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Refutation Systems in Modal Logic

Abstract. Complete deductive systems are constructed for the non–valid (refutable) formulae and sequents of some propositional modal logics. Thus, complete syntactic characterizations in the sense of Lukasiewicz are established for these logics and, in particular, purely syntactic decision procedures for them are obtained. The paper also contains some historical remarks and a general discussion on refutation systems.

Introduction: historical remarks

Formal logic traditionally deals with deductive systems for inference of statements valid according to certain semantics, and is not involved in the inference of the non–valid ones. As Lukasiewicz points out in [9], out of the two intellectual acts - acceptance and rejection of a statement, the latter one has been neglected in the modern formal logic. This neglect seems strange, moreover because the father of the formal logic, Aristotle, already noticed the importance of the systematic rejection of non–valid arguments. He realized that, in order to show that a syllogism was not universally valid, it was not necessary to construct a "refuting model", i.e. an example where the syllogism produces an obviously false conclusion from true premises. Instead, it was enough to infer that syllogism from others, the validity or non–validity of which had already been established, applying certain rules of inference. The typical rule used by Aristotle for that purpose was the so called "modus tollens": If A implies B, and B is rejected, then A is rejected too. Thus, he established a sort of deductive system for rejection of non–valid syllogisms. We shall call deductive systems which infer refutable statements instead of valid ones *refutation systems*. The statements which are inferred, i.e. provably refutable in such a system will be called *rejected* statements.

The history of refutation systems, unlike the history of the "orthodox" ones, according to the author's knowledge, is rather short and scanty. Lukasiewicz in [9], raising the general problem of the formal deduction of the non–valid statements of a given theory, suggested a complete refutation system for the non–valid classical propositions. The system is a very natural one: the only axiom is "p is rejected" where p is a fixed propositional variable,

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and there are two rules: the above mentioned modus tollens and "inverse substitution": if a substitution instance of a formula A is rejected then A itself is rejected. Further, Łukasiewicz pointed out that Aristotle's refutation system was incomplete. He added two additional non–valid syllogisms and showed that the obtained system could reject every non–valid syllogism, but not every non–valid meaningful expression in the language of the syllogistic. Moreover, Słupecki (see [9]) showed that no finite number of axioms added to Łukasiewicz's two rules would suffice for that purpose. He then invented a particular rule which did the job. Słupecki, Bryll, Wybraniec–Skardowska and others from Słupecki's school developed a general theory, in Tarski's style, of rejected propositions and investigated in detail the properties of the corresponding consequence relation (see [17],[18]) . Also, Słupecki and Bryll [16] constructed a complete refutation system for the propositional modal logic S5, and Bryll and Maduch [2] proposed refutation axioms for Łukasiewicz's many–valued logics.

Besides his work on refutable syllogisms and classical propositions Łukasiewicz tried to axiomatize the intuitionistically refutable propositions (see [8]). He conjectured that a complete deductive system for them could be obtained if the classical refutation system was extended with the "disjunction rule": if A is rejected and B is rejected then $A \vee B$ is rejected (a contraposition of the disjunction property of the intuitionistic logic: if $A \vee B$ is intuitionistically valid then either A or B is intuitionistically valid). Later on, Kreisel and Putnam ([7]) refuted Łukasiewicz's conjecture, producing an example of a proper extension of the intuitionistic logic in which the disjunction property holds and thus no tautology of this extension which is not a tautology of the intuitionistic logic could be rejected by the system proposed by Łukasiewicz. In 1957 D. Scott [10] proposed an infinite family of non–structural rules which, added to Łukasiewicz's classical refutation system, yielded a complete refutation system for the intuitionistic logic INT. Recently this result has been essentially improved by Skura [12] who introduced an infinite family of structural rules and thus obtained a complete refutation system in Hilbert style for INT. A closely related system, but based on semantic tableaux was introduced by Dutkiewicz [3]. Other results on refutation systems, including such a system for the propositional fragment of Leśniewski's ontology, are due to a group of Japanese logicians including Inoue, Ishimoto and Kobayashi ([5], [6]). Skura in [13] has introduced a general method for construction of a sort of refutation systems for equivalential logics, and in [14] he has discussed decision procedures rendered by refutation systems.

The purpose of the present paper is to introduce refutation systems for some propositional modal logics and to illustrate general methods for proving completeness of such refutation systems, based on suitable semantic characterization. In section 1 sentential refutation systems are proposed for **K**, **T**, **K4**, **KW** (the logic of provability in PA), **S4Grz** and others. In section 2 sequential refutation systems are introduced for the classical propositional calculus and for some modal logics including **K** and **KW**.

Hereafter, by modal logic we shall mean a propositional normal modal logic, i.e. an axiomatic extension of the minimal normal logic **K**. All background in modal logic, necessary for this paper can be found in the initial chapters of [1], [4] or [11].

1. Sentential refutation systems

A formal theory can be characterized syntactically, by means of certain deductive system, as the set of derivable (provable) formulae, or semantically, by means of a certain class of models and a notion of validity in them, as the set of all valid formulae. Accordingly, there are (at least) two ways to define "refutable" formulae of a theory: syntactically — those which are unprovable in the corresponding deductive system, or semantically — those which are not valid, i.e. are refuted in some model. When the deductive system for the theory is complete for the given semantics, the two ways give rise to the same notion of refutable formula. As the modal systems considered in the paper will be introduced syntactically, *refutable* will mean *unprovable*. As all of them are complete with respect to their Kripke semantics, *refutable* will also mean *non–valid* in this semantics. Thus, henceforth we shall use the notion "refutable" without specifying whether we mean its syntactic or semantic version.

The propositional language $\mathcal{L}$, which we deal with, consists of a countable set of propositional variables P and the logical symbols $\top, \bot, \wedge, \vee, \neg, \rightarrow$ and $\Box$. $\Diamond$ is introduced as $\Diamond = \neg\Box\neg$. The *modal depth* of a formula is defined as the maximal length of a chain of nested modalities occurring in the formula. The formulae with depth 0 (i.e. not containing modalities) are called $\Box$*–free*. We shall use the sign $\Vdash$ to denote validity at the root of a generated model, while $\models$ will denote the usual validity i.e. $\langle F, V\rangle \Vdash \varphi$ means $\langle F, V\rangle \models \varphi[r(F)]$ where $r(F)$ is the root of the frame F. Hereafter, by λ we shall denote an arbitrary $\Box$–free formula.

Definition. A (sentential) *refutation system* is every pair consisting of a set of axioms of the kind $\dashv \varphi$ and a set of *refutation rules* for inference of the kind

$$\frac{\vdash \varphi_1, \ldots, \vdash \varphi_k, \dashv \psi_1, \ldots, \dashv \psi_n}{\dashv \psi}.$$

Definition. Let S be a propositional (not necessarily modal) logic in the language $\mathcal{L}$, specified by some axiomatization of its tautologies.

1. A *refutation system* for S is a refutation system $\mathbf{R}$ in which $\vdash \theta$ is interpreted as "θ is provable in S" or "θ is a tautology of S" and $\dashv \theta$ is interpreted as "θ is refutable in (or rejected by) S".

2. Let $\mathbf{R}$ be a refutation system for S. An *inference in* $\mathbf{R}$ is every finite sequence of formulae $\alpha_0, \ldots, \alpha_n$, such that every α_i, is either an axiom of $\mathbf{R}$ or is obtained from $\alpha_0, \ldots, \alpha_{i-1}$, the axioms of $\mathbf{R}$ and the tautologies of S, accordingly applying some refutation rule of $\mathbf{R}$. The last formula α of every inference in $\mathbf{R}$ is called *S-rejected by* $\mathbf{R}$ *formula*, denoted $S \dashv_{\mathbf{R}} \alpha$. When the refutation system is fixed we only write $S \dashv \alpha$.

3. A refutation system $\mathbf{R}$ is *correct for* S if only non-tautologies of S are S-rejected by $\mathbf{R}$. We call a refutation system $\mathbf{R}$ *complete for S in sense of Lukasiewicz* or Ł-*complete for* S for short (Ł-decidable in [3] and [16]) if for every formula φ, exactly one of $S \vdash \varphi$ and $S \dashv \varphi$ holds.

Note, that the same refutation system can be attached to different logics. Moreover, it can be correct and even complete for more than one logic, since each logic carries its own interpretation of the inference operator $\vdash$, and different operators $\vdash$ applied to the same refutation system generate different sets of rejected formulae.

Here is the refutation system **CPC*** for the classical propositional calculus, essentially introduced by Lukasiewicz:

Axiom:

F: $\dashv \bot$;

Rules:

reverse substitution **RS**: $\frac{\dashv \sigma(\varphi)}{\dashv \varphi}$ for any uniform substitution σ,

modus tollens **MT**: $\frac{\vdash \varphi \to \psi, \quad \dashv \psi}{\dashv \varphi}$.

Obviously, the system $\mathbf{CPC}^*$ is correct for every consistent modal logic, i.e. it rejects only formulae that are non–valid in the logic. Moreover, the following holds:

THEOREM 1.1. *The systems* $\mathbf{CPC}^*$ *is L–complete for (and only for) the two maximal normal modal logics:* $\mathbf{K} + \Box\bot$ *and* $\mathbf{K} + p \leftrightarrow \Box p$.

(This result corresponds to the fact that $\mathbf{CPC}^*$, considered in the classical propositional language, is L–complete for (and only for) the classical logic, as shown by Lukasiewicz [9].)

PROOF. We inductively define two translations, $'$ and $''$, of the modal formulae into $\Box$–free formulae, as follows:

$(q)' = (q)'' = q$, if q is $\top, \bot$ or a propositional variable;

$(\neg\varphi)' = \neg\varphi'$, $(\neg\varphi)'' = \neg\varphi''$;

$(\varphi \circ \psi)' = \varphi' \circ \psi'$, $(\varphi \circ \psi)'' = \varphi'' \circ \psi''$, where $\circ$ is $\wedge, \vee$ or $\rightarrow$;

$(\Box\varphi)' = \top, (\Box\varphi)'' = \varphi''$.

Since $\mathbf{K}+\Box\bot \vdash \Box\theta \leftrightarrow \top$ for any θ, then (using the theorem of equivalent replacement) $\mathbf{K} + \Box\bot \vdash \varphi \leftrightarrow \varphi'$. Likewise $\mathbf{K} + p \leftrightarrow \Box p \vdash \varphi \leftrightarrow \varphi''$.

Now, if $\mathbf{K} + \Box\bot \not\vdash \varphi$ then $\mathbf{K} + \Box\bot \not\vdash \varphi'$ hence $\mathbf{CPC} \not\vdash \varphi'$ (every consistent classical modal logic is a conservative extension of $\mathbf{CPC}$) and therefore $\mathbf{CPC} \dashv \varphi'$ by completeness of $\mathbf{CPC}^*$. Moreover, $\mathbf{K} + \Box\bot \dashv \varphi'$ whence $\mathbf{K} + \Box\bot \dashv \varphi$, by $\mathbf{MT}$. By the same argument, if $\mathbf{K} + p \leftrightarrow \Box p \not\vdash \varphi$ then $\mathbf{K} + p \leftrightarrow \Box p \dashv \varphi$.

Now let $\mathbf{L}$ be any normal modal logic. $\mathbf{L}$ is contained in a maximal one, say in $\mathbf{K}+\Box\bot$, hence $\mathbf{CPC}^*$ applied to $\mathbf{L}$, will only derive refutable formulae for $\mathbf{K} + \Box\bot$, but no tautologies of $\mathbf{K} + \Box\bot$ refutable in $\mathbf{L}$. Therefore $\mathbf{CPC}^*$ is complete only for the two maximal normal modal logics. ■

Now we shall illustrate two more general methods for proving L–completeness. The first one (see the proof of Theorem 1.2 below) is syntactic, based on some uniform presentation of the formulas, and is suitable for particular cases. A similar idea is used in [16] where an L–complete system for **S5** is presented. The second method (Theorem 1.3) is semantic and is applicable in a more general situation. Its idea is close to the approach in [12] (where an L–complete system for the intuitionistic calculus is given), but uses Kripke semantics rather than an algebraic one. Both methods however, are essentially based on suitable semantic characterizations of the logics under consideration. Other approaches, using semantic tableaux are followed in [3] and [6].

THEOREM 1.2.

1) The system **CPC*** *extended with the axiom* $\dashv \Diamond\top$ *and the rule*

$$\mathbf{R_K}: \quad \frac{\dashv \lambda, \dashv \psi \vee \theta_1, \ldots, \dashv \psi \vee \theta_k}{\dashv \lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k \vee \Diamond\psi}$$

is L–complete for the logic **K**.

2) The system **CPC*** *extended with the axiom* $\dashv \Diamond\top$ *and the rule*

$$\mathbf{R_{KW}}: \quad \frac{\dashv \lambda, \dashv \Diamond\psi \vee \psi \vee \theta_1, \ldots, \dashv \Diamond\psi \vee \psi \vee \theta_k}{\dashv \lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k \vee \Diamond\psi}$$

is L–complete for the logic $\mathbf{KW} = \mathbf{K} + \Box(\Box p \to p) \to \Box p$.

PROOF. First we define a *normal modal form* (NMF for short):

i) every $\Box$–free formula is in NMF;

ii) every conjunction of formulae of the kind $\lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k \vee \Diamond\psi$ or $\lambda \vee \Diamond\psi$, where λ is $\Box$–free and ψ and θ's are in NMF, is itself in NMF.

One can prove by an easy induction on the depth of formulae that for every formula φthere exists a formula θ in NMF such that $\mathbf{K} \vdash \varphi \leftrightarrow \theta$.

Now we start with 2). The proof is hung on the fact that **KW** is complete with respect to all finite irreflexive transitive tree–like frames, or **KW**–trees, for short (see [11]). Let us give an exact definition:

i) Every frame of the kind $T_0 = \langle\{x\}, \emptyset\rangle$ is a **KW**–*tree* with a *root* $r(T_0) = x$ and a *length* $l(T_0) = 0$. We call such trees *trivial*; they will be freely identified with their roots.

ii) Let $T_1, \ldots, T_k$ be disjoint **KW**–trees, $T_i = \langle W_i, R_i\rangle$ for $i = 1, \ldots, k$, with corresponding roots $x_1, \ldots, x_k$ and let x not belong to any of $T_1, \ldots, T_k$. Then the frame

$$T = (x; T_1 \ldots, T_k) = \langle\{x\} \cup \bigcup_{i=1}^{k} W_i, \bigcup_{i=1}^{k} R_i \cup \{\langle x, y\rangle / y \in \bigcup_{i=1}^{k} W_i\}\rangle$$

is a **KW**–*tree* with a *root* $r(T) = x$ and a *length* $l(T) = max(l(T_1), \ldots, l(T_k)) + 1$.

The elements of a tree will be called *nodes*. Every node x in a tree T appears there as a root of a smaller tree, called a *subtree of T generated by* x, denoted here T/x. *Leaves* of a tree are those nodes which generate trivial subtrees.

KW $\nvdash \Diamond\top$ since $\Diamond\top$ is not valid in a trivial **KW**–tree, and it is easy to see that the rule $R_{\mathbf{KW}}$ is correct for **KW**. Indeed, if **KW** $\nvdash \lambda$ and for some disjoint trees $T_1, \ldots, T_k$,

$$\langle T_1, V_1\rangle \Vdash \Box\neg\psi \wedge \neg\psi \wedge \theta_1, \ldots, \langle T_k, V_k\rangle \Vdash \Box\neg\psi \wedge \neg\psi \wedge \theta_k,$$

we can construct a **KW**–tree $x(T_1, \ldots, T_k)$ and an appropriate valuation V in it so that

$$\langle (x; T_1, \ldots, T_k), V\rangle \Vdash \neg\lambda \wedge \Diamond\neg\theta_1 \wedge \cdots \wedge \Diamond\neg\theta_k \wedge \Box\neg\psi.$$

Now, to every **KW**–unprovable formula φ we can attach a natural number $l(\varphi)$ which is the least length of a **KW**–tree in which φ is refutable. Clearly, φ will be refuted at the root of such a tree.

For every such φ, we shall prove by induction on $l(\varphi)$ that

$$\mathbf{KW} \dashv \varphi.$$

i) Let $l(\varphi) = 0$. Then for some valuation in the trivial tree T_0, $\langle T_0, V\rangle \Vdash \neg\varphi$, hence for an appropriate substitution σ which replaces all variables by $\top$ or $\bot$ according to V, $T_0 \models \neg\sigma(\varphi)$. Then $\Box\bot \to \neg\sigma(\varphi)$ is valid in all **KW**–trees since $\Box\bot$ is satisfied only at the leaves. So, **KW** $\vdash \sigma(\varphi) \to \Diamond\top$. Thus **KW** $\dashv \sigma(\varphi)$ by $\Diamond\bot$ and **MT**, hence **KW** $\dashv \varphi$ by **RS**.

ii) Let $l(\varphi) = n + 1$ and for all **KW**–unprovable θ such that $l(\theta) \leqslant n$, **KW** $\dashv \theta$ holds. We can assume that φ is equivalent to some conjunction $\delta_1 \wedge \cdots \wedge \delta_m$ in NMF. Then for some i, **KW** $\nvdash \delta_i$ and $l(\delta_i) \leqslant n+1$.

a) $\delta_i = \lambda \vee \Diamond\psi$. **KW** $\vdash \delta_i \to (\lambda \vee \Diamond\top)$ and **KW** $\dashv \lambda \vee \Diamond\top$ since $K \vdash \neg\sigma(\lambda)$ for some suitable substitution σ. Thus **KW** $\dashv \delta_i$.

b) $\delta_i = \lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k \vee \Diamond\psi$. Let for some **KW**–tree T, with $l(T) \leqslant n + 1$,

$$\langle T, V\rangle \Vdash \neg\delta_i, \text{ i.e. } \langle T, V\rangle \Vdash \neg\lambda \wedge \Diamond\neg\theta_1 \wedge \cdots \wedge \Diamond\neg\theta_k \wedge \Box\neg\psi$$

hence $\langle T, V\rangle \Vdash \neg\lambda \wedge \Diamond\neg\theta_1 \wedge \cdots \wedge \Diamond\neg\theta_k \wedge \Box\neg\psi \wedge \Box\Box\neg\psi$ by the transitivity of the **KW**–trees. Then $\langle T, V\rangle \Vdash \neg\lambda$ and for some subtrees $T_1, \ldots, T_k$, $\langle T_j, V\rangle \Vdash \Box\neg\psi \wedge \neg\psi \wedge \neg\theta_j$, for $j = 1, \ldots, k$. Then **KW** $\dashv \lambda$ and **KW** $\dashv \Diamond\psi \vee \psi \vee \theta_j$, since $l(T_j) \leqslant n$, $j = 1, \ldots, k$. Now, applying the rule $\mathbf{R_{KW}}$ we get **KW** $\dashv \delta_i$. The induction is completed.

1) is proved much in the same way, using the fact that **K** is complete with respect to all finite irreflexive intransitive trees (see [11]). ■

THEOREM 1.3.

1) **CPC*** *extended with the rule*

$$\mathbf{R_T} : \frac{\dashv \lambda,\ \dashv \psi \vee \theta_1, \ldots, \dashv \psi \vee \theta_k}{\dashv \lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k \vee \Diamond((\lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k) \wedge \psi)}$$

is L–complete for the logic **T**.

2) **CPC*** *extended with the rule*

$$\mathbf{R_{S4Grz}} : \frac{\dashv \lambda,\ \dashv \Diamond\psi \vee \theta_1, \ldots, \dashv \Diamond\psi \vee \theta_k}{\dashv \lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k \vee \Diamond((\lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k) \wedge \psi)}$$

is L–complete for the logic $\mathbf{S4Grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$.

3) **CPC*** *extended with the axiom* $\dashv \Diamond\top$ *and the rule*

$$\mathbf{R_{K4.3W}} : \frac{\dashv \lambda,\ \dashv \Diamond\psi \vee \psi \vee \theta}{\dashv \lambda \vee \Box\theta \vee \Diamond\psi}$$

is L–complete for the logic $\mathbf{K4.3W} = \mathbf{KW} + \Box(\Box p \wedge p \to q) \vee \Box(\Box q \wedge q \to p)$.

4) **CPC*** *extended with the rule*

$$\mathbf{R_{S4.3Grz}} : \frac{\dashv \lambda,\ \dashv \Diamond\psi \vee \theta}{\dashv \lambda \vee \Box\theta \vee \Diamond((\lambda \vee \Box\theta) \wedge \psi)}$$

is L–complete for the logic $\mathbf{S4.3Grz} = \mathbf{S4.Grz} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$.

5) **CPC*** *extended with the rule*

$$\mathbf{R_C} : \frac{\dashv \lambda,\ \dashv \theta}{\dashv \lambda \vee \Box\theta}$$

is L–complete for the logic $\mathbf{K} + \Diamond p \leftrightarrow \Box p$.

6) **CPC*** *extended with the axiom* $\dashv \Diamond\top$ *and the rule* $\mathbf{R_C}$ *is L–complete for the logic* $\mathbf{K} + \Diamond p \to \Box p$.

PROOF. We shall prove in detail the most difficult case.

2) **S4Grz** is complete with respect to all finite reflexive and transitive trees (called within this proof simply trees), defined similarly to the **KW**–trees.

Let $T = \langle W, R\rangle$ be an arbitrarily fixed tree with a set of nodes $W = \{x_1, \ldots, x_n\}$. We attach to these nodes different propositional variables $q(x_1), \ldots, q(x_n)$ or for short $q_1, \ldots, q_n$. This set will be referred to as $var(T)$. A valuation V in the tree T will be called *suitable* if for every $x_i \in T$, $V(q_i) = \{x_i\}$. If V is a valuation in T, we may also consider it (its restriction) as a valuation in any subtree of T.

Let $\chi(x_i) = q_i \wedge \bigwedge\{\neg q_j / q_j \in var(T), j \neq i\}$.

Now we shall successively define for every subtree S of T formulae φ_S and ψ_S as follows:

If S is a leaf, $S = \langle\{x_i\}, \{\langle x_i, x_i\rangle\}\rangle$, put $\psi_S = \varphi_S = \chi(x_i) \wedge \Box\chi(x_i)$.
Let $S = (y; S_1, \ldots, S_k)$, $r(S_1) = y_1, \ldots, r(S_k) = y_k$ and suppose $\varphi_{S_1}, \ldots, \varphi_{S_k}$, $\psi_{S_1}, \ldots, \psi_{S_k}$ are already defined. Then put

$$\psi_S = \chi(y) \wedge \Diamond\psi_{S_1} \wedge \cdots \wedge \Diamond\psi_{S_k} \wedge \Box((\chi(y) \wedge \Diamond\psi_{S_1} \wedge \cdots \wedge \Diamond\psi_{S_k}) \vee \varphi_{S_1} \vee \cdots \vee \varphi_{S_k})$$

and

$$\varphi_S = \psi_S \vee \varphi_{S_1} \vee \cdots \vee \varphi_{S_k}.$$

Informally speaking, ψ_S is characteristic for the root of S while φ_S is characteristic for the whole S. Let us make some observations.

1. For every subtree S of T, **S4Grz** $\vdash \psi_S \to \Box\varphi_S$. Induction on the construction of the subtrees: for leaves this is clear. Let for $S_1, \ldots, S_k$ the above hold and $S = (y; S_1, \ldots, S_k)$. Denote

$$\alpha_S = \chi(y) \wedge \Diamond\psi_{S_1} \wedge \cdots \wedge \Diamond\psi_{S_k} \quad \text{and} \quad \beta_S = \varphi_{S_1} \vee \cdots \vee \varphi_{S_k}.$$

Then in **S4Grz**

$$\psi_S = \alpha_S \wedge \Box(\alpha_S \vee \beta_S) \vdash \Box(\alpha_S \vee \beta_S) \vdash \Box(\alpha_S \vee \beta_S) \wedge \Box\Box(\alpha_S \vee \beta_S) \vdash$$
$$\Box((\alpha_S \vee \beta_S) \wedge \Box(\alpha_S \vee \beta_S)) \vdash \Box((\alpha_S \wedge \Box(\alpha_S \vee \beta_S)) \vee (\beta_S \wedge \Box(\alpha_S \vee \beta_S))) \vdash$$
$$\Box(\psi_S \vee \beta_S) = \Box\varphi_S.$$

2. If S' and S'' are different subtrees of T then $\psi_{S'} \wedge \psi_{S''}$ is not satisfiable since $\chi(r(S')) \wedge \chi(r(S''))$ is not.

3. If S'' is a subtree of S' then **S4Grz** $\vdash \psi_{S'} \to \Diamond\psi_{S''}$. For immediate subtrees this is obvious. Now, suppose that Q'' is an immediate subtree of

Q' and Q' is an immediate subtree of Q. Then **S4Grz** $\vdash \psi_Q \to \Diamond\psi_{Q'}$ and **S4Grz** $\vdash \psi_Q \to \Diamond\psi_{Q''}$. Then **S4Grz** $\vdash \Diamond\psi_{Q'} \to \Diamond\Diamond\psi_{Q''}$, hence **S4Grz** $\vdash$ $\psi_{Q'} \to \Diamond\Diamond\psi_{Q''}$, so **S4Grz** $\vdash \psi_Q \to \Diamond\psi_{Q''}$ since **S4Grz** $\vdash \Diamond\Diamond\theta \to \Diamond\theta$.

4. An induction on subtrees shows simultaneously

$$\mathbf{S4Grz} \vdash \varphi_S \to \bigvee_{x\in S} \psi_{S/x} \text{ and } \mathbf{S4Grz} \vdash \psi_S \to \Box\left[\bigvee_{x\in S} \psi_{S/x}\right].$$

5. It easily follows from 2 and 4 that, if S'' is not a subtree of S', then **S4Grz** $\vdash \psi_{S'} \to \Box\neg\psi_{S''}$.

LEMMA 1.3.1. **S4Grz** $\dashv \neg\psi_T$ *for every tree* T.

PROOF. Let T be fixed. We shall prove that for every subtree S of T, **S4Grz** $\dashv \neg\psi_S$. The proof is inductive on the construction of S. If S is the leaf x_i then for an appropriate substitution σ (replacing q_i by $\top$ and all other variables by $\bot$), $\sigma(\psi_S)$ is equivalent to $\top$. Now, let $S = (r(S); S_1, \ldots, S_k)$ and suppose that for all immediate subtrees $S_1, \ldots, S_k$ of S:

(1) **S4Grz** $\dashv \neg\psi_{S_i}$. Then:

(2) **S4Grz** $\vdash (\psi_{S_i} \to \Box\varphi_{S_i}) \to ((\psi_{S_i} \to \neg\Box\varphi_{S_i}) \to \neg\psi_{S_i})$ (classical tautology)

(3) **S4Grz** $\vdash (\psi_{S_i} \to \neg\Box\varphi_{S_i}) \to \neg\psi_{S_i}$ (by 1., (2) and **MP**)

(4) **S4Grz** $\dashv \psi_{S_i} \to \neg\Box\varphi_{S_i}$ (by (3), (1) and **MT**)

Denote $\psi = \neg\varphi_{S_1} \wedge \cdots \wedge \neg\varphi_{S_k}$. Then:

(5) **S4Grz** $\vdash \Diamond\psi \to \Diamond\neg\varphi_{S_i}$ (normal modal tautology)

(6) **S4Grz** $\vdash (\psi_{S_i} \to \Diamond\psi) \to (\psi_{S_i} \to \Diamond\neg\varphi_{S_i})$ (by (5))

(7) **S4Grz** $\dashv \psi_{S_i} \to \Diamond\psi$, i.e. **S4Grz** $\dashv \neg\psi_{S_i} \vee \Diamond\psi$ (by (4), (6) and **MT**)

Denote $\lambda = \neg\chi(r(S))$ and $\theta_i = \neg\psi_{S_i}$, $i = 1, \ldots, k$. Then:

(8) **S4Grz** $\dashv \lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k \vee \Diamond((\lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_k) \wedge \psi)$ (by (7) and $\mathbf{R}_{\mathbf{S4Grz}}$)

Thus, **S4Grz** $\dashv \neg\psi_S$. The lemma is proved. ■

(In fact, $\langle T, V\rangle \Vdash \psi_T$ for every suitable valuation V.)

LEMMA 1.3.2 *For any trees* S *and* T *and a valuation* V *in* S, *if* $\langle S, V\rangle \Vdash \psi_T$ *then there exist valuations* V' *in* S *and* V'' *in* T *such that:*

i) V' coincides with V over $var(T)$;

ii) V'' is suitable for T;

iii) there exists a p–morphism f from $\langle S, V'\rangle$ onto $\langle T, V''\rangle$.

PROOF. Define V' and V'' as follows: $V'(q_i) = V(q_i)$ and $V''(q_i) = \{x_i\}$ for all $q_i \in var(T)$; $V'(p) = V''(p) = \emptyset$ for all other variables.

Now, let $\langle S, V\rangle \Vdash \psi_T$ and hence $\langle S, V'\rangle \Vdash \psi_T$. Then, by 4, for every $u \in S$, $\langle S/u, V'\rangle \Vdash \bigvee_{x \in T} \psi_{T/x}$ hence $\langle S/u, V'\rangle \Vdash \psi_{T/x}$ for some $x \in T$. This x is unique by 2; denote it by $f(u)$. We shall prove that $f : S \to T$, thus defined, is the desired p–morphism. Let the relations in S and T be R_S and R_T respectively.

— if uR_Sv then $\langle S/u, V'\rangle \Vdash \psi_{T/f(u)} \wedge \Diamond\psi_{T/f(v)}$ which is possible only if $f(u)R_Tf(v)$, by 5.

— if $f(u)R_Ty$ then $\mathbf{S4Grz} \vdash \psi_{T/f(u)} \to \Diamond\psi_{T/y}$ by 3.

Hence $\langle S/u, V'\rangle \Vdash \Diamond\psi_{T/y}$ i.e. there exists $v \in S$ such that uR_Sv and $\langle S/v, V'\rangle \Vdash \psi_{T/y}$, so $f(v) = y$.

— for every $x \in T$, $\mathbf{S4Grz} \vdash \psi_T \to \Diamond\psi_{T/x}$, hence $\langle S, V'\rangle \Vdash \Diamond\psi_{T/x}$.

Therefore f is onto.

— finally, for every variable p, $f^{-1}(V''(p)) = V'(p)$. ■

LEMMA 1.3.3. *If $\langle T, V\rangle \Vdash \neg\theta$ then there exists a substitution σ such that $\mathbf{S4Grz} \vdash \sigma(\theta) \to \neg\psi_T$.*

PROOF. Let $p_1, \ldots, p_m$ be the variables occurring in θ. We may regard $\{p_1, \ldots, p_m\}$ and $var(T)$ disjoint and hence V suitable. Define σ on $p_1, \ldots, p_m$ as follows:

$$\sigma(p_i) = \bigwedge\{q_j/x_j \in V(p_i)\} \wedge \bigwedge\{\neg q_j/x_j \notin V(p_i)\}.$$

All other variables are preserved by σ. Then for all p_i, $V(\sigma(p_i)) = V(p_i)$. Hence

$$(*) \qquad \langle T, V\rangle \Vdash \neg\sigma(\theta).$$

Now, suppose $\mathbf{S4Grz} \nvdash \sigma(\theta) \to \neg\psi_T$. Then there exists a tree–model $\langle S, V_1\rangle$ such that $\langle S, V_1\rangle \Vdash \sigma(\theta) \wedge \psi_T$. By lemma 1.3.2 we can harmlessly change V_1 into some V' and find a suitable valuation V'' in T such that $\langle S, V'\rangle \Vdash \sigma(\theta) \wedge \psi_T$ and there exists a p–morphism onto $\langle S, V'\rangle \to \langle T, V''\rangle$. Then, according to Segerberg's theorem for the p–morphism, $\langle T, V''\rangle \Vdash \sigma(\theta)$. But V and V'' coincide over $\sigma(\theta)$, hence $\langle T, V\rangle \Vdash \sigma(\theta)$ which contradicts $(*)$. ■

Now we turn to the main proof. By an easy semantic argument one can see that the rule $\mathbf{R_{S4Grz}}$ (and hence the whole system) is correct for **S4Grz** (for a hint, go back to the proof of Lemma 1.3.1).

To prove L–completeness, let **S4Grz** $\not\vdash \theta$. Then $\langle T, V\rangle \Vdash \neg\theta$ for some tree–model. Hence, by lemma 1.3.3, for some substitution σ, **S4Grz** $\vdash \sigma(\theta) \to \neg\psi_T$. By lemma 1.3.1 **S4Grz** $\dashv \neg\psi_T$, hence **S4Grz** $\dashv \sigma(\theta)$ by **MT**. Therefore **S4Grz** $\dashv \theta$ by **RS**.

All other statements of the theorem are proved much in the same way, but more easily. They are respectively based on the following facts (cf. [11]):

1) T is complete with respect to all finite reflexive intransitive trees.

3) **K4.3W** is complete with respect to all finite irreflexive (strict) linear orderings. (It is also the logic of $\langle \mathbb{N}, >\rangle$.)

4) **S4.3Grz** is complete with respect to all finite linear orderings. (It is also the logic of $\langle \mathbb{N}, \geqslant\rangle$.)

5) $\mathbf{K} + \Diamond p \leftrightarrow \Box p$ is complete with respect to all finite intransitive chains in which only the last element is reflexive. (It is also the logic of the infinite irreflexive intransitive chain $\langle \mathbb{N}, S\rangle$ where S is the relation "next": xSy iff $y = x + 1$.)

6) $\mathbf{K} + \Diamond p \to \Box p$ is complete with respect to all finite irreflexive intransitive chains. ■

REMARKS:

1. It is curious that the rule $\dashv \Box\varphi \;/\; \dashv \varphi$, which is admissible for all normal modal logics, turned out redundant in all systems considered here.

2. In all considered systems the rule **RS** can be specified (as it can be seen from the proofs) as follows: it is enough to admit only $\Box$–free substitutions, i.e. such that every variable is substituted by a $\Box$–free formula.

3. In fact some of the above introduced rules are rather rule schemata, since every number k yields a rule $\mathbf{R}(k)$. One can be easily persuaded that these schemata cannot be restricted to any fixed k. For instance, take $\mathbf{R_K}$. An easy induction on the $\dashv$–inference shows that if we restrict $\mathbf{R_K}$

to some $\mathbf{R}(k)$ then all $\mathbf{K}_{\dashv}$–deducible formulae would be refuted in trees in which every node has no more than k branches. But then the $\mathbf{K}$–unprovable formula $Alt_n = (\Diamond p_1 \wedge \cdots \wedge \Diamond p_n) \rightarrow \bigvee_{i \neq j} \Diamond(p_i \wedge p_j)$ for any $n > k$ will remain $\mathbf{K}_{\dashv}$–unprovable, too.

In [15] Skura proposes another, structural rule for **S4Grz**, obtained using the Gödel translation between this logic and INT. He also gives a rather complicated rule for S4. We shall finish this section with another refutation system for S4, employing a strengthened version of the refutation rule for **S5** introduced in [16] and a complicated version of $R_{\mathbf{S4Grz}}$. Still, it seems a bit more visible than the rule suggested in [15].

THEOREM 1.4. *The system* **CPC*** *extended with the rules*

$$\mathbf{R}^1_{\mathbf{S4}} : \frac{\dashv \lambda \vee \lambda_0, \ldots, \dashv \lambda \vee \lambda_m}{\dashv \lambda_0 \vee \Diamond\lambda \vee \Lambda^m_{\Box}}$$

and

$$\mathbf{R}^2_{\mathbf{S4}} : \frac{\dashv (\lambda_0 \wedge \cdots \wedge \lambda_m) \vee \Lambda^m_{\Box},\ \dashv \Diamond\psi \vee \theta_1, \ldots, \dashv \Diamond\psi \vee \theta_k}{\dashv \lambda_0 \vee \Lambda^m_{\Box} \vee \Theta^k_{\Box} \vee \Diamond(((\lambda_0 \wedge \cdots \wedge \lambda_m) \vee \Lambda^m_{\Box} \vee \Theta^k_{\Box}) \wedge \psi)}$$

(where: $\Lambda^m_{\Box} = \Box\lambda_0 \vee \cdots \vee \Box\lambda_m$, $\Theta^k_{\Box} = \Box\theta_1 \vee \cdots \vee \Box\theta_k$; in both rules $\lambda, \lambda_0, \ldots, \lambda_m$ are $\Box$-free) is L–complete for the logic **S4**.

PROOF. (Detailed sketch): The proof is a modification of that for **S4Grz**. First we shall present the semantic characterization of **S4**, employed here.

A quasi–ordered set $\langle W, R\rangle$ will be called a *cluster–tree* if the quotient–set $W/_{R^\sim}$, with the induced partial ordering, is a reflexive and transitive tree, where $R^\sim$ is the equivalence relation generated by R, i.e. $xR^\sim y$ iff xRy and yRx. *Length* of the cluster–tree is the length of the corresponding quotient–tree. Informally, a cluster–tree is a reflexive and transitive tree in which the nodes are replaced by sets (*cluster–nodes*) in which every two elements are R–accessible from each other. Accordingly are introduced the notions of *cluster–root, cluster leaf* and *cluster–subtree* of a cluster–tree. $(A; T_1, \ldots, T_n)$ will mean a cluster–tree with a cluster–root A and immediate cluster–subtrees $T_1, \ldots, T_n$.

LEMMA 1.4.0. **S4** *is sound and complete with respect to the class of finite cluster–trees.*

SKETCH OF THE PROOF. It is well–known that **S4** is characterized by the class of finite quasi–orderings. Let **S4** $\not\vdash \varphi$. Then φ is refuted at a point x of some finite quasi–ordered frame $F = \langle W, R\rangle$ under some valuation V. We may regard F to be generated by x, i.e. xRy holds for every $y \in W$, otherwise we consider the subframe of F generated by x. Thus F has a least cluster containing x. Therefore the corresponding quotient–tree $F^{\sim} = \langle W^{\sim}, \leqslant\rangle$, where $W^{\sim} = W/_{R^{\sim}}$ and $\leqslant$ is the partial ordering in $W^{\sim}$ generated by R, has a least element $x^{\sim}$. We shall show that $F^{\sim}$ is a p–morphic image of a finite reflexive and transitive tree. We set U to be the set of all $<$–chains in $W^{\sim}$ starting from $x^{\sim}$. If $c_1, c_2 \in U$ we say that $c_1 \prec c_2$ if c_1 is an initial segment of c_2. It is easy to verify that $\langle U, \prec\rangle$ is a finite reflexive and transitive tree. Now we define a mapping $f : U \to W^{\sim}$ as follows: $f(c)$ is the last element of the chain c. f is a p–morphism:

1) If $c_1 \prec c_2$ then $f(c_1) \leqslant f(c_2)$;

2) Let $f(c) \leqslant q \in W^{\sim}$. If $f(c) = q$ then for $c' = c$ $f(c) \leqslant f(c')$ and $f(c') = q$. If $f(c) < q$ then we set c' to be the chain c with q added as a last element. Again $f(c) \leqslant f(c')$ and $f(c') = q$.

Moreover, f is onto since every point of $W^{\sim}$ is a last element of a chain starting from $x^{\sim}$.

Thus, f is a surjective p–morphism: $\langle U, \prec\rangle \to \langle W^{\sim}, \leqslant\rangle$. It can be naturally uplifted to a surjective p–morphism $f' : \langle U', \prec'\rangle \to \langle W^{\sim}, \leqslant\rangle$ where U' is obtained from U by replacing each $c \in U$ by a copy of the cluster $f(c)$, and defining $\prec'$ to be the universal relation within each cluster, and the relation induced by $\prec$ between the points from different clusters. Thus $\langle W, R\rangle$ is a p–morphic image of the finite cluster–tree $\langle U', \prec'\rangle$. Therefore, by Segerberg's p–morphism theorem, φ is refuted in $\langle U', \prec'\rangle$. The lemma is proved. ■

Hereafter we shall consider only *finite* cluster trees.

Let $T = \langle W, R\rangle$ be a fixed cluster–tree. As in the proof of Th.1.3.2 we attach different variables to all nodes and introduce the characteristic formulae χ. We denote by $r(T)$ the cluster–root of T. Again, every node x generates a cluster–subtree T/x containing all nodes (and hence their cluster–nodes) R–accessible from x. Note, that nodes belonging to the same cluster–node generate the same cluster–subtree.

Now, inductively by the length of T, we shall define formulae ψ_x, θ_x, ψ_S and φ_S for every node x and cluster–subtree S of T, as follows:

If S is a cluster–leaf, $S = \{y_0, \ldots, y_m\}$ then we put

$\theta_S = \Box(\chi(y_0) \vee \cdots \vee \chi(y_m)) \wedge \Diamond\chi(y_0) \wedge \cdots \wedge \Diamond\chi(y_m)$;
$\psi_y = \chi(y) \wedge \theta_S$;
$\varphi_S = \psi_S = \psi_{y_0} \vee \cdots \vee \psi_{y_m}$.

Now, let $S = (r(S); S_1, \ldots, S_k)$ be a non–trivial cluster–subtree of T, and $\psi_{S_1}, \ldots, \psi_{S_k}, \varphi_{S_1}, \ldots, \varphi_{S_k}$ be already defined. Let $r(S) = y_0, \ldots, y_m$. Then put

$$\theta_S = \Diamond\chi(y_0) \wedge \cdots \wedge \Diamond\chi(y_m) \wedge \Diamond\psi_{S_1} \wedge \cdots \wedge \Diamond\psi_{S_k} \wedge$$
$$\Box\, [[(\chi(y_0) \vee \cdots \vee \chi(y_m)) \wedge \Diamond\chi(y_0) \wedge \cdots \wedge \Diamond\chi(y_m) \wedge \Diamond\psi_{S_1} \wedge \cdots \wedge \Diamond\psi_{S_k}]$$
$$\vee \varphi_{S_1} \vee \cdots \vee \varphi_{S_k}].$$

Now, for every $y \in r(S)$, put

$\psi_y = \chi(y) \wedge \theta_S$.

Finally,

$\psi_S = \psi_{y_0} \vee \cdots \vee \psi_{y_m}$ and
$\varphi_S = \psi_S \vee \varphi_{S_1} \vee \cdots \vee \varphi_{S_k}$.

Thus, ψ_y is characteristic for y, ψ_S is characteristic for the cluster–root of S and φ_S is characteristic for the whole S.

A few properties of these formulae, analogous to 1–5 from Th.1.3, 2), can be established and the three lemmata from the previous proof can be proved mutatis mutandis, whence the main proof follows. ∎

Still, it not clear how to adapt the method developed so far in order to obtain complete refutation systems for other modal logics with more sophisticated semantic characterizations. A possible way to extend that method would be to introduce several additional rules and impose some priority order in their application in the system.

2. Sequential refutation systems

An obvious drawback in the sentential refutation systems, introduced in the previous section, is that in most cases the specific refutation rules employed in them, although semantically well–motivated, are rather unhandy for practical purposes. Another common feature of these systems is that the inference of the refutable formulae involves inference of the acceptable, i.e. provable ones, because of the rule **MT**. This makes the refutation systems inferior to the orthodox ones (although, another point of view on that kind of refutation systems turns this flaw into an advantage; see the concluding remarks). A way to abolish this inequality is to construct refutation systems for refutable sequents rather than formulae, which will be done in this section. In [13] Skura introduces a uniform refutation system for the class of equivalential logics, for sequents of the kind $(\Gamma; \varphi)$. His system employs a class of axioms describing the semantic equivalence generated by the matrix semantics of the logic under consideration, and two rules: substitution rule and a rule for equivalent replacement of the sort

$$\frac{\Gamma,\ \alpha \leftrightarrow \beta \dashv \varphi}{\Gamma(\alpha/\beta) \dashv \varphi(\alpha/\beta)}.$$

Instead, we shall follow the more natural Gentzen–style approach, at the expense of concentrating on concrete logical systems. The author is not aware of any refutation systems of such kind existing in the literature, even for the classical propositional calculus.

Hereafter Γ, Δ will stand for finite (possibly empty) multisets of formulae and φ, ψ for single formulae. Γ, φ will stand for $\Gamma \cup \{\varphi\}$ and Γ, Δ for $\Gamma \cup \Delta$. By a *sequent* we shall mean any pair $(\Gamma; \Delta)$. We shall deal with two syntactical types of sequents: *acceptable* (i.e. provable or valid, depending on the way of specifying the logic under consideration), denoted $\Gamma \vdash \Delta$, and *refutable* (non–provable, non–valid) denoted $\Gamma \dashv \Delta$.

DEFINITION.

1. A *sequential refutation system* is any pair consisting of a set of refutable sequents (axioms) and a set of *refutation rules* of the kind:

$$\frac{\Gamma_0 \dashv \Delta_0, \ldots, \Gamma_k \dashv \Delta_k}{\Gamma \dashv \Delta}.$$

2. Let **R** be a sequential refutation system. An inference (refutation inference) in **R** is a sequence of refutable sequents $\Gamma_0 \dashv \Delta_0, \ldots, \Gamma_n \dashv \Delta_n$ such that:

   i) $\Gamma_0 \dashv \Delta_0$ is an axiom of **R**;

   ii) for every $i = 1, \ldots, n$ $\Gamma_i \dashv \Delta_i$ is either an obtained from $\Gamma_0 \dashv \Delta_0, \ldots, \Gamma_{i-1} \dashv \Delta_{i-1}$ by applying some of the rules of **R**. The last sequent of an inference in **R** is called **R**–*rejected sequent*.

DEFINITION. Let S be a propositional logic in the language $\mathcal{L}$, specified (anyhow) by its set of tautologies.

1. A sequent $(\Gamma; \Delta)$ is said to be *acceptable in S* if $\bigwedge\{\gamma : \gamma \in \Gamma\} \to \bigvee\{\delta : \delta \in \Delta\}$ (where $\bigwedge\emptyset = \top$ and $\bigvee\emptyset = \bot$) is a tautology of S, and is *refutable in S* otherwise.

2. A *sequential refutation system for S* is any sequential refutation system in which $\Gamma \dashv \Delta$ is interpreted as "$(\Gamma;\Delta)$ is refutable in S".

3. A sequential refutation system **R** is *correct* for S if only sequents, refutable in S, are rejected in **R**. **R** is *complete* for S if all sequents, refutable in S, are rejected in **R**. **R** is L–*complete* for S if it is both correct and complete.

We start with an L–complete sequential refutation system $\mathbf{CSC}_{\dashv}$ for the classical propositional sequential calculus **CSC**.

Axioms: $\Gamma \dashv \Delta$,

where Γ,Δ are disjoint finite sets of propositional variables.

Rules:

Structural:

$$(\mathrm{Contr}^{-1})\quad \frac{\Gamma, \varphi \dashv \Delta}{\Gamma, \varphi, \varphi \dashv \Delta}, \qquad \frac{\Gamma \dashv \Delta, \varphi}{\Gamma \dashv \Delta, \varphi, \varphi}$$

$$(\mathrm{Weak}^{-1})\quad \frac{\Gamma, \varphi \dashv \Delta}{\Gamma \dashv \Delta}, \qquad \frac{\Gamma \dashv \Delta, \varphi}{\Gamma \dashv \Delta}$$

Logical:

$$(l\wedge)\quad \frac{\Gamma, \varphi, \psi \dashv \Delta}{\Gamma, \varphi \wedge \psi \dashv \Delta} \qquad (r\wedge)\quad \frac{\Gamma \dashv \Delta, \varphi}{\Gamma \dashv \Delta, \varphi \wedge \psi}, \quad \frac{\Gamma \dashv \Delta, \psi}{\Gamma \dashv \Delta, \varphi \wedge \psi}$$

$$(l\vee)\quad \frac{\Gamma, \varphi \dashv \Delta}{\Gamma, \varphi \vee \psi \dashv \Delta}, \quad \frac{\Gamma, \psi \dashv \Delta}{\Gamma, \varphi \vee \psi \dashv \Delta} \qquad (r\vee)\quad \frac{\Gamma \dashv \Delta, \varphi, \psi}{\Gamma \dashv \Delta, \varphi \vee \psi}$$

$$(l\to)\quad \frac{\Gamma \dashv \Delta, \varphi}{\Gamma, \varphi \to \psi \dashv \Delta}, \quad \frac{\Gamma, \psi \dashv \Delta}{\Gamma, \varphi \to \psi \dashv \Delta} \qquad (r\to)\quad \frac{\Gamma, \varphi \dashv \Delta, \psi}{\Gamma \dashv \Delta, \varphi \to \psi}$$

$$(l\neg)\quad \frac{\Gamma \dashv \Delta, \varphi}{\Gamma, \neg\varphi \dashv \Delta} \qquad (r\neg)\quad \frac{\Gamma, \varphi \dashv \Delta}{\Gamma \dashv \Delta, \neg\varphi}$$

$$(\top)\quad \frac{\Gamma \dashv \Delta}{\Gamma, \top \dashv \Delta} \qquad (\bot)\quad \frac{\Gamma \dashv \Delta}{\Gamma \dashv \Delta, \bot}$$

REMARK: If we allow the axioms to be pairs of disjoint multisets rather than sets, then the structural rules become redundant.

THEOREM 2.1. **CSC**$_\dashv$ *is L–complete for the classical propositional calculus.*

PROOF. 1. (*Correctness*) Obviously, if $\Gamma \dashv \Delta$ is an axiom then $\Gamma \vdash \Delta$ is not provable in **CSC**. Now, for every rule $\frac{\Gamma_1 \dashv \Delta_1}{\Gamma_2 \dashv \Delta_2}$ of **CSC**$_\dashv$ we construct the *converse* rule $\frac{\Gamma_2 \vdash \Delta_2}{\Gamma_1 \vdash \Delta_1}$.

It is a routine task to show that all rules, converse to those from **CSC**$_\dashv$, are correct rules of **CSC**. Therefore if $\Gamma \dashv \Delta$ is inferred in **CSC**$_\dashv$ and $\Gamma \vdash \Delta$ is provable in **CSC**, then taking the converse sequence of the refutation inference of $\Gamma \dashv \Delta$, and replacing all rules applied in it by their converses, we eventually obtain a correct inference in **CSC** of some $\Gamma_0 \vdash \Delta_0$, where $\Gamma_0 \dashv \Delta_0$ is an axiom of **CSC**$_\dashv$, which contradicts to the correctness of **CSC**. Thus every rejected sequent in **CSC**$_\dashv$ corresponds to a non–provable $\dashv$–sequent in **CSC**.

2. (*Completeness*) Let $\Gamma \vdash \Delta$ be non–provable in **CSC**. Then, by completeness of **CSC** there is a valuation v of the set of variables $var(\Gamma;\Delta)$, occurring in $(\Gamma;\Delta)$, which makes all formulae from Γ true and all those from Δ false. We say that v *refutes* $(\Gamma;\Delta)$. Let $\{p_1,\dots,p_m\}$ be the variables from $var(\Gamma;\Delta)$ which are true under v, and $\{q_1,\dots,q_n\}$ be those which are false. Obviously $\{p,\dots,p\} \cap \{q_1,\dots,q_n\} = \emptyset$.

A (syntactic) *complexity* of a formula θ is the number $\chi(\theta)$ of the occurrences of logical symbols in θ. Complexity of a finite multiset of formulae Σ is the sum $\chi(\Sigma)$ of complexities of the members of Σ.

We shall prove that for every sequent $(\Gamma';\Delta')$ refuted by v and such that $var(\Gamma' \cup \Delta') \subseteq \{p_1,\dots,p_m,q_1,\dots,q_n\}$, the sequent

($) $$\Gamma',\{p_1,\dots,p_m\} \dashv \Delta',\{q_1,\dots,q_n\}$$

is inferred by **CSC**, at that the only axiom involved in the inference is $\{p_1,\dots,p_m\} \dashv \{q_1,\dots,q_n\}$.

The proof goes by induction on $\chi(\Gamma' \cup \Delta')$.

If $\chi(\Gamma' \cup \Delta') = 0$ then Γ' and Δ' consist of propositional variables, hence $\Gamma' \subseteq \{p_1,\dots,p_m\}$ and $\Delta' \subseteq \{q_1,\dots,q_n\}$, so that $(*)$ is rejected by applying (Contr^{-1}) to the axiom $\{p_1,\dots,p_m\} \dashv \{q_1,\dots,q_n\}$.

Let $\chi(\Gamma' \cup \Delta') > 0$. Then there is a formula $\theta \in \Gamma' \cup \Delta'$ with non–zero complexity.

Let us notice that for each logical rule of **CSC**$_\dashv$:

i) the complexity of the premise is less than that of the conclusion;

ii) all variables occurring in the premise, occur in the conclusion;

iii) if the conclusion is refuted by the valuation v, so is the premise, except for the rules $(r\wedge)$, $(l\vee)$, $(l\rightarrow)$ where there are two versions with the same conclusion. In them, if the conclusion is refuted by v so is the premise in at least one of the versions.

Therefore, depending on the main logical symbol of θ, we can construct the inference of ($) out of an inference of a $\dashv$–sequent with a lesser complexity for which the inductive hypothesis holds, by an application of a corresponding logical rule. For instance, if $\theta = \theta_1 \wedge \theta_2$ and θ occurs in Γ', i.e. $\Gamma' = \Gamma'', \theta$, then by the inductive hypothesis, $\Gamma'', \theta_1, \theta_2 \dashv \Delta'$ is rejected by $\mathbf{CSC}_{\dashv}$. Hence, applying $(l\wedge)$, $\Gamma' \dashv \Delta'$ is rejected, too. The other cases are dealt with analogously. This completes the induction.

In particular, $\Gamma, \{p_1, \ldots, p_m\} \dashv \Delta, \{q_1, \ldots, q_n\}$ is rejected, whence, applying (Weak^{-1}) we obtain $\Gamma \dashv \Delta$. ■

As in the sentential systems, it is clear that $\mathbf{CSC}_{\dashv}$ is correct for every consistent modal logic.

Now we shall introduce complete sequential refutation systems for several modal logics, extending $\mathbf{CSC}_{\dashv}$ with specific rules for the modal operator. Naturally, we should not expect these additional rules for $\Box$ to be so elegant, uniform and perspicuous as the rules for the truth–functional classical logical connectives, since the modality is an intensional operator whose behaviour crucially depends on the particular semantic characterization of the logic under consideration. That is why, we shall present a sample collection of concrete refutation systems for logics having suitable semantic characterizations, rather than a general construction. Still, this collection will illustrate the general idea of sequential refutation systems for modal logics.

For any finite set Γ of modal formulae we denote $\Box\Gamma = \{\Box\varphi : \varphi \in \Gamma\}$.

Our basic example will be a refutation system $\mathbf{SK}_{\dashv}$ for the minimal normal logic $\mathbf{K}$.

$\mathbf{SK}_{\dashv}$ is an extension of $\mathbf{CSC}_{\dashv}$ with the following axioms and rules:

Axioms:

$(Ax_{\mathbf{K}})$ $\quad \Box\Gamma \dashv$ for any finite set Γ.

Rules:

$MIX^1_{\mathbf{K}}: \quad \dfrac{\Gamma_0 \dashv \Delta_0, \Box\Gamma \dashv \Box\Delta}{\Gamma_0, \Box\Gamma \dashv \Delta_0, \Box\Delta}$ where $\Gamma_0 \dashv \Delta_0$ is an axiom of $\mathbf{CSC}_{\dashv}$;

$MIX^2_{\mathbf{K}}: \quad \dfrac{\Gamma \dashv \varphi, \Box\Gamma \dashv \Box\Delta}{\Box\Gamma \dashv \Box\Delta, \Box\varphi}.$

THEOREM 2.2. $\mathbf{SK}_{\dashv}$ *is L–complete for* $\mathbf{K}$.

PROOF. Again the crucial fact used in the proof is that $\mathbf{K}$ is sound and complete with respect to all finite irreflexive intransitive trees ($\mathbf{K}$–trees).

Correctness is easier, as usual. All axioms are refutable in a trivial $\mathbf{K}$–tree, since every $\Box\Gamma$ is satisfiable in such a tree. If $(\Box\Gamma; \Box\Delta)$ is refutable in $\mathbf{K}$, then it is refutable at the root of some $\mathbf{K}$–tree. We can amend the valuation at that root so that $(\Gamma_0; \Delta_0)$ is refutable there, without affecting the valuation of $\Box\Gamma$ and $\Box\Delta$, thus refuting $(\Gamma_0, \Box\Gamma; \Delta_0, \Box\Delta)$. Hence $MIX^1_{\mathbf{K}}$ is correct. As for $MIX^2_{\mathbf{K}}$, assume that $(\Box\Gamma; \Box\Delta)$ is refuted at the root of the $\mathbf{K}$–tree $(x; T_1, \ldots, T_k)$ and $(\Gamma; \{\varphi\})$ is refuted at the root of a $\mathbf{K}$–tree T. Then $(\Box\Gamma; \Box\Delta, \Box\varphi)$ is refuted at the root of the $\mathbf{K}$–tree $(x; T_1, \ldots, T_k, T)$.

Now, completeness. Let $(\Gamma; \Delta)$ be refutable in $\mathbf{K}$ and $l(\Gamma, \Delta)$ be the least length of a $\mathbf{K}$–tree at the root of which $(\Gamma; \Delta)$ is refuted under some valuation. We shall prove that $\Gamma \dashv \Delta$ is refuted in $\mathbf{SK}_{\dashv}$ by induction on $l(\Gamma, \Delta)$.

An *atom* of $(\Gamma; \Delta)$ is any propositional variable or a formula of the type $\Box\theta$, which occurs in a formula from $\Gamma \cup \Delta$, not in the scope of $\Box$. Thus, every formula form $\Gamma \cup \Delta$ is built from atoms, without using $\Box$.

Let $T = (x; T_1, \ldots, T_k)$ be a $\mathbf{K}$–tree of length $l(\Gamma, \Delta)$, at the root x of which $(\Gamma; \Delta)$ is refuted under a certain valuation V. Let $p_1, \ldots, p_m, \Box\varphi_1, \ldots, \Box\varphi_n$ be the different atoms of $(\Gamma; \Delta)$ which are true at x under V, and $q_1, \ldots, q_r, \Box\psi_1, \ldots, \Box\psi_s$ be those different atoms of $(\Gamma; \Delta)$ which are false there. Then it is enough to infer

$$(**) \qquad \{p_1, \ldots, p_m, \Box\varphi_1, \ldots, \Box\varphi_n\} \dashv \{q_1, \ldots, q_r, \Box\psi_1, \ldots, \Box\psi_s\},$$

since $\Gamma \dashv \Delta$ follows from (**) in $\mathbf{CSC}_{\dashv}$ due to the completeness of the latter. Indeed, substituting the occurrences of $\Box\varphi_1, \ldots, \Box\varphi_n, \Box\psi_1, \ldots, \Box\psi_s$ in (Γ, Δ) by corresponding different variables $x_1, \ldots, x_n, y_1, \ldots, y_s$, distinct from $p_1, \ldots, p_m, q_1, \ldots, q_r$, we obtain a sequent (Γ', Δ') refuted by the valuation V' which makes $p_1, \ldots, p_m, x_1, \ldots, x_n$ true and $q_1, \ldots, q_r, y_1, \ldots, y_s$ false. Then, by the proof of Theorem 2.1, there is an inference of $\Gamma', p_1, \ldots, p_m, x_1, \ldots, x_n \dashv \Delta', q_1, \ldots, q_r, y_1, \ldots, y_s$, and therefore of $\Gamma' \dashv \Delta'$ in $\mathbf{CSC}_{\dashv}$. At that, the only axiom involved in the inference of $\Gamma' \dashv \Delta'$ is $\{p_1, \ldots, p_m, x_1, \ldots, x_n\} \dashv \{q_1, \ldots, q_r, y_1, \ldots, y_s\}$. Now, substituting in that inference $\Box\varphi_1, \ldots, \Box\varphi_n, \Box\psi_1, \ldots, \Box\psi_s$ for $x_1, \ldots, x_n, y_1, \ldots, y_s$ respectively we obtain an inference of $\Gamma \dashv \Delta$.

Clearly, $\{p_1, \ldots, p_m\} \cap \{q_1, \ldots, q_r\} = \emptyset$, hence

$$(1) \qquad \{p_1, \ldots, p_m\} \dashv \{q_1, \ldots, q_r\}$$

is an axiom of $\mathbf{CSC}_{\dashv}$. Also

$$(2) \qquad \{\Box\varphi_1, \ldots, \Box\varphi_n\} \dashv$$

is an axiom of $\mathbf{SK}_{\dashv}$.

Now, if $l(\Gamma, \Delta) = 0$, then $(\Gamma; \Delta)$ is refuted in a trivial $\mathbf{K}$–tree and (**) must be of the kind

$$\{p_1, \ldots, p_m, \Box\varphi_1, \ldots, \Box\varphi_n\} \dashv \{q_1, \ldots, q_r\}$$

which is immediately inferred from (1) and (2) applying $MIX^1_{\mathbf{K}}$.

Let $l(\Gamma, \Delta) > 0$. Then each of the sequents

$$(\{\varphi_1, \ldots, \varphi_n\}; \{\psi_i\}), \quad i = 1, \ldots, s,$$

is refuted at the root of some immediate subtree T_{j_i} of T, of length $l(\Gamma, \Delta) - 1$. Therefore, by the inductive hypothesis,

$$(3) \qquad \{\varphi_1, \ldots, \varphi_n\} \dashv \{\psi_i\}, \quad i = 1, \ldots, s,$$

are inferred in $\mathbf{SK}_{\dashv}$.

Starting with (2) and applying $MIX^2_{\mathbf{K}}$ with (3) for $i = 1, \ldots, s$ we successively obtain

$$(4) \qquad \begin{array}{c} \{\Box\varphi_1, \ldots, \Box\varphi_n\} \dashv \{\Box\psi_1\} \\ \cdots \\ \{\Box\varphi_1, \ldots, \Box\varphi_n\} \dashv \{\Box\psi_1, \ldots, \Box\psi_s\}. \end{array}$$

Now, it remains to apply $MIX^1_{\mathbf{K}}$ to (1) and (4):

$$\frac{\{p_1, \ldots, p_m\} \dashv \{q_1, \ldots, q_r\}, \ \{\Box\varphi_1, \ldots, \Box\varphi_n\} \dashv \{\Box\psi_1, \ldots, \Box\psi_s\}}{\{p_1, \ldots, p_m, \Box\varphi_1, \ldots, \Box\varphi_n\} \dashv \{q_1, \ldots, q_r, \Box\psi_1, \ldots, \Box\psi_s\}}$$

and thus (**) is inferred and the proof is completed. ■

Appropriate simplifications of the above argument prove the following results:

THEOREM 2.3.

1) The extension of $\mathbf{CSC}_{\dashv}$ *with the axiom schema*

$$\Gamma_0, \Box\Gamma \dashv \Delta_0,$$

where $\Gamma_0 \dashv \Delta_0$ *is an axiom of* $\mathbf{CSC}_{\dashv}$*, is L–complete for* $\mathbf{K} + \Box\bot$.

2) The extension of $\mathbf{CSC}_{\dashv}$ *with the rule*

$$\frac{\Gamma_1, \Gamma \dashv \Delta_1, \Delta}{\Gamma_1, \Box\Gamma \dashv \Delta_1, \Box\Delta}$$

is L–complete for $\mathbf{K} + p \leftrightarrow \Box p$.

3) The extension of $\mathbf{CSC}_{\dashv}$ *with the rule*

$$\frac{\Gamma_0 \dashv \Delta_0, \ \Gamma \dashv \Delta}{\Gamma_0, \Box\Gamma \dashv \Delta_0, \Box\Delta}$$

where $\Gamma_0 \dashv \Delta_0$ *is an axiom of* $\mathbf{CSC}_{\dashv}$*, is L–complete for* $\mathbf{K} + \Diamond p \leftrightarrow \Box p$.

4) The previous system, extended with the axiom schema

(Ax$_{\mathbf{K}}$*)* $\qquad \Box\Gamma \dashv,$

is L–complete for $\mathbf{K} + \Diamond p \rightarrow \Box p$. ■

We finish this section with a sequential refutation system $\mathbf{SKW}_{\dashv}$ for the logic **KW** of provability in PA. $\mathbf{SKW}_{\dashv}$ is an extension of $\mathbf{CSC}_{\dashv}$ with:

Axioms: the axioms of $\mathbf{SK}_{\dashv}$,

$(Ax_{\mathbf{K}})$ $\quad \Box\Gamma \dashv$ $\quad$ for any finite set Γ.

Rules:

$$MIX^1_{\mathbf{K}}: \quad \frac{\Gamma_0 \dashv \Delta_0, \ \Box\Gamma \dashv \Box\Delta}{\Gamma_0, \Box\Gamma \dashv \Delta_0, \Box\Delta}$$

where $\Gamma_0 \dashv \Delta_0$ is an axiom of $\mathbf{CSC}_{\dashv}$,

$$MIX_{\mathbf{KW}}: \quad \frac{\Box\Gamma \dashv \Box\Delta, \ \Gamma, \Box\Gamma \dashv \psi}{\Box\Gamma \dashv \Box\Delta, \Box\psi}$$

THEOREM 2.4. $\mathbf{SKW}_{\dashv}$ *is L–complete for* **KW**.

PROOF. A slight modification of the proof of Th.2.2, accordingly using the semantic characterization of **KW** with the class of **KW**–trees. The only changes are that we replace the sequents (3) by

$$(3') \qquad \{\varphi_1, \ldots, \varphi_n, \Box\varphi_1, \ldots, \Box\varphi_n\} \dashv \{\psi_i\}, \ i = 1, \ldots, s,$$

and then starting from $\{\Box\varphi_1, \ldots, \Box\varphi_n\} \dashv$ and applying successively $MIX_{\mathbf{KW}}$ with (3′), for ψ being $\psi_1, \ldots, \psi_s$, we obtain (4). ■

3. Concluding remarks

As Lukasiewicz points out, a reason for the inequality between logical acceptance and rejection is that, while only universally valid arguments are accepted (inferred) in a logical theory, not only universally false ones are rejected, but all which are not universally valid. Thus, in any particular case of logical reasoning, the accepted argument yields a true statement, while nothing such can be said in general about a rejected arguments and therefore the latter are useless as logical schemata.

Besides this philosophically motivated reason, there are even more serious technical reasons which make the rejected arguments unhandy. It is often the case that a certain theory is recursively axiomatizable but not decidable, hence the set of its rejected statements is not recursively enumerable and thus not capturable by any practically reasonable deductive system. Moreover, even in the case when the theory is decidable, but has no suitable enough semantic characterization (say, has no finite model property) it may be extremely difficult to find a decent refutation system for it. So, the tradition in the modern formal logic, the valid statements to be produced syntactically while the refutable ones to be distinguished semantically by refuting models, is not accidental.

On the other hand, there are serious arguments in favour of refutation systems. Let us remind that refutation systems employing the rule modus tollens are actually combined deductive systems for both operators of provability (acceptance) and refutability (rejection) in the same theory. Once we have tolerated such systems, it is natural to admit in them not only (combined) refutation rules, but also combined "acceptance" rules of the kind:

$$\frac{\vdash \varphi_1, \ldots, \vdash \varphi_k, \dashv \psi_1, \ldots, \dashv \psi_n}{\vdash \varphi}.$$

Such deductive systems have a greater potential efficiency than the orthodox ones, since they can employ on a syntactical level self–reference to some of their meta–features, which are beyond the expressive abilities of the traditional systems. Trivial examples are:

consistency, expressed by the axiom $\dashv \bot$ or by the rule of inference

$$\frac{\vdash \varphi}{\dashv \neg\varphi};$$

completeness, expressed by the natural_deduction–like rules

$$\frac{\begin{array}{cc} \dashv \varphi & \vdash \varphi \\ \vdots & \vdots \\ \vdash \psi & \vdash \psi \end{array}}{\vdash \psi} \quad \text{and} \quad \frac{\begin{array}{cc} \dashv \varphi & \vdash \varphi \\ \vdots & \vdots \\ \dashv \psi & \dashv \psi \end{array}}{\dashv \psi}.$$

A more specific example is the disjunction property, mentioned in the introduction, which holds for the intuitionistic logic and some of its extensions. It cannot be expressed as an ordinary rule of inference, but can be expressed as a refutation rule:

$$\frac{\dashv \varphi, \dashv \psi}{\dashv \varphi \vee \psi}.$$

The last example comes from the non–monotonic logics. The default rule

$$\frac{\alpha(x) : M\beta_1(x), \ldots, M\beta_n(x)}{\gamma(x)}$$

can be formalized as a combined rule

$$\frac{\vdash \alpha(x), \dashv \Diamond\beta_1(x), \ldots, \dashv \Diamond\beta_n(x)}{\gamma(x)}.$$

Another merit of combined deductive systems is that (when L–complete) they render a decision procedure, at that purely syntactical, which might not be achieved by other conventional means. For discussion and results on this topic see [14].

Finally, it seems that combined deductive systems could shed a new light on the problem of "negation by failure" in logic programming and on the theory of automated deduction in general.

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References

[1] J. F. A. K. VAN BENTHEM, **Modal logic and classical logic**, Bibliopolis, Napoli, 1986.

[2] G. BRYLL and M. MADUCH, *Rejected axioms for Łukasiewicz's many-valued logics,*(in Polish), Zeszyty Naukowe Wyższej Szkoły Pedagogicznej w Opolu, **Mathematyka VI, Logika i algebra** (1969), pp. 3 – 19.

[3] R. DUTKIEWICZ, *The method of axiomatic rejection for the intuitionistic propositional logic,***Studia Logica, XLVIII**, 4, (1989), pp. 449 – 460.

[4] G. G. HUGHES, M. J. CRESSWELL, **A companion to modal logic**, Methuen, London, 1984.

[5] T. INOUE, *On Ishimoto's theorem in axiomatic rejection — the philosophy of unprovability* (in Japanese), **Philosophy of Science, 22** (Waseda University press, Tokyo, 1989), pp. 77 – 93.

[6] T. INOUE, A. ISHIMOTO and M. KOBAYASHI, *Axiomatic rejection for the propositional fragment of Leśniewski's ontology,* Soviet–Japan Symposium "Leśniewski's ontology and its applications", Moscow 1990.

[7] G. KREISEL and H. PUTNAM, *Unableitbarkeitsbeweismethode für den intuitionistischen Aussagenkalkul,* **Archiv Math. Logik Grundlagenforsch**, 3,(1957), pp. 74 – 78.

[8] J. ŁUKASIEWICZ, *On the intuitionistic theory of deduction,* **Indag. Math.**, 14,(1952), pp. 202 – 212.

[9] J. ŁUKASIEWICZ, **Aristotle's syllogistic from the standpoint of modern formal logic**, Oxford 1957.

[10] D. SCOTT, *Completeness proofs for the intuitionistic sentential calculus,* Summaries of talks presented at the Summer Institute for Symbolic Logic, Inst. for defense Analyses, Cornell University 1957, 2nd. ed., Communications Research division, Princeton 1960, pp. 231 – 241.

[11] K. SEGERBERG, **An essay in classical modal logic**, Filosofiska Studier 13, Uppsala, 1971.

[12] T. SKURA, *A complete syntactical characterization of the intuitionistic logic,* **Reports on mathematical logic, 23** (1989), pp. 75 – 80.

[13] T. SKURA, *On pure refutation formulations of sentential logics,* **Bull. of Sect. of Logic**, PAS, vol. 19, 3 (1990), pp. 102 – 107.

[14] T. SKURA, *On decision procedures for sentential logics,* **Studia Logica, L,** 2 (1991),pp. 173 – 179.

[15] T. SKURA, *Refutation rules for three modal logics,* **Bull. of Sect. of Logic, vol. 21**,1 (1992), pp. 31 – 32.

[16] J. SŁUPECKI and G. BRYLL, *Proof of L–decidability of Lewis system* **S5**, **Studia Logica, XXXII** (1973), pp. 99 – 105.

[17] J. SŁUPECKI, G. BRYLL and U. WYBRANIEC–SKARDOWSKA, *Theory of rejected propositions, I,* **Studia Logica XXIX** (1971), pp. 75 – 115; *II,* **Studia Logica, XXX** (1972), pp. 97 – 115.

[18] W. STASZEK, *On proofs of rejection,* **Studia Logica, XXIX** (1971), pp. 17 – 23.

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