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AXIOMS FOR ACTUALITY

The semantics of modal languages augmented by the operator "actually" have been understood for the last ten years; see, for example, [3] for a discussion of "now" – the analogue of "actually" in tense logic. Here I present a simple axiomatization of the logic of such languages, with an eye to determining their expressive power. For example, "There could be something which actually doesn't exist" is easily expressed with the actuality operator, though by results of [2], it cannot be expressed without it; however, I show that other interesting conditions, e.g., that for every possible world there could be something not existing in that world, are not expressible even with the actuality operator. If unexplained, all terminology and notation are as in [2].

We introduce the operator "@" and extend a modal language $L(C)$ to $L^{@}(C)$ by addition of this formation rule: if ϕ is a formula of $L^{@}(C)$, so is $@\phi$. We work only in S5; the notions of a frame and a structure for $L^{@}(C)$ are as in [2]. Where $\mathfrak{A}$ is a structure for $L^{@}(C)$, w and w' are from $\mathfrak{A}$ and $\bar{a}$ is an assignment for $\mathfrak{A}$, we define $(\mathfrak{A}, w, w') \models \phi[\bar{a}]$, "$\bar{a}$ satisfies ϕ at (w, w') in $\mathfrak{A}$", as follows.

$(\mathfrak{A}, w, w') \not\models \bot[\bar{a}]$;

$(\mathfrak{A}, w, w') \models P[\bar{a}]$ iff $V(w', P) = t$ for P 0-place;

$(\mathfrak{A}, w, w') \models P\sigma_1 \ldots \sigma_n[\bar{a}]$ iff $(a_1, \ldots, a_n) \in V(w', P)$ where $a_i = \text{den}(\mathfrak{A}, \bar{a}, \sigma_i)$ for $i = 1, \ldots, n$ and P n-place, $n \geqslant 1$;

$(\mathfrak{A}, w, w') \models \sigma = \sigma'[\bar{a}]$ iff $\text{den}(\mathfrak{A}, \bar{a}, \sigma) = \text{den}(\mathfrak{A}, \bar{a}, \sigma')$;

$(\mathfrak{A}, w, w') \models (\phi \supset \psi)[\bar{a}]$ iff $(\mathfrak{A}, w, w') \not\models \phi[\bar{a}]$ or $(\mathfrak{A}, w, w') \models \psi[\bar{a}]$;

$(\mathfrak{A}, w, w') \models (\forall\nu)\phi[\bar{a}]$ iff for every $a \in A(w')$, $(\mathfrak{A}, w, w') \models \phi[\bar{a}^{\nu}_{a}]$, where $\bar{a}^{\nu}_{a}$ is the variant of $\bar{a}$ assigning ν to a;

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$$(\mathfrak{A}, w, w') \models \Box\phi[\bar{a}] \text{ iff for every } u \in W, (\mathfrak{A}, w, u) \models \phi[\bar{a}];$$

$$(\mathfrak{A}, w, w') \models @\phi[\bar{a}] \text{ iff } (\mathfrak{A}, w, w) \models \phi[\bar{a}].$$

Where Γ is a set of formulae, $(\mathfrak{A}, w, w') \models \Gamma[\bar{a}]$ iff for all $\phi \in \Gamma$ $(\mathfrak{A}, w, w') \models \phi[\bar{a}]$. Let $(\mathfrak{A}, w) \models \phi[\bar{a}]$ iff $(\mathfrak{A}, w, w) \models \phi[\bar{a}]$. In determining whether $(\mathfrak{A}, w) \models \phi[\bar{a}]$, we unpack ϕ; w remains our "starting world"; modal operators send us to consider questions of the form $(\mathfrak{A}, w, w') \models \psi[\bar{a}]$; w' is then the "focus" world; "@" sends us back to our starting world.

In $L^{@}(C)$ we may express, for example, "There could be something which doesn't actually exist" by "$\Diamond(\exists x)@\neg Ex$". So $L^{@}(C)$ is more expressive than $L(C)$.

Where $\Gamma \cup \{\phi\}$ is a set of formulae of $L^{@}(C)$, Γ implies ϕ iff for all structures $\mathfrak{A}$ for $L^{@}(C)$, w from $\mathfrak{A}$ and $\bar{a}$ an assignment for $\mathfrak{A}$, if $(\mathfrak{A}, w) \models \Gamma[\bar{a}]$ then $(\mathfrak{A}, w) \models \phi[\bar{a}]$. Γ strongly implies ϕ iff for all such $\mathfrak{A}$, $\bar{a}$ and w, w' from $\mathfrak{A}$, if $(\mathfrak{A}, w, w') \models \Gamma[\bar{a}]$ then $(\mathfrak{A}, w, w') \models \phi[\bar{a}]$. ϕ is valid iff the empty set implies ϕ; ϕ is strongly valid iff the empty set strongly implies ϕ. Strong implication implies implication, but not conversely; for example, "$P \equiv @P$" is valid, but not strongly valid. Furthermore, the class of valid formulae is not closed under necessitation; for example, $P \supset @P$ is valid, but $\Box(P \supset @P)$ is not. The class of strongly valid formulae is closed under necessitation.

Let $\mathscr{A} = \{\Diamond @\phi \supset \phi \mid \phi \text{ a formula of } L^{@}(C)\}$. To axiomatize the class of strongly valid formulae, augment and axiomatization of quantified S5 presented in [1] by adding all formulae of the following forms to our list of axioms:

(@1) $\Diamond(\phi_1 \mathbin{\&} \ldots \mathbin{\&} \phi_n)$ for $\phi_1, \ldots, \phi_n \in \mathscr{A}$;

(@2) $(@\phi \supset @\psi) \supset @(\phi \supset \psi)$;

(@3) $@\bot \supset \bot$.

Theoremhood is defined by closing these axioms under Modus Ponens, Universal Generalization and Necessitation. Let $\Gamma \vdash \phi$ iff either $\vdash \phi$ or for some $\psi_1, \ldots, \psi_n \in \Gamma$, $\vdash (\psi_1 \mathbin{\&} \ldots \mathbin{\&} \psi_n) \supset \phi$.

We point out several sorts of theorems of this axiomatization.

(1) $\vdash \neg @\phi \supset @\neg\phi$; since $\vdash \bot \supset @\bot$, $\vdash \neg @\phi \supset (@\phi \supset @\bot)$; our claim follows from the axiom $(@\phi \supset @\bot) \supset @\neg\phi$.

(2) $\vdash @(\phi \supset \psi) \supset (@\phi \supset @\psi)$; we need these facts:

$$\vdash (@(\phi \supset \psi) \,\&\, @\phi \,\&\, \neg @\psi) \supset \Diamond(@(\phi \supset \psi) \,\&\, @\phi \,\&\, @\neg \psi),$$
using fact (1);

$$\vdash \Diamond(@(\phi \supset \psi) \,\&\, @\phi \,\&\, @\neg \psi) \supset \Box(\Diamond @(\phi \supset \psi) \,\&\, \Diamond @\phi \,\&\, \Diamond @\neg \psi);$$

using axiom (@1),

$$\vdash \Box(\Diamond @(\phi \supset \psi) \,\&\, \Diamond @\phi \,\&\, \Diamond @\neg \psi) \supset \Diamond((\phi \supset \psi) \,\&\, \phi \,\&\, \neg \psi);$$

but clearly $\vdash \neg\Diamond((\phi \supset \psi) \,\&\, \phi \,\&\, \neg \psi)$, proving (2);

(3) $\vdash @\neg\phi \equiv \neg @\phi$, putting together (1), (2), and (@3).

Clearly all axioms are strongly valid, and our rules preserve strong validity. Thus our axiomatization is sound with respect to strong validity: if $\vdash \phi$ then ϕ is strongly valid. As usual, Γ is consistent iff $\Gamma \nvdash \bot$ To prove completeness, we turn to the appropriate version of Henkin's lemma.

HENKIN'S LEMMA 1. If Γ is a consistent set of sentences of $L^{@}(C_0)$ then there is a structure $\mathfrak{A} = (W, A, V)$ for $L^{@}(C_0)$ and $w_0, w_1 \in W$ so that $(\mathfrak{A}, w_0, w_1) \models \Gamma$.

We use a version of the method of diagrams from [1].

Let $\kappa = \max \{\aleph_0, \text{card}(\text{Pred}), \text{card}(C_0)\}$. Fix sets W and $C \supseteq C_0$, $\text{card}(W) = \text{card}(C - C_0) = \kappa$; fix w_0, w_1, distinct members of W. A diagram is a set of ordered pairs (w, ϕ), $w \in W$ and ϕ a sentence of $L^{@}(C)$. A diagram D is consistent iff $\Diamond D = \cup \{\Diamond D(w) \mid w \in W\}$ is consistent, where $D(w) = \{\phi \mid (w, \phi) \in D\}$ and $\Diamond D(w) = \{\Diamond(\phi_1 \,\&\, \ldots \,\&\, \phi_n) \mid \phi_1, \ldots, \phi_n \in D(w)\}$. We review three familiar facts:

(1) if D is consistent then either $D \cup \{(w, \phi)\}$ or $D \cup \{(w, \neg\phi)\}$ is consistent;

(2) if D is consistent, $(\exists \nu)\phi \in D(w)$ and $c \in C$ does not occur in D then $D \cup \{(w, \phi(\nu/c)), (w, Ec)\}$ is consistent;

(3) if D is consistent, $\Diamond\phi \in D(w)$ and $w' \in W$ does not occur in D then $D \cup \{(w', \phi)\}$ is consistent.

Furthermore, if $\mathscr{A} \subseteq D(w_0)$, $@\phi \in D(w)$ and D is consistent then $\neg\phi \notin D(w_0)$; for in this case, $\Diamond D \vdash \Diamond @\phi$, so $\Diamond D \vdash \Box\Diamond @\phi$; if $\neg\phi \in D(w_0)$, since $\Diamond @\phi \supset \phi \in D(w_0)$, $\Diamond D \vdash \Diamond(\neg\phi \,\&\, \Diamond @\phi \,\&\, (\Diamond @\phi \supset \phi))$; so D is inconsistent.

Suppose Γ is consistent. Let $D_0 = (\{w_0\} \times \mathscr{A}) \cup (\{w_1\} \times \Gamma\}$. Since all members of $\Diamond A$ are axioms and Γ is consistent, D_0 is consistent. As in [1], we construct a sequence $\{D_\xi\}_{\xi \leqslant \kappa}$ of consistent diagrams over W, C, $D_\xi \subseteq D_\eta$ if $\xi \leqslant \eta$, so that D_κ is $\neg$-complete, $\exists$-complete and $\Diamond$-complete. D_κ may be converted into a structure $\mathfrak{A} = (W, A, V)$ for $L^@(C)$ so that for all $w \in W$ and ϕ a sentence of $L^@(C)$:

$$(\mathfrak{A}, w_0, w) \models \phi \text{ iff } \phi \in D_\kappa(w).$$

This follows as usual; we sketch the one novel case: ϕ is $@\psi$. Suppose $(\mathfrak{A}, w_0, w) \models @\psi$; then $(\mathfrak{A}, w_0, w_0) \models \psi$; so $\psi \in D_\kappa(w_0)$. By a previous remark, $@\neg\psi \notin D(w)$; thus $\neg @\psi \notin D(w)$; so $@\psi \in D(w)$. Suppose $@\psi \in D(w)$; since $\neg\psi \notin D(w_0)$, $\psi \in D(w_0)$; so $(\mathfrak{A}, w_0, w_0) \models \psi$; thus $(\mathfrak{A}, w_0, w) \models @\psi$.

Therefore $(\mathfrak{A}, w_0, w_1) \models \Gamma$; so the reduct of $\mathfrak{A}$ to $L^@(C_0)$ is as desired. Q.E.D.

COROLLARY. If ϕ is strongly valid then $\vdash \phi$.

To axiomatize the class of valid formulae, we show that ϕ is valid iff $\mathscr{A} \vdash \phi$. Members of $\mathscr{A}$ are axioms whose status differs from that of our other axioms. We don't have: if $\mathscr{A} \vdash \phi$ then $\mathscr{A} \vdash \Box\phi$. The soundness of this axiomatization is obvious. We show completeness.

HENKIN'S LEMMA 2. Let Γ be a set of sentences of $L^@(C_0)$. If $\Gamma \cup \mathscr{A}$ is consistent then there is a structure $\mathfrak{A}$ for $L^@(C_0)$ and $w_0 \in W$ so that $(\mathfrak{A}, w_0) \models \Gamma$.

Fix κ, W and C as before. Select $w_0 \in W$ and let $D_0 = \{w_0\} \times (\Gamma \cup \mathscr{A})$. Construct $\{D_\xi\}_{\xi \leqslant \kappa}$ as before. Again we have for all $w \in W$ and sentences ϕ of $L^@(C)$:

$$(\mathfrak{A}, w_0, w) \models \phi \text{ iff } \phi \in D_\kappa(w).$$

Thus $(\mathfrak{A}, w_0, w_0) \models \Gamma$. Q.E.D.

COROLLARY. If ϕ is valid then $\mathscr{A} \vdash \phi$.

Let T_0 be the set of universal closures of all formulae of $L^@(C_0)$ of the form:

$$\phi \supset \Box(\forall x_1)\Box \ldots \Box(\forall x_n)\Diamond(\phi \,\&\, Ex_1 \,\&\, \ldots \,\&\, Ex_n),$$

where $x_1, \ldots, x_n$ are not free in ϕ.

THEOREM 1. Let T be a set of sentences of $L^@(C_0)$.

(i) $T \cup T_0 \cup \{(\exists x)\ @\ \neg Ex\}$ is consistent iff there is a structure $\mathfrak{A} = (W, A, V)$ for $L^{@}(C_0)$ and $w_0, w_1 \in W$ so that $(\mathfrak{A}, w_0, w_1) \models T$ and $A(w_0) \subsetneq A(w_1) = \bar{A}$.

(ii) $T \cup \mathscr{A} \cup \Diamond(T_0 \cup \{(\exists x)\ @\neg Ex\})$ is consistent iff there is a structure $\mathfrak{A} = (W, A, V)$ for $L^{@}(C_0)$ and $w_0, w_1 \in W$ so that $(\mathfrak{A}, w_0) \models T$ and $A(w_0) \subsetneq A(w_1) = \bar{A}$.

Proof. For the "if" direction, observe that if $\mathfrak{A} = (W, A, V)$ is a structure for $L^{@}(C_0)$ and $A(w_0) \subsetneq A(w_1) = \bar{A}$, then $(\mathfrak{A}, w_0, w_1) \models T_0 \cup \{(\exists x) @ \neg Ex\}$.

For the "only if" direction, we use the technique of Theorem 1 of [2]. Fix κ, W and C as before. Select $w_0, w_1 \in W$, $w_0 \neq w_1$. To prove (i), suppose that $T \cup T_0 \cup \{(\exists x) @ \neg Ex\}$ is consistent. Let $D_0 = (\{w_0\} \times \mathscr{A}) \cup (\{w_1\} \times (T \cup T_0 \cup \{(\exists x) @ \neg Ex\}))$. D_0 is consistent. We define the usual sequence of consistent diagrams $\{D_\xi\}_{\xi \leqslant \kappa}$ over W, C as in the proof of Henkin's Lemma 1, except that we ensure that for every $c \in C$ occurring in D_ξ, $Ec \in D_\xi(w_1)$. The fact that $T_0 \subseteq D_0(w_1)$ makes it possible to do this without losing consistency. For details, see the proof of Theorem 1 in [1]. D_κ yields the desired structure.

To prove (ii), suppose that $T \cup \mathscr{A} \cup \Diamond(T_0 \cup \{(\exists x) @ Ex\})$ is consistent. Let

$$D_0 = (\{w_0\} \times (T \cup \mathscr{A})) \cup (\{w_1\} \times (T_0 \cup \{(\exists x) @ Ex\})).$$

D_0 is consistent. We define the usual sequence of diagrams $\{D_\xi\}_{\xi \leqslant \kappa}$ meeting the previously mentioned constraint; D_κ yields the desired structure.

Q.E.D.

We now construct a structure $\mathfrak{A} = (W, A, V)$ for $L^{@}(C_0)$ so that for some $w_0 \in W$ and any $w \in W$, $w \neq w_0$,

$(\mathfrak{A}, w_0, w) \models T_0 \cup \{(\exists x) @ \neg Ex\}$,
$A(w_0)$ is not a subset of $A(w)$.

Let Z be the set of integers, $w_0 \notin Z$; let $W = \{w_0\} \cup Z$. Select $A(w)$ for $w \in W$ so that:

for all $w \in W$, $A(w)$ is countably infinite;
$A(i) \subsetneq A(i+1)$ for $i \in Z$;
$A(w_0) \subseteq \cup\{A(i) | i \in Z\}$;
as i varies $\mathrm{card}(A(w_0) \cap (A(i+1) - A(i)))$ and

$$\text{card}((A(i+1) - A(i)) - A(w_0)) \text{ are constant and non-zero.}$$

Let $V''C_0 \subseteq \cap \{A(i) | i \in Z\}$, and $V(w, P) = f$ or be empty, for $P \in \text{Pred}$. Clearly $A(w_0) \nsubseteq A(i)$ for $i \in Z$. We show that for such i, $(\mathfrak{A}, w_0, i) \models T_0$. For $a_1, \ldots, a_m \in A(i)$ select an automorphism σ on $\bar{A}$ so that $\sigma''A(w_0) = A(w_0)$, $\sigma''A(i) = A(i+1)$ for all $i \in Z$ and σ is constant on $V''C_0 \cup \{a_1, \ldots, a_m\}$. Our constraints on A and V ensure that such a σ exists. We easily show that for any $b_1, \ldots, b_k \in \bar{A}$ and ϕ a formula of $L^{@}(C_0)$ with k free variables,

$$(\mathfrak{A}, w_0, i) \models \phi[b_1, \ldots, b_n] \text{ iff}$$
$$(\mathfrak{A}, w_0, i+1) \models \phi[\sigma(b_1), \ldots, \sigma(b_k)].$$

Suppose $(\mathfrak{A}, w_0, i_0) \models \phi[a_1, \ldots, a_m]$. Given $a_{m+1}, \ldots, a_{m+n} \in \bar{A}$ select $j \in \omega$ so that $a_{m+1}, \ldots, a_{m+n} \in A(i_0 + j)$. Then

$$(\mathfrak{A}, w_0, i_0 + j) \models \phi[\sigma^j(a_1), \ldots, \sigma^j(a_m)] \quad \text{and}$$
$$(\mathfrak{A}, w_0, i_0 + j) \models (Ex_1 \,\&\ldots\&\, Ex_n)[a_{m+1}, \ldots, a_{m+n}].$$

Since $\sigma^j(a_k) = a_k$ for $k = 1, \ldots, m$, we have shown that

$$(\mathfrak{A}, w_0, i_0) \models \Box(\forall x_1)\Box \ldots \Box(\forall x_n)\Diamond(\phi \,\&\, Ex_1 \,\&\ldots\&\, Ex_n)$$
$$[a_1, \ldots, a_m].$$

Thus $(\mathfrak{A}, w_0, i_0) \models T_0$.

THEOREM 2. (i) There is no set T of sentences of $L^{@}(C_0)$ so that for all structures $\mathfrak{A} = (W, A, V)$ for $L^{@}(C_0)$ and all $w_0, w \in W$, $(\mathfrak{A}, w_0, w) \models T$ iff $A(w_0) \subsetneqq A(w)$.

(ii) There is no set T of sentences of $L^{@}(C_0)$ so that for all structures $\mathfrak{A} = (W, A, V)$ for $L^{@}(C_0)$ and all $w_0 \in W$, $(\mathfrak{A}, w_0) \models T$ iff for some $w \in W$ $A(w_0) \subsetneqq A(w)$.

(ii) shows that "There could be something non-actual without there not being something actual" is not expressible in $L^{@}(C_0)$.

Proof of (i). Suppose the for any such $\mathfrak{A}$, w_0, w if $(\mathfrak{A}, w_0, w) \models T$ then $A(w_0) \subsetneqq A(w)$. Taking the $\mathfrak{A}$ and w_0 of our previous example, $(\mathfrak{A}, w_0, i) \not\models T$ for any $i \in Z$. Fix such an i and select $\phi \in T$ so that $(\mathfrak{A}, w_0, i) \models \neg\phi$. Thus $\{\neg\phi\} \cup T_0 \cup \{(\exists x) @ \neg Ex\}$ is consistent. Theorem 1(i) delivers a structure $\mathfrak{B}$ and u_0, u_1 from $\mathfrak{B}$ so that $B(u_0) \subsetneqq B(u_1)$ but $(\mathfrak{B}, u_0, u_1) \models \neg\phi$, and so $(\mathfrak{B}, u_0, u_1) \not\models T$.

Proof of (ii). Suppose that for any appropriate $\mathfrak{A}$ and w_0, if $(\mathfrak{A}, w_0) \models T$ then for some w, $A(w_0) \subsetneq A(w)$. Taking the $\mathfrak{A}$ and w_0 of our previous example, $(\mathfrak{A}, w_0) \not\models T$. Select $\phi \in T$ so that $(\mathfrak{A}, w_0) \models \neg\phi$. Thus $\{\neg\phi\} \cup \mathscr{A} \cup \Diamond(T_0 \cup \{(\exists x) @\neg Ex\})$ is consistent. Theorem 1(ii) delivers a structure $\mathfrak{B}$ and a u_0 from $\mathfrak{B}$ so that $(\mathfrak{B}, u_0) \not\models T$ but for some u_1, $B(u_0) \subsetneq B(u_1)$.

Q.E.D.

THEOREM 3. (i) There is no set T of sentences of $L^{@}(C_0)$ so that for all structures $\mathfrak{A} = (W, A, V)$ for $L^{@}(C_0)$ and all $w_0, w_1 \in W$, $(\mathfrak{A}, w_0, w_1) \models T$ iff for every $w \in W$ there is a $w' \in W$ so that $A(w')$ is not a subset of $A(w)$.

(ii) There is no set T of sentences of $L^{@}(C_0)$ so that for all structures $\mathfrak{A} = (W, A, V)$ for $L^{@}(C_0)$ and $w_0 \in W$, $(\mathfrak{A}, w_0) \models T$ iff for every $w \in W$ there is a $w' \in W$ so that $A(w')$ is not a subset of $A(w)$.

(ii) shows that the necessitation of the proposition expressed by $\Diamond(\exists x) @\neg Ex$ is unexpressible.

Proof of (i). Suppose that for any such $\mathfrak{A}$, w_0 and w_1, if for every $w \in W$ there is a $w' \in W$ so that $A(w') \not\subseteq A(w)$ then $(\mathfrak{A}, w_0, w_1) \models T$. Because the structure in our previous example has this property, $T \cup T_0 \cup \{(\exists x) @ \neg Ex\}$ is consistent. Theorem 1(i) yields a structure $\mathfrak{A} = (W, A, V)$ and $w_0, w_1 \in W$ with $(\mathfrak{A}, w_0, w_1) \models T$ although there is no w' so that $A(w') \not\subseteq A(w_1)$.

Proof of (ii). Suppose that for any such $\mathfrak{A}$ and w_0, if for every $w \in W$ there is a $w' \in W$ so that $A(w') \not\subseteq A(w)$ then $(\mathfrak{A}, w_0) \models T$. Our previous example then shows the consistency of $T \cup \mathscr{A} \cup \Diamond(T_0 \cup \{(\exists x) @\neg Ex\})$. Theorem 1(ii) then yields a structure $\mathfrak{A}$, w_0 and w_1 so that $(\mathfrak{A}, w_0) \models T$ but there is no w' from $\mathfrak{A}$ so that $A(w') \not\subseteq A(w_1)$. Q.E.D.

Similar arguments extend other inexpressibility results from [2] concerning $L(C_0)$ to $L^{@}(C_0)$.

One final observation on the expressive power of $L^{@}(C_0)$. Suppose we extend $L^{@}(C_0)$ to $L^{@,\dot\forall}(C_0)$ by introducing the "possibilist" universal quantifier $\dot\forall$; we define satisfaction with this additional clause:

$$(\mathfrak{A}, w, w') \models (\dot\forall v)\phi[\bar a] \quad \text{iff} \quad \text{for every } a \in \bar A, (\mathfrak{A}, w, w') \models \phi[\bar a^v_a].$$

Suppose ϕ is a formula of $L^{@,\dot\forall}(C_0)$ in which no occurrence of $\dot\forall$ is in the scope of an occurrence of $\Box$. Then there is a formula ϕ' of $L^{@}(C_0)$ equivalent to ϕ. To obtain ϕ', use this equivalence:

$$(\dot{\forall}\nu)\psi \text{ is equivalent to } \Box(\forall\nu) @ \psi.$$

Notice that the above constraint on occurrences of $\dot{\forall}$ is essential for this result, and that no similar result holds for strong equivalence.

REFERENCES

[1] Kit Fine, 'Model Theory for Modal Logic, Part 1', *Journal of Philosophical Logic* 7 (1978), 125–156.
[2] Harold Hodes, 'Some Theorems on the Expressive Limitations of Modal Languages', *Journal of Philosophical Logic* 13 (1984), 13–26 (this issue).
[3] Hans Kamp, 'Formal Properties of "Now"', *Theoria* 37 (1972), 227–273.

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