{3}{4} - \frac{1}{2}}}}$, and the second, shows $\frac{\sqrt{5 + \sqrt{7}}}{\sqrt{\sqrt{10}}}$.²¹





In the first figure the initial letter of *illā* (“less”) is omitted. It becomes

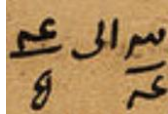
¹⁹ Mahdi Abdeljaouad’s translation.

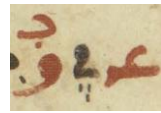
²⁰ (al-Mawāhidī MS, f. 69r = file 137).

²¹ *Guide for Neophytes on Operations with Irrational Roots* (Abī al-Faṭḥ al-Ṣūfī MS, ff. 27r, 33r = pages 59, 71). This author also puts the letter *kāf* (for *ka’b*, “cube”) above a number to indicate the cube root.

, making my translation ℓ for “less” visually close to the Arabic. These two examples also show that sometimes the “and” (wa, ) was inserted between terms.

Also beginning in the twelfth century signs for operations were sometimes used. In expressing an addition a number is added to a number. The Arabic word for “to” in this context is *ilā*, . Likewise, one subtracts a number *from* (*min*, ) a number, one multiplies a number *by* (*fī*, ) a number, and one divides a number *by* (*‘alā*, ) a number. The short words *ilā*, *min*, *fī*, and *‘alā* appear in the figures to indicate their respective operations. Al-Ḥaṣṣār represents

the addition of three fourths to four fifths like this: .²² Al-

Mawāhidī shows the example “multiply four in number by the root of nine” with this figure: .²³

²² (al-Ḥaṣṣār MS, f. 68v).

²³ (al-Mawāhidī MS, f. 73r = file 145). In some manuscripts we find a small circle used to indicate multiplication. (al-Ḥaṣṣār MS, f. 80v) and (Ibn al-Yāsamin 1993, 156) show two examples. I do not know the origin of this practice. (Like some other copyists, the copyist of this MS used red ink for numbers expressed with Hindi numerals and for the words of opening phrases.)

Apart from the numerals themselves, all of these notations are found mainly in western Arabic texts, and all were also used in conjunction with algebraic notation. In order to explain the notation for algebra I must place algebra in the context of arithmetical problem solving, and describe how the concepts of medieval algebra differ from those of modern algebra.

5. Problem solving in medieval Arabic arithmetic

Algebra was just one of several numerical problem solving techniques in medieval Islam. Its rules are explained not just in books dedicated to algebra, but also in the last chapters of many arithmetic books, usually just after the method of double false position. Worked-out problems are given in many of these books to illustrate a variety of problem solving methods. If asked to solve a problem like “A quantity (*māl*): you added its third and its fourth, so it yielded six. How much is the quantity?”,²⁴ we invariably turn to algebra. Naming the unknown quantity x , we set up and solve the equation $\frac{1}{3}x + \frac{1}{4}x = 6$. By contrast, in medieval Islam (and Europe) there were several ways to solve problems like this one. This particular problem, from Ibn al-Yāsamin’s

²⁴ (Ibn al-Yāsamin 1993, 195).

Grafting of Opinions, is worked out by four different methods: single false position (*al-‘amāl bi al-nisba*), algebra (*al-jabr*), *al-qiyās* (a variation on single false position), and double false position (*al-khaṭa’ayn*).

A solution by single false position begins with a guess for the answer. Ibn al-Yāsamīn tries twelve, since the fractions will vanish, and performs the operations: $\frac{1}{3}12 + \frac{1}{4}12 = 7$. The solution is then $6 \cdot 12 \div 7$, which is found to be “ten and two sevenths”. In double false position two guesses are made, and the answer is calculated from the guesses and their respective errors. Ibn al-Yāsamīn first picks 8, and calculates $\frac{1}{3}8 + \frac{1}{4}8 = 4\frac{2}{3}$, which yields an error of $1\frac{1}{3}$, since it should be 6. He then tries 12, calculating $\frac{1}{3}12 + \frac{1}{4}12 = 7$, which gives an error of 1. The answer is found by operating on the guesses and their errors: $(8 \cdot 1 + 12 \cdot 1\frac{1}{3}) \div (1 + 1\frac{1}{3}) \rightarrow 10\frac{2}{7}$. All of these calculations are expressed in words.

In other arithmetic books we find still more methods, like analysis (*al-tahlīl*), working backwards (*al-‘amāl bi-‘aks*), and proportion (*al-arba‘a al-mutanāsiba*).²⁵ Setting aside algebra, the answer in all of these methods is found by operating on the known, or given, numbers in the problem. An algebraic solution, by contrast, begins by *naming* an unknown, and operating on this unknown to set up an *equation*. The equation is then simplified to one of a handful of canonical, pre-solved equations.

²⁵ Solutions by these methods are given in (al-Ḥaṣṣār MS; al-Fārisī 1994; al-‘Āmilī 1976).

6. Arabic polynomials and equations²⁶

Textbooks dedicated to algebra typically divide the presentation into two parts. First is an explanation of the rules of algebra, covering the names of the powers, operations on polynomials, and solutions of the simplified equations. This is followed by a collection of worked-out problems. The enunciations of these problems are most often arithmetic questions, sometimes framed in terms of commerce.²⁷

In arithmetic books algebra is usually relegated to a chapter or two at the end, if it is included at all. There is just enough room to cover most of the rules, so worked-out problems are often absent. This is unfortunate, because it is in such books that the Maghrebian algebraic notation is commonly encountered.

The different powers of the unknown were given individual names in Arabic algebra. The first power, corresponding to our x , is called *shay*³ (“thing”), or sometimes *jidhr* (“root”). Its square is called *māl*

²⁶ Arabic polynomials and equations are described in more detail in (Oaks 2009).

²⁷ The well-known algebra texts of al-Khwārizmī, Abū Kāmil, al-Karajī (*al-Fakhrī*), and Ibn al-Bannā' have this structure. In addition, many books solve mensuration and inheritance problems by algebra.

(“sum of money”, “possession”).²⁸ Units were counted in dirhams, a unit of currency, or sometimes as *āhād* (“units”) or *min al-ʿadad* (“in number”), and often the name was dropped altogether. So “ten dirhams”, “ten units”, “ten in number” and “ten” all take the meaning of the number ten. The terms *jidhr* and *māl* were borrowed from arithmetical problem solving, where the *māl* is literally an amount of money or an abstract quantity, and the “root” is its square root. Many problems in Arabic arithmetic ask for an unknown *māl* satisfying some condition, sometimes involving its [square] root, like Ibn al-Yāsamin’s problem mentioned in the previous section.²⁹

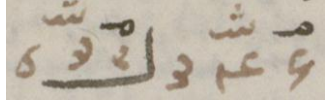
Al-Khwārizmī and Ibn Turk, writing in the early ninth century, work only with dirhams/numbers, roots/things, and *māls*. Higher powers first appear later in the century in Qusṭā ibn Lūqā’s translation of Diophantus’ *Arithmetica*. The cube of the “thing” is called *kaʿb* (“cube”), and higher powers were written as combinations of *māl* and *kaʿb*, like *māl māl* for the fourth power and *māl kaʿb* for the fifth. The reciprocals of the powers also date back to the Arabic translation of Diophantus’ book. “A part of a thing” expresses our $\frac{1}{x}$, “a part of a *māl*” our $\frac{1}{x^2}$, etc. Beginning with al-Karajī’s *al-Fakhrī* (early 11th c.) Arabic “polynomials” include these terms as well as the positive powers. The powers of the

²⁸ Because there is no adequate rendering of *māl* in English, I leave it untranslated, and I write its plural with the English suffix: *māls*.

²⁹ For a discussion of the different meanings of *māl* and *jidhr*, see (Oaks & Alkhateeb 2005).

unknown were called the “species” of number used in calculation by algebra. I have found the words *jins*, *ḍarb*, *nau^c* and *aṣl* in various Arabic texts meaning “species”, “kind”, or “type”.

Ibn Ghāzī gives this sample equation with algebraic notation in his 1471 *Aim of the Students in Commentary on Desire of Reckoners*: “five *māls* and four things and three in number equal two *māls* and three things

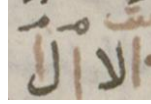
and six in number, and its figure is ”.³⁰ I will

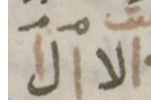
transcribe the notation as $\begin{matrix} m & t \\ 5 & 4 & 3 \end{matrix} = \begin{matrix} m & t \\ 2 & 3 & 6 \end{matrix}$. Starting from the right

in the figure, the “five *māls*” is shown as the letter *mīm* (the first letter in *māl*) above a 5. The second term shows a *shīn* (the first letter in *shay³*, “thing”) above a 4, and “three in number” is simply shown as a “3”. There are variations on the *shīn* in notation. Sometimes only the three dots (∴) are shown, and in one 18th c. manuscript the letter is written without the dots. An elongated letter *lām*, the last (written) letter in the words conjugated from *‘adala* (“equal”) indicates equality. It looks like a backwards “L”, which coincidentally is the last letter in “equal”. The polynomial on the left in the figure follows the same scheme. In my transcription I write the equation left-to-right, and I use an “*m*” for *māls* and a “*t*” for “things”. Here is how the equation “ten things less a *māl*

³⁰ (Ibn Ghāzī 1983, 229; Ibn Ghāzī MS, f. 83r = file 165).

equal a *māl*” is shown in Ibn Ghāzī’s book:



equal a *māl*” is shown in Ibn Ghāzī’s book: .³¹ I transcribe

this as $\frac{t}{10} \ell \frac{m}{1} = \frac{m}{1}$.

These two examples can be rewritten in modern notation as $5x^2 + 4x + 3 = 2x^2 + 3x + 6$ and $10x - x^2 = x^2$, but our modern symbols carry with them an understanding of polynomials which differs from that in medieval algebra. Medieval mathematicians did not conceive of polynomials like “five *māls* and four things and three in number” and “ten things less a *māl*” as modern linear combinations, with the implied operations of addition, subtraction, and scalar multiplication. As I argued in (Oaks 2009), they were seen instead as aggregations of the species, or names, of the powers of the unknown, with no operations present.

In a term like “five *māls*” the “five” is not a scalar multiple, but tells us instead *how many māls* we have. It is like saying “five bananas”. The “five” was called the “number” (*ʿadad*) of the term. There was no special word for “coefficient”. Arabic algebraists were careful not to let the number of any species be irrational. For example, Abū Kāmil (late 9th c.) multiplies “a thing” (x) by “the root of eight” ($\sqrt{8}$) to get “the root of eight *māls*” ($\sqrt{8x^2}$).³² While “three things” or “two thirds of a thing”

³¹ (Ibn Ghāzī 1983, 238; Ibn Ghāzī MS, f. 86v = file 172).

³² (Abū Kāmil 1986, 85.10; Abū Kāmil 2004, 138.10).

make sense, “the square root of eight things” ($\sqrt{8x}$) does not, so they would work with the square root of the entire term.³³

The polynomial “five *māls* and four things and three in number” is a collection or aggregation of twelve items of three species. It is like saying “five bananas and four apples and three pears”. The *wa* (“and”) between the terms in the text is the common conjunction, and does not take the meaning of the modern “plus”. In the notation, then, there is no symbol like our “+”. The terms are merely placed next to each other. In situations where the word *wa* (“and”) is included it is often to counterbalance the notation for “less”.

The polynomial “ten things less a *māl*” does not express the arithmetical operation of subtraction. Instead, the expression was regarded as a deficient “ten things”. The word *illā* (“less”) indicates that the ten things *lack* one *māl*. The *illā* functions like the word “short” in the English phrases “two people short of a quorum” or “three cents short of a euro”.³⁴

The difference between the operation of addition and the aggregation of two amounts is made explicit many times in algebra books. Abū Kāmil begins problem (53) by naming the unknown quantity “a thing”, then writes “So you *add* (*tazīd*) to it seven dirhams, so it yields

³³ We first find irrational “numbers” of a term in some 16th c. European texts (Oaks 2010, 50).

³⁴ See (Oaks & Alkhateeb 2007, §3.5) for more on this concept.

a thing *and* (*wa*) seven dirhams.”³⁵ The “thing and seven dirhams” is the *result* of adding seven to a thing. Likewise for subtraction. In problem (12) of his *Brief Book on Algebra* Ibn Badr (ca. 14th c.) has named an unknown a “thing”, and he writes “and you *subtract* (*taṭraḥ*) from it the two dirhams, leaving you with a thing *less* (*illā*) two dirhams.”³⁶ Again, “a thing less two dirhams” is the *result* of subtracting two dirhams from a thing. This distinction is reflected in the notation, too. The short words “to” (*ilā*) and “from” (*min*) indicate the operations of addition and subtraction. If there is a symbol for the sum of two quantities it is the one-letter word “and” (*wa*), and the result of a subtraction is indicated by “less” (*illā*).³⁷

So an expression of the form “*a* and *b*” is to be regarded as the aggregation of *a* together with *b*, and an expression of the form “*a* less *b*” signifies a diminished *a*, in which the *b* is absent or deleted from it. Think of a medieval polynomial as a list or inventory showing how many of each species are present and how many are lacking.

The two sides of an Arabic equation should be fixed amounts. In other words, they ideally should not contain any unresolved operations. This means that an algebraist is compelled to work out all the operations stated in the enunciation of a problem before setting up a

³⁵ (Abū Kāmil 1986, 98.18; Abū Kāmil 2004, 159.7).

³⁶ (Saidan 1986, 456.2).

³⁷ See (Oaks 2009, 195) for an example of notation for an addition with “to” yielding an aggregation with “and”.

polynomial equation. This leads us to the three basic stages in the solution of a problem in medieval algebra:

Stage 1. An unknown is named (usually as a “thing”), and the operations called for in the enunciation are manipulated and worked out in terms of the algebraic species to set up a polynomial equation.

Stage 2. The equation is simplified by “restoration” (*al-jabr*) and/or “confrontation” (*al-muqābala*).

Stage 3. The simplified equation is solved using the prescribed rule.

Because the rules for solving the equation in stage 3 take the “numbers” (coefficients) of the terms as parameters, there can be no deleted terms (or as we would say, “subtracted terms”). There are thus six simplified equations of first and second degree:³⁸

Type 1: *māls* equals roots (in modern notation, $ax^2 = bx$)

Type 2: *māls* equals numbers ($ax^2 = c$)

³⁸ Beginning in the second half of the ninth century Arabic algebraists worked with higher degree equations, but no solution based on the coefficients was found. Al-Khayyām classified all 25 types of equation up to the third degree and he solved irreducible cubic equations geometrically.

Type 3: roots equals numbers ($bx = c$)

Type 4: *māls* and roots equals numbers ($ax^2 + bx = c$)

Type 5: *māls* and numbers equals roots ($ax^2 + c = bx$)

Type 6: *māls* equals roots and numbers ($ax^2 = bx + c$)

Below is a typical worked-out problem illustrating the three stages. I took this one from ‘Alī al-Sulamī’s book *Sufficient Introduction to Calculation by Algebra and What One Can Learn from its Examples*³⁹ (before 13th c.). There is no notation in this book.

[Enunciation]

If [someone] said, ten: you divided it into two parts. You multiplied one of them by the other, then you divided that by the difference between the two parts. So it yielded twelve dirhams.

[*The enunciation is an arithmetic question, which will be worked out by algebra. Cast in modern algebraic notation, the enunciation says $\frac{ab}{a-b} = 12$, where $a+b=10$.*]

[Stage 1] One of the parts is a thing and the other is ten less a thing. Multiply one of them by the other: it yields

³⁹ (‘Alī al-Sulamī MS, f. 72a.12).

ten things less a *māl*.⁴⁰ Divide that by the difference between the two parts, which is ten less two things. It results in twelve dirhams. Multiply twelve by ten less two things. It yields: a hundred twenty less twenty-four things equals ten things less a *māl*.

[The solution begins by naming one of the parts “a thing” (like our x), making the other part “ten less a thing”. The operations are then worked out (the multiplication and subtraction) or manipulated (the division is taken care of by a multiplication) to arrive at the equation $120 \ell^t_{24} = \frac{t}{10} \ell^m_1$, or as we would write, $120 - 24x = 10x = x^2$.]

[Stage 2] Restore it by the *māl* and add it to the dirhams,⁴¹ and restore the dirhams by the things and add them to the ten things,⁴² so it becomes: thirty-four things equals a *māl* and a hundred twenty dirhams.

⁴⁰ The copyist mistakenly wrote “a hundred and a *māl* less twenty things” for the product. This was probably miscopied from another problem, as the incorrect phrase also appears in the next problem and in two problems back.

⁴¹ i.e., restore the ten things by the *māl*, and add the *māl* to the 120 dirhams on the other side. See (Oaks & Alkhateeb 2007, §3.5) for an explanation of the word *al-jabr* (restoration) in Arabic algebra.

⁴² i.e., restore the 120 dirhams by the 24 things, and add the 24 things to the 10 things.

[To simplify the equation the two diminished terms are restored.

This results in the type 5 equation $\frac{t}{34} = \frac{m}{120}$,

or $34x = x^2 + 120$.]

[Stage 3] Halve the things: it yields seventeen. Multiply it by itself: it yields two hundred eighty-nine. Subtract the hundred twenty from it. There remains one hundred sixty-nine. Take its root: it is thirteen. Subtract it from the seventeen, so there remains four, which is one of the two parts.

[The numerical procedure for solving a type 5 equation is followed. Some books give geometric proofs for these numerical rules, but the proofs have no place in problem solving.]

Where we would begin the solution to this problem by setting up an equation like $\frac{x(10-x)}{10-2x} = 12$, al-Sulamī first performs two multiplications and a subtraction in stage 1 so that when he states his equation it is in polynomial form: $120 \ell \frac{t}{24} = \frac{t}{10} \ell \frac{m}{1}$. From here the simplification in stage 2 is an easy matter. The ten things, lacking a *māl*, are restored, and the *māl* is added to the other side to balance the equation. Likewise the 120 dirhams are restored, and the 24 things are added to the other side. One noticeable difference between medieval and modern algebra is that we typically work out our operations after

setting up the equation, while medieval algebraists did this work in stage 1.

At this point I should say something about the difference between operations and equations in Arabic algebra. We write both with the same notation, like the operation $(10 - x)^2 = 100 + x^2 - 20x$ and the equation $4x = 10 - x$. In medieval algebra these were consciously differentiated both in rhetorical text and in notation. In the text an operation is stated with two verbs, the first indicating the operation and the second announcing the outcome. For example, Ibn al-Yāsamin (d. 1204) writes in the solution to one problem “*multiply* the ten less a thing by itself. It *yields* a hundred and a mal less twenty things.”⁴³ Likewise the enunciations to problems are usually expressed in terms of operations on an unknown number.⁴⁴ Sometimes these operations are restated in the solution in terms of the algebraic powers. A good example is found in Ibn Badr’s problem (T3). After naming an unknown a “thing” he writes “So *divide* ten less a thing by a thing. It *results* in four”.⁴⁵ Al-Sulamī also restates his enunciation in terms of the algebraic powers in the problem quoted above. Such restatements are not the norm.

⁴³ (Ibn al-Yāsamin 1993, 225.5).

⁴⁴ For a discussion of enunciations that do not express operations, see (Oaks 2010b, §4.5).

⁴⁵ (Saidan 1986, 444.-3). For my numbering of the problems, see (Oaks & Alkhateeb 2005, Appendix A).

By contrast, algebraic equations were written with the single verb “equal” (*ʿadala*). Continuing in the problem quoted above, Ibn Badr multiplies by a “thing” and sets up the equation “Four things *equal* (*taʿdil*, conjugated from *ʿadala*) ten less a thing”.⁴⁶ While operations are performed in time with a stated outcome, equations assert the equality of two static expressions using the one verb “equal”. In notation the letter *lām* (ل) was used only to indicate equality in an equation, while the result of an operation was usually placed underneath the operated quantities, with a line between them.

In some problems algebraists would run into difficulty trying to construct a polynomial equation from the conditions of the problem. This happens when the problem calls for divisions or square roots which are difficult to resolve. ʿAlī al-Sulamī took care of the division in his problem by multiplying the dividend by the quotient. But in some problems, particularly those involving the sum of a quotient and its reciprocal, it is not a simple task to follow the chain of operations to transform the enunciation into a polynomial equation. Several techniques were adopted to address this problem:

- to find a clever way to manipulate the operations to obtain a polynomial equation;⁴⁷

⁴⁶ (Saidan 1986, 445.1).

⁴⁷ For two examples from Abū Kāmil, see (Oaks 2009, 198-9).

- to give the quotients names of their own (usually “*fals*” and “*dīnār*”, two denominations of coin) and to construct an equation using them;⁴⁸
- to give a geometric proof that the enunciation can be recast in a different form that more easily allows for the construction of a polynomial equation,⁴⁹ and
- to simply allow for divisions or square roots in equations.

Ibn al-Hā'im is one algebraist who sometimes took the last option. In his 1387 *Commentary on the Poem of al-Yāsamīn* he worked with equations like “ten less a thing divided by a thing and a thing divided by ten less a thing, and that equals two dirhams and a sixth”⁵⁰

$\left(\frac{10 \ell \ t}{t \ 1} \frac{t}{10 \ell \ t} = 2\frac{1}{6} \right)$ and “the root of a mal and three things equals

a thing and a half” $\left(\sqrt{\frac{m \ t}{1 \ 3}} = 1\frac{1}{2} \right)$.⁵¹ The occasional admission of

roots and divisions to equations was merely a patch to get around a

⁴⁸ See (Oaks 2009, 200) for an example, again from Abū Kāmil.

⁴⁹ Abū Kāmil is the only algebraist I know who does this, in problems (4), (5), (6), (7), and (11). These are described in Appendix A of (Oaks 2011).

⁵⁰ (Ibn al-Hā'im 2003, 234.-3). This book contains no notation.

⁵¹ (Ibn al-Hā'im 2003, 224.6). Notation for the root, from an 18th c. manuscript, is shown below in §9.

shortcoming of the aggregations interpretation. It did not lead to a rethinking of the nature of equations. Down to the end of Arabic algebra all multiplications, additions, and subtractions were performed before setting up an equation, and the other tricks used to avoid divisions and roots continued to be applied.

7. Early/eastern notations for Arabic algebra

The problem quoted above from al-Sulamī is typical of Arabic algebra. The enunciation and the entire solution are presented rhetorically, with no notation at all. But of course algebraists did not solve problems by writing down the rhetorical version. The numerical calculations themselves were often too difficult for anyone to do mentally. To pick an extreme example, at one point in problem (56) Abū Kāmil squares $17\frac{27}{39} + \sqrt{315\frac{885}{1521}}$ to get $628\frac{912}{1521} + \sqrt{395130\frac{2057670}{2323441}}$ with no explanation, and all in words.⁵² Results like this would have been calculated using

⁵² (Abū Kāmil 1986, 110.7; Abū Kāmil 2004, 177.11). Further evidence is that the number under the last square root is written in Hindi form lower on the page of the Istanbul MS. This is not part of the text, and may have been put there so the copyist could check the rhetorical version, or so he could write it in words without making a mistake. Even so, in the course of the seven lines in which

the techniques explained in books on Hindi reckoning and then written into the text.

The same is true for algebraic calculations. Setting aside for the moment the few texts showing the Maghrebian notation, we have only limited evidence of how algebraic calculations might have been performed. In his early eleventh century *al-Badī* al-Karajī shows polynomials of high degree as a list of “coefficients”. As an example he writes “7 6 5 4 3 2 1” in Hindi notation to represent “a cube cube and two *māls* cube and three *māls māl* and four cubes and five *māls* and six roots and seven units.”⁵³ Later, al-Karajī’s successor al-Samaw’al (12th c.) represented operations on polynomials in tables. In his first example he divides $20x^6 + 2x^5 + 58x^4 + 75x^3 + 125x^2 + 96x + 94 + 140x^{-1} + 50x^{-2} + 90x^{-3} + 20x^{-4}$ by $2x^3 + 5x + 5 + 10x^{-1}$.⁵⁴ He sets up the problem in this table:

the operation is expressed, part of one line was scratched out and a forgotten phrase was added in the margin.

⁵³ (al-Karajī 1964, 53).

⁵⁴ (al-Samaw’al 1972, 44ff).

degree cube cube	degree <i>māl</i> cube	degree <i>māl</i> <i>māl</i>	degree of the cube	degree of the <i>māl</i>	degree of the thing	degree of the unit	degree part thing	degree part <i>māl</i>	degree part cube	degree part <i>māl</i> <i>māl</i>
20	2	58	75	125	96	94	140	50	90	20
2	0	5	5	10						

Then, by calculating, shifting, and erasing, he ends up with

degree of the cube	degree of the <i>māl</i>	degree of the thing	degree of the unit	degree part thing	degree part <i>māl</i>	degree part cube	degree part <i>māl</i> <i>māl</i>
10	1	4	10	0	8		
			4	0	10	10	20
			2	0	5	5	10

The answer $10x^3 + 1x^2 + 4x + 10 + 8x^{-2}$ is read off from the first line. In this calculation al-Samaw'al has adapted the algorithms for operating on the dustboard with Hindi numerals to polynomials. The powers of the unknown mimic the powers of ten in the algorithms for operating on numbers in Hindi form.

Al-Fārisī (d. ca. 1320) provides us with an example of the multiplication of polynomials in his *Foundation of Rules on Elements of Benefits*. He shows the multiplication of “four fifths of a thing and two and a half by itself”, $(\frac{4}{5}x + 2\frac{1}{2})^2$, in a sequence of three tables. In one of these he multiplies the $\frac{4}{5}x$ by $2\frac{1}{2}$, working only with the numbers.⁵⁵

⁵⁵ (al-Fārisī 1994, 563).

Four fifths	by	Two and a half	
5		2	10
4		5	20
Answer: two			

Other Eastern algebraists who show polynomials in tables or as a list of coefficients are al-Kāshī (d. 1429) in his *Key to Arithmetic*, and al-Yāzīdī (d. ca. 1637) in his commentary on the *Talkhīṣ* of Ibn al-Bannā'.⁵⁶ None of the notations in these books is truly algebraic because they work only with the numbers of the terms. The powers themselves are not indicated. We can only surmise that people outside the Maghreb or before the twelfth century would write the “numbers” of the terms to represent polynomials when working out problems by algebra.

8. Notation in Western Islam, 13th-15th centuries

It was in the Muslim west, in the Maghreb and probably also in al-Andalus, that a real notation for algebra finally appeared. This was not the creation of a group of scientifically minded algebraists, but instead came from the textbook tradition. Mahdi Abdeljaouad, building on

⁵⁶ (al-Kāshī 1969; Abdeljaouad, personal communication).

some remarks of İhsān Fazlıoğlu, conjectures that the creation of this notation was due in part from the multilingual environment of the Maghreb and al-Andalus.⁵⁷ The notation saw two phases of development. It is first found in a handful of texts dating from the late twelfth century down to the fifteenth century. Later, the notation was revived in mid-eighteenth century Constantinople under the direction of Muṣṭafā Şidqī ibn Şālih (d. 1769).

To date Abdeljaouad has identified twelve books by nine mathematicians of the first phase which show the notation.⁵⁸ Six of the books are commentaries the famous late thirteenth century *Talkhīṣ a‘māl al-ḥisāb* (*Condensed [Book] on the Operations of Arithmetic*) of Ibn al-Bannā⁵⁹ (d. 1321) which contains no notation itself. These six books, along with four others, are practical guides to arithmetic and problem-solving with Hindi numerals. They take the student through the operations of addition, subtraction, multiplication, division, and square root, with both whole and fractional numbers. Additionally, the methods of double false position and algebra are covered. The chapters on algebra in these books are usually rather brief, and they do not contain the dozens of worked-out problems that one finds in the earlier

⁵⁷ (Abdeljaouad 2011, 16-17).

⁵⁸ These are listed in the Appendix. The notation is also found in the Hebrew book *Treatise on Number* by Isaac ben Salomon, written in Syracuse at the end of the 14th c. There Arabic letters are replaced by Hebrew letters (Lévy 2003, 295).

⁵⁹ (Ibn al-Bannā’ 1969).

books of al-Khwārizmī, Abū Kāmil, and al-Karajī, or in the western algebra books of Ibn al-Bannā' and Ibn al-Hā'im.

The other two books with notation are commentaries on Ibn al-Yāsamin's brief poem on algebra.⁶⁰ The poem gives rules for solving the six equations and for multiplying and dividing monomials. The commentaries illustrate sample equations in notation.

The earliest known text showing the algebraic notation is Ibn al-Yāsamin's late twelfth century arithmetic book *Grafting of Opinions of the Work on Dust Figures*. It appears only twice in the book, as incidental additions to the rhetorical presentation. At one point the author multiplies "half of a *māl* less half of a root by half of a root". He adds

"To express the product you write it like this: $\frac{m}{\frac{1}{2}} \ell \frac{t}{\frac{1}{2}}$." ⁶¹ Earlier in the

book he writes the equation "a *māl* and a thing equals one hundred ten.

Its figure is $\frac{m}{1} \frac{t}{1} = 110$." ⁶² These two illustrations are casual

insertions, and show that the notation was already in use when Ibn al-Yāsamin wrote his book. The several worked-out problems by algebra in this book are entirely rhetorical.

⁶⁰ (Abdeljaouad 2005b) gives an English translation of the poem.

⁶¹ (Ibn al-Yāsamin 1993, 231).

⁶² (Ibn al-Yāsamin 1993, 137). Of course the "=" is shown in the MS as the letter *lām*.

Some later authors give more detailed instructions to the student on how to write the notation. Ibn Qunfudh al-Qustantīnī wrote his *Removal of the Veil from the Faces of Arithmetical Operations* ca. 1370. He explains:

Know that to represent *māls*, you write their number and above it you put one letter *mīm*. For example, the figure for three *māls* is $\overset{م}{3}$. For three *māl māls* it is $\overset{م}{\overset{م}{3}}$... To represent roots (i.e. “things”), you write their number, and above it you put the letter *shīn* once; for example, the figure for three things is $\overset{ش}{3}$ or $\overset{ش}{\overset{ش}{3}}$... Numbers are written, as was shown, without any modification. To represent cubes, you write their number and above it you put the letter *kāf*. For example, the figure for three cubes is $\overset{ك}{3}$... If you understood this, the figure for five *māls* and four things and three numbers equal three things and two *māls* and six numbers is $6 \overset{ك}{2} \overset{ش}{3} \text{ ل } 3 \overset{ش}{4} \overset{ك}{5}$. The letter ل comes from the word “equal” (*ta^cdil*, *تعديل*).⁶³

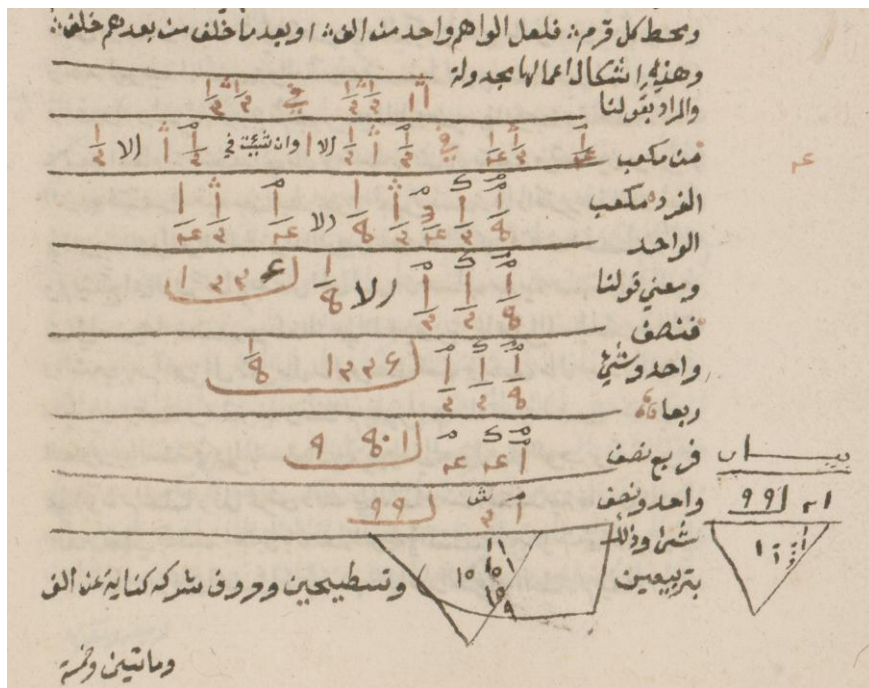
Still, the books with notation give us only glimpses of what must have been put down on the dust-board during the course of solving a

⁶³ Mahdi Abdeljaouad’s translation, from (Guergour 1990, 166-7). Note that this is the same sample equation shown above in §6 from Ibn Ghāzī’s book

problem. Most of them show only isolated equations and operations here and there. Although they sometimes explain how to write the notation, they rarely show how it was applied in algebraic problem solving. Three books, by Ibn al-Yāsamin, al-ʿUqbānī, and Ibn Ghāzī, give some worked-out problems, but with one exception in the last of these, the rhetorical solutions are only occasionally punctuated by an expression or equation in notation.

We are fortunate that Ibn Ghāzī delves a little more into algebra than the authors of the other arithmetic texts. In his 1471 *Aim of the Students in Commentary on Desire of Reckoners* he shows the entire solution to a problem in notation. In modern notation, the question asks for the value of n if $1^3 + 3^3 + 5^3 + 7^3 + \dots + n^3 = 1225$. Ibn Ghāzī gives a rhetorical solution by algebra, following the rule that the sum of the odd cubes from 1 to n is found by adding 1 to n , taking half the sum, squaring it, and then multiplying the result by its double less one. In modern terms, if we let $a = \left(\frac{n+1}{2}\right)^2$ then $1^3 + 3^3 + 5^3 + \dots + n^3 = a(2a - 1)$. After the rhetorical solution this figure is embedded in the text:⁶⁴

⁶⁴ (Ibn Ghāzī 1983, 302; Ibn Ghāzī MS, f. 108v = file 216). In place of the numbers in the stage 3 procedure Souissi's edition shows the solution to the simplified equation by completing the square, with the equations $x^2 + 2x + 1 = 100$, $x + 1 = 100$, and $x = 9$.



In my transcription below I write “c” for “cube”, and the fourth power is “mm” for “*māl māl*”. The word “by” is short for “multiplied by”. Our modern notation is on the right.

$$\begin{array}{c}
 t \\
 1 \ 1
 \end{array}
 \begin{array}{c}
 t \\
 \frac{1}{2} \ \frac{1}{2}
 \end{array}
 \text{ by }
 \begin{array}{c}
 t \\
 \frac{1}{2} \ \frac{1}{2}
 \end{array}
 \qquad
 x + 1 \quad \left(\frac{1}{2}x + \frac{1}{2}\right)\left(\frac{1}{2}x + \frac{1}{2}\right)$$

$$\begin{array}{c}
 m \ t \\
 \frac{1}{4} \ \frac{1}{2} \ \frac{1}{4}
 \end{array}
 \text{ by }
 \begin{array}{c}
 m \ t \\
 \frac{1}{2} \ 1 \ \frac{1}{2}
 \end{array}
 \ell \ 1, \text{ and if you wished, by }
 \begin{array}{c}
 m \ t \\
 \frac{1}{2} \ 1 \ \frac{1}{2}
 \end{array}
 \ell \ \frac{1}{2}$$

$$\left(\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}\right)\left(\frac{1}{2}x^2 + x + \frac{1}{2} - 1\right), \text{ and if you wished, by } \frac{1}{2}x^2 + x - \frac{1}{2}$$

$$\begin{array}{cccccccc} mm & c & m & t & \frac{1}{8} & \ell & m & t & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array} \quad \frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{8} - \left(\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}\right)$$

$$\begin{array}{cccc} mm & c & m & \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{2} & \ell \frac{1}{8} \end{array} = 1225 \quad \frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{2}x^2 - \frac{1}{8} = 1225$$

$$\begin{array}{ccc} mm & c & m \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{2} \end{array} = 1225\frac{1}{8} \quad \frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{2}x^2 = 1225\frac{1}{8}$$

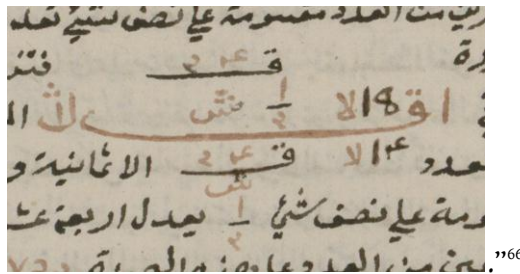
$$\begin{array}{ccc} mm & c & m \\ 1 & 4 & 4 \end{array} = 9801 \quad x^4 + 4x^3 + 4x^2 = 9801$$

$$\begin{array}{cc} m & t \\ 1 & 2 \end{array} = 99 \quad x^2 + 2x = 99$$

In the triangular region below the last equation Ibn Ghāzī writes the numbers obtained in solving the simplified equation in stage 3: 1 (half the number of “things”), 1 (its square), 100 (adding 1 to 99), 10 (its square root), and 9 (subtracting 1 from 10).⁶⁵ Just above the triangle in the right margin we see the last equation written in what is probably the Eastern notation, with just coefficients: $1 \ 2 = 99$, and the same solution is written inside the triangle.

⁶⁵ There are a couple extra 1's in the triangle that I cannot account for.

Ibn Ghāzī also provides us with the only pre-eighteenth century example of the notation for division in an equation: “one in number undivided, and eighteen in number less twenty-four in number divided by half a thing, all that divided by fourteen numbers less twenty-four in number divided by half a thing, equal a thing, as in this figure:



The elongated letter *qāf* (with the two dots on the right end) is the first letter in *qasama* (to divide), and serves as the division bar.⁶⁷ My

⁶⁶ (Ibn Ghāzī 1983, 311.-1; Ibn Ghāzī MS, f. 112r = file 223). Souissi’s edition omits part of the text, and the corresponding part of the symbolic equation.

⁶⁷ This elongated letter *qāf* is seen by some historians (Mahdi Abdeljaouad, Ahmed Djebbar) as being the possible origin of the plain fraction bar in arithmetic.

transcription of the equation is $1 \frac{18 \ell \frac{24}{\frac{1}{2}}}{14 \ell \frac{24}{\frac{1}{2}}} = \frac{t}{1}$, or in modern notation,

$$1 + \frac{18 - \frac{24}{\frac{1}{2}x}}{14 - \frac{24}{\frac{1}{2}x}} = x .$$

No pre-eighteenth century text I have seen shows square roots in conjunction with the notation for algebra, but in all likelihood the letter *jīm* was used on polynomials. Also, we do not yet know if the other notations common to arithmetic and algebra were developed first in one field then applied to the other, or if they were applied simultaneously to both Hindi notation and algebra.

9. The eighteenth century Ottoman revival⁶⁸

In the mid-eighteenth century the Ottoman scientist and military administrator Muṣṭafā Şidqī ibn Şālih (d. 1769) collected or copied numerous old Arabic manuscripts on mathematics and astronomy as part of an effort to glorify Muslim achievements in these areas. Two of

⁶⁸ See (Abdeljaouad 2011) for an account of the revival.

his students, Seker-Zāde and Ibrāhīm al-Halabī, studied the arithmetical works of Ibn al-Bannā³, Ibn al-Hā³im, and al-Qalaṣādī under his supervision, and they produced manuscripts exhibiting extensive use of the algebraic notation. A third mathematician, Muḥammad Hamūd al-Bāz al-Tūnisī, studied under Ibrāhīm al-Halabī.

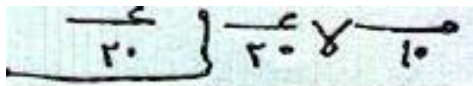
Seker-Zāde (d. 1787) wrote the book *Examples from the Talkhīṣ of Ibn al-Bannā³ and the Hawā³ī of Ibn al-Hā³im*.⁶⁹ This remarkable work is composed of solved problems in which the enunciations are written rhetorically, and the solutions are given entirely in notation. (Abdeljaouad 2011, 29) shows a page from MS Esad Effendi 3150/2, with the enunciation on two lines at the top, and the algebraic solution, all in notation, filling the rest of the page.

Ibrāhīm al-Halabī and his student Muḥammad Hamūd al-Bāz both produced copies of Ibn al-Hā³im's *Commentary on the Poem of al-Yāsamin*. This book was originally written without notation, so al-Halabī added solutions to all the problems in notation in the margins. Muḥammad al-Bāz's manuscript, written in 1747, is a copy of al-Halabī's, and was examined in (Abdeljaouad 2002).

The notation in the eighteenth century books is slightly modified from the earlier notation. One change is that numbers now receive the letter *ʿayn*, the first letter in *ʿadad* (“number”) above them. Below is an

⁶⁹ *ʿAmthilatu al-talkhīṣ li Ibn al-Bannā³ wa'l-hawā³ī li Ibn al-Hā³im*. The *Hawā³ī* of Ibn al-Hā³im is his *Guidebook for the Science of Mental Reckoning (Al-Maʿūnah fi ʿilm al-ḥisāb al-hawā³ī)*.

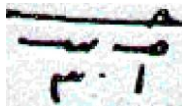
example from Muḥammad al-Bāz. In the transcriptions I write “n” for “number”, and I also give our modern equivalents:



$$10x^2 - 20 = 20$$

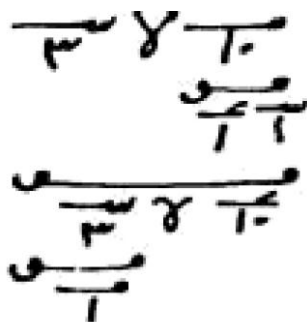
$$10x^2 - 20 = 20.$$

Another feature that we had not seen in the earlier manuscripts is the use of the *jīm* for the square root of a polynomial. Here is an example from Muḥammad al-Bāz. (the *shīn* is written here without the three dots):



$$\sqrt{1x^2 + 3x}$$

One last example from this manuscript shows the versatility of the division bar, here shown as an elongated *mīm-qāf*, the first two letters in *maqsūm* (“divided”):



$$\begin{array}{r} \frac{m}{t} \frac{10}{n} \ell \frac{t}{3} \\ \hline 1 \quad 1 \\ \frac{n}{10} \ell \frac{t}{\frac{3}{m}} \\ 1 \end{array}$$

$$\frac{\frac{10x^2}{x+1} - 3x}{10 - \frac{3x}{x^2}} .$$

10. Assessment of the notation

I will focus on the notation of the first phase, from the 12th-15th centuries. Like the notation for calculating with Hindi numerals it was used to work out problems on some ephemeral surface like a dust-board. The notation was not used to communicate algebra to other people, for which purpose a rhetorical text was composed. The dual way of writing algebra appears to be a consequence of oral culture, and contrasts with modern algebra, where we work out problems in the same notation found in books.

The algebraic notation is described in only a few elementary textbooks, and there only in scattered examples. It was clearly regarded as a calculation aid, and was not crucial to understanding the method. Taking into account also the total absence of notation in the

more advanced works of Ibn al-Bannā' and Ibn al-Hā'im,⁷⁰ the “proper form” of expressing algebra (and in fact all knowledge) was rhetorical.

The Arabic notation is not an abbreviated version of rhetorical algebra. Its value lies with its visual, two-dimensional layout. The vertical dimension links the power to its number, and the horizontal dimension links individual terms. Also, because the notation was not intended to be read by others, there would have been no motive to pronounce it. While it is possible to translate the notation to rhetorical form, one perceives its meaning directly without resorting to audible or silent speech. It is symbolic like the Hindi numerals from which it sprung.

This symbolism is quite flexible, and can handle complex expressions better than rhetorical text. There is often ambiguity relating to the grouping of terms when the algebra is written out in words, but in notation there is no need for a device like parentheses. The symbols for square root and the division bar extend across all the affected terms and can be used recursively, just like we do in expressions like $\sqrt{x^2 - \sqrt{2x + 1}}$ and $\frac{5x + \frac{2}{x}}{x^3 + 3x + 6}$. By contrast, the rhetorical text is often awkwardly worded. One example is Ibn Ghāzī's division quoted above in section 8, and one with roots is this phrase

⁷⁰ Ibn al-Bannā's *Kitāb al-jabr wa'l-muqābala* (*Book of Algebra*), published in (Saidan 1986), and Ibn al-Hā'im's *Commentary on the Poem of al-Yāsamin* (Ibn al-Hā'im 2003).

from Abū Kāmil: “the root of twenty *māls māl* and two *māls*, whose root is taken”.⁷¹ In my transcribed notation this is $\sqrt{\sqrt{\frac{mm}{20} \frac{m}{2}}}$. The

other operations have no need for a notation for grouping. Because of the aggregations interpretation, the operations of addition, subtraction, and multiplication are worked out one at a time as expressions are formed.

Just as the medieval concept of polynomial differs from the modern, their notation is not merely a variation of ours. In a term like $\frac{t}{20}$ the “*t*” indicates the kind or species of algebraic number, and the 20 tells us how many are present. By contrast, in our “20*x*” the 20 and the *x* are in a symmetric relation. They are both numbers, their only difference being that one is known and the other is not. While it would be unconventional, we could reverse them and write “*x*20” with the same meaning, such as when we substitute *y* with 20 in the expression *xy*. In the medieval notation $\frac{20}{t}$ is nonsense. And while “*x*” can be written alone to indicate one *x*, the Arabic notation does not allow a lone “*t*” or “*m*”, since it leaves unanswered the question “how many?”. This is why we always find a “1” under the species, so $\frac{m}{1}$ indicates one *māl*, $\frac{t}{1}$ one thing, etc.

Said another way, the letter *mīm* (my “*m*”), for example, does not stand for the *value* of the *māl*. It is a label for the number below it,

⁷¹ (Abū Kāmil 1986, 118.7; Abū Kāmil 2004, 191.4).

indicating to what power the “coefficient” belongs. Because these labels do not stand for unknown numbers the way our x and y do, it would have been impossible in the Arabic notation to represent general polynomials like our $ax^2 + bx + c$. The potential figure $\begin{matrix} m & t \\ a & b & c \end{matrix}$ is a semiotic monstrosity mixing two kinds of symbols.

Can the Arabic notation adequately represent modern formulas like $F = MA$ or $V = \frac{1}{3}\pi r^2 h$? Both formulas go beyond simple polynomials by representing *multiplications* with *independent* variables. One could borrow the short word *fī* (“by”) from arithmetic for multiplication, and Arabic algebraists occasionally worked with independent named unknowns, like Abū Kāmil, who named four unknowns in one problem *shay*⁷² (thing), *dīnār* (a gold coin), *fals* (a copper coin), and *khātam* (seal).⁷² We have no examples of notation applied to such solutions prior to the eighteenth century, but it would have been easy to adapt the notation to fit them. But writing the formulas as $\frac{F}{1} = \frac{M}{1} \text{ by } \frac{A}{1}$ and $\frac{V}{1} = \frac{R}{\frac{\pi}{3}} \text{ by } \frac{h}{1}$ (where R is a symbol for the square of r) is awkward at best. Although Arabic algebra could be extended to allow for divisions, square roots, multiple independent unknowns, and even multiplications, it was most effective in writing and operating on polynomials conceived as aggregations in a single unknown. The operations implied by our notation allow us to create expressions like $\frac{x^3(1-2x)-x(1+2x)}{\sqrt{3x}\sqrt{5x}}$, but in the Arabic notation these become unmanageable. The advantage of modern

⁷² See (Sesiano 2000, 149).

notation, which traces its roots back to Viète, is that it is well-suited for expressing operations and for working with known and unknown quantities equally.

11. Conclusion

The Oxford English Dictionary defines “algebra” as “The department of Mathematics which investigates the relations and properties of numbers by means of general symbols...” This definition reflects the *modern* practice of algebra, and disregards the fact that throughout most of its history algebra was written in books without any symbols at all. Because Arabic algebra was written both rhetorically in texts and in symbols on the dust-board, the form in which it is expressed is not an essential feature of the art. What is essential in both medieval and modern algebraic problem solving is the *naming* of an unknown and the setting up and solution of an *equation*.

There was no single notation for solving problems over the whole course of Arabic algebra. We have sporadic illustrations in some Eastern texts which suggest that polynomials were written as lists of coefficients, or that they were arranged in tables. The one truly algebraic notation of which we are aware flourished in North Africa and

possibly in al-Andalus from the 12th to the 15th centuries.⁷³ Some of the books with notation produced in this period, particularly those by al-Qalaṣādī and Ibn Ghāzī, were taught and commented on in the Ottoman Empire, and were used as sources in the eighteenth century revival of the notation in Constantinople.

The two modes of writing algebra, the rhetorical and the symbolic, served two different purposes. One would work out a problem alone using the notation, and, if needed, a rhetorical version would be composed to communicate the results to other people in a book. We are lucky, then, that some textbook authors desired to instruct students in how to use the notation on the dust-board or *lawḥa*. Just enough examples in notation are preserved in manuscripts to show us how operations and equations were represented in symbols.

The notation is truly symbolic. Although the signs for the powers of the unknown, for operations, and for “less” and “and” all originated in short words or abbreviations, they take on a symbolic role in the notation. The letter for “thing” was often reduced to three dots, and the word *illā* (“less”) was usually truncated. The two-dimensional layout discourages any conversion to speech, especially in recursive uses of the notations for square root and division.

The symbolism was created as a way of expressing *medieval* algebra, an algebra in which polynomials are aggregations of the named powers,

⁷³ Because some of the textbooks showing notation continued to be used down to the latter 19th c., the notation may have remained in use even longer.

and in which equations should ideally contain no operations. It is true that a notation can affect the way its users conceive of the objects represented. But one should not overlook the fact that notation is created with certain concepts in mind. In the case of Arabic algebra, the aggregations concept preceded and informed the creation of the notation, and no detectable developments occurred in Arabic algebra after the introduction of the symbolism. This lack of development may be partially due to the fact that algebra was still tied to rhetorical presentation, but it may also be a consequence of the ossification of Arabic education in the centuries after the symbols were introduced. Arithmetic books written after roughly the fourteenth century in the Maghreb are largely epitomes, commentaries, super-commentaries, and even epitomes of commentaries and commentaries of epitomes of earlier works. Mahdi Abdeljaouad writes of the “terrible and desolating consequence” of a textual tradition which turned in on itself, with an aim more to preserve knowledge than to expand it.⁷⁴

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⁷⁴ (Abdeljaouad 2006, end of section 1).

Appendix. Maghrebi books containing algebraic notation, 12th-15th centuries.⁷⁵

Ibn al-Yāsamin (d. 1204. L #347; RI #521)

(1); [M3] *Talqīh al-afkār fī l-‘ilm bi-rusūm al-ghubār* (*Grafting of Opinions of the Work on Dust Figures*). (Ibn al-Yāsamin 1993)

al-Ghurbī (ca. 1350. L #399; RI #998)

Takhṣiṣ ūlī al-albāb fī sharḥ talkhiṣ a‘māl al-ḥisāb (*First Appointment for Minds in Commentary on the Condensed [Book] on the Operations of Arithmetic*).

al-Mawāhidī (1350's-1380's. L #414)

Taḥṣīl al-munā fī sharḥ talkhiṣ Ibn al-Bannā' (*Strengthening of Meaning in Ibn al-Bannā's Condensed [Book]*). (al-Mawāhidī MS)

Ibn Qunfudh al-Qustantīnī (1339-1407. L #425; RI #780)

(1); [M1] *Ḥaṭṭ al-niqāb ‘an wujūh a‘māl al-ḥisāb* (*Removing the Veil from the Faces of Arithmetical Operations*) (1370). (Guergour 1990)

(3) *Mabādī' al-sālikīn fī sharḥ urjūzat Ibn al-Yāsamin* (*Foundations for Beginning the Commentary of Ibn al-Yāsamin's Poem*).

al-‘Uqbānī (d. 1408. L #428; RI #781)

⁷⁵ “L #347; RI #521” means author #347 in (Lamrabet 1994) and author #521 in (Rosenfeld & Ihsanoglu 2003). “(1); [M3]” refers to text (1) in (Lamrabet 1994) and text M3 in (Rosenfeld & Ihsanoglu 2003).

(1); [M1] *Sharḥ al-talkhīṣ* (Commentary on the Condensed [Book]). (Harbili 1997)

Ibn Haydūr al-Tādilī (d. 1413. L #429)

(1) *Al-Tamhīs fi sharḥ al-talkhīṣ* (Clarifying the Commentary on the Condensed [Book]).

al-Qaṭrawānī (end of 14th c. L #430)

Rashfat al-ruḍāb min [or 'an] thughūr a'mal al-ḥisāb (Sucking the Nectar from the Mouths of the Operations of Arithmetic).

al-Qalaṣādī (d. 1486. L #454; RI #865)

(1); [M3] *Kashf al-asrār 'an 'ilm hurūf al-ghubār* (Unveiling the Mysteries of the Science of Dust[-Reckoning]). (al-Qalaṣādī 1988)

(12); [M7] *Sharḥ Talkhīṣ a'māl al-ḥisāb* (Commentary on the Condensed [Book] on the Operations of Arithmetic). (al-Qalaṣādī 1999)

Tuḥfat al-nāshi'in 'alā urjūzat Ibn al-Yāsamin (Delight for Beginning [the study] of Ibn al-Yāsamin's poem).

Ibn Ghāzī (1437-1513. L #468; RI #913)

(2); [M2] *Bughyat al-ṭullāb fi sharḥ munityat al-ḥussāb* (Aim of the Students in Commentary on Desire of Reckoners) (1471). (Ibn Ghāzī 1983; Ibn Ghāzī MS)

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