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Tools, Objects, and Chimeras: Connes on the Role of Hyperreals in Mathematics

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Abstract We examine some of Connes' criticisms of Robinson's infinitesimals starting in 1995. Connes sought to exploit the Solovay model $\mathcal{S}$ as ammunition against non-standard analysis, but the model tends to boomerang, undercutting Connes' own earlier work in functional analysis. Connes described the hyperreals as both a "virtual theory" and a "chimera", yet acknowledged that his argument relies on the transfer principle. We analyze Connes' "dart-throwing" thought experiment, but reach an opposite conclusion. In $\mathcal{S}$, all definable sets of reals are Lebesgue measurable, suggesting that Connes views a theory as being "virtual" if it is not *definable* in a suitable model of ZFC. If so, Connes' claim that a theory of the hyperreals is "virtual" is refuted by the existence of a definable model of the hyperreal field due to Kanovei and Shelah. Free ultrafilters aren't definable, yet Connes exploited such ultrafilters both in his own earlier work on the classification of factors in the 1970s and 80s, and in *Noncommutative Geometry*, raising the question whether the latter may not be vulnerable to Connes' criticism of virtuality. We analyze the philosophical underpinnings of Connes' argument based on Gödel's incompleteness theorem, and detect an apparent circularity in Connes' logic. We document the reliance on non-constructive foundational material, and specifically on the Dixmier trace $⨍$ (featured on the front cover of Connes' *magnum opus*)

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and the Hahn–Banach theorem, in Connes' own framework. We also note an inaccuracy in Machover's critique of infinitesimal-based pedagogy.

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Contents

1 Infinitesimals from Robinson to Connes via Choquet

A theory of infinitesimals claiming to vindicate Leibniz's calculus was developed by Abraham Robinson in the 1960s (see Robinson 1966). In France, Robinson's lead was followed by G. Reeb, G. Choquet, [1] and others. Alain Connes started his work under Choquet's leadership, and published two texts on the hyperreals and ultrapowers (Connes 1969/70, 1970).

In 1976, Connes used ultraproducts (exploiting in particular free ultrafilters on $\mathbb{N}$) in an essential manner in his work on the classification of factors (Connes 1976). (See Remark 8.1 for Connes' use of ultrafilters in *Noncommutative geometry*.)

During the 1970s, Connes reportedly discovered that Robinson's infinitesimals were not suitable for Connes' framework. A quarter of a century later, in 1995, Connes unveiled an alternative theory of infinitesimals (Connes 1995). Connes' presentation of his theory is usually not accompanied by acknowledgment of an intellectual debt to Robinson. Instead, it is frequently accompanied by criticism of Robinson's framework, exploiting epithets that range from "inadequate" to "end of the rope for being 'explicit' " (see Table 1 in Sect. 3). We will examine some of Connes' criticisms, which tend to be at tension with Connes' earlier work. A related challenge to the hyperreal approach was analyzed by Herzberg (2007). Another challenge by E. Bishop was analyzed by Katz and Katz (2011b, 2012b). For a related analysis see Katz and Leichtnam (2013).

In Sect. 2, we examine the philosophical underpinnings of Connes' position. In Sect. 3, we analyze the Connes character and its relation to ultrafilters, and present a chronology of Connes' criticisms of NSA. In Sect. 4, we examine some meta-mathematical implications of the definable model of the hyperreal field constructed by Kanovei and Shelah. Machover's critique is analyzed in Sect. 5. The power of the Łoś-Robinson transfer principle is sized up in Sect. 6. The foundational status of the Dixmier trace and its role in noncommutative geometry are analyzed in Sect. 7.

2 Tools and Objects

Connes' variety of Platonism can be characterized more specifically as a *prescriptive* Platonism, whereby one not merely postulates the existence of abstract objects, but proceeds to assign "hierarchical levels" (see Connes et al. 2001, p. 31) of realness to them, and to issue value judgments based on the latter. Thus, non-standard numbers and Jordan algebras get flunking scores (see Sect. 8.3). Connes mentions such "hierarchical levels" in the context of a dichotomy between "tool" and "object". In Connes' view, only *objects* enjoy a full Platonic existence, while *tools* (such as ultrafilters and non-standard numbers) serve merely the purpose of investigating the properties of the objects.

As a general methodological comment, we note the following. There is indisputably a kind of aprioriness about the natural numbers and other concepts in mathematics, that is not accounted for by a "formalist" view of mathematics as a game of pushing symbols around. Such aprioriness requires explanation. However, Platonism and Formalism are not the only games in town, which is a point we will return to at the end of the section.

To take a historical perspective on this issue, Leibniz sometimes described infinitesimals as "useful fictions", similar to imaginary numbers (see Katz and Sherry 2012a,b for more details). Leibniz's take on infinitesimals was a big novelty at the time and in fact displeased

[1] See e.g., Choquet's work on ultrafilters (Choquet 1968). Choquet's constructions were employed and extended by Mokobodzki (1967/68).

his disciples Bernoulli, l'Hôpital, and Varignon. But Leibniz, while clearly rejecting what would be later called a platonist view, certainly did not think of mathematics as a meaningless game of symbols. One can criticize certain forms of Platonism while adhering to the proposition that mathematics has meaning.

2.1 Tool/Object Dichotomy

Connes' approach to the tool/object dichotomy is problematic, first and foremost, because it does not do justice to the real history of mathematics. Mathematical concepts may start their career as mere tools or instruments for manipulating concepts already given or accepted as full-fledged objects, but later they (the tools) may themselves become recognized as full-fledged objects. Historical examples of such processes abound. The ancient Greeks did not think of the rationals as numbers, but rather as relations among natural numbers (see e.g., Błaszczyk et al. 2012, Section 2.1). Wallis and others in the 17th century were struggling with the ontological expansion involved in incorporating irrational (transcendental) numbers beyond the algebraic ones in the number system. Ideal points and ideal lines at infinity in projective geometry had to face an uphill battle before joining the ranks of *objects* that can be mentioned in ontologically polite company (see e.g., Wilson 1992). G. Cantor's cardinals started as indices and notational subscripts for sets, and only gradually came to be thought of as objects in their own right. Certain well-established objects still bear the name *imaginary* because they were once characterized as not possessing the same reality as genuine objects. Hersh (1997, p. 74) describes some striking cases, including Fourier analysis, of a historical evolution of tools into objects.

The distinction between "tools" and "real objects" is not only blurred by the ongoing conceptual evolution of mathematics. It is also *relative* to the perspective one takes. For instance, set-theoretic topology considers points as the basic building blocks of its objects, to wit, topological spaces. From this perspective, nothing is a more robust and solid object than a point. On the other hand, from the perspective of "point-free" (lattice-theoretical) topology, the points of set-theoretic topology appear as highly "chimerical" entities the existence of which can only be ensured by relying on the axiom of choice or some similar lofty principle (cf. Gierz et al. 2003). More precisely, the situation can be described as follows. The basic objects of point-free topology are complete Heyting algebras (locales) which correspond to the Heyting algebras of open sets of topological spaces. The prime elements of these algebras may be considered as their "points". The existence of sufficiently many points can only be secured by relying on the Hausdorff maximality principle. Under some mild assumptions on the Heyting algebras and the topological spaces involved, one can show that there is a 1-1 correspondence between set-theoretical points of spaces and constructed points of the corresponding Heyting algebras (cf. ibid., Proposition V-5.20, p. 423).

2.2 The Results of Solovay and Shelah

The perspectival relativity of the tool/object distinction and the mutual dependence between its components do not pose a problem for an account that recognizes both tools and objects as *complementary* components of mathematics (that would perhaps make both of them "primordial" in Connes' terminology; see Sect. 2.3).

This may be elaborated as follows. As in any other realm of knowledge, also in mathematics, object and tool of knowledge are connected through the activity of mathematical research and application: the one does not make sense without the other. The dynamics of knowledge requires that both components are not only related, but also opposed to each other. Objects are, as the etymological roots of this word reveal, “resistances” or “obstacles” for knowledge (similarly for the Greek *problema* and the German *Gegenstand*). Tools should therefore not be disparaged as mere subjective “chimeras” but should be conceived of, together with objects, as constitutive ingredients of the evolution of mathematical knowledge (cf. Otte 1994, ch. X).[2]

But for Connes such an “ecumenical” option is not available. This leads him into difficulties. On the one hand, he relies upon the Solovay model where all sets of real numbers are Lebesgue measurable (see Sect. 4.1), so as to relegate non-standard numbers to the chimerical realm of mere *tools*:

> tout réel non standard détermine canoniquement un sous-ensemble non Lebesgue mesurable de l’intervalle [0, 1] de sorte qu’il est impossible [Ste] d’en exhiber un seul (Connes 1997, p. 211).

Here the reference “[Ste]” cited by Connes is an article by Stern (1985). The main subject of Stern’s article is a result of Shelah (1984). Shelah proved that the assumption of the consistency of the proposition that all sets of real numbers are Lebesgue measurable implies the consistency of inaccessible cardinals. Connes’ citation of Stern indicates that Connes was aware of Shelah’s 1984 result.

On the other hand, Connes ignores the fact that for the consistency of the proposition that all sets of real numbers are Lebesgue measurable, Solovay (see Theorem 4.1) had to assume the existence of inaccessible cardinals, and S. Shelah showed that one cannot remove the hypothesis of inaccessible cardinal from Solovay’s theorem. Meanwhile, Connes’ meta-mathematical speculations, such as the claim that “noone will ever be able to name, etc.” (see Sect. 3.1) rely on Solovay’s theorem. Therefore ultimately Connes’ meta-mathematical speculations rely on inaccessible cardinals, as well. The linchpin that keeps Connesian Platonism from unraveling turns out to be an inaccessible cardinal, yet another chimera.

What kind of evidence does Connes present in favor of his approach? It is of two kinds:

(1) Gödel’s incompleteness theorem and Goodstein’s theorem;
(2) feelings of eternity.

We will examine these respectively in Sects. 2.3 and 2.4.

2.3 The Incompleteness Theorem: Evidence for Platonism?

There is an instance of apparent circular reasoning in one of Connes’ arguments in favor of his philosophical approach in the *La Recherche* interview (Connes 2000d).[3] More specifically, Connes claims that Gödel’s incompleteness theorem furnishes evidence in favor of Connes’ philosophical approach, in that it asserts the existence of “true” propositions about natural numbers that cannot be proved:

> Or le théorème de Gödel est bien plus méchant que cela. Il dit qu’il y a aura toujours une proposition vraie qui ne sera pas démontrable dans le système. Ce qui est beaucoup plus dérangeant (Connes 2000d).

[2] In a related vein, Marquis (1997, 2006) pointed out the ever-growing importance of complex conceptual tools for modern mathematics by characterizing generalized (co)homology theories like K-theories as a kind of knowledge-producing “machines”. Probably most mathematicians would agree in that these machines had so many useful applications that it seems a bit unfair to describe them as mere chimeras.

[3] The discussion in this subsection was inspired by I. Hacking’s *The Mathematical Animal* (Hacking 2013, chapter 5).

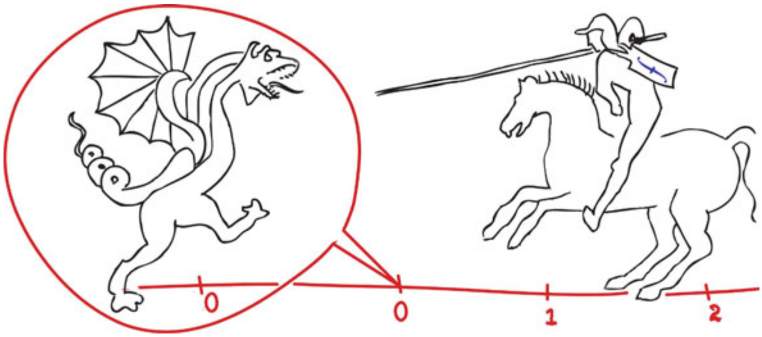

Fig. 1 A virtual view of primordial mathematical reality: An attempted slaying of a hyperreal chimera, following P. Uccello

Such "true" propositions, undecidable in Peano Arithmetic (PA), are taken by Connes to furnish evidence in favor of the hypothesis of a mind-independent (Platonic) primordial mathematical reality (PMR), referred to as *réalité mathématique archaïque* in the interview.[4]

However, the "truth" of such propositions refers to truth relative to an *intended interpretation* of natural numbers, such as the one built in Zermelo–Fraenkel set theory (ZF) or a fragment ZF_0 thereof. Relative to such an interpretation, the said propositions are "true" but not provable in PA. At variance with Connes, K. Kunen presents Gödel's theorem (in the context of ZF) in a philosophically neutral way as follows:

> if T is any consistent set of axioms extending ZF, [5] then [the set] $\{\varphi : T \vdash \varphi\}$ is not recursive … A consequence of this is Gödel's First Incompleteness Theorem—namely, that if such a T is recursive, then it is incomplete in the sense that there is a sentence φ such that $T \nvdash \varphi$ and $T \nvdash \neg\varphi$ (Kunen 1980, p. 38).[6]

With regard to Platonism, Kunen specifically mentions that Gödel's theorem, as well as the closely related Tarski's theorem on non-definability of truth, admit of platonist *interpretations* (rather than furnishing *evidence* in favor of Platonism):

> The platonistic interpretation of [Tarski's theorem] is that no formula $\chi(x)$ can say "x is a true sentence"[7] (Kunen 1980, p. 41).

While Connes' argument appears to rely on an unspoken hypothesis of an imbedding of such a fragment ZF_0 in his PMR, he is certainly free to believe in the hypothesis of such an imbedding

$$ZF_0 \hookrightarrow \text{PMR}. \tag{2.1}$$

[4] An attempt to illustrate this concept graphically may be found in Fig. 1, and further discussion in Sect. 8.2.

[5] Actually it is sufficient to assume that T is consistent and contains a suitable small set of axioms governing addition and multiplication of natural numbers.

[6] The string $T \vdash \varphi$ denotes the statement "sentence φ is provable in theory T", while the string $T \nvdash \varphi$ denotes the statement "φ is *not* provable in T".

[7] In some rare cases, it is possible to document a kind of "model-theoretic failure" of the Tarski truth undefinability theorem. Thus Kanovei (1980) (and independently L. Harrington, unpublished) showed that in a suitable model of ZFC, the set of all analytically definable reals is defined analytically; namely, it is equal to the set of Gödel-constructible reals.

Our goal here is to argue neither in favor nor against Connes' hypothesis (2.1), but rather to point out an apparent circularity inherent in Connes' argument. Connes seeks to argue in favor of Platonism based on Gödel's result, but an unspoken hypothesis of his argument is… Platonism itself, about some fragment ZF_0 properly containing PA, betraying an apparent circularity in his logic.

When Postel-Vinay (the *La Recherche* interviewer) pressed Connes for examples of statements that are "true" but not provable, Connes fell back on what he called "La fable du lièvre et de la tortue" ("the hare and the turtle" phenomenon). What Connes describes here is in fact Goodstein's theorem (Goodstein 1944). As its name suggests, this "true" theorem does admit of a proof, namely Goodstein's. The proof takes place not in PA but rather in a fragment assuming ϵ_0-transfinite induction. Relative to such a widely accepted infinitary hypothesis, Goodstein's theorem is provable and therefore true.

Davis (2006) argued that Π^0_1 sentences such as Cons (PA) are equivalent to checking specific Diophantine problems and therefore their truth value should be determinate, and described such a viewpoint as *pragmatic Platonism* (Davis 2012a). Meanwhile, Connes is characteristically evasive as to the scope of his platonist beliefs, but his categorical tone suggests a rather broad Platonism. What is clear, at any rate, is that his Platonism transcends the Π^0_1 class of the arithmetic hierarchy (since Goodstein's theorem falls outside that class) and is probably much broader. In terms of Shapiro's distinction between *realism in ontology* and *realism in truth-value* (Shapiro 1997, p. 37), Davis may be described as a truth-value realist while Connes, an ontological one.

2.4 Premonitions of Eternity

Connes' additional argument invokes "a feeling of eternity" in connection with his PMR:

> La différence essentielle … c'est qu'elle échappe à toute forme de localisation dans l'espace ou dans le temps. Si bien que lorsqu'on en dévoile ne serait-ce qu'une infime partie, on éprouve un *sentiment d'éternité*. Tous les mathématiciens le savent (Connes 2000d) [emphasis added–the authors].

Taking such a *"sentiment d'eternité"* as the ultimate litmus test for one's reflection on what mathematics is and what mathematicians do is a powerful means of effectively cutting off any further reflection on the nature and the aim of mathematics and its role in the context of culture and society at large. After all, my "sentiments" may be different from yours, and there is no room for rational argumentation. To take this road, one must invoke other means of deciding which sentiments are justified and which are not, such as appeals to the *great mathematicians*: *their* "sentiments" are taken to need no justification at all, as they are the only ones taken to have a legitimate say on what mathematics in its essence really is (see, however, Sect. 2.6 for the anti-Platonist sentiments of M. Atiyah).

However, relying on "sentiments" when dealing with ontological issues concerning mathematics not only has damaging effects on the discourse about mathematics in general. It also affects rather concrete issues concerning the history of mathematics. Arguably, a brand of prescriptive Platonism about the real number continuum may, in fact, be at the root of historical misconceptions concerning key figures and pivotal mathematical developments. Thus, consider the issue of Fermat's technique of *adequality* (stemming from Diophantus's $\pi\alpha\rho\iota\sigma\acute{o}\tau\eta\varsigma$) for solving problems of tangents and maxima and minima. Fermat's technique involves an aspect of approximation and "smallness" in an essential way, as shown by its applications to transcendental curves and variational problems such as Snell's law (see Cifoletti 1990; Katz et al. 2013). This aspect of Fermat's technique is, however, oddly denied by such

Fermat scholars as Breger (1994) and Barner (2011). Similarly, the non-Archimedean nature of Leibniz's infinitesimals is routinely denied by some modern scholars (see Ishiguro 1990; Levey 2008), inspite of ample evidence is Leibniz's writings (see Jesseph 2012; Katz and Sherry 2012a,b). A close textual analysis of Cauchy's foundational writings reveals the existence of a Cauchy–Weierstrass *dis*continuity rather than continuity, *pace* Grabiner (1981) (see Błaszczyk et al. 2012; Borovik and Katz 2012; Bråting 2007; Katz and Katz 2012b, 2011a; Sinaceur 1973).

2.5 Cantor's Dichotomy

Cantor may be said to have opened Pandora's box of the "chimeras" of modern mathematics. It appears that Cantor had a more elaborate and flexible concept of mathematical reality than does Connes. In his *Foundations of a general theory of manifolds* (Cantor 1932), Cantor pointed out that we may speak in two distinct ways of the reality or existence of mathematical concepts.

First, we may consider mathematical concepts as *real* insofar as they, due to their definitions, occupy a fully determined place in our mind whereby they can be distinguished perfectly from all other components of our thought to which they stand in certain relations. Thereby they are real since they may modify the substance of our mind in certain ways. Cantor called this kind of mathematical reality *intrasubjective* or *immanent reality*.

On the other hand, one may ascribe reality to mathematical concepts insofar as they can be considered as expressions or images of processes and relations of the outside world. Cantor referred to this kind of reality as *transient reality*. Cantor had no doubt that these two kinds of reality eventually came together. Namely, concepts with solely *immanent reality* would, in the course of time, acquire *transient reality*, as well. By this two-tiered concept of the reality or "Wirklichkeit" of mathematical entities Cantor thought to have done justice to the idealist as well as to the realist aspects of mathematics and mathematized sciences.[8]

Our analysis of Connes' approach should not be misunderstood. We do not deny that the distinction between *tool* and *object* is an eminently useful one. The point is that one has to take into account the historical and relative character of this distinction. Exactly this Connes' Platonism does not do. Thereby it is blinded to certain essential features of modern mathematical knowledge. The manifest historical evolution of the domain of mathematical objects and the emergence of new tools, which depend on the *changing* character of the object domain, points to a dynamism of the ontological realm of mathematics to which Connes' vision of a "primordial mathematical reality" (PMR) is directly opposed. Connes' account of mathematical knowledge implies a static ontology. The innate weakness of Connes' vision of PMR is that it ignores the inevitable interaction between tools and objects in science.

Furthermore, such an interaction between tools and objects brings into play the institution of a *subject* that is actively using and creating both tools and objects for its specific purposes that may change over time and historical context. In Connes' account, the subject (that is

[8] Cantor's actions did not always faithfully reflect his professed flexible and tolerant attitude toward immanent "chimeras". As is well-known, he was eagerly hunting down infinitesimals of all kinds as allegedly noxious chimeras to be eliminated. One of his strategies of elimination was the publication of a "proof" of an alleged inconsistency of infinitesimals. Accepting Cantor's analysis on faith, Russell declared infinitesimals to be inconsistent (Russell 1903, p. 345), influencing countless other philosophers and mathematicians. The errors in Cantor's "proof" are analyzed by Ehrlich (2006). It is interesting to note that Cantor's contemporary B. Kerry was apparently unconvinced by either Cantor's feelings of eternity or by his "proof", and tried to put up an argument, but was scornfully rebuffed by Cantor, who condemned Kerry's alleged "deplorable psychologistic blindness" (see Proietti 2008, p. 356) and concluded: "*Dixi et salvavi animam meam*. I think I did my best to dissuade you from your deplorable mistakes" (ibid.).

engaged in "doing" mathematics) fatally resembles the ideal, non-empirical subject of classical philosophy for which finiteness and other empirical limitations of the real empirical subjects were philosophically irrelevant.

Despite his platonist preferences, history as well as subject-with-a-history is surreptiously introduced by Connes himself, however. The talk of *tools* only makes sense if a *subject*, i.e., an agent is presupposed that *employs* these tools for its purposes. Connes' subject is a transmundane and very abstract entity. A more convincing choice of the subject would be a historically situated subject. After all, it can hardly be denied that mathematics as every other scientific discipline has undergone a historical development; *our* mathematics is not the same as Greek mathematics, and it is hardly plausible that the mathematics of the future will be "essentially the same" as present-day mathematics. The line between *tools* and *objects* is moving. A tool may gain the status of an object and, conversely, an object may become a tool in a suitable context.

2.6 Atiyah's Anti-platonist Realism

Not all great contemporary mathematicians share Connes' philosophical position. Thus, Sir Michael Atiyah confided:

> I consider myself as a realist. I think the mathematics we use is derived from the outside world by observation and abstraction. If we didn't live in the outside world and see things, we wouldn't have invented things and thought of things as we do. I think that much of what we do is based on what we see, but then abstracted and simplified, and in that sense they become the ideal things of Plato, but they have an origin in the outside world and that's what brings them close to physics. … You can't separate the human mind from the physical world. And therefore everything we think of, in some sense or other, derives from the physical world (Atiyah 2006, p. 38).

Atiyah's outline of a realistic conception of mathematics is not, of course, without problems. For instance, one may object that we do not spend our life time by merely "seeing the outside world". Rather, we are beings in a material world and have to come to terms with the multifarious challenges that the world poses to us. Hence, rather than describing our contact with the outside world as "seeing", it may be more appropriate to adopt a broader approach that emphasizes the multifaceted totality of the various activities in which cognizing beings like us are engaged. One may object that Atiyah does not elaborate much on the profound issue of what exactly is meant by "deriving mathematics from the outside world" and how this is carried out. We think that such a criticism would be a bit unfair. One may well argue that these issues are not, properly speaking, mathematical issues and therefore are not a primary concern for mathematicians.

2.7 Mac Lane's form and Function

A more elaborate account of how "mathematics is derived from the outside world" can be found in Saunders Mac Lane's *Mathematics, Form and Function* (Mac Lane 1986). This book recorded Mac Lane's

> efforts … to capture in words a description of the form and function of Mathematics, as a background for the Philosophy of Mathematics" (Preface).

Here Mac Lane compiled a list of rather mundane activities such as collecting, counting, comparing, observing, moving and others that can be considered as the modest origins of the high-brow concepts of contemporary mathematics (ibid., p. 35). An interesting elaboration of Mac Lane's account may be found in *Where Mathematics Comes From. How the Embodied Mind Brings Mathematics into Being* (Lakoff and Núñez 2000).

The details of the processes underlying the historical evolution of mathematics may not be fully understood yet. However, cutting off any further discussion on these issues by falling back on "feelings of eternity" does not seem the best way to meet such challenges. Mathematics, as any other intellectual endeavor, cannot be considered as an autonomous domain totally cut off from other areas of knowledge. As Atiyah put it explicitly:

> The idea that there is a pure world of mathematical objects (and perhaps other ideal objects) totally divorced from our experience, which somehow exists by itself is obviously inherent nonsense (Atiyah 2006, p. 38).

A PMR-free perspective on mathematics is gaining momentum. In fact, Connes' feelings of eternity may be misdirected. Scholars from many a discipline converge to a view that thinking about mathematics should not treat the latter as an isolated endeavor, separate from other areas of knowledge.

2.8 Margenau and Dennett: To be or …

Connes' radical Platonism with its postulation of a strict separation of the sphere of mathematics from the rest of the world is, in a sense, radically anti-modern. Modernity in the sciences began with a turn toward epistemological and semantical questions, leaving aside classical ontological questions such as "What is the essence of the world?", "What is the essence of Man?", or, more to the point of the present paper, "What is the essence of number or space?". Instead, in the modern perspective, semantical and epistemological questions such as "What is the meaning of this or that scientific concept in this or that context?", "What is scientific knowledge?", or "Can one make sense of the progress in science?" take centerstage. In this way, ontology, epistemology, and semantics get inextricably intertwined. In particular, ontology became theory-dependent. For the mathematized sciences of nature, the neo-Kantian philosopher Ernst Cassirer expressed this observation explicitly as follows:

> [Scientific] concepts are valid not in that they copy a fixed, given being, but insofar as they contain a plan for possible constructions of unity, which must be progressively verified in practice … (Cassirer 1957, p. 476).

What we need is not the objectivity of absolute concepts (it seems difficult to give convincing arguments to account for how one could have cognitive access to such concepts), but rather objective methods which determine the rational and reliable practice of our intersubjective empirical science. As Cassirer put it,

> What we need is not the objectivity of absolute objects, but rather the objective determinacy of the *method of experience* (ibid.)

Cassirer's characterisation of scientific concepts as applied to mathematical concepts amounts to the contention that mathematical concepts should not be conceived of as intending to copy a pre-existing platonic universe but "contain plans for possible constructions of unity". This characterization would match quite well with Hilbert's dictum "By their fruits ye will known them". If this is true, a "theory of chimeras" à la Connes hardly provides a promising framework for dealing with these problems.

Rather, what is needed is an investigation of the entire spectrum of the various meanings of the concept of *being* as it is used in modern science. The need for such an investigation was pointed out by Cassirer's friend and colleague, the renowned physicist Henry Margenau, by means of the following provocative question:

> Do masses, electrons, atoms, magnetic field strengths etc., exist? Nothing is more surprising indeed than the fact that … most of us still expect an answer to this question in terms of yes or no. … Almost every term that has come under scientific scrutiny has lost its initally absolute significance and acquired a range of meaning of which even the boundaries are often variable. Apparently the word *to be* has escaped this process (Margenau 1935, p. 164).

Margenau argued in favor of a nuanced concept of "the real" based on an elaborate theory of theoretical constructs in which "tools" and "objects" interact in complex ways (cf. Margenau 1935, 1950).

Sixty years later, Margenau's question was taken up and generalized to the object of other sciences by Daniel Dennett:

> Are there really beliefs? Or are we learning (from neuro-science and psychology, presumably) that strictly speaking, beliefs are figments of our imagination, items in a superseded ontology. Philosophers generally regard such ontological questions as admitting just two possible answers: either beliefs exist or they do not. (Dennett 1991, p. 27).

Dennett argued that an ontological account centered around the concept of "patterns" may be helpful to develop an "intermediate" (Dennett's term) position that conceives of beliefs and other questionable abstract entities as patterns of some data. Taking data as a bit stream, a pattern is said to exist in some data, i.e., is real if there is a description of the data that is more efficient than the bit map, whether or not anyone can concoct it. Thereby centers of gravity exist in physicalist ontologies because they are good abstract concepts that perform some useful work. Meanwhile, bogus concepts such as "Dennett's lost socks center" (defined as "the center of the smallest sphere that can be circumscribed around all the socks Dennett ever lost in his life") do not obtain this status but remain meaningless "chimeras" (ibid., 28).

In a somewhat analogous way, Michael Resnik and other philosophers of mathematics are working on a project of describing "mathematics as a science of patterns", in which Resnik defends the thesis that mathematical structures obtain their reality as "patterns of reality" (Resnik 1994).

This section is not the place to engage in an in-depth study of these and similar attempts to clarify the murky issue of the ontology and epistemology of mathematics. Our goal is merely to evoke some possibly fruitful directions of inquiry that may help overcome the limitations of the traditional accounts of formalism, intuitionism, and platonism. In the long run it is unsatisfying (to put it mildly) to play off against each other these classical positions over and over again, by manufacturing unappealing and unrealistic strawmen of the other party. Such dated ideas on the nature of mathematics do not exhaust the spectrum of possible approaches to the epistemology and ontology of mathematics.

Connes' views on non-standard analysis are inseparable from his philosophical position, as we discuss in Sect. 3.

3 "Absolutely Major Flaw" and "Irremediable Defect"

Having clarified the philosophical underpinnings of Connes' views in Sect. 2, we now turn to the details of his critique. Connes published his magnum opus *Noncommutative geometry* (an expanded English version of an earlier French text) in 1994. Shortly afterwards, Connes published his first criticism of non-standard analysis (NSA) in 1995, describing the non-standard framework as being "inadequate". In 1997, the adjective was *"décevante"* (see Connes 1997). By 2000, Connes was describing non-standard numbers as "chimeras". Such criticisms have appeared in his books, research articles, interviews, and a blog.

It is instructive to compare two papers Connes wrote around 2000. The paper (Connes 2000c) in *Journal of Mathematical Physics* (JMP) presents Connes' theory of infinitesimals without a trace of any reference to either NSA or the Solovay model. The other text from the same period (see Connes 2000a,b, 2004) presents the – by then – familiar meta-mathematical speculations around the Solovay model (see Sect. 4 for details), and proceeds to criticize NSA. The JMP text demonstrates that Connes is perfectly capable of presenting his approach to infinitesimals (which he claims to be entirely different from Robinson's) without criticizing NSA.

Connes was familiar with the ultrapower construction $\mathbb{R}^{\mathbb{N}}/\mathcal{F}$ of the hyperreals, having authored the 1970 articles (Connes 1969/70, 1970). At least on one occasion, Connes described ultraproducts as "very efficient",[9] which adds another dimension to the puzzle. To understand Connes' position, one may have to examine the historical context of his changing attitude toward non-standard analysis. After Robinson's death in 1974, many voices were heard that were critical of Robinson's theory. Active in this area were Paul Halmos and his student Bishop (1977) (see Katz and Katz 2011b). Some of the criticisms were plain incoherent, such as Earman's (1975) (see Katz and Sherry 2012a, Section 11.2), suggesting that for a time, it was sufficient to criticize Robinson to get published. It may have become difficult starting in the mid-1970s to be a supporter of Robinson, and it would have been natural for young researchers to seek to distance themselves from him. The objection to hyperreal numbers on the part of many mathematicians may be due, consciously or unconsciously, to their attitude that the traditional model of the real numbers in the context of ZF is a true representation of Reality itself[10] and that hyperreal numbers are therefore a contrived model that does not represent anything of interest, even if it provides a solution to some paradox. E. Nelson, however, turned the tables on this attitude, by introducing an enriched syntax into ZF, building the "usual" real line $\mathbb{R}$ in ZF with the enriched syntax, and exhibiting infinitesimals within the real line $\mathbb{R}$ itself (see Nelson 1977). Related systems were elaborated by Hrbáček (1978), Kawai (1983) and Kanovei (1991).

3.1 The Book

The 2001 book Connes et al. (2001) was ostensibly authored by Connes, A. Lichnerowicz, and M. Schützenberger. Lichnerowicz and Schützenberger died several years prior to the book's publication. A reviewer notes:

> The main contributions to the conversations come from Connes [...] and the fact that some of Connes' contributions look relatively polished may indicate that they have been edited to some extent [...] Connes often explains a topic in a more or less systematic way; Schützenberger makes interesting comments, often from a very different

9 See main text at footnote 43.

10 An alternative view is explored in Katz and Katz (2012a).

Table 1 Connes' epithets for NSA arranged chronologically

Date	Epithet	Source
1995	"inadequate"	Connes (1995, p. 6207)
1997	"*décevante*" [disappointing]	Connes (1997, p. 211)
2000	"very bad obstruction"	Connes (2004, p. 20)
2000	"chimera"	Connes (2004, p. 21)
2001	"absolutely major flaw"; "irremediable defect"; "the theory remains virtual"	Connes et al. (2001, p. 16)
2007	"I have found a catch in the theory"; "it seemed utterly doomed to failure to try to use non-standard analysis to do physics"	Connes (2007, p. 26)
2007	"the promised land for 'infinitesimals' "; "the end of the rope for being 'explicit' "	Connes (2007)

angle while introducing many side-subjects, Lichnerowicz interjects skeptical remarks (Dieks 2002).

The book's discussion of NSA in the form of an exchange with Schützenberger appears on pp. 15–21. Here Connes expresses himself as follows on the subject of non-standard analysis:

> A.C. - […] I became aware of an absolutely major flaw in this theory, an irremediable defect. It is this: in nonstandard analysis, one is supposed to manipulate infinitesimals; yet, if such an infinitesimal is given, starting from any given nonstandard number, a subset of the interval *automatically* arises which is not measurable in the sense of Lebesgue.
>
> M.P.S. - Aha!
>
> A.C. - Yes, a nonstandard number yields in a simple *canonical* way, a subset of [0, 1] which is not measurable in the sense of Lebesgue […] What conclusion can one draw about nonstandard analysis? This means that, since noone will ever be able to name a nonstandard number, the theory remains *virtual*, and has absolutely no significance except as a tool to understand "primordial mathematical reality"[11] (Connes et al. 2001, p. 16) [emphasis added–the authors]

Connes goes on[12] to criticize the role of the axiom of choice in non-standard analysis (ibid., p. 17).

Connes' criticisms of non-standard analysis have appeared in numerous venues, and have been repeatedly discussed.[13] Some of the epithets he used for NSA, arranged by year, appear in Table 1.

Some of Connes' criticisms are more specific than others. Thus, the precise meaning of his terms such as "virtual theory" and "primordial mathematical reality" is open to discussion (see Sect. 2). We will focus on the more mathematically identifiable claim of a *canonical* derivation of a Lebesgue nonmeasurable set from a non-standard number, as well as the role of Solovay's models in Connes' criticism.

[11] See Sect. 2.3 for an analysis of the term "primordial mathematical reality".

[12] The continuation of the discussion is dealt with in Sect. 3.7.

[13] See, e.g., http://mathoverflow.net/questions/57072/a-remark-of-connes.

Note that a construction of a nonmeasurable set starting from a hyperinteger was described decades earlier by Luxemburg (1963) and Luxemburg (1973, Theorem 10.2, p. 66), Stroyan and Luxemburg (1976), and Davis (1977, pp. 71–74).[14]

3.2 Skolem's Non-standard Integers

Before going into the *mathematical* details of Connes' criticism of non-standard numbers, we would like to comment on its *historical* scope. Connes' criticism of non-standard integers is worded in such a general fashion that one wonders if it would encompass also the non-standard integers constructed by T. Skolem in the 1930s (see Skolem 1933, 1934; an English version may be found in Skolem 1955). Skolem's accomplishment is generally regarded as a major milestone in the development of 20th century logic.

Scott (1961, p. 245) compares Skolem's predicative approach with the ultrapower approach (Skolem's nonstandard integers are also discussed by Bell and Slomson 1969 and Stillwell 1977, pp. 148–150). Scott notes that Skolem used the ring DF of algebraically (first-order) definable functions from integers to integers. The quotient DF/P of DF by a minimal prime ideal P produces Skolem's non-standard integers. The ideal P corresponds to a prime ideal in the Boolean algebra of idempotents. Note that the idempotents of DF are the characteristic functions of (first-order) definable sets of integers. Such sets give rise to a denumerable Boolean algebra $\mathfrak{P}$ and therefore can be given an *ordered basis*. Such a basis for $\mathfrak{P}$ is a nested sequence[15]

$$X_n \supset X_{n+1} \supset \dots$$

such that $Y \in \mathfrak{P}$ if and only if $Y \supset X_n$ for a suitable n. Choose a sequence (s_n) such that

$$s_n \in X_n \backslash X_{n+1}.$$

Then functions $f, g \in DF$ are in the same equivalence class if and only if

$$(\exists N)(\forall n \geq N)\, f(s_n) = g(s_n).$$

The sequence (s_n) is the *comparing function* used by Skolem to partition the definable functions into congruence classes. Note that, even though Skolem places himself in a context limited to definable functions, a key role in the theory is played by the comparing function which is *not* definable.

3.3 The Connes Character

In Sect. 3.1, we cited Connes to the effect that a nonmeasurable set "automatically" arises, and that a non-standard number "canonically" produces such a set. Challenged to elaborate on his claim, Connes expressed himself as follows:

> Pour exhiber un ensemble non-mesurable a partir d'un entier non-standard n il suffit de prendre le caractère de $G = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ qui est donné par l'evaluation de la composante a_n …On obtient un caractère non continu de G et il est donc non-mesurable (Connes 2009).

Similar remarks appear at Connes' non-standard blog (Connes 2007).

14 Davis (2012b) noted recently that he based his construction on (Luxemburg 1964), by filling in the proof of Theorem 9.1 in Davis (1977, p. 72) and otherwise following Luxemburg.

15 We reversed the inclusions as given in Scott (1961, p. 245) so as to insist on the analogy with a filter.

In more detail, consider the natural numbers $\mathbb{N}$, and form the infinite product $G = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ (when equipped with the product topology, it is homeomorphic to the Cantor set). Each $n \in \mathbb{N}$ gives rise to a homomorphism $\chi_n : G \to \mathbb{Z}/2\mathbb{Z}$ given by evaluation at the nth component. Each element $x \in G$ can be thought of as a map

$$x : \mathbb{N} \to \mathbb{Z}/2\mathbb{Z} = \{e, a\}, \tag{3.1}$$

where e is the additive identity element and a is the multiplicative identity element. Consider the set

$$A_x = x^{-1}(a) \subset \mathbb{N}. \tag{3.2}$$

Then x can be thought of as an "indicator" function of the set A_x. In non-standard analysis, the map x of (3.1) has a natural extension ${}^*x$ whose domain is the ring of hypernatural numbers, ${}^*\mathbb{N}$:

$${}^*x : {}^*\mathbb{N} \to \mathbb{Z}/2\mathbb{Z}. \tag{3.3}$$

Now let $n \in {}^*\mathbb{N}\backslash\mathbb{N}$ be an infinite hypernatural. The evaluation of the map ${}^*x$ of (3.3) at n gives the value ${}^*x(n) \in \mathbb{Z}/2\mathbb{Z}$ of ${}^*x$ at n. This again produces a homomorphism from ${}^*G$ to $\mathbb{Z}/2\mathbb{Z}$. Its restriction to $G \subset {}^*G$ is denoted

$$\chi_n : G \to \mathbb{Z}/2\mathbb{Z}, \quad x \mapsto {}^*x(n). \tag{3.4}$$

Thus, the character[16] χ_n maps G to $\mathbb{Z}/2\mathbb{Z} = \{e, a\}$. Here

$$\chi_n(x) = a \text{ if and only if } n \in {}^*A_x, \tag{3.5}$$

where ${}^*A_x \subset {}^*\mathbb{N}$ is the natural extension of the set $A_x \subset \mathbb{N}$ of (3.2). Connes notes that the character χ_n is nonmeasurable. He describes the passage from n to the character as "canonical", and alleges that non-standard analysis introduces entities that lead "canonically" to nonmeasurable objects.[17]

3.4 From Character to Ultrafilter

The Connes character χ_n carries the same information as an ultrafilter. Indeed, consider the inverse image of $a \in \mathbb{Z}/2\mathbb{Z}$ under the character χ_n of (3.4), namely, $\chi_n^{-1}(a) \subset G$. To each $x \in \chi_n^{-1}(a)$, we can associate the subset $A_x \in \mathcal{P}(\mathbb{N})$ of (3.2).[18] If $n \in {}^*\mathbb{N} \setminus \mathbb{N}$ is a fixed hypernatural, then the collection

$$\left\{A_x \in \mathcal{P}(\mathbb{N}) : \chi_n(x) = a\right\}$$

yields a free ultrafilter on $\mathbb{N}$. By (3.5), Connes' construction can be canonically identified with the following construction.

[16] A *character* is generally understood to have image in $\mathbb{C}$; if one wishes to think of (3.4) as a character, one identifies $\mathbb{Z}/2\mathbb{Z}$ with $\{\pm 1\} \subset \mathbb{C}$.

[17] Another interpretation: $G = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is the standard product which is a compact metrizable group. Each element $x \in G$ has an internal extension ${}^*x$ defined on ${}^*\mathbb{N}$. Thus, if n is a standard or non-standard hypernatural, then ${}^*x$ can be evaluated at n. Now the continuous dual of G, by Pontryagin duality, is the algebraic direct sum of countably many copies of $\mathbb{Z}/2\mathbb{Z}$ with the discrete topology. Thus, the evaluation at a non-standard integer n is not continuous and therefore not measurable, and cannot be equal a.e. to a Borel function.

[18] Here $\mathcal{P}(\mathbb{N})$ denotes the set of subsets of $\mathbb{N}$.

Construction 3.1 *Choose an unlimited hypernatural $n \in {}^*\mathbb{N}$, and construct the ultrafilter $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ consisting of subsets $A \subset \mathbb{N}$ whose natural extension ${}^*A \subset {}^*\mathbb{N}$ contains n:*

$$\mathcal{F} = \left\{A \in \mathcal{P}(\mathbb{N}) : n \in {}^*A\right\}. \tag{3.6}$$

The important remark at this stage is that Connes' construction exploits a new principle of reasoning introduced by Robinson, called the *transfer principle*.[19] The reliance of the construction on the transfer principle was acknowledged[20] by Connes (2012a).

Remark 3.2 If one applies Construction 3.1 to the hypernatural

$$n = [(1, 2, 3, \ldots)], \tag{3.7}$$

i.e., the equivalence class of the sequence listing all the natural numbers, then one recovers precisely the ultrafilter $\mathcal{F}$ used in the ultrapower construction of a hyperreal field as the quotient[21]

$${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{F}. \tag{3.8}$$

3.5 A Forgetful Functor

Connes has repeatedly used the terminology of "canonical" in his publications, as in the claim that "a hyperreal number *canonically* produces" a nonmeasurable entity. To an uninformed reader, this may sound similar to an assertion that "to every rational number one can *canonically* associate a pair of integers" (reduce to lowest terms), or "to every real number one can *canonically* associate a unique Dedekind cut" on $\mathbb{Q}$. Both of these statements are true if the field is given up to isomorphism, with no additional structure.

It is not entirely clear if Connes means to choose an element from a specific model of a hyperreal field, or an element[22] from an isomorphism type of such a model (i.e., its class up to isomorphism). We will therefore examine both possibilities:

(1) element of an isomorphism type of a hyperreal field; or
(2) element of a particular non-standard model.

Briefly, we argue that in the former case Connes' claim is false. Meanwhile, in the latter case, the complaint is moot as we already have an ultrafilter $\mathcal{F}$, namely the one used to build the model as in (3.8). Thus, Connes' "canonical" procedure is canonically equivalent to a black box[23] that canonically returns its input (namely, the original ultrafilter $\mathcal{F}$; see Remark 3.2). More precisely, it is a forgetful functor Φ from the category $\mathcal{E}$ of hyperreal enlargements to the category $\mathcal{U}$ of ultrafilters:

$$\Phi : \mathcal{E} \to \mathcal{U}, \quad \Phi\left(\mathbb{R}; \mathcal{F}; {}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{F}; *\right) = \mathcal{F}. \tag{3.9}$$

[19] The transfer principle for ultraproduct-type nonstandard models follows from Łoś's theorem dating from 1955 (see Łoś 1955).

[20] See Sect. 3.7 at footnote 28 for a further discussion of the role of the transfer principle.

[21] More precisely, we form the quotient of $\mathbb{R}^{\mathbb{N}}$ by the space of real sequences that vanish on members of $\mathcal{F}$. The notational ambiguity is widespread in the literature.

[22] More precisely, the orbit of an element under field automorphisms.

[23] See also main text in Sect. 3.8 at footnote 29.

3.6 P-points and Continuum Hypothesis

We argue that to produce a canonical ultrafilter from a hyperreal, an isomorphism type of ${}^*\mathbb{R}$ does not suffice. To see this, assume for the sake of simplicity the truth of the continuum hypothesis (CH); note that a procedure claimed to be "canonical" should certainly work in the assumption of CH, as well. Now in the traditional Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC) together with the assumption of CH, we have the following theorem (see Erdös et al. 1955).

Theorem 3.3 (Erdös et al.) *In ZFC+CH, all models of* ${}^*\mathbb{R}$ *of the form* $\mathbb{R}^{\mathbb{N}}/\mathcal{F}$ *are isomorphic as ordered fields.*[24]

Meanwhile, the ultrafilter $\mathcal{F}$ may or may not be of a type called a "P-point". The most relevant property of an ultrafilter $\mathcal{F}$ of this type is that every infinitesimal in $\mathbb{R}^{\mathbb{N}}/\mathcal{F}$ is representable by a null sequence, i.e., a sequence tending to zero (see Cutland et al. 1988). Meanwhile, not all ultrafilters are P-points.[25]

Thus, the isomorphism type of ${}^*\mathbb{R}$ does not retain the information as to which ultrafilter was used in the construction thereof. If $\mathcal{F}$ is a P-point, then the hypernatural (3.7) fed into (3.6) will return the P-point ultrafilter $\mathcal{F}$ itself, but also *every* choice of a hyperinteger $n \in {}^*\mathbb{N} \setminus \mathbb{N}$ would yield a P-point (this follows from the properties of the Rudin–Keisler order on the ultrafilters).

If a P-point $\mathcal{F}$ were used in the construction of ${}^*\mathbb{R}$, any imaginable "canonical" construction (such as Connes', exploiting the transfer principle) would have to yield a P-point, as well. But if all one knows is the isomorphism class of ${}^*\mathbb{R}$, the nature of the ultrafilter used in the construction cannot be detected; it may well have been a *non-P*-point ultrafilter. We thus obtain the following:

> *There does not exist a canonical construction of a nonprincipal ultrafilter from an element*[26] *in an isomorphism type of a hyperreal field.*

Such a construction could not exist unless one is working with *additional* data (i.e., in addition to the isomorphism type), such as a specific enlargement $\mathbb{R} \to {}^*\mathbb{R}$ with a transfer principle, where we can apply Construction 3.1. However, the construction of such an enlargement requires an ultrafilter to begin with! This reveals a circularity in Connes' claim.

3.7 Contrasting infinitesimals

We continue our analysis of the discussion between Schützenberger and Connes started above in Sect. 3.1. Connes contrasts *his* infinitesimals with Robinson's infinitesimals in the following terms:

> An infinitesimal [in Connes' theory] is a certain type of operator which I am not going to define. What I want to emphasize is that in the critique of the nonstandard model, the axiom of choice plays an extremely important role that I would like to make explicit. In

[24] In fact, the uniqueness up to isomorphism of this ordered field is equivalent to CH (see Farah and Shelah 2010).

[25] Thus, Rudin (1956) proved the following results assuming CH. Recall that a space is called homogeneous if for any two points, there is a homeomorphism taking one to the other.
Theorem 4.4: $\beta\mathbb{N} - \mathbb{N}$ is not homogeneous; Theorem 4.2: $\beta\mathbb{N} - \mathbb{N}$ has 2^c P-points; Theorem 4.7: for any two P-points of $\beta\mathbb{N} - \mathbb{N}$, there is a homeomorphism of $\beta\mathbb{N} - \mathbb{N}$ that carries one to the other.

[26] See footnote 22.

logic, when one constructs a nonstandard model, for example of the integers, or of the real line, one tacitly uses the axiom of choice. It is applied in an *uncountable* situation (Connes et al. 2001, p. 17) [emphasis added–the authors].

The comment appears to suggest that Robinson's theory relies on uncountable choice but Connes' does not. The validity or otherwise of this suggestion will be discussed below (see end of this Subsection). The discussion continues as folows:

M.P.S. - What you are saying is fantastic. I had never paid attention to the fact that the countable axiom of choice differed from the uncountable one. I must say that I have nothing to do with the axiom of choice in daily life.
A.L. - Of course not! (Connes et al. 2001, p. 17).

What emerges from Schützenberger's comments is that he "never paid attention" to the distinction between the countable case of the axiom of choice and the general case. The continuation of the discussion reveals that Schützenberger is similarly ignorant of the concept of a *well ordering*:

M.P.S. What do you mean by "well ordering"!?

A.C. Well ordering! The integers have the property that …[there follows a page-long introduction to well ordering.]
M.P.S. Amazing!
A.L. [Lichnerowicz] So the countable and uncountable axioms of choice are different.
A.C. Absolutely. It is worth noting that most mainstream mathematics only requires the countable axiom of choice[27] […] (Connes et al. 2001, p. 20–21).

Connes' discussion of the distinction between the countable axiom of choice (AC) and the general AC appears to suggest that one of the shortcomings of non-standard analysis is the reliance on the uncountable axiom of choice.

Such a suggestion is surprising, since Connes' own framework similarly exploits nonprincipal ultrafilters which cannot be obtained with merely the countable AC (see Remark 4.4, Sect. 7, and Remark 8.1). The impression created by the discussion that Connes' theory relies on countable AC alone, is therefore spurious.

3.8 A Virtual Discussion

Shützenberger was not in a position to challenge any of Connes' claims due to ignorance of basic concepts of set theory such as the notion of a well ordering. Had he been more knowledgeable about such subjects, the discussion may have gone rather differently.

M.P.S. - I have the following question concerning the evaluation at a nonstandard integer. Why does this produce a character?

A.C. - The recipe is very simple to get a character from a nonstandard integer:

[27] It is difficult to argue with a contention that "mainstream mathematics only requires the countable axiom of choice", since the term *mainstream mathematics* is sufficiently vague to accomodate a suitable interpretation with respect to which the contention will become accurate. Note, however, that such an interpretation would have to relegate Connes' work in functional analysis on the classification of factors (for which Connes received his Fields medal) to the complement of "most mainstream mathematics", as his work exploited ultrafilters in an essential manner, whereas ZF+DC is not powerful enough to prove the existence of ultrafilters (see Remark 4.2).

(1) View an element x of the compact group $C^{\mathbb{N}}$ as a map $n \to x(n)$ from the integers $\mathbb{N}$ to the group C with two elements ± 1.
(2) Given a non-standard integer n the evaluation ${}^*x(n)$ gives an element of ${}^*C$, but since C is finite one has ${}^*C = C$.
(3) The map $x \to {}^*x(n)$ is a character of the compact group $C^{\mathbb{N}}$ since it is a multiplicative map from $C^{\mathbb{N}}$ to ± 1.
(4) This character cannot be measurable, since otherwise it would be continuous and hence n would be standard.

M.P.S. - I was precisely asking why it is true that, as you mention in step (3), the map $x \to {}^*x(n)$ is a multiplicative map.
A.C. - Just because the product xy of two elements x, y in the group $C^{\mathbb{N}}$ is defined by the equality $(xy)(n) = x(n)y(n)$ for all n, and this equality is first order and holds hence also for non-standard integers.
M.P.S. - Then you are using the transfer principle to conclude that we have an elementary extension?
A.C. - Yes, I am using the transfer principle[28] to get that if $z(n) = x(n)y(n)$ for all n then one has also ${}^*z(n) = {}^*x(n)\,{}^*y(n)$ for all non-standard n.
M.P.S. - Exploiting the transfer principle presupposes a *model* where such a principle applies, such as [for example] the ultrapower one constructed using an ultrafilter, say a selective one. With such a model in the background, seeking to exhibit a character in a canonical fashion would seem to be canonically equivalent to seeking to exhibit an ultrafilter. But why not pick the selective one we started with?[29]
A.C. -

Needless to say, Schützenberger never challenged Connes as above. However, the exchange is not entirely virtual: it reproduces an exchange of emails in june 2012, between Connes and the second-named author.[30] Connes never replied to the last question about ultrafilters [see the discussion of the forgetful functor at (3.9)].

4 Definable Model of Kanovei and Shelah

In 2004, Kanovei and Shelah constructed a definable model of the hyperreals. In this section, we explore some of the meta-mathematical ramifications of their result.

4.1 What's in a Name?

Let us consider in more detail Connes' comment on *naming* a hyperreal:

> What conclusion can one draw about nonstandard analysis? This means that, since noone will ever be able *to name* a nonstandard number, the theory remains *virtual* (Connes et al. 2001, p. 16) [emphasis added–the authors]

The exact meaning of the verb "to name" used by Connes here is not entirely clear. Connes provided a hint as to its meaning in 2000, in the following terms:

28 Connes' acknowledgment of his use of the transfer principle was mentioned in Sect. 3.4 (see footnote 20).

29 The point about choosing the ultrafilter that one started with is related to the metaphor of the black box that canonically returns its input, mentioned in Sect. 3.5 at footnote 23.

30 The email exchange is reproduced here with the consent of Connes (2012a).

> if you are given a non standard number you can canonically produce a subset of the interval which is not Lebesgue measurable. Now we know from logic (from results of Paul Cohen and Solovay) that it will forever be impossible to produce explicitely [sic] a subset of the real numbers, of the interval [0, 1], say, that is not Lebesgue measurable (Connes 2000a, p. 21, 2004, p. 14).

The hint is the name *Solovay* (Robert M. Solovay). Apparently Connes is relying on the following result, which may be found in Solovay (1970, p. 3, Theorem 2).

Theorem 4.1 (Solovay (1970, Theorem 2)) *There is a model $\mathcal{S}$ of set theory ZFC, in which (it is true that) every set of reals definable from a countable sequence of ordinals is Lebesgue measurable.*

4.2 The Solovay and Gödel Models

The model $\mathcal{S}$ mentioned in Theorem 4.1 is referred to as *the Solovay model* by set theorists. The assumption of "definability from a countable sequence of ordinals" includes definability from a real (and hence such types of definable pointsets as Borel and projective sets, among others), since any real can be effectively represented as a countable sequence of ordinals — natural numbers, in this case.

Remark 4.2 The model $\mathcal{S}$ contains a submodel $\mathcal{S}'$ of all sets x that are *hereditarily definable from a countable sequence of ordinals*. This means that x itself, all elements $y \in x$, all elements of elements of x, *etc.*, are definable from a countable sequence of ordinals. This submodel $\mathcal{S}'$ is sometimes called *the second Solovay model*. It turns out that $\mathcal{S}'$ is a model of ZF in which the full axiom of choice AC fails. Instead, *the axiom DC of countable dependent choice*[31] holds in $\mathcal{S}'$, so that $\mathcal{S}'$ is a model of ZF + DC.

The following is an immediate consequence of Theorem 4.1.

Corollary 4.3 (Solovay (1970, Theorem 1)) *It is true in the second Solovay model $\mathcal{S}'$ that every set of reals is Lebesgue measurable.*

Remark 4.4 A free ultrafilter on $\mathbb{N}$ yields a set in $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ which is nonmeasurable in the sense of the natural uniform probability measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. Meanwhile, the second Solovay model $\mathcal{S}'$ of ZF + DC contains no such sets, and therefore no such ultrafilters, either. It follows that one cannot prove the existence of a free ultrafilter on $\mathbb{N}$ in ZF + DC.

The *constructible model* **L**, introduced by Gödel (1940), is another model of ZFC, opposite to the Solovay model in many of its features, including the existence of *definable non-measurable* sets of reals. Indeed, it is true in **L** that there is a non-measurable set in $\mathbb{R}$ which is not merely definable, but definable in a rather simple way which places it in the effective class Δ^1_2 of the projective hierarchy (see Novikov 1963). With these two models in mind, it is asserted that the existence of a definable Lebesgue non-measurable set is *independent* of the axioms of set theory.

[31] Given a sequence of nonempty sets $\langle X_n : n \in \mathbb{N}\rangle$, the axiom DC postulates the existence of a countable sequence of choices $x_0, x_1, x_2, \ldots$ in the case when, for each n, the domain X_n of the nth choice $x_n \in X_n$ may depend not only on n but also on the previously made choices $x_0, x_1, \ldots, x_{n-1}$. It is considered to be the strongest possible version of "countable choice".

4.3 That Which we Call a Non-sequitur

If, in Connes' terminology, "to name" is "to define", then Connes' remark to the effect that

> since noone will ever be able to name a nonstandard *number*, the *theory* remains virtual (Connes et al. 2001, p. 16) [emphasis added–the authors]

is that which we call a *non-sequitur*. Namely, while an ultrafilter (associated with a non-standard number by means of the transfer principle) cannot be *defined*, a *definable* (countably saturated) model of the hyperreals was constructed by Kanovei and Shelah (2004). Their construction appeared later than Connes' "virtual" comment cited at the beginning of Sect. 4.1. However, three years after the publication of Kanovei and Shelah (2004), Connes again came back to an alleged "catch in the theory":

> I had been working on non-standard analysis but after a while I had found a catch in the theory…The point is that as soon as you have a non-standard number, you get a non-measurable set. And in Choquet's circle, having well studied the Polish school, we knew that every set you can *name* is measurable (Connes 2007, p. 26) [emphasis added–the authors].

An ultrafilter associated with a non-standard *number* cannot be "named" or, more precisely, *defined*; however, the *theory* had been shown (three years prior to Connes' 2007 comment) to admit a *definable* model. Connes' reference to Solovay suggests that, to escape being *virtual*, a theory needs to have a *definable* model. If so, his "virtual" allegation concerning non-standard analysis is erroneous, by the result of Kanovei and Shelah.

Connes' claim that "every set you can name is measurable" is similarly inaccurate, by virtue of the Gödel constructible model **L**, as discussed in Sect. 4.2. A correct assertion would be the following: if you "name" a set of reals then you cannot prove (in ZFC) that it is nonmeasurable, and moreover, one can "name" a set of reals (a Gödel counterexample) regarding which you cannot prove that it is measurable, either.

Connes elaborated a distinction between countable AC and uncountable AC, and criticized NSA for relying on the latter (see Sect. 3.7). He invoked the Solovay model to explain why he feels NSA is a "virtual" theory. Now the second Solovay model $\mathcal{S}'$ of ZC+DC demonstrates that ultrafilters on $\mathbb{N}$ cannot be shown to exist without uncountable AC (see Remark 4.2). Thus, no ultrafilters, chimerical or otherwise, can be produced by means of the countable axiom of choice alone; yet Connes exploited ultraproducts (and ultrafilters on $\mathbb{N}$) in an essential manner in his work on the classification of factors (Connes 1976).

5 Machover's Critique

In 1993, M. Machover analyzed non-standard analysis and its role in teaching, expanding on a discussion in Bell and Machover (1977, p. 573). We will examine Machover's criticism in this section.

5.1 Is There a Best Enlargement?

In 1993, Machover wrote:

> The [integers, rationals, reals] can be characterized (informally or within set theory) *uniquely up to isomorphism* by virtue of their mathematical properties …But there is

> no ...known way of singling out a particular enlargement that can plausibly be regarded as canonical, nor is there any reason to be sure that a method for obtaining a canonical enlargement will necessarily be invented (Machover 1993) [emphasis added–the authors]

The problem of the uniqueness of the nonstandard real line is discussed in detail in an article by (Keisler 1994), to which we refer an interested reader. Meanwhile, Machover emphasizes

(A) the *uniqueness up to isomorphism* of the traditional number systems (integers, rationals, reals), allegedly *unlike* the hyperreals; and
(B) an absence of a preferred enlargement.

As we will see, he is off-target on both points (though the latter became entirely clear only after his text was published). We start with three general remarks.

(1) A methodological misconception on the part of some critics of NSA is an insufficient appreciation of the fact that the hyperreal approach does *not* involve a claim to the effect that hyperreals $^*\mathbb{R}$ are "better" than $\mathbb{R}$. Rather, one works with the *pair* $(\mathbb{R}, {}^*\mathbb{R})$ together with, say, the standard part function from limited hyperreals to $\mathbb{R}$. It is the interplay of the *pair* that bestows an advantage on this approach. The real field is still present in all its unique complete Archimedean totally ordered glory.
(2) Noone would dismiss an algebraic number field on the grounds that it is not as good as $\mathbb{Q}$ because of a lack of uniqueness. It goes without saying that the usefulness of an algebraic number field is not impaired by the fact that there exist other such number fields.
(3) The specific technical criticism of Machover's that the hyperreal enlargement is not unique and therefore one needs to prove that the notion of "continuity", for example, is model-independent, is answered by the special enlargement constructed by Morley and Vaught (1962) for any uncountable cardinality κ satisfying $2^\alpha \leq \kappa$ for all $\alpha < \kappa$ (see Sect. 5.2 for more details) and providing a unique such enlargement up to isomorphism.

Remark 5.1 Under the assumption of GCH, the condition on κ holds for all infinite cardinals κ. If GCH is not assumed, then it still holds for *unboundedly many* uncountable cardinals[32] κ, one of which (not necessarily the least one) can be defined by $\kappa = lim_n a_n$, where $a_0 = \aleph_0$ and $a_{n+1} = 2^{a_n}$.

5.2 Aesthetic and Pragmatic Criticisms

Machover's critique of NSA actually contains two separate criticisms even though he tends to conflate the two. The first criticism is an *aesthetic* one, mainly addressed to traditionally trained mathematicians: the reals are unique up to isomorphism, the hyperreals aren't. The second criticism is a *pragmatic* one, and is addressed to workers in NSA: hyperreal definitions of standard concepts apparently depend on the particular extension of $\mathbb{R}$ chosen, and therefore necessitate additional technical work. We will comment further on the two criticisms below.

Machover expressed his *aesthetic* criticism by noting that if we choose a system of real numbers

[32] Namely, for every cardinal κ there is one of this kind larger than κ (note that this is more than merely "infinitely many").

> in which the Continuum Hypothesis holds, and another in which it does not [hold], then for each such choice there are still infinitely many non-isomorphic enlarged systems of [hyper]reals, none of which has a claim to be 'the best one' (Machover 1993, p. 210).

How cogent is Machover's aesthetic criticism? The CH-part of his claim is dubious as it does not accord with what we observed above. Indeed, as noted in Sect. 3.6, all models of ${}^*\mathbb{R}$ of the form $\mathbb{R}^{\mathbb{N}}/\mathcal{F}$ are isomorphic in ZFC+CH (see Theorem 3.3). The uniqueness of the isomorphism type of such a hyperreal field parallels that of the traditional structures (integers, rationals, reals) emphasized by Machover in item (A) above.

The non-CH part of Machover's claim is similarly dubious. Although all models of ${}^*\mathbb{R}$ are not necessarily isomorphic under the ZFC axioms, still uniqueness up to isomorphism is attainable within the category of *special* models, that is, those represented in the form of limits of certain increasing transfinite sequences of successive saturated elementary extensions of ${}^*\mathbb{R}$. (See a detailed definition in Chang and Keisler 1992, 5.1.) The following major theorem is due to Morley and Vaught (1962), see also 5.1.8 and 5.1.17 in Chang and Keisler (1992).

Theorem 5.2 *Suppose that an uncountable cardinal κ satisfies the implication $\alpha < \kappa \Longrightarrow 2^\alpha \le \kappa$. Then*

(1) *there are special models of ${}^*\mathbb{R}$ of cardinality κ, and*
(2) *all those models are pairwise isomorphic.*

Thus, for any cardinal κ as in the theorem, there is a uniquely defined isomorphism type of nonstandard extensions of $\mathbb{R}$ of cardinality κ. Cardinals of this type do exist independently of GCH (see Remark 5.1) and can be fairly large, but at any rate one does have uniquely defined isomorphism types of models of ${}^*\mathbb{R}$ in suitable infinite cardinalities.

Remark 5.3 A decade after the publication of Machover's article, (Kanovei and Shelah 2004) proved the existence of a definable individual model of the hyperreals (not just a definable isomorphism type), contrary to all expectation (including Machover's, as the passage cited above suggests). Further research by Kanovei and Uspensky (2006) proved that all Morley–Vaught isomorphism classes given by Theorem 5.2 likewise contain definable individual models of ${}^*\mathbb{R}$.[33]

Remark 5.4 If one works in the Solovay model $\mathcal{S}$ as a background ZFC universe, then the definable Kanovei–Shelah model of ${}^*\mathbb{R}$ does not contain a *definable nonstandard integer*, as any such would imply a definable non-measurable set, contrary to Theorem 4.1. The apparent paradox of a non-empty definable set with no definable element is an ultimate expression of a known mathematical phenomenon when a simply definable set has no equally simply definable elements.[34]

As to Machover's *pragmatic* criticism addressed to NSA workers, we note that requiring suitable properties of saturation in a given cardinal, one in fact does obtain a unique model of the hyperreals. Therefore the criticism concerning the dependence on the model becomes moot.

[33] We note that a *maximal* class hyperreal field (in the von Neumann-Bernays-Gödel set theory) was recently analyzed by Kanovei and Reeken (2004, Theorem 4.1.10(i)) in the framework of axiomatic nonstandard analysis, and by Ehrlich (2012) from a different standpoint. In each version, it is similarly unique, and, in the second version, isomorphic to a maximal surreal field.

[34] For instance, one can define in a few lines what a transcendental real number is, but it would require a number of pages to prove for an average math student that π, e, or any other favorite trancendental number is in fact trancendental.

5.3 Microcontinuity

Machover recalls a property of a function f that we will refer to as *microcontinuity* at a point $r \in \mathbb{R}$ following Davis (1977, p. 96):

$$f(x) \approx f(r) \text{ for every hyperreal } x \approx r. \tag{5.1}$$

Here "$\approx$" stands for equality up to an infinitesimal. Property (5.1) is equivalent[35] to the usual notion of continuity of a real function f at r. Machover goes on to assert that

> in order to legitimize [(5.1)] as a definition ..., we must make sure that it is independent of the choice of enlargement. (Otherwise, what is being defined would be a *ternary* relation between f, r and the enlargement.) (Machover 1993, p. 208).

Microcontinuity formally depends on the enlargement. Machover concludes that it cannot replace (ϵ, δ) definitions altogether:

> Therefore, [(5.1)] cannot displace the old standard $[\epsilon, \delta]$ definition altogether, if one's aim is to achieve proper rigour and methodological correctness ...There is a long tradition of teaching *first-year calculus* in a way that sacrifices a certain amount of rigour in order to make the material more intuitive. There is, of course, nothing wrong or dishonourable about this–provided the students are told that what they are getting is a version that does not satisfy the highest standards of rigour and glosses over some problems requiring closer consideration (ibid.) [emphasis added–the authors].

Granted, we need to be truthful toward our students. However, Machover's argument is unconvincing, as he misdiagnozes the educational issue involved. The issue is *not* whether the (ϵ, δ) definition should be replaced *altogether* by a microcontinuous definition as in (5.1). Rather, the issue revolves around *which* definition should be the *primary* one. Thus, Keisler's textbook does present the (ϵ, δ) definition (Keisler 1986, p. 286), once continuity has been thoroughly explained via microcontinuity.[36] The (ϵ, δ) definition is an elementary formula, which shows that continuity is expressible in first order logic, a fact not obvious from the microcontinuous definition (5.1) dependent as it is on an external relation "$\approx$". Since the (ϵ, δ) definition needs to be mentioned in any case, the apparent dependence of (5.1) on the choice of an enlargement is a moot point.

6 How Powerful is the Transfer Principle?

The back cover of the 1998 hyperreal textbook by R. Goldblatt describes non-standard analysis as

> a wellspring of powerful new principles of reasoning (transfer, overflow, saturation, enlargement, hyperfinite approximation, etc.) (see Goldblatt 1998).

Of the examples mentioned here, we are particularly interested in *transfer*, i.e., the transfer principle whose roots go back to Łoś's theorem (Łoś 1955). The back cover describes the transfer principle as a *powerful new principle of reasoning*.

[35] Strictly speaking f should be replaced by ${}^*f$ in (5.1). Note that, modulo replacing the term "hyperreal" by the expression "variable quantity", definition (5.1) is Cauchy's definition of continuity, contrary to a widespread Cauchy–Weierstrass tale concerning Cauchy's definition (see Borovik and Katz 2012 as well as Katz and Katz 2012b, 2011a).

[36] Pedagogical advantages of microcontinuity were discussed in Błaszczyk et al. (2012, Appendix A.3).

On the other hand, a well-established tradition started by P. Halmos holds that the said principle is not powerful at all. Thus, Halmos described non-standard analysis as

> a special tool, too special, and other tools can do everything it does (Halmos 1985, p. 204).

Are we to conclude that the 1998 back cover contains a controversial assertion and/or a well-meaning exaggeration? Hardly so. The term "powerful" is being used in different senses. In this section we will try to clarify some of the meanings of the term.

6.1 Klein–Fraenkel Criterion

In 1908, Felix Klein formulated a criterion of what it would take for a theory of infinitesimals to be successful. Namely, one must be able to prove a mean value theorem (MVT) for arbitrary intervals, including infinitesimal ones:

> The question naturally arises whether [...] it would be possible to modify the traditional foundations of infinitesimal calculus, so as to include actually infinitely small quantities in a way that would satisfy modern demands as to rigor; in other words, to construct a non-Archimedean system. The first and chief problem of this analysis would be to prove the mean-value theorem
>
> $$f(x+h) - f(x) = h \cdot f'(x + \vartheta h)$$
>
> from the assumed axioms. I will not say that progress in this direction is impossible, but it is true that none of the investigators have achieved anything positive (Klein 1908, p. 219).

In 1928, Fraenkel (1946, pp. 116–117) formulated a similar criterion in terms of the MVT.

Such a Klein–Fraenkel criterion is satisfied by the framework developed by Hewitt, Łoś, and Robinson. Indeed, the MVT is true for the natural extension ${}^*f$ of every real smooth function f on an arbitrary hyperreal interval, by the transfer principle. Fraenkel's opinion of Robinson's theory is on record:

> my former student Abraham Robinson had succeeded in saving the honour of infinitesimals - although in quite a different way than Cohen[37] and his school had imagined (Fraenkel 1967, p. 107).

The hyperreal framework is the only modern theory of infinitesimals that satisfies the Klein-Fraenkel criterion. The fact that it satisfies the criterion is due to the transfer principle. In this sense, the transfer principle can be said to be a "powerful new principle of reasoning".

One could object that the classical form of the MVT is not a key result in modern analysis. Thus, in L. Hörmander's theory of partial differential operators (Hörmander 1976, p. 12–13), a key role is played by various multivariate generalisations of the following Taylor (integral) remainder formula:

$$f(b) = f(a) + (b-a)f'(a) + \int_a^b (b-x)f''(x)dx. \tag{6.1}$$

[37] The reference is to Hermann Cohen (1842–1918), whose fascination with infinitesimals elicited fierce criticism by both G. Cantor and B. Russell. For an analysis of Russell's *non-sequiturs*, see Ehrlich (2006) and Katz and Sherry (2012a,b).

Denoting by $\mathcal{D}$ the differentiation operator and by $\mathcal{I} = \mathcal{I}(f, a, b)$ the definite integration operator, we can state (6.1) in the following more detailed form for a function f:

$$\begin{aligned}&(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})\\ &\quad f(b) = f(a) + (b-a)(\mathcal{D}f)(a) + \mathcal{I}\left((b-x)(\mathcal{D}^2 f), a, b\right)\end{aligned} \tag{6.2}$$

Applying the transfer principle to the elementary formula (6.2), we obtain

$$\begin{aligned}&(\forall a \in {}^*\mathbb{R})(\forall b \in {}^*\mathbb{R})\\ &\quad {}^*f(b) = {}^*f(a) + (b-a)({}^*\mathcal{D}\,{}^*f)(a) + {}^*\mathcal{I}\left((b-x)({}^*\mathcal{D}^2\,{}^*f), a, b\right)\end{aligned} \tag{6.3}$$

for the natural hyperreal extension ${}^*f$ of f. The formula (6.3) is valid on every hyperreal interval of ${}^*\mathbb{R}$. Multivariate generalisations of (6.1) can be handled similarly.

We focused on the MVT (and its generalisations) because, historically speaking, it was emphasized by Klein and Fraenkel. The transfer principle applies far more broadly, as can be readily guessed from the above.

6.2 Logic and Physics

There is another sense of the term *powerful* that is more controversial than the one discussed in Sect. 6.1. Namely, how powerful are the hyperreals as a research tool and an engine of discovery of *new* mathematics? The usual litany of impressive breakthroughs achieved using NSA includes progress on the invariant subspace problem, canards, hydrodynamics and Boltzmann equation, non-standard proof of Gromov's theorem on groups of polynomial growth, Hilbert's fifth problem (see Hirschfeld 1990 and Goldbring 2010), etc.[38]

However, declaiming such a list does little more than encourage the partisans while further antagonizing the critics. We will therefore comment no further other than clarifying that this is *not* the meaning of the term *powerful* when we use it in reference to the transfer principle. Namely, we use it solely in the sense explained in Sect. 6.1.

The significance of the back cover comment cited at the beginning of Sect. 6 is that Robinson's theory introduces new perspectives and intuitions into mathematics, similarly to physics. [39] When E. Witten informally wrote down a pair of equations on the board at MIT a couple of decades ago, he was motivated by physical intuitions. The resulting Seiberg-Witten theory caused a revolution in gauge theory, and in particular resulted in much shorter proofs of theorems that S. Donaldson received his Fields medal for (see e.g., Katz 1995). Logic, similarly, introduces new intuitions and techniques. Today logicians like Hrushovski (1996) obtain results in "ordinary mathematics" by model-theoretic means.

Interesting recent uses of non-standard methods as applied to the structure of approximate groups may be found in Hrushovski (2012) and Breuillard et al. (2011).

7 How Non-constructive is the Dixmier Trace?

This section deals with the foundational status of the Dixmier trace, and with the role of Dixmier trace in noncommutative geometry.

[38] For additional examples see the book (van den Berg and Neves 2007).

[39] Such an analogy between logic and physics is due to David Kazhdan.

7.1 Front Cover

The front cover of the book *Noncommutative geometry* features an elaborate drawing, done by Connes himself (according to the copyright page). The drawing contains only three formulas. One of them is the expression

$$⨍ |dZ|^p.$$

The barred integral symbol $⨍$ is Connes' notation for the trace constructed by Dixmier (1966). The notation first occurred in print in Connes (1995, p. 6213, formula (2.34)), i.e., the year after its appearance on the front cover of Connes' book. The appearance of Dixmier's trace on the book cover indicates not only that Connes was already thinking of the Dixmier trace as a kind of "integration" (this idea is already found in Connes 1990), but also that Connes himself thought of the trace as an important ingredient of noncommutative geometry.

7.2 Foundational Status of Dixmier Trace

The Dixmier trace is a linear functional on the space of compact operators whose characteristic values have a specific rate of convergence to 0. In Connes' framework, the Dixmier trace can be thought of as a kind of an "integral" of infinitesimals. An analogous concept in Robinson's framework is the functional

$$\mathrm{st}(n\epsilon).$$

Here $n \in {}^*\mathbb{N} \setminus \mathbb{N}$ is a *fixed* hypernatural, and the functional is defined for a *variable* infinitesimal ϵ constrained by the condition that $n\epsilon$ is finite.

Dixmier exploited ultrafilters in constructing his trace. Dixmier traces can also be constructed using universally measurable *medial limits*, independently constructed by Christensen (1974) and Mokobodzki in the assumption of the continuum hypothesis (CH). Mokobodzki's work was explained by Meyer (1973). Meyer's text is cited in Connes' book (Connes 1994), but not in the section dealing with Dixmier traces (Connes 1994, p. 303–308), which does not use medial limits and instead relies on the Hahn–Banach theorem (Connes 1994, p. 305, line 8 from bottom).[40]

Medial limits have been shown *not* to exist in the assumption of the filter dichotomy (FD) by Larson (2009). FD is known to be consistent (Blass and Laflamme 1989). The assumption of CH (exploited in the construction of medial limits) is generally considered to be a very strong foundational assumption, more controversial than the axiom of choice (see e.g., Hamkins 2012a,b; Isaacson 2011).

Indeed, while all the major applications of the "uncountable" AC outside of set theory proper[41] can be reduced to the assumption that the continuum of real numbers can be wellordered, CH requires, in addition, the existence of a wellordering of $\mathbb{R}$ specifically of length ω_1 (which is the shortest possible length of such a wellordering).

[40] Note that, in the spirit of reverse mathematics, the Hahn–Banach theorem is sufficient to generate a Lebesgue nonmeasurable set (see Foreman and Wehrung 1991; Pawlikowski 1991).

[41] This includes such constructions as the Vitali non-measurable set, Hausdorff's gap, ultrafilters on $\mathbb{N}$, the Hamel basis, the Banach–Tarski paradox, nonstandard models, etc. Sierpiński (1934) gives many additional examples.

Moreover, CH implies the existence of P-point ultrafilters[42] on $\mathbb{N}$, and Shelah (1982) showed that the existence of P-points cannot be established in ZFC, again indicating the controversial nature of CH.

Furthermore, Connes notes that the results he is interested in happen to be *independent of the choice* of the Dixmier trace (Connes 1994, p. 307, line 14 from bottom). Thus the strong assumption of CH appears superfluous, and the nonconstructive nature of the ultrafilter construction of the Dixmier trace, a paper tiger. Namely, Dixmier trace is constructive or non-constructive in a sense similar to that of a hyperreal number being constructive or non-constructive: both rely on nonconstructive foundational material (be it AC, CH, or Hahn-Banach), but yield results *independent of choices* made. For instance, differentiating x^2 yields $2x$ regardless of the variety of infinitesimals exploited in defining the derivative. Similarly, the notion of continuity, when defined via microcontinuity, is independent of the hyperreal model used (see Sect. 5.3).

7.3 Role of Dixmier Trace in Noncommutative Geometry

At a recent conference (see Gayral et al. 2012) on singular traces (such as the Dixmier trace), a majority of the speakers mentioned the Dixmier trace in their abstracts, while none of them mentioned (or cited) either Mokobodzki or medial limits. Recent work by the conference speakers dealing with Dixmier traces includes: Carey et al. (2003), Engliš and Zhang (2010), Lord and Sukochev (2010, 2011), Lord et al. (2010), Kalton et al. (2011), Sukochev and Zanin (2011a,b).

Most speakers also cite Connes' *Noncommutative Geometry*. Ever since its appearance on the front cover of Connes' book (see Sect. 7.1), the Dixmier trace has played a major role in Connes' framework and related fields.

8 Of Darts, Infinitesimals, and Chimeras

In this section we will be concerned with a somewhat elusive issue of what is real and what is chimerical.

8.1 Darts

Connes outlined a game of darts in 2000 in the following terms:

> You play a game of throwing darts at some target called Ω … what is the probability $dp(x)$ that actually when you send the dart you land exactly at a given point $x \in \Omega$? … what you find out is that $dp(x)$ is smaller than any positive real number ε. On the other hand, if you give the answer that $dp(x)$ is 0, this is not really satisfactory, because whenever you send the dart it will land *somewhere* (Connes 2004, p. 13) [emphasis added–the authors].

As Connes points out, no satisfactory interpretation of such intuitions seems to exist in a real number system devoid of infinitesimals. But if one interprets the "p" to be an infinitesimal interval rather than a point, there is a consistent theory that can capture the intuitions Connes spoke of. Namely, assume for the sake of simplicity that the target is the unit interval [0, 1].

[42] See footnote 25.

A more satisfactory answer than the one above is provided in terms of a hyperfinite grid

$$\text{Grid}_H = \left\{0, \tfrac{1}{H}, \tfrac{2}{H}, \tfrac{3}{H}, \ldots, \tfrac{H-1}{H}, 1\right\} \tag{8.1}$$

defined by a hypernatural $H \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then the probability of the dart hitting an infinitesimal interval $[\frac{k}{H}, \frac{k+1}{H}] \subset [0, 1]$ can be taken to be precisely $\frac{1}{H}$. The hypernatural H can be chosen to be the explicit unchimerical one appearing in (3.7).

Similarly, the probability of the dart hitting a real set $A \subset [0, 1]$ can be computed as follows. Roughly speaking, one counts the number of points in the intersection $\text{st}^{-1}(A) \cap \text{Grid}_H$ and divides by H, where st is the standard part function on limited hyperreals, and Grid_H is the hyperfinite grid of (8.1), yielding a probability of

$$\frac{|\text{st}^{-1}(A) \cap \text{Grid}_H|}{H}; \tag{8.2}$$

more precisely, since $\text{st}^{-1}(A)$ is not an internal set, one takes the infimum of $\text{st}(|X|/H)$ over all internal sets X containing $\text{st}^{-1}(A) \cap \text{Grid}_H$ (see Goldblatt 1998, Lemma 16.5.1 on p. 210, and Theorem 16.8.2 on p. 217).

8.2 Chimeras

Probability theory and measure theory over the hyperreals are today vast research fields (see e.g., Benci et al. 2011; Wenmackers and Horsten 2012). Meanwhile, Connes comments as follows:

> A nonstandard number is some sort of chimera[43] which is impossible to grasp and certainly not a concrete object. In fact when you look at nonstandard analysis you find out that except for the use of ultraproducts, which is *very efficient*, it just shifts the order in logic by one step; it's not doing much more (Connes 2004, p. 14) [emphasis added–the authors]

Connes describes ultraproducts as "very efficient", apparently in contrast to the rest of non-standard analysis. Meanwhile, the special case of an ultraproduct used in the construction of ${}^*\mathbb{R}$ as in (3.8) exploits an ultrafilter $\mathcal{F}$ described by Connes as a "chimera". Are we to conclude that we are dealing with a *very efficient chimera*?

Remark 8.1 Connes exploits a nonprincipal ultrafilter ω in constructing the ultraproduct von Neumann algebra N^ω containing a von Neumann algebra N in *Noncommutative geometry*:

> Definition 11. For every ultrafilter $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ let N^ω be the ultraproduct, $N^\omega =$ the von Neumann algebra $\ell^\infty(\mathbb{N}, N)$ divided by the ideal of sequences $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\omega} \|x_n\|_2 = 0$ (Connes 1994, ch. V, sect. 6.δ, Def. 11).[44]

Perhaps Connes' intention is similar to that of Leibniz, who sometimes described infinitesimals as "useful fictions" (see Katz and Sherry 2012a,b and Sect. 2 below). But Leibniz's position is generally thought to be close to a formalist one, akin to Robinson's, whereas Connes is known as a Platonist (see Sect. 2.3).[45]

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43 See Fig. 1. The reader may be amused to find similar terminology in Karl Marx, who commented as follows: "The closely held belief of some rationalising mathematicians that dy and dx are quantitatively actually only infinitely small, only approaching 0/0, is a chimera" (Marx cited in Fahey et al. 2009, p. 260).

44 The definition appears on page 495 in the pdf version available from Connes' homepage, and on page 483 in the published book.

45 See also footnote 46 on a comment by Davies.

Connes goes on to argue that *his* infinitesimal framework does provide an adequate framework for solving the dart problem (see Connes 1997, formula (2.35)). However, Connes' noncommutative infinitesimals do not form a division ring, do not possess a total order, lack a transfer principle, and would have difficulty handling the dart problem as smoothly as (8.2).

8.3 Shift

What is the meaning of the phrase

> "nonstandard analysis …just shifts the order in logic by one step; it's not doing much more"

penned by Connes (see Sect. 8.2)? The phrase is characteristically evasive (cf. the discussion of his use of the verb "to name" in Sect. 4.1), but perhaps he is referring to the fact that non-standard analysis permits one to express formulas in *second* order logic as formulas in *first* order logic over the hyperreals (hence "shifts the order in logic by one step"). In this context, it is instructive to consider what Fields medalist T. Tao has to say concerning the expressive power of non-standard analysis:

> [it] allows one to rigorously manipulate things such as "the set of all small numbers", or to rigorously say things like "η_1 is smaller than anything that involves η_0", while greatly reducing epsilon management issues by automatically concealing many of the quantifiers in one's argument (Tao 2008, p. 55).

The 2009 Abel prize winner M. Gromov said in 2010:

> After proving the theorem about polynomial growth using the limit and looking from infinity, there was a paper by Van den Dries and Wilkie giving a much better presentation of this using ultrafilters (Gromov cited in Raussen and Skau 2010).

Other authors have taken note of Connes' sweeping judgments of mathematical subjects not to his liking. Thus, E. B. Davies writes:

> In 2001 Alain Connes, a committed Platonist,[46] who has spent a lifetime working on C*-algebras and their applications, nevertheless excluded the theory of *Jordan algebras* from the Platonic world of mathematics …How do mathematicians make such value judgments, and are their opinions more than prejudices? (Davies 2011, p. 1456) [emphasis added–the authors].

Here Davies is referring to the following comment by Connes:

> I would say that the exceptional algebra of three-by-three matrices on Cayley octonions definitely exists because of its connections to the Lie group F_4. As for the general notion of Jordan algebra, it is difficult to assert that it really holds water (Connes et al. 2001, p. 30).

Connes finds it "difficult to assert" that the theory of Jordan algebras "holds water". Meanwhile, E. Zelmanov wrote that I. Kantor's work on Jordan algebras (see, e.g., the influential text Kantor 1972)

[46] In the context of Davies' comment on Connes' Platonism, see also main text at footnote 45 which examines the possibility that Connes may also hold views close to Formalism.

> played a crucial role in [Zelmanov's] proof of the Restricted Burnside problem (Zelmanov 2008, p. 111),

work for which Zelmanov was awarded the Fields medal in 1994.

8.4 Continuum in Quantum Theory

Quantum physicists Časlav Brukner and 2010 Wolf prize winner Anton Zeilinger speculate that

> the concept of an infinite number of complementary observables and therefore, indirectly, the assumption of continuous variables, are just mathematical constructions which might not have a place in a final formulation of quantum mechanics …continuous variables are devoid of operational and therefore physical meaning in quantum mechanics" (Brukner et al. 2005, p. 59).

I. Durham concurs:

> This latter proposal[47] is similar to coarse-graining arguments in thermodynamic and quantum systems which have been used by Brukner and Zeilinger to argue that the continuum is nothing but a mathematical construct, a view I wholeheartedly endorse (Durham 2011).

In 1994, Wolf prize winner John A. Wheeler wrote:

> The space continuum? Even continuum existence itself? Except as *idealization* neither the one entity nor the other can make any claim to be a primordial category in the description of nature (Wheeler 1994, p. 308) [emphasis added–the authors].

There appears to be an identifiable view in the quantum physics community that the mathematical continuum is an idealisation, or to borrow Connes' terminology, it is a "virtual theory" or "chimera", though undoubtedly an "efficient" one.

A mathematician need not ordinarily be concerned about opinions found in a separate scientific community. However, Connes' motivation for his framework is drawn from quantum theory, and he frequently mentions quantum mechanics as the inspiration for his noncommutative solution of the dart problem (see Sect. 8.1). His references to alleged "absolutely major flaw" and "irredemiable defect" in Robinson's infinitesimals emanate from their status as an idealisation. But in quantum theory, the same observation would apply to Connes' framework based as it is on the continuum, creating tensions with Connes' Platonism about the latter (see Sect. 2).

Connes claimed that "it seemed utterly doomed to failure to try to use non-standard analysis to do physics" (Connes 2007, p. 26). Such a claim is particularly dubious coming as it does two decades after the publication of the 500-page monograph *Nonstandard Methods in Stochastic Analysis and Mathematical Physics* by the 1992 Max-Planck-Award recipient Albeverio and others (Albeverio et al. 1986), where just such applications were developed in great detail.

[47] I.e., a proposal to resolve the paradox of quantum behavior of light.

9 Conclusion

The use of non-constructive foundational material such as the axiom of choice in the hyperreal context is similar to the use of non-constructive foundational material in Connes' theory. Thus, Connes exploits the Dixmier trace (Connes 1995, p. 6208), the Hahn–Banach theorem (Connes 1994, p. 305), as well as ultrafilters (Connes 1994, p. 483, see our Remark 8.1 above). Such concepts rely on non-constructive foundational material and are unavailable in the framework of the Zermelo–Fraenkel axioms alone.

Connes claims to provide "substantial and calculable" results based on his theory exploiting the Dixmier trace (Connes 1997, p. 211), and laments the allegedly non-exhibitable nature of Robinson's infinitesimals. Meanwhile, Dixmier's construction of the trace relies on the choice of a nonprincipal ultrafilter on the integers (Dixmier 1966), while an alternative construction requires the continuum hypothesis (see Sect. 7). Connes exploits ultrafilters in classifying factors and in constructing von Neumann algebras, but there are no ultrafilters in the second Solovay model $\mathcal{S}'$ of the set-theoretic universe ZFC + DC (countable choice only) that Connes professes to favor. Connes proclaims himself to be an adherent of countable AC (see Sect. 3.7 above), but $\mathcal{S}'$ is a model of ZFC + DC containing no ultrafilters, so that Connes' philosophical advocacy of countable AC is divorced from the facts on the ground of his scientific practice.

Thus, Connes' claims to the effect that his theory produces computationally meaningful results, allegedly *unlike* Robinson's theory, are unconvincing. There is in fact strong similarity between the two nonconstructivities involved.

Given powerful[48] tools such as non-standard enlargements and the transfer principle, one is able to associate an ultrafilter to a hyperinteger. But such ability is a spin-off of the power of the new principles of reasoning developed in Robinson's approach, and is a reflection, not of a shortcoming, but rather of the strength of Robinson's method.

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48 See Sect. 6 for a discussion of the term.

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