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Iterated privation and positive predication

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ABSTRACT

The standard rule of *single privative modification* replaces privative modifiers by Boolean negation. This rule is valid, for sure, but also simplistic. If an individual a instantiates the privatively modified property (MF) then it is true that a instantiates the property of *not* being an F, but the rule fails to express the fact that the properties (MF) and F have something in common. We replace Boolean negation by property negation, enabling us to operate on *contrary* rather than contradictory properties. To this end, we apply our theory of *intensional essentialism*, which operates on properties (intensions) rather than their extensions. We argue that each property F is necessarily associated with an essence, which is the set of the so-called *requisites* of F that jointly define F. Privation deprives F of *some but not all* of its requisites, replacing them by their contradictories. We show that properties formed from iterated *privatives*, such as being an *imaginary fake banknote*, give rise to a *trifurcation* of cases between returning to the original root property or to a property contrary to it or being semantically undecidable for want of further information. In order to determine which of the three forks the bearers of particular instances of multiply modified properties land upon we must examine the requisites, both of unmodified and modified properties. Requisites underpin our *presuppositional* theory of *positive predication*. Whereas privation is about being deprived of certain properties, the assignment of requisites to properties makes positive predication possible, which is the predication of properties the bearers must have because they have a certain property formed by means of privation.

1. Introduction

There are large amounts of natural-language text data that need to be analyzed and formalized, because we want to build up question-answering systems over these data. We want not only to convey information explicitly recorded in these texts but also to derive implicit information entailed by these explicit data so as to answer questions in an intelligent way. In other words, we want to apply logical *reasoning* to these

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natural-language corpuses. To this end, we must analyze natural-language sentences in a fine-grained way. Since adjectives that denote property modifiers are part and parcel of our everyday vernaculars as well as artificial languages, we need to logically analyze property modifiers as well. *Privation* being the most complicated kind of property modification, the goal of this paper is a fine-grained analysis of privation accompanied by rules governing reasoning about sentences that contain such modifiers.

Privative modification is an operation that forms negated properties from properties. It is one among three kinds of negation:

- *privation*, which applies to *properties*
- the *complement function*, which applies to *sets*
- the *Boolean not*, which applies to propositions-in-extension, i.e. *truth-values*.

When *propositions* are identified with (or at least modelled as) *sets*, then the complement function subsumes propositional negation as a special case. Nothing in this paper hinges on this. What matters is the contrast between privation, which is property negation and therefore an operation on *intensions*, and set-theoretic negation, which takes a set to its complement and is therefore an operation on *extensions*.

The standard theory of modifiers is Montague Grammar, which is a typed version of model-theoretic intensional logic. This paper provides an extension of this framework such that it is now possible to analyze a particular sort of properties (or predicates, in the formal mode) that would previously fall outside the purview of the framework. The paper also offers reasons for revising one of the existing rules; however, the extension we provide can be incorporated without revising anything. We are building upon the work of not least Coulson and Fauconnier [3], Horn [13–15], Iwańska [16], Jespersen [17], Kamp [22], Montague [25], Partee [27], Primiero and Jespersen [28], while the background theory is based on Duží et al. [6,8].

Montague Grammar comes with a well-entrenched logic for *single privation*. This framework states its logic for the various modifiers in the form of elimination rules. The rule of single privation amounts to replacing the privative modifier by Boolean negation:

$$\frac{a \text{ is a } \textit{fake} \text{ banknote}}{a \text{ is } \textit{not} \text{ a banknote}} \quad \textit{single privation}$$

This rule is valid, because all that is required for validity is that the property (here, *banknote*) modified by the privative modifier (here, *fake*) not be predicated of a, and the conclusion achieves at least this much. However, the above rule misses the internal link between the property of being a banknote and the property of being a fake banknote. We will probe further into this point below, but the basic idea is that a fake banknote is not just some object or other that fails to be a banknote, but rather it is an object that must have a host of properties in common with banknotes. Though both forged banknotes and, say, weathercocks and zebras are not banknotes, there is an intuitive sense in which forged banknotes are somehow 'closer to' banknotes than are weathercocks and zebras. The challenge before us is to define privation in such a way that it is made explicit what banknotes and forged banknotes have, and must have, in common.

Another problem with the rule of single privation is that it fails to extend to *iterated privation*, as Boolean negation can replace a privative modifier only once. Here are some examples of predicates that express iterated privation:

- 'is an imaginary fake banknote'
- 'is a former heir apparent'
- 'is a former fallen angel'
- 'weighs almost half a kilo'
- 'is anything but a false friend'

- ‘is a theory of non-antisymmetric mereology’
- ‘is an imaginary, burned fake banknote’

For instance, Horn in [13, pp. 296–308], [14,15] ponders the logic and rhetoric of *double negatives*, e.g. as expressed by ‘not un-F’ (‘not unhappy’, ‘not impolite’, etc.).[1] Is a *not impolite remark* a polite remark, perhaps even a very polite remark, as per litotes (cf. [14, pp. 86ff]; or a remark that is neither polite nor impolite, ending up in the neutral mid-interval? For a further example, consider so-called superdollars, which are not US dollars, but counterfeit 100-dollar bills manufactured in, e.g., North Korea that are materially (though not conceptually) well-nigh indiscernible from their genuine originals.[2] This particular occurrence of ‘super’ in ‘superdollar’ has a privative effect, so the predicate ‘is a fake superdollar’ expresses double privation.[3] A fake superdollar is a fraudulent imitation of what is already a fraudulent imitation. If somebody collects first-degree counterfeit banknotes then they want a superdollar, and not a fake superdollar, which exemplifies second-degree forgery by being a fake.[4] We all know that faking a fake will not return us to the genuine original; but how do we know that? There is also the opposite direction: although you start out with a 100-dollar bill, successfully passing it off as a fake superdollar to a collector of forged banknotes of any degree, your 100-dollar bill has not transmogrified into a fake superdollar, despite being accepted as one. But how do we know that? The answer we will be pursuing is that we know that because we know the meaning of the respective predicates.

But what to replace Boolean negation with in order to develop a logic of iterated privation? We suggest replacing Boolean negation with property negation. First, property negation operates on properties, just as property modifiers do, so the intensional character of modification is carried through to negation. Second, property negation obeys a logic of contraries rather than contradictories, which provides the kind of rule that privative modification requires.

Let us take a closer look at privation. There are two material sources of privation. One is resultative and hence diachronic: individual a once was an F, but is no longer an F.[5] A recaptured fugitive (cf. [11]) once was a fugitive, but is no longer one. Finished meals, burnt (not just charred) pieces of meat, and obsolete banknotes all exemplify resultative privation. Given the actual laws of nature, neither a finished meal, nor a burnt piece of meat can again become a meal or a piece of meat, whereas an obsolete banknote might be restored to its previous glory as a banknote should the social institutions so favour it. The other source is achronic: a did not start out as an F and might never become an F, although it is possible that a might in fact become an F, as when the relevant social institutions decree that such-and-such fake banknotes shall henceforth acquire the status of valid tender, thus turning them into banknotes. Only this latter property is extraneous to the property of being a fake banknote.

There are two formal sources of privation: either by way of first-degree or higher-degree modification. Either a privative modifier modifies a property that has already been modified by a privative modifier, as when *imaginary* is applied to *fake banknote*. Or a privative modifier modifies another privative modifier, and the resulting modifier is applied to a property, as when *anything but* is applied to *false* and the resulting modifier, *anything but false*, is applied to *friend*.[6] (In this paper we shall consider only first-degree iterated

[1] See Horn [13, pp. 38–41] for a historical survey of various takes on contrariety and predicate term negation.

[2] Cf. http://en.wikipedia.org/wiki/Superdollar.

[3] It is not an open-and-shut issue whether some modifiers are absolutely privative while the rest are context-sensitive by being privative only with respect to some argument properties. *Fake* might be an example of the former, though we are issuing no guarantee. Examples of the latter would include *Nordic gold*, which is not gold (but an alloy); *fides punica*, which is not trust (but treachery); a *baker's dozen*, which is not a dozen (but thirteen); and *Rocky Mountain oysters*, which are not oysters. See also [16] on context-sensitive privatives.

[4] We could shift both the real McCoys and the fakes one level up with collectors collecting second-degree fakes and being fooled by third-degree fakes; and so on up.

[5] Other dynamic examples of ‘stages of loss in the privative process’ and ‘incomplete realizations of possible privational histories’ (Martin [24, p. 439, 441, resp.]) would include going bald, i.e. progressing (or perhaps regressing) toward being almost or entirely without hair.

[6] *Anything but* is a privative intensifier, just like *very* is a subsective intensifier, as in *very good*.

privation in the interest of brevity.) What we just described is double privation, but the theory readily generalizes to triple, quadruple (etc.) privation, as when *imaginary* is applied to *burned fake banknote*.

In the light of the fact that privative modifiers can be nested, one may wonder: could we avail ourselves of a rule that would calculate, for an arbitrary string of privative modifiers of two or more, whether the root property F is true of a? Our research shows that no such rule is forthcoming. There can be no unique rough-and-ready rule for iterated privation.

Iterated privation issues instead in a *trifurcation* of cases:

(i) a is an F
(ii) a fails to be an F
(iii) it is semantically indeterminate whether a is an F

The fact that this trifurcation emerges reflects the nature of privation. The first of two general points bearing on privation is the negative one that privation is about what something is *not*, or *fails* to be. It is about one or more properties that an object is *deprived of*. In particular, no theory of privatives should predict that fake banknotes are extracted from sets of banknotes: fake banknotes are not banknotes that are fake.[7]

But the second point is the positive one that there is substantially more to privation than deprivation. Let F be a property, M_p a modifier privative with respect to F, and $[M_pF]$ the privatively modified property that results from applying the modifier to the root property. The intuition we wish to capture is that when an object has the property $[M_pF]$ then the object is—in some sense yet to be made clear—'closer' to having F than are many or most other objects that lack the privatively modified property $[M_pF]$.[8] By way of an example, a fake banknote is 'almost' a banknote, definitely barred from being one, yet it has a greater overlap in terms of properties with a banknote than have most other objects. Fake banknotes must share a host of properties with banknotes; otherwise they could not be fake banknotes in the first place, but would be merely, say, colourful slips of paper. For instance, a banknote must mention the issuing authority, a currency and a denomination. Therefore, a fake banknote must also mention an issuing authority, a currency and a denomination. If a fake banknote sports, for instance, the words 'ECB', 'EURO', '100' and is printed on cotton-based paper with the look and feel of garden-variety banknotes then it lends itself to several instances of what we call *positive predication*. Positive predication predicates properties of an object which the object must instantiate and which are not privatively modified.

Positive predication appears to be less complicated vis-à-vis achronic privation than diachronic privation. A burnt piece of meat is ash (inorganic matter) and in this second state not at all close to being meat (organic matter), whereas a fake banknote must be close to being a banknote. However, we are able to put forward a theory of positive predication with regard to objects that exemplify privatively modified properties of either kind, because we offer a *presuppositional* theory of privation. The theory is presuppositional because, for an object to exemplify a privatively modified property, it is presupposed that the object should already exemplify other properties. By way of illustration, the property of being a former smoker comes with the presupposition that, as a matter of analytic necessity, whoever currently instantiates it previously, but no longer, instantiated the property of being a smoker. Or if a has the property of being a Vatican cardinal then a has also the property of being fluent in Latin. Beyond the well-rehearsed example of former smokers, the theory extends to not only social artefacts like positions in a hierarchy of institutional power, but also

[7] This is to say that we do not stretch the meaning of 'is a banknote' so as to include fake banknotes among the banknotes. Partee [27] suggests using coercion to do just that, such that it becomes meaningful to inquire whether a banknote is a real banknote or a fake banknote. Jespersen [17, p. 544, fn. 14] and Duží et al. [6, p. 400, fn. 52] argue against Partee's suggestion.

[8] Coulson and Fauconnier [3] and Iwańska [16] also think of privation both as the elimination of some, but not all, properties (or concepts, features, etc.) inhering in privatively modified properties, and as the 'blending with' or 'introduction of' additional properties so as to form new, hybrid properties like *stone lion* or *toy elephant*.

technical artefacts like tools and scientific frameworks like taxonomies.[9] The presuppositional theory enables us to infer conclusions about, say, dead whales (a dead whale being a dead mammal), disassembled watches (a disassembled watch not being a timekeeping device) and burnt pieces of meat (a burnt piece of meat having been previously a piece of meat). Importantly, each and every property we countenance has a host of other properties associated with it. Thus, each instance of $[M_p[M_pF]]$ also comes with a host of adjacent properties that their respective bearers must also bear.

We call the adjacent properties *requisites*.[10] Our thesis is that privation is the deprivation of some, but not all, requisites. The surviving requisites, together with some added ones, form the basis of positive predication. The above trifurcation arises because the root property F will be one of the requisites of some multiply privatively modified properties, while *non-F* will be one of the requisites of other multiply privatively modified properties, whereas neither F, nor *non-F* is among the requisites of still other multiply privatively modified properties. In the third and final case, as far as the semantics of such properties goes, there is no semantic fact of the matter as to which side of the fence a comes down on. Extra-semantic, empirical investigation must, in each individual case, determine which side a given individual comes down on. To give a taste of the trifurcation, here is an example of each of its three horns.

- If somebody is *anything but a false friend* then they are a friend (and that to a very high degree) (i).
- If something weighs *almost half a kilo* then it weighs less than a kilo and, therefore, does not weigh a kilo (ii).[11]
- If someone is a *former heir apparent* then either they are now the incumbent monarch or they are no longer being even considered for the throne (iii).

The requisites of a given property enable valid reasoning from assumptions about privatively modified properties. Philosophically speaking, associating requisites with properties amounts, in the case of privation, to laying down at least some of what goes into being a wooden horse, a burnt wooden horse, a burnt fake wooden horse, a fake burnt wooden horse, etc. Achieving the latter, philosophical, objective comes with a fair amount of idealization while still requiring substantial philosophical justification.[12] In this paper we rest content with setting out the formal features of the framework within which we discuss iterated privation and positive predication. Just to be clear, while we will be arguing for a particular *elimination* rule for privatives, we will not attempt to put forward any *introduction* rules for privatives in the vein of:

$$\frac{P_1, \ldots, P_n}{a \text{ is an } [M_pF]}$$

For particular instances of P_i and $[M_pF]$, such a rule would make explicit what the conditions are for being, e.g., a fake banknote, or a wooden horse, or a malfunctioning toothbrush.[13] Philosophy of technology would make a great leap forward if particular instances of P_i could be spelt out with the rigour required

[9] For the relevant theory of presuppositions, see [7,9,8].

[10] The notion of requisite was conceived by Tichý and introduced in [30, p. 408]. It has subsequently been turned into a theory of intensional essentialism. See Jespersen and Materna [21], Duží et al. [6, Ch. 3].

[11] We are making the fairly uncontroversial assumption that when something weighs almost half a kilo then it weighs no more than that. We want to blot out the kind of scenario where something that weighs exactly a kilo weighs also 900 grams, almost half a kilo, etc., in virtue of a simple argument of downward monotonicity that also validates the 'countdown' inference that if you have five fingers on your hand then you also have four (three, ..., zero) fingers, which still does not entail that you have fifteen fingers.

[12] See [2].

[13] It is not a matter of course that *malfunctioning* is privative. It is on the causal-role theory of technical function (what cannot hammer cannot be a hammer), whereas it is subsective on the proper-function theory (a malfunctioning hammer was still designed to hammer as its proper function). See Jespersen and Carrara [20]. An interesting study on malfunctioning software has been recently provided by Floridi, Fresco and Primiero [12]. The authors distinguish between two kinds of malfunctioning software, namely in terms of 'negative' dysfunction and 'positive' misfunction. They argue that while dysfunction is the core property of malfunctioning technical artefacts, an executed software token cannot dysfunction, because it will always work in accordance with its design. Yet it can, and often will, misfunction, because the design does not completely live up to the intended specification.

for an introduction rule. But, although the notion of requisite would come in handy, this very ambitious enterprise is beyond the compass of this paper.[14]

The fundamental distinction among modifiers is typically considered to be one between the *subsectives* and the *non-subsectives*.[15] The former group would consist of the *pure subsectives* (that are governed by the upwardly monotonic rule of *right subsectivity*, which amounts to eliminating the modifier and predicating the surviving property) and the *intersectives* (that are governed by the rule of *right subsectivity* and a rule of *left subsectivity*).[16]

Here is a brief comparison in prose of the four standard types of modifiers, where an index is an index of empirical evaluation, such as a possible world or a world/time pair.

- *Pure subsectives.* At every index a skillful surgeon is a surgeon.
- *Intersectives.* At every index a round peg is round and is a peg.
- *Privatives.* At no index is a fake banknote a banknote.
- *Modals.* At some indices an alleged assassin is an assassin, and at some other indices an alleged assassin is not an assassin.

Montague [25, p. 211] seeks to provide a uniform theory of modifiers (strictly speaking, of adjectival phrases). Each modifier, according to Montague, is a property-to-property mapping.[17] We subscribe to this uniform account of the corpus of modifiers. We depart, however, from Montague's contention that these functions are *meaning-to-meaning* functions (ibid.). The contention, of course, makes perfect in Montague's *intensional* framework in which intensions (functions whose domain are the logical space of possible worlds) count as meanings.[18] In our framework, *meaning-to-meaning* functions would be *hyperintension-to-hyperintension* functions. We have such functions, but we do not need them here. We do need hyperintensions, however, when working with modifiers: we need hyperintensions (meanings) when defining a couple of key notions that go into defining modifiers. In a word, we are using hyperintensions in order to operate on intensions.

It is relevant to compare modal and iterated privative modification, for in neither case is only one conclusion possible. The modals require *extra-semantic*, empirical inquiry to establish, for each particular instance, which of two ways the facts happen to go. Only empirical inquiry can decide which allegations of being an assassin are true and which ones are false. The iterated privatives require *intra-semantic* inquiry to establish, for each particular instance, which of the three ways the meanings go. If we land on the third fork, then we need to get out of the armchair and into the field to establish which way the facts happen to go.

Privation is literally radical modification, because the root property is modified away. Subsective modification, by contrast, enriches the root property, whether the modifier be intersective (e.g. *round*) or purely subsective (e.g. *skillful*). A peg, say, is qualified as a round peg, or conversely, something round is qualified as a round peg; and a surgeon as a skillful surgeon. A layer of modification is added on top of the existing requisites of the root property. Privation goes in the opposite direction by purging the root property of some of its requisites. This is the crucial step toward explaining why a fake banknote fails to be a banknote. One property that drops out is that of being valid tender, which comes with requisites of its own. Yet privation

14 See Del Frate [5] for conceptual discussion of a catalogue of engineering conceptions of malfunction.

15 See, e.g., Makinson [23, pp. 64–65] on the distinction between *qualifiers* and *proper modifiers*.

16 See Jespersen [17] for two rules of left subsectivity.

17 A topic we will not be delving further into here is how to decide for a given token of a given adjective whether it denotes a property or a modifier. See, however, Siegel [29], Kamp [22], Montague [25], Beesley [1]. Schematically put, Montague pairs all adjectives off with modifiers, Beesley pairs all adjectives off with properties, and Kamp pairs some adjectives off with modifiers and the rest with properties.

18 We are glossing over the facts that Montague did not fully commit to s (i.e. combined world/time pairs) as a stand-alone type on an equal footing with e (i.e. 'entity'), t (i.e. truth-value), etc., and that Montague's empirical indices were combined world/time pairs.

not only detracts, but also adds something. One property that gets added is that of being a forgery (i.e. a fraudulent imitation of something or other), which also comes with requisites of its own. The crucial step toward the presuppositional theory required for positive predication is that privation adds new requisites to the purged set of requisites. Moreover, some of these new requisites contradict some of the original purged requisites. This explains why we can predicate several properties of fake banknotes that they must have.

If we did not assign requisites to properties, we would be left with an exceptionally minimalist logic of iterated privation. First of all, the replacement of privatives by Boolean negation can occur only once, as we announced at the outset. Here is why. The standard rule of single privation lays down what to do when it is *true* that a has property $[M_pF]$. The rule fails to state what to do when the premise is the negation that a has property $[M_pF]$. This inference, therefore, is *invalid*:

$$\frac{\neg[[M_pF]\ a]}{\neg\neg[Fa]}$$

If, counterfactually, the rule for privation had specified logical equivalence between $[[M_pF]\,a]$ and $\neg[Fa]$ *then* the above argument would have come out valid. However, the rule of privation does not specify equivalence; rather it specifies that $[[M_pF]\,a]$ entails $\neg[Fa]$. It is also intuitive enough that the above inference must come out invalid. If it did not, all instances of double privation would land on the first fork. Thus, a fake rhinestone diamond would emerge as a diamond. So not only would the inference fail to be truth-preserving by over-generating instances of the first fork, it would also leave no room for the other two forks.

Secondly, therefore, in the interest of setting up a logic of iterated privation, we suggest replacing Boolean negation by property negation, denoted by '*non*'. This replaces contradictories by contraries, which makes for a sufficiently weakened form of negation. Applied to single privation, the result is:

$$\frac{a \text{ is a } \textit{fake} \text{ banknote}}{a \text{ is a } \textit{non}\text{-banknote}} \quad \textit{single privation}^*$$

When a is a fake banknote at some index then a is sent to the complement set of the set of banknotes at the same index, though not to just anywhere in the complement, but to its particular subset of fake banknotes. The good news is that we can reiterate *non* so as to form the property *non-non-banknote.* The bad news is that $[non\ [non\,F]]$ would be the final word on iterated privation in the absence of requisites. The above trifurcation would remain, but it would be impossible to decide which particular fork a particular instance of iterated privation landed on. A logic of iterated privation that amounted to replacing privatives by *non* would grind to a halt after having established the general insight that pairs of privatives yield contraries.

The thesis, then, that we are arguing for can be condensed thus. A logic of iterated privation that invokes requisite properties of privatively modified properties enables positive predication and is in a position to land particular instances of multiply privatively modified properties on the right fork.

The rest of the paper is organized as follows. Section 2 sets out the relevant portions of our formal semantic theory. Section 3 compares the logic of subsectives against the logic of privatives, introduces property negation, and offers case studies of each of the forks of the trifurcation.

2. Logical foundations

In this section, we set out the formal framework within which we raise and solve the problem of iterated privation. The framework is a fragment of Tichý's Transparent Intensional Logic (TIL). The relevant fragment is more or less continuous with Montague's intensional logic and its myriad extensions. However, TIL has added a theory of modal modifiers (see Primiero and Jespersen [28]) to the Montagovian corpus, and a spelt-out logic, including an introduction rule, for intersective modifiers (see Jespersen [17]), as well as a general rule of left subsectivity applying to privatives and modals, intersectives and pure subsectives (see Jespersen [17]; Duží et al. [6, §4.4]), which extends to single privation. The present paper is the third and final of a trilogy of papers on how to model various states (especially malfunction) of technical artefacts by means of property modifiers. The two preceding papers are Jespersen and Carrara [19,20].

2.1. Key definitions of transparent intensional logic

We need definitions of the following basic notions:

- *Simple type theory.* We need this definition in order to define both intensional and extensional entities. Properties of individuals are typed as functions from possible worlds to functions from times to sets of individuals, where sets are identified with their characteristic functions. Property modifiers are typed as property-to-property functions. (Modifier modifiers are typed as modifier-to-modifier functions.)
- *Constructions.* We need this definition for the following reasons. Constructions are (fine-grained and structured) *meanings*; we define four of the altogether six constructions that make up the full inductive definition of constructions. Furthermore, the definition introduces the *formalism* of TIL, which is based on λ-abstracts.
- *Requisite.* The requisite relation *Req* is a relation-in-extension between two properties R and P, such that, necessarily, whatever is (in) the extension of P must, as a matter of analytic necessity, also be (in) the extension of R, though not necessarily conversely. We say that R is a *requisite* of P.
- *Essence.* The essence of a property P is the set of its requisites which together define P.
- *Property negation.* Property negation, *non*, allows *iteration* and obeys a logic of *contraries*.

Note that our theory is based on what we call *intensional essentialism*. The analytically necessary relation of being a requisite of P and being an element of the essence of P obtains between intensional entities such as properties, and not between extensional entities (such as individuals) and intensional entities. Consequently, we subscribe to *individual anti-essentialism*: no individual has any purely contingent property necessarily. By 'purely contingent property' we mean a non-constant property that does not have what we call an essential core; e.g., the property of having exactly as many inhabitants as Prague is necessarily exemplified by Prague, whatever number of inhabitants Prague may happen to have.[19]

We define the essence of a property as a set of its requisites that jointly define the property. For instance, the property of being a mammal is related by the requisite relation to the property of being a whale. Thus, necessarily, if the individual a happens to be a whale at a world/time index of evaluation then a is also a mammal at this world/time. It is an open question (epistemologically and ontologically speaking) whether a is a whale. Establishing whether it is one requires investigation *a posteriori*. On the other hand, establishing whether a must be a mammal in case a happens to be a whale is *a priori*, the requisite relation being in-extension and as such independent of what is true at any particular state of affairs. Comparing the essences of a root property and a modified property enables us to define subsective and privative modifiers in a new way that is an extension of previous definitions.

Definition 1 *(Simple type theory).* Let B be a *base*, where a base is a collection of pair-wise disjoint, non-empty sets. Then:

i) Every member of B is an elementary *type of order 1 over B.*

[19] See Duží et al. [6, §1.4.2.1] for a classification of empirical properties and (ibid.: 68) for the notion of essential core.

ii) Let $\alpha, \beta_1, ..., \beta_m$ $(m > 0)$ be types of order 1 over B. Then the collection $(\alpha\ \beta_1...\beta_m)$ of all m-ary partial mappings from $\beta_1 \times ... \times \beta_m$ into α is a functional *type of order 1 over B*.
iii) Nothing is a *type of order 1 over B* unless it so follows from (i) and (ii). □

Notation. That an object O is of type α, i.e. belongs to the type α, will be denoted '$O : \alpha$'.

Remark. For the purposes of natural-language analysis TIL uses the following so-called objectual base B consisting of the following atomic types:

- o: the set of truth-values **T, F**;
- ι: the set of individuals (the universe of discourse);
- τ: the set of real numbers (doubling as discrete times);
- ω: the set of logically possible worlds (the logical space).

Definition 2 *(Constructions).*

(i) *Variables* x, y, ... are *constructions* that construct objects (elements of their respective ranges) dependently on a valuation v; they *v-construct.*
(ii) Where X is an object whatsoever (an extension, an intension or a *construction*), ${}^0X$ is the *construction Trivialization.* ${}^0X$ constructs X without any change in X.
(iii) Let $X, Y_1, ..., Y_n$ be *constructions.* Then *Composition* $[X\ Y_1...Y_m]$ is the following *construction.* If X *v-constructs* a function f of type $(\alpha\beta_1...\beta_m)$, and $Y_1, ..., Y_m$ v-construct entities $\mathrm{B}_1, ..., \mathrm{B}_m$ of types β_1, ..., β_m, respectively, then $[X\ Y_1...Y_m]$ *v-constructs* the value (an entity, if any, of type α) of f on the tuple-argument $\langle \mathrm{B}_1, ..., \mathrm{B}_m \rangle$. Otherwise $[X\ Y_1...Y_m]$ does not *v-construct* anything and so is *v-improper.*
(iv) The *Closure* $[\lambda x_1...x_m Y]$ is the following *construction.* Let $x_1, x_2, ..., x_m$ be pair-wise distinct variables v-constructing entities of types $\beta_1, ..., \beta_m$, respectively, and Y a construction typed to v-construct an α-entity. Then $[\lambda x_1...x_m Y]$ is the *construction Closure* (or λ-*Closure*). It *v-constructs* the following function $f : (\alpha\beta_1 \dots \beta_m)$. Let $v(\mathrm{B}_1/x_1, ..., \mathrm{B}_m/x_m)$ be a valuation identical with v at least up to assigning objects $\mathrm{B}_1 : \beta_1, ..., \mathrm{B}_m : \beta_m$ to variables $x_1, ..., x_m$. If Y is $v(\mathrm{B}_1/x_1, ..., \mathrm{B}_m/x_m)$-improper (see iii), then f is undefined on $\langle \mathrm{B}_1, ..., \mathrm{B}_m \rangle$. Otherwise the value of f on $\langle \mathrm{B}_1, ..., \mathrm{B}_m \rangle$ is the α-entity $v(\mathrm{B}_1/x_1, ..., \mathrm{B}_m/x_m)$-constructed by Y.
(v) Nothing is a *construction*, unless it so follows from (i) through (iv). □

Remark. That a variable x v-constructs entities of a type α will be referred to as 'ranging over α', denoted by '$x \rightarrow_v \alpha$'. We model sets and relations by their characteristic functions. Thus, for instance, $(o\iota)$ is the type of a set of individuals, while $(o\iota\iota)$ is the type of binary relations-in-extension between individuals. Empirical expressions denote *empirical conditions* that may or may not be satisfied at some world/time pair of evaluation. We model these empirical conditions as possible-world-semantic (PWS) *intensions.* PWS intensions are entities of type $(\beta\omega)$: mappings from possible worlds to an arbitrary type β. The type β is frequently the type of the *chronology* of α-objects, i.e., a mapping of type $(\alpha\tau)$. Thus α-intensions are frequently functions of type $(\alpha(\tau\omega))$, abbreviated as '$\alpha_{\tau\omega}$'. *Extensional entities* are entities of the arbitrary type α where $\alpha \neq (\beta\omega)$ for any type β. Where w ranges over ω and t over τ, the following logical form essentially characterizes the logical syntax of empirical language: $\lambda w \lambda t\ [\dots w \dots t \dots]$.

Examples of frequently used PWS intensions are:

- *propositions* of type $o_{\tau\omega}$
- *properties* of individuals of type $(o\iota)_{\tau\omega}$

- *binary relations-in-intension* between individuals of type $(o\iota\iota)_{\tau\omega}$
- *individual offices* (or roles) of type $\iota_{\tau\omega}$

Logical objects like *truth-functions* and *quantifiers* are extensional: $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) are of type (ooo), and $\neg$ (Boolean negation) of type (oo). Since TIL has no syncategorematic symbols, all the symbols in the TIL formalism denote functions, including quantifiers. The *quantifiers* $\forall^\alpha$, $\exists^\alpha$ are type-theoretically polymorphic total functions, just as in Montague Grammar, of type $(o(o\alpha))$, for an arbitrary type α, and are defined as follows.

Definition 3 *(Quantifiers).* The universal quantifier $\forall^\alpha$ is a function of type $(o(o\alpha))$ that takes a class A of α-elements to **T** if A contains all elements of the type α, otherwise to **F**. The existential quantifier $\exists^\alpha$ is a function of type $(o(o\alpha))$ that takes a class A of α-elements to **T** if A is a non-empty class, otherwise to **F**. □

Notational conventions.

- '$\forall x \ldots$' serves as a shorthand for '$[{}^0\forall\lambda x \ldots]$'; similarly for '$\exists y$': all variable-binding is λ-binding, and universal (existential) quantification is presented by means of Trivialization.
- Below all type indications will be provided outside the formulae in order not to clutter the notation.
- The outermost brackets will be omitted whenever no confusion can arise.
- While '$X : \alpha$' means that an object X is (a member) of type α, '$X \rightarrow_v \alpha$' means that X is typed to v-construct an object of type α, if any. We write '$X \rightarrow \alpha$' if no confusion concerning valuation arises.
- $w \rightarrow_v \omega$ and $t \rightarrow_v \tau$.
- If $C \rightarrow_v \alpha_{\tau\omega}$ then the frequently used Composition $[[C\ w]\ t]$, which is the intensional descent (a.k.a. extensionalization) of the α-intension v-constructed by C, will be encoded as 'C_{wt}'.

Predication is an instance of Composition.[20] An empirical predicate such as 'is a planet' denotes the property of being a planet; it is subsequently extensionalized in order to obtain the set of planets at the empirical indices of evaluation; the characteristic function of the set is applied, by way of Composition, to the individual of which the property of being a planet is predicated; the result (a truth-value) is finally abstracted over by means of w and t variables in order to construct an empirical truth-condition of type $o_{\tau\omega}$. The form of the predication of being a planet of an individual a is this[21]:

$$\lambda w \lambda t\, [{}^0 Planet_{wt}\, {}^0 a]$$

The form of the predication of the subsectively modified property of being a gas planet is this:

$$\lambda w \lambda t [[{}^0 Gas\, {}^0 Planet]_{wt}\, {}^0 a]$$

To begin, construct, by way of Composition, the property of being a gas planet and then follow the same steps as above.

Types: $Planet : (o\iota)_{\tau\omega}$; $a : \iota$; $Gas : (((o\iota)_{\tau\omega})((o\iota)_{\tau\omega}))$;
$[{}^0 Gas\, {}^0 Planet] \rightarrow (o\iota)_{\tau\omega}$; $[[{}^0 Gas\, {}^0 Planet]_{wt}\, {}^0 a] \rightarrow_v o$; $\lambda w \lambda t\, [[{}^0 Gas\, {}^0 Planet]_{wt}\, {}^0 a] \rightarrow o_{\tau\omega}$: the proposition that a is a gas planet.

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[20] See Duží et al. [6, §2.4.2] on predication.

[21] We apply the method of analysis according to which semantically simple predicates like 'is a planet' are associated with the Trivialization of the denoted object; ${}^0 Planet$, in this case. See Duží et al. [6, §2.1].

2.2. Requisites

The requisite relations *Req* are a family of relations-in-extension between two intensions, so they are of the polymorphous type $(o\,\alpha_{\tau\omega}\,\beta_{\tau\omega})$, with the possibility that $\alpha = \beta$.[22] Infinitely many combinations of *Req* are possible, but for our purpose we will need just this one:

$$Req\text{: } (o\,(o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$$

Req is a relation between two properties of individuals, such that one is a requisite of the other.

TIL embraces *partial functions*.[23] Partiality gives rise to the following complication. The requisite relation obtains necessarily, i.e. for all worlds w and times t, and so the values at this or that $\langle w, t\rangle$ of particular intensions are irrelevant. But the extensions of properties (i.e. sets) are isomorphic to characteristic functions, and these functions are amenable to truth-value gaps. As already mentioned, the property of having stopped smoking comes with a bulk of requisites including not least the property of being a former smoker. Thus, the predication of such a property P of a may also fail, causing $[{}^0P_{wt}\,{}^0a]$ to be v-improper. There is a straightforward remedy, however, namely the propositional property of being true at $\langle w, t\rangle$; *True*: $(o\,o_{\tau\omega})_{\tau\omega}$. Given a proposition *Prop*, $[{}^0True_{wt}\,{}^0Prop]$ v-constructs **T** if *Prop* is true at $\langle w, t\rangle$; otherwise (i.e., if *Prop* is false or else undefined at $\langle w, t\rangle$) it v-constructs **F**.

Definition 4 *(Requisite relation between ι-properties).* Let X, Y be constructions such that $X, Y\colon *_n \to (o\,\iota)_{\tau\omega}$; $x \to \iota$. Then

$$[{}^0Req\,Y\,X] = \forall w\forall t[\forall x[[{}^0True_{wt}\,\lambda w\lambda t\,[X_{wt}\,x]] \supset [{}^0True_{wt}\,\lambda w\lambda t\,[Y_{wt}\,x]]]].$$

Gloss *definiendum* as, "Y is a requisite of X", and *definiens* as, "Necessarily, i.e. at every $\langle w, t\rangle$, any x that instantiates X at $\langle w, t\rangle$ also instantiates Y at $\langle w, t\rangle$."

Example. Let the property of being a person be a requisite of the property of being a student. Then the hyperproposition that all students are persons is an analytic truth. It constructs the proposition *TRUE*, which is the necessary proposition, which takes value **T** at all world/time pairs. Wherever and whenever somebody is a student they are also a person. Formally:

$$[{}^0Req\,{}^0Person\,{}^0Student] = \forall w\forall t\,[\forall x\,[[{}^0True_{wt}\,\lambda w\lambda t\,[{}^0Student_{wt}\,x]] \supset [{}^0True_{wt}\,\lambda w\lambda t\,[{}^0Person_{wt}\,x]]]]$$

Claim 1. *Req is a quasi-order on the set of ι-properties.*

Proof. Let $X, Y \to (o\iota)_{\tau\omega}$. Then *Req* belongs to the class QO:$(o(o(o\iota)_{\tau\omega}(o\iota)_{\tau\omega}))$ of quasi-orders over the set of individual properties:

Reflexivity. $[{}^0Req\,X\,X] = \forall w\forall t\,[\forall x\,[[{}^0True_{wt}\,\lambda w\lambda t\,[X_{wt}\,x]] \supset [{}^0True_{wt}\,\lambda w\lambda t\,[X_{wt}\,x]]]]$.

Transitivity. We want to prove that $[[[{}^0Req\,Y\,X] \wedge [{}^0Req\,Z\,Y]] \supset [{}^0Req\,Z\,X]]$.

[22] For comparison, Jespersen [18] offers a detailed study of a requisite relation, of type $(o\,\iota_{\tau\omega}\,\iota_{\tau\omega})$, where one individual office is a requisite of another individual office, the way the office of Commander-in-Chief is a requisite of the office of President of the United States. The paper analyses "Superman is Clark Kent" as expressing that this particular requisite relation obtains between one office denoted by 'Superman' and another office denoted by 'Clark Kent'. If you occupy the office of Superman you must co-occupy the office of Clark Kent, but you can occupy the Clark Kent office without occupying the Superman office. This goes to show that TIL offers an intensional analysis (based on intensional essentialism) of "Superman is Clark Kent", contrary to the prevalent 'Millian' extensional analyses.

[23] See Duží et al. [6, 276–78] for a philosophical justification of partiality in spite of the associated technical complications.

1. $[[^0 Req\, Y\, X] \wedge [^0 Req\, Z\, Y]]$ $\emptyset$
2. $[\forall w \forall t\, [\forall x\, [[^0 True_{wt}\, \lambda w \lambda t\, [X_{wt}\, x]] \supset [^0 True_{wt}\, \lambda w \lambda t\, [Y_{wt}\, x]]]] \wedge \forall w \forall t\, [\forall x\, [[^0 True_{wt}\, \lambda w \lambda t\, [Y_{wt}\, x]] \supset [^0 True_{wt}\, \lambda w \lambda t\, [Z_{wt}\, x]]]]]$ 1, Definition 4
3. $[[^0 True_{wt}\, \lambda w \lambda t\, [X_{wt}\, x]] \supset [^0 True_{wt}\, \lambda w \lambda t\, [Y_{wt}\, x]]]$ 2, $\forall E$, $\wedge E$
4. $[[^0 True_{wt}\, \lambda w \lambda t\, [Y_{wt}\, x]] \supset [^0 True_{wt}\, \lambda w \lambda t\, [Z_{wt}\, x]]]$ 2, $\forall E$, $\wedge E$
5. $[[^0 True_{wt}\, \lambda w \lambda t\, [X_{wt}\, x]] \supset [^0 True_{wt}\, \lambda w \lambda t\, [Z_{wt}\, x]]]$ 3, 4,*
6. $[\forall w \forall t\, [\forall x\, [[^0 True_{wt}\, \lambda w \lambda t\, [X_{wt}\, x]] \supset [^0 True_{wt}\, \lambda w \lambda t\, [Z_{wt}\, x]]]]]$ 5, $\forall I$
7. $[[^0 Req\, Z\, X]$ 6, Definition 4
8. $[[[^0 Req\, Y\, X] \wedge [^0 Req\, Z\, Y]] \supset [^0 Req\, Z\, X]]$ 7, $\supset I$

Remark. In line (5) '*' denotes the theorem of the transitivity of implication.

In order for a requisite relation to be a weak partial order, it would need to be also anti-symmetric. The *Req* relation is, however, not anti-symmetric. If properties X, Y are mutually in the *Req* relation, i.e. if

$$[[^0 Req\, Y\, X] \wedge [^0 Req\, X\, Y]]$$

then at every $\langle w, t\rangle$ the two properties are true of exactly the same individuals. This does not entail, however, that X, Y are identical. It may be the case that there is an individual a such that $[X_{wt}\, a]$ v-constructs $\mathbf{F}$ whereas $[Y_{wt}\, a]$ is v-improper. For instance, the following properties X, Y differ only in truth-value for those individuals who never smoked. Let *StopSmoke*:$(o\iota)_{\tau\omega}$ be the property of having stopped smoking. Whereas X yields truth-value *gaps* on such individuals, Y is *false* of them:

$$X = \lambda w \lambda t\, \lambda x\, [^0 StopSmoke_{wt}\, x]$$
$$Y = \lambda w \lambda t\, \lambda x\, [^0 True_{wt}\, \lambda w \lambda t\, [^0 StopSmoke_{wt}\, x]]$$

This makes for a negligible difference that can be abstracted away, so we introduce the equivalence relation $\approx : (o\, (o\iota)_{\tau\omega} (o\iota)_{\tau\omega})$ on the set of individual properties; $p, q \rightarrow (o\iota)_{\tau\omega}$; $= : (ooo)$; $=_{df}$: $(o(o\, (o\iota)_{\tau\omega} (o\iota)_{\tau\omega})(o\, (o\iota)_{\tau\omega} (o\iota)_{\tau\omega}))$, i.e. the identity of binary relations between properties.

$$^0{\approx} = \lambda pq\, [\forall x\, [[^0 True_{wt}\, \lambda w \lambda t\, [p_{wt}\, x]] = [^0 True_{wt}\, \lambda w \lambda t\, [q_{wt}\, x]]]]$$

Now we can define the *Req'* relation on the factor set of the set of ι-properties as follows.[24] Let $[p]_{\approx} = \lambda q\, [^0 \approx p\, q]$ and $[Req'\, [p]_{\approx} [q]_{\approx}] = [Req\, p\, g]$. Then:

Claim 2. *Req' is a partial order on the factor set of the set of ι-properties with respect to the relation $\approx$.*

Proof. It is sufficient to prove that *Req'* is well-defined. Let p, q be ι-properties such that $[^0 \approx p\, p']$ and $[^0 \approx q\, q']$. Then:

$$[Req'\, [p]_{\approx} [q]_{\approx}] = [Req\, p\, g]$$
$$= \forall w \forall t\, [\forall x\, [[^0 True_{wt}\, \lambda w \lambda t\, [p_{wt}\, x]] \supset [^0 True_{wt}\, \lambda w \lambda t\, [q_{wt}\, x]]]]$$
$$= \forall w \forall t\, [\forall x\, [[^0 True_{wt}\, \lambda w \lambda t\, [p'_{wt}\, x]] \supset [^0 True_{wt}\, \lambda w \lambda t\, [q'_{wt}\, x]]]] = [Req'\, [p']_{\approx} [q']_{\approx}]$$

[24] The definition of *Req'* was first introduced in Duží et al. [6, 363–364].

Now, obviously, the relation *Req'* is anti-symmetric:

$$[[{}^0Req'\,[p]_{\approx}\,[q]_{\approx}] \wedge [{}^0Req'\,[q]_{\approx}\,[p]_{\approx}]] \supset [[p]_{\approx} = [q]_{\approx}]$$

To make the exposition easier to follow, in what follows we will neglect this minor difference between properties $\lambda w \lambda t\, \lambda x[{}^0True_{wt}\, \lambda w \lambda t\, [p_{wt}\, x]]$ and p so that instead of the former we will write simply 'p'. □

2.3. Intensional essentialism

Next, we are going to define the essence of a property. Our essentialism is based on the idea that since no purely contingent property can be essential of any individual, essences are borne by intensions rather than by individuals exemplifying intensions. That a property F has an essence means that a relation-in-extension obtains *a priori* between F and a set *Essence* of other properties such that, as a matter of analytic necessity, whenever an individual (an ι-entity) instantiates F at some $\langle w, t \rangle$ then the same individual also instantiates all the properties belonging to *Essence* at the same $\langle w, t \rangle$. Hence our essentialism is based on the requisite relation, couching essentialism in terms of a priori interplay between properties, regardless of who or what exemplifies a given property. The essence of a property F is identical to the set of requisites that jointly define F. The $\langle w, t \rangle$-relative extensions of a given property are irrelevant, as we said; but so are the various equivalent constructions of the property.

Definition 5 *(Essence of a property).* Let $p, q \rightarrow (o\iota)_{\tau\omega}$; *Ess*: $((o(o\iota)_{\tau\omega})(o\iota)_{\tau\omega})$, i.e. a function assigning to a given property p the set of its requisites defined as follows:

$${}^0Ess = \lambda p \lambda q\, [{}^0Req\, q\, p]$$

Then the essence of a property p is the set of its requisites:

$$[{}^0Ess\, p] = \lambda q\, [{}^0Req\, q\, p]$$

Each property has many requisites. The question is: how do we know which properties are the requisites of a given property? The answer requires an analytic *definition* of the given property. For instance, consider the property of being a (domestic) cat. A classification according to biological taxonomy can serve as such a definition:

Kingdom: *Animalia*
Phylum: *Chordata*
Clade: *Synapsia*
Class: *Mammalia*
Order: *Carnivora*
Family: *Felidae*
Subfamily: *Felinae*
Genus: *Felis*
Species: *Felis Catus*

Thus, we can define a cat as an animal belonging to all of the above categories.[25] From this definition it follows that, for instance, the sentence "Cats are mammals" comes out analytically true:

$$\forall w \forall t\,[\forall x\,[[{}^{0}Cat_{wt}\,x] \supset [{}^{0}Mammal_{wt}\,x]]]$$

Hence the property of being a mammal is a requisite of the property of being a cat. All the above properties defined by a given taxonomy belong to the essence of the property of being a cat.

3. Subsectives, privatives, property negation, and case studies

3.1. Subsectives and privatives

With the above definitions in place, we can go on to compare two kinds of subsectives against privatives[26]:

- A modifier M is *non-trivially subsective* with respect to property F iff the modified property $[M\,F]$ has all the requisites of F and at least one additional requisite that is not a requisite of F. In other words, the essence of F is a proper subset of the essence of $[M\,F]$.

For instance, a skillful surgeon is a surgeon because the property of being a skillful surgeon must have all the requisites of the property of being a surgeon, and the additional property of being skillful with respect to the property of being a surgeon.

- A modifier M is *trivially subsective* with respect to F iff the modified property $[M\,F]$ has exactly the same requisites as the property F, i.e. if $[M\,F]$ and F share the same essence, hence are identical properties. The trivial subsectives are trivial in that the modification has no effect on the modified property and so might just as well not have taken place.

For instance, there is no semantic or logical (but perhaps rhetorical) difference between the property of being a diamond and the property of being a *genuine* diamond. Trivial modifiers such as *genuine* and *real* are pure subsectives: genuine diamonds are not located in the intersection of diamonds and objects that are genuine, for there is no such property as being genuine, pure and simple. Genuine diamonds form a subset, though not a proper one, of a given set of diamonds.[27]

- A modifier M is *privative* with respect to F iff the modified property $[M\,F]$ lacks at least one, but not all, of the requisites of the property F. Moreover, the essence of $[M\,F]$ contains at least one other requisite that does not belong to the essence of F, and contradicts at least one of the requisites of F. Hence, M is privative with respect to F iff the essence of $[M\,F]$ has a non-empty intersection with the essence of F, and this intersection is a proper subset of both the essences of F and of $[MF]$.

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[25] Contra Kripke, it is not a discovery (*a posteriori*, yet 'metaphysically' necessary) that a domestic cat belongs to any of the categories above. The definition of domestic cat in virtue of the conjunction of the above categories is a *stipulative* definition, which is conceptually prior to any empirical discovery of the further properties of various domestic cats (such as weighing seven pounds, basking on a hot tin roof, or having grey stripes). Our stance is at odds with Kripkean essentialism, as we find anyone conducting empirical inquiry in the animal kingdom needs a conceptual steer on what deserves to be called a domestic cat in the first place before they can claim to have had any sort of causal interaction with domestic cats. (These remarks barely scrape the surface of a deep philosophical issue, but they serve at least to indicate where we stand.)

[26] We are disregarding intersective modification in order not to clutter the exposition. However, intersectives are controlled by the same rule of *right subsectivity* that applies to the subsectives, together with the special rule of *left subsectivity* defined in [17].

[27] Iwańska [16, p. 350] refers to 'ideal', 'real', 'true', and 'perfect' as *type-reinforcing* adjectives, which seems to get the pragmatics right of what are semantically pleonastic adjectives. Trivial subsectives should not be confused with subsective intensifiers, as in 'is real pain', when real pain does not contrast with imaginary pain, but with slight pain.

For instance, *forged banknote* has *almost* the same requisites as does *banknote*, but it has also another requisite, namely the property of not being issued by an instance endowed with issuing authority.

To formally define the difference between subsective and privative modification, we need the TIL definition of the relation of being a subset between sets and the operation of the intersection of two sets. The relation of being a *subset* between α-sets, $\subseteq : (o(o\alpha)(o\alpha))$, is defined for any type α as follows. Let $a, b \rightarrow_v (o\alpha)$, $x \rightarrow_v \alpha$. Then:

$$^0\!\subseteq\, = \lambda ab\,[\forall x\,[[a\,x] \supset [b\,x]]]$$

The relation of being a *proper subset*, $\subset : (o(o\alpha)(o\alpha))$, is then defined as usual:

$$^0\!\subset\, = \lambda ab\,[[\forall x\,[[a\,x] \supset [b\,x]]] \wedge \neg[^0\!=a\,b]]$$

For instance, that the set of primes, *Prime*: $(o\tau)$, is a subset of the naturals, *Natural*: $(o\tau)$, is captured by this construction:

$$[^0\subseteq {}^0Prime\,{}^0Natural] = [\forall x\,[^0Prime\,x] \supset [^0Natural\,x]]$$

Similarly, that the set of primes is a proper subset of the naturals is captured by this construction:

$$[^0\!\subset {}^0Prime\,{}^0Natural] = [[\forall x[^0Prime\,x] \supset [^0Natural\,x]] \wedge [^0Prime \neq {}^0Natural]]$$

The operation of intersection, $\cap : ((o\alpha)(o\alpha)(o\alpha))$, is defined as follows:

$$^0\!\cap\; = \;\lambda ab\,\lambda x\,[[ax] \wedge [bx]]$$

For instance, that the intersection of primes and even numbers, *Even*: $(o\tau)$, is equal to the singleton 2 is captured by this construction:

$$[^0\!\cap {}^0Prime\,{}^0Even] = \lambda x\,[[^0Prime\,x] \wedge [^0Even\,x]] = \lambda x\,[x = {}^02]$$

In what follows we will use classical (infix) set-theoretical notation for any sets A, B; hence instead of '$[^0\!\subset A\,B]$' we will write '$[A \subset B]$', and instead of '$[^0\!\cap A\,B]$' we will write '$[A \cap B]$'. Since we will be comparing sets of properties, the type α is here the type of an individual property, $(o\iota)_{\tau\omega}$.

We are now able to provide the following two definitions.

Definition 6 *(Subsective vs. privative modifiers).* Let $M \rightarrow ((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$; $F, p \rightarrow (o\iota)_{\tau\omega}$. Then

- A modifier M is *subsective* with respect to a property F iff

$$[^0Ess\,F] \subseteq [^0Ess\,[M\,F]]$$

- A modifier M is *non-trivially subsective* with respect to a property F iff

$$[^0Ess\,F] \subset [^0Ess\,[M\,F]]$$

- A modifier M is *privative* with respect to a property F iff

$$\begin{aligned}&[[^0Ess\,F] \cap [^0Ess\,[M\,F]]] \neq \emptyset\\ &\quad \wedge \exists p\,[[[^0Ess\,F]\,p] \wedge [[^0Ess\,[M\,F]]\,\lambda w \lambda t\,[\lambda x\,\neg[p_{wt}x]]]]\end{aligned}$$

Remark. The second conjunct defining *privative modifier* is to be read as follows: "There is a property p such that it is a requisite of the property F ($[[^0Ess\, F]\, p]$), and among the requisites of the modified property $[M\, F]$ there is a property that *contradicts* p: $[[^0Ess\, [M\, F]]\, \lambda w\lambda t\, [\lambda x\, \neg[p_{wt}x]]]$." This follows from the semantics of privative modification. The privative modifier M not only deprives the property F of one or more of its requisites, it also adds at least one requisite that causes privation.

Remark. The above definition of subsective and privative modifiers is a novel contribution of this paper. It is an improvement over the corresponding definitions in Primiero and Jespersen [28] and Duží et al. [6, §4.4]. As for subsective modifiers, the new definition differentiates between non-trivially and trivially subsective modifiers. As for privatives, the original definition is a logical consequence of this new one, as we are going to prove below. It not only stipulates that among the requisites of the privatively modified property F is the property of not being an F, but also explains why it is so. Furthermore, the new definition also specifies what the modified property and the root property have in common. Privation deprives the root property of some *but not all* of its requisites. The more requisites of the root property F are preserved, the closer a relative the modified property is to F. Thus, we are able to keep track of the root property in the modified property, which in turn makes it possible to prove that, for instance, a demolished damaged house is not a demolished damaged bridge (see below for this example).

Example. The modifier *Wooden*: $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ is subsective with respect to the property of being a table, *Table*: $(o\iota)_{\tau\omega}$, but privative with respect to the property of being a horse, *Horse*: $(o\iota)_{\tau\omega}$. Of course, a wooden table is a table, but the essence of the property $[^0Wooden\, ^0Table]$ is enriched by the property of being wooden. Being wooden is a requisite of the property of being a wooden table, but it is not a requisite of the property of being a table, because tables can be instead made of stone, iron, glass, etc.

$$[^0Ess\, ^0Table] \subset [^0Ess\, [^0Wooden\, ^0Table]]$$

But a wooden horse is not a horse. The modifier *Wooden*, the same modifier that just modified *Table*, deprives the essence of *Horse* of many requisites, for instance of the property of being a living thing, or having a bloodstream, or having kidneys, etc. Hence among the requisites of the property $[^0Wooden\, ^0Horse]$ there are properties like *not being a living thing*, *not having a bloodstream*, etc., which are contradictory (not just contrary) to some of the requisites of the property *Horse*. On the other hand, the property $[^0Wooden\, ^0Horse]$ shares many requisites with the property of being a horse, like the outline of the body, resemblance of a horse, etc., and has the additional requisite of being made of wood. Thus, we have (LT: $(o\iota)_{\tau\omega}$, the property of being a living thing, HB: $(o\iota)_{\tau\omega}$, the property of having blood):

$$\begin{aligned}
&[[^0Ess\, ^0Horse] \cap [^0Ess\, [^0Wooden\, ^0Horse]]] \neq \emptyset\, \wedge \\
&[[[^0Ess\, ^0Horse]\, ^0LT] \wedge [[^0Ess\, [^0Wooden\, ^0Horse]]\lambda w\lambda t\, \lambda x\, \neg[^0LT\, x]]]\, \vee \\
&[[[^0Ess\, ^0Horse]\, ^0HB] \wedge [[^0Ess\, [^0Wooden\, ^0Horse]]\lambda w\lambda t\, \lambda x\, \neg[^0HB\, x]]]\, \vee \\
&\text{etc.}
\end{aligned}$$

At the outset of this paper we characterized the difference between subsective and privative modifiers by means of the rule of *right subsectivity*, which holds for subsective but not privative modifiers: a skillful surgeon is a surgeon; a fake banknote fails to be a bank note.

When $M_s \to ((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ is a construction of a modifier subsective with respect to the property v-constructed by $F \to (o\iota)_{\tau\omega}$, then necessarily and for all individuals x the following rule of right subsectivity (RS) is valid:

$$\frac{[[M_s F]_{wt}\, x]}{[F_{wt}\, x]} \quad RS$$

By Definition 6 it holds that $[{}^0Ess\, F] \subseteq [{}^0Ess\, [M_s F]]$. Hence each requisite of F is also a requisite of $[M_s F]$, but not vice versa, provided M_s is non-trivially subsective. By Definition 4 and Claim 1, since each property is a requisite of itself, it follows that F is a requisite of $[M_s F]$:

$$\forall w \forall t\, [\forall x[[{}^0True_{wt}\, \lambda w \lambda t[[M_s F]_{wt}\, x]] \supset [{}^0True_{wt}\, \lambda w \lambda t[F_{wt}\, x]]]]$$

which proves the rule of right subsectivity (RS).

For privatives, we already suggested replacing Boolean negation by *property negation*, denoted by '*non*', to specify the rule governing privatives. Let $M_p \rightarrow ((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ be a construction of a modifier privative with respect to the property v-constructed by $F \rightarrow (o\iota)_{\tau\omega}$. Then:

$$\frac{[[M_p F]_{wt}\, x]}{[[non\, F]_{wt}\, x]} \quad Priv$$

Of course, it also holds that if x is an $[M_p\, F]$ then it is not the case that x is an F:

$$\frac{[[M_p F]_{wt}\, x]}{\neg[F_{wt}\, x]} \quad Single\ Privation$$

The reason for replacing Boolean negation by property negation is this. For each individual x and for each property F, it is either true that x is an F or it is not true. Yet there are many individuals that are neither an F nor a $[non\, F]$. For instance, each individual either is or is not a banknote. Yet most individuals are neither a banknote nor a fake banknote, because a fake banknote must still have something in common with a banknote. A well-forged banknote is almost a banknote, because the property of being a well-forged banknote is a 'close relative' of the property of being a banknote, sharing many requisites with this property. Hence the property $[non\, F]$ is not contradictory but only contrary to F. Due to the difference between contradictory and contrary properties, the *Priv* rule is indeterministic between the three forks with the third fork having a further measure of indeterminacy, whereas the standard rule of single privation is deterministic. Our strategy being that the *non*-based rule of privation ought to be extended to all instances of single privation, the discrepancy between indeterministic and deterministic rules will vanish, as both the rule of single and the rule of iterated privation will now be indeterministic.

We are now going to define the property negation *non* and prove that *Priv* is valid for privative modifiers.

3.2. Property negation

The philosophical source of inspiration is Aristotle's observations that:

> The sentences "It is a not-white log" and "It is not a white log" do not imply one another's truth. For if "It is a not-white log" is true, it must be a log: but that which is not a white log need not be a log at all. (*Prior Analytics* I, 46, 1.)

That is, in modern parlance, a set of logs divides into those that are white and those that are non-white, whereas a set of non-(white logs) divides into those elements that are non-white logs and those that are not even logs (though perhaps white). More specifically, this quotation has inspired us to adopt property negation. And directly relevant for our present purpose:

From the fact that John is not dishonest we cannot conclude that John is honest, but only that he is possibly so. ([26, p. 255])

The alternative is namely that John is neither dishonest, nor honest, so "John is not dishonest", if true, tells us what John *fails* to be and what the *alternatives* are: (i) being honest, (ii) neither being honest nor being dishonest. The contradictory property is that it is not the case that it is not the case that John is honest, which is logically equivalent to him being honest. More specifically, this quotation has inspired us to introduce the trifurcation of cases presented in the Introduction. This trifurcation is epistemic rather than ontological, as it bears on the (in-)validity of various inferences.

The definition of property negation must encapsulate the *contrariety clause* that the intensional negation of one of two conjuncts that are *mutually exclusive* does not entail the truth of the other conjunct.

Definition 7 *(Contrary properties).* Let $x \to \iota$; $F, G \to (o\iota)_{\tau\omega}$. Then the properties F, G are mutually *contrary* iff

$$\forall w \forall t\, \forall x\, [[F_{wt}\, x] \supset \neg[G_{wt}\, x]] \wedge \exists w \exists t\, \exists x\, [\neg[F_{wt}\, x] \wedge \neg[G_{wt}\, x]]$$

The definition states that it is not possible for x to co-instantiate F and G, and possibly x instantiates neither F, nor G. The left-hand conjunct,

$$\forall w \forall t\, \forall x\, [[F_{wt}\, x] \supset \neg[G_{wt}\, x]]$$

is the clause that F and G are *mutually exclusive*. The second conjunct,

$$\exists w \exists t\, \exists x\, [\neg[F_{wt}\, x] \wedge \neg[G_{wt}\, x]]$$

is the *contrariety* clause that the negation of one of the conjuncts $[F_{wt}\, x]$, $[G_{wt}\, x]$ does not entail the truth of the other one.

Next, we want to show that any property $[M_p F]$ formed from a property F by a modifier M_p privative with respect to F is a property contrary to F. First, we prove the left-hand conjunct:

$$\forall w \forall t\, \forall x\, [[[M_p F]_{wt}\, x] \supset \neg[F_{wt}\, x]]$$

To this end, we apply the second clause of the definition of privative modifiers (Definition 6): $\exists p\, [[[{}^0 Ess\, F]\, p] \wedge$ $[[{}^0 Ess\, [M_p F]]\, \lambda w \lambda t\, [\lambda x\, \neg[p_{wt}\, x]]]]$. Hence the property $[M_p F]$ has among its requisites at least one property contradictory to a requisite of the property F. Let these properties be P and $\lambda w \lambda t\, [\lambda x\, \neg[P_{wt}\, x]]$, respectively. Then at no $\langle w, t\rangle$ is there an individual x that would satisfy both $[[M_p F]_{wt}\, x]$ and $[F_{wt}\, x]$; if there were such an x, then according to Definitions 4 and 5, x would also have to satisfy both $[P_{wt}\, x]$ and $\neg[P_{wt}\, x]$, which is logically impossible.

Remark. This proves that the previous definition found in [6, §4.4] and [28] is a corollary of the new Definition 6.

The contrariety clause $\exists w \exists t\, \exists x\, [\neg[[M_p F]_{wt}\, x] \wedge \neg[F_{wt}\, x]]$ holds due to the thesis of *individual anti-essentialism* which we subscribe to: no individual has any purely contingent property necessarily.

We should not forget, however, the limiting case where F is a *trivial*, non-contingent property with a constant extension, such as *being self-identical*. In this case, necessarily, when the type is (say) ι, F_{wt} is the entire type ι and $\lambda x\, \neg[F_{wt}\, x]$ is an empty ι-set, because at no $\langle w, t\rangle$ is there an individual that would be neither identical with itself nor non-identical with itself. Another example of a non-contingent property is the property *being identical to a or b*.[28] At all $\langle w, t\rangle$ the extension of this property is the set $\{a, b\}$, and at no $\langle w, t\rangle$ is there an individual that would be neither identical with a or b, nor non-identical with a or b, for both a and b are necessarily around to instantiate this property. The upshot is that non-contingent properties do not lend themselves to being modified by privative modifiers on pain of necessary falsehood. Hence, if *Tri* is such a trivial non-contingent property then the extension of $[M_p\, Tri]$ is necessarily the empty ι-set for any modifier privative with respect to *Tri*. Such a modifier turns *Tri* into an 'idle property' that has necessarily an empty extension.

Definition 8 *(General modifier privative with respect to a property f).* Let $=: (o((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})((o\iota)_{\tau\omega}(o\iota)_{\tau\omega}))$ be the identity relation defined over first-order modifiers, $non \to ((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ a variable ranging over first-order modifiers, $f \to (o\iota)_{\tau\omega}$, $Con : (o\,(o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ the relation of contrariety between properties. Then:

$$^0Non = \lambda f \lambda w \lambda t\, [\lambda x\, \exists non\, [[[non\, f]_{wt}\, x] \wedge [^0Con\, [non\, f]\, f]]]$$

is the *general modifier privative with respect to f.*

Remark. Any of the modifiers *non* meeting the condition specified by Definition 8 are privative with respect to the property F. Property negation takes a particular property F to an arbitrary property contrary to it, $[non\, F]$.[29]

Non is thus the unique general privative modifier, and it takes a property F to the general contrary property $[^0Non\, F]$. For instance, $[^0Non\, Banknote]$ is the general property contrary to the property of being a banknote. Necessarily, the extension of $[^0Non\, Banknote]_{wt}$ includes the extensions of the properties *forged banknote*, *banknote dissolved in acid*, *Monopoly banknote*, etc., some of the extensions possibly being empty. One might worry that it is too much to claim that, necessarily, the extension contains the full panoply of non-banknotes. But it follows from Definition 8 that the full panoply is indeed involved. At any $\langle w, t\rangle$, for any individual x and the property constructed by $F \to (o\iota)_{\tau\omega}$, this holds:

$$[[^0Non\, F]_{wt}\, x] = \exists non\, [[[non\, F]_{wt}\, x] \wedge [^0Con\, [non\, F]\, F]]$$

Hence, individual a has the property $[^0Non\, F]$ iff a has any property $[non\, F]$ for some *non* privative with respect to F. Thus, the set $[^0Non\, F]_{wt}$ is *almost* as large as the complement $\overline{F_{wt}}$ of the set F_{wt}. At *some*, but not all, $\langle w, t\rangle$ it is the case that $[^0Non\, F]_{wt} = \overline{F_{wt}}$. Or, when F_{wt} happens to be the entire type ι, then $[^0Non\, F]_{wt}$ must be the empty ι-set, i.e. the union of all empty sets $[^0Non\, F]_{wt}$. Definition 8 does not exclude such modifier functions as do not even have a name in our vernacular.

Contrariety provides the weaker form of negation that is suitable for privative modifiers as explained above. Definition 8 thus justifies the *elimination rule Priv* for modifiers M_p privative with respect to property F stated above:

$$[[M_p F]_{wt}\, x] \vdash [[^0Non\, F]_{wt}\, x]$$

The conclusion of *Priv* states that the predication of F eludes x: F does not get to be predicated of x. For instance, if the premise is that a is a *fake banknote* then the conclusion is that a is a *Non-banknote*, therefore the property banknote is not predicated of a. Or if b is a *wooden horse* then b is a *Non-horse*. But if c is

[28] TIL comes with a constant domain. See [6, 378–379].

[29] Martin [24, p. 449] says, "Semantically [infinite negation, e.g. *non-human*] converts a term into one that stands for its non-empty complement ...". Our property negation does not come with an ontological restriction such as non-emptiness. However, more importantly, our '*non-F*' does not denote a complement set, but a contrary *property*; what denotes a set is '*non-F_{wt}*'.

a *wooden bird* then it follows neither that c is a *Non-horse*, nor that c is a *horse*, because the properties *Non-horse* and *horse* necessarily have still something in common (at least one common requisite), unlike the properties *wooden bird* and *horse* or *Non-horse*.

Moreover, the partial order defined on sets of requisites makes it possible to compare how close the privatively modified property $[M_pF]$ is to F. Since $[M_pF]$ and F have some requisites in common, they are relatives. For instance, a fake banknote is not a fake passport or even a *Non*-passport; they are not relatives. Yet a fake banknote is a close relative of banknotes, closer than, for instance, a Monopoly banknote or a burnt banknote. From this point of view the most distant relative of the property F is thus the property $[{}^0Non\,F]$.

Let us run a test case. Can a paradox be deduced from our theory? Consider this example:

(1) Individual a is a €10 banknote
(2) Whatever is a €10 banknote is a banknote

---

(3) a is a banknote

(a) a is a forged €100 banknote created by adding a '0' to '10' to form '100'
(b) Whatever is a forged banknote is a non-banknote

(c) a is a non-banknote

Contradiction: (3) and (c).

This does not follow, however. One fact is that a is a tampered-with €10 banknote. Yet having a zero add to '10' does not have to undermine a's property as a €10 banknote. Hence, a may remain a banknote, for the modifier *€10* is subsective with respect to the property of being a banknote. Another fact is that a is a forged €100 banknote. From this, however, it does not follow that a is no longer a banknote. It only follows that a is not a €100 banknote. The property that has been compromised by the attempted forgery is that of being a €100 banknote, not the property of being a banknote per se. The apparent paradox arises, because premise (b) fails to state that a is a forged €100 banknote and hence a non-€100 banknote. Therefore, premise (b) becomes irrelevant. Hence, a can be a €10 banknote (and thus a banknote) while being a non-€100 banknote.[30]

3.3. Double privation

We turn next to double privation, which has this form:

$$[M'_p\,[M_p\,F]]$$

Since M_p is privative with respect to $[M_p\,F]$, the intersection of the essences of $[M'_p\,[M_p\,F]]$ and $[M_p\,F]$ must be non-empty. And since M_p is privative with respect to F, the intersection of the essences of $[M_p\,F]$ and F must also be non-empty. One may then wonder whether the respective essences of $[M'_p\,[M_p\,F]]$ and F can be disjunctive. We think not. There must be an overlap of requisites, and not just of any old properties, but of carefully chosen ones.

Recall the earthquakes in central Italy in 2016. Many houses, bridges and other buildings and constructions were damaged, some beyond repair. A demolished damaged house is surely not a house, but debris: a particular object goes through the stages of being a house, then a damaged house and finally a demolished damaged house, which is in material terms nothing but debris. Yet a demolished damaged house is different

[30] We are indebted to Nikolaj Nottelmann and Lars Binderup for discussion of this example.

from a demolished damaged bridge. A demolished damaged house shares requisites with houses that it does not share with demolished damaged bridges.

It may so happen that the essence of $[M'_p\,[M_p\,F]]$ is a superset of the essence of F. In such a case, if x instantiates $[M'_p\,[M_p\,F]]$ then x also instantiates F. For instance, a repaired damaged house is again a house. To repair a damage is to undo the damage and in so doing returning the previously damaged artefact to its still earlier state of functioning properly; such is the semantics of the verb 'to repair' and the adjective 'repaired'. So here we have come full circle back to F. This particular instance of the modifier *repaired* is privative with respect to *damaged house*, because what is a non-house turns into a house. (We are presupposing, to get the example off the ground, that a damaged house is so damaged that it no longer qualifies as a house.) Being a repaired damaged house is one way of being a house. Formally:

$$[[^0 Ess\,^0 House] \subset [^0 Ess\,[^0 Repaired\,[^0 Damaged\,^0 House]]]]$$

Yet it may also so happen that the essence of $[M'_p\,[M_p\,F]]$ and the essence of F have a non-empty intersection, but neither is a subset of the other. For instance, a demolished damaged house is neither a damaged house, nor a house, but something altogether different, namely a pile of rubble. The modifier *demolished*, like *repaired* above, is privative with respect to *damaged house*, but the logical effect of applying it to *damaged house* is the opposite. The semantics of the verb 'to demolish' puts it in opposition to 'to repair' or 'to restore'. Nonetheless, a demolished damaged house must possess the requisite of having previously been a house.

As is seen, the property of being a demolished damaged house spans three states: first, being a house; second, being a damaged house; third, being a demolished damaged house. Formally:

$$[^0 Ess\,[^0 Demolished\,[^0 Damaged\,^0 House]]] \cap [^0 Ess\,^0 House] \neq \emptyset$$

Absent the requisite property of having been previously a house, there is nothing to block the inference that a demolished damaged house is (say) a demolished damaged bridge.

3.4. Three case studies

Here we revisit three examples that were broached above. For better readability of the following formulae, we will now abbreviate formulae for constructions of the form '$\lambda w \lambda t\,[\lambda x\,\neg[p_{wt}\,x]]$' as '*not-p*'.

3.4.1. First fork

Since *damaged* is privative with respect to *house*, we have (as per Definition 6):

$$[[^0 Ess\,^0 House] \cap [^0 Ess\,[^0 Damaged\,^0 House]]] \neq \emptyset$$
$$\wedge\,\exists p\,[[[^0 Ess\,^0 House]\,p] \wedge [[^0 Ess\,[^0 Damaged\,^0 House]]\,\textit{not-p}]]$$

Hence *damaged* has turned some of the requisites of *house* into their opposites. For instance, if one of the requisites of being a house is the property of being a place to live in, then *damaged* turns this property into the property of not being a place to live in. Since *repaired* is privative with respect to the property *damaged house*, we have:

$$[[^0 Ess\,[^0 Damaged\,^0 House]] \cap [^0 Ess\,[^0 Repaired\,[^0 Damaged\,^0 House]]]] \neq \emptyset$$
$$\wedge\ \exists q\,[[[^0 Ess\,[^0 Damaged\,^0 House]]\,q]$$
$$\wedge\ [[^0 Ess\,[^0 Repaired\,[^0 Damaged\,^0 House]]]\,\textit{not-q}]]$$

Now *repaired* cancels the effect of *damaged*; it must turn all those opposites *not-p* of *damaged house* back into the original requisites p of *House*. Thus, among those properties q that are contained in the essence of $[{}^0Damaged\,{}^0House]$ and appear as *not-q* in the essence of $[{}^0Repaired\,[{}^0Damaged\,{}^0House]]$ there must be all those properties p which are contained in the essence of *house* and their opposites *not-p* in the essence of $[{}^0Damaged\,{}^0House]$. As a result, among these properties q there are the properties $\lambda w\lambda t\,[\lambda x\,\neg\neg[p_{wt}\,x]]$, hence p. We obtain:

$$[[{}^0Ess\,{}^0House] \subset [{}^0Ess\,[{}^0Repaired\,[{}^0Damaged\,{}^0House]]]]$$

The property *repaired damaged house* has all the requisites of *house*, being again a place to live in.

3.4.2. Second fork

Contrast the above property with the property *demolished damaged house*. Whatever is a demolished damaged house cannot be a house, for the same reason that a demolished house cannot be a house. As soon as we understand the meaning of the predicate ‘is a demolished damaged house’, we are able to calculate which way it goes, and that we must land on the second fork. So, we know that a demolished damaged house is a non-house. But we know something positive about it, too: we know that it is now a pile of rubble. A demolished damaged house has been physically reduced to its raw matter (wood, steel, brick, etc.), just like a melted-down statue is reduced to its raw matter (bronze, clay, etc.). The internal link between being a demolished damaged house and being a pile of rubble is that that pile of rubble has a noble past as a damaged house and before that as a house.

3.4.3. Third fork

Consider again *former heir apparent*. This combination of privatives is doubly dynamic due to the backward-looking aspect of *former* and the forward-looking aspect of *apparent*, in the special sense of ‘apparent’ as ‘designated to become’. Someone who is a designated F is currently not yet an F, though they are supposed to become one. We are deploying the strict interpretation of ‘former’ as a privative rather than a modal modifier to get the example of former heir apparent off the ground.[31] With that in place, someone who is a former heir apparent is not an heir, for one of two reasons: either the person succeeded in succeeding the previous monarch (promotion), or the person is no longer being even considered for the throne (demotion).

Accordingly, one requisite which *heir apparent* comes with is that any bearer must lack the property of being the successor (where it is understood which is the relevant royal position, e.g. the office of King of Denmark): this requisite property is due to the modification provided by *apparent*. Another requisite which *former heir apparent* comes with is that any bearer must lack the property of being any longer the prospective heir. The backward-looking aspect of *former* voids the forward-looking aspect of *apparent*, which brings us to the present time where the bearer of the property of being a former heir apparent may, or may not, be sitting on the throne.

3.4.4. Summary

To sum up these three case studies, which fork is the right one depends on the *semantics* of the modifiers involved. When faced with iterated privatives, the agents who operate within some interactive system for reasoning on the basis of natural-language texts can request additional information about particular modifiers.[32] The appropriate answer will be a refinement of the modifier in question. For instance, an appropriate refinement of *repaired* would be this:

[31] On the privative reading, from “a is a former F” it can be inferred that a is no longer an F, hence is not an F. On the modal reading, it cannot be excluded that a has been reinstated as an F.

[32] A particular such system is investigated in, e.g., [4] and [10].

$$\forall p\,[[[[^0Ess\,F]\,p] \wedge [[^0Ess[M_pF]]\,not\text{-}p]] \supset [[^0Ess\,[^0Repaired\,[M_pF]]]\,p]]$$

Thus, we can infer that whatever x is a repaired $[M_pF]$ is also an F. Similarly, a supplementary piece of information about the semantics of *demolished* might be this:

$$\exists p\,[[[[^0Ess\,F]\,p] \wedge [[^0Ess[M_pF]]\,not\text{-}p]] \wedge [[^0Ess\,[^0Demolished\,[M_pF]]]\,not\text{-}p]]$$

Then we can infer that a demolished $[M_pF]$ is not an F. If no such refinement can be supplied, then we cannot decide which of the first two forks an individual x lands on, and so we know that we are facing a case of the third fork.

4. Conclusion

The results obtained in this paper amount to an extension of the standard theory of property modification by adding a logic of iterated privation to it. We started out with the problem that the received rule of single privation is too crude, because it turns the root property into the contradictory property. To start solving the problem, we replaced Boolean negation by *property negation*, enabling us to operate on *contrary* rather than contradictory properties.

We then assigned so-called *requisites* to properties, and defined the *essence* of a property as the set of all its requisites. Also, properties formed by means of iterated privation are equipped with requisites. They underpin our *presuppositional* theory of *positive predication*, which is the predication of properties an object must have, as a matter of analytic necessity, if it has a particular privatively modified property.

The notion of requisite properties enabled us to show that properties formed from iterated privatives give rise to a *trifurcation* of cases between returning to the original root property or to a property contrary to it or being semantically undecidable for want of further information. We have thereby exceeded the general insight that pairs of privatives yield contraries rather than contradictories, because we are in a position to calculate which of the forks of the trifurcation we land on.

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Further reading

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