

WILEY

Pattern and Chaos: New Images in the Semantics of Paradox

Author(s): Gary Mar and Patrick Grim

Source: *Noûs*, Vol. 25, No. 5 (Dec., 1991), pp. 659-693

Published by: [Wiley](#)

Stable URL: <http://www.jstor.org/stable/2215637>

Accessed: 13/02/2015 10:00

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



Wiley is collaborating with JSTOR to digitize, preserve and extend access to *Noûs*.

<http://www.jstor.org>

Pattern and Chaos: New Images in the Semantics of Paradox

GARY MAR AND PATRICK GRIM

Group for Logic and Formal Semantics
Department of Philosophy
SUNY at Stony Brook

Dedicated to the Memory of Hector-Neri Castañeda.

The paradox of the Liar and its kin are well known as recalcitrant puzzles, perennially resistant to perennial attempts at solution. There is also a tradition, however, in which such paradoxes are treated as more than mere puzzles: a tradition running at least from Russell (1903) through Gödel (1931) and Tarski (1935) to Kripke (1975), Herzberger (1982), Gupta (1982), and Barwise and Etchemendy (1987). In such approaches, patterns of paradox are respectfully treated as possible keys to a better understanding of incompleteness phenomena and semantics in general. Our work here is intended as a part of this latter tradition.

Given certain standard assumptions—that particular sentences are meaningful, for example, and do genuinely self-attribute their own falsity—the paradoxes appear to show intriguing patterns of generally unstable semantic behavior. In what follows we want to concentrate on those patterns themselves: the pattern of the Liar, for example, which if assumed either true or false appears to oscillate endlessly between truth and falsehood. We see the work of the present paper, then, as very much in the spirit of Hans Herzberger's 'naive semantics':

Rather than attempting to resolve the paradoxes by rendering critical statements truth-valueless or otherwise neutralizing them, naive semantics undertakes to exhibit and characterize their specific patterns and degrees of instability. (Herzberger (1982), p. 135 of Martin)

We won't here be concerned with proposed *solutions* to the paradoxes. The standard assumptions mentioned above may of course

NOÛS 25 (1991) 659-693
© 1991 by Noûs Publications

be assumptions critically challenged in one or another attempt at solving the paradoxes, and the goal of attempts at solution may be precisely to show that the apparent semantic instability of such paradoxes is *merely* apparent—that despite appearances, for example, the Liar does have some single and stable third truth-value. What we want to explore, however, are the apparent semantic patterns themselves, regardless of whether these are portrayed as merely apparent within one or another attempt at solution.

In what follows we offer some new ways of using computer graphics to analyze the apparent semantic behavior of a variety of paradoxes, both old and new.¹ When considered within the range of an infinite-valued logic and on the mathematical model of iterated functions studied in chaos theory,² in particular, familiar paradoxes often show elegant, unexpected, and visually beautiful semantic patterns, and exhibit moreover a number of surprising structural relations.

In section I below we outline the parametric-operator development of infinite-valued logic, due to Nicholas Rescher, that we will be using throughout, and indicate by example some basic tools of graphic analysis.

Section II is an extended tour of some of the remarkable images generated under such graphic analysis by both familiar paradoxes—including the Liar, the Heterological paradox, the Curry paradox, and Dualist forms of the Liar—and a range of new variations suggested by tools of the analysis itself. Here we introduce a sentence we call the Chaotic Liar which exhibits genuinely chaotic semantic behavior and proves to be of quite central importance. Our analysis of variations on the Dualist takes us into both strange attractors and fractals.

The graphic exhibition and analysis of this range of paradoxes is admittedly a main purpose of the present paper, and in this regard we must confess to a fascination with the beauty of the images themselves. In section III, however, we also offer a brief example of philosophical and metamathematical applications. Here we use a strengthening of the Chaotic Liar to illustrate an intriguing route into limitative results regarding chaos theory itself in the tradition of Gödel (1931), Tarski (1935), Church (1936), and Turing (1936).

I.

In what follows we will treat the paradoxes, new and old, in the context of an infinite-valued logic. What we will be asking is what the semantic behavior of various sentences looks like if we are not restricted to merely two values—true and false, or 1 and 0—but are allowed any real number between 0 and 1 as a semantic value.³

There are of course arguments—of various types and of various plausibilities—that we should in general think of sentences or statements as infinite-valued: as having not merely two possible values, true or false, but a continuum of possible intermediate values as well.⁴

One consideration is that of vague statements. Take for example:

(1) It is cold today .

Is (1) either absolutely true or absolutely false? Is

(2) Alvin looks like Abraham Lincoln

or

(3) Oklahoma is a lovely state

either absolutely true or absolutely false? In many cases, at least, the common unprompted and untutored response is that such examples are not simply true or false, but *more or less* true—that (3) is less true than (2), perhaps, itself less true than (1), which is *fairly* true. And this common response, at least for sentences such as these, may ultimately be the right one.⁵

It is also possible to view the values assigned to statements within an infinite-valued logic as something other than genuine truth-values. Even those most uncompromising in their bivalence with regard to truth and falsity, for example, are quite willing to admit that some propositions may be more or less *accurate*. On a second interpretation, then, the assignment of a value of .7 to a statement might be taken to indicate not a measure of partial truth but simply a measure of accuracy.

Our treatment throughout is perhaps more in accord with the first approach to infinite-valued logics: we will often speak as if sentences can genuinely take any of a continuum of semantic values between truth and falsity.⁶ Nonetheless we don't consider our task here to be one of arguing for infinite-valued logics in general, and we don't feel ourselves in any way committed philosophically to the ultimate rightness of infinitely many semantic values. Our approach is hypothetical: what does the semantic behavior of certain sentences look like if we do assume an infinite-valued logic?

Assuming a range of real values between 0 and 1, and representing the value of a sentence p as $/p/$, we will take the value of $\sim p$ to be $1-/p/$. For sentences p and q with values $/p/$ and $/q/$, we take the value of $(p \ \& \ q)$ to be $\min\{/p/,/q/\}$ and the value of $(p \ \vee \ q)$ to be $\max\{/p/,/q/\}$. All of this is a straightforward generalization from standard finitely many-valued logics. The value of $(p \rightarrow q)$ we take to be $\max\{1-/p/,/q/\}$.⁷

We will also use Nicholas Rescher’s development of infinite-valued logics in terms of a parametrized propositional operator. Following Rescher, we take the value of a proposition $V_{\mathbf{v}\mathbf{p}}$ asserting that a proposition \mathbf{p} has value \mathbf{v} to be given by:

$$/V_{\mathbf{v}\mathbf{p}}/ = 1 - \text{abs}(\mathbf{v} - / \mathbf{p} /) ,$$

where $\text{abs}(\mathbf{v} - / \mathbf{p} /)$ indicates the absolute difference between \mathbf{v} and $/ \mathbf{p} /$ (Rescher, 1969, 81-82).⁸ Intuitively, such a formula states that the proposition that \mathbf{p} has the value \mathbf{v} is untrue to the extent that the value of \mathbf{p} differs from \mathbf{v} . The standard Tarskian \mathbf{T} schema can be seen as a special instance of this formula in which \mathbf{v} and $/ \mathbf{p} /$ are restricted to values of 0 and 1.

At this point we offer a simple illustration of the kinds of *formal* lessons regarding paradox that such an infinite-valued logic may have to teach, regardless of whether such a logic is taken to be fully defensible *philosophically* or not.

We start with merely the two classical values and the Simple Liar:

(4) This sentence is false .

Given the standard \mathbf{T} schema, by a standard pattern of reasoning the assumption that (4) is true leads to the conclusion that it is false and the assumption that (4) is false leads to the conclusion that it is true. It is therefore quite natural to think of the classical semantic behavior of the Liar as an oscillation between truth-values \mathbf{t} and \mathbf{f} :

$\mathbf{t} \ \mathbf{f} \ \mathbf{t} \ \mathbf{f} \ \mathbf{t} \ \mathbf{f} \ \mathbf{t} \ \mathbf{f} \ \dots$

Using 1 for truth and 0 for falsity, then, we can model the classical semantic behavior of the Liar in terms of a sequence of values x_n , where x_0 is some initial assumed truth-value and

$$x_{n+1} = 1 - x_n .$$

Starting with an initial estimate of either 1 or 0, we obtain a sequence of alternating 1s and 0s. In a simple graph:

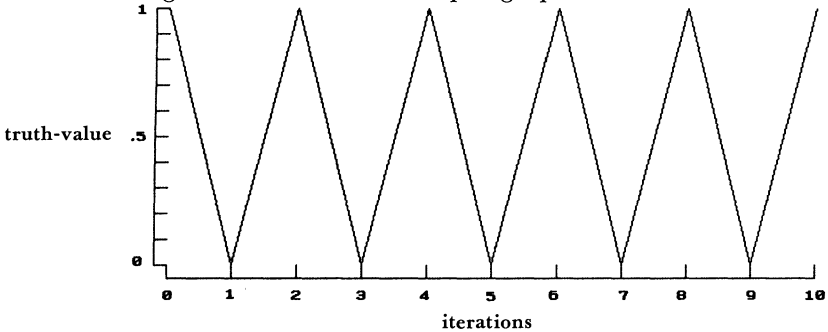


Figure 1

Consider now the Simple Liar within the context of the infinite-valued logic outlined above.

If (4) is assigned a value of 0, the **Vvp** schema above forces us to a revised estimate of $1 - \text{abs}(0 - 0)$, or 1. Given an estimated value of 1, the **Vvp** schema forces us to conclude that (4) has a value of $1 - \text{abs}(0 - 1)$, or 0. Thus if assigned either 0 or 1 the Liar will still give us the familiar oscillation between 0 and 1.

If we propose that the Liar is .25 true, on the other hand, using the **Vvp** schema above, we are forced to a revised estimate of $1 - \text{abs}(0 - .25)$, or .75. With an estimate of .75 the **Vvp** schema forces us to a revised estimate of .25. Starting with an initial estimate of .25 the Liar thus gives us the following series of values:

.25 .75 .25 .75 .25 .75 . . .⁹

Graphed as before:

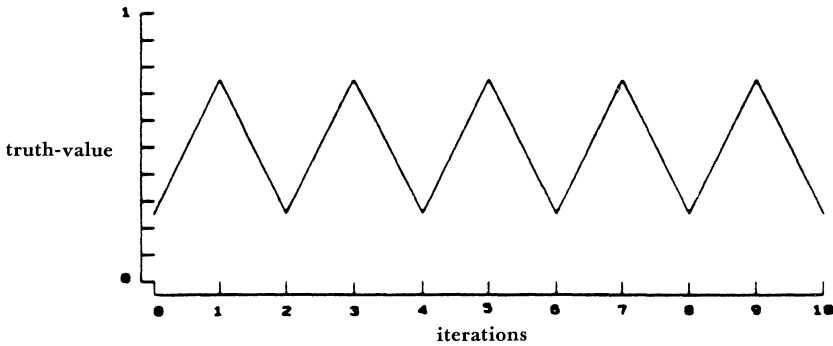
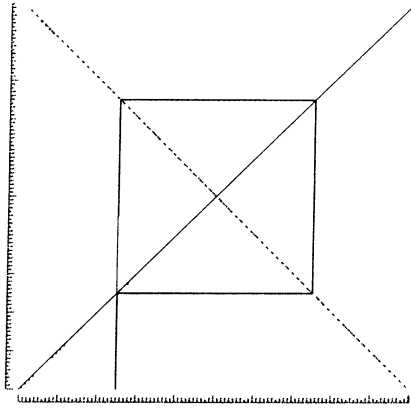


Figure 2

Here let us also introduce an alternative form of graphic analysis, known as a web diagram, capable of showing patterns of behavior through indefinitely many iterations. In a graph such as that below our initial value **a**—.25, in this case—is plotted as **(a,0)**. A line is drawn vertically to meet the plotted function **f(x)** at point **(a, f(a))**. In order to graphically represent the iteration of the function, we draw a horizontal (to the right or to the left) to a point **(f(a), f(a))** on the diagonal **y = x** and then draw a vertical line from there to our function at **(f(a), f(f(a)))**. This process is repeated.

In this second form of graphic analysis an initial estimated value of .25 for the Liar gives us a simple box, indicating the infinite oscillation between .25 and .75:

Figure 3



With an initial estimate of .86, on the other hand, the characteristic Liar-like oscillation appears between .86 and .14:

.86 .14 .86 .14 .86 .14 . . .

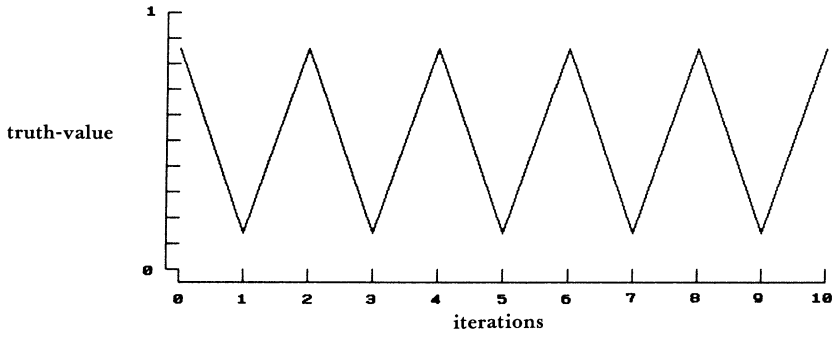
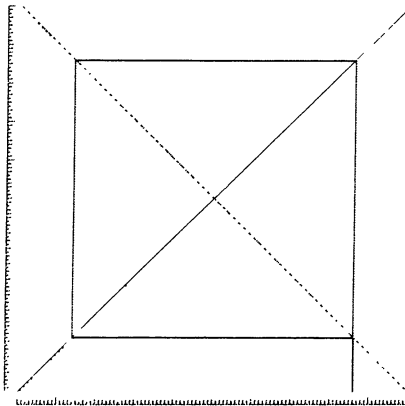


Figure 4

Figure 5



Only a value of .5 serves as a fixed point, generating a constant series of .5s as iterated values.

So much, for the moment, for the Simple Liar. Consider now another sentence, the 'Minimalist', which refers not merely to its own truth-value but to its *estimated* truth-value:

- (5) The actual value of this sentence is whichever is smaller:
its estimated value or the opposite of its estimated value .

Here we take the opposite of a value v , fairly naturally, to be $1-v$.

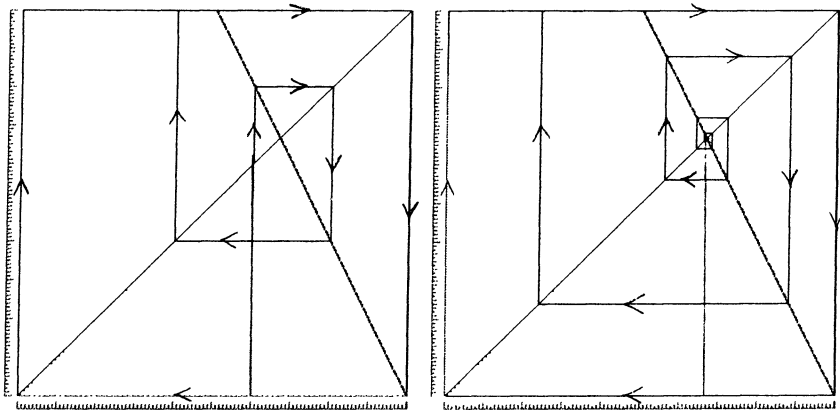
Is (5) true or false? Given an estimated value of 'true', or 1, what (5) says is that its actual value is whichever is smaller: 1 or 0. With an estimated value of 'true', in other words, what (5) asserts is that it is false. Given our initial estimate, then—that (5) is true—we are forced to conclude that (5) is false. Take this as a revised estimate. With an estimated value of 'false' for (5), what (5) asserts is that its actual value is 0. But with an estimated value of 'false', if (5) asserts that it's false, we're forced to a further revised estimate of 'true' . . . , and so forth, endlessly oscillating between 0 and 1.¹⁰

Notice that if we restrict ourselves to two values—'true' and 'false' or 0 and 1—what (5) gives us is behavior identical to that of the Simple Liar. Within an infinite-valued context, however, the behavior of the Minimalist and the Simple Liar diverge sharply.

An initial estimate of .25 for the Simple Liar, we've seen, gives us a periodic oscillation between .25 and .75. Consider the behavior of the Minimalist with the same initial estimate. Given an estimate of .25, what the Minimalist asserts is that its actual value is whichever is smaller: .25 or .75. What it asserts, in other words, is that its actual value is .25. Using the **Vvp** schema above, we can then calculate its 'actual' value as $1-\text{abs}(.25-.25)$, or 1. Starting with an initial estimate of .25 for the Minimalist, then, we are forced to a revised estimate of 1. But given an estimate of 1, what the Minimalist asserts is that its actual value is whichever is smaller: 0 or 1. By the **Vvp** schema we are forced to a further revised estimate of $1-\text{abs}(0-1)$, or 0, and this in turn leads to a series of revised estimates 1, 0, 1, 0, 1, 0,

An initial estimate of .6 for the Minimalist, on the other hand, gives us the following series of values: .6, .8, .4, 1, 0, 1, 0, 1, 0, . . . , again converging on an infinite oscillation between 0 and 1. An initial value of .66 gives us .68, .64, .72, .56, .88, .24, 1, 0, 1, 0, 1, 0, Web diagrams with arrows to indicate direction appear below. (Figures 6 and 7)

With but one exception, all initial values for the Minimalist converge ultimately on an oscillation between 0 and 1. The exception



initial value .6

Figure 6

initial value .66

Figure 7

is the point $2/3$. Although any terminal decimal approximation to $2/3$ will give us the same oscillation, an easy calculation regarding the **Vvp** schema shows that an initial value of $2/3$ will result in a revised estimate of $2/3$ as well: $2/3$ is the single fixed point for the Minimalist.

Although the behavior of the Minimalist is indistinguishable from that of the Simple Liar within a two-valued logic, then, the semantic patterns of the two diverge sharply within an infinite-valued context. For initial values v between 0 and 1 the Simple Liar gives us a Liar-like oscillation between v and $1-v$. For any initial value v between 0 and 1 other than $2/3$, the Minimalist gives a series of values which converge on an oscillation between 0 and 1. Within an infinite-valued context it is thus the Minimalist, rather than the Simple Liar itself, that converges on the characteristic behavior of the classical two-valued Liar.

Whatever its philosophical status, then, an infinite-valued logic is capable of exhibiting clear formal differences between certain sentences which a standard two-valued logic is not. One way of thinking of the matter, of course, is this: that employment of an infinite-valued logic creates illusory images of semantic-like patterns where there are in fact none—on the conviction, say, that both the Minimalist and the Simple Liar can in reality have only one of two values. An alternative way of thinking of the matter, however, is the following: that these sentences have latent ranges of semantic behavior some of which are visible for the first time only when considered within an infinite-valued context. Here we aren't prepared to argue for this second approach as a philosophical thesis, though we must confess that such a view has in fact guided our exploration of patterns of paradox and related metamathematical results.

II.

In introducing some tools of analysis above we illustrated the semantic behavior of the Simple Liar and the Minimalist in the context of an infinite-valued logic. In this section we want to do the same for both traditional relatives of the Liar and a range of new variations—variations in many cases suggested by the tools of the analysis itself.

1. THE TRUTH-TELLER AND THE SAMESAYER

Long discussed as a companion to the Liar is the Truth-Teller, which asserts not its own falsity but its own truth:

(6) This sentence is true .

Although not paradoxical in the sense of forcing a contradiction given an assumption of either truth or falsity, the Truth-Teller is still semantically peculiar: (6) can consistently be assumed either true or false, but there seem to be no grounds for either assignment. For this reason the Truth-Teller is often treated in ways similar to the Liar within attempted solutions for the paradoxes. Kripke (1975), for example, diagnoses both the Truth-Teller and the Liar as ungrounded and so without truth-value.

Within an infinite-valued logic using the V_{vp} schema above, (6) will take a series of iterated values

$$T(x_{n+1}) = 1 - \text{abs}(1 - x_n) ,$$

or, since the values of our sentences are restricted to the interval $[0,1]$, simply $T(x_{n+1}) = x_n$. The behavior of the Truth-Teller within the infinite-valued context is thus an extension of its behavior in the two-valued case: *any* estimated truth-value for (6) proves consistent, in the sense that (6)'s actual value, computed by means of the V_{vp} schema, will in all cases match that estimated value.

Consider also the Samesayer, a sentence superficially similar to the Truth-Teller but which says that its value is as estimated:

(7) The value of this sentence is precisely what it has been estimated to be .

Given an estimate of x_n , what (7) *says* is that its value is x_n . Its series of revised values, then, computed by means of our V_{vp} schema, is:

$$S(x_{n+1}) = 1 - \text{abs}(x_n - x_n) ,$$

or simply $S(x_{n+1}) = 1$.

Perhaps contrary to initial intuitions, the Truth-Teller differs quite sharply from the Samesayer within both infinite-valued and

two-valued contexts. The Truth-Teller can consistently be assigned any of the available values. The Samesayer cannot: it proves tautologous in the sense of forcing in all cases a revised estimate of perfect truth.

Beneath these differences, however, lies an intriguing structural symmetry between the Truth-Teller and the Samesayer.

Let $*T(x)*$ be the value that the Truth-Teller *says* it has—that is, 1. Let $*S(x)*$ be the value that the Samesayer says it has— x_n , for any previous estimate x_n . Then it happens that the actual value $/T(x)/$ of the Truth-Teller, computed by means of the **Vvp** schema, is identical to what the Samesayer *says* its value is:

$$/T(x)/ = *S(x)* .$$

Conversely, the actual value $/S(x)/$ of the Samesayer is identical to what the Truth-Teller says *its* value is:

$$/S(x)/ = *T(x)* .$$

2. THE HALF-SAYER

As a variation on the Samesayer consider the Half-Sayer, which says not that its actual value is its estimated value but that it is *half* its estimated value:

(8) The actual truth of this sentence is half its estimated truth .

If we start with an initial estimate of .5 for the truth of (8), what (8) asserts is that its actual truth is half our estimate—that is, .25. But how true *is* (8) then? Given our initial estimate and the **Vvp** schema, the value of (8) will be $1 - \text{abs}(.5 - .25)$, or .75. This of course qualifies as a new estimate. But what (8) says is that its actual truth is half its estimated truth. Given our new estimate,

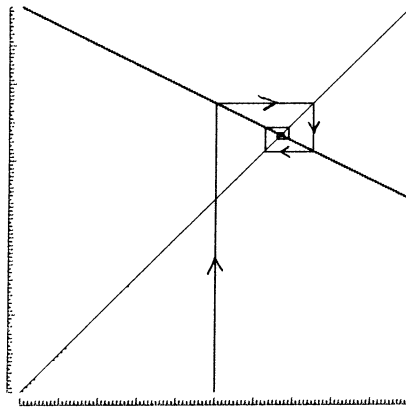


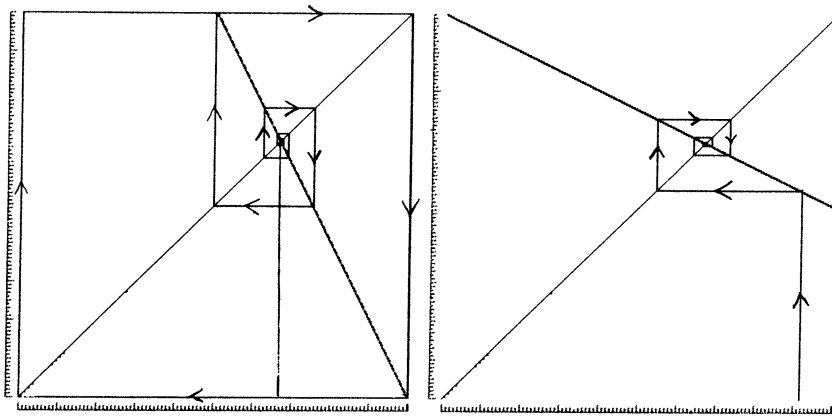
Figure 8

then, what (8) says is that its value is .375. How true is (8)? Using the **Vvp** schema, we are forced to a further revised estimate of $1 - \text{abs}(.75 - .375)$, or .625.

Continuing this pattern of reasoning we are forced to successive estimates of .6875, .65625, .671875, .6640625, .6660125, . . . converging ultimately on $2/3$. (Figure 8)

It happens, in fact, that any initial estimate will give us the same result: revised estimates for (8) converge inexorably on a fixed point of $2/3$.

The dynamical behavior of the Half-Sayer is thus precisely the opposite of the Minimalist. For the Minimalist $2/3$ is a fixed point repeller. For the Half-Sayer $2/3$ is a fixed point attractor:



The Minimalist with an initial estimate of .665
Figure 9

The Half-Sayer with an initial estimate of .916
Figure 10

What of a ‘Quarter-Sayer’, giving us

$$x_{n+1} = 1 - \text{abs}((1/4)x_n - x_n) ,$$

or a ‘Third-Sayer’, giving us

$$x_{n+1} = 1 - \text{abs}((1/3)x_n - x_n) ?$$

Each of these converges to its own fixed point: the ‘Quarter-Sayer’ has a fixed point of $4/7$, the ‘Third-Sayer’ a fixed point of $3/5$. In general, for any k between zero and one, the attractor fixed point for the ‘ k -Sayer’ will be $1/(2-k)$.¹¹

3. THE CHAOTIC LIAR

Given an estimated truth \mathbf{v} for a sentence, we will speak of its estimated falsehood as $(1-\mathbf{v})$. Consider:

(9) The actual truth of this sentence is its estimated falsehood or alternatively

(10) This sentence is as true as it is estimated to be false , which we shall call the Chaotic Liar. Other renderings include:

(11) The actual value of this sentence is precisely the opposite of its estimate ,

(12) The actual value of this sentence is precisely the opposite of what you estimate it to be ,

or still more informally

(13) I'm as true as you think I'm false .

What the Chaotic Liar *asserts*, of course, is that its value is $(1-v)$, where v is its estimated value. In terms of the **Vvp** schema, then, successive values x_{n+1} for the sentence will be given by

$$x_{n+1} = 1 - \text{abs}((1-x_n) - x_n) .$$

For an initial value .32, for example, the Chaotic Liar gives us a graph which begins as follows:

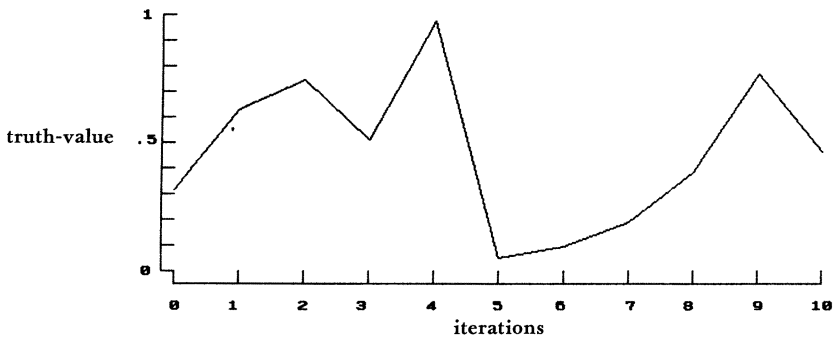
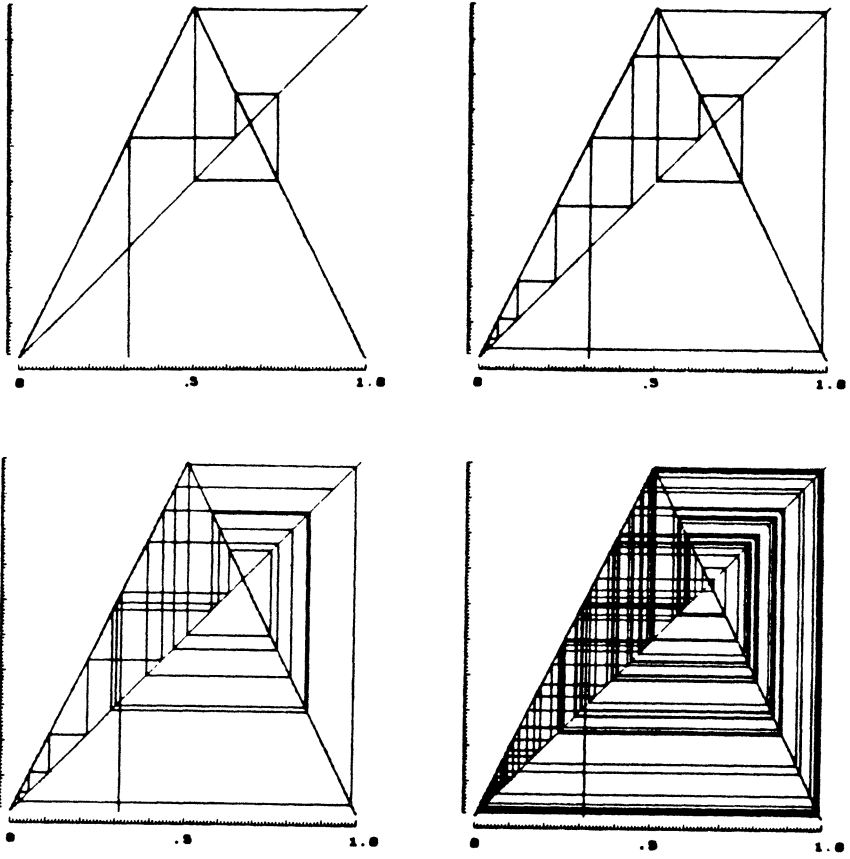


Figure 11

and a web diagram something like the following, represented at four successive stages:



Figures 12-15

Iterated values for sentence (9) exhibit the sensitivity to initial conditions that is the hallmark of chaos.¹² Initial values .314 and .3141, for example, give us quickly divergent iterations:

Iteration	.314	.3141			
1	.628	.6282	11	.928	.7232
2	.744	.7436	12	.144	.5536
3	.512	.5128	13	.288	.8928
4	.976	.9744	14	.576	.2144
5	.048	.0512	15	.848	.4288
6	.096	.1024	16	.304	.8576
7	.192	.2048	17	.608	.2848
8	.384	.4096	18	.784	.5696
9	.768	.8192	19	.432	.8608
10	.464	.3616	20	.864	.2784

It happens that $x_{n+1} = 1 - \text{abs}((1 - x_n) - x_n)$ is a very simple but paradigmatically chaotic function on the interval $[0, 1]$.¹³ Since this is precisely the interval of semantic values within our infinite-valued logic, (9) through (13) will be sentences with genuinely chaotic *semantics*. In section III we use a strengthened relative of the Chaotic Liar in order to introduce a family of limitative results regarding chaos theory itself.

Here we also note an intriguing relationship between the Chaotic Liar and the Simple Liar.

Let us first introduce the useful notion of the **Vvp** of a value. A sentence which self-attributes a value s , we've seen, takes a series of values

$$x_{n+1} = 1 - \text{abs}(s - x_n) ,$$

computed in terms of the **Vvp** schema. It thus seems natural to think of $1 - \text{abs}(s - x_n)$ as 'the **Vvp** of' the value s . In general, for any value function $s(x)$, we will take the formula

$$x_{n+1} = 1 - \text{abs}(s(x_n) - x_n)$$

to be the **Vvp** of $s(x)$.

The value of the Chaotic Liar, in these terms, is precisely the **Vvp** of the formula for the Simple Liar. Values for the Simple Liar, it will be remembered, are given by:

$$x_{n+1} = 1 - \text{abs}(0 - x_n) ,$$

or more simply, for positive x ,

$$x_{n+1} = 1 - x_n .$$

This function is not itself chaotic: as indicated above, it produces a pair of oscillating values which appear as a simple box within a web diagram. Its **Vvp**, on the other hand,

$$x_{n+1} = 1 - \text{abs}((1 - x_n) - x_n)$$

does give us a fully chaotic semantics.

4. THE HETEROLOGICAL PARADOX

The Heterological paradox (Grelling and Nelson (1908)) concerns adjectival phrases which do not apply to themselves. An adjectival phrase is said to be *autological* if it has the property it expresses, and is said to be *heterological* otherwise. Letting π be any adjectival expression and ' π ' the name of that adjectival expression, we may express these properties as follows:

$$(14) \text{Aut}(' \pi ') \leftrightarrow \pi(' \pi ') ,$$

and

$$(15) \text{Het}(' \pi ') \leftrightarrow \sim \pi(' \pi ') .$$

But is the adjectival phrase 'is heterological' itself heterological or not? Instantiating (15), we quickly obtain a contradiction:

$$(16) \text{Het}(' \text{Het} ') \leftrightarrow \sim \text{Het}(' \text{Het} ') .$$

Strictly speaking, the Heterological paradox depends not on sentential self-reference, as does the Liar, but on adjectival self-application. Quine (1962) offers a form of the Liar which is structurally similar to the Heterological paradox, however, employing the notion of falsity within an adjectival expression. When

$$(17) \text{ is false when appended to its own quotation}$$

is applied to itself—appended to its own quotation—we get:

$$(18) \text{'is false when appended to its own quotation' is false when appended to its own quotation .}$$

To generalize these ideas to the infinite-valued case, we work with the following natural assumption: that a predicate π self-applies fully iff the value of $\pi(' \pi ') = 1$, and in general applies with a value \mathbf{v} iff the value of $\pi(' \pi ')$ is \mathbf{v} ; it applies with the value of $.4$, say, iff the value of $\pi(' \pi ') = .4$.

Now consider

$$(19) \text{'applies to itself precisely as much as you estimate it does not' applies to itself precisely as much as you estimate it does not .}$$

Let us start with an estimate of $.3$, say, regarding the degree to which 'applies to itself precisely as much as you estimate it does not' applies to itself. What (19) says is that its value is the opposite of our estimate—that is, $(1-.3)$ or $.7$. Using the **Vvp** schema, we are then forced to a revised estimate of $1-\text{abs}(1-.3)-.3$ or $.6$. Note that $.6$ is both a revised estimate for the value of (19) and—by our assumption above—a revised estimate of the self-applicability of the adjectival phrase incorporated in (19).

In general, the formula for revised values of (19) will be given by

$$x_{n+1} = 1-\text{abs}((1-x_n)-x_n) .$$

This is of course precisely the formula for the Chaotic Liar.¹⁴ Just as the simple Heterological paradox parallels the Simple Liar, then,

a chaotic version of the Heterological paradox parallels the Chaotic Liar.

Here we outline some intriguing relationships in somewhat more detail.

5. THE CURRY PARADOX

The Curry paradox¹⁵ is generated by a conditional

C: If C is true, then P ,

where P is some arbitrarily chosen proposition.

In a standard two-valued logic and given the Tarskian principle that C is true iff what it says is the case, from C we can derive P simpliciter. The paradox, of course, is that P may be any proposition whatsoever; we have seemingly proven any arbitrarily chosen proposition by pure logic alone.

Consider an infinite-valued variant of the Curry sentence which says

This is as true as ‘If I am true, then P’ .

Using a conditional ($p \rightarrow q$) definable as $(\sim p \vee q)$, which takes a value within our logic of $\max\{1-p, q\}$, what this variation on the Curry says is that its value is $\max\{1-C, P\}$.¹⁶ Using the Vvp schema, then, this gives us a series of values

$$x_{n+1} = 1 - \text{abs}(\max\{1 - x_n, P\} - x_n) .$$

With a value of P greater than .5, values for C seem to ‘stair-case’ up to a periodic oscillation. With a P of .75, for example, and an initial value of .2, we get a series of values that converges on an oscillation between .85 and .9:

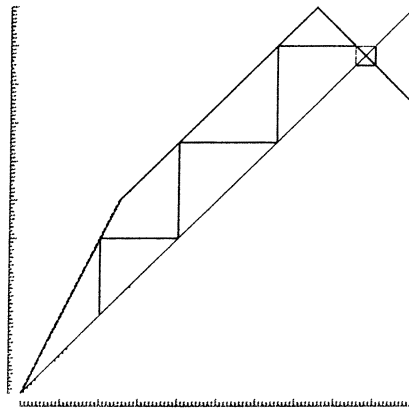
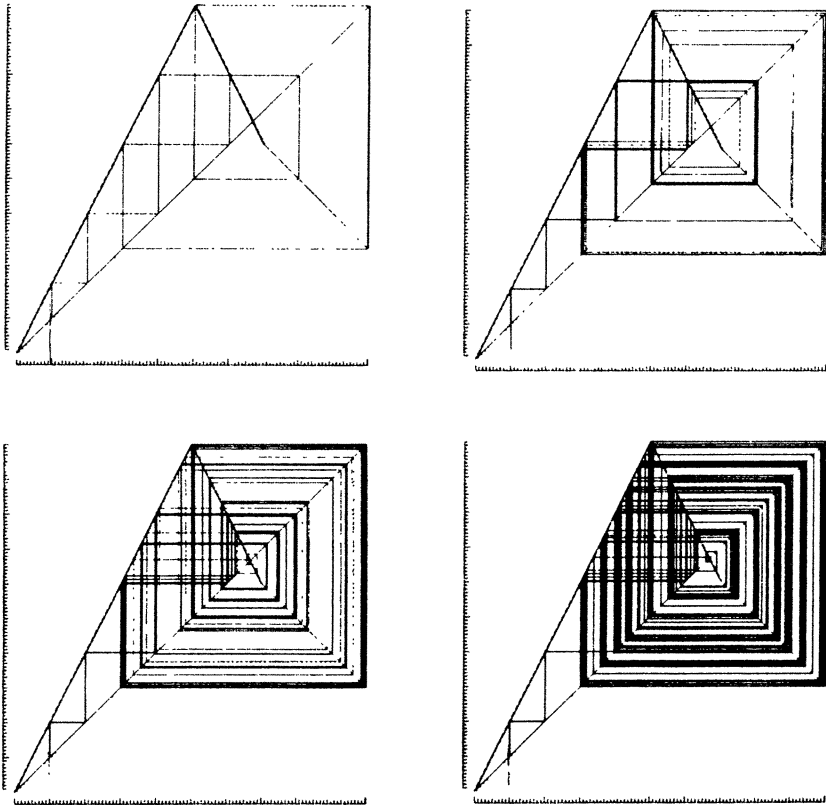


Figure 16

With a value of 1 for P , we get 1 as a fixed point for all initial estimates: our Curry variation converges on the Samesayer.

For P less than .5, on the other hand, we seem to get a region of chaotic behavior governed by the extent of a central 'tent' in the graphing of the function. For a P of .3, for example, with an initial value of .1, we get the following:



Figures 17-20

As the value of P approaches 0, chaotic behavior increases to the full unit interval; with the value of P equal to 0 we obtain a function identical to the Chaotic Liar.

To see how the behavior of our Curry sentence changes with different values for P we can also use an alternative form of graphing. The following, a variant of what is known as an orbit diagram, shows the range of values taken by our sentence for values of P between 0 and 1. Here in all cases we use an initial input of .23:

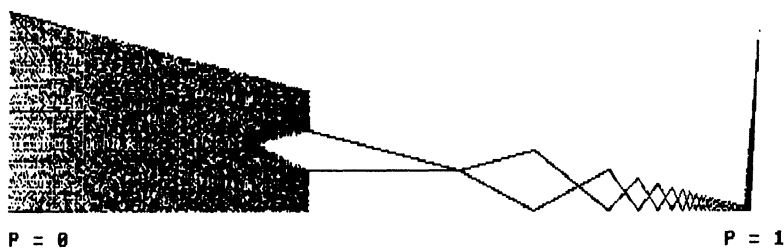


Figure 21

6. THE DUALIST

As has been clear from at least the medieval period,¹⁷ beyond the Simple Liar is an infinite series of Liar cycles in which indirect self-reference replaces the direct self-reference of the Liar. The simplest of these is the Dualist, which combines features of both the Liar and the Truth-Teller:¹⁸

Socrates: What Plato is about to say is false

Plato: Socrates speaks truly

or simply

X: Y is false

Y: X is true .

Within a two-valued logic the reasoning of the Dualist is as follows. If X is true, then Y is false, but then it is false that X is true, and thus X is *not* true. If X is false, on the other hand, then it is false that Y is false; Y is then true, and thus X is true rather than false.

A fairly natural infinite-valued representation of iterated values for these sentences is given by

$$\begin{aligned} x_{n+1} &= 1 - \text{abs}(0 - y_{n+1}) \\ y_{n+1} &= 1 - \text{abs}(1 - x_n) . \end{aligned}$$

Here it is assumed that we start with an estimate x_n for X, calculate y_{n+1} as an estimate for Y in terms of that x_n , and then re-evaluate our original estimate for X in light of our last estimate for Y.¹⁹ Successive values for X then exhibit semantic behavior identical to that of the Simple Liar within an infinite-valued context: given an initial estimate of .25 for X, for example, we are forced to revised estimates of .75, .25, .75, .25,

Consider also a variation of the Dualist:

X': This sentence is as true as Y' is false
 Y': X' is true ,

which on a similar pattern of reasoning gives us

$$\begin{aligned} x_{n+1} &= 1-\text{abs}([1-y_{n+1}]-x_n) \\ y_{n+1} &= 1-\text{abs}(1-x_n) \end{aligned} .$$

Note that if we substitute the right hand side of the second equation for y_{n+1} in the first we obtain

$$x_{n+1} = 1-\text{abs}([1-(1-\text{abs}(1-x_n))]-x_n) .$$

Consequently, for x_n between 0 and 1, we derive

$$x_{n+1} = 1-\text{abs}((1-x_n)-x_n) ,$$

which is precisely the formula for the Chaotic Liar.

7. SOME DUALIST STRANGE ATTRACTORS

Here we offer a further variation on the Dualist, in which both sentences speak of each other in tones akin to that of the Chaotic Liar:

X'': X'' is true to the extent that Y'' is true
 Y'': Y'' is true to the extent that X'' is false .

Alternatively put:

X'': X'' is as true as Y''
 Y'': Y'' is as true as X'' is false .

What X'' says is that its truth-value is that of Y''. Using the **Vvp** schema, then, we can compute its value as $1-\text{abs}(/X''/-/Y'')$. Given estimates of x_n and y_n for X'' and Y'', the value of X'' at the next estimate is thus given by:

$$x_{n+1} = 1-\text{abs}(y_n-x_n) .$$

What Y'' says, on the other hand, is that it is true to the extent that X'' is false, or that its value is the opposite of that of X''. Given the same x_n and y_n , in other words,

$$y_{n+1} = 1-\text{abs}((1-x_n)-y_n) .$$

With initial values of 1/8 and 1/8 for X'' and Y''—(.125,.125)—these formulae give us revised values of (1,.25), (.25,.75), (.5,1), (.5,.5), (1,1), (1,0), (0,1), (0,1), (0,1), Graphically represented, these value pairs (x,y) outline the triangular upper half of the unit square as they move toward a final fixed point of (0,1):

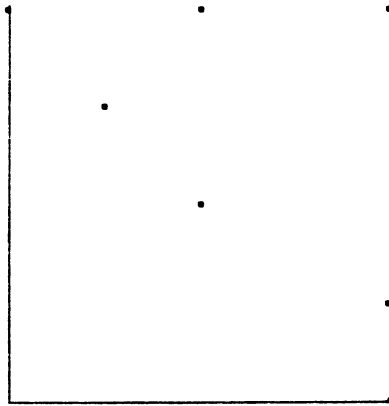


Figure 22

Other pairs of points give us periodic behavior: initial estimates of .4 and .6 for X'' and Y'' , for example, give us as successive value pairs $(.8, 1)$, $(.8, .2)$, $(.4, 1)$, $(.4, .6)$, . . ., with a repeating period of four points. Throughout the $[0, 1]$ interval, however, the triangular upper half of the unit square appears as a persistent constraint. The following is an overlay of graphs for initial points (x, y) where x and y range from 0 to 1 in increments of .05:²⁰

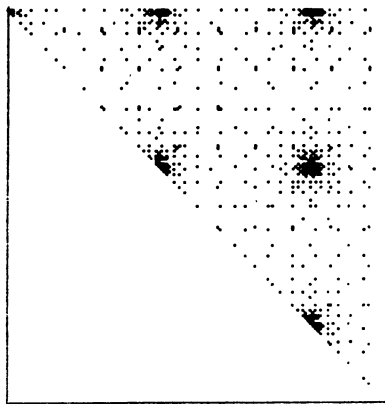


Figure 23

It should be noted that what our formulae above actually capture is not merely two sentences but a particular pattern of reason-

ing with regard to them. Starting with a pair of estimates for sentences X'' and Y'', we have in effect calculated revised estimates for X'' and Y'' *simultaneously*. But here one might also consider an alternative pattern of reasoning with respect to the Dualist sentences. On this second pattern of reasoning, one would start with a pair of estimates for X'' and Y'', calculate a revised estimate for the first sentence in terms of those two estimates, but then go on to calculate a revised estimate for the second sentence in terms of the initial estimate for Y'' together with the most recently *revised* estimate for X''.

This 'sequential' rather than 'simultaneous' pattern of reasoning with respect to X'' and Y'' can be represented by a pair of formulae:

$$\begin{aligned} x_{n+1} &= 1 - \text{abs}(y_n - x_n) \\ y_{n+1} &= 1 - \text{abs}((1 - x_{n+1}) - y_n) \end{aligned} \quad ,$$

in which we replace the previous x_n of the second formula with x_{n+1} .

For a wide range of initial values (x,y), what these formulae yield is a very persistent strange attractor.²¹ Initial values (.1,.9), for example, give us the following pattern of successive values:

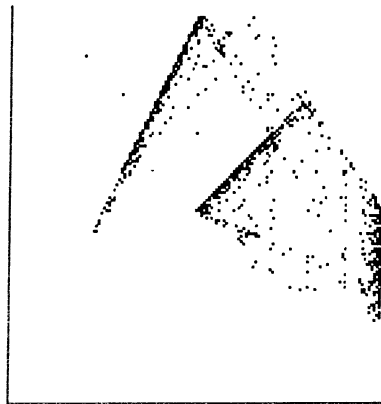


Figure 24

The persistence of such an attractor is clearly evident in an overlay diagram for initial points (x,y) in increments of .05 as before:

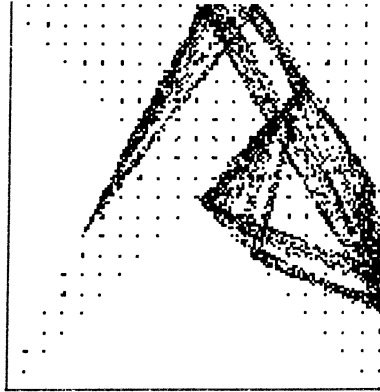


Figure 25

An entirely different strange attractor appears if the first member of our Dualist pair is replaced with a sentence reminiscent of the Half-Sayer:

X''' : X''' is true to half the extent that Y''' is true
 Y''' : Y''' is true to the extent that X''' is false

Following the successive pattern of reasoning outlined above,

$$\begin{aligned} x_{n+1} &= 1 - \text{abs}(.5y_n - x_n) \\ y_{n+1} &= 1 - \text{abs}((1 - x_{n+1}) - y_n) \end{aligned} .$$

In the case of X''' and Y''' the persistent attractor takes the form of two ellipses. For initial values $(.7, .3)$:

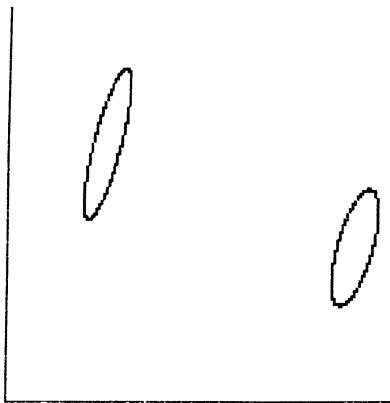


Figure 26

An overlay diagram using initial values (x,y) in increments of .1 for this attractor shows ellipses in much the same position but of differing sizes depending on initial values. For some values, only a four-fold scattering of dots or a central cross-pattern emerges:

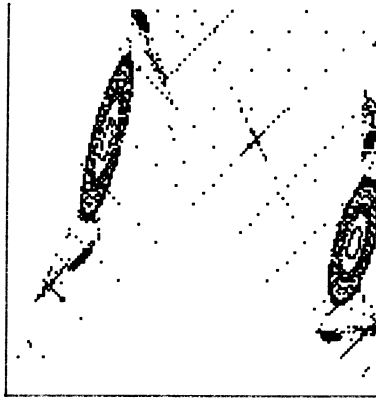


Figure 27

8. FRACTALS IN THE SEMANTICS OF PARADOX

Here we finally want to offer another way of graphing the behavior of the Dualist functions sketched in terms of attractors immediately above. Though we consider the results that follow to be intriguingly beautiful, we cannot at this point claim to understand fully the semantic lessons they may have to teach.

Standard escape-time diagrams show, for each pair of points (x,y) on the Cartesian plane, the number of iterations through a given function that is required before a certain specified result is achieved. An escape-time diagram may show us, for example, whether the point $(.5,.5)$ cycled through a particular function gives a result (x,y) such that $\sqrt{x^2 + y^2} > 1$ within one iteration, within two iterations, within three iterations, or more Those points on the plane which reach the chosen threshold in one iteration can be colored with one shading, those in two iterations with another, and so forth. Alternatively, as below, we can emphasize the interfaces between different areas: points at which the number of required iterations changes.

Here we offer an escape-time diagram for the simplest of the chaotic Dualist variations sketched above, in which X'' and Y'' are calculated simultaneously for input values (x,y) :²²

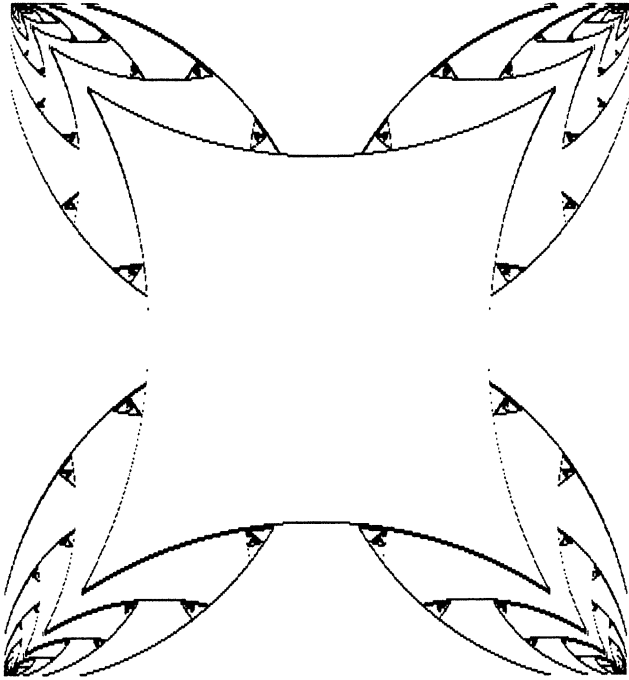


Figure 28

This fragile tracery of lines indicates those points at which there is a change in the number of iterations required for the ‘hypotenuse’ $\sqrt{x^2 + y^2}$ of a result (x,y) to exceed 1.03.

We’ve confined the image above to the unit square, reflecting the fact that semantic values for x and y within our logic are confined to the interval $[0,1]$. Formally, however, this image is a section of the larger one below, for values x and y between -1.4 and $+2.4$. A central box indicates the unit square:

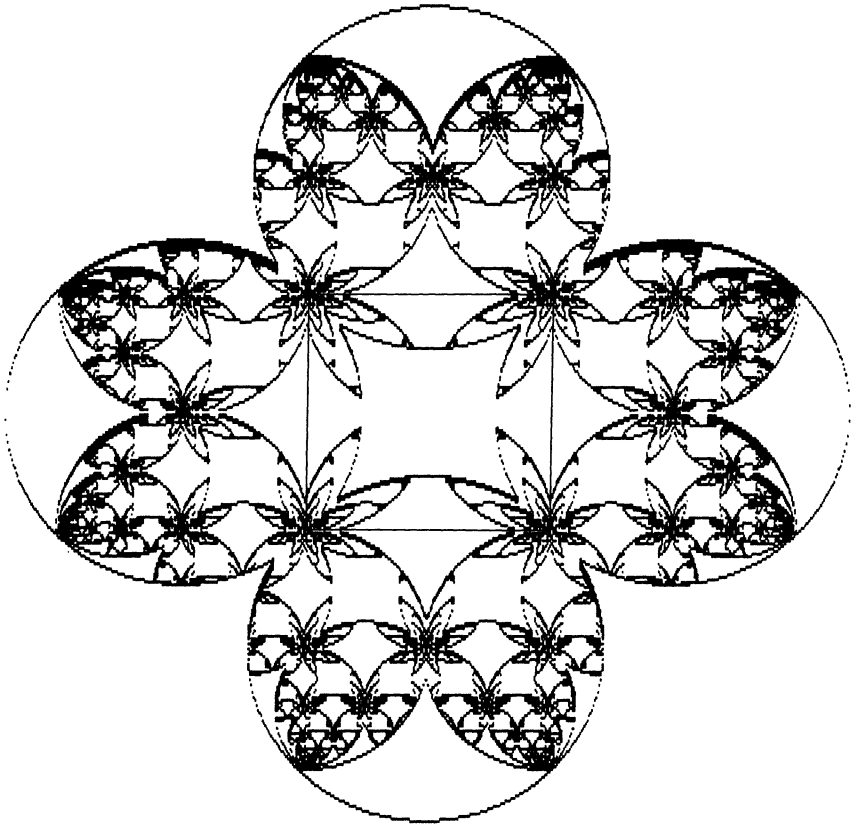


Figure 29

These images clearly exhibit an intricate fractal character, or self-similarity under magnification.²³ Nonetheless what is being graphed within the unit square is simply information regarding the semantic behavior for different inputs of a pair of English sentences:

X'' : X'' is true to the extent that Y'' is true

Y'' : Y'' is true to the extent that X'' is false

The following is an escape-time diagram for the second variation of the Dualist—that in which values for X'' and Y'' are computed successively and which gave us the first of the strange attractors above. Both this diagram and the next are for values between -2 and $+6$, with the unit square indicated as before:

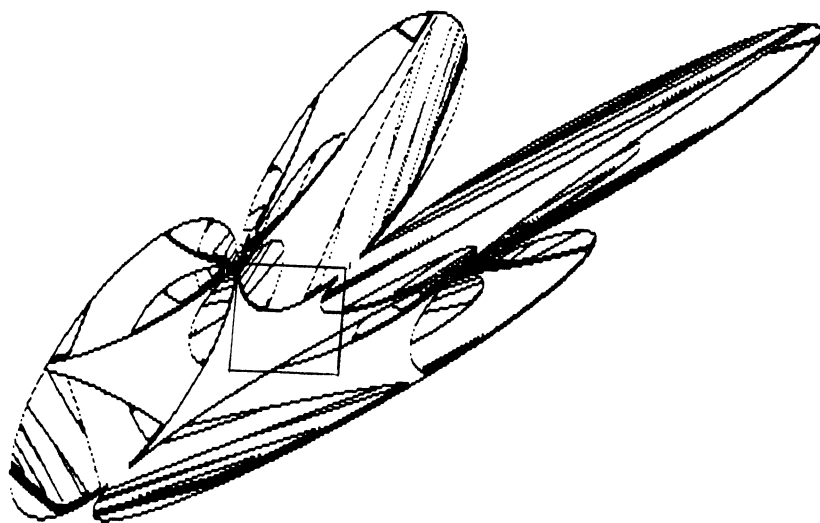


Figure 30

An escape-time diagram for the third variation of the Dualist—which gave us the double ellipse attractor—appears as follows:



Figure 31

Note that the general shape of the second two escape-time diagrams—despite the fact that the corresponding attractors are quite different—are obviously related both to each other and to the first escape-time diagram, though it is also true that the three differ in an infinite range of details.

The existence of such fractal images within an infinite-valued semantic analysis of paradoxical sentences seems to offer beauty and an intriguing promise of some deep truths. Nonetheless at present we cannot say precisely what they may have to tell us about truth and paradox.

In the next section, however, we do want to offer a clear illustration of at least one area in which the semantical work above offers a route into important philosophical and metamathematical results regarding chaos theory itself.

III.

In this section we use a strengthening of the Chaotic Liar to introduce a family of limitative results in the tradition of Gödel (1931), Tarski (1935), Church (1936) and Turing (1936) but here regarding chaos theory. Though the results we offer can also be reached by other means,²⁴ the route through the Chaotic Liar seems particularly elegant and appropriate with regard to formal limitations on chaos theory.

1. A MOTIVATING ANTINOMY

We start with a motivating antinomy. The familiar Strengthened Liar, it will be remembered, is as follows:

(20) This sentence is false or neither true nor false ,

and proves useful in showing that the move to truth-value gaps is at best a temporary expedient with regard to Liar-like paradoxes.

The Chaotic Liar, outlined in section II.3 above, introduces the possibility of a sentence with genuinely chaotic semantic behavior: such that its value is dependent on its previously estimated value and the pattern of its iterated values is chaotic on the interval $[0, 1]$.

Here we use the pattern of (20) with the general notion of chaotic semantic behavior in order to introduce what might be considered a strengthened form of the Chaotic Liar:

(21) Either this sentence has chaotic semantic behavior or its actual truth is its estimated falsehood .

Rather than ask whether (21) is true or not, however—the stan-

dard question asked of the Strengthened Liar—we will ask whether (21) has chaotic semantic behavior or not.

Note that the semantic behavior of a sentence that is simply *true* will not qualify as chaotic. If thought of on the model of an iterated function at all, it will simply take the value ‘true’ regardless of previous estimates, and can thus be thought of as giving us merely a constant series of 1s.

If (21) *does* have chaotic semantic behavior, however, it will be simply true in virtue of its first disjunct. By our reasoning above, then, it *won't* be semantically chaotic, and we have derived a contradiction.

If (21) *does not* have chaotic semantic behavior, on the other hand, its truth-value will depend entirely on its second disjunct and will thus parallel the values of the Chaotic Liar:

- (9) The actual truth of this sentence is its estimated falsehood .

The semantic behavior of *that* sentence, however, we know to be chaotic. Thus if (21) is *non*-chaotic, its semantic behavior *will* be chaotic, and we have again derived a contradiction.²⁵

Just as the Gödel results can be seen as employing a Liar-like sentence within formal systems (and Gödel himself alludes to the Richard paradox and the Liar in motivating his (1931)), the results that follow can be seen as employing a formal analogue to this antinomy.

2. SOME COMPLICATIONS

In what follows we will be concerned with formal systems intended to deal with real arithmetic and adequate for number theory. Systems of real arithmetic include, for example, Rogers' system **R** (Rogers (1971)), taken from Montague's formulation in Kalish, Montague, and Mar (1980) and equivalent to Tarski's theory of real closed fields in Tarski (1951). The condition ‘adequate for number theory’ requires merely three additional axioms for ‘is an integer’. Our reason for concentrating on systems of real arithmetic is to illustrate some interesting limitative results regarding chaos theory, and paradigmatically chaotic functions such as $x_{n+1} = 1 - \text{abs}((1-x_n) \cdot x_n)$ are most commonly thought of as functions on the reals—as chaotic on the real interval $[0,1]$, say.

Formal systems of real arithmetic such as those at issue, however—precisely because they are formal systems—contain only denumerably many expressions, and thus cannot for example con-

tain as many numerals as there are reals.²⁶ One difficulty this creates is that the notion of *representation* of a function standard within number theory cannot simply be carried over to real arithmetic without qualification. Within number theory an n -place function f on natural numbers is said to be represented by a formula $A(x_1, \dots, x_n, x_{n+1})$ just in case for any natural numbers p_1, \dots, p_n, j , if $f(p_1, \dots, p_n) = j$, then

$$\vdash \forall x_{n+1} (A(p_1, \dots, p_n, x_{n+1}) \leftrightarrow x_{n+1} = j) \quad ,$$

where p_1, \dots, p_n and j are numerals within the system at issue for p_1, \dots, p_n and j respectively (see Boolos and Jeffrey (1980)). Within formal systems for the reals, on the other hand, there simply won't be numbers p_1, \dots, p_n and j for all real numbers p_1, \dots, p_n and j .²⁷

One way to accommodate this cardinality problem is to follow Tarski (1931). We continue to address functions genuinely on the reals, but use a notion of functions *determined* within formal systems for real arithmetic instead of a notion of functions *represented* within such systems. Tarski outlines 'definable' sets of reals as follows:

A set X is a *definable set* (or a *definable set of order n*) if there is a sentential function (or a sentential function of order n at most) which contains some variable of order 1 as its only free variable, and which satisfies the condition that, for every real number x , $x \in X$ if and only if x satisfies this function. (Tarski (1931), p. 118 of Corcoran)

Using 'determined' in place of Tarski's 'definable' for the sake of clarity, and treating one-place functions on the reals as sets of ordered pairs of reals, we can similarly speak of a function X on the reals as *determined* by a functional expression f^e within a formal system just in case for every pair of reals x , $x \in X$ if and only if x satisfies f^e .

3. THE UNDEFINABILITY OF CHAOS

With this notion of functions on the reals determined within formal systems, and on the pattern of the antinomy above, we can now illustrate a limitative result regarding formal theories of chaos: given any consistent formal system of real arithmetic \mathbf{T} adequate for number theory, the set Γ of gödel numbers of expressions which determine functions $f(x)$ chaotic on the interval $[0,1]$ is undefinable in \mathbf{T} .

Theorem I on the undefinability of chaos. There is no function c representable in \mathbf{T} such that

$$c(\#f(x)) = \begin{cases} 1 & \text{if } \#f(x) \in \Gamma \\ 0 & \text{if } \#f(x) \notin \Gamma \end{cases} .$$

Proof. Suppose, for a proof by contradiction, that such a function c is represented in \mathbf{T} . There will then be a class of expressions which determine a class of functions g such that, for a fixed gödel number $\#f_0(x)$ of an expression determining a one-place function,

$$g(y) = \begin{cases} 1-(y-y) & \text{if } c(\#f_0(x)) = 1 \\ 1-\text{abs}((1-y)-y) & \text{otherwise} \end{cases} .$$

Different numbers $\#f_0(x)$ in such a schema will give us different functions g , of course. If $\#f_0(x)$ is the gödel number of an expression which determines a function which is chaotic on $[0,1]$, assuming c , we will have a $g(y)$ that will simply give us a constant series of 1s for all iterations. If on the other hand $\#f_0(x)$ is the gödel number of an expression which determines a function *not* chaotic on $[0,1]$, assuming c , we will have a $g(y)$ which gives us $1-\text{abs}((1-y)-y)$ as output. Here the particular formula we have chosen is that for the Chaotic Liar, which as noted above is paradigmatically chaotic on the real interval $[0,1]$.

Now on the assumptions above, by the diagonal lemma, there will be an expression which determines a function G where $\#G(x)$ is the gödel number of the expression at issue:

$$G(y) = \begin{cases} 1-(y-y) & \text{if } c(\#G(x)) = 1 \\ 1-\text{abs}((1-y)-y) & \text{otherwise} \end{cases} .^{28}$$

But will $G(y)$ be chaotic on the interval $[0,1]$ or not?

Suppose that it is. In that case, on the assumption of a represented function c and since $\#G(x)$ is the gödel number of an expression which determines $G(x)$, it will be the case that $c(\#G(x)) = 1$. By the specifications of G , then, G will give us a constant output of 1s for any y . Since for every natural number $n > 0$, $G^n(y) = 1$, G will then *not* be chaotic on the interval $[0,1]$, and we have derived a contradiction.

Suppose instead that $G(y)$ is not chaotic on $[0,1]$. Assuming function c represented, $c(\#G(x)) = 0$. By the specification of G , then, G gives us $1-\text{abs}((1-y)-y)$. But G will then be chaotic on the interval $[0,1]$; here again we have derived a contradiction.

Within any consistent system of real arithmetic adequate for number theory, then, there can be no function c represented. It follows that within any such system the set Γ of gödel numbers of

expressions which determine functions $f(x)$ chaotic on the interval $[0,1]$ is undefinable.

As related results it should perhaps be noted that Γ will be nonrecursive and undecidable. Assuming Church's thesis, then, there can be no effective method for deciding whether an arbitrary expression of a system such as \mathbf{T} determines a function chaotic on the interval $[0,1]$.

These formal limitations on chaos theory can also be seen—along with the halting problem and for that matter Gödel's and Tarski's theorems—as special applications of Rice's Theorem in recursion theory.²⁹ With regard to chaos theory in particular, however, we think the route through the Chaotic Liar a particularly intriguing and instructive one.

Submitted June 1990.

NOTES

¹We are grateful to Paul St. Denis for programming invention and assistance throughout the project. We leave a more complete outline of relevant programming techniques to another context.

²Basic notions of chaos theory, also known as the mathematics of dynamical systems, are introduced informally in section II.3. For a formal outline see footnote 12.

³An alternative here of course would be an infinite-valued logic which uses as values, say, merely the rationals rather than the full reals in the unit interval. How much difference that might make seems to us a question worthy of further investigation. For Łukasiewiczian many-valued propositional logics Rescher shows that these alternatives give us essentially equivalent systems; i.e., systems which share the same tautologies (Rescher (1969), 37-38).

⁴In general we use 'sentences', 'statements', and 'propositions' interchangeably, putting aside for present purposes the philosophical controversy as to which should properly be considered the bearers of truth.

⁵J.A. Goguen (1969) proposes an infinite-valued logic as a solution to the sorites paradoxes, and arguments for and applications of infinite-valued 'fuzzy set theory' and 'fuzzy logics' appear in Zadeh (1965) and Zadeh, Fu, Tanaka, and Shimura (1975). Some arguments against philosophical applications of many-valued logics, on the other hand, appear in Urquhart (1986) and Haack (1974).

⁶The \mathbf{Vvp} schema outlined below, it appears, may also be more appropriate to a genuine truth-value interpretation of infinite-valued logics than a probabilistic one. We are obliged to Jordan Howard Sobel for calling our attention to this point.

⁷This last specification, regarding the conditional, distinguishes ours—an infinite-valued Kleene strong system—from a Łukasiewiczian alternative in which

$$/(p \rightarrow q)/ = \begin{cases} 1 & \text{if } /p/ \leq /q/ \\ (1-/p/) + /q/ & \text{if } /p/ > /q/ \end{cases} .$$

The only difference this would make for our work here is in the treatment of the Curry paradox.

⁸Alternatives to Rescher's \mathbf{Vvp} schema are of course possible, and well worthy of investigation. We won't, however, pursue them here.

⁹The behavior of the Liar within this infinite-valued context can in fact be given the same characterization as in the two-valued case above since $1-\text{abs}(0-x_n) = 1-x_n$ for the interval $[0,1]$.

¹⁰Reference within the Minimalist to ‘its estimated value’, of course, serves as an indexical element; given a revised estimate, what (5) effectively asserts changes. Such indexicals also appear in a number of the sentences we consider. In working through sample calculations, we talk about a sentence’s ‘actual’ value in a similar way.

¹¹For $k = 0$, intriguingly enough, we get the Simple Liar.

¹²There are many outlines for chaos in the literature, stronger and weaker, ranging from measure theoretic notions of randomness in ergodic theory to topological characterizations such as that immediately below. For definiteness, ‘chaotic’ here and throughout can be understood as follows, using a slight modification of Devaney (1989):

Let J be a set. $f: J \rightarrow J$ is said to be chaotic on J if

- (a) f has sensitive dependence on initial conditions,
- (b) f is topologically transitive, and
- (c) the set of periodic points is dense in J .

Here we use the notation $f^n(x)$ to stand for the composition or iteration of the function $f(x)$ n times, i.e.

$$f^n(x) = \underbrace{f \circ \dots \circ f}_{n \text{ times}}(x)$$

(a) $f: J \rightarrow J$ has sensitive dependence on initial conditions if there exist points arbitrarily close to any $x \in J$ which eventually separate from x by at least δ under iteration of f , i.e. there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x , there exists $y \in N$ and $n \geq 0$ such that $\text{abs}(f^n(x) - f^n(y)) > \delta$.

(b) $f: J \rightarrow J$ is said to be topologically transitive if it has points which eventually move under iteration from one arbitrarily small neighborhood to any other, i.e. for any pair of open sets $U, V \subseteq J$ there exists $k > 0$ such that $f^k(U) \cap V$ is non-empty.

(c) The set of periodic points of J , $\text{PER}(J)$, is the set of all $x \in J$ such that $f^n(x) = x$ for some natural number n , i.e. $\text{PER}(J) = \{x \in J : \exists n f^n(x) = x\}$. $\text{PER}(J)$ is dense in J if $\text{PER}(J)$ together with all its limit points is equal to J , i.e. $J = \overline{\text{PER}(J)}$.

The proof offered for Theorem I below, it should be noted, doesn’t demand any features of this definition in particular.

¹³In a mathematically more familiar guise the function for the iterated values of the Chaotic Liar may be expressed as a tent function

$$x_{n+1} = \begin{cases} 2x_n & \text{for } 0 \leq x < .5 \\ 2(1-x_n) & \text{for } .5 \leq x \leq 1 \end{cases}$$

Though well known as a particularly simple chaotic function, however, this function was not known to have the kind of natural semantic interpretation offered for it here. It is relegated to the class of mere “mathematical curiosities”, for example, in Robert May’s classic (1976).

One peculiarity of this particular function is that standard rounding off within the binary arithmetic of computers in fact disguises its chaoticity: although it is provably chaotic on the interval $[0,1]$, it doesn’t show up as such on the computer screen. In order to graph something closer to its true behavior it is thus standard to ‘cancel out’ the effect of the rounding-off by introducing a small element of randomness. Here we are obliged to John Milnor for discussion.

¹⁴The affinity of this function to the Heterological paradox is perhaps more evident when expressed in the alternative form of footnote 13.

¹⁵Curry (1941), (1942). The paradox is also known as the Kleene-Rosser antinomy (see however Church (1942)) and is called Löb’s paradox in Barwise and Etchemendy (1987).

¹⁶As indicated above (footnote 3), the use of such a conditional distinguishes ours as an infinite-valued Kleene strong system as opposed to a Łukasiewiczian alternative. We leave a treatment of the Curry using this alternative to another context.

¹⁷See for example Buridan (1489), pp. 200-201 of Scott.

¹⁸Liar cycles are nicely outlined in Herzberger (1982) and showcased in Barwise and Etchemendy (1987).

Barwise and Etchemendy characterize Liar cycles in general as combining features of the Liar and Truth-Teller (p. 22). This however is clearly inessential: Liar cycles can be produced without any shadow of the Truth-Teller using any odd number of sentences all of which except for the last say ‘the next sentence in the series is false,’ and the last of which says ‘the first sentence in the series is false’. Here we are obliged to Stephen Bae.

¹⁹Another alternative is to represent the successive values of the two sentences at issue as:

$$\begin{aligned} x_{n+1} &= 1 - \text{abs}(1 - y_n) \\ y_{n+1} &= 1 - \text{abs}(0 - x_n) \end{aligned}$$

This would reflect a procedure which started with simultaneous estimates for the two sentences, calculated a new estimate for each independently in terms of the last estimate for the other, and so forth. This second form of reasoning about the Dualist has an important role to play with regard to fractal images discussed below.

²⁰This graph should however be viewed with a measure of skepticism with regard to the reality of some calculated points. Because of the computer limitations indicated in footnote 13, for example, (.4, .6), though truly periodic, appears when graphed to converge to (0,1).

²¹See David Ruelle and Floris Takens (1971). As Ruelle remarks in Ruelle (1980), however, “the mathematical theory of strange attractors is difficult and, in part, still in its infancy.” Our function here, of course, is not continuous.

²²The idea of adapting escape-time diagrams to the present case is due to Paul St. Denis.

²³Although fractal geometry was given its name by Benoit Mandelbrot (1977), the subject has a long and interesting mathematical history extending back to Peano, Cantor, and Hausdorff. See Devaney (1990), p. 129.

²⁴They can, for example, be seen as applications of Rice’s Theorem in recursion theory. See Rice (1953) and Rogers (1967), 34.

²⁵Within the reasoning of this antinomy, it should perhaps be noted, we have assumed that only the two classical values will be possible for ‘has chaotic semantic behavior’: that (21) either will or will not be chaotic, and thus that its first disjunct will have a value of either 1 or 0. At least for the present, then, we are not attempting to apply an infinite-valued logic to degrees of chaoticity itself.

²⁶It is also because such systems contain only denumerably many expressions, of course, that they will be gödel numberable using only the natural numbers.

²⁷We are grateful to Robert F. Barnes for helpful correspondence on this point.

A related issue regarding extension of the notions of ‘recursive’ and ‘recursively enumerable’ to sets of complex or real numbers is addressed in Penrose (1989), 124-129.

²⁸The precise form of the diagonal lemma required is that for a formula $\mathbf{B}(x,y)$ of our language there will be a formula $\mathbf{G}(y)$ such that

$$\vdash_{\mathbf{T}} \mathbf{G}(y) \leftrightarrow \mathbf{B}(\#\mathbf{G}(y), y)$$

Here Boolos and Jeffrey’s proof for the diagonal lemma can simply be extended, given appropriate restrictions, to formulae with one additional variable. See Boolos and Jeffrey (1980), 172-173. The generalized diagonal lemma also appears in Boolos (1979), 49-50.

²⁹See Rice (1953) and Rogers (1967), 34.

REFERENCES

- Barwise, Jon, and John Etchemendy
 1987 *The Liar*, New York: Oxford University Press.
- Boolos, George S.
 1979 *The Unprovability of Consistency*, New York: Cambridge University Press.
- Boolos, George S., and Richard C. Jeffrey
 1980 *Computability and Logic*, second edition, Cambridge University Press.
- Buridan, John
 1489 *John Buridan: Sophisms on Meaning and Truth*, trans. T. K. Scott, New York: Appleton-Century-Crofts, 1966, and *John Buridan on Self-Reference: Chapter Eight of Buridan’s Sophismata*, trans. and with a commentary by G. E. Hughes, Cambridge: Cambridge University Press, 1982.

- Church, Alonzo
 1936 "A note on the entscheidungsproblem," *Journal of Symbolic Logic* 1, 40-41, 101-102.
 1942 "Review of H. B. Curry's 'The Inconsistency of Certain Formal Logics,'" *Journal of Symbolic Logic* 7, 170-171.
- Curry, Haskell B.
 1941 "The Paradox of Kleene and Rosser," *Transactions of the American Mathematical Society* 50, 454-516.
 1942 "The Inconsistency of Certain Formal Logics," *Journal of Symbolic Logic* 7, 115-117.
- Devaney, Robert L.
 1989 *An Introduction to Chaotic Dynamical Systems*, second edition, Menlo Park, Ca.: Addison-Wesley.
 1990 *Chaos, Fractals, and Dynamics*, Menlo Park, Ca.: Addison-Wesley.
- Gödel, Kurt
 1931 "Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I," *Monatshefte für Mathematik und Physik* 38, 173-198, tr. as "On formally undecidable propositions of *Principia Mathematica* and related systems I" by Jean van Heijenoort in *Kurt Gödel: collected works, vol. I*, Solomon Feferman, et. al. (eds.), New York: Oxford University Press, 1986, 144-195.
- Goguen, J.A.
 1969 "The Logic of Inexact Concepts," *Synthese* 19, 325-373.
- Grelling, Kurt, and Leonard Nelson
 1908 "Bemerkungen zu den Paradoxien von Russell und Burali-Forti," *Abhandlungen der Preussischen Schule* 2 (new series), 301-324.
- Gupta, Anil
 1982 "Truth and Paradox," *Journal of Philosophical Logic* 11, 1-60, reprinted in Robert L. Martin (ed.) *Recent Essays on Truth and the Liar Paradox*, New York: Oxford University Press, 175-235.
- Haack, Susan
 1974 *Deviant Logic*, New York: Cambridge University Press.
- Herzberger, Hans
 1982 "Notes on naive semantics," *Journal of Philosophical Logic* 11, 61-102, reprinted in Robert L. Martin (ed.) *Recent Essays on Truth and the Liar Paradox*, New York: Oxford University Press, 1984, 133-174.
- Kalish, Donald, Richard Montague, and Gary Mar
 1980 *Logic: Techniques of Formal Reasoning*, second edition, New York: Harcourt Brace Jovanovich.
- Kripke, Saul
 1975 "Outline of a Theory of Truth," *Journal of Philosophy* 72, 690-716. Reprinted in Robert L. Martin (ed.) *Recent Essays on Truth and the Liar Paradox*, New York: Oxford University Press, 1984, 53-81.
- Li, Tien-Yien and James Yorke
 1975 "Period Three Implies Chaos," *American Mathematical Monthly* 82, 985-992.
- Mandelbrot, Benoit
 1977 *The Fractal Geometry of Nature*, New York: Freeman Press.
- May, Robert
 1976 "Simple mathematical models with very complicated dynamics," *Nature* 261, 459-467.
- Penrose, Roger
 1989 *The Emperor's New Mind*, Oxford: Oxford University Press.
- Quine, W. V. O.
 1962 "Paradox," *Scientific American* 206, reprinted as "The Ways of Paradox" in *The Ways of Paradox and Other Essays*, Cambridge Mass.: Harvard University Press, 1966, 1-18.

- Rescher, Nicholas
1969 *Multi-valued Logic*, McGraw-Hill.
- Rice, H. Gordon
1953 "Classes of recursively enumerable sets and their decision problems," *Transactions of the American Mathematical Society* 95, 341-360.
- Rogers, Hartley, Jr.
1967 *Theory of Recursive Functions and Effective Computability*, New York: McGraw-Hill.
- Rogers, Robert
1971 *Mathematical Logic and Formalized Theories*, Amsterdam: North-Holland.
- Ruelle, David
1980 "Strange Attractors," *Mathematical Intelligencer* 2, 126-137.
- Ruelle, David and Floris Takens
1971 "On the Nature of Turbulence," *Communications in Mathematical Physics* 20, 167-192.
- Russell, Bertrand
1903 *The Principles of Mathematics*, New York: W. W. Norton, 1964.
- Tarski, Alfred
1931 "Sur les ensembles définissables des nombres réels. I.," *Fundamenta Mathematicae* 17, 210-239, tr. as "On Definable Sets of Real Numbers" by J. H. Woodger in John Corcoran (ed.), *Logic, Semantics, Meta-mathematics*, second edition, Indianapolis: Hackett Pub. Co., 1983, 110-151.
1935 "Der wahrheitsbegriff in den formalisierten sprachen," *Studien Philosophica* I, 261-405, tr. as "The Concept of Truth in Formalized Languages" by J. H. Woodger in John Corcoran (ed.), *Logic, Semantics, Meta-mathematics*, second edition, Indianapolis: Hackett Pub. Co., 1983, 152-278.
1951 *A Decision Method for Elementary Algebra and Geometry*, second edition, Berkeley and Los Angeles.
- Turing, Alan
1936 "On computable numbers, with an application to the entscheidungsproblem," reprinted in Martin Davis (ed.) *The Undecidable*, Hewlett, New York: Raven Press, 1965, 116-303.
- Urquhart, Alasdair
1986 "Many-valued Logic," in D. Gabbay and F. Guentner (eds.) *Handbook of Philosophical Logic*, vol. III, Dordrecht: D. Reidel, 71-116.
- Zadeh, L.
1965 "Fuzzy sets," *Information and Control* 8, 338-353.
- Zadeh, L., K.-S. Fu, K. Tanaka, and M. Shimura
1975 *Fuzzy Sets and their Application to Cognitive and Decison Processes*, New York: Academic Press.